Excluding the questions on past exams which ask to reproduce derivations, Question 5 from the 2019 Semester 2 exam was one of the most difficult (in my opinion). Here is a solution.

> Find a solution of the diffusion equation
>
> $$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$$
>
> on the infinite domain $-\infty < x < +\infty$ of the following form
>
> $$u(x,t) = A(t)e^{-\left(\frac{x}{w(t)}\right)^2}$$
>
> by determining the unknown functions $A(t)$ and $w(t)$.

We abbreviate $A = A(t)$ and $w = w(t)$. Let $H = -x^2/w^2$, so $u(x,t) = Ae^H$. We have

$$\begin{aligned}
H_x &= \frac{-2x}{w^2} = \frac{2H}{x} \\
H_{xx} &= \frac{-2}{w^2} = \frac{2H}{x^2} \\
H_t &= -x^2 \cdot -2w^{-3} \cdot w' = \frac{2x^2 w'}{w^3} = \frac{-2w'}{w}H.
\end{aligned}$$

Since we want to solve $u_t = Du_{xx}$, we find those derivatives:

$$\begin{aligned}
u_t &= \frac{d}{dt}\left(Ae^H\right) \\
&= A'e^H + AH_t e^H \\
&= e^H(A' + AH_t) \\
&= e^H\left(A' + A\cdot\frac{-2w'}{w}H\right).
\end{aligned}$$

Next,

$$\begin{aligned}
u_{xx} &= A\frac{\partial^2}{\partial x^2}(e^H) \\
&= A\frac{\partial}{\partial x}\left(H_x e^H\right) \\
&= A\left(H_{xx}e^H + H_x H_x e^H\right) \\
&= Ae^H\left(\frac{2H}{x^2} + \left(\frac{2H}{x}\right)^2\right) \\
&= Ae^H\left(\frac{2H}{x^2} + \frac{4H^2}{x^2}\right) \\
&= Ae^H \cdot \frac{2H + 4H^2}{x^2}.
\end{aligned}$$

Now, we substitute into the PDE.

$$\begin{aligned}
u_t &= Du_{xx} \\
e^H\left(A' + A \cdot \frac{-2w'}{w}H\right) &= DAe^H \cdot \frac{2H + 4H^2}{x^2} \\
A' + A \cdot \frac{-2w' \cdot -x^2}{w^3} &= DA \cdot \frac{2H}{x^2} + DA \cdot \frac{4H^2}{x^2} \\
A' + \frac{2x^2 w'}{w^3}A &= DA \cdot \frac{2 \cdot -x^2}{w^2 \cdot x^2} + DA \cdot 4 \cdot \frac{x^4}{w^4} \cdot \frac{1}{x^2} \\
A' + \frac{2x^2 w'}{w^3}A &= \frac{-2}{w^2}DA + \frac{4x^2}{w^4}DA.
\end{aligned}$$

From here, we express this as a polynomial in x:

$$\underbrace{\left(A' + \frac{2}{w^2}DA\right)}_{(1)} + \underbrace{\left(\frac{-4}{w^4}DA + \frac{2w'}{w^3}A\right)}_{(2)} x^2 = 0,$$

so (1) and (2) must vanish *separately*. Starting with (2),

$$\begin{aligned}
\frac{-4}{w^4}DA + \frac{2w'}{w^3}A &= 0 \\
\frac{A}{w^4}\left(-4D + 2w'w\right) &= 0.
\end{aligned}$$

Seeking a non-trivial solution, assume $A \neq 0$ and $w^4 \neq 0$. Then,

$$\begin{aligned}
-4D + 2w'w &= 0 \\
4D &= 2\frac{dw}{dt}w \\
2D &= \frac{dw}{dt}w \\
\int 2D\,dt &= \int w\,dw \\
2Dt + k_1 &= \frac{1}{2}w^2 \\
w^2 &= 4Dt + k_1 \\
w &= \pm\sqrt{4Dt + k_1},\ k_1 \in \mathbb{R}.
\end{aligned}$$

Then, from (1),

$$\begin{aligned}
A' + \frac{2}{w^2}DA &= 0 \\
A' + \frac{2}{4Dt + k_1}DA &= 0 \\
\frac{dA}{dt} &= \frac{-2DA}{4Dt + k_1} \\
\int \frac{1}{A}\,dA &= \int \frac{-2}{4Dt + k_1}\,dt \\
\ln|A| &= \frac{-2D}{4D}\ln|4Dt + k_1| + c \\
\ln|A| &= \frac{-1}{2}\ln|4Dt + k_1| + c \\
A &= (4Dt + k_1)^{\frac{-1}{2}}\,k_2 \\
&= \frac{k_2}{\sqrt{4Dt + k_1}}.
\end{aligned}$$

Thus,

$$u(x,t) = A(t)e^{\frac{-x^2}{[w(t)]^2}} = \frac{k_2}{\sqrt{4Dt + k_1}}\exp\left(\frac{-x^2}{4Dt + k_1}\right),$$

for $k_1, k_2 \in \mathbb{R}$.