

## Coupled ODEs

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{r}$$

$$\ddot{y} + p(t)y + q(t) = r(t) \longrightarrow \begin{pmatrix} y \\ \dot{y} \end{pmatrix} = \dot{\mathbf{y}} = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix} \mathbf{y} + \begin{pmatrix} 0 \\ r(t) \end{pmatrix}$$

The eigenvalues of  $\mathbf{A}$  found by solving  $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$  gives the same characteristic equation for the uncoupled system. General solutions for a second-order system with real constant coefficients, with eigenvalues  $\lambda_1, \lambda_2$  and eigenvectors  $\mathbf{x}_1, \mathbf{x}_2$ , are given by:

$$\lambda_1 \neq \lambda_2: \mathbf{y} = c_1 \mathbf{x}_1 e^{\lambda_1 t} + c_2 \mathbf{x}_2 e^{\lambda_2 t}$$

$$\lambda_1, \lambda_2 \in \mathbb{C}: \mathbf{y} = c_1 \operatorname{Re}(\mathbf{x}_1 e^{\lambda_1 t}) + c_2 \operatorname{Im}(\mathbf{x}_1 e^{\lambda_1 t})$$

$$\lambda_1 = \lambda_2: \mathbf{y} = c_1 \mathbf{x} e^{\lambda t} + c_2 (t\mathbf{x} + \mathbf{p}) e^{\lambda t}, \text{ where } (\mathbf{A} - \lambda\mathbf{I})\mathbf{p} = \mathbf{x}.$$

## Phase Portraits of 2D Autonomous Systems

Let  $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$  with  $\mathbf{y} \in \mathbb{R}^2$ ,  $\mathbf{f}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ , where  $\mathbf{f}$  does not depend on  $t$ .  $\mathbf{f}$  defines a direction field in the  $y_1$ - $y_2$  plane, and a trajectory of the system is a curve following the field.  $\mathbf{f}$  being continuous with continuous partials implies existence and uniqueness for these trajectories, so they do not cross. By the chain rule, the slope of a trajectory is

$$\frac{dy_2}{dy_1} = \frac{\dot{y}_2}{\dot{y}_1} = \frac{f_2}{f_1}.$$

A nullcline is a curve in phase space where the slope of the trajectory is either 0 or  $\infty$ . Set  $\dot{y}_2 = 0$  for horizontal nullclines and  $\dot{y}_1 = 0$  for vertical nullclines, and solve for  $y_2$ . Solving for the trajectories themselves is possible in specific cases, usually when the ODE in  $y_2$  from the slope formula is separable.

For the lines in phase space in the direction of the eigenvectors (called eigendirections), there are straight line solutions either flowing towards or away from the origin, determined by the sign of the corresponding eigenvalue (– stable, + unstable).

$\lambda_1 \neq \lambda_2$  and both have the same sign:

- Negative  $\lambda$  gives a *stable improper node*, where trajectories curve like a parabola and approach the eigendirection with smallest absolute value, since that exponential term dominates at large  $t$ .
- Positive  $\lambda$  gives an *unstable improper node*, where trajectories approach eigendirection with largest absolute value.

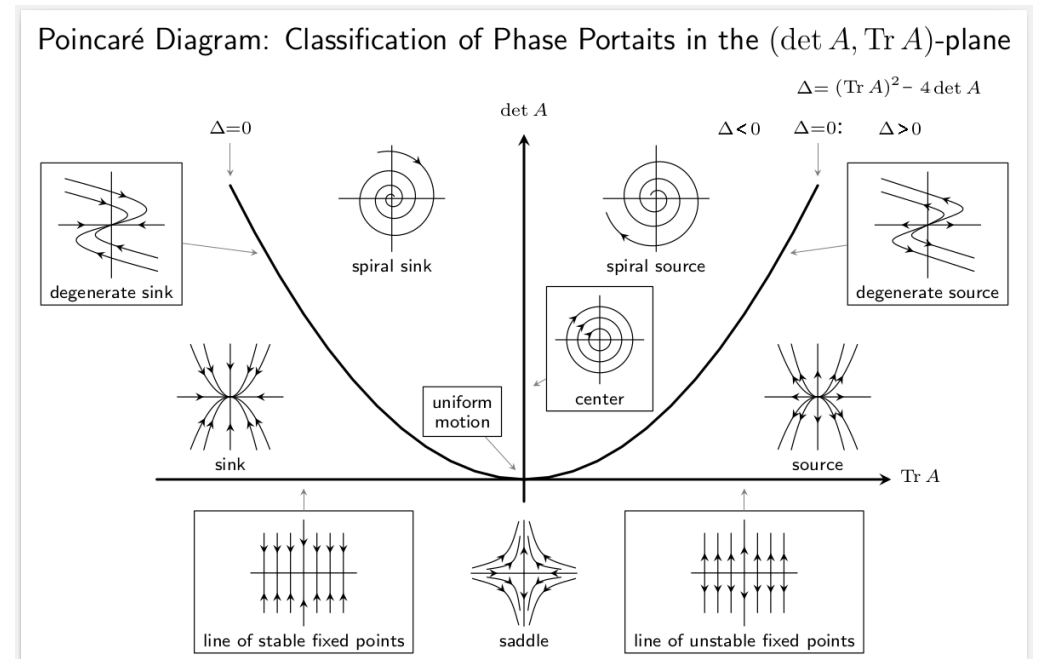
$\lambda_1 > 0, \lambda_2 < 0$ :

- This gives a *saddle point*. Trajectories come in parallel to the stable eigendirection then turn around and become parallel to the unstable direction.

$\lambda_1 = \lambda_2$ :

- If there are 2 linearly independent eigenvectors (eigenspace of  $\lambda$  has dimension 2), all trajectories travel in straight lines either towards or away from the origin, called a *proper node*. For a 2D system, this is only possible if  $\mathbf{A}$  is some multiple of the identity matrix, i.e.  $\mathbf{A} = k\mathbf{I}$  for  $k \in \mathbb{R}$ .
  - If there is only 1 eigenvector, trajectories come tangent to the eigendirection near the origin and do a U-turn away, called a *degenerate node*.
- $\lambda = \alpha \pm \beta i$ :
- If  $\alpha = 0$ , trajectories are ellipses around the origin, and the origin is called a *centre*. To determine the direction of flow, consider the sign of  $\dot{y}_1$  at a point in the positive quadrant.
  - If  $\alpha \neq 0$ , trajectories trace out a logarithmic spiral  $r(t) = e^{\alpha t}$  shown by converting to polar coordinates. This is called a *spiral*. Plotting nullclines helps with sketching trajectories.

The following stability chart allows for the system to be classified given values of  $\operatorname{tr} \mathbf{A}$  and  $\det \mathbf{A}$  [2]. The discriminant  $\Delta$  is defined as  $\Delta = (\operatorname{tr} \mathbf{A})^2 - 4 \det \mathbf{A}$ .



## Non-Linear or Inhomogeneous Systems

A system  $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$  has a critical point at  $\mathbf{y}^*$  if  $\mathbf{f}(\mathbf{y}^*) = \mathbf{0}$ . Linear systems always have one at  $\mathbf{0}$ , and only one unless  $\det \mathbf{A} = 0$ . An inhomogeneous linear system  $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{p}$  is a linear translation away from a homogeneous system. Shifting the critical point at  $\mathbf{p}$  to  $\mathbf{0}$  via the change of variables  $\mathbf{x} = \mathbf{y} - \mathbf{p}$  gives the homogeneous  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ , so techniques discussed earlier are applicable.

For a nonlinear system  $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$ , the nature of the system around the critical point is approximated via the Jacobian matrix  $\mathbf{Df}$ . The stability diagram can be used to classify these critical points by evaluating  $\mathbf{Df}$  at each  $\mathbf{y}^*$ . The linearised system near  $\mathbf{y}^*$  is then

$$\dot{\mathbf{y}} = \mathbf{Df}|_{\mathbf{y}^*}(\mathbf{y} - \mathbf{y}^*), \quad \mathbf{Df} = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{pmatrix}.$$

## Special Functions

The *Dirac Delta function*  $\delta(t)$  is an “impulse” with infinite value at  $t = 0$  and zero value everywhere else, satisfying  $\int_0^\infty \delta(t) dt = 1$ .  $\delta(t - a)$  represents an impulse at  $t = a$ .

The *unit step function*  $u(t)$  is defined by

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0. \end{cases}$$

To write a piecewise-continuous function in terms of  $u$ , if  $f(t)$  changes from  $g_1(t)$  to  $g_2(t)$  at  $t = c$ , add a  $u(t - c)(g_2(t) - g_1(t))$  term to the new expression of  $f$  [1].

The *Gamma function*  $\Gamma(z)$  generalises the factorial by  $\Gamma(n + 1) = n!$  for  $n \in \mathbb{N}$ , and is defined for all positive  $z$  by

$$\Gamma(z) = \int_0^\infty e^{-x} x^{z-1} dx,$$

In this course the *error function* is defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du.$$

It is odd and monotone increasing. The following formulas may be useful for computation:

$$\int_0^\infty e^{-u^2} du = \int_{-\infty}^0 e^{-u^2} du = \frac{\sqrt{\pi}}{2}, \quad \frac{1}{\sqrt{\pi}} \int_a^b e^{-u^2} du = \operatorname{erf}(b) - \operatorname{erf}(a).$$

For the heat equation with parameter  $c$ , the *Gaussian function*  $G(x, t)$  is defined as

$$G(x, t) = \frac{1}{\sqrt{4\pi c^2 t}} \exp\left(\frac{-x^2}{4c^2 t}\right).$$

$\int_{-\infty}^\infty G(x, t) dx = 1$  for  $t > 0$ ,  $\lim_{t \rightarrow 0^+} G(x, t) = 0$  for  $x \neq 0$  and  $\lim_{t \rightarrow 0^+} G(0, t) = +\infty$ .

## Laplace Transforms

The following formulas are given in the formula sheet for the exam, but have been added here for completeness.

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st} dt$$

$f(t)$	$F(s)$	$f(t)$	$F(s)$
$c$	$\frac{c}{s}$	$ag(t) + bh(t)$	$aG(s) + bH(s)$
$t^n$	$\frac{n!}{s^{n+1}}$	$e^{at}f(t)$	$F(s - a)$
$t^a$	$\frac{\Gamma(a+1)}{s^{a+1}}$	$f(t - k)u(t - k)$	$e^{-ks}F(s)$
$\cos \alpha t$	$\frac{s}{s^2 + \alpha^2}$	$f'(t)$	$sF(s) - f(0)$
$\sin \alpha t$	$\frac{\alpha}{s^2 + \alpha^2}$	$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$\delta(t - k)$	$e^{-ks}$	$tf(t)$	$-F'(s)$

The convolution theorem states that:

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t f(\tau)g(t - \tau) d\tau = (f * g)(t).$$

If  $f(t)$  has period  $p$ , then the Laplace integral can be simplified:

$$\mathcal{L}\{f(t)\} = \frac{1}{1 - e^{-sp}} \int_0^p f(t)e^{-st} dt.$$

The following formulas are not included in the formula sheet, but may be helpful to know.

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{s}{(s^2 + \alpha^2)^2}\right\} &= \frac{1}{2\alpha} t \sin \alpha t \\ \mathcal{L}\{f(t)u(t - k)\} &= e^{-ks} \mathcal{L}\{f(t + k)\} \\ \mathcal{L}\{f^{(n)}(t)\} &= s^n F(s) - \sum_{k=1}^n s^{n-k} f^{(k-1)}(0) \\ \mathcal{L}\{t^n f(t)\} &= (-1)^n F^{(n)}(s). \end{aligned} \tag{1}$$

## Differential Operators and Change of Variables

A homogeneous linear differential equation has the form  $L(u) = 0$  where  $L$  is a *linear differential operator* satisfying  $L(u + v) = L(u) + L(v)$  and  $L(cu) = cL(u)$  for  $c \in \mathbb{R}$  and functions  $u, v$ . This can be used to verify a DE is linear using the linearity of the derivative. A common differential operator is the del operator  $\nabla$ , defined in 3 dimensions by  $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$ . It can be applied to a scalar field  $f$  to get  $f$ 's gradient field, or be dotted or crossed with a vector field  $\mathbf{v}$  to get its *divergence* or *curl*:

$$\text{grad } f = \nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right), \quad \text{div } \mathbf{v} = \nabla \cdot \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z},$$

$$\text{curl } \mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Additionally, the *Laplacian* is defined as  $\nabla^2 = \nabla \cdot \nabla$ . For scalar fields,

$$\nabla^2 f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2},$$

and for a vector field  $\mathbf{v}$  in cartesian coordinates, the Laplacian is just defined elementwise, so  $\nabla^2 \mathbf{v} = (\nabla^2 v_1, \nabla^2 v_2, \nabla^2 v_3)$ . Equivalently, it can be defined as

$$\nabla^2 \mathbf{v} = \nabla(\nabla \cdot \mathbf{v}) - \nabla \times (\nabla \times \mathbf{v}).$$

The above formulas only apply for *cartesian coordinates*; other coordinate systems have different expressions. Importantly, the Laplacian in polar coordinates is given by

$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}.$$

Changing variables that a PDE is expressed in may simplify the equation and give different types of solutions not apparent from the initial coordinate system. The goal is to express derivatives in the *initial* coordinates in terms of derivatives in the *new* coordinates. If  $f(x, y)$  is changed to  $f(u, v)$  via  $u(x, y)$  and  $v(x, y)$ , then by the chain rule,

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial x},$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial y}.$$

Note that  $f$  can be removed from the above equations to get a relationship purely involving the differential operators  $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial u}$  and  $\frac{\partial}{\partial v}$ . This can then be substituted into the original PDE to get a new equation.

An example is with the PDE  $u_t + cu_x = 0$  for  $u(x, t)$ . Changing variables to  $x' = x - ct$  and  $t' = t$  allows the equation to be written as  $u_{t'} = 0$ , thus  $u(x', t') = f(x')$  and  $u(x, t) = f(x - ct)$  for differentiable  $f$ . The solution is defined on a plane in  $x$ - $t$  space.

## Fourier Series

If a function  $f(x)$  has period  $2L$ , i.e.  $f(x + 2L) = f(x)$  for all  $x$ , then it can be written as a *Fourier series*:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right),$$

with Fourier coefficients given by

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx,$$

$$a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx.$$

In the case that  $f$  is *even* ( $f(-x) = f(x)$ ) or *odd* ( $f(-x) = -f(x)$ ), the formulas for the coefficients can be simplified, only changing the fraction and integral bounds:

$$a_0 = \frac{1}{L} \int_0^L (\dots) dx, \quad a_n = \frac{2}{L} \int_0^L (\dots) dx, \quad b_n = \frac{2}{L} \int_0^L (\dots) dx.$$

Odd  $f$  makes  $a_n = 0$  for all  $n$ , and even  $f$  makes  $b_n = 0$  for all  $n$ .

These simplified formulas can also be used when extending  $f$  only defined on  $[0, L]$  to be odd or even periodic over period  $2L$ . This is called a *half-range expansion* of  $f$ ; odd periodic extension makes  $a_n = 0$  and even period extension makes  $b_n = 0$  for all  $n$ .

## References

- [1] Andrew D. Loveless. *Laplace Transform Practice Problems*. 2019. URL: <https://sites.math.washington.edu/~aloveles/Math307Fall2019/m307LaplacePractice.pdf>.
- [2] Freesodas. *Stability Diagram*. 2018. URL: [https://commons.wikimedia.org/wiki/File:Stability\\_Diagram.png](https://commons.wikimedia.org/wiki/File:Stability_Diagram.png).

