Differential Equations

$$y' + a(x)y = b(x) \longrightarrow h(x) = e^{\int a(x)\,\mathrm{d}x}$$

$$\begin{aligned} \text{2 distinct } \lambda\text{:} \quad & y_H = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ \text{1 repeated } \lambda\text{:} \quad & y_H = c_1 x e^{\lambda x} + c_2 e^{\lambda x} \\ \lambda = \alpha \pm \omega i\text{:} \quad & y_H = e^{\alpha x}\left(c_1 \cos \omega x + c_2 \sin \omega x\right). \end{aligned}$$

$$\begin{aligned} k e^{\alpha x} &\longrightarrow c e^{\alpha x} \\ k x^n &\longrightarrow \sum_{i=0}^{n} c_i x^i \\ k \cos \alpha x \text{ or } k \sin \alpha x &\longrightarrow c_1 \cos \alpha x + c_2 \sin \alpha x \\ (\cdots)\, e^{\alpha x} &\longrightarrow (\cdots)\, e^{\alpha x} \end{aligned}$$

$$W(y_1, y_2)(x) = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = y_1 y_2' - y_2 y_1'$$

$$r = y'' + py' + qy \qquad y_P = u y_1 + v y_2$$

$$u = -\int \frac{y_2 r}{W}\,\mathrm{d}x \qquad v = \int \frac{y_1 r}{W}\,\mathrm{d}x$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh ix = \cos x \qquad \sinh ix = i \sin x$$

Projections and Orthonormal Bases

$$P_{\beta' \to \beta''} P_{\beta \to \beta'} = P_{\beta \to \beta''}$$

$$\begin{aligned} \|\mathbf{v}\| &= \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} \\ d(\mathbf{u}, \mathbf{v}) &= \|\mathbf{u} - \mathbf{v}\| \\ |\langle \mathbf{u}, \mathbf{v} \rangle| &\le \|\mathbf{u}\|\,\|\mathbf{v}\| \\ \|\mathbf{u} + \mathbf{v}\| &\le \|\mathbf{u}\| + \|\mathbf{v}\| \end{aligned}$$

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \iff \langle \mathbf{u}, \mathbf{v} \rangle = 0$$

$$\begin{aligned} U^{\perp} &= \{\mathbf{v} \in V \mid \langle \mathbf{v}, \mathbf{u} \rangle = 0\ \forall \mathbf{u} \in U\} \\ \mathrm{Proj}_U(\mathbf{v}) &= \langle \mathbf{v_1}, \hat{e}_1 \rangle \hat{e}_1 + \cdots + \langle \mathbf{v}, \hat{e}_k \rangle \hat{e}_k \\ \mathrm{Proj}_{U\perp}(\mathbf{v}) &= \mathbf{v} - \mathrm{Proj}_U(\mathbf{v}) \end{aligned}$$

$$A\mathbf{x} = \mathbf{b} \longrightarrow A^{\mathrm{T}} A \mathbf{x} = A^{\mathrm{T}} \mathbf{b} \longrightarrow \mathbf{x} = \left(A^{\mathrm{T}} A\right)^{-1} A^{\mathrm{T}} \mathbf{b}$$

Matrix-Related Computation

$$A\mathbf{x} = \lambda \mathbf{x} \quad \det(A - \lambda I) = 0 \quad (A - \lambda I)\mathbf{x} = \mathbf{0}$$

$AP = PD$, where $A = PDP^{-1}$ for diagonalisation and $A = PDP^{\mathrm{T}}$ for orthogonal diagonalisation.

$$P = \begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} & \cdots \end{bmatrix} \quad D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{pmatrix}$$

P's vectors need to be *orthonormal* when orthogonally diagonalising.

$$ax^2 + by^2 + cxy \longrightarrow \begin{pmatrix} a & c/2 \\ c/2 & b \end{pmatrix}$$

$$\begin{gathered} ax^2 + by^2 + cz^2 + dxy \\ + exz + fyz \end{gathered} \longrightarrow \begin{pmatrix} a & d/2 & e/2 \\ d/2 & b & f/2 \\ e/2 & f/2 & c \end{pmatrix}$$

$$\mathbf{x}^{\mathrm{T}} A \mathbf{x} + K\mathbf{x} + c = 0$$

When orthogonally diagonalising coefficient matrix of quadratic form, arrange column vectors of P such that $\det P = +1$.

Taylor Series and Critical Points

$$H_f = \begin{pmatrix} f_{x_1 x_1} & \cdots & f_{x_1 x_n} \\ \vdots & \ddots & \vdots \\ f_{x_n x_1} & \cdots & f_{x_n x_n} \end{pmatrix}$$

$$\begin{aligned} f(\mathbf{x}) \approx{} & f(\mathbf{x_0}) + \left(\boldsymbol{\nabla} f(\mathbf{x_0})\right)^{\mathrm{T}} (\mathbf{x} - \mathbf{x_0}) \\ & + \frac{1}{2}(\mathbf{x} - \mathbf{x_0})^{\mathrm{T}} H_f(\mathbf{x_0})(\mathbf{x} - \mathbf{x_0})^{\mathrm{T}} + \cdots \end{aligned}$$

$$f(\mathbf{x} + \mathbf{h}) \approx \sum_{l=0}^{\infty} \frac{1}{l!} \left(\mathbf{h} \cdot \boldsymbol{\nabla}\right)^l f(\mathbf{x})$$

- Minimum: $\lambda_i > 0\, \forall i$
- Maximum: $\lambda_i < 0\, \forall i$
- Saddle: there are λ with different signs
- Inconclusive: there exist $\lambda_i = 0$ and non-zero λ have same sign

Double and Triple Integrals

For $a, b, c, d \in \mathbb{R}$,

$$\int_a^b \int_c^d f(x) g(y)\,\mathrm{d}y\,\mathrm{d}x = \left(\int_a^b f(x)\,\mathrm{d}x\right)\left(\int_c^d g(y)\,\mathrm{d}y\right)$$

$$\text{area} = \iint_D 1\,\mathrm{d}A \qquad \text{average} = \frac{\iint_D f(x,y)\,\mathrm{d}A}{\iint_D 1\,\mathrm{d}A}$$

$$\text{volume} = \iiint_V 1\,\mathrm{d}V \qquad \text{average} = \frac{\iiint_V f(x,y,z)\,\mathrm{d}V}{\iiint_V 1\,\mathrm{d}v}$$

2D case:

2D case

$$\begin{aligned} m &= \iint_D \rho(x,y)\,\mathrm{d}A \\ M_y &= \iint_D x\rho(x,y)\,\mathrm{d}A \\ M_x &= \iint_D y\rho(x,y)\,\mathrm{d}A \end{aligned}$$

3D case:

$$\begin{aligned} m &= \iiint_V \rho(x,y,z)\,\mathrm{d}V \\ M_{yz} &= \iiint_V x\rho(x,y,z)\,\mathrm{d}V \\ M_{xz} &= \iiint_V y\rho(x,y,z)\,\mathrm{d}V \\ M_{xy} &= \iiint_V z\rho(x,y,z)\,\mathrm{d}V \end{aligned}$$

$$\bar{x} = \frac{M_{yz}}{m} \quad \bar{y} = \frac{M_{xz}}{m} \quad \bar{z} = \frac{M_{xy}}{m}$$

For 2D, $M_{yz} = M_y$, $M_{xz} = M_x$, $M_{xy} = 0$.

Coordinate Systems

Polar: $x = r\cos\theta, y = r\sin\theta$ so $r^2 = x^2 + y^2$ and $|J| = r$.

Cylindrical: same as polar but with $z = z$, $|J| = r$.

Spherical: $x = r\cos\theta\sin\varphi, y = r\sin\theta\sin\varphi, z = r\cos\varphi$ so $r^2 = x^2 + y^2 + z^2$, $x^2 + y^2 = r^2\sin\varphi$ and $|J| = r^2\sin\varphi$. θ is angle along x-y plane, φ is angle from point to z-axis.

$$\text{jacobian} = |J| = \left|\det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}\right|$$

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Vector Calculus

$$\boldsymbol{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$$D_{\hat{\mathbf{u}}}(f) = (\boldsymbol{\nabla} f) \cdot \hat{\mathbf{u}}$$

$$W = \int_C \mathbf{F} \cdot \, \mathrm{d}\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, \mathrm{d}t$$

$$\int_C \boldsymbol{\nabla} f \cdot \, \mathrm{d}\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

$$\iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \, \mathrm{d}A = \oint_{\partial D} \mathbf{F} \cdot \, \mathrm{d}\mathbf{r}$$

$$\text{flux} = \int_C \mathbf{v} \cdot \mathbf{n} \, \mathrm{d}S = \int_a^b \mathbf{v}(\mathbf{r}(t)) \cdot \underbrace{\left(\mathbf{r}'(t) \times \hat{\mathbf{k}}\right)}_{\text{check orientation}} \, \mathrm{d}t$$

$$\operatorname{div} \mathbf{v} = \boldsymbol{\nabla} \cdot \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

$$\oint_{\partial D} \mathbf{v}(x, y) \cdot \mathbf{n} \, \mathrm{d}S = \iint_D \boldsymbol{\nabla} \cdot \mathbf{v}(x, y) \, \mathrm{d}A$$

$$(\mathbf{r}_{\mathbf{u}}(a, b) \times \mathbf{r}_{\mathbf{v}}(a, b)) \cdot \left(\begin{pmatrix} x & y & z \end{pmatrix} - \mathbf{r}(a, b)\right) = 0$$

$$\iint_S f(x, y, z) \, \mathrm{d}S = \iint_D f(\mathbf{r}(u, v)) \left\| \mathbf{r}_{\mathbf{u}} \times \mathbf{r}_{\mathbf{v}} \right\| \, \mathrm{d}A$$

$$\text{flux} = \iint_S \mathbf{v} \cdot \mathbf{n} \, \mathrm{d}S = \iint_D \mathbf{v} \cdot (\mathbf{r}_{\mathbf{u}} \times \mathbf{r}_{\mathbf{v}}) \, \mathrm{d}A$$

$$\oiint_S \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S = \iiint_V \boldsymbol{\nabla} \cdot \mathbf{F} \, \mathrm{d}V$$

$$\operatorname{curl} \mathbf{v} = \boldsymbol{\nabla} \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$\iint_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \mathbf{n} \, \mathrm{d}S = \oint_{\partial S} \mathbf{F} \cdot \, \mathrm{d}\mathbf{r}$$

this is a test $A^{\mathrm{T}} A^{-\mathrm{T}}$