

Differential Equations

$$y' + a(x)y = b(x) \longrightarrow h(x) = e^{\int a(x) dx}$$

$$2 \text{ distinct } \lambda: y_H = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

$$1 \text{ repeated } \lambda: y_H = c_1 x e^{\lambda x} + c_2 e^{\lambda x}$$

$$\lambda = \alpha \pm \omega i: y_H = e^{\alpha x} (c_1 \cos \omega x + c_2 \sin \omega x).$$

$$k e^{\alpha x} \longrightarrow c e^{\alpha x}$$

$$k x^n \longrightarrow \sum_{i=0}^n c_i x^i$$

$$k \cos \alpha x \text{ or } k \sin \alpha x \longrightarrow c_1 \cos \alpha x + c_2 \sin \alpha x$$

$$(\dots) e^{\alpha x} \longrightarrow (\dots) e^{\alpha x}$$

$$W(y_1, y_2)(x) = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = y_1 y_2' - y_2 y_1'$$

$$r = y'' + p y' + q y \quad y_P = u y_1 + v y_2$$

$$u = - \int \frac{y_2 r}{W} dx \quad v = \int \frac{y_1 r}{W} dx$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh ix = \cos x \quad \sinh ix = i \sin x$$

Projections and Orthonormal Bases

$$P_{\beta' \rightarrow \beta''} P_{\beta \rightarrow \beta'} = P_{\beta \rightarrow \beta''}$$

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\|$$

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \iff \langle \mathbf{u}, \mathbf{v} \rangle = 0$$

$$U^\perp = \{\mathbf{v} \in V \mid \langle \mathbf{v}, \mathbf{u} \rangle = 0 \forall \mathbf{u} \in U\}$$

$$\text{Proj}_U(\mathbf{v}) = \langle \mathbf{v}_1, \hat{\mathbf{e}}_1 \rangle \hat{\mathbf{e}}_1 + \dots + \langle \mathbf{v}, \hat{\mathbf{e}}_k \rangle \hat{\mathbf{e}}_k$$

$$\text{Proj}_{U^\perp}(\mathbf{v}) = \mathbf{v} - \text{Proj}_U(\mathbf{v})$$

$$A\mathbf{x} = \mathbf{b} \longrightarrow A^T A\mathbf{x} = A^T \mathbf{b} \longrightarrow \mathbf{x} = (A^T A)^{-1} A^T \mathbf{b}$$

Matrix-Related Computation

$$A\mathbf{x} = \lambda \mathbf{x} \quad \det(A - \lambda I) = 0 \quad (A - \lambda I)\mathbf{x} = \mathbf{0}$$

$AP = PD$, where $A = PDP^{-1}$ for diagonalisation and $A = PDP^T$ for orthogonal diagonalisation.

$$P = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \dots] \quad D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{pmatrix}$$

P 's vectors need to be *orthonormal* when orthogonally diagonalising.

$$ax^2 + by^2 + cxy \longrightarrow \begin{pmatrix} a & c/2 \\ c/2 & b \end{pmatrix}$$

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz \longrightarrow \begin{pmatrix} a & d/2 & e/2 \\ d/2 & b & f/2 \\ e/2 & f/2 & c \end{pmatrix}$$

$$\mathbf{x}^T A \mathbf{x} + K \mathbf{x} + c = 0$$

When orthogonally diagonalising coefficient matrix of quadratic form, arrange column vectors of P such that $\det P = +1$.

Taylor Series and Critical Points

$$H_f = \begin{pmatrix} f_{x_1 x_1} & \dots & f_{x_1 x_n} \\ \vdots & \ddots & \vdots \\ f_{x_n x_1} & \dots & f_{x_n x_n} \end{pmatrix}$$

$$f(\mathbf{x}) \approx f(\mathbf{x}_0) + (\nabla f(\mathbf{x}_0))^T (\mathbf{x} - \mathbf{x}_0) + \frac{1}{2} (\mathbf{x} - \mathbf{x}_0)^T H_f(\mathbf{x}_0) (\mathbf{x} - \mathbf{x}_0) + \dots$$

$$f(\mathbf{x} + \mathbf{h}) \approx \sum_{l=0}^{\infty} \frac{1}{l!} (\mathbf{h} \cdot \nabla)^l f(\mathbf{x})$$

- Minimum: all $\lambda > 0$
- Maximum: all $\lambda < 0$
- Saddle: there are λ with different signs
- Inconclusive: there are some λ which equal 0 and the rest (non-zero λ) have the same sign

Double and Triple Integrals

For $a, b, c, d \in \mathbb{R}$,

$$\int_a^b \int_c^d f(x)g(y) dy dx = \left(\int_a^b f(x) dx \right) \left(\int_c^d g(y) dy \right)$$

$$\text{area} = \iint_D 1 dA \quad \text{average} = \frac{\iint_D f(x, y) dA}{\iint_D 1 dA}$$

$$\text{volume} = \iiint_V 1 dV \quad \text{average} = \frac{\iiint_V f(x, y, z) dV}{\iiint_V 1 dV}$$

$$m = \iiint_V \rho(x, y, z) dV \quad M_{yz} = \iiint_V x \rho(x, y, z) dV$$

$$M_{xz} = \iiint_V y \rho(x, y, z) dV \quad M_{xy} = \iiint_V z \rho(x, y, z) dV$$

$$\bar{x} = \frac{M_{yz}}{m} \quad \bar{y} = \frac{M_{xz}}{m} \quad \bar{z} = \frac{M_{xy}}{m}$$

For 2D, use $\iint_A (\dots) dA$ with $\rho(x, y)$ and $z = 0$. Then use $M_{yz} = M_y$, $M_{xz} = M_x$, $M_{xy} = 0$.

Coordinate Systems

Polar	$x = r \cos \theta$	$y = r \sin \theta$
	$ J = r$	$r^2 = x^2 + y^2$
Cylindrical	Polar but with z parameter	
	$ J = r$	
Spherical	$x = r \cos \theta \sin \varphi$	$y = r \sin \theta \sin \varphi$
	$z = r \cos \varphi$	$r^2 = x^2 + y^2 + z^2$
	$ J = r^2 \sin \varphi$	

$$\text{Jacobian} = |J| = \left| \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix} \right|$$

Vector Calculus

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

$$D_{\hat{\mathbf{u}}}(f) = (\nabla f) \cdot \hat{\mathbf{u}}$$

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$$

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

$$\operatorname{div} \mathbf{v} = \nabla \cdot \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

$$\operatorname{curl} \mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$(\mathbf{r}_{\mathbf{u}}(a, b) \times \mathbf{r}_{\mathbf{v}}(a, b)) \cdot ((x, y, z) - \mathbf{r}(a, b)) = 0$$

$$\iint_S f(x, y, z) dS = \iint_D f(\mathbf{r}(u, v)) \|\mathbf{r}_{\mathbf{u}} \times \mathbf{r}_{\mathbf{v}}\| dA$$

$$\text{flux} = \int_C \mathbf{v} \cdot \mathbf{n} dS = \int_a^b \mathbf{v}(\mathbf{r}(t)) \cdot \underbrace{(\mathbf{r}'(t) \times \hat{\mathbf{k}})}_{\text{check orientation}} dt$$

$$\text{flux} = \iint_S \mathbf{v} \cdot \mathbf{n} dS = \iint_D \mathbf{v} \cdot (\mathbf{r}_{\mathbf{u}} \times \mathbf{r}_{\mathbf{v}}) dA$$

$$\iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} dA = \oint_{\partial D} \mathbf{F} \cdot d\mathbf{r}$$

$$\oint_{\partial D} \mathbf{v}(x, y) \cdot \mathbf{n} dS = \iint_D \nabla \cdot \mathbf{v}(x, y) dA$$

$$\oiint_S \mathbf{F} \cdot \mathbf{n} dS = \iiint_V \nabla \cdot \mathbf{F} dV$$

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r}$$

$$\mathbf{F} \text{ conservative} \iff \nabla \times \mathbf{F} = \mathbf{0} \left(\text{or } \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \right)$$

Miscellaneous

$$\int \frac{1}{\sqrt{x^2 + 1}} dx = \operatorname{arsinh}(x) + c$$

$$\int \frac{1}{\sqrt{x^2 - 1}} dx = \operatorname{arcosh}(x) + c, x > 1$$

$$\operatorname{arsinh}(x) = \ln \left(x + \sqrt{x^2 + 1} \right)$$

$$\operatorname{arcosh}(x) = \ln \left(x + \sqrt{x^2 - 1} \right), x \geq 1$$

$$\operatorname{artanh}(x) = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right), x \in (-1, 1)$$

For some $A \in M_{n \times n}(\mathbb{F})$, the following statements are equivalent:

- A is non-singular (A^{-1} exists)
- Only $\mathbf{x} = \mathbf{0}$ satisfies $A\mathbf{x} = \mathbf{0}$
- The row-echelon form of A does not have a row of zeroes
- $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b} \in \mathbb{R}^n$
- $\det A \neq 0$
- Columns of A are linearly independent
- Rows of A are linearly independent
- $\dim(\operatorname{NS}(A)) = 0$
- $\operatorname{rank} A = n$
- $\lambda = 0$ is not an eigenvalue of A

The Gram-Schmidt process constructs an orthonormal basis $\{\hat{\mathbf{e}}_1, \dots, \hat{\mathbf{e}}_n\}$ out of n linearly independent $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$. First let $\hat{\mathbf{e}}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}$. Then, for each $i = 1, \dots, n$:

1. Let $U_i = \operatorname{span}\{\hat{\mathbf{e}}_1, \dots, \hat{\mathbf{e}}_i\}$.
2. Calculate $\mathbf{w}_{i+1} = \mathbf{v}_{i+1} - \operatorname{Proj}_{U_i}(\mathbf{v}_{i+1})$.
3. Set $\hat{\mathbf{e}}_{i+1} = \frac{\mathbf{w}_{i+1}}{\|\mathbf{w}_{i+1}\|}$ and increment i if required.

$A \in M_{n \times n}(\mathbb{F})$ is diagonalisable ($A = PDP^{-1}$)

iff A has n linearly independent eigenvectors

iff algebraic and geometric multiplicities are equal for every eigenvalue

$A \in M_{n \times n}(\mathbb{R})$ is orthogonally diagonalisable ($A = PDP^T$)

iff A is symmetric, i.e. $A = A^T$

iff eigenvectors corresponding to different eigenvalues are orthogonal with respect to the dot product

The complex inner product for $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$ is defined by

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1 \bar{v}_1 + \dots + u_n \bar{v}_n = \mathbf{v}^* \mathbf{u} \in \mathbb{C}$$

where \bar{z} is the conjugate of z and \mathbf{v}^* is the conjugate transpose of \mathbf{v} . It satisfies $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$.

- Ellipse or circle: $\frac{x^2}{k^2} + \frac{y^2}{l^2} = 1$ for $k, l > 0$
- Hyperbola: $\frac{x^2}{k^2} - \frac{y^2}{l^2} = 1$ or $\frac{y^2}{c^2} - \frac{x^2}{k^2} = 1$ for $k, l > 0$
- Parabola: $x^2 = ky$ or $y^2 = kx$ for $k \neq 0$