Coupled ODEs

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{r}$$

$$\ddot{y} + p(t)y + q(t) = r(t) \longrightarrow \begin{pmatrix} y \\ \dot{y} \end{pmatrix} = \dot{\mathbf{y}} = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix} \mathbf{y} + \begin{pmatrix} 0 \\ r(t) \end{pmatrix}$$

The eigenvalues of $\mathbf{A}$ found by solving $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$ gives the same characteristic equation for the uncoupled system. General solutions for a second-order system with real constant coefficients, with eigenvalues λ_1, λ_2 and eigenvectors $\mathbf{x_1}, \mathbf{x_2}$, are given by:

$$\begin{aligned} \lambda_1 \neq \lambda_2 : &\quad \mathbf{y} = c_1 \mathbf{x_1} e^{\lambda_1 t} + c_2 \mathbf{x_2} e^{\lambda t} \\ \lambda_1, \lambda_2 \in \mathbb{C} : &\quad \mathbf{y} = c_1 \operatorname{Re}\left(\mathbf{x_1} e^{\lambda_1 t}\right) + c_2 \operatorname{Im}\left(\mathbf{x_1} e^{\lambda_1 t}\right) \\ \lambda_1 = \lambda_2 : &\quad \mathbf{y} = c_1 \mathbf{x} e^{\lambda t} + c_2 (t\mathbf{x} + \mathbf{p}) e^{\lambda t}, \text{ where } (\mathbf{A} - \lambda \mathbf{I})\mathbf{p} = \mathbf{x}. \end{aligned}$$

Phase Portraits of 2D Autonomous Systems

Let $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$ with $\mathbf{y} \in \mathbb{R}^2$, $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$, where $\mathbf{f}$ does not depend on t. $\mathbf{f}$ defines a direction field in the y_1-y_2 plane, and a trajectory of the system is a curve following the field. $\mathbf{f}$ being continuous with continuous partials implies existence and uniqueness for these trajectories, so they do not cross. By the chain rule, the slope of a trajectory is

$$\frac{dy_2}{dy_1} = \frac{\dot{y}_2}{\dot{y}_1} = \frac{f_2}{f_1}.$$

A nullcline is a curve in phase space where the slope of the trajectory is either 0 or ∞. Set $\dot{y}_2 = 0$ for horizontal nullclines and $\dot{y}_1 = 0$ for vertical nullclines, and solve for y_2. Solving for the trajectories themselves is possible in specific cases, usually when the ODE in y_2 from the slope formula is separable.

For the lines in phase space in the direction of the eigenvectors (called eigendirections), there are straight line solutions either flowing towards or away from the origin, determined by the sign of the corresponding eigenvalue ($-$ stable, $+$ unstable).

$\lambda_1 \neq \lambda_2$ and both have the same sign:

- Negative λ gives a *stable improper node*, where trajectories curve like a parabola and approach the gendirection with smallest absolute value, since that exponential term dominates at large t.
- Positive λ gives an *unstable improper node*, where trajectories approach eigendirection with largest absolute value.

$\lambda_1 > 0$, $\lambda_2 < 0$:

- This gives a *saddle point*. Trajectories come in parallel to the stable eigendirection then turn around and become parallel to the unstable direction.

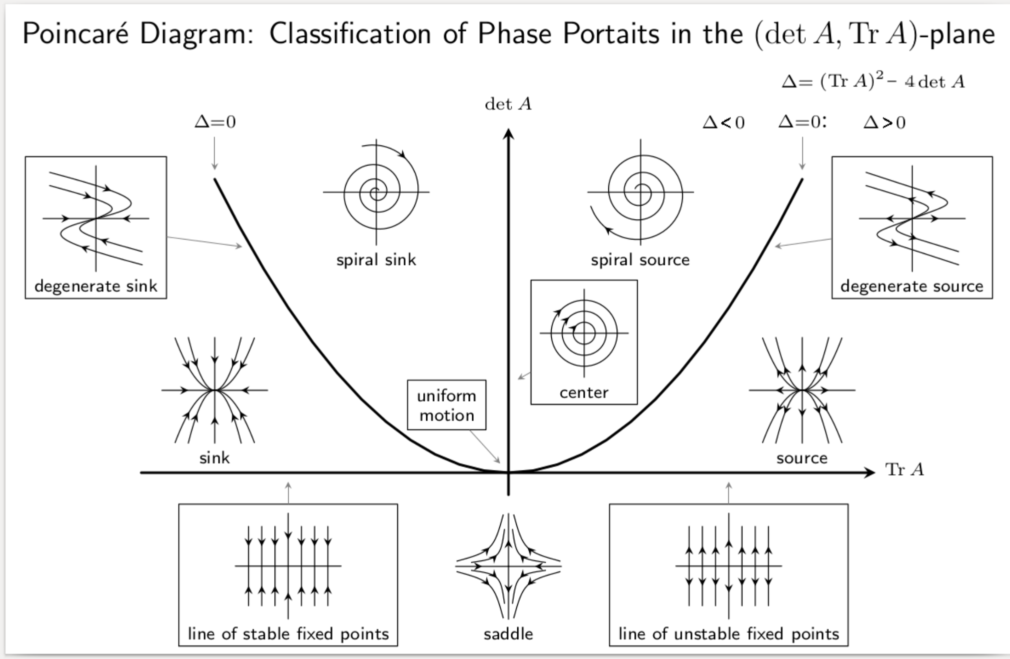
$\lambda_1 = \lambda_2$:

- If there are 2 linearly independent eigenvectors (eigenspace of λ has dimension 2), all trajectories travel in straight lines either towards or away from the origin, called a *proper node*. For a 2D system, this is only possible if $\mathbf{A}$ is some multiple of the identity matrix, i.e. $\mathbf{A} = k\mathbf{I}$ for $k \in \mathbb{R}$.
- If there is only 1 eigenvector, trajectories come tangent to the eigendirection near the origin and do a U-turn away, called a *degenerate node*.

$\lambda = \alpha \pm \beta i$:

- If $\alpha = 0$, trajectories are ellipses around the origin, and the origin is called a *centre*. To determine the direction of flow, consider the sign of $\dot{y}_1$ at a point in the positive quadrant.
- If $\alpha \neq 0$, trajectories trace out a logarithmic spiral $r(t) = e^{\alpha t}$ shown by converting to polar coordinates. This is called a *spiral*. Plotting nullclines helps with sketching trajectories.

The following stability chart allows for the system to be classified given values of $\operatorname{tr}\mathbf{A}$ and $\det \mathbf{A}$ [2]. The discriminant Δ is defined as $\Delta = (\operatorname{tr}\mathbf{A})^2 - 4\det\mathbf{A}$.



Non-Linear or Inhomogeneous Systems

A system $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$ has a critical point at $\mathbf{y}^*$ if $\mathbf{f}(\mathbf{y}^*) = \mathbf{0}$. Linear systems always have one at $\mathbf{0}$, and only one unless $\det \mathbf{A} = 0$. An inhomogeneous linear system $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{p}$ is a linear translation away from a homogeneous system. Shifting the critical point at $\mathbf{p}$ to $\mathbf{0}$ via the change of variables $\mathbf{x} = \mathbf{y} - \mathbf{p}$ gives the homogeneous $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, so techniques discussed earlier are applicable.

For a nonlinear system $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$, the nature of the system around the critical point is approximated via the Jacobian matrix $\mathbf{Df}$. The stability diagram can be used to classify these critical points by evaluating $\mathbf{Df}$ at each $\mathbf{y}^*$. The linearised system near $\mathbf{y}^*$ is then

$$\dot{\mathbf{y}} = \mathbf{Df}|_{\mathbf{y}^*}(\mathbf{y} - \mathbf{y}^*), \quad \mathbf{Df} = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{pmatrix}.$$

Special Functions

The *Dirac Delta function* $\delta(t)$ is an "impulse" with infinite value at $t = 0$ and zero value everywhere else, satisfying $\int_0^\infty \delta(t) = 1$. $\delta(t - a)$ represents an impulse at $t = a$.

The *unit step function* $u(t)$ is defined by

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0. \end{cases}$$

To write a piecewise-continuous function in terms of u, if $f(t)$ changes from $g_1(t)$ to $g_2(t)$ at $t = c$, add a $u(t - c)(g_2(t) - g_1(t))$ term to the new expression of f [1].

The *Gamma function* $\Gamma(z)$ generalises the factorial by $\Gamma(n + 1) = n!$ for $n \in \mathbb{N}$, and is defined for all positive z by

$$\Gamma(z) = \int_0^\infty e^{-x} x^{z-1} \, \mathrm{d}x,$$

In this course the *error function* is defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} \, \mathrm{d}u.$$

It is odd and monotone increasing. The following formulas may be useful for computation:

$$\int_0^\infty e^{-u^2} \, \mathrm{d}u = \int_{-\infty}^0 e^{-u^2} \, \mathrm{d}u = \frac{\sqrt{\pi}}{2}, \quad \frac{1}{\sqrt{\pi}} \int_a^b e^{-u^2} \, \mathrm{d}u = \operatorname{erf}(b) - \operatorname{erf}(a).$$

For the heat equation with parameter c, the *Gaussian function* $G(x, t)$ is defined as

$$G(x, t) = \frac{1}{\sqrt{4\pi c^2 t}} \exp\left(\frac{-x^2}{4c^2 t}\right).$$

$\int_{-\infty}^\infty G(x, t) \, \mathrm{d}x = 1$ for $t > 0$, $\lim_{t \to 0^+} G(x, t) = 0$ for $x \neq 0$ and $\lim_{t \to 0^+} G(0, t) = +\infty$.

Laplace Transforms

The following formulas are given in the formula sheet for the exam, but have been added here for completeness.

$$\mathscr{L}\{f(t)\} = F(s) = \int_0^\infty f(t) e^{-st} \, \mathrm{d}t$$

$f(t)$	$F(s)$	$f(t)$	$F(s)$
c	$\frac{c}{s}$	$ag(t) + bh(t)$	$aG(s) + bH(s)$
t^n	$\frac{n!}{s^{n+1}}$	$e^{at} f(t)$	$F(s - a)$
t^a	$\frac{\Gamma(a+1)}{s^{a+1}}$	$f(t - k)u(t - k)$	$e^{-ks} F(s)$
$\cos \alpha t$	$\frac{s}{s^2 + \alpha^2}$	$f'(t)$	$sF(s) - f(0)$
$\sin \alpha t$	$\frac{\alpha}{s^2 + \alpha^2}$	$f''(t)$	$s^2 F(s) - sf(0) - f'(0)$
$\delta(t - k)$	e^{-ks}	$tf(t)$	$-F'(s)$

The convolution theorem states that:

$$\mathscr{L}^{-1}\{F(s)G(s)\} = \int_0^t f(\tau) g(t - \tau) \, \mathrm{d}\tau = (f * g)(t).$$

If $f(t)$ has period p, then the Laplace integral can be simplified:

$$\mathscr{L}\{f(t)\} = \frac{1}{1 - e^{-sp}} \int_0^p f(t) e^{-st} \, \mathrm{d}t.$$

The following formulas are not included in the formula sheet, but may be helpful to know.

$$\mathscr{L}^{-1}\left\{\frac{s}{(s^2 + \alpha^2)^2}\right\} = \frac{1}{2\alpha} t \sin \alpha t$$

$$\mathscr{L}\{f(t)u(t - k)\} = e^{-ks} \mathscr{L}\{f(t + k)\} \quad [1]$$

$$\mathscr{L}\{f^{(n)}(t)\} = s^n F(s) - \sum_{k=1}^n s^{n-k} f^{(k-1)}(0)$$

$$\mathscr{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s).$$

Differential Operators and Change of Variables

A homogeneous linear differential equation has the form $L(u) = 0$ where L is a *linear differential operator* satisfying $L(u+v) = L(u) + L(v)$ and $L(cu) = cL(u)$ for $c \in \mathbb{R}$ and functions u, v. This can be used to verify a DE is linear using the linearity of the derivative. A common differential operator is the del operator $\boldsymbol{\nabla}$, defined in 3 dimensions by $\boldsymbol{\nabla} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$. It can be applied to a scalar field f to get f's gradient field, or be dotted or crossed with a vector field $\mathbf{v}$ to get its *divergence* or *curl*:

$$\operatorname{grad} f = \boldsymbol{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right), \qquad \operatorname{div} \mathbf{v} = \boldsymbol{\nabla} \cdot \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z},$$

$$\operatorname{curl} \mathbf{v} = \boldsymbol{\nabla} \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Additionally, the *Laplacian* is defined as $\boldsymbol{\nabla}^2 = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}$. For scalar fields,

$$\boldsymbol{\nabla}^2 f = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2},$$

and for a vector field $\mathbf{v}$ in cartesian coordinates, the Laplacian is just defined elementwise, so $\boldsymbol{\nabla}^2 \mathbf{v} = (\boldsymbol{\nabla}^2 v_1, \boldsymbol{\nabla}^2 v_2, \boldsymbol{\nabla}^2 v_3)$. Equivalently, it can be defined as

$$\boldsymbol{\nabla}^2 \mathbf{v} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{v}) - \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{v}).$$

The above formulas only apply for *cartesian coordinates*; other coordinate systems have different expressions. Importantly, the Laplacian in polar coordinates is given by

$$\boldsymbol{\nabla}^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

Changing variables that a PDE is expressed in may simplify the equation and give different types of solutions not apparent from the initial coordinate system. The goal is to express derivatives in the *initial* coordinates in terms of derivatives in the *new* coordinates. If $f(x, y)$ is changed to $f(u, v)$ via $u(x, y)$ and $v(x, y)$, then by the chain rule,

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}, \\ \frac{\partial f}{\partial y} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}. \end{aligned}$$

Note that f can be removed from the above equations to get a relationship purely involving the differential operators $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$. This can then be substituted into the original PDE to get a new equation.

An example is with the PDE $u_t + cu_x = 0$ for $u(x, t)$. Changing variables to $x' = x - ct$ and $t' = t$ allows the equation to be written as $u_{t'} = 0$, thus $u(x', t') = f(x')$ and $u(x, t) = f(x - ct)$ for differentiable f. The solution is defined on a plane in x-t space.

Fourier Series

If a function $f(x)$ has period $2L$, i.e. $f(x + 2L) = f(x)$ for all x, then it can be written as a *Fourier series*:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right),$$

with Fourier coefficients given by

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x,$$
$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\, \mathrm{d}x.$$

In the case that f is *even* ($f(-x) = f(x)$) or *odd* ($f(-x) = -f(x)$), the formulas for the coefficients can be simplified, only changing the fraction and integral bounds:

$$a_0 = \frac{1}{L}\int_0^L (\cdots)\, \mathrm{d}x, \quad a_n = \frac{2}{L}\int_0^L (\cdots)\, \mathrm{d}x, \quad b_n = \frac{2}{L}\int_0^L (\cdots)\, \mathrm{d}x.$$

Odd f makes $a_n = 0$ for all n, and even f makes $b_n = 0$ for all n.

These simplified formulas can also be used when extending f only defined on $[0, L]$ to be odd or even periodic over period $2L$. This is called a *half-range expansion* of f; odd periodic extension makes $a_n = 0$ and even period extension makes $b_n = 0$ for all n.

The following identities may be useful for computation.

$$\begin{aligned} 2\sin A \sin B &= \cos(A - B) - \cos(A + B) \\ 2\cos A \cos B &= \cos(A - B) + \cos(A + B) \\ 2\sin A \cos B &= \sin(A - B) + \sin(A + B) \end{aligned}$$

If f has period $2L$ and is piecewise continuous within that period, then its Fourier series converges to $\frac{1}{2}(f(x^+) + f(x^-))$ at every x where f has left and right derivatives.

To derive formulas for the coefficients, define functions f_1 and f_2 to be orthogonal if $\int_{-L}^{L} f_1(x) f_2(x)\, \mathrm{d}x = 0$. It can be shown that all functions $\left\{1, \cos\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)\right\}$ are mutually orthogonal by integration by parts. Both sides of the Fourier Series equation can then be integrated and solved for the coefficients, where orthogonal terms vanish. This orthogonality is further discussed in MATH2001.

Important PDEs

Name	1D	2D	General
Wave Equation	$u_{tt} = c^2 u_{xx}$	$u_{tt} = c^2(u_{xx} + u_{yy})$	$u_{tt} = c^2 \boldsymbol{\nabla}^2 u$
Heat Equation	$u_t = c^2 u_{xx}$	$u_t = c^2(u_{xx} + u_{yy})$	$u_t = c^2 \boldsymbol{\nabla}^2 u$
Laplace's Equation	$u_{xx} = 0$	$u_{xx} + u_{yy} = 0$	$\boldsymbol{\nabla}^2 u = 0$

These equations are all linear and homogeneous.

1D Wave, Infinite Domain

The general solution to the 1D wave equation $u_{tt} = c^2 u_{xx}$ can be found with the change of variables $v = x + ct$ and $w = x - ct$, giving $utt - c^2 u_{xx} = u_{vw} = 0$. This results in $u(x,t) = F(x+ct) + G(x-ct)$ for arbitrary twice-differentiable F, G.

From here, consider an infinite string with $u(x,0) = f(x)$ as initial deflection and $u_t(x,0) = g(x)$ as initial velocity profile. From the general solution derived earlier, taking derivatives gives $u_t = cF'(x+ct) - cG'(x-ct)$ where the prime denotes a derivative with respect to the entire argument. After substituting initial conditions and solving for F and G, the *D'Alembert's solution* to the IVP is

$$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2C}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s.$$

A *plucked* string is modelled with $g(x) = 0$ and $f(x)$ arbitrary. A *struck* string is modelled with $f(x) = 0$ and $g(x) = Ae^{-x^2}$ for $A \in \mathbb{R}$. The resulting integral can be written in terms of the error function,

$$u(x,t) = B\operatorname{erf}(x+ct) - B\operatorname{erf}(x-ct), \quad B = \frac{A\sqrt{\pi}}{4c}.$$

1D Wave, Finite Domain

A finite string is modelled with $u_{tt} - c^2 u_{xx} = 0$ on $x \in [0, L]$ for $t \geq 0$, and again has initial deflection $u(x,0) = f(x)$ and initial velocity profile $u_t(x,0) = g(x)$. Boundary conditions are $u(0,t) = u(L,t) = 0$.

By Fourier's method (discussed in a later section), the non-trivial solution to this IVP is

$$\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty}\left(A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right)\sin\left(\frac{n\pi x}{L}\right) \\
A_n &= \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) f(x)\,\mathrm{d}x \\
B_n &= \frac{2}{n\pi c}\int_0^L \sin\left(\frac{n\pi x}{L}\right) g(x)\mathrm{d}x.
\end{aligned}$$

Note that A_n are just the coefficients of the half-range sin series of f.

1D Heat, Finite Domain

Consider the 1D heat equation $u_t = c^2 u_{xx}$ on $x \in [0, L]$ for $t \geq 0$, with ICs $u(x,0) = f(x)$ and BCs $u_x(0,t) = u_x(L,t) = 0$, indicating that the endpoints are insulated. Again using Fourier's method, one possible solution is given by $u(x,t) = A_0 \in \mathbb{R}$. A non-constant solution is given by

$$\begin{aligned}
u(x,t) &= \sum_{n=0}^{\infty} A_n e^{-\left(\frac{n\pi c}{L}\right)^2 t}\cos\left(\frac{n\pi x}{L}\right) \\
A_0 &= \frac{1}{L}\int_0^L f(x)\,\mathrm{d}x \\
A_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x.
\end{aligned}$$

Note that A_n are just the coefficients of the half-range cos series of f. There is an exp term in the series instead of sin and cos due to the first derivative u_t in the PDE.

1D Heat, Infinite Domain

For $u_t = c^2 u_{xx} = 0$ on $x \in \mathbb{R}$ for $t \geq 0$, with IC $u(x,0) = f(x)$ and no boundary conditions, the Fourier method will not be useful. Instead, the Gaussian function $G(x,t)$ is a special solution to the equation, and so is $G(x-y,t)$ with $y \in \mathbb{R}$ as a parameter. It represents an injection of 1 unit of heat energy at $x = y, t = 0$. Due to the superposition principle, the following formula can be verified as the solution to the IVP:

$$u(x,t) = \int_{-\infty}^{\infty} G(x-y,t) f(y)\,\mathrm{d}y = \frac{1}{\sqrt{4\pi c^2 t}}\int_{-\infty}^{\infty} \exp\left(-\frac{(x-y)^2}{4c^2 t}\right) f(y)\,\mathrm{d}y.$$

This is verified by distributing partial derivatives into the integral, and taing the limit as $t \to 0$ to show the initial condition holds.

1D Heat, Positive Domain

For $u_t = c^2 u_{xx}$ on $x, t \geq 0$, with IC $u(x,0) = F(x)$ for $x \geq 0$ and BC $u(0,t) = 0$, this is converted into a problem over all of $\mathbb{R}$ considering values $x < 0$ to be *unphysical* via symmetry. Define an odd extension of F:

$$f(x) = \begin{cases} F(x), & x \geq 0 \\ -F(-x), & x < 0. \end{cases}$$

The solution to the infinite domain problem with $u(x,0) = f(x)$ on $\mathbb{R}$ is a solution to the original problem for $x \geq 0$. The BC is satisfied since f odd $\implies f(0) = 0$. Using such symmetric extensions is called the *method of images*. The solution given above for the infinite case can be simplified to the following formula:

$$u(x,t) = \int_0^{\infty} \left[G(x-y,t) - G(x+y,t)\right] F(y)\,\mathrm{d}y.$$

1D Heat, Positive Domain (cont.)

For the case that an *insulation condition* $u_x(0,t) = 0$ is given instead of $u(0,t) = 0$, extend F with even symmetry:

$$f(x) = \begin{cases} F(x), & x \geq 0 \\ F(-x), & x < 0. \end{cases}$$

By a similar derivation as before, the solution is given by

$$u(x,t) = \int_0^\infty \left[G(x-y,t) + G(x+y,t)\right] F(y)\,\mathrm{d}y.$$

Note the $+$ instead of $-$ in the square brackets.

For a non-zero boundary condition $u(0,t) = u_0 \neq 0$, the problem can be modified to have the solution $\hat{u}(x,t) = u(x,t) - u_0$, which also satisfies $\hat{u}_t = c^2 \hat{u}_{xx}$ and has BC $\hat{u}(0,t) = u(0,t) - u_0 = 0$, so the previous formulas can be used to obtain $\hat{u}(x,t)$ using

$$\hat{F}(x) = F(x) - u_0.$$

This can then be transformed back to the original problem by adding u_0 to $\hat{u}(x,t)$.

1D Heat, Finite, Sinusoidal BC

uhh

1D Heat, Finite, Inhomogeneous, General Case

If there is a heat source present, the 1D heat equation is modified to $u_t(x,t) - c^2 u_{xx}(x,t) = q(x,t)$.

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