Real Number Systems

Classification of Numbers

Real Numbers [$\mathbb{R}$]

Irrational Numbers [I] Rational Numbers [Q]

Integers [Z] Non-rational

Set Notation

$A = \{\ldots\}$	set named A with elements "....."
$\in$	element of (a set)
$\notin$	not element of

Recurring Decimal Fractions

Rational numbers can be written as a fraction.

a) Example $0.\dot{4}\dot{4}$ as fraction

Let $x = 0.\dot{4}\dot{4}$	
$10x = 4.\dot{4}\dot{4}$	Multiply by 10.
$10x - x = 4.\dot{4}\dot{4} - 0.\dot{4}\dot{4}$	Subtract ① from ②
$9x = 4$	Simplify
$\therefore x = 4/9$	Rearrange

b) $1.\overline{285}$ as fraction

Let $x = \overline{1.285}$	
$1000x = 1285.\overline{285}$	Multiply to make the original whole
$1000x - x = 1285.\overline{285} - 1.\overline{285}$	Subtract ① from ②
$999x = 1284$	Simplify
$\therefore x = 1284/999$	Rearrange

Binary Numbers

Binary numbers are just like decimal numbers except they're in base 2.

To convert to binary:

25_{10} needs to be factorised by 2.

$\div 2$	25_{10}	(remainder)
2	12_{10}	1
2	6_{10}	0
2	3_{10}	0
2	1_{10}	1
2	0_{10}	1

~~Read this~~ upward for Dec → Bin

Convert from binary:

$2^4\ 2^3\ 2^2\ 2^1\ 2^0$

$1\ 0\ 0\ 1\ 1$

$16 + 0 + 0 +$

$2^4\ 2^3\ 2^2\ 2^1\ 2^0$

$1\ 1\ 0\ 0\ 1$

$16 + 8 + 0 + 0 + 1$

$= 25$

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Addition

$$\begin{array}{cccccccccc}
 & & 2^6 & 2^5 & 2^4 & 2^3 & 2^2 & 2^1 & 2^0 & \\
{}^1 & {}^1 & 1 & {}^1 0 & {}^1 1 & {}^1 0 & {}^1 0 & 1 & 0 & \\
+ & & 1 & 1 & 0 & 1 & 1 & 1 & 1 & \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & & {}_2
\end{array}$$

$1 + 1 = 2_{10} = 10_2$

$0 + 1 = 1$

Rationalising the Denominator

See Grade 20 Book

1. $\dfrac{\sqrt{6}}{\sqrt{13}}$

$= \dfrac{\sqrt{6}}{\sqrt{13}} \times \dfrac{\sqrt{13}}{\sqrt{13}}$

$= \dfrac{\sqrt{6} \times \sqrt{13}}{13}$

$= \dfrac{\sqrt{78}}{13}$

2. $\dfrac{2\sqrt{12}}{3\sqrt{54}}$

$= \dfrac{2\sqrt{4 \times 3}}{3\sqrt{9 \times 6}}$

$= \dfrac{4\sqrt{3}}{9\sqrt{6}} \times \dfrac{\sqrt{6}}{\sqrt{6}}$

$= \dfrac{4\sqrt{18}}{54}$

$= \dfrac{2\sqrt{18}}{27}$ ← $9 \times 2 = 3^2 \times 2$

$= \dfrac{6\sqrt{2}}{27}$

$= \dfrac{2\sqrt{2}}{9}$

3. $\dfrac{\sqrt{6} + 3\sqrt{2}}{3 + \sqrt{3}}$

$= \dfrac{\sqrt{6} + 3\sqrt{2}}{3 + \sqrt{3}} \times \dfrac{3 - \sqrt{3}}{3 - \sqrt{3}}$ conjugate

$= \dfrac{(\sqrt{6} + 3\sqrt{2})(3 - \sqrt{3})}{9 - 3}$

$= \dfrac{3\sqrt{6} - \sqrt{18} + 9\sqrt{2} - 3\sqrt{6}}{6}$

$= \dfrac{-3\sqrt{2} + 9\sqrt{2}}{6}$

$= \dfrac{6\sqrt{2}}{6}$

$= \sqrt{2}$

Absolute Values of a Number

Represents the distance of the number from the origin.
Modules (pl. modulus) are denoted as $|x|$ and always gives a +ve answer.

$$|a| = a \qquad |-a| \neq -a$$

Solving Inequations / Inequalities

Look at examples.

Matrices

A matrix is an array of numbers.

Example matrix -

$$\begin{bmatrix} 1 & 3 & 9 \\ 2 & 4 & 5 \end{bmatrix}$$

This is a "2 by 3" matrix, as there are 2 rows and 3 columns.

The number "1" is in row 1, column 1. If the matrix is named "a", this number is represented as a_{11}.

The number "4" is in row 2, column 2. It is a_{22}.

The number "9" is in row 1, column 3. It is a_{13}.

In general, the elements of matrix A are ~~refto~~ referred to as a_{ij} where i is the row position and j is the column position.

$$A = \begin{bmatrix} 2 & 5 \\ 3 & 6 \\ 4 & 7 \end{bmatrix} \quad 3 \times 2 \text{ matrix}$$

$a_{32} = 7$

a_{23} = No number.

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Addition of Matrices

Add each element to each corresponding element in the matrices. This can only be done if the matrices are the same size (for now, at the current level).

Example

a) $A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ $C = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$

$A + B$

$= \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$= \begin{bmatrix} 4 & 5 \\ 3 & 8 \end{bmatrix}$

b) $A - B$

$= \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$= \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}$

c) $A + C$

$= \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} + \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$

= Different size. Can't add.

Matrix Multiplication by a Scalar

Multiply each element in the matrix by the scalar.

Example

a) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

$3A$

$= 3\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$= \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$

b) $-2B$

$= -2\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

$= \begin{bmatrix} -2 & -4 & -6 \\ -8 & -10 & -12 \end{bmatrix}$

Simple Matrix Equations

a) $5A = \begin{bmatrix} 50 & 35 \\ -15 & 20 \end{bmatrix}$

$A = \begin{bmatrix} 10 & 7 \\ -3 & 4 \end{bmatrix}$

b)

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Multiplying Matrices

If A is in order $m \times n$ and B is $n \times p$, then $A \times B$ can occur.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad (2 \times 3) \qquad B = \begin{bmatrix} 2 & -1 \\ 0 & 4 \\ 5 & 3 \end{bmatrix} \quad (3 \times 2)$$

AB exists, as it will have order mp (2×2)

BA exists, as it will have order mp (3×3)

Multiply all the elements in row 1 of A by column 1 of B. This is element 11 of AB.

Example

$$A \times B = \begin{bmatrix} (1 \times 2) + (2 \times 0) + (3 \times 5) & \cdots \\ \cdots & \cdots \end{bmatrix}$$

Multiply the elements in row 2 of A by column 2 of B. This is element 12 of AB. Continue until all the columns of B have been multiplied by A.

$$A \times B = \begin{bmatrix} (1 \times 2) + (2 \times 0) + (3 \times 5) & (1 \times -1) + (2 \times 4) + (3 \times 3) \\ \cdots & \cdots \end{bmatrix}$$

Multiply the elements in row 2 of A with column 1 of B. Continue until all the columns of B have been multiplied by row 2 of A.

$$A \times B = \begin{bmatrix} (1 \times 2) + (2 \times 0) + (3 \times 5) & (1 \times -1) + (2 \times 4) + (3 \times 3) \\ (4 \times 2) + (5 \times 0) + (6 \times 5) & (4 \times -1) + (5 \times 4) + (6 \times 3) \end{bmatrix}$$

Evaluate each element of $A \times B$.

$$A \times B = \begin{bmatrix} 2 + 0 + 15 & -1 + 8 + 9 \\ 8 + 0 + 30 & -4 + 20 + 18 \end{bmatrix}$$

$$= \begin{bmatrix} 17 & 16 \\ 38 & 34 \end{bmatrix}$$

Recap

1. $a_{11} \quad a_{12} \quad a_{13}$ → b_{11}, b_{21}, b_{31}
2. $a_{11} \quad a_{12} \quad a_{13}$ → b_{12}, b_{22}, b_{32} and so on.
3. $a_{21} \quad a_{22} \quad a_{23}$ → b_{11}, b_{21}, b_{31}
4. $a_{21} \quad a_{22} \quad a_{23}$ → b_{12}, b_{22}, b_{32} and so on.
5. Evaluate.

Powers of Matrices

$$A^2 = A \times A$$

If $A = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$

$= \begin{bmatrix} -2 & 9 \\ -3 & 1 \end{bmatrix}$

$A^3 = A^2 \times A$

$= \begin{bmatrix} -2 & 9 \\ -3 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$

$= \begin{bmatrix} -2+-9 & -6+18 \\ -3+-1 & -9+2 \end{bmatrix}$

$= \begin{bmatrix} -11 & 12 \\ -4 & -7 \end{bmatrix}$

Identity Matrix

$$I = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

continues with leading diagonal

$$AI = IA = A$$

where A is a square matrix.

Multiplicative Inverse of Matrices

A^{-1} is the multiplicative inverse of A if

$$AA^{-1} = A^{-1}A = I$$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$\left(A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$

$ad-bc$ is called the determinant

or $\det A$

Types of Matrices

If det A = 0 or does not exist, A is singular.

If $A^2 = 0$, A is nilpotent.

If $A^2 = A$, A is indempotent.

Solving Equations using Matrices

If $AX = B$ A comes first

then $X = \frac{B}{A}$

$= A^{-1}B$ Order of multip. matters

If $XA = B$ X comes first

then $X = BA^{-1}$

Transposing Matrices

To transpose a matrix is to interchange the rows and columns.

A' is said to be the transpose of A

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A' = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

e.g. $A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$ Find A'

$A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$

$A' = \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix}$

Rules for transposing:

1. $(A')' = A$
2. $(A+B)' = A' + B'$
3. $(kA)' = kA'$ where k is a scalar
4. $(AB)' = B'A'$
5. $A \times A'$ results in a symmetrical matrix

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Applications of Matrices

Simultaneous Equations

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$$ Standard Form

This can be used to solve 2 simultaneous equations with standard forms of $ax+by=u$ and $cx+dy=v$.

Example $\underset{a\ \ b\ \ u}{\overset{3\ \ -1\ \ 16}{3x-y=16}}$ $\quad \underset{c\ \ d\ \ v}{\overset{2\ \ 5\ \ 5}{2x+5y=5}}$

$$\begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 16 \\ 5 \end{bmatrix}$$

What I need to solve

$AX = B$

$X = A^{-1} \times B$

$= \frac{1}{17}\begin{bmatrix} 5 & 1 \\ -2 & 3 \end{bmatrix} \times \begin{bmatrix} 16 \\ 5 \end{bmatrix}$

$= \frac{1}{17}\begin{bmatrix} 85 \\ -17 \end{bmatrix}$

$= \begin{bmatrix} 5 \\ -1 \end{bmatrix}$

$\therefore x=5,\ y=-1$

$A^{-1} = \frac{1}{ad-cb}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$= \frac{1}{17}\begin{bmatrix} 5 & 1 \\ -2 & 3 \end{bmatrix}$

Sequences and Series

Arithmetic

General Form

$$T_n = a + (n-1)d$$

Where T_n is the n-th term

$d = T_2 - T_1 = T_n - T_{n-1}$ which is the constant difference

$a = T_1$ which is the first term.

Example $T = 1, 3, 5, 7, 9 \ldots$

Find T_{100}

~~T_{100}~~ $T_n = a + (n-1)d$

$T_{100} = (1) + (100-1)(2)$

$= 1 + 198$

$= 199$

Also, $T_a - T_b = (a-b)d$

Example $T_3 = -1$ $\quad T_5 = 11$ $\quad$ Find T_{50}

$T_a - T_b = (a-b)d$

$T_5 - T_3 = (5-3)d$

$2d = 11 - (-1)$

$2d = 12$

$\therefore d = 6$

$T_3 = a + (n-1)d$

$-1 = a + (3-1)(6)$

$a = -13$

$T_{50} = a + (n-1)d$

$= -13 + (50-1)(6)$

$= 281$

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Sum of Arithmetic sequence

If the last term (l) is given,

$$S_n = \frac{n}{2}(a + l)$$

Where S_n is the sum of the sequence up to term n

If the last term (l) is not given,

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

Example Sequence = 4, 10, 16 ... Find S_{10}

$S_n = \frac{n}{2}(2a + (n-1)d)$

$S_{10} = \frac{10}{2}(2(4) + (10-1)(6))$

$= 5(8 + 54)$

$= 310$

Geometric Sequence

General Form

$$T_n = ar^{n-1}$$

Where T_n is the n-th term

$a = T_1$ which is the first term

$r = \frac{T_2}{T_1} = \frac{T_3}{T_2}$ which is the common ratio.

Sum $S_n = \frac{a(r^n - 1)}{r - 1}$

Where S_n is the sum of the sequence up to term n.

Also, $\frac{T_a}{T_b} = r^{a-b}$

Example $T_2 = 8$ $T_5 = 512$ Find T_{10}

$\frac{T_a}{T_b} = r^{a-b}$

$\frac{T_5}{T_2} = r^{5-2}$

$\frac{512}{8} = r^3$

$64 = r^3$

$r = \sqrt[3]{64}$

$= 4$

$T_2 = ar^{n-1}$

$8 = a(4)^{2-1}$

$8 = 4a$

$a = 2$

$T_{10} = ar^{n-1}$

$= (2)(4)^{10-1}$

$= 2 \times 4^9$

$= 524288$

Sum to Infinity $S_\infty = \frac{a}{1-r}$

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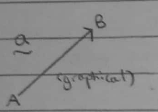
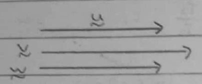
Vectors

A vector is a quantity that has a magnitude and direction.

Notation



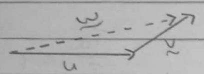
2 vectors are equal if they have the same direction and magnitude.



$u = w$

Addition

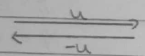
$w = u + v$



- Join the tail end of the second vector to the head end of the first vector.

Negative

$-u$ is u flipped



We can use this to subtract vectors. Add the negative.

Magnitude and Angle/Bearing Calculations

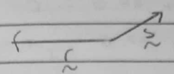
Use Pythagoras' Theorem and inverse trig ratios.

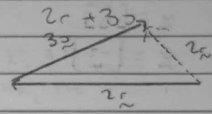
$|u| = \sqrt{x^2 + y^2}$ if $u = xi + yj$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

Scalar Multiplication

This only affects the magnitude of the vector. Multiply the magnitude by the scalar. If the scalar is negative, flip the vector.



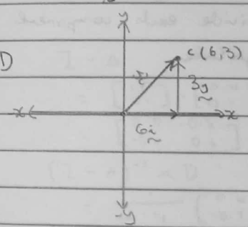


Position Vectors in 2D and 3D

Let i be a vector along the x-axis

Let j be a vector along the y-axis.

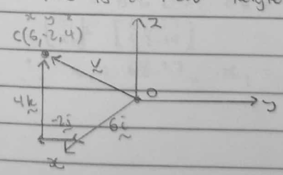
2D



u is the position vector of point c.

$\therefore u = 6i + 3j$

3D There is a new 'height' dimension, z. k is the new unit vector.



v is the position vector of point c.

$\therefore v = 6i - 2j + 4k$

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Relationship between vectors and matrices

The vector $\underset{\sim}{v} = a\underset{\sim}{i} + b\underset{\sim}{j} + c\underset{\sim}{k}$ can be represented as

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

More on this later.

e.g. $\underset{\sim}{v} = 6\underset{\sim}{i} + 3\underset{\sim}{j}$ $\quad \underset{\sim}{v} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$

$\underset{\sim}{v} = 6\underset{\sim}{i} - 2\underset{\sim}{j} + 4\underset{\sim}{k}$ $\quad \underset{\sim}{v} = \begin{bmatrix} 6 \\ -2 \\ 4 \end{bmatrix}$

Unit Vectors

$\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ are known as unit vectors. They lie along the axes and have a magnitude of 1.

To find the unit vector of any vector, divide each component of the original vector by the magnitude.

$$\underset{\sim}{\hat{a}} = \frac{\underset{\sim}{a}}{|\underset{\sim}{a}|}$$

Check Worked Example 4, p.315

Matrices and Applications

Leontief Matrix

$$X = (I - A)^{-1} \times D$$

output matrix, identity matrix, technology matrix, final demand matrix

eg. $x_1 = 0.4x_1 + 0.3x_2 + 15$

$x_2 = 0.5x_1 + 0.4x_2 + 18$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.4 & 0.3 \\ 0.5 & 0.4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 15 \\ 18 \end{bmatrix}$$

Find I − A.

$I - A$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.4 & 0.3 \\ 0.5 & 0.4 \end{bmatrix}$

$= \begin{bmatrix} 0.6 & -0.3 \\ -0.5 & 0.6 \end{bmatrix}$

Invert I − A and mult. D

$(I - A)^{-1} \times D$

$= \frac{1}{ad - bc} \begin{bmatrix} 0.6 & -0.3 \\ -0.5 & 0.6 \end{bmatrix} \times \begin{bmatrix} 15 \\ 18 \end{bmatrix}$

$= \frac{1}{0.36 - 0.15} \begin{bmatrix} 0.6 & -0.3 \\ -0.5 & 0.6 \end{bmatrix} \times \begin{bmatrix} 15 \\ 18 \end{bmatrix}$

$= \frac{1}{0.21} \begin{bmatrix} 0.6 & -0.3 \\ -0.5 & 0.6 \end{bmatrix} \times \begin{bmatrix} 15 \\ 18 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 68.57 \\ 87.14 \end{bmatrix}$

$\therefore x_1 = 68.57, \; x_2 = 87.14$

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Gaussian Elimination

2 simultaneous linear equations:

$$ax + by = u \qquad cx + dy = v$$

e.g. $$x + 2y = 3 \qquad 2x + 3y = 5$$

With Gaussian Elimination, you need to use row operations to convert the technology matrix to an identity matrix. The row operations affect the entire row, and can reference elements from other rows.

$$\begin{matrix} ax + by = u \\ cx + dy = v \end{matrix} \longrightarrow \left[\begin{array}{cc|c} a & b & u \\ c & d & v \end{array}\right]$$

Standard Form — Augmented Matrix

Continuing on with the example:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & 3 & 5 \end{array}\right]$$

$$R_2 - 2R_1: \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -1 & -1 \end{array}\right]$$ changes a_{21} to 0

$$-R_2: \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 1 \end{array}\right]$$ changes a_{22} to 1

$$R_1 - 2R_2: \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right]$$ changes a_{12} to 0

therefore $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $x = 1, y = 1$

3×3 Determinants

If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ then

$$\det A = \underbrace{a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}}_{\text{'minors'}}$$

Example:

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ -1 & 2 & 0 \end{bmatrix}$$

$$\det A = 2\begin{vmatrix} -1 & 2 \\ 2 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} + 3\begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix}$$
$$= 2(-1\cdot 0 - 2\cdot 2) - 1(1\cdot 0 - 2\cdot -1) + 3(1\cdot 2 - -1\cdot -1)$$
$$= -7$$

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3×3 Determinants - Alternate Method

Steps:

1. Rewrite the first two columns of the matrix to the right.
2. Multiply diagonally downward from the top of each column and diagonally upward from the bottom of each column.
3. Add the downward numbers together, and the upward numbers together.
4. Subtract the upward sum from the downward sum to get the determinant. (Down - Up)

Example: ①

$$\left|\begin{matrix} -3 & 3 & 2 \\ 5 & 4 & -1 \\ 2 & 1 & 4 \end{matrix}\right|\begin{matrix} -3 & 3 \\ 5 & 4 \\ 2 & 1 \end{matrix}$$

$$= \left|\begin{matrix} -3 & 3 & 2 \\ 5 & 4 & -1 \\ 2 & 1 & 4 \end{matrix}\right|\begin{matrix} -3 & 3 \\ 5 & 4 \\ 2 & 1 \end{matrix}$$

$-3 \cdot 4 \cdot 4 = -48$ $\quad 3 \cdot -1 \cdot 2 = -6$ $\quad 2 \cdot 5 \cdot 1 = 10$ ②

$2 \cdot 4 \cdot 2 = 16$ $\quad 1 \cdot -1 \cdot -3 = 3$ $\quad 4 \cdot 5 \cdot 3 = 60$

$$= \left|\begin{matrix} -3 & 3 & 2 \\ 5 & 4 & -1 \\ 2 & 1 & 4 \end{matrix}\right|\begin{matrix} -3 & 3 \\ 5 & 4 \\ 2 & 1 \end{matrix}$$

$-48 + (-6) + 10 = -44$

$16 + 3 + 60 = 79$ ③

$-44 - 79 = -123$ ④

Co-Factor Matrices

Replace each element of A with its corresponding co-factor. Change the signs of each element.

If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ then

$$C = \begin{bmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & \begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ \begin{vmatrix} b & c \\ e & f \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} & \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix}$$

Sign pattern: $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

Note: the elements to be put inside each minor are given by the position of that element.

e.g. The elements in the minor belonging to C_{11} (e, f, h and i) are found by ignoring the elements in row 1 and column 1 of A.

e.g. The elements in C_{32}'s minor (a, c, d and f) are found by ignoring row 3 and column 2 of A.

See Worked Example 7, p. 226

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The Inverse of a 3×3 Matrix

The adjoint matrix of A, written adj A, is the transpose of A's co-factor matrix.

Unlike the determinant, adj A is not a scalar.

To find the inverse of a 3×3 matrix, we divide its adjoint matrix by the determinant.

$$A^{-1} = \frac{\text{adj} A}{\det A}$$

Complex Numbers

Defined as a quantity consisting of a real number part added to a multiple of the imaginary unit i.

$$z = a + bi$$

a : Real part of z, written as $Re(z) = a$

b : Imaginary part of z, written as $Im(z) = b$

$$\therefore i = \sqrt{-1}, \quad i^2 = -1, \quad i^3 = -i, \quad i^4 = 1$$

Examples:

1. $\sqrt{-16}$

$= \sqrt{-1} \times \sqrt{16}$

$= 4i$

2. Find the real and imaginary parts of $z = -3+2i$.

$Re(z) = -3$

$Im(z) = 2$

3. Write $i^8 + i^5$ in cartesian form.

$i^8 + i^5$

$= (i^4)^2 + i^4 i$

$= (1)^2 + (1)i$

$= 1 + i$

4. $z = i^4 - 2i^2 + 1$, $w = i^6 - 3i^4 + 3i^2 - 1$. Show that $z + w = -4$.

$i^4 - 2i^2 + 1 + (i^6 - 3i^4 + 3i^2 - 1)$

$= i^4 - 2i^2 + 1 + i^6 - 3i^4 + 3i^2 - 1$

$= (1) - 2(-1) + 1 + (-1) - 3(1) + 3(-1) - 1$

$= \cancel{1} + 2 + \cancel{1} \cancel{-1} - 3 - 3 \cancel{-1}$

$= -4$

Multiplication of Complex Numbers

Use the FOIL method. remember $i^2 = -1$.

Examples :

$z = 3 + 5i$, $w = 4 - 2i$, $v = 6 + 10i$

1. $4z - 3w + 2v$

$= 4(3+5i) - 3(4-2i) + 2(6+10i)$

$= 12 + 20i - 12 + 6i + 12 + 20i$

$= 12 + 46i$

2. $z \times w$

$= (3+5i)(4-2i)$

$= 12 - 6i + 20i - 10i^2$

$= 12 + 14i - 10(-1)$

$= 22 + 14i$

Complex Conjugates $(\bar{z})$

If $z = x + yi$ then $\bar{z} = x - yi$.

To find the conjugate of a complex number, change the sign of the imaginary part.

Division of Complex Numbers

To divide two complex numbers, multiply both the numerator and the denominator with the conjugate of the denominator.

If $z = a + bi$ and $w = c + di$ then:

$$\frac{z}{w} \times \frac{\bar{w}}{\bar{w}}$$ (multiply both top & bottom by conjugate)

$$= \frac{ac + bd + (bc - ad)i}{c^2 + d^2}$$ (explicit formula)

Multiplicative Inverse of Complex Numbers

$$z^{-1} = \frac{1}{z}$$

Use conjugate process.

If $z = a + bi$ then:

$$z^{-1} = \frac{a - bi}{a^2 + b^2}$$ (explicit formula)

Complex Numbers in Polar Form

Coordinate transform:

$x = r\cos\theta \qquad y = r\sin\theta$

$$z = x + yi$$
$$= r\cos\theta + r\sin\theta\, i$$
$$= r\,\overline{\text{cis}}\,\theta$$

shorthand for $\cos\theta + i\sin\theta$

Modulus: $|z| = \sqrt{x^2 + y^2}$

$|z| = r$

Distance of point from origin

Argument: Angle from line segment to the real axis

$$\text{Arg } z = \tan^{-1}\left(\frac{y}{x}\right)$$

	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{6}$
sin	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
cos	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
tan	1	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$

$\sin(\theta) = \sin(180 - \theta)$, $\cos\theta = -\cos(180-\theta)$

$\cos(\theta) = \cos(-\theta)$, $\sin(-\theta) = -\sin(\theta)$

Vectors: Dot Product

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = |\underset{\sim}{u}||\underset{\sim}{v}| \cos\theta$$

where θ is the angle between $\underset{\sim}{u}$ and $\underset{\sim}{v}$.

If $\underset{\sim}{u} \cdot \underset{\sim}{v} = 0$ then $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are perpendicular.

If $\underset{\sim}{u} = k\underset{\sim}{v}$, $k \in \mathbb{R}$ then $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are parallel.

If $\underset{\sim}{u} = a\underset{\sim}{i} + b\underset{\sim}{j}$ and $\underset{\sim}{v} = c\underset{\sim}{i} + d\underset{\sim}{j}$,

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = ac + bd$$

Alternatively,

$$\cos\theta = \frac{ac + bd}{|\underset{\sim}{u}| + |\underset{\sim}{v}|}$$

$\underset{\sim}{i} \cdot \underset{\sim}{i} = 1$, $\underset{\sim}{j} \cdot \underset{\sim}{j} = 1$, $\underset{\sim}{k} \cdot \underset{\sim}{k} = 1$

$\underset{\sim}{i} \cdot \underset{\sim}{j} = 0$, $\underset{\sim}{j} \cdot \underset{\sim}{k} = 0$, $\underset{\sim}{i} \cdot \underset{\sim}{k} = 0$

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Permutations and combinations

1. Multiplication Principle: $m \times n$ ways to perform A and B
2. Addition Principle: $m + n$ ways to perform A or B
3. Arrangement: Selection where order is important

Factorials

No. of ways n objects can be arranged is $n!$ where:

$n! = n \times (n-1) \times (n-2) \dots \times 3 \times 2 \times 1$, $0! = 1$

Permutation

No. of different arrangements when r things are chosen from n things and order is important is ${}^nP_r$ where:

$${}^nP_r = \frac{n!}{(n-r)!}, \quad {}^nP_n = n!, \quad {}^nP_0 = 1$$

No. of ways n people can be seated, r at a time, in a circle is:

$$\frac{{}^nP_r}{r}$$

Combinations

No. of different arrangements when r things are chosen from n things and order is not important is ${}^nC_r$ where:

$${}^nC_r = \frac{{}^nP_r}{r!} = \frac{n!}{(n-r)!\,r!}, \quad {}^nC_r = {}^nC_{n-r}$$

Arrangements

No. of different ways of arranging n things made up of groups of indistinguishable things, n_1 in the first group, n_2 in the second etc. is:

$$\frac{n!}{n_1!\,n_2!\,n_3!\dots n_r!}$$

De Moivre's Theorem

Use this to assist in multiplying and dividing complex numbers.

If $z_1 = r_1 \operatorname{cis} \theta_1$ and $z_2 = r_2 \operatorname{cis} \theta_2$

then $z_1 \times z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$

and $\frac{z_1}{z_2} = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2)$

Also, if $z = r \operatorname{cis} \theta$ then

$$z^n = (r \operatorname{cis} \theta)^n = r^n \operatorname{cis} n\theta$$

Factorisation of Polynomials over C

Factor Theorem:

If $P(a) = 0$ then $(x-a)$ is a factor of $P(x)$ and conversely.

Fundamental Theorem of Algebra:

If $P_n(z)$ is a polynomial of degree n over C, then there exists a $z_0 \in C$ such that $P_n(z_0) = 0$.

These two theorems combined prove there are n linear factors for all $P_n(z)$ over C.

Conjugate Root Theorem:

If a polynomial has real coefficients, its roots are either real numbers or occur as pairs of conjugate complex numbers.

In other words,

If $P(z) = 0$ then $P(\bar{z}) = 0$.

Use long division to reduce polynomials of degree 3 or higher to quadratics, and then use completing the square reducing the constant in the form $-(\sqrt{c})^2$.

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Solving Equations over C

A polynomial equation $P(z)=0$ can be solved by first factorising $P(z)$ so that:

If $P(z)=(z-z_1)(z-z_2)\dots(z-z_n)=0$

then $z=z_1, z_2, \dots z_n$

Solving Quadratics

Use the quadratic formula.

Solving cubics/Higher-order polynomials

Find a real factor using the factor theorem, then use long division and quadratic formula.

Solving radical equations

with the equation $z=\sqrt{a+bi}$ use the formula:
$$x^2=\frac{a+\sqrt{a^2+b^2}}{2}$$
to find x. Use the formula:
$$y^2=x^2-a$$
to find y. Answers will be expressed in $z_1, z_2 = x+yi$ form.

Note: evaluating the square root to isolate x^2 and y^2 gives two $\pm$ answers.

Solving $z^n=w$ where $w \in C$

1. Express w in polar form,
$$w=r\operatorname{cis}(\theta+2k\pi)$$
2. Let:
$$z^n=r\operatorname{cis}(\theta+2k\pi)$$
increment k until n solns. are found
3. Hence,
$$z=r^{\frac{1}{n}}\operatorname{cis}\left(\frac{\theta+2k\pi}{n}\right)$$ (de Moivre's Theorem)

Solving $z^n=a$ where $a \in R$

If $z^n=a$, $a \in R$,

then $z=a^{\frac{1}{n}}\operatorname{cis}\frac{2k\pi}{n}$

where $k \in J$ until n solutions are found.

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Trigonometric Links to Complex Numbers

$\cos n\theta = \mathrm{Re}\,(\cos\theta + i\sin\theta)^n$

$\sin n\theta = \mathrm{Im}\,(\cos\theta + i\sin\theta)^n$

Example: Prove that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

$\sin 3\theta = \sin n\theta,\ n = 3$

$\sin(3)\theta = \mathrm{Im}(\cos\theta + i\sin\theta)^n$

(Binomial Theorem) $= {}^3C_0\cos^3\theta + {}^3C_1\cos^2\theta(i\sin\theta) + {}^3C_2\cos\theta(i\sin\theta)^2 + {}^3C_3(i\sin\theta)^3$

(Eval. ${}^nC_r$) $= \cos^3\theta + 3\cos^2\theta\, i\sin\theta + 3\cos\theta\, i^2\sin^2\theta + i^3\sin^3\theta$

(Group Re and Im) $= \cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\sin\theta - \sin^3\theta)$

Since $\sin 3\theta = \mathrm{Im}(\cos\theta + i\sin\theta)^n$

(Im part) $\sin 3\theta = 3\cos^2\sin\theta - \sin^3\theta$

(Pythag. Id.) $= 3(1 - \sin^2\theta)\sin\theta - \sin^3\theta$

(Expand) $= 3\sin\theta - 3\sin^3\theta - \sin^3\theta$

(Simplify) $= 3\sin\theta - 4\sin^2\theta$ ∎

Multi-Angled Formula

$\cos n\theta = \dfrac{z^n + z^{-n}}{2}$

$\sin n\theta = \dfrac{z^n - z^{-n}}{2i}$

Example: Prove that $2\cos 3\theta \sin 2\theta = \sin 5\theta - \sin\theta$.

$2\cos 3\theta \sin 2\theta$

$= 2 \times \dfrac{z^3 + z^{-3}}{2} \times \dfrac{z^2 - z^{-2}}{2i}$

(cancel 2's) $= \dfrac{1}{2i}(z^3 + z^{-3})(z^2 - z^{-2})$

$= \dfrac{1}{2i}(z^5 - z^1 + z^{-1} - z^{-5})$

$= \dfrac{1}{2i}\left((z^5 - z^{-5}) - (z^1 - z^{-1})\right)$

$= \dfrac{z^5 - z^{-5}}{2i} - \dfrac{z^1 - z^{-1}}{2i}$

$= \sin 5\theta - \sin\theta$

$\therefore 2\cos 3\theta \sin 2\theta = \sin 5\theta - \sin\theta$ ∎

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Integration/Antiderivative Techniques

Derive $\longrightarrow$

Integrate $\longleftarrow$

$ax^n \rightleftharpoons \frac{ax^{n+1}}{n+1} + c$

$\frac{1}{x} \rightleftharpoons \ln x + c$

$e^{kx} \rightleftharpoons \frac{e^{kx}}{k} + c$

$\sin kx \rightleftharpoons \frac{-\cos kx}{k} + c$

$\cos kx \rightleftharpoons \frac{\sin kx}{k} + c$

$\tan x \rightleftharpoons -\ln|\cos x| + c$

$\sec x \rightleftharpoons \ln|\sec x + \tan x| + c$

$\csc x \rightleftharpoons \ln|\csc x - \cot x| + c$

$\cot x \rightleftharpoons \ln|\sin x| + c$

$\sec^2 x \rightleftharpoons \tan x + c$

$\csc^2 x \rightleftharpoons -\cot x + c$

$\ln x \rightleftharpoons x(\ln x - 1) + c$

$\sec x \tan x \rightleftharpoons \sec x + c$

$\csc x \cot x \rightleftharpoons -\csc x + c$

$\frac{1}{\sqrt{a^2-x^2}} \rightleftharpoons \sin^{-1}\left(\frac{x}{a}\right) + c$

$\frac{-1}{\sqrt{a^2-x^2}} \rightleftharpoons \cos^{-1}\left(\frac{x}{a}\right) + c$

$\frac{a}{a^2+x^2} \rightleftharpoons \tan^{-1}\left(\frac{x}{a}\right) + c$

$\frac{1}{x\sqrt{1-x^2}} \rightleftharpoons \sec^{-1}(x) + c$

$a^x \rightleftharpoons \frac{1}{\ln a} a^x + c$

Substitution where the derivative is present in the integrand

1. Identify u, the part to be substituted, in the integral.
2. Find du, then dx in terms of du.
3. Substitute u and dx.
4. Evaluate the integral.
5. Substitute u back for x.

e.g. $\int (x+3)^7 \cdot dx$

1. $u = x+3$

   $\frac{du}{dx} = 1$

2. $dx = du$

3. $\int u^7 \cdot du$

   $= \frac{u^{7+1}}{7+1} + c$

4. $= \frac{u^8}{8} + c$

5. $= \frac{(x+3)^8}{8} + c$

$\int \frac{x+3}{(x^2+6x)^3} \cdot dx$

1. $u = x^2 + 6x$

   $\frac{du}{dx} = 2x + 6$

2. $dx = \frac{du}{2x+6}$

3. $\int \frac{x+3}{u^3} \cdot \frac{du}{2x+6}$

   $= \int \frac{x+3}{u^3} \cdot \frac{du}{2(x+3)}$

   $= \int \frac{1}{2} u^{-3} \cdot du$

   $= \frac{1}{2}\left(\frac{u^{-3+1}}{-3+1}\right) + c$

4. $= \frac{-1}{4u^2} + c$

5. $= \frac{-1}{4(x^2+6x)^2} + c$

$\int (x^2-1)\cos(3x-x^3) \cdot dx$

1. $u = 3x - x^3$

   $\frac{du}{dx} = 3 - 3x^2$

2. $dx = \frac{du}{3-3x^2}$

3. $\int (x^2-1)\cos u \cdot \frac{du}{3-3x^2}$

   $= \int (x^2-1)\cos u \cdot \frac{du}{-3(x^2-1)}$

   $= \int \frac{-\cos u}{3} \cdot du$

4. $= \frac{-\sin u}{3} + c$

5. $= \frac{-\sin(3x-x^3)}{3} + c$

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Integration by Substitution

1. Identify u, the part to be substituted, in the integral.
2. Find du, then dx in terms of du.
3. Find x, or a function to be substituted, in terms of du.
4. Substitute u, dx and the function.
5. Algebraically manipulate the integral to get an expression with only u.
6. Evaluate the integral.
7. Substitute u back for x.

e.g. $\int \frac{e^{2x}}{e^x+1} \cdot dx$

1. $u = e^x + 1$

   $\frac{du}{dx} = e^x$
2. $dx = \frac{du}{e^x}$
3. $e^x = u - 1$
4. $F(x) = \int \frac{e^{2x}}{u} \cdot \frac{du}{e^x}$

   $= \int \frac{e^x}{u} \cdot du$
5. $= \int \frac{u-1}{u} \cdot du$

   $= \int 1 - u^{-1} \cdot du$
6. $= u - \ln u + C$
7. $= (e^x+1) - \ln(e^x+1) + C$

Integration by Trigonometric substitution

$$\left.\begin{aligned} \sin^2 ax &= \tfrac{1}{2}(1 - \cos 2ax) \\ \cos^2 ax &= \tfrac{1}{2}(1 + \cos 2ax) \\ \sin ax \cos ax &= \tfrac{1}{2} \sin 2ax \end{aligned}\right\} \text{Can integrate } \sin^n x,\ \cos^n x \text{ where } n \text{ is even}$$

1. Find the value for a to be used in the identity.
2. Substitute the identity into the integral.
3. Evaluate the integral.

Note: take care that the trigonometric functions that are being substituted cannot be used with the first technique. If they can, that technique takes priority.

e.g. $\int \sin^2\left(\frac{x}{2}\right) \cdot dx$

1. $= \int \sin^2\left(\frac{1}{2} \times x\right) \cdot dx$
2. $= \int \frac{1}{2}\left(1 - \cos 2 \times \frac{1}{2} x\right) \cdot dx$

   $= \frac{1}{2} \int 1 - \cos x \cdot dx$
3. $= \frac{1}{2}(x - \sin x) + C$

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$\cot^2\theta + 1 = \csc^2\theta$

$\int \sin^n x \cos^m x \, dx$

1. n odd: Take out 1 sin, convert $\cos^2 x \to 1 - \sin^2 x$, $u = \cos x$
2. m odd: Take out 1 cos, convert $\sin^2 x \to 1 - \cos^2 x$, $u = \sin x$
3. n, m odd: Do 1 or 2
4. n, m even: Use $\sin ax \cos ax = \frac{1}{2} \sin 2ax$

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Integration by Partial Fractions

$$\frac{f(x)}{(ax+b)(cx+d)} \rightarrow \frac{A}{(ax+b)} + \frac{B}{(cx+d)}$$

$$\frac{f(x)}{(ax+b)^2} \rightarrow \frac{A}{(ax+b)^2} + \frac{B}{(ax+b)}$$

1. Express the rational expression as one of the two partial fraction ~~identities~~ forms shown above.
2. Rewrite the partial fractions with the original common denominator.
3. Equate the numerator of the original expression with the numerator of the new expression.
4. Solve for A and B by letting $x =$ the value ~~which~~ which will cancel the opposite bracket.
5. Substitute A and B into the partial fraction form.
6. Integrate as usual.

e.g. $\int \frac{x+7}{(x+2)(x-3)} \cdot dx$

① $\frac{x+7}{(x+2)(x-3)} = \frac{A}{x+2} + \frac{B}{x-3}$

② $= \frac{A(x-3)+B(x+2)}{(x+2)(x-3)}$

③ $x+7 = A(x-3)+B(x+2)$

④ If $x=-2$, $-2+7 = A(-2-3)$

$5 = -5A$

$A = -1$

If $x = 3$, $3+7 = B(3+2)$

$10 = 5B$

$B = 2$

⑤ $F(x) = \int \frac{-1}{x+2} + \frac{2}{x-3}$

⑥ $= -\ln(x+2) + 2\ln(x-3) + C$

$= \ln\left(\frac{(x-3)^2}{x+2}\right) + C$

Note: logarithm laws are usually required after partial fraction decomposition.

Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

How to pick the right u:

- Logarithmic
- Inverse trigonometric
- Arithmetic
- Trigonometric
- Exponential

1. Pick u using LIATE and the corresponding dv.
2. Substitute u, v, du, dv into the integration by parts formula.
3. Evaluate the new integral $\int v\,du$. Repeat the process if required.
4. If the resulting integral is a constant multiple of the original, group the identical integrals together and divide by its coefficient. This leaves the answer on the other side.

Note: Sometimes, manipulating the integrand to get a nicer u and dv is preferable. Setting either u or dv to 1 can also work in some situations.

If the resulting integral has to be done by parts again, it is <u>imperative</u> that the same u and dv are chosen as the previous integral. Otherwise, this results in a meaningless answer like $1=1$.

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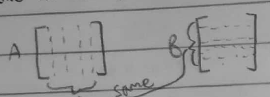
Matrix Multiplication Revision

1. The product of AB is only possible if matrix A has the same number of columns as B has rows.



2. If A is an $m \times n$ matrix and B is a $n \times p$ matrix, then the product of $A \times B$ is an $n \times p$ matrix.
3. Matrix multiplication is not commutative.
4. The 2x2 identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is written as I, and has the property that $IA = AI = A$.
5. To calculate the product $A \times B$, each row of A is multiplied by each column of B.

Determinant and Inverse

A square matrix has equal numbers of rows and columns.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then its determinant, $|A|$ or $\det A = ad - bc$

If $\det A = 0$ the matrix is singular.

The inverse of A, $A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

The inverse of a singular matrix does not exist.

Markov Chains and Eigenvalues

Markov Chain - Technique that calculates the probability associated with the state of a system as it makes various transitions.

Transition Matrix - Used to simplify calculations. The columns of a transition matrix always add up to 1.

Eigenvalue - Represents the unchanged object in a situation.

$AX = \lambda X$ where X is the eigenvector of matrix A, λ is the eigenvalue of matrix A

so, $\underbrace{AX - \lambda X = 0}_{\text{for eigenvector}}$, $\underbrace{\det(A - \lambda I) = 0}_{\text{for eigenvalue}}$ $X = \begin{bmatrix} a \\ b \end{bmatrix}$

$\underset{n^{th}\text{ state}}{P_n} = \underset{\text{tst matrix}}{M^n} \underset{\text{initial state}}{P_0}$

Example to calculate eigenvalues and eigenvectors:

$A = \begin{bmatrix} 1 & 6 \\ 2 & 2 \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} 1 & 6 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

$= \begin{bmatrix} 1-\lambda & 6 \\ 2 & 2-\lambda \end{bmatrix}$

$\det(A - \lambda I) = 0$

$0 = (1-\lambda)(2-\lambda) - 12$

$0 = \lambda^2 - 3\lambda - 10$

$0 = (\lambda - 5)(\lambda + 2)$

$\therefore \lambda = 5, \lambda = -2$

$AX = \lambda X$, $\lambda = 5$, $\lambda = -2$

Let $X = \begin{bmatrix} a \\ b \end{bmatrix}$

$\begin{bmatrix} 1 & 6 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 5\begin{bmatrix} a \\ b \end{bmatrix}$, $\begin{bmatrix} 1 & 6 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = -2\begin{bmatrix} a \\ b \end{bmatrix}$

$a + 6b = 5a$ | $a + 6b = -2a$

$6b = 4a$ | $6b = -3a$

$b = \frac{2a}{3}$ | $b = \frac{-a}{2}$

$\therefore X = \begin{bmatrix} a \\ \frac{2a}{3} \end{bmatrix}$, $X = \begin{bmatrix} a \\ \frac{-a}{2} \end{bmatrix}$

See ~~Ex. 10~~ Example 8, 9 (p. 59-62) for transition matrix.

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Leontief and Leslie Matrices

Leontief matrices - relates inputs and outputs of a system

$$\text{Consumption Matrix} \quad \begin{array}{c} \begin{array}{ccc} E & R & M \end{array} \\ \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \end{array} \begin{array}{c} \\ E \\ R \\ M \end{array}$$

Element C_{ij} means it takes C_{ij} units of row i to make 1 unit of j

Each column describes the resources required to make 1 unit of that resource (column heading)

To find production matrix, use formula

$P = (I - C)^{-1} \times d$ where d is demand.

Example: p71 Example 13

$$\underset{\text{input}}{q} = \underset{C}{C}\,\underset{\text{output}}{p}$$

Leslie matrices

$$L = \begin{bmatrix} F_1 & F_2 & F_3 \\ S_1 & 0 & 0 \\ 0 & S_2 & 0 \end{bmatrix}$$

F_i is fecundity rate of i^{th} age group

S_i is survival rate of i^{th} age group

$$N_1 = \begin{bmatrix} \text{pop'n of 1yo} \\ \text{pop'n of 2yo} \\ \text{pop'n of 3yo} \end{bmatrix}$$

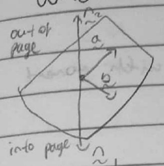
, N_1 is initial population

For future predictions of populations, use:

$$N_r = L \times N_{r-1} \qquad N_r = L^{r-1} \times N_1$$

Vector Product

The vector product of two vectors describes a vector that is perpendicular to both. The order which the vector product is taken is important;



if $\underset{\sim}{a} \times \underset{\sim}{b}$ is clockwise, the resulting vector product will be in a downward direction. The opposite is true for $\underset{\sim}{b} \times \underset{\sim}{a}$.

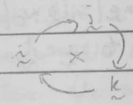
The vector product can be calculated using:

$$\underset{\sim}{a} \times \underset{\sim}{b} = |\underset{\sim}{a}|\,|\underset{\sim}{b}| \sin\theta \,\underset{\sim}{\hat{n}}$$

where θ is the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$

$\hat{n}$ is the unit vector $\perp$ to $\underset{\sim}{a}$ and $\underset{\sim}{b}$ (see above)

To find vector products of $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$, use this:



A product in clockwise dir. gives +ve value

A product in ccw dir. gives -ve value

e.g. $\underset{\sim}{i} \times \underset{\sim}{j} = \underset{\sim}{k}$, $\underset{\sim}{j} \times \underset{\sim}{i} = -\underset{\sim}{k}$

note: $\underset{\sim}{i} \times \underset{\sim}{i} = 0$, $\underset{\sim}{j} \times \underset{\sim}{j} = 0$, $\underset{\sim}{k} \times \underset{\sim}{k} = 0$

To find vector product with $\underset{\sim}{a}$ and $\underset{\sim}{b}$ in $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ form:

1. Multiply together using distributive law
2. Convert into matrix form where

$$\underset{\sim}{a} \times \underset{\sim}{b} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

and find determinant:

$$\underset{\sim}{a} \times \underset{\sim}{b} = \underset{\sim}{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \underset{\sim}{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \underset{\sim}{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

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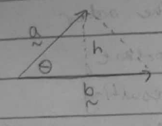
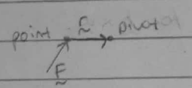
Applications of Vector Product

Area of Triangle - Area of triangle formed by $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is:



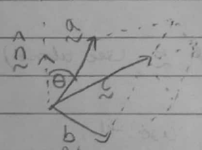
$$A = \frac{1}{2}|\underset{\sim}{a}||\underset{\sim}{b}|\sin\theta$$
$$= \frac{1}{2}|\underset{\sim}{a} \times \underset{\sim}{b}|$$

Torque - The torque acting on a point with moment arm $\underset{\sim}{r}$ and force $\underset{\sim}{F}$ is:



$$\tau = |\underset{\sim}{r} \times \underset{\sim}{F}|$$

Scalar Triple Product



Three vectors that are not coplanar form a parallelepiped. Its volume is the product of the area of the base and the ⊥ height. The area of the base is double the area of the triangle formed by $\underset{\sim}{b}$ and $\underset{\sim}{c}$: $A = 2 \times \frac{1}{2}|\underset{\sim}{b} \times \underset{\sim}{c}| = |\underset{\sim}{b} \times \underset{\sim}{c}|$

The perpendicular height is $|\underset{\sim}{a}|\cos\theta$, also equal to $\underset{\sim}{a} \cdot \underset{\sim}{\hat{n}}$.

Thus, the volume is:

$$V = |\overbrace{\underset{\sim}{a} \cdot \underset{\sim}{b} \times \underset{\sim}{c}}^{\text{scalar triple product}}|$$

Note: $\underset{\sim}{b} \times \underset{\sim}{c}$ must be calculated first.

The triple product can also be calculated using determinants:

$$\underset{\sim}{a} \cdot \underset{\sim}{b} \times \underset{\sim}{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

where the elements are the components of the vectors.

Eccentricity - Conics

e represents the eccentricity of a conic section.

If	$e = 0$	the conic is a circle
	$0 < e < 1$	the conic is an ellipse
	$e = 1$	the conic is a parabola
	$e > 1$	the conic is a hyperbola

Parabolas

Refer to formula sheet for equations and features of parabolas. An equation of a parabola can be rewritten to fit the standard form by completing the square with the variable of order 2.

Converting an Ellipse from Complex to Cartesian Form

1. Substitute $z = x + yi$.
2. Separate the real and imaginary components of both complex numbers.
3. Take the modulus of both numbers, $|x + yi| = \sqrt{x^2 + y^2}$
4. Take one square root term to the other side.
5. Square both sides.
6. Expand quadratic terms outside of the $\sqrt{\ }$ and simplify fully.
7. Isolate $\sqrt{\ }$ term and divide by common factor of coefficients.
8. Square both sides.
9. Expand fully.
10. Take all constant terms to one side and evaluate.
11. Factorise the opposite side for both x and y.
12. Divide by the constant and simplify.

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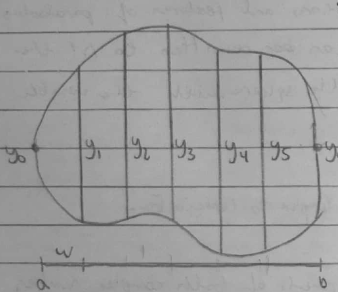
Definite Integration

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

A definite integral calculates the area under the curve between two bounds, a and b.

Example 22, 23 p.169

Simpson's Rule



Simpson's Rule approximates the area of a curve/shape using the heights of evenly cut strips. To use this rule, the number of strips (n) must be even.

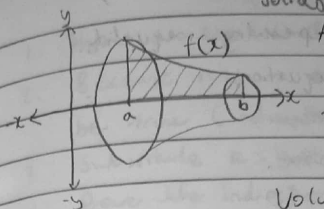
$$w = \frac{b-a}{n}$$

$$A = \int_a^b f(x)\,dx \approx \frac{w}{3}\left[y_0 + y_n + 4(y_1 + y_3 + \cdots) + 2(y_2 + y_4 + \cdots)\right]$$

f(a) f(b) height of odd strips, excluding 1st and last; height of even strips, excluding 1st and last

a1 - consumption matrix 2 - eigenvalues
3 - transition matrix 4 - leslie matrix
5 - sketch ellipse, identify pts 6 - find solid of rev. volume 7 - eval. definite integral
8 - find eqn of ellipse 9 - $\frac{1}{2}$ ellipse arc eqn 10 - vector area 11 - parallelopiped vector V
12 - Simpson's Rule 13 - Volume, Simpson's Rule 14 - Orbits (elliptic)

Solids of Revolution



A solid of revolution is formed by rotating the area bounded by a curve and an axis around that axis. In this case, $\int_a^b f(x)\,dx$ is rotated around the x-axis.

Volume Formulae:

1. Around x-axis: $V = \pi \int_a^b [f(x)]^2\,dx$
2. Around y-axis: $V = \pi \int_a^b [f(y)]^2\,dy$
3. Area is bounded by two curves: $V = \pi \int_a^b [f(x)]^2 - [g(x)]^2\,dx$

Trig Substitution II: Electric Boogaloo

$\sqrt{b^2x^2 - a^2}$: $\sec^2\theta - 1 = \tan^2\theta$, $x = \frac{a}{b}\sec\theta$, $0 \le \theta < \frac{\pi}{2}$, $\frac{\pi}{2} < \theta \le \pi$

$\sqrt{a^2 - b^2x^2}$: $1 - \sin^2\theta = \cos^2\theta$, $x = \frac{a}{b}\sin\theta$, $\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$

$\sqrt{a^2 + b^2x^2}$: $\tan^2\theta + 1 = \sec^2\theta$, $x = \frac{a}{b}\tan\theta$, $\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$

Form Identity Substitution Limits

For indef. integration, assume abs. val is +ve ~~say~~ and remove it.

For def. integration, check new terminals (in θ) are within range of limits. If range is +ve for abs. function (inside), clear abs. val. If range is -ve, clear abs. and make negative.

For indef. integration, turn $\theta \to x$ in the answer. Isolate trig function in initial substitution and use right triangle trig to find value of θ and other trig functions (use pythag. for unknown side).

Simplex Algorithm

Slack variable - placeholder variable representing inequality. changes inequation to equation.

If variables $\leq \#$, slack is positive.

If variables $\geq \#$, slack is negative.

e.g. $2p - 3q \geq 20 \rightarrow 2p - 3q \boxed{-s} = 20$

$p + q + r < 5 \rightarrow p + q + r + s = 5$

Simplex tableau - "matrix" holding coefficients of linear system

e.g.

$v \geq 8 \rightarrow v + 0w + 0p - s_1 = 8$

$3v - 2w + p \geq 8 \rightarrow 3v - 2w + p - s_2 = 8$

$w - p - v \leq 10 \rightarrow -v + w - p + s_3 = 10$

$p + w < 12 \rightarrow 0v + w + p + s_4 = 12$

$$
\begin{array}{c|cccccccc}
 & v & w & p & s_1 & s_2 & s_3 & s_4 & T \\
\hline
s_1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 8 \\
s_2 & 3 & -2 & 1 & 0 & -1 & 0 & 0 & 8 \\
s_3 & -1 & 1 & -1 & 0 & 0 & 1 & 0 & 10 \\
s_4 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 12
\end{array}
$$

Use row operations to convert to RREF.

Proof By Mathematical Induction

1. Show an event is true for $n=1$ (base case).
2. Because $n=1$ is true, assume $n=k$ will also be true (induction hypothesis).
3. Substitute $k = k+1$ (induction step).
4. Prove the induction step to be true, by making use of the induction hypothesis.

e.g. Prove, using mathematical induction, that:

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

1. Let $n=1$

   $\therefore 1 = \frac{1(1+1)}{2} = \frac{2}{2} = 1$

   Base case is true.

2. Assume $n=k$

   $\therefore 1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2}$

3. $k = k+1$

   $\therefore \underbrace{1 + 2 + 3 + \ldots + k}_{\text{Induction Hypothesis}} + k + 1 = \frac{(k+1)(k+1+1)}{2}$

4. $\therefore \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}$

   $\frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$

   $\frac{(k+2)(k+1)}{2} = \frac{(k+1)(k+2)}{2}$ QED.

Parametric Equations

Convert from parametric form → cartesian form

Examples:

a) $x = t,\ y = t+2$

Substitute $x = t \rightarrow y = t+2$

$\therefore y = x+2$

b) $x = \frac{2t}{1+t^2},\ y = \frac{1-t^2}{1+t^2}$

$x^2 = \frac{(2t)^2}{(1+t^2)^2}\quad y^2 = \frac{(1-t^2)^2}{(1+t^2)^2}$ — Square both sides

$x^2+y^2 = \frac{(2t)^2}{(1+t^2)^2} + \frac{(1-t^2)^2}{(1+t^2)^2}$ — Add x^2+y^2

$x^2+y^2 = \frac{4t^2+1-2t^2+t^4}{(1+t^2)^2}$ — Make common denominator

$x^2+y^2 = \frac{t^4+2t^2+1}{(1+t^2)^2}$ — Evaluate

$x^2+y^2 = \frac{(1+t^2)^2}{(1+t^2)^2}$ — Simplify, factorise

$\therefore x^2+y^2 = 1$

c) $x = \cos\theta,\ y = \sin\theta$

$x^2 = \cos^2\theta,\ y^2 = \sin^2\theta$

$x^2+y^2 = \cos^2\theta + \sin^2\theta$

$\therefore x^2+y^2 = 1$

a) → Isolate then substitute t

b) → Use algebra to get x and y

c) → Use trig identities to get x and y

Tangents and Normals w/ Parametric Eqns

To find gradient of normal from gradient of tangent:

$$M_{normal} = \frac{-1}{M_{tangent}}$$

With parametric eqns, convert to cartesian form then apply derivative to calculate gradient. Then, use $y = mx + c$ w/ given pt to calculate equation of line.

e.g. $x = t,\ y = t^2$ @ $x = 1$

$\therefore y = x^2$ $\qquad$ $y = x^2,\ x = 1$

$\frac{dy}{dx} = 2x,\ x = 1$ $\qquad$ $= 1 \quad pt(1,1)$

$\therefore M = 2$

Tangent: $y = mx + c$

$1 = 2 \times 1 + c$

$1 = 2 + c$

$\therefore c = -1$

$\therefore y = 2x - 1$

Normal: $m = \frac{-1}{2}$

$y = mx + c$

$1 = \frac{-1}{2} \times 1 + c$

$1 = \frac{-1}{2} + c$

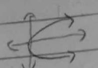
$c = \frac{3}{2}$

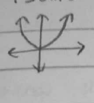
$\therefore y = \frac{-1}{2}x + \frac{3}{2}$

Alternatively, use implicit differentiation if given equation form.

Parabolas in Polar Form

$r = \frac{d}{1 \pm \cos\theta}$ or $r = \frac{d}{1 \pm \sin\theta}$

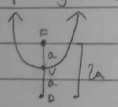




Case	Vertex	Directix*	Focus
$\cos\theta$	$(\frac{d}{2}, 0)$	$(d, 0)$	
$-\cos\theta$	$(\frac{d}{2}, \pi)$	(d, π)	All
$\sin\theta$	$(\frac{d}{2}, \frac{\pi}{2})$	$(d, \frac{\pi}{2})$	$(0,0)$
$-\sin\theta$	$(\frac{d}{2}, \frac{3\pi}{2})$	$(d, \frac{3\pi}{2})$	

* pt of intersection btwn directix & axis of symmetry

Expressing as cartesian form:



$(x-h)^2 = 4a(y-k)$ $h = r\cos\theta$

$2a = d$ ↑ vertical $k = r\sin\theta$

$(y-k)^2 = 4a(x-h)$

e.g. $r = \frac{10}{1 - \sin\theta}$

$d = 10 = 2a$

$\therefore a = 5$

$h = r\cos\theta = \frac{d}{2}\cos\frac{3\pi}{2} = \frac{10}{2}\cos\frac{3\pi}{2} = 0$

$k = r\sin\theta = \frac{d}{2}\sin\frac{3\pi}{2} = \frac{10}{2}\sin\frac{3\pi}{2} = -5$

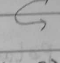
$\therefore (x-0)^2 = 20(y-(-5))$

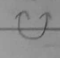
$x^2 = 20(y+5)$

$\therefore y = \frac{1}{20}x^2 - 5$

Hyperbolas in Polar Form

$r = \frac{de}{1 \pm e\cos\theta}$ or $r = \frac{de}{1 \pm e\sin\theta}$



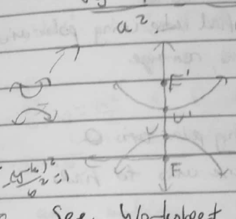


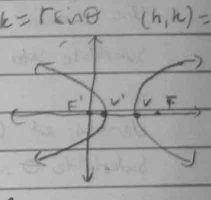
Focus	$(0,0)$	$(0,0)$
Vertices	Subst. 0 and π	Subst. $\frac{\pi}{2}$ and $\frac{3\pi}{2}$

Expressing as Cartesian Form:

$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$ $h = r\cos\theta$ $2a = VV'$

$k = r\sin\theta$ $(h,k) = \text{midp}(VV')$





$b^2 = a^2(e^2 - 1)$

$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

See Worksheet Example 18.

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Polar Form Reference Sheet

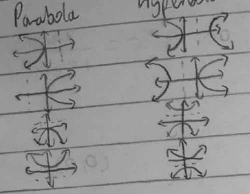
Case | Parabola | Hyperbola

$\cos\theta$

$-\cos\theta$

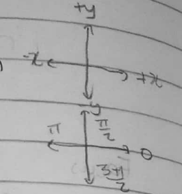
$\sin\theta$

$-\sin\theta$

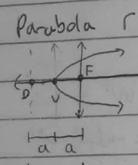


Cartesian

Polar



Parabola $r=\frac{d}{1\pm\cos\theta} \rightarrow (y-k)^2=4a(x-h)$, $r=\frac{d}{1\pm\sin\theta} \rightarrow (x-h)^2=4a(y-k)$



Convert to Cartesian Form:

1. Use $d=2a$ to find a
2. Use $h=a\cos\theta, k=a\sin\theta$ to find ~~centre~~ vertex using polar axis θ
3. Substitute into appropriate form, rearrange

Graph:

1. Vertex at $(\frac{d}{2},\theta)$ using polar axis θ
2. Substitute θ not on transverse axis to find intercepts
3. Graph in polar plane

Directrices can be given in polar form.

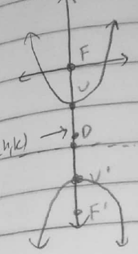
$r\sin\theta=c \rightarrow y=c$, $r\cos\theta=a \rightarrow x=a$

Distance between directrix and focus $(0,0)$ is d.

To make polar form eqn, parabola requires directrix.

hyperbola requires directrix and eccentricity.

Hyperbola $r=\frac{de}{1\pm e\cos\theta} \rightarrow \frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$, $r=\frac{de}{1\pm e\sin\theta} \rightarrow \frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$



Graph:

1. Substitute θ not on transverse axis to find intercepts
2. Substitute θ on transverse axis to find vertices.
   Negative r means +ve r on opposite side.
3. Find F' using reflection, F' = FU' + FU
4. Graph in polar plane

Convert to Cartesian Form:

D is not midpoint of UU'.

1. Find e, de, d from general form
2. Find a^2 using $2a=UU'$ (square this)
3. Find b^2 using $b^2=a^2(e^2-1)$
4. Find (h,k) by calculating midpoint of UU'
5. Substitute into appropriate form, rearrange

~~Directrix:~~
$x=\frac{a^2}{c}$, $y=\frac{a^2}{c}$
horiz. vert.
$c=\sqrt{a^2+b^2}$

I eat babies

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