Differential Equations

$$y' + a(x)y = b(x) \longrightarrow h(x) = e^{\int a(x)\,\mathrm{d}x}$$

$$\begin{aligned} \text{2 distinct } \lambda\text{:} \;\; y_H &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ \text{1 repeated } \lambda\text{:} \;\; y_H &= c_1 x e^{\lambda x} + c_2 e^{\lambda x} \\ \lambda = \alpha \pm \omega i\text{:} \;\; y_H &= e^{\alpha x}\left(c_1 \cos \omega x + c_2 \sin \omega x\right). \end{aligned}$$

$$\begin{aligned} k e^{\alpha x} &\longrightarrow c e^{\alpha x} \\ k x^n &\longrightarrow \sum_{i=0}^{n} c_i x^i \\ k \cos \alpha x \text{ or } k \sin \alpha x &\longrightarrow c_1 \cos \alpha x + c_2 \sin \alpha x \\ (\cdots)\, e^{\alpha x} &\longrightarrow (\cdots)\, e^{\alpha x} \end{aligned}$$

$$W(y_1, y_2)(x) = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = y_1 y_2' - y_2 y_1'$$

$$r = y'' + py' + qy \qquad y_P = u y_1 + v y_2$$

$$u = -\int \frac{y_2 r}{W}\,\mathrm{d}x \qquad v = \int \frac{y_1 r}{W}\,\mathrm{d}x$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh ix = \cos x \qquad \sinh ix = i \sin x$$

Projections and Orthonormal Bases

$$\mathbf{P}_{\beta' \to \beta''} \mathbf{P}_{\beta \to \beta'} = \mathbf{P}_{\beta \to \beta''}$$

$$\begin{aligned} \|\mathbf{v}\| &= \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} \\ d(\mathbf{u}, \mathbf{v}) &= \|\mathbf{u} - \mathbf{v}\| \\ |\langle \mathbf{u}, \mathbf{v} \rangle| &\leq \|\mathbf{u}\|\,\|\mathbf{v}\| \\ \|\mathbf{u} + \mathbf{v}\| &\leq \|\mathbf{u}\| + \|\mathbf{v}\| \end{aligned}$$

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \iff \langle \mathbf{u}, \mathbf{v} \rangle = 0$$

$$\begin{aligned} U^{\perp} &= \{\mathbf{v} \in V \mid \langle \mathbf{v}, \mathbf{u} \rangle = 0 \; \forall \mathbf{u} \in U\} \\ \mathrm{Proj}_U(\mathbf{v}) &= \langle \mathbf{v_1}, \hat{\mathbf{e}}_\mathbf{1} \rangle \hat{\mathbf{e}}_\mathbf{1} + \cdots + \langle \mathbf{v}, \hat{\mathbf{e}}_\mathbf{k} \rangle \hat{\mathbf{e}}_\mathbf{k} \\ \mathrm{Proj}_{U^{\perp}}(\mathbf{v}) &= \mathbf{v} - \mathrm{Proj}_U(\mathbf{v}) \end{aligned}$$

$$\mathbf{A}\mathbf{x} = \mathbf{b} \longrightarrow \mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{x} = \mathbf{A}^{\mathrm{T}}\mathbf{b} \longrightarrow \mathbf{x} = (\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{b}$$

Matrix-Related Computation

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x} \quad \det(\mathbf{A} - \lambda \mathbf{I}) = 0 \quad (\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = \mathbf{0}$$

$\mathbf{AP} = \mathbf{PD}$, where $\mathbf{A} = \mathbf{PDP}^{-1}$ for diagonalisation and $\mathbf{A} = \mathbf{PDP}^{\mathrm{T}}$ for orthogonal diagonalisation.

$$\mathbf{P} = \begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} & \cdots \end{bmatrix} \quad \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{pmatrix}$$

$\mathbf{P}$'s vectors need to be *orthonormal* when orthogonally diagonalising.

$$ax^2 + by^2 + cxy \longrightarrow \begin{pmatrix} a & c/2 \\ c/2 & b \end{pmatrix}$$

$$\begin{aligned} ax^2 + by^2 + cz^2 + dxy \\ + exz + fyz \end{aligned} \longrightarrow \begin{pmatrix} a & d/2 & e/2 \\ d/2 & b & f/2 \\ e/2 & f/2 & c \end{pmatrix}$$

$$\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} + \mathbf{K}\mathbf{x} + c = 0$$

When orthogonally diagonalising coefficient matrix of quadratic form, arrange column vectors of $\mathbf{P}$ such that $\det \mathbf{P} = +1$.

Taylor Series and Critical Points

$$\mathbf{H}_f = \begin{pmatrix} f_{x_1 x_1} & \cdots & f_{x_1 x_n} \\ \vdots & \ddots & \vdots \\ f_{x_n x_1} & \cdots & f_{x_n x_n} \end{pmatrix}$$

$$\begin{aligned} f(\mathbf{x}) \approx f(\mathbf{x_0}) &+ \left(\boldsymbol{\nabla} f(\mathbf{x_0})\right)^{\mathrm{T}} (\mathbf{x} - \mathbf{x_0}) \\ &+ \frac{1}{2}(\mathbf{x} - \mathbf{x_0})^{\mathrm{T}} \mathbf{H}_f(\mathbf{x_0})(\mathbf{x} - \mathbf{x_0}) + \cdots \end{aligned}$$

$$f(\mathbf{x} + \mathbf{h}) \approx \sum_{l=0}^{\infty} \frac{1}{l!} \left(\mathbf{h} \cdot \boldsymbol{\nabla}\right)^l f(\mathbf{x})$$

- Minimum: all $\lambda > 0$
- Maximum: all $\lambda < 0$
- Saddle: there are λ with different signs
- Inconclusive: there are some λ which equal 0 and the rest (non-zero λ) have the same sign

Double and Triple Integrals

For $a, b, c, d \in \mathbb{R}$,

$$\int_a^b \int_c^d f(x) g(y)\,\mathrm{d}y\,\mathrm{d}x = \left(\int_a^b f(x)\,\mathrm{d}x\right)\left(\int_c^d g(y)\,\mathrm{d}y\right)$$

$$\text{area} = \iint_D 1\,\mathrm{d}A \qquad \text{average} = \frac{\iint_D f(x,y)\,\mathrm{d}A}{\iint_D 1\,\mathrm{d}A}$$

$$\text{volume} = \iiint_V 1\,\mathrm{d}V \qquad \text{average} = \frac{\iiint_V f(x,y,z)\,\mathrm{d}V}{\iiint_V 1\,\mathrm{d}v}$$

$$m = \iiint_V \rho(x,y,z)\,\mathrm{d}V \qquad M_{yz} = \iiint_V x\rho(x,y,z)\,\mathrm{d}V$$

$$M_{xz} = \iiint_V y\rho(x,y,z)\,\mathrm{d}V \qquad M_{xy} = \iiint_V z\rho(x,y,z)\,\mathrm{d}V$$

$$\bar{x} = \frac{M_{yz}}{m} \quad \bar{y} = \frac{M_{xz}}{m} \quad \bar{z} = \frac{M_{xy}}{m}$$

For 2D, use $\iint_A (\cdots)\,\mathrm{d}A$ with $\rho(x,y)$ and $z = 0$. Then use $M_{yz} = M_y$, $M_{xz} = M_x$, $M_{xy} = 0$.

Coordinate Systems

Polar	$x = r\cos\theta$	$y = r\sin\theta$
	$\lvert J\rvert = r$	$r^2 = x^2 + y^2$
Cylindrical	Polar but with z parameter	
	$\lvert J\rvert = r$	
Spherical	$x = r\cos\theta\sin\varphi$	$y = r\sin\theta\sin\varphi$
	$z = r\cos\varphi$	$r^2 = x^2 + y^2 + z^2$
	$\lvert J\rvert = r^2\sin\varphi$	

$$\text{Jacobian} = |J| = \left| \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix} \right|$$

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Vector Calculus

$$\boldsymbol{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$$D_{\hat{\mathbf{u}}}(f) = (\boldsymbol{\nabla} f) \cdot \hat{\mathbf{u}}$$

$$W = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,\mathrm{d}t$$

$$\int_C \boldsymbol{\nabla} f \cdot \mathrm{d}\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

$$\operatorname{div} \mathbf{v} = \boldsymbol{\nabla} \cdot \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

$$\operatorname{curl} \mathbf{v} = \boldsymbol{\nabla} \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$(\mathbf{r_u}(a,b) \times \mathbf{r_v}(a,b)) \cdot ((x,y,z) - \mathbf{r}(a,b)) = 0$$

$$\iint_S f(x,y,z)\,\mathrm{d}S = \iint_D f(\mathbf{r}(u,v))\,\|\mathbf{r_u} \times \mathbf{r_v}\|\,\mathrm{d}A$$

$$\text{flux} = \int_C \mathbf{v} \cdot \mathbf{n}\,\mathrm{d}S = \int_a^b \mathbf{v}(\mathbf{r}(t)) \cdot \underbrace{\left(\mathbf{r}'(t) \times \hat{\mathbf{k}}\right)}_{\text{check orientation}}\,\mathrm{d}t$$

$$\text{flux} = \iint_S \mathbf{v} \cdot \mathbf{n}\,\mathrm{d}S = \iint_D \mathbf{v} \cdot (\mathbf{r_u} \times \mathbf{r_v})\,\mathrm{d}A$$

$$\iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\,\mathrm{d}A = \oint_{\partial D} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

$$\oint_{\partial D} \mathbf{v}(x,y) \cdot \mathbf{n}\,\mathrm{d}S = \iint_D \boldsymbol{\nabla} \cdot \mathbf{v}(x,y)\,\mathrm{d}A$$

$$\oiint_S \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}S = \iiint_V \boldsymbol{\nabla} \cdot \mathbf{F}\,\mathrm{d}V$$

$$\iint_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \mathbf{n}\,\mathrm{d}S = \oint_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

$$\mathbf{F} \text{ conservative} \iff \boldsymbol{\nabla} \times \mathbf{F} = \mathbf{0} \left(\text{or } \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}\right)$$

Miscellaneous

$$\int \frac{1}{\sqrt{x^2+1}}\,\mathrm{d}x = \operatorname{arsinh}(x) + c$$

$$\int \frac{1}{\sqrt{x^2-1}}\,\mathrm{d}x = \operatorname{arcosh}(x) + c, x > 1$$

$$\operatorname{arsinh}(x) = \ln\left(x + \sqrt{x^2+1}\right)$$

$$\operatorname{arcosh}(x) = \ln\left(x + \sqrt{x^2-1}\right), x \geq 1$$

$$\operatorname{artanh}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), x \in (-1,1)$$

For some $\mathbf{A} \in M_{n\times n}(\mathbb{F})$, the following statements are equivalent:

- $\mathbf{A}$ is non-singular ($\mathbf{A}^{-1}$ exists)
- Only $\mathbf{x} = \mathbf{0}$ satisfies $\mathbf{Ax} = \mathbf{0}$
- The row-echelon form of $\mathbf{A}$ does not have a row of zeroes
- $\mathbf{Ax} = \mathbf{b}$ has a solution for all $\mathbf{b} \in \mathbb{R}^n$
- $\det \mathbf{A} \neq 0$
- Columns of $\mathbf{A}$ are linearly independent
- Rows of $\mathbf{A}$ are linearly independent
- $\dim(\mathrm{NS}(\mathbf{A})) = 0$
- $\operatorname{rank} \mathbf{A} = n$
- $\lambda = 0$ is not an eigenvalue of $\mathbf{A}$

The Gram-Schmidt process constructs an orthonormal basis $\{\mathbf{\hat{e}_1}, \cdots, \mathbf{\hat{e}_n}\}$ out of n linearly independent $\{\mathbf{v_1}, \cdots, \mathbf{v_n}\}$. First let $\mathbf{\hat{e}_1} = \frac{\mathbf{v_1}}{\|\mathbf{v_1}\|}$. Then, for each $i = 1, \cdots, n$:

1. Let $U_i = \operatorname{span}\{\mathbf{\hat{e}_1}, \cdots, \mathbf{\hat{e}_i}\}$.
2. Calculate $\mathbf{w_{i+1}} = \mathbf{v_{i+1}} - \operatorname{Proj}_{U_i}(\mathbf{v_{i+1}})$.
3. Set $\mathbf{\hat{e}_{i+1}} = \frac{\mathbf{w_{i+1}}}{\|\mathbf{w_{i+1}}\|}$ and increment i if required.

$\mathbf{A} \in M_{n\times n}(\mathbb{F})$ is diagonalisable ($\mathbf{A} = \mathbf{PDP}^{-1}$)

iff $\mathbf{A}$ has n linearly independent eigenvectors

iff algebraic and geometric multiplicities are equal for every eigenvalue

$\mathbf{A} \in M_{n\times n}(\mathbb{R})$ is orthogonally diagonalisable ($\mathbf{A} = \mathbf{PDP}^{\mathrm{T}}$)

iff $\mathbf{A}$ is symmetric, i.e. $\mathbf{A} = \mathbf{A}^{\mathrm{T}}$

iff eigenvectors corresponding to different eigenvalues are orthogonal with respect to the dot product

The complex inner product for $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$ is defined by

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1\overline{v_1} + \cdots + u_n\overline{v_n} = \mathbf{v}^*\mathbf{u} \in \mathbb{C}$$

where $\bar{z}$ is the conjugate of z and $\mathbf{v}^*$ is the conjugate transpose of $\mathbf{v}$. It satisfies $\overline{\langle \mathbf{u}, \mathbf{v} \rangle} = \langle \mathbf{v}, \mathbf{u} \rangle$.

- Ellipse or circle: $\frac{x^2}{k^2} + \frac{y^2}{l^2} = 1$ for $k, l > 0$
- Hyperbola: $\frac{x^2}{k^2} - \frac{y^2}{c^2}$ or $\frac{y^2}{c^2} - \frac{x^2}{k^2}$ for $k, l > 0$
- Parabola: $x^2 = ky$ or $y^2 = kx$ for $k \neq 0$