Differential Equations

$$y' + a(x)y = b(x) \longrightarrow h(x) = e^{\int a(x)\,\mathrm{d}x}$$

$$\begin{aligned} \text{2 distinct } \lambda\text{:}\ \ y_H &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ \text{1 repeated } \lambda\text{:}\ \ y_H &= c_1 x e^{\lambda x} + c_2 e^{\lambda x} \\ \lambda = \alpha \pm \omega i\text{:}\ \ y_H &= e^{\alpha x}\left(c_1 \cos\omega x + c_2 \sin\omega x\right). \end{aligned}$$

$$\begin{aligned} k e^{\alpha x} &\longrightarrow c e^{\alpha x} \\ k x^n &\longrightarrow \sum_{i=0}^{n} c_i x^i \\ k \cos\alpha x \text{ or } k \sin\alpha x &\longrightarrow c_1 \cos\alpha x + c_2 \sin\alpha x \\ (\cdots)\, e^{\alpha x} &\longrightarrow (\cdots)\, e^{\alpha x} \end{aligned}$$

$$W(y_1, y_2)(x) = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = y_1 y_2' - y_2 y_1'$$

$$r = y'' + py' + qy \qquad y_P = u y_1 + v y_2$$

$$u = -\int \frac{y_2 r}{W}\,\mathrm{d}x \qquad v = \int \frac{y_1 r}{W}\,\mathrm{d}x$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh ix = \cos x \qquad \sinh ix = i \sin x$$

Projections and Orthonormal Bases

$$P_{\beta' \to \beta''} P_{\beta \to \beta'} = P_{\beta \to \beta''}$$

$$\begin{aligned} \|\mathbf{v}\| &= \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} \\ d(\mathbf{u}, \mathbf{v}) &= \|\mathbf{u} - \mathbf{v}\| \\ |\langle \mathbf{u}, \mathbf{v} \rangle| &\le \|\mathbf{u}\|\,\|\mathbf{v}\| \\ \|\mathbf{u} + \mathbf{v}\| &\le \|\mathbf{u}\| + \|\mathbf{v}\| \end{aligned}$$

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \iff \langle \mathbf{u}, \mathbf{v} \rangle = 0$$

$$\begin{aligned} U^\perp &= \{\mathbf{v} \in V \mid \langle \mathbf{v}, \mathbf{u} \rangle = 0\ \forall \mathbf{u} \in U\} \\ \mathrm{Proj}_U(\mathbf{v}) &= \langle \mathbf{v_1}, \hat{e}_1 \rangle\, \hat{e}_1 + \cdots + \langle \mathbf{v}, \hat{e}_k \rangle\, \hat{e}_k \\ \mathrm{Proj}_{U\perp}(\mathbf{v}) &= \mathbf{v} - \mathrm{Proj}_U(\mathbf{v}) \end{aligned}$$

$$A\mathbf{x} = \mathbf{b} \longrightarrow A^{\mathrm{T}} A \mathbf{x} = A^{\mathrm{T}} \mathbf{b} \longrightarrow \mathbf{x} = \left(A^{\mathrm{T}} A\right)^{-1} A^{\mathrm{T}} \mathbf{b}$$

Matrix-Related Computation

$$A\mathbf{x} = \lambda \mathbf{x} \quad \det(A - \lambda I) = 0 \quad (A - \lambda I)\mathbf{x} = \mathbf{0}$$

$AP = PD$, where $A = PDP^{-1}$ for diagonalisation and $A = PDP^{\mathrm{T}}$ for orthogonal diagonalisation.

$$P = \begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} & \cdots \end{bmatrix} \quad D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{pmatrix}$$

P's vectors need to be *orthonormal* when orthogonally diagonalising.

$$ax^2 + by^2 + cxy \longrightarrow \begin{pmatrix} a & c/2 \\ c/2 & b \end{pmatrix}$$

$$\begin{aligned} ax^2 + by^2 + cz^2 + dxy \\ + exz + fyz \end{aligned} \longrightarrow \begin{pmatrix} a & d/2 & e/2 \\ d/2 & b & f/2 \\ e/2 & f/2 & c \end{pmatrix}$$

$$\mathbf{x}^{\mathrm{T}} A \mathbf{x} + K\mathbf{x} + c = 0$$

When orthogonally diagonalising coefficient matrix of quadratic form, arrange column vectors of P such that $\det P = +1$.

Taylor Series and Critical Points

$$H_f = \begin{pmatrix} f_{x_1 x_1} & \cdots & f_{x_1 x_n} \\ \vdots & \ddots & \vdots \\ f_{x_n x_1} & \cdots & f_{x_n x_n} \end{pmatrix}$$

$$\begin{aligned} f(\mathbf{x}) \approx{}& f(\mathbf{x_0}) + \left(\boldsymbol{\nabla} f(\mathbf{x_0})\right)^{\mathrm{T}} (\mathbf{x} - \mathbf{x_0}) \\ &+ \frac{1}{2}(\mathbf{x} - \mathbf{x_0})^{\mathrm{T}} H_f(\mathbf{x_0})(\mathbf{x} - \mathbf{x_0})^{\mathrm{T}} + \cdots \end{aligned}$$

$$f(\mathbf{x} + \mathbf{h}) \approx \sum_{l=0}^{\infty} \frac{1}{l!} \left(\mathbf{h} \cdot \boldsymbol{\nabla}\right)^l f(\mathbf{x})$$

- Minimum: all $\lambda > 0$
- Maximum: all $\lambda < 0$
- Saddle: there are λ with different signs
- Inconclusive: there are some λ which equal 0 and the rest (non-zero λ) have the same sign

Double and Triple Integrals

For $a, b, c, d \in \mathbb{R}$,

$$\int_a^b \int_c^d f(x) g(y)\,\mathrm{d}y\,\mathrm{d}x = \left(\int_a^b f(x)\,\mathrm{d}x\right)\left(\int_c^d g(y)\,\mathrm{d}y\right)$$

$$\text{area} = \iint_D 1\,\mathrm{d}A \qquad \text{average} = \frac{\iint_D f(x,y)\,\mathrm{d}A}{\iint_D 1\,\mathrm{d}A}$$

$$\text{volume} = \iiint_V 1\,\mathrm{d}V \qquad \text{average} = \frac{\iiint_V f(x,y,z)\,\mathrm{d}V}{\iiint_V 1\,\mathrm{d}v}$$

$$m = \iiint_V \rho(x,y,z)\,\mathrm{d}V \qquad M_{yz} = \iiint_V x\rho(x,y,z)\,\mathrm{d}V$$

$$M_{xz} = \iiint_V y\rho(x,y,z)\,\mathrm{d}V \qquad M_{xy} = \iiint_V z\rho(x,y,z)\,\mathrm{d}V$$

$$\bar{x} = \frac{M_{yz}}{m} \quad \bar{y} = \frac{M_{xz}}{m} \quad \bar{z} = \frac{M_{xy}}{m}$$

For 2D, use $\iint_A (\cdots)\,\mathrm{d}A$ with $\rho(x,y)$ and $z = 0$. Then use $M_{yz} = M_y$, $M_{xz} = M_x$, $M_{xy} = 0$.

Coordinate Systems

Polar	$x = r\cos\theta$	$y = r\sin\theta$
	$r^2 = x^2 + y^2$	$\lvert J\rvert = r$
Cylindrical	Polar but with z parameter	
	$\lvert J\rvert = r$	
Spherical	$x = r\cos\theta\sin\varphi$	$y = r\sin\theta\sin\varphi$
	$z = r\cos\varphi$	$r^2 = x^2 + y^2 + z^2$
	$\lvert J\rvert = r^2\sin\varphi$	

$$\text{Jacobian} = |J| = \left|\det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}\right|$$

ducky's MATH2001 cheat sheet — 12th July, 2021

Vector Calculus

$$\boldsymbol{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$
$$D_{\hat{\mathbf{u}}}(f) = (\boldsymbol{\nabla} f) \cdot \hat{\mathbf{u}}$$

$$W = \int_C \mathbf{F} \cdot \,\mathrm{d}\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \,\mathrm{d}t$$
$$\int_C \boldsymbol{\nabla} f \cdot \,\mathrm{d}\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

$$\operatorname{div} \mathbf{v} = \boldsymbol{\nabla} \cdot \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$
$$\operatorname{curl} \mathbf{v} = \boldsymbol{\nabla} \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$
$$(\mathbf{r_u}(a,b) \times \mathbf{r_v}(a,b)) \cdot ((x,y,z) - \mathbf{r}(a,b)) = 0$$
$$\iint_S f(x,y,z) \,\mathrm{d}S = \iint_D f(\mathbf{r}(u,v)) \left\|\mathbf{r_u} \times \mathbf{r_v}\right\| \,\mathrm{d}A$$

$$\text{flux} = \int_C \mathbf{v} \cdot \mathbf{n} \,\mathrm{d}S = \int_a^b \mathbf{v}(\mathbf{r}(t)) \cdot \underbrace{\left(\mathbf{r}'(t) \times \hat{\mathbf{k}}\right)}_{\text{check orientation}} \,\mathrm{d}t$$
$$\text{flux} = \iint_S \mathbf{v} \cdot \mathbf{n} \,\mathrm{d}S = \iint_D \mathbf{v} \cdot (\mathbf{r_u} \times \mathbf{r_v}) \,\mathrm{d}A$$

$$\iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \,\mathrm{d}A = \oint_{\partial D} \mathbf{F} \cdot \,\mathrm{d}\mathbf{r}$$
$$\oint_{\partial D} \mathbf{v}(x,y) \cdot \mathbf{n} \,\mathrm{d}S = \iint_D \boldsymbol{\nabla} \cdot \mathbf{v}(x,y) \,\mathrm{d}A$$
$$\oiint_S \mathbf{F} \cdot \mathbf{n} \,\mathrm{d}S = \iiint_V \boldsymbol{\nabla} \cdot \mathbf{F} \,\mathrm{d}V$$
$$\iint_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \mathbf{n} \,\mathrm{d}S = \oint_{\partial S} \mathbf{F} \cdot \,\mathrm{d}\mathbf{r}$$

Miscellaneous

$$\int \frac{1}{\sqrt{x^2+1}} \,\mathrm{d}x = \operatorname{arsinh}(x) + c$$
$$\int \frac{1}{\sqrt{x^2-1}} \,\mathrm{d}x = \operatorname{arcosh}(x) + c, x > 1$$
$$\operatorname{arsinh}(x) = \ln\left(x + \sqrt{x^2+1}\right)$$
$$\operatorname{arcosh}(x) = \ln\left(x + \sqrt{x^2-1}\right), x \geq 1$$
$$\operatorname{artanh}(x) = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right), x \in (-1,1)$$

For some $A \in M_{n\times n}(\mathbb{F})$, the following statements are equivalent:

- A is non-singular (A^{-1} exists)
- Only $\mathbf{x} = \mathbf{0}$ satisfies $A\mathbf{x} = \mathbf{0}$
- The row-echelon form of A does not have a row of zeroes
- $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b} \in \mathbb{R}^n$
- $\det A \neq 0$
- Columns of A are linearly independent
- Rows of A are linearly independent
- $\dim(\operatorname{NS}(A)) = 0$
- $\operatorname{rank} A = n$
- $\lambda = 0$ is not an eigenvalue of A

$A \in M_{n\times n}(\mathbb{F})$ is diagonalisable ($A = PDP^{-1}$)

- iff A has n linearly independent eigenvectors
- iff algebraic and geometric multiplicities are equal for every eigenvalue

$A \in M_{n\times n}(\mathbb{R})$ is orthogonally diagonalisable ($A = PDP^{\mathrm{T}}$)

- iff A is symmetric, i.e. $A = A^{\mathrm{T}}$
- iff eigenvectors corresponding to different eigenvalues are orthogonal with respect to the dot product

The complex inner product for $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$ is defined by

$$\mathbf{u} \cdot \mathbf{v} = u_1\bar{v}_1 + \cdots + u_n\bar{v}_n = \mathbf{v}^*\mathbf{u} \in \mathbb{C}$$

where $\bar{z}$ is the conjugate of z and $\mathbf{v}^*$ is the conjugate transpose of $\mathbf{v}$. Now, $\overline{\langle \mathbf{u}, \mathbf{v} \rangle} = \langle \mathbf{v}, \mathbf{u} \rangle$.