Coupled ODEs

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{r}$$

$$\ddot{y} + p(t)y + q(t) = r(t) \longrightarrow \begin{pmatrix} y \\ \dot{y} \end{pmatrix} = \dot{\mathbf{y}} = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix} \mathbf{y} + \begin{pmatrix} 0 \\ r(t) \end{pmatrix}$$

The eigenvalues of $\mathbf{A}$ found by solving $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$ gives the same characteristic equation for the uncoupled system. General solutions for a second-order system with real constant coefficients, with eigenvalues λ_1, λ_2 and eigenvectors $\mathbf{x_1}, \mathbf{x_2}$, are given by:

$$\begin{aligned} \lambda_1 \neq \lambda_2: &\quad \mathbf{y} = c_1 \mathbf{x_1} e^{\lambda_1 t} + c_2 \mathbf{x_2} e^{\lambda t} \\ \lambda_1, \lambda_2 \in \mathbb{C}: &\quad \mathbf{y} = c_1 \operatorname{Re}\left(\mathbf{x_1} e^{\lambda_1 t}\right) + c_2 \operatorname{Im}\left(\mathbf{x_1} e^{\lambda_1 t}\right) \\ \lambda_1 = \lambda_2: &\quad \mathbf{y} = c_1 \mathbf{x} e^{\lambda t} + c_2 (t\mathbf{x} + \mathbf{p}) e^{\lambda t}, \text{ where } (\mathbf{A} - \lambda \mathbf{I})\mathbf{p} = \mathbf{x}. \end{aligned}$$

Phase Portraits of 2D Autonomous Systems

Let $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$ with $\mathbf{y} \in \mathbb{R}^2$, $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$, where $\mathbf{f}$ does not depend on t. $\mathbf{f}$ defines a direction field in the y_1-y_2 plane, and a trajectory of the system is a curve following the field. $\mathbf{f}$ being continuous with continuous partials implies existence and uniqueness for these trajectories, so they do not cross. By the chain rule, the slope of a trajectory is

$$\frac{dy_2}{dy_1} = \frac{\dot{y}_2}{\dot{y}_1} = \frac{f_2}{f_1}.$$

A nullcline is a curve in phase space where the slope of the trajectory is either 0 or ∞. Set $\dot{y}_2 = 0$ for horizontal nullclines and $\dot{y}_1 = 0$ for vertical nullclines, and solve for y_2. Solving for the trajectories themselves is possible in specific cases, usually when the ODE in y_2 from the slope formula is separable.

For the lines in phase space in the direction of the eigenvectors (called eigendirections), there are straight line solutions either flowing towards or away from the origin, determined by the sign of the corresponding eigenvalue ($-$ stable, $+$ unstable).

$\lambda_1 \neq \lambda_2$ and both have the same sign:

- Negative λ gives a *stable improper node*, where trajectories curve like a parabola and approach the gendirection with smallest absolute value, since that exponential term dominates at large t.
- Positive λ gives an *unstable improper node*, where trajectories approach eigendirection with largest absolute value.

$\lambda_1 > 0$, $\lambda_2 < 0$:

- This gives a *saddle point*. Trajectories come in parallel to the stable eigendirection then turn around and become parallel to the unstable direction.

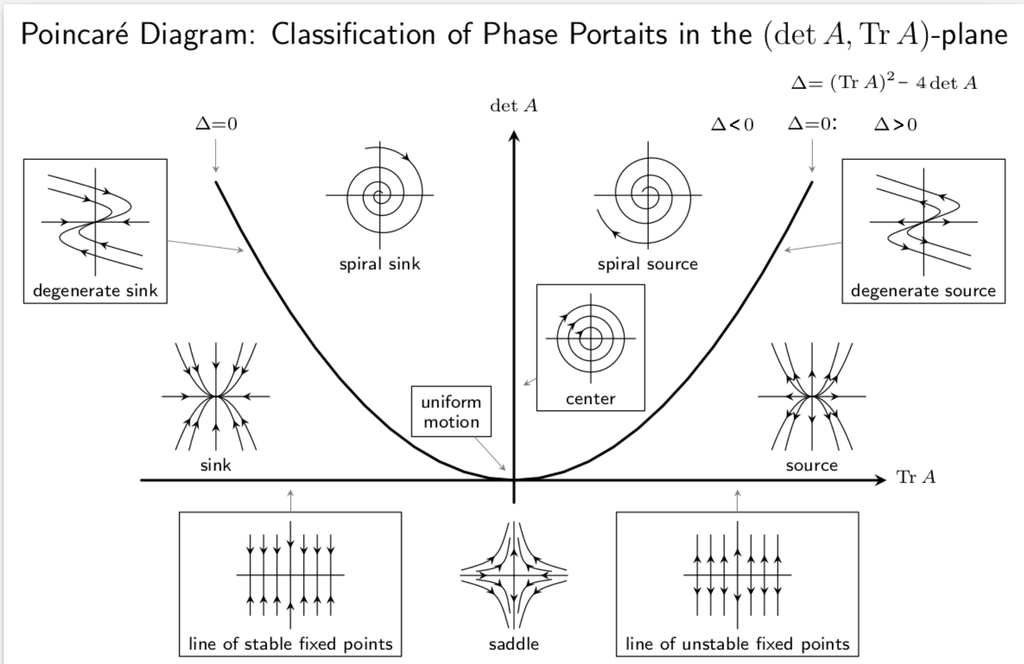
$\lambda_1 = \lambda_2$:

- If there are 2 linearly independent eigenvectors (eigenspace of λ has dimension 2), all trajectories travel in straight lines either towards or away from the origin, called a *proper node*. For a 2D system, this is only possible if $\mathbf{A}$ is some multiple of the identity matrix, i.e. $\mathbf{A} = k\mathbf{I}$ for $k \in \mathbb{R}$.
- If there is only 1 eigenvector, trajectories come tangent to the eigendirection near the origin and do a U-turn away, called a *degenerate node*.

$\lambda = \alpha \pm \beta i$:

- If $\alpha = 0$, trajectories are ellipses around the origin, and the origin is called a *centre*. To determine the direction of flow, consider the sign of $\dot{y}_1$ at a point in the positive quadrant.
- If $\alpha \neq 0$, trajectories trace out a logarithmic spiral $r(t) = e^{\alpha t}$ shown by converting to polar coordinates. This is called a *spiral*. Plotting nullclines helps with sketching trajectories.

The following stability chart allows for the system to be classified given values of $\operatorname{tr}\mathbf{A}$ and $\det \mathbf{A}$ [2]. The discriminant Δ is defined as $\Delta = (\operatorname{tr}\mathbf{A})^2 - 4\det\mathbf{A}$.



Non-Linear or Inhomogeneous Systems

A system $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$ has a critical point at $\mathbf{y}^*$ if $\mathbf{f}(\mathbf{y}^*) = \mathbf{0}$. Non-linear systems may have more than 1 critical point, but linear $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}$ always has one at $\mathbf{0}$, and this is the only critical point unless $\det \mathbf{A} = 0$.

An inhomogeneous linear system $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{p}$ is a linear translation away from a homogeneous system. Shifting the (unique) critical point at $\mathbf{p}$ to $\mathbf{0}$ via the change of variables $\mathbf{x} = \mathbf{y} - \mathbf{p}$ gives the homogeneous $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, so the previous techniques are applicable.

For a nonlinear system $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$, the nature of the system around the critical point is approximated via the Jacobian matrix $\mathbf{Df}$. The stability diagram can be used to classify these critical points by evaluating $\mathbf{Df}$ at each $\mathbf{y}^*$. The linearised system near $\mathbf{y}^*$ is then

$$\dot{\mathbf{y}} = \mathbf{Df}|_{\mathbf{y}^*}(\mathbf{y} - \mathbf{y}^*), \quad \mathbf{Df} = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{pmatrix}.$$

Special Functions

The Dirac Delta function $\delta(t)$ is an "impulse" with infinite value at $t = 0$ and zero value everywhere else. It satisfies $\int_0^\infty \delta(t) = 1$. An impulse applied at $t = a$ is represented by $\delta(t - a)$.

The unit step function $u(t)$ is defined by

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0. \end{cases}$$

Many piecewise-continuous functions can be written in terms of u. As described in [1], if $f(t)$ changes from $g_1(t)$ to $g_2(t)$ at $t = c$, add a $u(t - c)(g_2(t) - g_1(t))$ term to the unit step representation of f.

The Gamma function is defined as

$$\Gamma(z) = \int_0^\infty e^{-x} x^{z-1} \, \mathrm{d}x,$$

which generalises the factorial: $\Gamma(n + 1) = n!$ for $n \in \mathbb{N}$.

Although there are a few estabilished conventions, the error function as defined in MATH2100 is given by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} \, \mathrm{d}u.$$

It is odd and monotone increasing. The following formulas may help when doing computation with erf:

$$\int_0^\infty e^{-u^2} \, \mathrm{d}u = \int_{-\infty}^0 e^{-u^2} \, \mathrm{d}u = \frac{\sqrt{\pi}}{2}, \quad \frac{1}{\sqrt{\pi}} \int_a^b e^{-u^2} \, \mathrm{d}u = \operatorname{erf}(b) - \operatorname{erf}(a).$$

For the heat equation with parameter c, the Gaussian function $G(x, t)$ is defined as

$$G(x, t) = \frac{1}{\sqrt{4\pi c^2 t}} \exp\left(\frac{-x^2}{4c^2 t}\right).$$

A few properties are $\int_{-\infty}^\infty G(x, t) \, \mathrm{d}x = 1$ for $t > 0$, $\lim_{t \to 0^+} G(x, t) = 0$ for $x \neq 0$ and $\lim_{t \to 0^+} G(0, t) = +\infty$.

Laplace Transforms

The following formulas are given in the formula sheet for the exam, but have been added here for completeness.

$$\mathscr{L}\{f(t)\} = F(s) = \int_0^\infty f(t) e^{-st} \, \mathrm{d}t$$

$f(x)$	$F(s)$	$f(x)$	$F(s)$
t^n	$\frac{n!}{s^{n+1}}$	t^a	$\frac{\Gamma(a+1)}{s^{a+1}}$
$\cos \alpha t$	$\frac{s}{s^2 + \alpha^2}$	$\sin \alpha t$	$\frac{\alpha}{s^2 + \alpha^2}$

The following formulas are not included in the formula sheet, but may be helpful to know.

$$\mathscr{L}^{-1}\left\{\frac{s}{(s^2 + \alpha^2)^2}\right\} = \frac{1}{2\alpha} t \sin \alpha t$$

$$\mathscr{L}\{f(t) u(t - k)\} = e^{-ks} \mathscr{L}\{f(t + k)\}.$$

Differential Operators and Change of Variables

References

[1] Andrew D. Loveless. *Laplace Transform Practice Problems*. 2019. URL: https://sites.math.washington.edu/~aloveles/Math307Fall2019/m307LaplacePractice.pdf.

[2] Freesodas. *Stability Diagram*. 2018. URL: https://commons.wikimedia.org/wiki/File:Stability_Diagram.png.