ALGEBRAIC FORMULATION AND GRAPHICAL REPRESENTATION

Dr. Natalia Summerville

Linear Programming (LP)

- Linearity in objective function and constraints
- Continuous decision variables

$$\max x_1 + x_2 \\ st. \\ x_1 \le 5 \\ x_2 \le 4$$

$$min\, x_1 + x_2 \\ st. \\ x_1 \ge 1 \\ x_2 \ge 2$$

$$max\, x_1 + x_2 \\ st. \\ x_1 \ge 1 \\ x_2 \ge 2$$

Unbounded

$$min\, x_1 \\ st. \\ x_1 \ge 1 \\ x_2 \ge 2$$

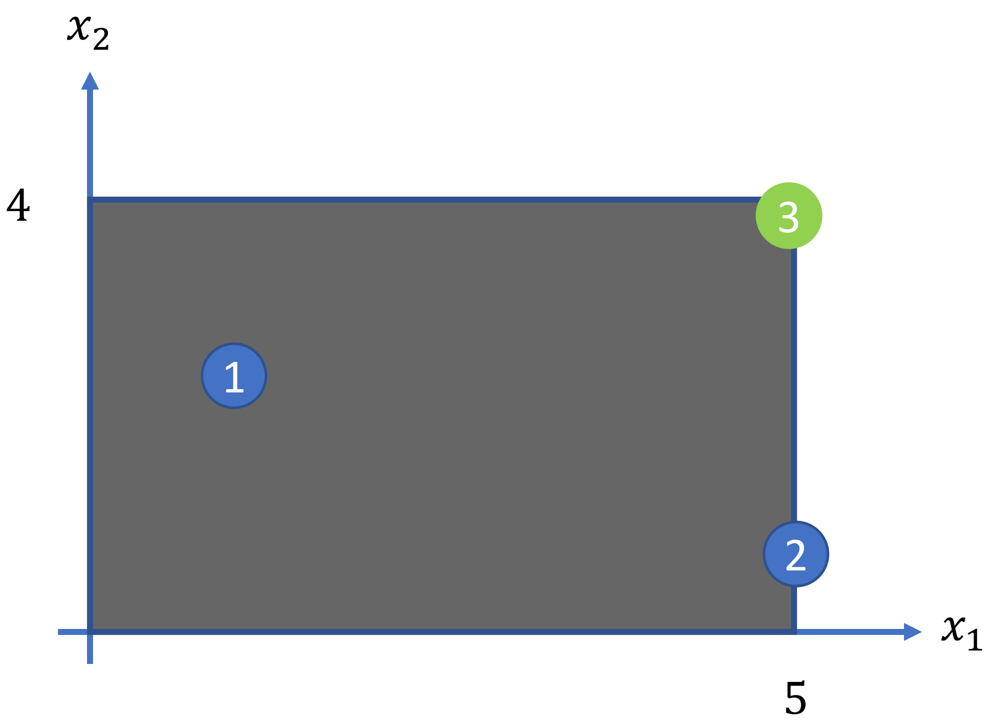
Degenerate

$$min\, x_1 \\ st. \\ x_1 \le 1 \\ x_1 \ge 2$$

Infeasible

Linear Programming

$$\max z = x_1 + x_2$$
$st.$
$$x_1 \le 5$$
$$x_2 \le 4$$
$$x_i \ge 0 \; \forall \; i$$



Solution 1:	Solution 2:	Solution 3:
$x_1 = 1.2$	$x_1 = 5$	$x_1 = 5$
$x_2 = 2.9$	$x_2 = .9$	$x_2 = 4$
$z = 4.1$	$z = 5.9$	$z = 9$
Feasible Sub-Optimum	Feasible Sub-Optimum	Optimum

Mixed Integer Linear Programming (MILP)

- Linearity in objective function and constraints
- Continuous and integer/binary decision variables

$$\max z = x_1 + x_2$$

$$st.$$

$$x_1 \leq 3.3$$

$$x_2 \leq 3$$

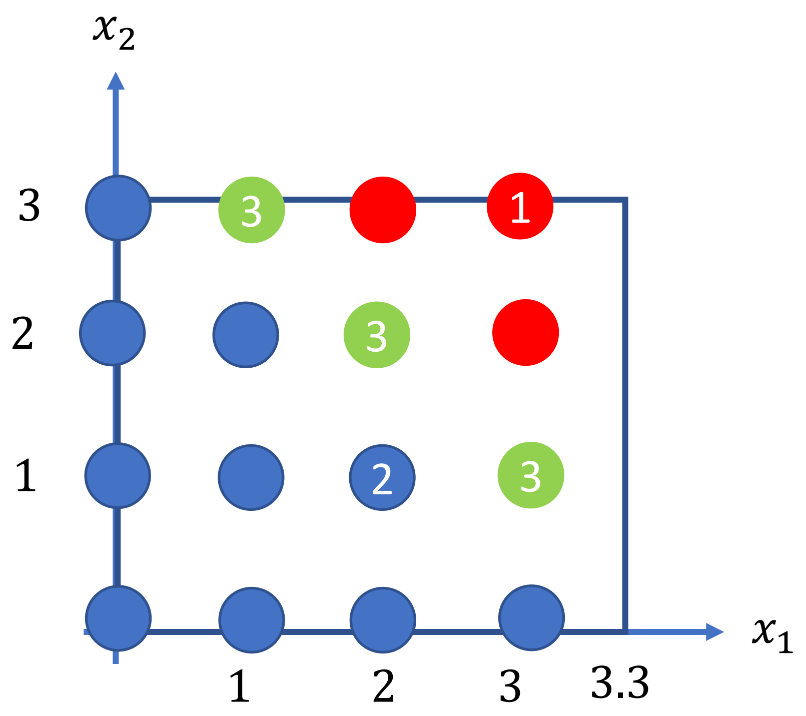
$$x_1 + x_2 \leq 4$$

$$x_i \geq 0 \; \forall \; i$$

$$x_i \in \mathbb{I} \; \forall i$$

Mixed Integer Linear Programming

$$\max z = x_1 + x_2$$
$st.$
$$x_1 \leq 3.3$$
$$x_2 \leq 3$$
$$x_1 + x_2 \leq 4$$
$$x_i \geq 0 \ \forall\, i$$
$$x_i \in \mathbb{I} \ \forall i$$



Solution 1:
$x_1 = 3$
$x_2 = 3$
$x_1 + x_2 = 6$
$z = 6$

Infeasible

Solution 2:
$x_1 = 2$
$x_2 = 1$
$x_1 + x_2 = 3$
$z = 3$

Sub-Optimum

Solutions 3:
$x_1 + x_2 = 4$
$z = 4$

Optimum & Degenerate

Goal Programming

- Objective is to achieve a certain target value
- Minimize penalty associated of not achieving the value

$$
\begin{aligned}
&\min pen \\
&st. \\
&\quad \text{pen} \geq 10 - x \\
&\quad \text{pen} \geq x - 10 \\
&\quad \text{pen} \geq 0
\end{aligned}
$$

Goal = 10

Solution 1:
if $x = 9$
then $pen = 1$

Sub-Optimum

Solution 2:
if $x = 12$
then $pen = 2$

Sub-Optimum

Solution 3:
if $x = 10$
then $pen = 0$

Optimum

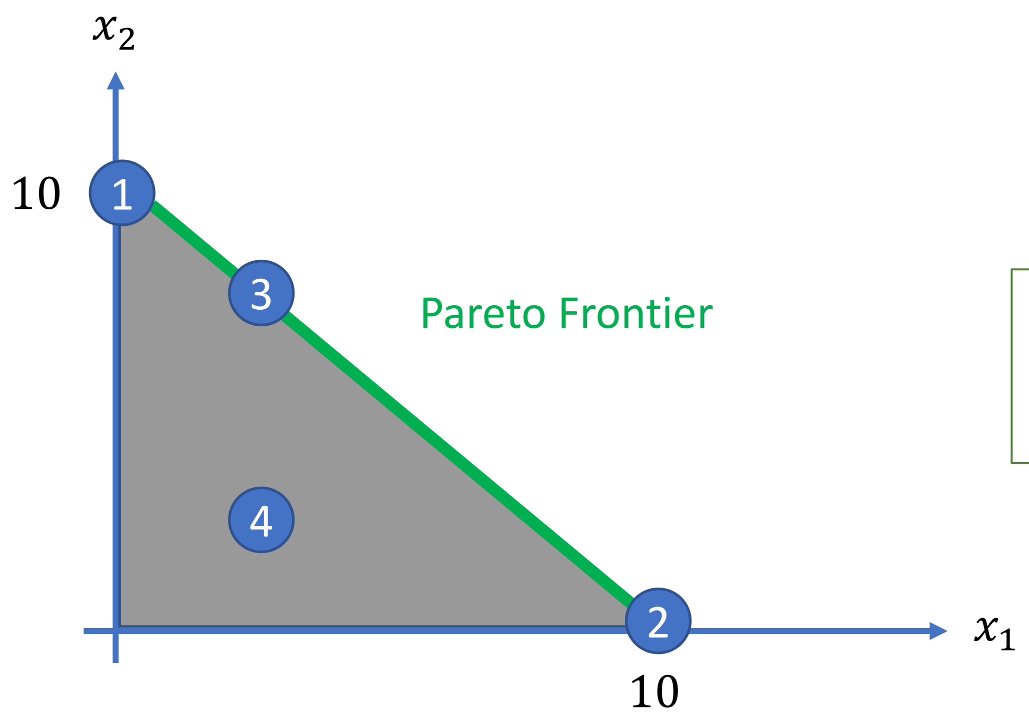
Multicriteria Programming

- Optimize several objectives at the same time

$$
\begin{aligned}
&\max x_1 \\
&\max x_2 \\
&st. \\
&\quad x_1 + x_2 \leq 10 \\
&\quad x_i \geq 0, \forall i
\end{aligned}
$$

Multicriteria Programming

$$\max x_1$$
$$\max x_2$$
$$st.$$
$$x_1 + x_2 \leq 10$$
$$x_i \geq 0, \forall i$$



Solution 1:	Solution 2:	Solution 3:	Solution 4:
$x_1 = 0$	$x_1 = 10$	$x_1 = 8$	$x_1 = 2$
$x_2 = 10$	$x_2 = 0$	$x_2 = 2$	$x_2 = 2$
Pareto Optimum	Pareto Optimum	Pareto Optimum	Dominated

Non-Linear Programming

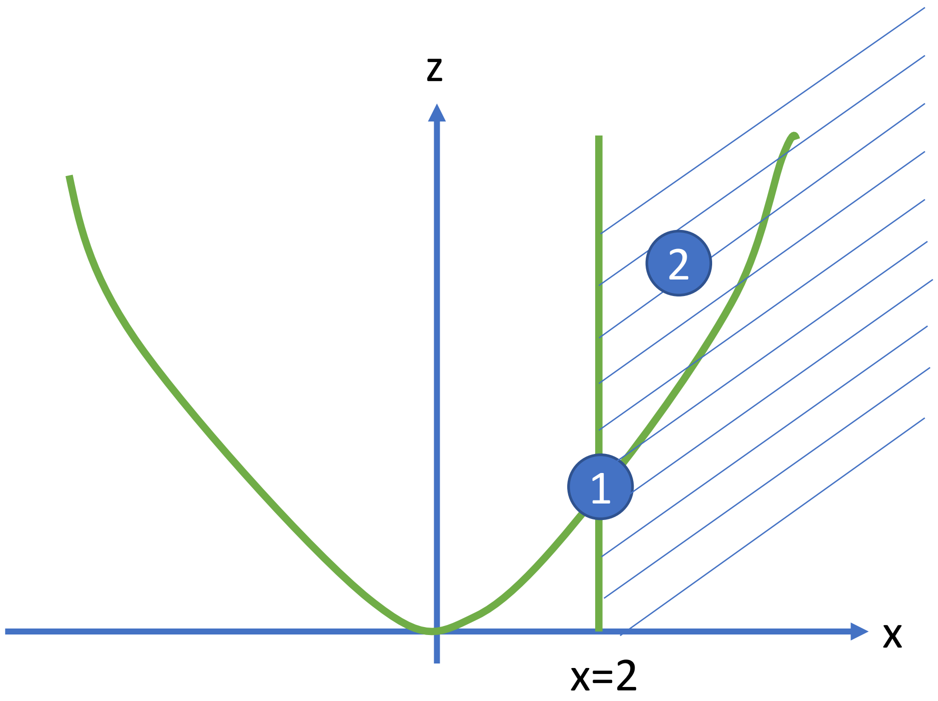
- Nonlinearity either in objective function or in constraints

$$\begin{aligned} &\min z = x^2 \\ &st. \\ &\quad x \geq 2 \end{aligned}$$

Non-Linear Programming

$$\min z = x^2$$
$$st.$$
$$x \geq 2$$

- Nonlinearity either in objective function or in constraints



Solution 1:	Solution 2:
$x = 2$	$x = 2.5$
$z = 4$	$z = 6.25$
Optimum	Feasible Sub-Optimum

Mathematical Formulation General Form - LP

$$\max(\min) \quad c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

$$s.t.$$

$$a_{11}x_1 + a_{21}x_2 + \cdots + a_{n1}x_n \leq (\geq) b_1$$
$$a_{12}x_1 + a_{22}x_2 + \cdots + a_{n2}x_n \leq (\geq) b_2$$
$$\vdots$$
$$a_{1m}x_1 + a_{2m}x_2 + \cdots + a_{nm}x_n \leq (\geq) b_m$$

$$x_1 \geq 0$$
$$x_2 \geq 0$$
$$\vdots$$
$$x_n \geq 0$$

$$\max(\min) \quad \sum_{i=1}^{n} c_i x_i$$

$$s.t.$$

$$\sum_{i=1}^{n} a_{ij}x_i \leq (\geq) b_j \quad \forall\, j = 1 \ldots m$$
$$x_i \geq 0 \quad \forall i = 1 \ldots n$$

[Refresher on Linear Algebra]

Building Blocks for Problem Solving:

- Sets
- Summations

Mathematical Formulation General Form - MILP

$$\max(\min)\quad c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

$s.t.$

$$a_{11}x_1 + a_{21}x_2 + \cdots + a_{n1}x_n \le (\ge) b_1$$
$$a_{12}x_1 + a_{22}x_2 + \cdots + a_{n2}x_n \le (\ge) b_2$$
$$\vdots$$
$$a_{1m}x_1 + a_{2m}x_2 + \cdots + a_{nm}x_n \le (\ge) b_m$$

$$x_i \in \mathbb{I} \ \ \forall\, i \in I$$
$$x_i \in \{0,1\} \ \forall\, i \in B$$
$$x_i \ge 0 \ \forall\, i \in C$$

$$\max(\min)\quad \sum_{i=1}^{n} c_i x_i$$

$s.t.$

$$\sum_{i=1}^{n} a_{ij} x_i \le (\ge) b_j \quad \forall\, j = 1 \ldots m$$

$$x_i \in \mathbb{I} \ \ \forall\, i \in I$$
$$x_i \in \{0,1\} \ \forall\, i \in B$$
$$x_i \ge 0 \ \forall\, i \in C$$

Steps to Formulate Optimization Problems

1. Define Decision Variables
2. State Objective Function
3. State Constraints
4. Identify bounds on Decision Variables