PhD Bootcamp: Distributions and Inference

Glenn Palmer

Duke University – Department of Statistical Science

August 18, 2023

Acknowledgment: These slides and the accompanying exercises contain materials created by Alex Dombowsky and Jennifer Kampe.

Hello!

Welcome to the department! Today's bootcamp session is structured as follows:

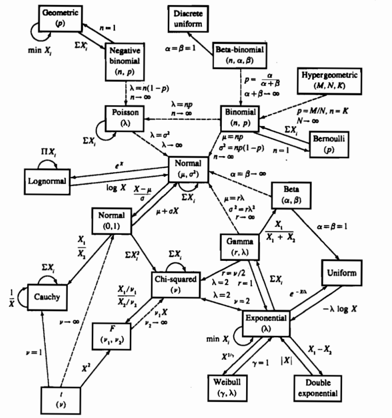
- A review of concepts that you should be comfortable with before starting classes (with a heavy focus on basic distribution theory).
- A list of review exercises that *you should do* to warm-up your stats knowledge before the school year begins.

Distribution reference sheet from STA 711

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\, e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{Z}_+$	q/p	q/p^2	$(q=1-p)$
		$f(y)=p\,q^{y-1}$	$y\in\{1,\ldots\}$	$1/p$	q/p^2	$(y=x+1)$
HyperGeo.	$\mathsf{HG}(n,A,B)$	$f(x)=\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P=\frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\, q^x$	$x\in\mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
		$f(y)=\binom{y-1}{y-\alpha}p^\alpha\, q^{y-\alpha}$	$y\in\{\alpha,\ldots\}$	α/p	$\alpha q/p^2$	$(y=x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\epsilon)$	$f(x)=(\alpha/\epsilon)(1+x/\epsilon)^{-\alpha-1}$	$x\in\mathbb{R}_+$	$\frac{\epsilon}{\alpha-1}$ if $\alpha>1$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$ if $\alpha>2$	
		$f(y)=\alpha\,\epsilon^\alpha/y^{\alpha+1}$	$y\in(\epsilon,\infty)$	$\frac{\epsilon\,\alpha}{\alpha-1}$ if $\alpha>1$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$ if $\alpha>2$	$(y=x+\epsilon)$
Poisson	$\mathsf{Po}(\lambda)$	$f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$	$x\in\mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x\in\mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$ if $\nu_2>2$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$ if $\nu_2>4$	
Student t	$t(\nu)$	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0 if $\nu>1$	$\frac{\nu}{\nu-2}$ if $\nu>2$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta)$	$f(x)=\alpha\beta\, x^{\alpha-1}\, e^{-\beta\, x^\alpha}$	$x\in\mathbb{R}_+$	$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$	$\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$	

Important relationships between distributions

TABLE OF COMMON DISTRIBUTIONS 627



Relationships among common distributions. Solid lines represent transformations and special cases, dashed lines represent limits. Adapted from Leemis (1986). [1]

[1] Casella, G., Berger, R. L. (2021). Statistical inference. Cengage Learning.

Cumulative Distribution Functions

- The distribution of a real-valued random-variable X is defined by its **cumulative distribution function**, $F_X(x) = P(X \leq x)$. The CDF is right continuous, non-decreasing, and has limits 0 and 1 as x tends to $-$ or $+\infty$.
- The CDF is usually represented as an integral over another function, so that

$$F_X(x) = \int_{-\infty}^{x} dF_X(t).$$

Probability Density and Mass Functions

- While the dF_X notation may be unfamilar, it is defined as

$$\int_{-\infty}^{x} dF_X(t) = \begin{cases} \int_{-\infty}^{x} f_X(t)dt: & \text{continuous rv} \\ \sum_{t=-\infty}^{x} f_X(t): & \text{discrete rv} \end{cases}.$$

- $f_X(x)$ is the **probability mass** (discrete) or **density** (continuous) function. It is often convenient to write

$$f_X(x) = \frac{h(x)}{c}$$

for **kernel** h and **normalizing constant** $c < \infty$. We assume $h(x) \geq 0$ and

$$c = \begin{cases} \int_{-\infty}^{\infty} h(x)dx: & \text{continuous rv} \\ \sum_{x=-\infty}^{\infty} h(x): & \text{discrete rv} \end{cases}.$$

Multivariate Random Variables

- Random variables $\boldsymbol{X}$ can be defined on $\mathbb{R}^m$. The multivariate CDF is

$$F_{\boldsymbol{X}}(\boldsymbol{x}) = P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_m \leq x_m)$$

$$= \begin{cases} \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_m} f_{\boldsymbol{X}}(\boldsymbol{t}) dt_1 \cdots dt_m : & \text{continuous rvs} \\ \sum_{t_1=-\infty}^{x_1} \cdots \sum_{t_m=-\infty}^{x_m} f_{\boldsymbol{X}}(\boldsymbol{t}) : & \text{discrete rvs} \end{cases}.$$

- This can be generalized to random vectors consisting of discrete and continuous rvs. To recover the PDF/PMF of, say, X_1, we merely integrate out all other variables:

$$f_{X_1}(x_1) = \begin{cases} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{\boldsymbol{X}}(x_1, t_2, \ldots, t_m) dt_2 \cdots dt_m : & \text{cont.} \\ \sum_{t_2=-\infty}^{\infty} \cdots \sum_{t_m=-\infty}^{\infty} f_{\boldsymbol{X}}(x_1, t_2, \ldots, t_m) : & \text{disc.} \end{cases}.$$

Independence and Covariance

- Two random variables are **independent** if their joint density/mass function factorizes into the product of their marginal distributions, i.e.

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) \; \forall \, x, y.$$

- The **covariance** of two random variables is

$$\mathrm{Cov}(X,Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y].$$

- Covariance says something about the relationship between X and Y (note that the outer expectation is with respect to their *joint* distribution).

Does covariance tell us anything about independence?

- We can roughly describe how two random variables affect each other with **correlation**:

$$\rho_{XY} = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}} \in [-1, 1].$$

- $\rho_{XY} > 0$ implies X and Y are *positively correlated*, ie. an increase in X *tends to result* in an increase in Y (and X and Y are dependent).
- $\rho_{XY} < 0$ implies X and Y are *negatively correlated*, ie. an increase in X *tends to result* in an decrease in Y (and X and Y are dependent).
- What about $\rho_{XY} = 0$?

Example:

Let $X \sim \text{Unif}(-1, 1)$, and let $Y = X^2$.

- Are X and Y independent?
- Are X and Y correlated?

Zero Correlation

- **The problem with correlation**: it describes (approximately) linear relationships.
- In a sense, ρ_{XY} may be interepreted as the sign of a in a linear equation $Y = aX + \epsilon$.
- But what if the relationship between X and Y is *not linear* (eg. quadratic, cubic, sinusoidal, step functions, etc.).
- As it turns out,

$$\rho_{XY} = 0 \not\Rightarrow X \text{ and } Y \text{ are independent.}$$

- So, correlation can only tell us about the dependence structure if it is non-zero.

Conditional Distributions

- The conditional PDF/PMF of $X \mid Y = y$ is

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{f_{X,Y}(x,y)}{\int f_{X,Y}(x,y)dx}.$$

- **Bayes theorem** gives us a way to do "backward conditioning"

$$f_{\Theta|X}(\theta \mid x) = \frac{f_{X,\Theta}(x,\theta)}{f_X(x)} = \frac{f_{X|\Theta}(x \mid \theta) f_\Theta(\theta)}{f_X(x)} = \frac{f_{X|\Theta}(x \mid \theta) f_\Theta(\theta)}{\int f_{X|\Theta}(x \mid \theta) f_\Theta(\theta) d\theta}.$$

- Note that the denominator does not depend on θ.

Example: Finding a posterior distribution

Let $\Theta \sim Beta(a, b)$, with density

$$f_\Theta(\theta) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\theta^{a-1}(1-\theta)^{b-1},$$

and let $X|\Theta = \theta \sim Binomial(n, \theta)$ so that

$$f_{X|\Theta}(x|\theta) = \binom{n}{x}\theta^x(1-\theta)^{n-x}.$$

Show that $\Theta|X = x \sim Beta(a + x, n + b - x)$.

Conditional Expectations

- The conditional expectation of $X \mid Y = y$ is

$$\mathbb{E}[X \mid Y = y] = \begin{cases} \int_{-\infty}^{\infty} x f_{X|Y}(x \mid y) dx \\ \sum_{x=-\infty}^{\infty} x f_{X|Y}(x \mid y) \end{cases}$$

  and will be a function of y.
- As such, we can define the random variable $\mathbb{E}[X \mid Y]$.
- **Law of Total Expectation**:

$$\mathbb{E}[\mathbb{E}[X \mid Y]] = \mathbb{E}[X].$$

- **Law of Total Variance**:

$$\mathbb{E}[\mathrm{Var}(X \mid Y)] + \mathrm{Var}(\mathbb{E}[X \mid Y]) = \mathrm{Var}(X).$$

Example: Iterated expectations and variances

A store has N customers in a day, where $N \sim \text{Poisson}(\lambda)$. The amount of money spent by the jth customer is denoted X_j. Assume the X_j's are iid with mean μ and variance σ^2, and that they are independent of N. Let X be the total revenue for the day, so that

$$X = \sum_{j=1}^{N} X_j.$$

Find $\mathbb{E}[X]$ and $\text{Var}(X)$.

Moment Generating Functions

For a random variable X, the **moment generating function** (MGF) is the real-valued function

$$M_X(t) = \mathbb{E}[e^{tX}]$$

for all $t \in \mathbb{R}$. If the MGF is finite for an open interval around 0,

$$\mathbb{E}[X^n] = \left.\frac{dM_X(t)}{dt^n}\right|_{t=0}.$$

MGF Properties

1. **Uniqueness property**: If $M_X(t) = M_Y(t)$ for all $t \in \mathbb{R}$, then $F_X(x) = F_Y(x)$ for all $x \in \mathbb{R}$ (ie, $X \stackrel{d}{=} Y$).
2. **Linear transformations**: For all $a, b \in \mathbb{R}$,

$$M_{aX+b} = e^{bt} M_X(at).$$

3. **Linear combinations**: Let $X_1, \dots, X_n$ be *independent*, $a_i \in \mathbb{R}$, and $S_n = \sum_{i=1}^n a_i X_i$. Then

$$M_{S_n}(t) = \prod_{i=1}^n M_{X_i}(a_i t).$$

Example: Find the MGF

A chi-squared random variable with k degrees of freedom (usually written as χ^2_k) has support $[0, \infty)$ and pdf:

$$f(x) = \frac{1}{2^{k/2}\Gamma(k/2)} x^{k/2-1} e^{-x/2}.$$

Let $X \sim \chi^2_1$, and find the moment generating function $M_X(t)$.

Change of Variables

Motivation: Let X be a real-valued random variable with pdf $f_X(x)$ and let $Y = g(X)$ for some one-to-one differentiable function $g(x)$. Then Y will also have a continuous distribution - what is it?

One Dimension: let $Y = g(X)$, g monotone with $X = g^{-1}(Y) = h(Y)$, then

$$X \sim f_X(x) \implies f_Y(y) = f_X(h(y))|dh/dy|$$

Example: 1D change of variables

Let $X \sim N(0, 1)$, and let $Y = a + bX$. Find the pdf of Y, and specify the support.

Proof of 1D change-of-variables formula

Let $Y = g(X)$, g monotone increasing with $X = g^{-1}(Y) = h(Y)$. Then if the CDF for X is given by $F_X(x)$, then the CDF for Y is:

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(g(X) \leq y) \\ &= P(X \leq g^{-1}(y)) \\ &= F_X(h(y)) \end{aligned}$$

Taking the derivative w.r.t. y yields

$$f_Y(y) = f_X(h(y))\frac{dh}{dy}$$

Finally, repeat the above for g monotone decreasing – the result will be $f_Y(y) = -f_X(h(y))\left(\frac{dh}{dy}\right)$, where dh/dy is negative, hence the absolute value in the formula.

What if g isn't one-to-one? (Part 1)

Two other strategies are:

- Try to express the CDF of Y in terms of the CDF of X, and then take a derivative. (Similar to the proof of the change-of-variables formula.)
- Try finding the MGF of $g(X)$ and see if you recognize it as the MGF of a known distribution.

Change of Variables: d-Dimensions

Let $\boldsymbol{X} = (X_1, \cdots, X_{d1})$ be a collection of random variables with support $\mathbb{X}^{(d_1)}$ and joint pdf $f_{X_1,\cdots,X_{d_1}}$, and let

$$\boldsymbol{Y} = g(\boldsymbol{X}) \leftrightarrow (Y_1, \cdots, Y_{d_2}) = (g_1(\boldsymbol{X}), \cdots, g_{d_2}(\boldsymbol{X})),$$

where $g : \mathbb{X}^{d_1} \to \mathbb{R}^{d_2}$ and $h = g^{-1} : \mathbb{R}^{d_1} \to \mathbb{X}^{d_2}$
Then $\boldsymbol{Y}$ has joint pdf:

$$f_{\boldsymbol{Y}}(\boldsymbol{y}) = f_{\boldsymbol{X}}\left(h_1(\boldsymbol{Y}), \cdots, h_{d1}(\boldsymbol{Y})\right) \times |J(\boldsymbol{Y})|$$

Change of Variables: Step-by-Step

1. Note the set of transformation functions $g = (g_1, \cdots, g_{d_2})$:

$$
\begin{aligned}
Y_1 &= g_1(X_1, \cdots, X_{d_1}) \\
&\vdots \\
Y_{d_2} &= g_{d_2}(X_1, \cdots, X_{d_1})
\end{aligned}
$$

2. Find the set of inverse functions, $h = g^{-1}(\boldsymbol{X})$:

$$
\begin{aligned}
X_1 &= h_1(Y_1, \cdots, Y_{d_2}) \\
&\vdots \\
X_{d_1} &= h_{d_1}(Y_1, \cdots, Y_{d_2})
\end{aligned}
$$

3. Identify the joint support of the new variables, $\mathbb{Y}^{d_2}$

4. Compute the Jacobian of the inverse transformation $h(\boldsymbol{Y})$ in Step 2: form the matrix of partial derivatives and take its determinant.

$$D_y = \left[\frac{\partial x_i}{\partial y_j}\right]_{ij} = \begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_{d_2}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_{d_1}}{\partial y_1} & \frac{\partial x_{d_1}}{\partial y_2} & \cdots & \frac{\partial x_{d_1}}{\partial y_{d_2}} \end{bmatrix}$$

Set $J(y_1, \cdots, y_{d_2}) = \det D_y$. Alternately, note $J(\boldsymbol{Y}) = \frac{1}{J(\boldsymbol{X})}$

5. The joint pdf of $(Y_1, \cdots, Y_{d_1})$ is

$$f_{\boldsymbol{Y}}(\boldsymbol{y}) = f_{\boldsymbol{X}}\left(h_1(\boldsymbol{Y}), \cdots, h_{d1}(\boldsymbol{Y})\right) \times |J(\boldsymbol{Y})|$$

What if g is not one-one? (Part 2)

Make it one-to-one! **For example**:

1. Let $g : \mathbb{R}^2 \to \mathbb{R}$ and suppose we know the distribution of (X_1, X_2) (and at least one of the marginal distributions).
2. Set up a one-to-one transformation:

$$Y_1 = g(X_1, X_2) \text{ and } Y_2 = X_1 \text{ (or } X_2)$$

   and find the distribution of (Y_1, Y_2).
3. Then use marginalization:

$$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{Y_1,X_1}(y_1, x_1)dx_1 = \int_{-\infty}^{\infty} f_{Y_1,X_2}(y_1, x_2)dx_2.$$

Examples: change of variables when g isn't one-to-one

1. Let $X \sim N(0,1)$ and let $Y = X^2$. Find the distribution of Y.
2. Let $X_1 \sim Gamma(a, \xi)$ and let $X_2 \sim Gamma(b, \xi)$, and let X_1 and X_2 be independent. What is the distribution of $W = \frac{X_1}{X_1 + X_2}$?

Example: Multivariate Normal Distribution

Let $Z_1, ..., Z_p \overset{iid}{\sim} N(0,1)$, and let $Z = (Z_1, ..., Z_p)^T \in \mathbb{R}^p$. Let $Y = \mu + AZ$, where $\mu \in \mathbb{R}^p$ and $A \in \mathbb{R}^{p \times p}$ are constants and A is full rank.

Defining $\Sigma = AA^T$, show that the pdf of Y is

$$f_Y(y) = (2\pi)^{-p/2}|\Sigma|^{-1/2} \exp\left(-1/2(y-\mu)^T\Sigma^{-1}(y-\mu)\right).$$

Thanks!

- The rest of these slides also have some great info in them (some of which might be helpful for a few parts of the exercises). You should definitely read them if you have time, but in my opinion they're not quite as critical for day 1 of your fall courses.
- Let me know if you have any questions about anything in the slides or the exercises!

Characteristic Functions

Similarly, the **characteristic function** (CF) is the complex function

$$\varphi_X(t) = \mathbb{E}[e^{itX}] = \mathbb{E}[\cos(tX) + i\sin(tX)]$$

for $t \in \mathbb{R}$. For all t such that $M_X(t)$ is finite,

$$\varphi_X(-it) = M_X(t).$$

The CF has many of the same properties as the MGF. However, the CF *always exists* for all $t \in \mathbb{R}$ and, in some cases, is easier to calculate than the MGF.

The Likelihood Function

- If $X_1, \cdots, X_n$ are and i.i.d. sample from a population with pdf/pmf $f(x \mid \boldsymbol{\theta})$ the **likelihood function** is

$$L(\boldsymbol{\theta}|x_1 \cdots, x_n) = \prod_{i=1}^{n} f(x_i \mid \boldsymbol{\theta})$$

- Density function versus likelihood: the density function $f(x \mid \boldsymbol{\theta})$ is a non-negative function of the data x that integrates to 1. The likelihood function is a function of the parameters $\boldsymbol{\theta}$ and typically will not integrate to 1

Maximum Likelihood Estimation

- **Maximum likelihood estimation** finds values of the parameters that maximize the likelihood function:
  $\hat{\theta} = \arg\max_{\theta \in \Theta} L(\boldsymbol{\theta}|\boldsymbol{x})$
- If the likelihood function is differentiable, then candidates for the MLE satisfy $\frac{\partial}{\partial \theta_i} L(\boldsymbol{\theta}|\boldsymbol{X}) = 0, i = 1, \cdots, k$.
- Since $log(t)$ is a monotonically increasing function of t, for any positive valued function f, $\arg\max_\theta f(x) = \arg\max_\theta \log f(x)$.
- Verify that the identified root is a local max by checking that the Hessian matrix is negative semi-definite at $\hat{\boldsymbol{\theta}}$.
- **Invariance property**: if $\hat{\theta}$ is the MLE for θ, then $g(\hat{\theta})$ is the MLE for $g(\theta)$

Convergence in Probability and Distribution

- Suppose we have an infinite sequence of random variables $X_1, X_2, \ldots$. What happens as $n \to \infty$? Can it "converge" like a sequence of real numbers? It turns out it can... in several ways!
- The sequence X_n **converges in probability** to an rv X (denoted $X_n \xrightarrow{p} X$) if for all $\epsilon > 0$,

$$P(|X_n - X| > \epsilon) \to 0 \text{ as } n \to \infty.$$

- The sequence X_n with corresponding sequence of CDFs F_n **converges in distribution** to an rv X (denoted $X_n \xrightarrow{d} X$) with cdf F if

$$F_n(x) \to F(x) \text{ for all continuity points } x \text{ of } F.$$

Large Sample Theory: Key Theorems

Under some conditions, the sample mean $\bar{X} = n^{-1} \sum_{i=1}^{n} X_i$ has some interesting properties as the sample size gets arbitrarily large.

1. **The Central Limit Theorem**: Let $X_1, X_2, \ldots$ be an infinite sequence of iid rvs, with $\mathbb{E}[X_i] = \mu$ and $\mathrm{Var}[X_i] = \sigma^2 < \infty$. Then

$$\sqrt{n}(\bar{X} - \mu) \xrightarrow{d} \mathcal{N}(0, \sigma^2) \text{ as } n \to \infty.$$

2. **Weak Law of Large Numbers**: Let $X_1, X_2, \ldots$ be an infinite sequence of iid rvs, with $\mathbb{E}[X_i] = \mu < \infty$. Then

$$\bar{X} \xrightarrow{p} \mu \text{ as } n \to \infty.$$

Large Sample Theory: Useful Tools

1. **Slutsky's Theorem**: Let X_n, Y_n be sequences of rvs with $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{p} c$, a constant. Then:
   - $X_n + Y_n \xrightarrow{d} X + c$;
   - $X_n Y_n \xrightarrow{d} Xc$;
   - $X_n / Y_n \xrightarrow{d} X/c$ if $c \neq 0$.
2. **Continuous Mapping Theorem**: Let $X_n \xrightarrow{p} X$ and h be any continuous function on $\mathbb{R}$. Then
$$h(X_n) \xrightarrow{p} h(X).$$
3. $X_n \xrightarrow{p} X \implies X_n \xrightarrow{d} X$ and $X_n \xrightarrow{p} c \iff X_n \xrightarrow{d} c$.

Example: Bivariate Normal Distribution

$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix}, \begin{bmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{bmatrix}\right)$$

- Describes a two-dimensional vector that takes values in $\mathbb{R}^2$.
- $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ and $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$.
- $\mathrm{Cov}(X, Y) = \sigma_{XY}$.
- $X \mid Y$ and $Y \mid X$ are also normal.
- For any $a, b \in \mathbb{R}$,

$$aX + bY \sim \mathcal{N}(a\mu_X + b\mu_Y, a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\sigma_{XY}).$$

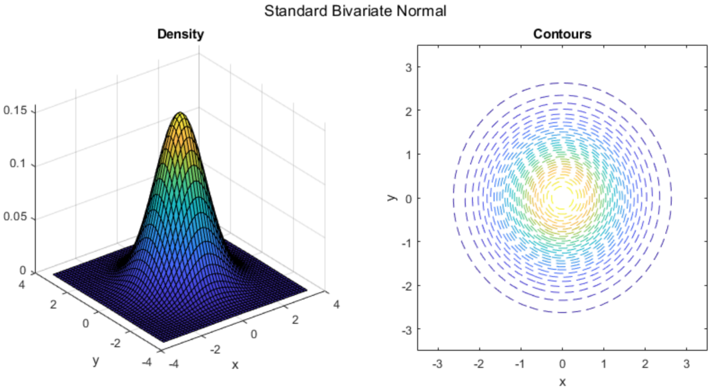


Figure 1: Density and contours of the standard bivariate normal distribution.

Super(!!) useful property of MVN distribution

Let $X \sim N_p(\mu, \Sigma)$ and partition $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ where $X_1 \in \mathbb{R}^k$ and $X_2 \in \mathbb{R}^{p-k}$. Partition μ and Σ accordingly so that

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}.$$

If we observe $X_2 = x_2$, the conditional distribution of X_1 is still multivariate normal, and can be written as

$$X_1 | (X_2 = x_2) \sim N_k(\mu_{1|2}, \Sigma_{1|2})$$

where

$$\mu_{1|2} = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2)$$

and

$$\Sigma_{1|2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

Example: Gamma Distribution

A positive random variable $X \sim \text{Gamma}(\alpha, \beta)$ with PDF:

$$
\begin{aligned}
f_X(x) &= \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-x\beta}; \\
E[X] &= \frac{\alpha}{\beta}; \\
V[X] &= \frac{\alpha}{\beta^2};
\end{aligned}
$$

where $x \in (0, \infty)$, $\alpha, \beta > 0$.
Note: this is referred to as the shape-rate parameterization. You may also see the shape-scale parameterization with scale $\theta = 1/\beta$

Gamma Distribution - Important Properties

Here are some properties that will come in handy throughout the first year:

- If $X \sim \text{Gamma}(\alpha, \beta)$ with $\alpha = 1$, $X \sim \text{Exponential}(\lambda = \beta)$
- If $X \sim \text{Gamma}(v/2, 1/2)$, then $X \sim \chi^2_v$
- If $X \sim \text{Gamma}(\alpha_1, \beta)$ and $Y \sim \text{Gamma}(\alpha_2, \beta)$, then $X + Y \sim \text{Gamma}(\alpha_1 + \alpha_2, \beta)$
- If $X \sim \text{Gamma}(\alpha, \beta)$ (shape-rate parameterization), then $1/X \sim \text{Inverse Gamma}(\alpha, \beta)$ with expectation $\frac{\beta}{\alpha - 1}$
- If $X \sim \text{Gamma}(\alpha, \theta)$ (shape-scale parameterization), then $1/X \sim \text{Inverse Gamma}(\alpha, 1/\theta)$ with expectation $\frac{\beta}{\alpha - 1}$
- If $X \sim \text{Gamma}(\alpha, \beta)$, then $X/n \sim \text{Gamma}(\alpha, n\beta)$

Miscellaneous Useful Facts about Distributions

- If $X_1, \cdots, X_n$ are iid with CDF $F(x)$, then $X_{(1)}$ has CDF $1-(1-F(x))^n$
- If $X_1, \cdots, X_n$ are iid with CDF $F(x)$, then $X_{(n)}$ has CDF $F(x)^n$
- If $X_1, \cdots, X_n \overset{iid}{\sim}$ Poisson(λ), then $\sum_{i=1}^n X_i \sim$ Poisson$(n\lambda)$
- f $X_1, \cdots, X_n \overset{iid}{\sim}$ Exponential$(\lambda) \leftrightarrow$ Gamma$(1, \lambda)$, then $\sum_{i=1}^n X_i \sim$ Gamma(n, λ)
- If $\beta|\phi \sim N(m, \Sigma/\phi)$ and $\phi \sim$ Gamma$(v/2, v\sigma^2/2)$ then the marginal distribution of β is $t_v(m, \sigma^2\Sigma)$
- Mins, maxes, and CDF counts of random variables are binomial random variables