Exercises – Probability and Inference

Bootcamp 2023

August 18, 2023

1 Exercise 1: Normal Distribution and Moments

Let $X \sim N(0,1)$.

(a) Give the pdf for the distribution of X^2.

(b) Give an expression for $\mathbb{E}[X^k]$ for all odd k.

(c) Let $Y = X^2$. Calculate $\text{Cov}(X, X^2)$.

(d) The *moment generating function* of X is defined as $M_X(t) = \mathbb{E}[e^{tX}]$ for all $t \in \mathbb{R}$. Using integrals, find the closed form expression for $M_X(t)$.

(e) Let $Z \sim N(\mu, \sigma^2)$. Using part (d), find $M_Z(t) = \mathbb{E}[e^{tZ}]$ for all t.

(f) **Optional Challenge:** Assume $\sigma^2 = 1$, so $Z \sim N(\mu, 1)$. The *characteristic function* of Z is defined as $\phi_Z(t) = \mathbb{E}[e^{itZ}]$, where $i = \sqrt{-1}$. Fortunately, $\phi_Z(t) = M_Z(it)$. Find an unbiased estimator for $(-1)^\mu$. *Hint:* you may find *Euler's Identity* to be helpful here, which is $e^{i\pi} = -1$.

2 Exercise 2: Classic Transformations of Continuous Random Variables

Let $X_1 \sim Gamma(a, \xi)$ and $X_2 \sim Gamma(b, \xi)$, where $a, b, \xi > 0$ and $X_1 \perp\!\!\!\perp X_2$. For each problem, give the pdf (*don't forget the support*) and the name of the distribution.

(a) What is the distribution of $W = \frac{X_1}{X_1 + X_2}$?

(b) Set $a = b = 1$. For $\lambda > 0$, what is the distribution of $Y = -\frac{1}{\lambda} \log W$?

(c) What is the distribution of $Z = Y^{1/\alpha}$ for $\alpha > 0$?

(d) Let $U_1 \sim \chi^2_{\nu_1}$ and $U_2 \sim \chi^2_{\nu_2}$ with $U_1 \perp\!\!\!\perp U_2$. Show that $F = \frac{U_1/\nu_1}{U_2/\nu_2}$ has an F_{ν_1,ν_2} distribution.

(e) Let X be a continuous random variable with support $\mathbb{R}$ with cdf $F_X(x)$. What is the distribution of $V = F_X(X)$?

3 Exercise 3: Fun with Exponentials

Let $X_1, ..., X_n \overset{\text{iid}}{\sim} Exp(\lambda)$.

(a) What is the distribution of $X_{(1)} = \min(X_1, ..., X_n)$?

(b) Derive the moment generating function for X_i.

(c) What is the distribution of $Y = \sum_{i=1}^{n} X_i$?

(d) Let $W \sim Poisson(\mu)$, $X \sim Exp(\lambda)$, $X \perp\!\!\!\perp W$. Is $Z = X - W$ continuous, discrete, or neither?

(e) Give an expression for $P(Z < z)$ for $z \in \mathbb{R}$.

(f) **Optional Challenge:** Let $n = 2$. Show that $T_1 = \min(X_1, X_2)$ and $T_2 = X_1 - X_2$ are independent.

4 Exercise 4: Heavy Tailed Distributions

Let $X_1, X_2 \overset{\text{iid}}{\sim} N(0, 1)$, $W \sim \chi^2_\nu, W \perp\!\!\!\perp X_1, X_2$.

(a) What is the distribution of $Y = \frac{X_1}{X_2}$? What is $\mathbb{E}[Y]$?

(b) What is the distribution of $T = \frac{X_1}{\sqrt{W/\nu}}$?

(c) Show that $T \overset{d}{=} Y$ if $\nu = 1$. That is, $\frac{X_1}{X_2} \overset{d}{=} \frac{X_1}{\sqrt{W}}$. Conceptually, why does this make sense?

5 Exercise 5: Large Sample Theory

(a) Let $X_1, ..., X_n \overset{\text{iid}}{\sim} Exp(1)$. Let $M_n = \max(X_1, ..., X_n)$. Find the limiting distribution (you can just state the CDF) of $M_n - \log(n)$.

(b) Let $X_1, ..., X_n$ be *iid* continuous random variables with pdf f and $\mathbb{E}[X_i] = \mu < \infty$. Let g be another pdf (with the same support as f). Show that

$$W_n = \frac{\sum_{i=1}^{n} X_i}{\sum_{i=1}^{n} g(X_i)/f(X_i)} \xrightarrow{p} \mu$$

as $n \to \infty$.

(c) Let $f(x|\theta_1, \theta_2) = \frac{1}{\theta_2} \exp(-\frac{x-\theta_1}{\theta_2})$, $x \geq \theta_1$, $\theta_1 \in \mathbb{R}$, $\theta_2 > 0$. Find the maximum likelihood estimators for θ_1, θ_2.

(d) **Optional Challenge:** Let $X_1, ..., X_n \overset{\text{iid}}{\sim} Cauchy(0, 1)$. What is the distribution of $\frac{1}{n}\sum_{i=1}^{n} X_i$? *Hint: this requires the CF of the standard Cauchy distribution, which is* $\phi(t) = \exp(-|t|)$. *This is a nice counterexample for when the* L_1 *requirement of the Central Limit Theorem does not hold.*

6 Exercise 6: True or False Questions

For each statement, provide a short answer or proof to why it is true or false.

(a) For a random variable X, the moment generating function $M_X(t)$ exists (i.e., is finite) for all $t \in \mathbb{R}$.

(b) Let $X \sim N(0, 1)$. Then $Y = \mathbb{1}(X > 0) \sim Bernoulli(1/2)$.

(c) If X and Y are two random variables such that $X \sim N(\mu_x, \sigma_x^2)$, $Y \sim N(\mu_y, \sigma_y^2)$, then the pair (X, Y) has a joint normal distribution.

(d) If X and Y are two independent random variables such that $X \sim N(\mu_x, \sigma_x^2)$, $Y \sim N(\mu_y, \sigma_y^2)$, then the pair (X, Y) has a joint normal distribution.

(e) If X and Y are two normally distributed random variables that are uncorrelated, then they are independent.

(f) If X and Y are jointly normal random variables that are uncorrelated, then they are independent.

(g) Let X be a discrete random variable and $g : \mathbb{R} \to \mathbb{R}$ be some (well-defined) function. Then $Y = g(X)$ is discrete.

(h) Let X be a continuous random variable and $g : \mathbb{R} \to \mathbb{R}$ be some function. Then $Y = g(X)$ is continuous.