Exercise: Calculus and Measure Theory

Bootcamp 2022

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1 Taylor Series

Let $f(x) = \exp\left(-1/x^2\right)$ when $x \neq 0$.

(1) Define $f(0)$ so that f is continuous at 0.

(2) Show that f is infinitely differentiable at 0, and $f^{(n)}(0) = 0$, $\forall n \in \mathbb{N}$. (**Hint:** *You might not need to find the explicit form of $f^{(n)}(x)$. Can you show that $f^{(n)}(x)$ has the form of $P^{(3n)}(1/x)\exp(-1/x^2)$ where $P^{(k)}(x)$ is some polynomial of degree k?*)

(3) Show that the Taylor expansion of f is convergent for all $x \in \mathbb{R}$. However, the limit does not converge to f.

2 p-series

Consider the p-series

$$H(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}, \quad p > 0.$$

(1) Use the fact that

$$\frac{1}{(n+1)^p} \leq \frac{1}{x^p} \leq \frac{1}{n^p}$$

for $x \in [n, n+1]$, $n \in \mathbb{N}$ to show that

$$H(p) - 1 \leq \int_1^{\infty} \frac{1}{x^p}\,\mathrm{d}x \leq H(p).$$

(2) Prove that $H(p) < \infty$ if and only if $p > 1$.

3 Harmonic Series

We have shown that the harmonic series $H(1) = \sum_{n=1}^{\infty} 1/n$ is divergent. Actually, the finite sum $S_n = \sum_{k=1}^{n} 1/k$ grows roughly at the rate of $\ln n$.

(1) Show that $\ln(1+x) \leq x$. Hence, $\{S_n - \ln n\}$ is monotonically decreasing.

(2) By using the fact in problem 2(1), show that $S_n - \ln n \geq 0$. Argue that the limit $\lim_{n\to\infty} S_n - \ln n$ exists.

(3) Denote the above limit by γ. Consider the alternating series

$$T_n = \sum_{k=1}^{n} \frac{(-1)^k}{k}.$$

Show that $\lim T_n$ exists and $\lim_{n\to\infty} T_n = \ln 2$. (**Hint:** *How can you express T_{2n} in terms of S_{2n} and S_n?*)

4 Gaussian distribution

A random variable X is said to be a gaussian distribution with mean μ and variance σ^2 if the probability density function of X satisfies

$$f(x) \propto \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

For simplicity, we also use the notation $X \sim \mathcal{N}(\mu, \sigma^2)$.

(1) Show that

$$\int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) = \sqrt{2\pi}.$$

Hint: *Note that*

$$\int_0^{\infty}\int_0^{\infty} \exp\left(-\frac{x^2+y^2}{2}\right) \mathrm{d}x\mathrm{d}y = \left[\int_0^{\infty} \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x\right]^2.$$

Can you transform (x, y) to polar coordinates (r, θ) to evaluate the double integral?

(2) Suppose that $X \sim \mathcal{N}(0, 1)$, $Y \mid X = x \sim \mathcal{N}(x, 1)$. Determine the distribution of $X \mid Y = y$. (**Hint:** *Any quadratic function $ax^2 + bx + c$ can be written in the form of $a(x-h)^2 + p$. Determine the value of h.*)

5 Gamma Function

The Gamma function is defined on $(0, +\infty)$ by

$$\Gamma(a) = \int_0^{\infty} x^{a-1} e^{-x} \,\mathrm{d}x.$$

(1) Using integration by parts, show that $\Gamma(a+1) = a\Gamma(a)$, $\forall a > 0$. Hence, show that $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$.

(2) Show that $\Gamma(1/2) = \sqrt{\pi}$. (**Hint:** *You may use the conclusion of Problem 4(1).*)

(3) Using integration by substitution, show that

$$\int_0^{\infty} x^{a-1} e^{-bx} \,\mathrm{d}x = \frac{\Gamma(a)}{b^a}.$$

The gamma distribution $\mathrm{Gamma}(\alpha, \beta)$ is a distribution with probability density function

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \mathbf{1}_{x\in(0,\infty)}.$$

Find the expectation and variance of this distribution.

(4) Let $X \sim \text{Exp}(1)$ (this is actually $\text{Gamma}(1,1)$). Find $\mathbb{E}[\ln X]$. (**Hint:** *What do you get when you take the derivative of* $\Gamma(a)$*? The derivative* $\Gamma'(a)$ *is an interesting function with* $\Gamma'(1) = -\gamma$*, where* γ *is exactly the limit defined in Problem 3(3)!*)

6 Beta distribution

We say that X follows a $\text{Beta}(\alpha, \beta)$ distribution if the probability density function of X is given by

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}\mathbf{1}_{x\in(0,1)}.$$

One soccer player takes n penalty kicks with the probability of a goal being p. Assume that these kicks are independent. Hence, the number of goals $X \sim \text{Bin}(n, p)$. From a Bayesian point of view (which will be discussed in more details in STA 702), we may treat p as a random variable to represent the a priori belief of p before the player takes any penalty kicks. After the kicks, one updates such belief on p by calculating the posterior distribution $p \mid X$.

(1) Assume that $p \sim \text{Beta}(\alpha, \beta)$. Show that the posterior distribution $p \mid X$ is also a Beta distribution. Therefore, the Beta distribution is a **conjugate prior** for the probability parameter of a binomial distribution.

(2) Show that the expectation of $\text{Beta}(\alpha, \beta)$ is $\alpha/(\alpha+\beta)$. Therefore, $\mathbb{E}[p \mid X]$ can be written as a weighted average of $\mathbb{E}[p]$ and X/n. This indicates that the "average belief" on the goal probability after the penalty kicks is a weighted average of that before the penalty kicks and the proportion of goals during these kicks.

7 Measure Theory

Let $\mathbb{N} = \{1, 2, \dots\}$. Consider the **probability measure space** $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$ with $\mu(\{\omega\}) \propto \omega^{-p}$, $p > 1$. For $k \in \mathbb{N}$, define $k\mathbb{N} = \{kn : n \in \mathbb{N}\}$, the subset of $\mathbb{N}$ consisting of all multiples of k.

(1) Show that $\mu(k\mathbb{N}) = k^{-p}$.

(2) Explicitly write out the elements of $\mathcal{F}$, the smallest sigma field on $\mathbb{N}$ that contains $2\mathbb{N}$ and $3\mathbb{N}$.

(3) Now let $p = 2$. Suppose one selects a positive integer randomly with the probabilities given by μ. Find the probability that such an integer is co-prime with 6. (**Hint:** *An integer that is co-prime with 6 can be written in the form of* $6k+1$ *or* $6k+5$*,* $k \in \mathbb{Z}$*.*)

8 Matrix Calculus

(1) Consider a linear model given by $Y_i \mid \mathbf{x}_i \overset{\text{iid}}{\sim} N(\mathbf{x}_i^\intercal \beta, \sigma^2)$ for $i = 1, \dots, n$, where σ^2 is fixed and known. We are interested in estimating β. First, we can write this model in matrix form: $\mathbf{Y} \mid \mathbf{X} \sim (\mathbf{X}\beta, \sigma^2 I_n)$. The least squares (LS) estimator is defined by:

$$\widehat{\beta} = \arg\min_{\beta} \|\mathbf{Y} - \mathbf{X}\beta\|_2^2$$

Compute the LS estimator for this model.

(2) Now, consider the linear model given by $Y_i \mid \mathbf{x}_i \overset{\text{iid}}{\sim} N(\mathbf{x}_i^\intercal \beta, \sigma_i^2)$ for $i = 1, \ldots, n$, where the σ_i^2 are fixed and known but not necessarily equal. Write this model in matrix form. Then suggest a linear transformation $\mathbf{A}$ of $\mathbf{Y}$ so that the covariance matrix of $\mathbf{AY}$ is the identity matrix. Finally, write down the LS estimator $\widetilde{\beta}$ for this transformed model and solve for the LS estimator. (**Hint:** *If* $\mathbf{X} \sim N(\mu, \Sigma)$*, then* $\mathbf{AX} \sim N(\mathbf{A}\mu, \mathbf{A}\Sigma\mathbf{A}^\intercal)$.)

9 Moment Generating Functions

Consider a random variable X. Its moment generating function (MGF) is defined by $M_X(t) := \mathbb{E}[e^{tX}]$ for values of $t \in \mathbb{R}$ such that the right-hand side exists.

(1) Suppose X is a continuous random variable and that $M(t)$ exists for all $t \in \mathbb{R}$. Compute $M_X'(0), M_X''(0)$, and more generally, $M_X^{(n)}(0)$. What fact from the notes allows you to evaluate these expressions?

(2) Consider the case of the "standard" Cauchy random variable X with probability density function given by $f(x) = \frac{1}{\pi(1+x^2)}$ for $x \in \mathbb{R}$. For what values of t does $M_X(t)$ exist?

(3) A closely related function to the MGF is the cumulant generating function $\psi_X(t) := \log \mathbb{E}[e^{tX}]$. Compute $\psi'(0)$ and $\psi''(0)$.

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