

# Explore Geometric Anisotropy for Point-Referenced Spatial Data

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## Abstract

Isotropic covariance functions are routinely adopted in specifying models for point-referenced spatial data. Implicit in such modeling is the assumption that spatial dependence is not directional. Geometric anisotropic models offer a class of specifications which incorporate directional dependence. They have received little attention in the literature because their associated parameters increase and become difficult to identify as we increase richness of the function. This paper aims to illuminate when and how much such models for random effects in geostatistical settings improve predictive performance, and to explore a Bayesian framework to facilitate fitting such models. We show that geometric anisotropy yields better predictive performance when the data significantly departs from isotropy (anisotropy ratio is much greater than one), and the improvement is more prominent when spatial variance is greater than pure error. The improvement in predictive performance is illustrated through modeling scallop catches data. We perform full Bayesian inference on all model parameters using Metropolis algorithm.

## 1 Introduction

Researchers in diverse areas such as climatology, ecology and environmental health are increasingly interested in analyzing data that are geographically referenced. For example, epidemiologists may be interested in studying the number of lung cancer incidents by county and state. Ecologists may be interested in the location of a particular species of trees in a forest. Researchers are often interested in statistical inference tasks, such as modeling of trends and correlation structures,

estimation of underlying model parameters, hypothesis testing, model comparison and selection, and prediction of observations at unobserved locations (kriging).

With the advancement of Markov chain Monte Carlo (MCMC) computing, Bayesian modeling approaches have become increasingly common, as they enable hierarchical model structures where prior belief is updated with new information, as well as natural quantification of uncertainty through sampling schemes. The Banerjee et al. (2014) text presents a thorough treatment of hierarchical Bayesian approaches for a variety of complex spatial data problems.

One common type of spatial data is *point-referenced* data, often referred to as *geostatistical* data, where we observe realizations of a spatial stochastic process at a fixed set of locations. We are often interested in a geographical distribution of the realizations that account for spatial correlation and perhaps underlying covariates. The simplest choice for modeling spatial correlation are *isotropic* covariance functions, where we assume spatial correlation between locations depends only on the distance between locations. In cases where this assumption does not hold, i.e. spatial correlation varies by direction, we use *anisotropic* covariance functions which depend on the separation vector between locations.

The most common form of anisotropy is *Geometric Anisotropy*, where coordinates are linearly transformed, i.e., rotated and stretched, to allow for different magnitudes of correlation in different directions. Budrikaite and Ducinskas (2005) explored different forms of geometric anisotropic variograms. Eriksson and Siska (2000) provided the geometrical details for modeling various types of anisotropy (range, sill, power, slope, nugget) on an ellipse. Allard et al. (2016) derived a directional representation of anisotropies to build a large class of models that include and go beyond classical anisotropies such as the geometric and zonal ones. Porcu et al. (2006) incorporated anisotropy into spatio-temporal covariance models. Ecker and Gelfand (1999) proposed a Bayesian methodology for simultaneously estimating the linear transformation of the coordinates and other variogram parameters, which also allows full inference for any characteristic of the geometrically anisotropic model. Following Ecker and Gelfand (1999) who proposed to use objective, independent priors for model parameters, Kazianka (2013) developed default priors and studied their posterior propriety.

This paper aims to illuminate when and how much geometric anisotropic models for random effects in geostatistical settings improve predictive performance. We use a transformation matrix parametrized by a decay parameter, rotation angle and anisotropy ratio. We compare the predictive performance of isotropic and anisotropic models for data generated under anisotropy. We also test the sensitivity of the models to different parameter values of the data generating model,

including different sample sizes, different choices of anisotropy ratio, and different scales and ratios of spatial variance and pure error. We find that geometric anisotropy yields better predictive performance when the data significantly departs from isotropy (anisotropy ratio is much greater than one), and the improvement is more prominent when spatial variance is greater than pure error. We use Metropolis Hastings algorithm to perform full Bayesian inference for all model parameters. We then fit isotropic and anisotropic models to scallop catches data used in Ecker and Gelfand (1999) which have been shown to exhibit anisotropic behavior. We show that the anisotropic model performs better in terms of empirical coverage, predictive mean squared error and continuous rank probability score.

The paper proceeds as follows. Section 2 formally defines the isotropic and geometric anisotropic models we use for point-referenced data. Section 3 details our model fitting algorithm and distribution theory for making predictions. Section 4 lays out the metrics we use for model comparison. Section 5 demonstrates simulation setting and results. Section 6 presents modeling results for scallop catches data. Finally, section 7 discusses future work.

## 2 Models

### 2.1 Point-Level Models

Suppose  $\{Y(s) : s \in D\}$  is a stochastic process where  $D$  is a fixed subset of  $r$ -dimensional Euclidean space. Data generated by such spatial process  $Y(s)$  where  $s$  varies continuously over  $D$  is *point-referenced* data. When  $r = 2$ ,  $Y(s)$  is a *spatial process*. Even though  $Y(s)$  exists on an infinite dimensional function space, in reality we can only observe data at a finite number of locations. The data is a partial realization of the spatial process. For example,  $Y(s)$  may represent pollutant level over a region, and we observe pollutant measurements at a finite set of locations  $\{s_1, \dots, s_n\}$  where there are monitoring stations. The problem facing statisticians is to make inference about the spatial process  $Y(s)$  and prediction at new locations based on this partial realization.

We model  $Y(s)$  with a *Gaussian Process*. Suppose our spatial process has a mean  $\mu(s) = E(Y(s))$  associated with it and the variance of  $Y(s)$  exists for all  $s \in D$ .  $Y(s)$  is said to be *Gaussian* if for any  $n \geq 1$ , any site of  $n$  sites  $\{s_1, \dots, s_n\}$ ,  $Y = (Y(s_1), \dots, Y(s_n))^T$  has a multivariate normal distribution. The remarkable feature of Gaussian Process models is that, despite only observing the

spatial surface at a finite number of locations, we can infer about the process at an uncountable number of locations by specifying the association between locations through *structured* dependence. Suppose we assume the random variables at two locations depend on the *distance* of the locations. One commonly used covariance specification is the exponential model, where the covariance between measurements at two locations is an exponential function of the distance between the two locations. If we assume spatial correlation is a function solely of the distance  $d_{ii'}$  between  $s_i$  and  $s_{i'}$ , the covariance function is *isotropic*.

Suppose we have observations  $\mathbf{Y} = \{Y(s_1), \dots, Y(s_n)\}$ . We assume a multivariate normal model where

$$\mathbf{Y} \sim N_n(\mu \mathbf{1}, \Sigma(\theta)) \quad (1)$$

where  $N_n$  denotes the  $n$ -dimensional normal distribution,  $\mu$  is the global mean, and  $(\Sigma(\theta))_{ii'}$  gives the covariance between  $Y(s_i)$  and  $Y(s_{i'})$ .

If we use an exponential correlation function,

$$\Sigma(\theta)_{ii'} = \begin{cases} \sigma^2 + \tau^2 & \text{if } d_{ii'} = 0 \\ \sigma^2 \exp(-\phi d_{ii'}) & \text{if } d_{ii'} > 0 \end{cases} \quad (2)$$

where  $d_{ii'}$  is the distance between site  $s_i$  and  $s_{i'}$ ,  $\phi > 0$  is the *decay parameter* ( $1/\phi$  is the *range* parameter),  $\sigma^2 > 0$  is *partial sill* or spatial variance,  $\tau^2 > 0$  is *nugget* or pure error, and  $\tau^2 + \sigma^2$  is *sill*. The parameters of the covariance matrix are  $\theta = (\sigma^2, \tau^2, \phi)^T$ .

## 2.2 Isotropy

Suppose we observe  $\{Y(s) : s \in D \subseteq \mathbb{R}^r\}$ . For any  $h \in \mathbb{R}^r$ , *intrinsic stationarity* assumes that  $E(Y(s+h) - Y(s)) = 0$  and that  $\text{Var}(Y(s+h) - Y(s))$  is a function of the separation vector  $h$ , denoted by  $2\gamma(h)$ , the *variogram*. The stronger assumption of *weak stationarity* asserts that  $E(Y(s)) = \mu$  (the mean is constant) and  $\text{Cov}(Y(s), Y(s+h)) = C(h)$ , and implies *intrinsic stationarity* with  $\gamma(h) = C(0) - C(h)$ . Under *isotropy*, the *semivariogram* function  $\gamma(h)$  depends on the separation vector only through its length  $\|h\|$ . That is, the variogram is a function of the Euclidean distance  $d$  between two sites. The *sill* is  $\lim_{d \rightarrow \infty} 2\gamma(d)$ . The *range* is the distance at which the sill is reached; sites separated by distances larger than the range are uncorrelated. The *effective range* is defined as the distance such that  $\text{Corr}(Y(s_i), Y(s_j)) = 0.05$ . The *nugget* is  $\lim_{d \rightarrow 0} 2\gamma(d)$ , which need not be zero, to allow for measurement error and microscale variability.

## 2.3 Geometric Anisotropy

Isotropic variograms are popular because of their simplicity and interpretability. However, in many cases correlation function does not simply depend on the distance between locations, but depends on separation vector between locations. As a result, association that depends upon direction. Here we explore covariance functions that are stationary but not isotropic. When the variogram is a function of both length and orientation of the separation vector  $h$ , the process  $Y(s)$  is said to be *anisotropic*. That is, the semivariogram is  $\gamma(h)$  rather than  $\gamma(\|h\|)$ .

A commonly used set of anisotropic covariance functions are geometric covariance functions where

$$\text{Cov}(s - s') = \sigma^2 \rho((s - s')^T B (s - s')) \quad (3)$$

where  $B$  is a positive semidefinite matrix and  $\rho$  is a valid correlation function in  $R^r$ . When  $r = 2$ ,  $B$  is  $2 \times 2$ , and the correlation specification will have three parameters rather than one decay parameter.  $B$  indicates a linear transformation to isotropy which corresponds to when  $B = cI$ . The contour corresponding to  $\rho = 0.05$  provides the range in each spatial direction, and it falls on an ellipse.

As detailed in Eriksson and Siska (2000), we define the covariance function as:

$$\Sigma(\theta)_{ii'} = \sigma^2 \exp(-\phi(h_{ij}^T B h_{ij})^{1/2}) + \tau^2 I(i = i') \quad (4)$$

where

$$B = AA^T, A = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix} \begin{bmatrix} 1/r & 0 \\ 0 & 1 \end{bmatrix} \quad (5)$$

where  $h_{ij} = s_i - s_j$ ,  $\alpha$  is orientation of the associated ellipse and  $r$  is anisotropy ratio (ratio of the major axis to the minor axis of the ellipse). The  $B$  matrix rotate and stretch the range of the covariance. When  $B$  is an identity matrix, the covariance function is isotropic.

The model given by (1), (4) and (5) can be written as a random effects model.

$$\begin{aligned} y(s_i) &= \mu + w(s_i) + \epsilon(s_i) \\ W(s_i) &\sim N(0, \sigma^2 \exp(-\phi(h_{ij}^T B h_{ij})^{1/2})) \\ \epsilon(s_i) &\sim N(0, \tau^2) \end{aligned}$$

where  $\mu$  is the global mean and  $W(s_i)$  captures spatial correlation, independent of  $\epsilon(s_i)$ .

We can obtain the range  $c$  in direction  $\theta$  is obtained by taking a unit vector in direction  $\theta$ ,  $h_\theta$ , and then solving  $\rho(c_\theta^2 h_\theta^T B h_\theta) = .05$  for  $c_\theta$ . In the case of exponential correlation function, we have

$$\exp(-c_\theta(h_\theta^T B h_\theta)^{1/2}) = 0.05 \quad (6)$$

where  $h_\theta = (\cos(\theta), \sin(\theta))$ .

Figure 1 shows spatial ranges in every direction from  $0^\circ$  to  $360^\circ$  with different combinations of the rotation angle and anisotropy ratio. The right panel shows the range in polar coordinates, forming ellipses. We can see that when the anisotropy axis ratio is closer to 1, the ellipse is smoother and closer to a circle.

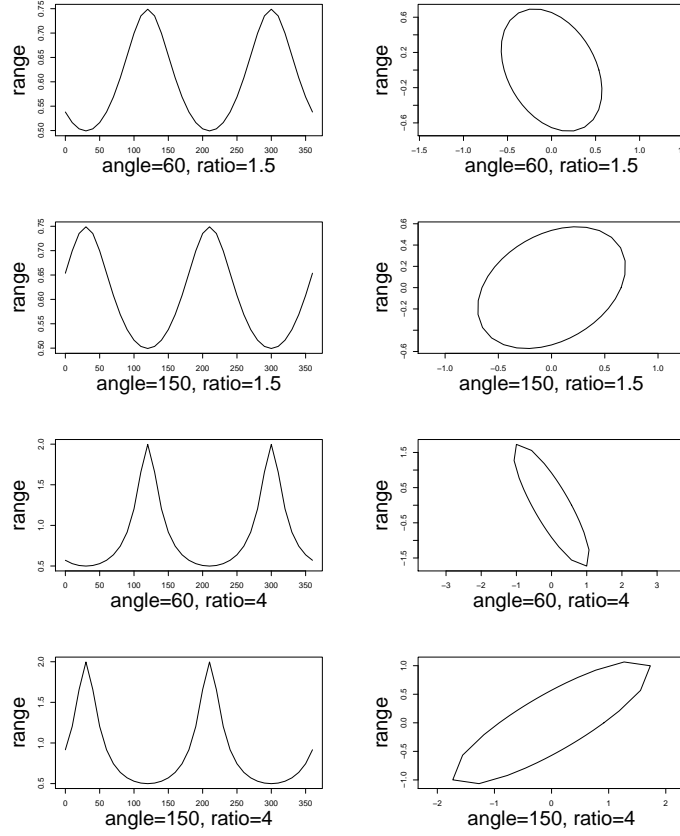


Figure 1: Spatial range with different specifications of rotation angle and anisotropy ratio

## 3 Methodology

### 3.1 Model Fitting

We use the Bayesian framework to perform inference on model parameters and predictions for new locations. Not only do we simultaneously estimate for the ratio of major axis to minor axis of the ellipse, the angle of orientation of the ellipse with respect to the x-axis, the decay parameter and the additional variogram parameters, but we provide complete inference in the form of a posterior distribution for each parameter. In addition, the Bayesian framework enables incorporation of prior information. If a priori, the process is expected to exhibit geometric anisotropy whose characteristics we can quantify, this information is easily built into the prior specifications and the model.

We model the data with a Gaussian Process

$$\mathbf{Y} \sim N_n(\mu \mathbf{1}, \Sigma(\theta)) \quad (7)$$

where  $\Sigma(\theta)$  is specified by (4) and (5), and  $\theta = (\sigma^2, \tau^2, \phi, \alpha, r)^T$ . Under the likelihood implied by (7), we complete the Bayesian formulation by assuming the prior takes the form

$$\pi(\mu, \theta) = \pi(\mu, \sigma^2, \tau^2, \phi, \alpha, r) = \pi(\mu)\pi(\sigma^2)\pi(\tau^2)\pi(\phi)\pi(\alpha)\pi(r) \quad (8)$$

assuming all the parameters are independent.

We use non-informative  $N(0, 1000)$  prior for  $\mu$ . We use non-informative inverse gamma distributions for the variances  $\sigma^2$  and  $\tau^2$  and anisotropy ratio  $r$  with shape and scale parameters both equal to one, implying no prior center and infinite prior variance. We put a uniform  $(0, \pi)$  prior on rotation angle  $\alpha$  and a uniform  $(0.2D, 0.7D)$  on range  $1/\phi$ , where  $D$  is the maximum distance between locations.

We use Metropolis-Hastings sampling algorithm to approximate the posterior distribution of all parameters  $p(\theta|y)$ . The Metropolis-Hastings algorithm is a Markov chain Monte Carlo (MCMC) method for obtaining a sequence of random samples from a probability distribution for which direct sampling is difficult. As more and more sample values are produced, the distribution of values more closely approximates the desired posterior distribution  $p(\theta|y)$ .

Intuitively, assume we have a collection of  $\{\theta^{(1)}, \dots, \theta^{(s)}\}$ . To add a new value  $\theta^{(s+1)}$ , we sample a new value  $\theta^*$  that is nearby  $\theta^{(s)}$  and consider whether to accept the sampled  $\theta^*$  into the collection of  $\theta$ 's. We sample  $\theta^*$  from a proposal distribution  $J$  centered on  $\theta^{(s)}$ , and accept  $\theta^*$  if  $p(\theta^*|y) > p(\theta^{(s)}|y)$ . If  $p(\theta^*|y) < p(\theta^{(s)}|y)$ ,

we accept  $\theta^*$  with some probability. In Metropolis algorithm, the proposal distribution  $J$  is symmetric. That is,  $J(\theta^*|\theta^{(s)}) = J(\theta^{(s)}|\theta^*)$ . In Metropolis-Hastings,  $J$  may not be symmetric.

The Metropolis-Hastings algorithm proceeds as follows.

1. Sample  $\theta^* \sim J(\theta|\theta^{(s)})$

2. Compute acceptance ratio  $r$ :

$$r = \frac{p(\theta^*|y)}{p(\theta^{(s)}|y)} \times \frac{J(\theta^{(s)}|\theta^*)}{J(\theta^*|\theta^{(s)})} = \frac{p(y|\theta^*)p(\theta^*)}{p(y|\theta^{(s)})p(\theta^{(s)})} \times \frac{J(\theta^*|\theta^{(s)})}{J(\theta^{(s)}|\theta^*)}$$

(In the case of Metropolis,  $\frac{J(\theta^*|\theta^{(s)})}{J(\theta^{(s)}|\theta^*)} = 1$ )

3. Sample  $u \sim \text{Uniform}(0, 1)$ . Set  $\theta^{(s+1)} = \theta^*$  if  $u < r$  and set  $\theta^{(s+1)} = \theta^s$  otherwise.

We use appropriate proposal distributions for different parameters. For rotation angle  $\alpha$  we generate proposals from normal distribution, and then take the remainder of the proposals divided by  $\pi$  so that the samples will be between 0 and  $\pi$ . For anisotropy ratio  $r$ , we generate proposals from truncated normal distribution from 0 to  $\infty$  as  $r$  can only be positive. We also experimented with using log normal proposal for  $r$ , but the sampler behaved poorly as it frequently generated extremely big values. For  $\sigma^2$ ,  $\tau^2$  and  $\phi$  we generate proposals from log normal distribution. Finally, we generate  $\mu$  from a normal distribution. Table 1 lists the proposal and prior distributions we use for all the parameters.

Because  $\alpha$  and  $r$  are highly correlated, we update them jointly. We tune the variances of the proposal distributions so that we accept around 25% of all generated samples to achieve optimal efficiency of the sampler. If proposals vary too much, the sampler will reject too many samples which is inefficient. If proposals vary too little, the chain of samples will not move very much and might get stuck in a local mode.



	Proposal Distribution	Prior Distribution
$\alpha$	Normal (mod $\pi$ )	Uniform(0, $\pi$ )
$r$	Truncated Normal $_{(0,+\infty)}$	Inverse Gamma(1, 1)
$\sigma^2$	Log Normal	Inverse Gamma(1, 1)
$\tau^2$	Log Normal	Inverse Gamma(1, 1)
$\phi$	Log Normal	Uniform (3/0.7, 3/0, 2)

Table 1: Proposal and prior distributions for Metropolis algorithm

### 3.2 Kriging

Spatial prediction in the point-referenced data setting is often referred to as *kriging*. Given observations  $Y = (Y(s_1), \dots, Y(s_n))$ , how do we predict  $Y(s_0)$ , where  $s_0$  is a site that has not been observed? An observation at location we want to predict  $s_0$  follows the following distribution:

$$y(s_o) = \mu + w(s_0) + \epsilon(s)$$

There are two ways of formulating the posterior predictive distribution. The first way samples spatial random effect  $w$ :

$$\begin{aligned}
[Y(s_0)|Y] &= \int [Y(s_0)|w(s_0), \theta][w(s_0)|w, \theta][w|\theta, Y][Y|\theta] \\
[w|\theta, Y] &= N\left(\left(\frac{1}{\tau^2}I + \Sigma^{-1}\right)^{-1} \frac{1}{\tau^2}(Y - \mu\mathbf{1}), \left(\frac{1}{\tau^2}I + \Sigma^{-1}\right)^{-1}\right) \\
[w(s_0)|w] &= N(r^T \Sigma^{-1} w, \sigma^2 - r^T \Sigma^{-1} r) \\
[Y(s_0)|w(s_0), \theta] &= N(\mu + w(s_0), \tau^2)
\end{aligned}$$

where  $r$  is the  $n \times 1$  covariance matrix between the new location  $s_0$  and all other observed locations,  $\text{cov}(w(s_0), w(s_i)), i = 1, 2, \dots, n$ .  $\Sigma$  is the  $n \times n$  covariance matrix between all observed locations  $(w(s_1), \dots, w(s_n))$ .

The second way marginalizes out spatial random effect w:

$$[Y(s_0)|Y] = \int [Y(s_0)|Y, \theta][\theta|Y]$$

$$[Y(s_0)|Y, \theta][\theta|Y] = N(\mu + r^T(\Sigma + \tau^2 I)^{-1}(Y - \mu \mathbf{1}), \sigma^2 + \tau^2 - r^T(\Sigma + \tau^2 I)^{-1}r)$$

We use the second approach in our algorithm as it has been shown to yield better sampling behavior.

## 4 Model Comparison

We use the following three metrics for model comparison.

### 4.1 Predictive Mean Squared Error

The predictive mean squared error measures the mean squared difference between the predicted value and the true value averaged over all hold out locations.

$$PMSE = \frac{1}{n} \sum (\hat{Y} - Y_{true})^2$$

In the Bayesian context, we have posterior samples from each parameter and prediction for each hold out location. We average the squared difference between the predicted value and the true value over all posterior samples and over all hold out locations.

### 4.2 Empirical Coverage

Empirical coverage assesses how well credible intervals, derived from the posterior predictive distributions of the predictions, capture the true values of the hold out sample. Suppose we take 90% credible intervals from the posterior predictive distributions for all hold out locations. If the credible intervals capture the true value for <90% hold out locations, we have *undercoverage*. If the credible intervals capture the true value for >90% hold out locations, we have *overcoverage*.

### 4.3 Continuous Rank Probability Score (CRPS)

To examine how concentrated the predictive distribution of  $Y(s_0)$  is around the true value, we use the metric Continuous Rank Probability Score (CRPS), the squared integrated distance between the predictive distribution and the degenerate distribution at the observed value,

$$CRPS(F, y) = \int_{-\infty}^{\infty} (F(u) - I(u \geq y))^2 du$$

where  $F$  is the predictive distribution and  $y$  is the observed value. In our case,  $Y(s_0)$  is the observation and  $F$  is the posterior predictive distribution for  $Y(s_0)$ . With a collection of hold out observations and associated predictive distributions, we would sum the CRPS over these observations for model comparison.

## 5 Simulation Study

We design a set of simulations to compare the predictive performance of isotropic and anisotropic models for data generated under anisotropy. We test the sensitivity of the models to different parameter values of the data generating model, including different sample sizes, different choices of anisotropy ratio, and different scales of spatial variance and pure error.

We randomly generate locations on a unit square. We generate observations following a multivariate normal distribution with constant global mean 0 and exponential covariance function as specified in (4) and (5). Spatial range is 0.5, around half of the maximum distance between the locations. The decay parameter  $\phi$  is  $3/0.5 = 6$ . We fit the model and predict on hold out sample using the methodology outlined in section 3. We fit the isotropic model using the R package `spBayes` (Finley et al., 2007, 2015). We compare the predictive performance of isotropic model and anisotropic model using empirical coverage, PMSE and CRPS as defined in section 4.

We choose three different anisotropy ratios, 1.5, 4 and 8, as well as four different combinations of spatial variance  $\sigma^2$  and pure error  $\tau^2$ . We hypothesize that, when anisotropy ratio is big, spatial variance of simulated data is high, and spatial variance is significantly higher compared to pure error, the predictive performance of the anisotropic model will significantly improve over the isotropic model. We use two different sample sizes 100 (with 40 additional observations for hold out) and 500 (with 100 additional observations for hold out). We hypothesize that the two models will be more distinguishable with bigger sample size.

Table 2-7 display the comparisons of empirical coverage, PMSE, and CRPS of anisotropy and isotropy under different data generation scenarios. When sample size is 100, anisotropy yields smaller MSE than isotropy under all but one scenario when anisotropy ratio is 1.5. As we expect, the improvement is more prominent when anisotropy ratio is 8 (10%) than when it is 4 (1%), when spatial variance is greater than non-spatial variance (11%) than when they are equal (3%) and when both variances are small (7%) than when they are big (6%). The CRPS confirms these results. The empirical coverage of the anisotropic model is closer to the true 90% under all scenarios. These conclusions are confirmed when sample size is 500. Sample size does not seem to make a difference in distinguishing the models.

$r = 1.5$	(0.9275, 0.93)		$\sigma^2/\tau^2 = 1$	$\sigma^2/\tau^2 = 5$
$r = 4$	(0.92, 0.9275)	$\sigma^2 = 0.2$	(0.9175, 0.9175)	(0.9575, 0.96)
$r = 8$	(0.9125, 0.925)	$\sigma^2 = 1$	(0.89, 0.8775)	(0.9125, 0.925)

Table 2: Model comparison for simulated data, 90% Empirical Coverage (Anisotropy, Isotropy), sample size = 100

$r = 1.5$	(0.925, 0.9)		$\sigma^2/\tau^2 = 1$	$\sigma^2/\tau^2 = 5$
$r = 4$	(0.916, 0.906)	$\sigma^2 = 0.2$	(0.905, 0.904)	(0.928, 0.912)
$r = 8$	(0.9, 0.902)	$\sigma^2 = 1$	(0.896, 0.898)	(0.9, 0.902)

Table 3: Model comparison for simulated data, 90% empirical coverage (anisotropy, isotropy), sample size = 500

$r = 1.5$	(0.588, 0.528)		$\sigma^2/\tau^2 = 1$	$\sigma^2/\tau^2 = 5$
$r = 4$	(0.497, 0.504)	$\sigma^2 = 0.2$	(0.265, 0.275)	(0.082, 0.093)
$r = 8$	(0.411, 0.457)	$\sigma^2 = 1$	(1.377, 1.407)	(0.411, 0.457)

Table 4: Model comparison for simulated data, predictive mean squared error (anisotropy, isotropy), sample size = 100

$r = 1.5$	(0.487, 0.399)		$\sigma^2/\tau^2 = 1$	$\sigma^2/\tau^2 = 5$
$r = 4$	(0.334, 0.338)	$\sigma^2 = 0.2$	(0.226, 0.239)	(0.061, 0.068)
$r = 8$	(0.318, 0.345)	$\sigma^2 = 1$	(1.141, 1.197)	(0.318, 0.345)

Table 5: Model comparison for simulated data, predictive mean squared error (anisotropy, isotropy), sample size = 500

$r = 1.5$	(0.431, 0.409)		$\sigma^2/\tau^2 = 1$	$\sigma^2/\tau^2 = 5$
$r = 4$	(0.397, 0.399)	$\sigma^2 = 0.2$	(0.292, 0.299)	(0.165, 0.176)
$r = 8$	(0.364, 0.385)	$\sigma^2 = 1$	(0.664, 0.672)	(0.364, 0.385)

Table 6: Model comparison for simulated data, continuous rank probability score (anisotropy, isotropy), sample size = 100

$r = 1.5$	(0.391, 0.355)		$\sigma^2/\tau^2 = 1$	$\sigma^2/\tau^2 = 5$
$r = 4$	(0.325, 0.326)	$\sigma^2 = 0.2$	(0.268, 0.275)	(0.139, 0.147)
$r = 8$	(0.317, 0.329)	$\sigma^2 = 1$	(0.603, 0.617)	(0.317, 0.329)

Table 7: Model comparison for simulated data, continuous rank probability score (anisotropy, isotropy), sample size = 500

## 6 Real Data Example: Scallop Catches

We apply the models to data of sampling of scallop abundance on the continental shelf off the coastline of the northeastern U.S. The data comes from a survey conducted by the Northeast Fisheries Science Center of the National Marine Fisheries Service. Figure 2 shows the sampling sites for 1993 and Figure 3 shows the spatial surface as well as contours of the fitted sites. There are a total of 148 sampling sites. We use 118 for model fitting and 30 as hold out data for kriging. We see a lot more sampling in southwest to northeast direction than in the northwest to southeast direction. Evidently it's more appropriate to follow the coastline in searching for scallops.

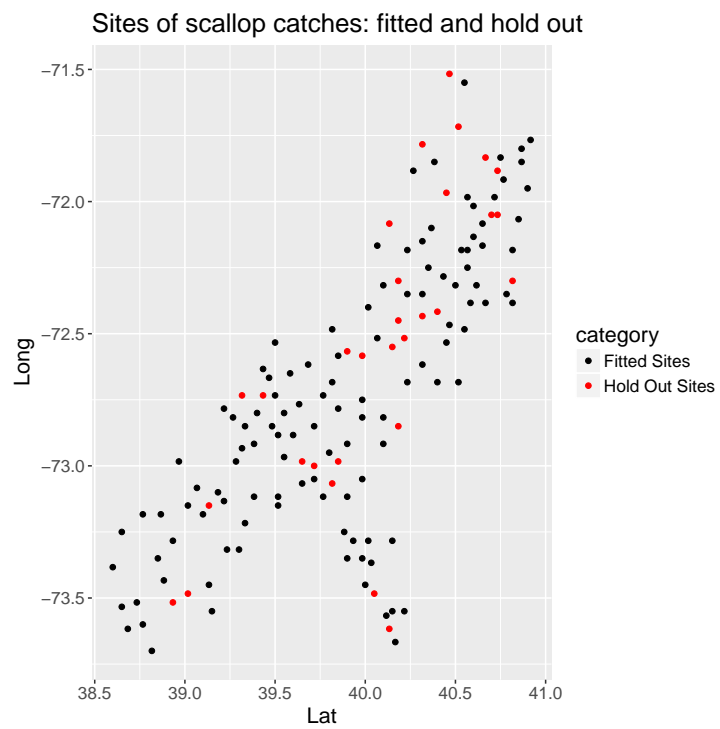


Figure 2: Sites sampled in the Atlantic Ocean for 1993 scallop catch data (fitted and hold out)

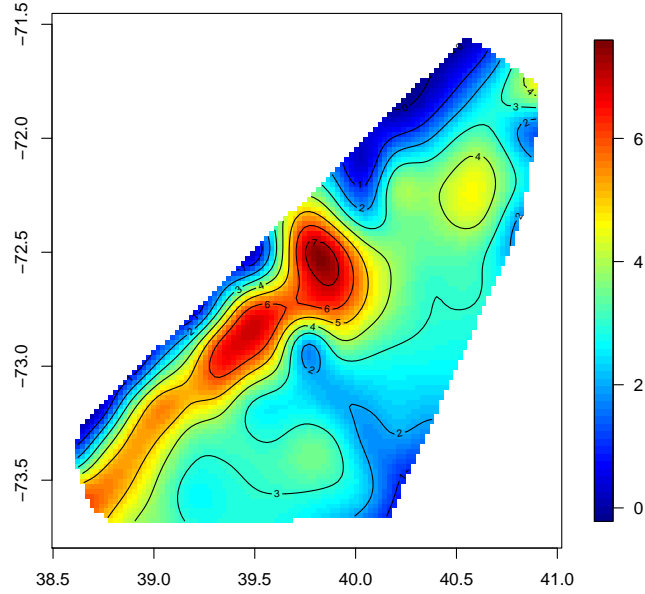


Figure 3: Surface and contour plot for 1993 scallop data (fitted)

Again, we fit the anisotropic model using methodology detailed in section 3, and the isotropic model using spBayes. In both cases, we run 30,000 iterations, using a burn-in of 20,000 and a thinning rate of 1/20 for the remaining 10,000 samples, and we retain 500 posterior samples of all model parameters. Figure 4 shows the density plot of the remaining posterior samples under anisotropy. Table 8 shows the posterior means and 95% credible intervals for all model parameters under isotropy and anisotropy. shows the mean posterior range plotted as a function of angle with associated individual 95% intervals.



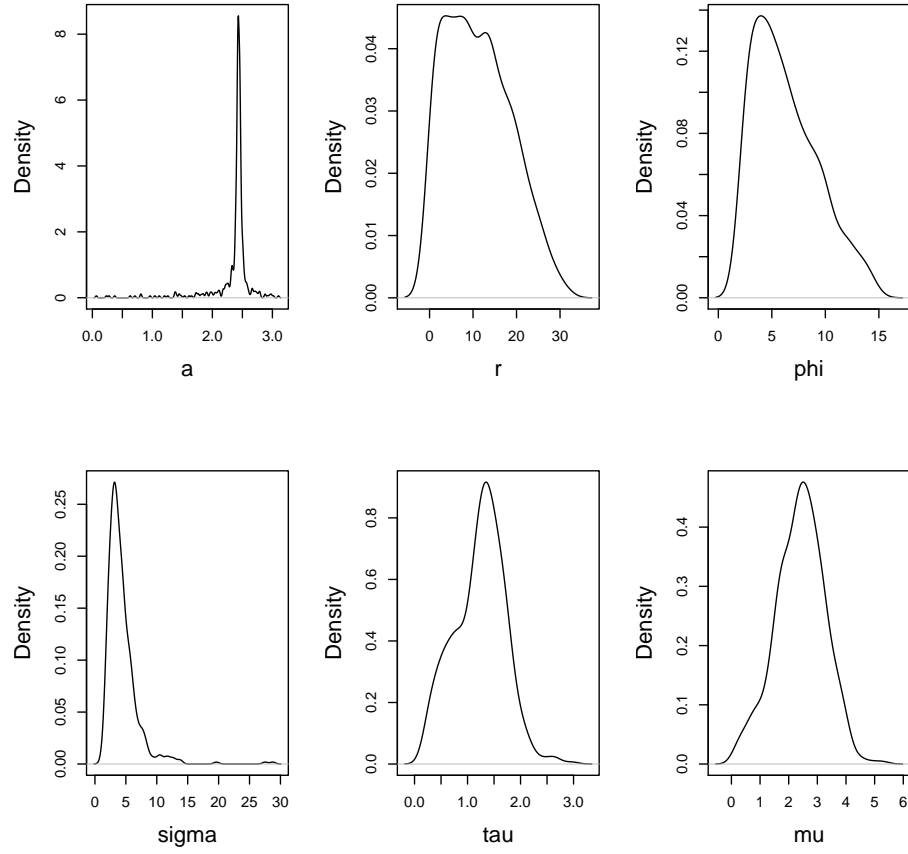


Figure 4: Density plots of posterior samples for all parameters under anisotropy (after burn-in and thinning)

Parameters	$\mu$	$\phi$	$\sigma^2$	$\tau^2$	$\alpha$	$r$
Isotropy	NA	4.92 (2.49, 8.71)	5.20 (3.10, 8.94)	0.49 (0.19, 1.15)	NA	NA
Anisotropy	2.40 (0.57, 3.96)	6.3 (2.23, 13.24)	4.36 (1.77, 10.81)	1.23 (0.30, 2.07)	2.35 (1.31, 2.74)	11.2 (0.49, 26.27)

Table 8: Posterior means and 95% credible intervals for all model parameters under isotropy and anisotropy

To show evidence for departure from isotropy, we obtain the posterior distribution for range in each direction. Figure shows the mean posterior range plotted as a function of angle with associated individual 95% credible intervals. The plot on the right shows the range in polar coordinates which forms an ellipse.

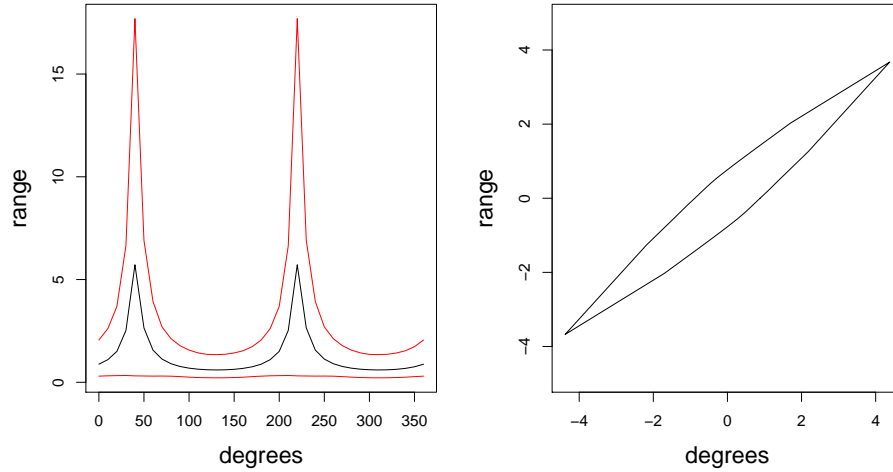


Figure 5: Posterior range as a function of angle under geometric anisotropy. The plot on the right shows the range in polar coordinates.

We evaluate the predictive performance of the two models on the 30 hold-out sites using empirical coverage, PMSE and CRPS, as displayed in Table 9. The anisotropic model reduces average PMSE by 12% and reduces average CRPS by 7% over the 30 sites. Anisotropy also has higher empirical coverage than isotropy.

Model	EC	PMSE	CRPS
Isotropy	86.7%	1.792	0.781
Anisotropy	96.7%	1.581	0.725

Table 9: Model comparison of isotropy and anisotropy for scallops data: 90% empirical coverage, PMSE, and CRPS

Figure 6 shows the empirical coverage of isotropy and anisotropy, where the dots represent the true observed values and the grey lines represent the 90% credible interval of the posterior predictive samples. We can see the credible intervals produced by isotropy fail to capture the observed value for 4 of the 30 sites, while anisotropy fails to capture 1 site.

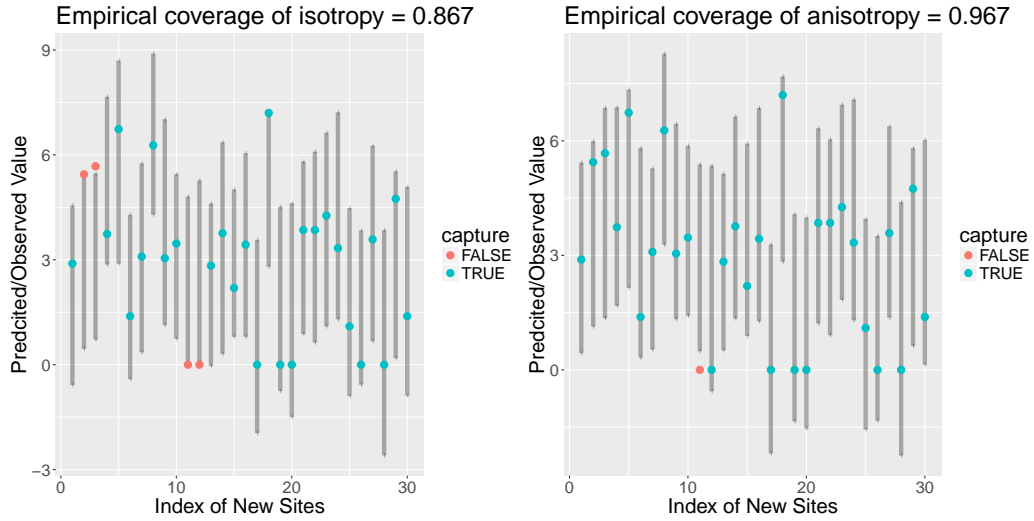


Figure 6: Model comparison: empirical coverage of isotropy and anisotropy

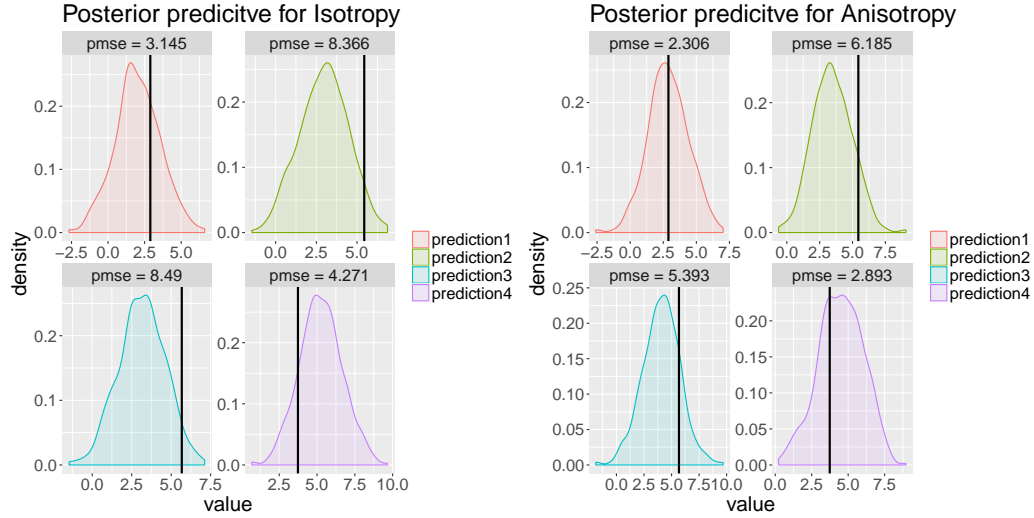


Figure 7: Model comparison: posterior predictive distribution and PMSE for 4 hold out sites under isotropy and anisotropy. Vertical line represents the true value.

Figure 7 shows posterior predictive distribution and PMSE for 4 randomly selected hold out sites under isotropy and anisotropy. The vertical line represents the true value. Under anisotropy, the predictive distributions are more closely concentrated around the true value, and the PMSE's are smaller.

## 7 Discussion and Future Work

This paper explored when and how much geometric anisotropic models for random effects in geostatistical settings improve predictive performance over isotropic models. We compared the predictive performance of isotropic and anisotropic models for data generated under anisotropy, using different parameter values of the data generating model, including different sample sizes, different choices of anisotropy ratio, and different scales and ratios of spatial variance and pure error. We found that anisotropy yields better predictive performance when the data significantly departs from isotropy (anisotropy ratio is much greater than one), and the improvement is more prominent when the anisotropy ratio is higher and when spatial variance is higher compared to pure error, regardless of sample size. The anisotropic model yields much better predictive results on scallop catches data which have been shown to exhibit anisotropic behavior. We performed full Bayesian inference on all model parameters using Metropolis algorithm.

Our future work involves extending the geometric anisotropic model to multivariate and space-time settings. We will explore different ways of constructing anisotropy such as using different covariance structures. In particular we are interested in product covariance function, i.e. the product of one-dimensional covariance function in the x-coordinate and a one-dimensional covariance function in the y-coordinate, which takes the following form:

$$C(d) = \begin{cases} \tau^2 + \sigma^2 & \text{if } d = 0 \\ \sigma^2 \exp(-\phi_x d_x) \exp(-\phi_y d_y) & \text{if } d > 0 \end{cases}$$

where  $d_x$  is the distance between x-coordinates and  $d_y$  is the distance between y-coordinates.

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