CT-1 17.10.19 (EEE)

(2)

Solution of 1st order differential equations by various Method:

First order & 1st degree differential equ^n:

A differential equation of the form $M + N\frac{dy}{dx} = 0$ or $Mdx + Ndy = 0$ is called first order and first degree differential equation, where both M & N are functions of x & y. They are divided mainly into 6 catagories:

(i) Seperation of variables

(ii) Homogeneous equation

(iii) Equation reducible to homogeneous

(iv) Exact equation

(v) Linear equation

(vi) Reducible to linear equation.

Solution by integration:

$$\frac{dy}{dx} = g(x)$$

$$\Rightarrow \int dy = \int g(x)dx$$

$$\therefore y = G(x) + c$$

Example:

$$\frac{dy}{dx} = 1 + e^{2x}$$

$$\Rightarrow dy = (1 + e^{2x})dx$$

$\Rightarrow y = \int (1+e^{2x})\,dx + c$

$\therefore y = x + \frac{1}{2}e^{2x} + c.$

Separation of variable:

If the equation of $M(x,y)dx + N(x,y)d$ can be written in this form $f(x)dx + g(y)dy = 0$ then it can be solved easily term by term

the solution is,

$$\int f(x)dx + \int g(y)dy = c$$

CSE. 1 Solve $\frac{dy}{dx} = \frac{x^2+x+1}{y^2+y+1}$

Solution:

$$\frac{dy}{dx} = \frac{x^2+x+1}{y^2+y+1}$$

$$\Rightarrow (y^2+y+1)\,dy = (x^2+x+1)\,dx$$

$$\Rightarrow \frac{y^3}{3} + \frac{y^2}{2} + y = \frac{x^3}{3} + \frac{x^2}{2} + x + c$$

$$\Rightarrow \frac{1}{3}(y^3 - x^3) + \frac{1}{2}(y^2 - x^2) + y - x = c$$

CSE

(2) C.W Solve the 1st order differential eqn $\frac{dy}{dx} = \frac{2y}{x}$

Solution:

Here, $\frac{dy}{dx} = \frac{2y}{x}$

$\Rightarrow \frac{dy}{y} = \frac{2dx}{x}$

Now integrating both sides,

$\int \frac{dy}{y} = 2\int \frac{dx}{x} + \ln c$

$\Rightarrow \ln y = 2\ln x + \ln c$

$\Rightarrow \ln y = \ln x^2 + \ln c$

$\Rightarrow \ln y = \ln(x^2 c)$

$\Rightarrow y = cx^2$ (taking exponential on both sides)

C.W CSE TEX

(3) Solve $(x-y)^2 \frac{dy}{dx} = a^2$

Solution:

Given that,

$(x-y)^2 \frac{dy}{dx} = a^2$ —— (1)

Let $x-y = z$

$\therefore y = x - z$

$\frac{dy}{dx} = 1 - \frac{dz}{dx}$

From (1) we get,

$z^2 \left(1 - \frac{dz}{dx}\right) = a^2$

$\Rightarrow \left(1 - \frac{dz}{dx}\right) = \frac{a^2}{z^2}$

$\Rightarrow \frac{dz}{dx} = 1 - \frac{a^2}{z^2}$

$$\Rightarrow \frac{dz}{dx} = \frac{z^2 - a^2}{z^2}$$

$$\Rightarrow \frac{z^2}{z^2 - a^2}\, dz = dx$$

Now integrating both sides,

$$\int \frac{z^2 dz}{z^2 - a^2} = \int dx + c$$

$$\Rightarrow \int \frac{(z^2 - a^2 + a^2)dz}{z^2 - a^2} = \int dx + c$$

$$\Rightarrow \int \frac{z^2 - a^2}{z^2 - a^2}\, dz + \int \frac{a^2 dz}{z^2 - a^2} = \int dx + c$$

$$\Rightarrow \int dz + a^2 \int \frac{dz}{z^2 - a^2}\, dz = \int dx + c$$

$$\Rightarrow z + a^2 \frac{1}{2a} \ln\left|\frac{z-a}{z+a}\right| = x + c$$

$$\Rightarrow z + \frac{a}{2} \ln\left|\frac{z-a}{z+a}\right| = x + c$$

$$\Rightarrow x - y + \frac{a}{2} \ln\left|\frac{x-y-a}{x-y+a}\right| = x + c$$

$$\Rightarrow \frac{a}{2} \ln\left|\frac{x-y-a}{x-y+a}\right| - y = c.$$

Ans.

B.W CSE TEX

4. Solve $3e^x \tan y\, dx + (1 - e^x)\sec^2 y\, dy = 0$

Solution:

Given,

$$3e^x \tan y\, dx + (1 - e^x)\sec^2 y\, dy = 0$$

$$\Rightarrow 3e^x \tan y\, dx = (e^x - 1)\sec^2 y\, dy$$

$$\Rightarrow \frac{e^x dx}{e^x - 1} = \frac{\sec^2 y}{3\tan y}\, dy$$

Now integrating both sides,

$$\int \frac{e^x dx}{e^x - 1} = \int \frac{\sec^2 y\, dy}{3\tan y} + \ln c$$

$$\Rightarrow \ln(e^x - 1) = \frac{1}{3}\ln(\tan y) + \ln c$$

$$\Rightarrow \ln(e^x - 1) = \ln(\tan y)^{1/3} + \ln c$$

$$\Rightarrow \ln(e^x - 1) = \ln\{c(\tan y)^{1/3}\}$$

$$\Rightarrow e^x - 1 = c(\tan y)^{1/3}$$

$$\therefore e^x = c(\tan y)^{1/3} + 1. \quad \text{Ans.}$$

$e^x - 1 = z$
$e^x dx = dz$
$\tan y = \theta$
$\sec^2 y = \frac{d\theta}{dy}$
$\sec^2 y\, dy = d\theta$
$\int \frac{dz}{z} = \frac{1}{3}\int \frac{d\theta}{\theta}$

C.W CSE TEX TEE

5. Solve $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$

Solution:

Given that,

$$\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$$

$$\Rightarrow \frac{dy}{dx} = e^{-y}(e^x + x^2)$$

$$\Rightarrow e^y dy = (e^x + x^2)\, dx$$

Integrating both sides,

$$\int e^y dy = \int (e^x + x^2)\, dx + c$$

$$\Rightarrow \int e^y dy = \int e^x dx + \int x^2 dx + c$$

$$\Rightarrow e^y = e^x + \frac{x^3}{3} + c$$

$$\therefore e^y - e^x - \frac{x^3}{3} = c. \quad \text{Ans.}$$

TE H.W (6). Solve $\log\left(\frac{dy}{dx}\right) = ax + by$

Solution:

Given, $\log\left(\frac{dy}{dx}\right) = ax + by$

$$\Rightarrow \frac{dy}{dx} = e^{ax+by}$$

$$\Rightarrow \frac{dy}{dx} = e^{ax} \cdot e^{by}$$

$$\Rightarrow e^{-by} dy = e^{ax} dx$$

Integrating bothsides,

$$\int e^{-by} dy = \int e^{ax} dx + c$$

$$\Rightarrow -\frac{1}{b} e^{-by} = \frac{1}{a} e^{ax} + c$$

$$\Rightarrow \frac{1}{a} e^{ax} + \frac{1}{b} e^{-by} + c = 0$$

Ans.

H.W

H.W CSE (7). Solve $5e^{-5x} \sin y \, dx + (e^{-5x} - 3) \cos y \, dy = 0$

Solution:

Given, $5e^{-5x} \sin y \, dx + (e^{-5x} - 3) \cos y \, dy = 0$

$$\Rightarrow 5e^{-5x} \sin y \, dx = (3 - e^{-5x}) \cos y \, dy$$

$$\Rightarrow \frac{e^{-5x} dx}{3 - e^{-5x}} = \frac{\cos y}{5 \sin y} dy$$

$$\Rightarrow \int \frac{e^{-5x}}{3 - e^{-5x}} dx = \int \frac{\cos y}{5 \sin y} dy \quad \text{--- (1)}$$

Let $3 - e^{-5x} = z$ & $\sin y = \theta$

$\Rightarrow 5e^{-5x}\,dx = dz$ $\qquad \Rightarrow \cos y\,dy = d\theta$

$\therefore$ ① $\Rightarrow$

$$\frac{1}{5}\int \frac{dz}{z} = \frac{1}{5}\int \frac{d\theta}{\theta}$$

$$\Rightarrow \frac{1}{5}\ln z = \frac{1}{5}\ln\theta + \ln c$$

$$\Rightarrow \ln z = \ln\theta + \ln c$$

$$\Rightarrow z = \theta c$$

$$\therefore 3 - e^{-5x} = c\sin y$$

Ans.

② Solve: $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$

H.W.

Solution:

Given,

$$y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$$

$$\Rightarrow y - ay^2 = a\frac{dy}{dx} + x\frac{dy}{dx}$$

$$\Rightarrow y(1-ay) = (a+x)\frac{dy}{dx}$$

$$\Rightarrow \frac{dx}{a+x} = \frac{dy}{y(1-ay)}$$

$$\Rightarrow \frac{dx}{a+x} = \left(\frac{a}{1-ay} + \frac{1}{y}\right)dy$$

Integrating both sides w.r.t. x

$$\int \frac{dx}{x+a} = \int \frac{a}{1-ay}\,dy + \int \frac{1}{y}\,dy + \ln c$$

$\Rightarrow \ln(x+a) = -\ln(1-ay) + \ln y + \ln c$

$\Rightarrow \ln(x+a) = -\ln(1-ay) + \ln(cy)$

$\Rightarrow \ln(x+a) = \ln \dfrac{cy}{1-ay}$

$\Rightarrow x+a = \dfrac{cy}{1-ay}$ [taking exponential on both sides]

$\Rightarrow (x+a)(1-ay) = cy.$

Ans.

H.W

CSE TEST

Solve : $\dfrac{dy}{dx} = e^{x+y} + x^2 e^{x^3+y}$

Solution:

Given that,

$\dfrac{dy}{dx} = e^{x+y} + x^2 e^{x^3+y}$

$\Rightarrow \dfrac{dy}{dx} = e^{y}[e^{x} + x^2 e^{x^3}]$

$\Rightarrow e^{-y}\dfrac{dy}{dx} = e^{x} + x^2 e^{x^3}$

$\Rightarrow e^{-y}\,dy = [e^{x} + x^2 e^{x^3}]\,dx$

Now integrating both sides,

$\int e^{-y}dy = \int e^{x}dx + \int x^2 e^{x^3} dx$

$\Rightarrow -e^{-y} = e^{x} + \int x^2 e^{x^3} dx$ ——— (1)

Let,

$x^3 = t$

$\Rightarrow 3x^2 dx = dt$

$\therefore x^2 dx = dt/3$

from equ^n (1)

$-e^{-y} = e^x + \int e^t \frac{dt}{3}$

$\Rightarrow -e^{-y} = e^x + \frac{1}{3} e^t + c$

$\Rightarrow e^x + e^{-y} + \frac{1}{3} e^t + c = 0$

$\therefore e^x + e^{-y} + \frac{1}{3} e^{x^3} + c = 0$. Ans.

10. Solve $\frac{dy}{dx} + \frac{y^2+y+1}{x^2+x+1} = 0$

Solution:

Given that,

$$\frac{dy}{dx} + \frac{y^2+y+1}{x^2+x+1} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y^2+y+1}{x^2+x+1}$$

$$\Rightarrow \frac{dy}{y^2+y+1} = -\frac{dx}{x^2+x+1}$$

Integrating both sides,

$$\int \frac{dy}{y^2+y+1} = -\int \frac{dx}{x^2+x+1}$$

$$\Rightarrow \int \frac{dy}{y^2+2\cdot y\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} = -\int \frac{dx}{x^2+2\cdot x\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$\Rightarrow \int \frac{dy}{\left(y+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} = -\int \frac{dx}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$\Rightarrow \frac{1}{\sqrt{3}/2}\tan^{-1}\frac{y+\frac{1}{2}}{\sqrt{3}/2}+c=-\frac{1}{\sqrt{3}/2}\tan^{-1}\frac{x+1/2}{\sqrt{3}/2}$$

$$\Rightarrow \frac{2}{\sqrt{3}}\tan^{-1}\frac{\left(\frac{2y+1}{2}\right)}{\sqrt{3}/2}+c=-\frac{2}{\sqrt{3}}\tan^{-1}\frac{\frac{2x+1}{2}}{\sqrt{3}/2}$$

$$\Rightarrow \frac{2}{\sqrt{3}}\tan^{-1}\frac{2y+1}{\sqrt{3}}+c=-\frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}$$

$$\Rightarrow \frac{2}{\sqrt{3}}\tan^{-1}\frac{2y+1}{\sqrt{3}}+c+\frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}=0$$

$$\therefore \frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{2y+1}{\sqrt{3}}+\tan^{-1}\frac{2x+1}{\sqrt{3}}\right]+c=0$$

C.W

11 CSE Solve $\sin^{-1}\left(\frac{dy}{dx}\right)=x+y$

Solution:

Given,

$$\sin^{-1}\left(\frac{dy}{dx}\right)=x+y$$

$$\Rightarrow \frac{dy}{dx}=\sin(x+y) \quad \text{--- (i)}$$

let $x+y=z$

$$\Rightarrow 1+\frac{dy}{dx}=\frac{dz}{dx}$$

$$\Rightarrow \frac{dy}{dx}=\frac{dz}{dx}-1 \quad \text{--- (ii)}$$

from equ^n (i)

$$\frac{dz}{dx}-1=\sin z$$

$$\Rightarrow \frac{dz}{dx}=1+\sin z$$

$$\Rightarrow \frac{dz}{1+\sin z} = dx$$

Integrating both sides,

$$\int \frac{dz}{1+\sin z} = \int dx$$

$$\Rightarrow \int \frac{(1-\sin z)dz}{(1+\sin z)(1-\sin z)} = \int dx$$

$$\Rightarrow \int \frac{(1-\sin z)dz}{1-\sin^2 z} = \int dx$$

$$\Rightarrow \int \frac{(1-\sin z)dz}{\cos^2 z} = \int dx$$

$$\Rightarrow \int \{\sec^2 z\, dz - \tan z \sec z\, dz\} = \int dx$$

$$\Rightarrow \tan z - \sec z = x + c$$

$$\Rightarrow \tan(x+y) - \sec(x+y) = x + c.$$ Ans.

(12) Assign. Solve $\frac{dy}{dx} = (4x+y+1)^2$

Solution: Given, $\frac{dy}{dx} = (4x+y+1)^2$ —— (1)

let $4x+y+1 = z$

$$\Rightarrow 4 + \frac{dy}{dx} = \frac{dz}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dz}{dx} - 4$$

So from (1) we get,

$$\frac{dz}{dx} - 4 = z^2$$

$$\Rightarrow \frac{dz}{dx} = z^2+4$$

$$\Rightarrow \frac{dz}{z^2+4} = dx$$

Integrating bothsides we get,

$$\int \frac{dz}{z^2+4} = \int dx$$

$$\Rightarrow \int \frac{dz}{z^2+2^2} = \int dx$$

$$\Rightarrow \frac{1}{2} \tan^{-1} \frac{z}{2} = x+c$$

$$\Rightarrow \tan^{-1} \frac{z}{2} = 2x+2c$$

$$\Rightarrow \tan^{-1} \frac{z}{2} = 2x+c'$$

$$\Rightarrow \frac{z}{2} = \tan(2x+c')$$

$$\Rightarrow z = 2\tan(2x+c')$$

$$\Rightarrow 4x+y+1 = 2\tan(2x+c')$$

$$\therefore 4x+y+1-2\tan(2x+c')=0$$

Ans.

H.W Assignment :

CSV

1. $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$

Solution: Given that,

$$\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$$

$$\Rightarrow \sec^2 x \tan y \, dx = -\sec^2 y \tan x \, dy$$

$$\Rightarrow \frac{\sec^2 x \, dx}{\tan x} = -\frac{\sec^2 y \, dy}{\tan y}$$

Integrating both sides,

$$\int \frac{\sec^2 x \, dx}{\tan x} = -\int \frac{\sec^2 y \, dy}{\tan y} \quad \text{——} (1)$$

Let $\tan x = z$ & $\tan y = m$

$\Rightarrow \sec^2 x dx = dz$ $\Rightarrow \sec^2 y dy = dm$

From (1) we get,

$$\int \frac{dz}{z} = -\int \frac{dm}{m}$$

$$\Rightarrow \ln z = -\ln m + \ln c$$

$$\Rightarrow \ln z + \ln m = \ln c$$

$$\Rightarrow \ln (zm) = \ln c$$

$$\Rightarrow zm = c$$

$$\therefore \tan x \tan y = c.$$

Assign Exam unseen

2. $x\sqrt{1-y^2} \, dx + y\sqrt{1-x^2} \, dy = 0$

Solution: Given that,

$$x\sqrt{1-y^2} \, dx + y\sqrt{1-x^2} \, dy = 0$$

$$\Rightarrow x\sqrt{1-y^2} \, dx = -y\sqrt{1-x^2} \, dy$$

$$\Rightarrow \frac{x\,dx}{\sqrt{1-x^2}} = -\frac{y\,dy}{\sqrt{1-y^2}}$$

Integrating bothsides,

$$\int \frac{x}{\sqrt{1-x^2}}\,dx = -\int \frac{y}{\sqrt{1-y^2}}\,dy \quad \text{--- (1)}$$

Let $1-x^2 = z^2$ & $1-y^2 = m^2$

$\Rightarrow -2x\,dx = 2z\,dz$

$\Rightarrow -x\,dx = z\,dz$

$\therefore x\,dx = -z\,dz$

$\Rightarrow -2y\,dy = 2m\,dm$

$\Rightarrow y\,dy = -m\,dm$

from equ^n (1) we get,

$$-\int \frac{z\,dz}{\sqrt{z^2}} = \int \frac{m\,dm}{\sqrt{m^2}}$$

$$\Rightarrow -\int \frac{z\,dz}{z} = \int \frac{m\,dm}{m}$$

$$\Rightarrow -\int dz = \int dm$$

$$\Rightarrow -z = m + c$$

$$\Rightarrow -(\sqrt{1-x^2}) = \sqrt{1-y^2} + c$$

Ans.

3. $(1+x)\,dy - y\,dx = 0$

Solution: Given that,

$$(1+x)\,dy - y\,dx = 0$$

$$\Rightarrow (1+x)\,dy = y\,dx$$

$$\Rightarrow \frac{dy}{y} = \frac{dx}{1+x}$$

$$\Rightarrow \int \frac{dy}{y} = \int \frac{dx}{1+x}$$

$$\Rightarrow \ln|y| = \ln|1+x| + c_1$$

$$\Rightarrow y = e^{\ln|1+x| + c_1}$$

$$\Rightarrow y = e^{\ln|1+x|} \cdot e^{c_1}$$

$$\Rightarrow y = |1+x| e^{c_1}$$

$$\Rightarrow y = \pm e^{c_1} (1+x)$$

$$\therefore y = c(1+x) \quad [c = \pm e^{c_1}]$$

Some related problems:

1. $\frac{dy}{dx} = \sin 5x$
2. $dx + e^{3x} dy = 0$
3. $x \frac{dy}{dx} = 4y$ TEX
4. $\frac{dy}{dx} = e^{3x+2y}$ TEX
5. $\frac{dy}{dx} = (x+1)^2$
6. $\tan x \, dy = \cot y \, dx$
7. $\sqrt{1-x^2}\, dy + \sqrt{1-y^2}\, dx = 0$
8. $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

9. $\frac{dy}{dx} = e^{x+y} + x^2 e^{-y}$

10. $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$

11. $\frac{dy}{dx} = \frac{y(x-1)}{x^2}$

12. $\frac{dy}{dx} = (4x+y+1)^2$