CHAPTER 7

Applications of First-Order Differential Equations

GROWTH AND DECAY PROBLEMS

Let $N(t)$ denote the amount of substance (or population) that is either growing or decaying. If we assume that dN/dt, the time rate of change of this amount of substance, is proportional to the amount of substance present, then $dN/dt = kN$, or

$$\frac{dN}{dt} - KN = 0 \tag{7.1}$$

where k is the constant of proportionality. (See Problems 7.1–7.7.)

We are assuming that $N(t)$ is a differentiable, hence continuous, function of time. For population problems, where $N(t)$ is actually discrete and integer-valued, this assumption is incorrect. Nonetheless, (*7.1*) still provides a good approximation to the physical laws governing such a system. (See Problem 7.5.)

TEMPERATURE PROBLEMS

Newton's law of cooling, which is equally applicable to heating, states that *the time rate of change of the temperature of a body is proportional to the temperature difference between the body and its surrounding medium.* Let T denote the temperature of the body and let T_m denote the temperature of the surrounding medium. Then the time rate of change of the temperature of the body is dT/dt, and Newton's law of cooling can be formulated as $dT/dt = -k(T - T_m)$, or as

$$\frac{dT}{dt} + kT = kT_m \tag{7.2}$$

where k is a *positive* constant of proportionality. Once k is chosen positive, the minus sign is required in Newton's law to make dT/dt negative in a cooling process, when T is greater than T_m, and positive in a heating process, when T is less than T_m. (See Problems 7.8–7.10.)

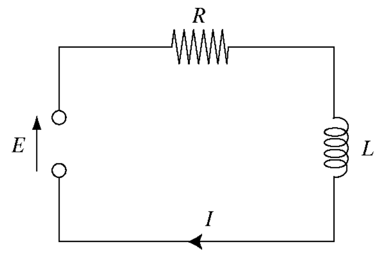

Fig. 7-3

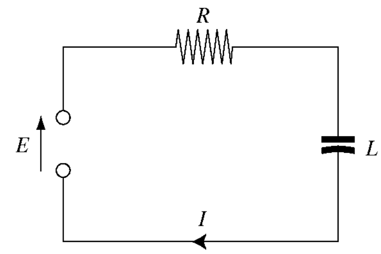

Fig. 7-4

ORTHOGONAL TRAJECTORIES

Consider a one-parameter family of curves in the xy-plane defined by

$$F(x, y, c) = 0 \tag{7.12}$$

where c denotes the parameter. The problem is to find another one-parameter family of curves, called the *orthogonal trajectories* of the family (*7.12*) and given analytically by

$$G(x, y, k) = 0 \tag{7.13}$$

such that every curve in this new family (*7.13*) intersects at right angles every curve in the original family (*7.12*).

We first implicitly differentiate (*7.12*) with respect to x, then eliminate c between this derived equation and (*7.12*). This gives an equation connecting x, y, and y', which we solve for y' to obtain a differential equation of the form

$$\frac{dy}{dx} = f(x, y) \tag{7.14}$$

The orthogonal trajectories of (*7.12*) are the solutions of

$$\frac{dy}{dx} = -\frac{1}{f(x, y)} \tag{7.15}$$

(See Problems 7.23–7.25.)

For many families of curves, one cannot explicitly solve for dy/dx and obtain a differential equation of the form (*7.14*). We do not consider such curves in this book.

Solved Problems

7.1. A person places \$20,000 in a savings account which pays 5 percent interest per annum, compounded continuously. Find (*a*) the amount in the account after three years, and (*b*) the time required for the account to double in value, presuming no withdrawals and no additional deposits.

Let $N(t)$ denote the balance in the account at any time t. Initially, $N(0) = 20{,}000$. The balance in the account grows by the accumulated interest payments, which are proportional to the amount of money in the account. The constant of proportionality is the interest rate. In this case, $k = 0.05$ and Eq. (*7.1*) becomes

$$\frac{dN}{dt} - 0.05N = 0$$

This differential equation is both linear and separable. Its solution is

$$N(t) = ce^{0.05t} \tag{1}$$

At $t = 0$, $N(0) = 20{,}000$, which when substituted into (*1*) yields

$$20{,}000 = ce^{0.05(0)} = c$$

With this value of c, (*1*) becomes

$$N(t) = 20{,}000e^{0.05t} \tag{2}$$

Equation (*2*) gives the dollar balance in the account at any time t.

(*a*) Substituting $t = 3$ into (*2*), we find the balance after three years to be

$$N(3) = 20{,}000e^{0.05(3)} = 20{,}000(1.161834) = \$23{,}236.68$$

(*b*) We seek the time t at which $N(t) = \$40{,}000$. Substituting these values into (*2*) and solving for t, we obtain

$$40{,}000 = 20{,}000e^{0.05t}$$
$$2 = e^{0.05t}$$
$$\ln|2| = 0.05t$$
$$t = \frac{1}{0.05}\ln|2| = 13.86 \text{ years}$$

7.2. A person places \$5000 in an account that accrues interest compounded continuously. Assuming no additional deposits or withdrawals, how much will be in the account after seven years if the interest rate is a constant 8.5 percent for the first four years and a constant 9.25 percent for the last three years?

Let $N(t)$ denote the balance in the account at any time t. Initially, $N(0) = 5000$. For the first four years, $k = 0.085$ and Eq. (*7.1*) becomes

$$\frac{dN}{dt} - 0.085N = 0$$

Its solution is

$$N(t) = ce^{0.085t} \qquad (0 \le t \le 4) \tag{1}$$

At $t = 0$, $N(0) = 5000$, which when substituted into (*1*) yields

$$5000 = ce^{0.085(0)} = c$$

and (*1*) becomes

$$N(t) = 5000e^{0.085t} \qquad (0 \le t \le 4) \tag{2}$$

Substituting $t = 4$ into (*2*), we find the balance after four years to be

$$N(t) = 5000e^{0.085(4)} = 5000(1.404948) = \$7024.74$$

This amount also represents the beginning balance for the last three-year period.

Over the last three years, the interest rate is 9.25 percent and (*7.1*) becomes

$$\frac{dN}{dt} - 0.0925N = 0 \qquad (4 \le t \le 7)$$

Its solution is

$$N(t) = ce^{0.0925t} \qquad (4 \le t \le 7) \tag{3}$$

At $t = 4$, $N(4) = \$7024.74$, which when substituted into (*3*) yields

$$7024.74 = ce^{0.0925(4)} = c(1.447735) \qquad \text{or} \qquad c = 4852.23$$

and (*3*) becomes

$$N(t) = 4852.23e^{0.0925t} \qquad (4 \le t \le 7) \tag{4}$$

Substituting $t = 7$ into (*4*), we find the balance after seven years to be

$$N(7) = 4852.23e^{0.0925(7)} = 4852.23(1.910758) = \$9271.44$$

7.3. **What constant interest rate is required if an initial deposit placed into an account that accrues interest compounded continuously is to double its value in six years?**

The balance $N(t)$ in the account at any time t is governed by (*7.1*)

$$\frac{dN}{dt} - kN = 0$$

which has as its solution

$$N(t) = ce^{kt} \tag{1}$$

We are not given an amount for the initial deposit, so we denote it as N_0. At $t = 0$, $N(0) = N_0$, which when substituted into (*1*) yields

$$N_0 = ce^{k(0)} = c$$

and (*1*) becomes

$$N(t) = N_0\, e^{kt} \tag{2}$$

We seek the value of k for which $N = 2N_0$ when $t = 6$. Substituting these values into (*2*) and solving for k, we find

$$2N_0 = N_0 e^{k(6)}$$

$$e^{6k} = 2$$

$$6k = \ln |2|$$

$$k = \frac{1}{6}\ln |2| = 0.1155$$

An interest rate of 11.55 percent is required.

7.4. **A bacteria culture is known to grow at a rate proportional to the amount present. After one hour, 1000 strands of the bacteria are observed in the culture; and after four hours, 3000 strands. Find (*a*) an expression for the approximate number of strands of the bacteria present in the culture at any time *t* and (*b*) the approximate number of strands of the bacteria originally in the culture.**

(*a*) Let $N(t)$ denote the number of bacteria strands in the culture at time t. From (*6.1*), $dN/dt - kN = 0$, which is both linear and separable. Its solution is

$$N(t) = ce^{kt} \tag{1}$$

At $t = 1$, $N = 1000$; hence,

$$1000 = ce^{k} \tag{2}$$

At $t = 4$, $N = 3000$; hence,

$$3000 = ce^{4k} \tag{3}$$

Solving (*2*) and (*3*) for k and c, we find

$$k = \frac{1}{3}\ln 3 = 0.366 \quad \text{and} \quad c = 1000e^{-0.366} = 694$$

Substituting these values of k and c into (*1*), we obtain

$$N(t) = 694e^{0.366t} \tag{4}$$

as an expression for the amount of the bacteria present at any time t.

(*b*) We require N at $t = 0$. Substituting $t = 0$ into (*4*), we obtain $N(0) = 694e^{(0.366)(0)} = 694$.