—ution of 1st order differential equation:
Integrating factor:

A 1st order ordinary differential equation is linear in dependent variable y where x is independent variable if it can be written as,

$$\frac{dy}{dx} + P(x)y = Q(x) \quad \text{—(1)}$$

A linear equation has an integrating factor,

$$I.F. = e^{\int P(x)dx}$$

Multiplying (1) by I.F., we may write

$$\frac{dy}{dx} e^{\int P(x)dx} + P(x)y e^{\int P(x)dx} = Q(x) e^{\int P(x)dx}$$

$$\Rightarrow \frac{d}{dx}\left[y e^{\int P(x)dx}\right] = Q(x) e^{\int P(x)dx} \quad \text{—(2)}$$

Integrating (2) we get,

$$y e^{\int P(x)dx} = \int Q(x) e^{\int P(x)dx} dx + c$$

$$\Rightarrow y = e^{-\int P(x)dx} \int Q(x) e^{\int P(x)dx} dx + c e^{-\int P(x)dx}$$

where c is constant.

1. Solve $\frac{dy}{dx} + \frac{2x+1}{x} y = e^{-2x}$

Solution: Here, $P(x) = \frac{2x+1}{x}$

$\therefore$ I.F. $= e^{\int \frac{2x+1}{x} dx} = e^{\int (2 + \frac{1}{x}) dx} = e^{2x + \ln x}$

$= e^{2x} . e^{\ln x}$

$= x e^{2x}$

Multiplying the given ODE we get,

$$x e^{2x} \frac{dy}{dx} + x e^{2x} . \frac{2x+1}{x} y = x e^{2x} . e^{-2x}$$

$$\Rightarrow x e^{2x} \frac{dy}{dx} + e^{2x} (2x+1) y = x$$

$$\Rightarrow \frac{d}{dx} (x e^{2x} y) = x$$

$$\Rightarrow x e^{2x} y = \frac{x^2}{2} + c$$

$$\Rightarrow y = \frac{x}{2} e^{-2x} + \frac{c}{x} e^{-2x} \quad \text{Ans.}$$

Q. 2. $(1-x^2) \frac{dy}{dx} - xy = 1$

Solution:

Given that,

$$(1-x^2) \frac{dy}{dx} - xy = 1$$

$$\Rightarrow \frac{dy}{dx} - \frac{xy}{1-x^2} = \frac{1}{1-x^2} \quad \text{—(1)}$$

Integrating factor $= e^{\int - \frac{x}{1-x^2} dx}$

$= e^{\frac{1}{2} \int \frac{-2x}{1-x^2} dx}$

$\frac{1}{2} \ln (1-x^2)$

$$= e^{\ln(1-x^2)^{1/2}}$$

$$= (1-x^2)^{1/2}$$

Multiplying bothsides by $\sqrt{1-x^2}$ we get,

$$\sqrt{1-x^2}\,\frac{dy}{dx} - \frac{xy}{1-x^2}\sqrt{1-x^2} = \frac{\sqrt{1-x^2}}{1-x^2}$$

$$\Rightarrow \sqrt{1-x^2}\,\frac{dy}{dx} - \frac{xy}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{d}{dx}\left[y\sqrt{1-x^2}\right] = \frac{1}{\sqrt{1-x^2}}$$

Integreating w.r.to x we get,

$$y\sqrt{1-x^2} = \int \frac{dx}{\sqrt{1-x^2}} + c$$

$$\Rightarrow y\sqrt{1-x^2} = \sin^{-1}x + c.$$

$$\therefore y = \sin^{-1}x\,(1-x^2)^{-1/2} + c\,(1-x^2)^{-1/2}$$

12.
3. $(1+x^2)\frac{dy}{dx} + y = \tan^{-1}x$

Solution: Given that,

$$(1+x^2)\frac{dy}{dx} + y = \tan^{-1}x$$

$$\Rightarrow \frac{dy}{dx} + \frac{y}{1+x^2} = \frac{\tan^{-1}x}{1+x^2} \quad \text{——} (1)$$

$$\therefore \text{I.F.} = e^{\int \frac{dx}{1+x^2}} = e^{\tan^{-1}x}$$

Multiplying bothsides by $e^{\tan^{-1}x}$ we get,

$$\frac{dy}{dx}e^{\tan^{-1}x} + \frac{y}{1+x^2}e^{\tan^{-1}x} = \frac{\tan^{-1}x}{1+x^2}e^{\tan^{-1}x}$$

$$\Rightarrow \frac{d}{dx}\left[y e^{\tan^{-1}x}\right] = \frac{\tan^{-1}x \, e^{\tan^{-1}x}}{1+x^2}$$

Integrating w.r.to x

$$y e^{\tan^{-1}x} = \int \frac{\tan^{-1}x \, e^{\tan^{-1}x}}{1+x^2} dx \quad \text{—— (2)}$$

$$\Rightarrow y e^{\tan^{-1}x} = \int z e^z dz$$

Let, $\tan^{-1}x = z$

$\Rightarrow \frac{dx}{1+x^2} = dz$

$$\Rightarrow y e^{\tan^{-1}x} = z\int e^z dz - \int\left\{\frac{dz}{dz}\int e^z dz\right\}dz$$

$$\Rightarrow y e^{\tan^{-1}x} = ze^z - \int e^z dz$$

$$\Rightarrow y e^{\tan^{-1}x} = ze^z - e^z + c$$

$$\Rightarrow y e^{\tan^{-1}x} = (z-1)e^z + c$$

$$\Rightarrow y e^{\tan^{-1}x} = (\tan^{-1}x - 1)e^{\tan^{-1}x} + c$$

$$\therefore y = \tan^{-1}x - 1 + ce^{-\tan^{-1}x}$$

Ans.

H.W. 4. $(2+y^2)dx = (xy + 2y + y^3)dy$

Solution: Given that,

$$(2+y^2)\,dx = (xy+2y+y^3)\,dy$$

$$\Rightarrow \frac{dx}{dy} = \frac{xy+2y+y^3}{2+y^2}$$

$$\Rightarrow \frac{dx}{dy} = \frac{xy}{2+y^2} + \frac{y(2+y^2)}{2+y^2}$$

$$\Rightarrow \frac{dx}{dy} = \frac{xy}{2+y^2} + y$$

$$\Rightarrow \frac{dx}{dy} - \frac{xy}{2+y^2} = y \quad \text{——— (1)}$$

$$\therefore \text{I.F.} = e^{-\int \frac{y}{2+y^2}dy} = e^{-\frac{1}{2}\int \frac{2y\,dy}{2+y^2}}$$

$$= e^{-\frac{1}{2}\ln(2+y^2)}$$

$$= (2+y^2)^{-1/2}$$

$$= \frac{1}{\sqrt{2+y^2}}$$

Multiplying bothsides of (1) by $\frac{1}{\sqrt{2+y^2}}$ we get,

$$\frac{d}{dy}\left[x/\sqrt{2+y^2}\right] = y/\sqrt{2+y^2}$$

Integrating w.r.to. y we get.

$$\frac{x}{\sqrt{2+y^2}} = \int \frac{y\,dy}{\sqrt{2+y^2}}$$

$$\Rightarrow \frac{x}{\sqrt{2+y^2}} = \frac{1}{2}\int \frac{2y\,dy}{\sqrt{2+y^2}}$$

$$\Rightarrow \frac{x}{\sqrt{2+y^2}} = \frac{1}{2}\cdot 2\sqrt{2+y^2} + c$$

$$\therefore x = 2+y^2 + c\sqrt{2+y^2}$$

Ans.

5. $\frac{dy}{dx} + \frac{1}{x}\sin 2y = x^3\cos^2 y$

Solution:

Given that,

$$\frac{dy}{dx} + \frac{1}{x}\sin 2y = x^3\cos^2 y$$

$$\Rightarrow \frac{dy}{dx} + \frac{1}{x}\cdot 2\sin y\cos y = x^3\cos^2 y$$

$$\Rightarrow \frac{1}{\cos^2 y}\frac{dy}{dx} + \frac{1}{x}\frac{2\sin y\cos y}{\cos^2 y} = x^3$$

$$\Rightarrow \sec^2 y\frac{dy}{dx} + \frac{2\tan y}{x} = x^3 \quad \text{--- (1)}$$

let, $\tan y = z$

$$\Rightarrow \sec^2 y\frac{dy}{dx} = \frac{dz}{dx}$$

$$\therefore (1) \Rightarrow \frac{dz}{dx} + \frac{2z}{x} = x^3 \quad \text{--- (2)}.$$

$\therefore$ I.F. $= e^{\int \frac{2}{x} dx} = e^{2 \ln x} = e^{\ln x^2} = x^2$

Multiplying bothsides of (2) by x^2 we get,

$$\frac{d}{dx}\left[z x^2\right] = x^5$$

Integrating w.r.to x we get,

$$z x^2 = \frac{x^6}{6} + \frac{c}{6}$$

$$\Rightarrow x^2 \tan y = \frac{x^6}{6} + \frac{c}{6}$$

$$\therefore 6x^2 \tan y = x^6 + c \text{ . Ans.}$$

6. $\frac{dy}{dx} + y = y^2 e^x$

Solution: Given that,

$$\frac{dy}{dx} + y = y^2 e^x$$

$$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} + \frac{1}{y} = e^x \quad \text{——— (1)}$$

Let, $\frac{1}{y} = z$

$$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} = -\frac{dz}{dx} \quad \text{——— (2)}$$

$\therefore$ (1) $\Rightarrow$

$$-\frac{dz}{dx} + z = e^x$$

$$\Rightarrow \frac{dz}{dx} - z = -e^x \quad \text{——— (3)}$$

$\therefore$ I.F. $= e^{-\int dx} = e^{-x}$

Multiplying both sides of (2) by e^{-x} we get,

$$\frac{d}{dx}[ze^{-x}] = -e^{x}e^{-x}$$

$$\Rightarrow \frac{d}{dx}[ze^{-x}] = -1$$

Integreating w.r.to x we get,

$$ze^{-x} = -x + c$$

$$\Rightarrow \frac{1}{y}e^{-x} = -x + c$$

$$\Rightarrow \frac{1}{ye^{x}} + x = c$$

$$\therefore 1 + xye^{x} = cye^{x}. \quad \text{Ans.}$$

7. $\frac{dy}{dx} - 3y = 6$

Solution: Given that,

$$\frac{dy}{dx} - 3y = 6 \quad \text{——— (1)}$$

$$\text{I.F.} = e^{\int -3dx} = e^{-3x}$$

Multiplying bothsides by e^{-3x} we get,

$$e^{-3x}\frac{dy}{dx} - 3ye^{-3x} = 6.e^{-3x}$$

$$\Rightarrow \frac{d}{dx}\left[e^{-3x} y \right] = 6e^{-3x}$$

$$\Rightarrow e^{-3x} y = -2e^{-3x} + c$$

$$\therefore y = -2 + ce^{3x} \quad \text{Ans.}$$

Homework:

1. $\frac{dy}{dx} + 2xy = e^{-x^2}$

2. $\cos^2 x \frac{dy}{dx} + y = \tan x$

3. $\frac{dy}{dx} + y \cot x = 2\cos x$

4. $\frac{dy}{dx} + 2xy = x$

5. $\frac{dy}{dx} = 5y$

6. $\frac{dy}{dx} + 2y = 0$

7. $\frac{dy}{dx} + y = e^{3x}$

8. $y' + 2xy = x^3$

9. $x\frac{dy}{dx} + 4y = x^3 - x$

10. $(1-x^2)\frac{dy}{dx} - xy = 1$

11. $x\frac{dy}{dx} + 2y = x^2 \log x$

12. $x\frac{dy}{dx} + 2y = x^4$

13. $(x^3-x)\frac{dy}{dx}-(3x^2-1)y=x^5-2x^3+x$

14. $x\frac{dy}{dx}+2y=\frac{dy}{dx}+4$

15. $\frac{dy}{dx}-\frac{2}{x}y=x+\frac{1}{x}\cdot\sin\frac{1}{x^2}$

8. $\frac{dy}{dx}+xy=x^3y^3$

Solution:

Given that,

$$\frac{dy}{dx}+xy=x^3y^3$$

Dividing by y^3 we get,

$$\frac{1}{y^3}\frac{dy}{dx}+x\cdot\frac{1}{y^2}=x^3 \quad \text{—①}$$

let,

$$\frac{1}{y^2}=v$$

$$\Rightarrow -2y^{-3}\frac{dy}{dx}=\frac{dv}{dx}$$

$$\Rightarrow -\frac{2}{y^3}\frac{dy}{dx}=\frac{dv}{dx}$$

$$\Rightarrow \frac{1}{y^3}\frac{dy}{dx}=-\frac{1}{2}\frac{dv}{dx}$$

So we get from ①

$$-\frac{1}{2}\frac{dv}{dx}+xv=x^3$$

$$\Rightarrow \frac{dv}{dx}-2xv=-2x^3$$

$$\therefore I.F.=e^{\int -2x\,dx}=e^{-x^2}$$

So we get,

$$\frac{d}{dx}\left[ve^{-x^2}\right]=\int -2x^3e^{-x^2}dx$$

$$\Rightarrow ve^{-x^2}=\int -2x^3e^{-x^2}dx+c$$

$$=\int x^2(-2x)e^{-x^2}dx+c$$

So we get,

$$v e^{-x^2} = \int -t e^t \, dt + c$$

Let $t = -x^2$
$\Rightarrow dt = -2x\,d$

$$= -te^t + e^t + c$$

$$= -e^{-x^2}(x^2 - 1) + c$$

$$\Rightarrow v = 1 - x^2 + ce^{x^2}$$

$$\Rightarrow \frac{1}{y^2} = 1 - x^2 + ce^{x^2}$$

Bernoulli equations:

An equation of the form

$$\frac{dy}{dx} + P(x)y = \phi(x)y^n \quad \text{— (I)}$$

is called a Bernoulli equation.

Suppose $n \neq 0$ or 1. Then the transformatio
$v = y^{1-n}$ reduces the Bernoulli equation

$$\frac{dy}{dx} + P(x)\, y = \phi(x) y^n$$

to a linear equation in v.

Proof: We first multiply (I) by y^{-n}.

$$\therefore y^{-n} \frac{dy}{dx} + P(x)\, y^{1-n} = \phi(x) \quad \text{— (II)}$$

If we let $v = y^{1-n}$, then

$$\frac{dv}{dx} = (1-n) y^{-n} \frac{dy}{dx}$$

$\therefore$ (2) $\Rightarrow$

$$\frac{1}{1-n} \frac{dv}{dx} + P(x) v = \phi(x)$$

$$\Rightarrow \frac{dv}{dx} + (1-n) P(x) v = (1-n) \phi(x)$$

Putting $P_1(x) = (1-n) P(x)$

$Q_1(x) = (1-n) \phi(x)$

we get,

$$\frac{dv}{dx} + P_1(x) v = Q_1(x)$$

which is linear in v.

1. $\frac{dy}{dx} + y = xy^3$

Solution:

Given that,

$$\frac{dy}{dx} + y = xy^3$$

This is a bernoulli defferential equation where $n = 3$. Multiplying both sides by y^{-3} we get,

$$y^{-3}\frac{dy}{dx}+y^{-2}=x$$

let $v=y^{1-n}=y^{-2}$

$$\frac{dv}{dx}=-2y^{-3}\frac{dy}{dx}$$

$\therefore$ ~~$\frac{1}{2}\frac{dv}{dx}$~~

$$\therefore -\frac{1}{2}\frac{dv}{dx}+v=x$$

Writting this linear equation in the standard form,

$$\frac{dv}{dx}-2v=-2x \quad \text{—— (2)}$$

$$\therefore \text{I.F.}=e^{\int P(x)dx}=e^{-\int 2dx}=e^{-2x}$$

Multiplying (2) by I.F.

$$e^{-2x}\frac{dv}{dx}-2e^{-2x}v=-2xe^{-2x}$$

$$\Rightarrow \frac{d}{dx}(e^{-2x}v)=-2xe^{-2x}$$

Integrating bothsides we get,

$$e^{-2x}v=\frac{1}{2}e^{-2x}(2x+1)+c$$

$$\Rightarrow v=x+\frac{1}{2}+ce^{2x}$$

$$\Rightarrow \frac{1}{y^2}=x+\frac{1}{2}+ce^{2x}$$

Ans.