1 Partial Differential Equations(P.D.E.)

Formation of partial differential equations - Lagrange's Linear equation Solution of standard types of first order partial differential equations - Linear partial differential equations of second and higher order with constant coefficients.

1.1 Introduction

In a differential equation if there are two or more independent variables and the derivatives are partial derivatives then it is called a ***partial differential equation***.

Examples:

1. $\left(\frac{\partial^2 z}{\partial x^2}\right)^2 + \left(\frac{\partial^2 z}{\partial y^2}\right)^3 = xy$ ($z-$ dependent variable; x and $y-$ independent variables)

2. $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + t\frac{\partial z}{\partial t} = xyt$ ($z-$ dep. variable; x, y and $t-$ independent variables)

The order of a partial differential equation if the order of the highest partial derivative occurring in the equation.

The degree of a partial differential equation is the greatest exponent of the highest order.

In the above, e.g.1 is a second order & third degree equation, e.g.2 is a first order equation & first degree equation.

Note : Throughout this chapter we use the following notations; z will be taken as a dependent variable which depends on two independent variables x and y so that $z = f(x, y)$.

We write

$$\frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q, \frac{\partial^2 z}{\partial x^2} = r, \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = s \text{ and } \frac{\partial^2 z}{\partial y^2} = t.$$

Thus $p^2 + q^2 = x^2 + y^2$ is a partial differential equation of order 1 and $r + xt = s^2 - p$ is a partial differential equation of order 2.

1.2 Formation of Partial Differential Equation by Elimination of Arbitrary Constants

Let $f(x, y, z, a, b) = 0.$ (1)
be an equation which contains two arbitrary constants 'a' and 'b'.

Partially differentiating (1) with respect to 'x' and 'y' we get two more equations.

Eliminating a and b from these three equations, we get $\phi(x, y, z, p, q) = 0$ which is a partial differential equation of order 1.

In this case the number of arbitrary constants to be eliminated is equal to the number of independent variables and we obtain a first order partial differential equation. If the number of arbitrary constants to be eliminated is more than the number of independent variables, we get partial differential equations of second or higher order.

1.2.1 Examples of Formation of P.D.E. by Elimination of Arbitrary Constants

Example 1.1. *Form the partial differential equation by eliminating the arbitrary constants from* $z = ax + by + a^2 + b^2$.

Solution: Given $z = ax + by + a^2 + b^2$ (1)

Differentiating (1) partially w.r.t 'x'

$$\frac{\partial z}{\partial x} = a \qquad \text{i.e., } p = a \tag{2}$$

Differentiating (1) partially w.r.t 'y'

$$\frac{\partial z}{\partial y} = b \qquad \text{i.e., } q = b \tag{3}$$

From (2) and (3)

$$a = p \text{ and } b = q$$

Substituting in (1), we get $z = px + qy + p^2 + q^2$.

Example 1.2. *Eliminating the arbitrary constants* a *and* b *from* $z = a^2x + b^2y + ab$.

Solution: Given $z = a^2x + b^2y + ab.$ (1)

Differentiating (1) partially w.r.t 'x'

$$\frac{\partial z}{\partial x} = a^2 \qquad \text{i.e., } p = a^2 \tag{2}$$

Differentiating (1) partially w.r.t 'y'

$$\frac{\partial z}{\partial y} = b^2 \qquad \text{i.e., } q = b^2 \tag{3}$$

From (2) and (3)

$$a^2 = p \text{ and } b^2 = q$$

Substituting in (1), we get $z = px + qy + \sqrt{pq}$.

Example 1.3. *Form the PDE by eliminating the arbitrary constants a and b from $z = (x + a)(y + b)$.*

Solution: Given $z = (x + a)(y + b)$ (1)

Differentiating (1) partially w.r.t 'x'

$$\frac{\partial z}{\partial x} = y + b \qquad \text{i.e., } p = y + b \tag{2}$$

Differentiating (1) partially w.r.t 'y'

$$\frac{\partial z}{\partial y} = x + a \qquad \text{i.e., } q = x + a \tag{3}$$

From (2) and (3)

$$x + a = q \text{ and } y + b = p$$

Substituting in (1), we get $z = pq$.

Example 1.4. *Form the PDE by eliminating a, b from $u = (x^2 + a^2)(y^2 + b^2)$.*

Solution: Given $u = (x^2 + a^2)(y^2 + b^2)$ (1)

Differentiating (1) partially w.r.t 'x'

$$\frac{\partial z}{\partial x} = 2x(y^2 + b^2) \qquad \text{i.e., } p = 2x(y^2 + b^2) \tag{2}$$

Differentiating (1) partially w.r.t 'y'

$$\frac{\partial z}{\partial y} = 2y(x^2 + y^2) \qquad \text{i.e., } q = 2y(x^2 + y^2) \tag{3}$$

From (2) and (3)

$$x^2 + a^2 = \frac{q}{2y} \text{ and } y^2 + b^2 = \frac{p}{2x}$$

Substituting in (1), we get $4xyu = pq$.

Example 1.5. *Find the differential equation of all sphere whose centres lie on the $z-axis$.* *[UQ]*

Solution: Any point on the $z-axis$ is of the form $(0,0,c)$. The equation of the sphere with centre $(0,0,c)$ and radius 'a' is given by

$$(x-0)^2+(y-0)^2+(z-c)^2=a^2 \tag{1}$$

Differentiating (1) partially w.r.t 'x'

$$2x+2(z-c)\frac{\partial z}{\partial x}=0 \qquad \text{i.e., } x+p(z-c)=0 \tag{2}$$

Differentiating (1) partially w.r.t 'y'

$$2y+2(z-c)\frac{\partial z}{\partial y}=0 \qquad \text{i.e., } y+q(z-c)=0 \tag{3}$$

From (2) and (3)

$$z-c=\frac{-x}{p} \text{ and } z-c=\frac{-y}{q} \Rightarrow \frac{-x}{p}=\frac{-y}{q}$$
$$\therefore qx=py.$$

Example 1.6. *Form the PDE from* $x^2+y^2+(z-c)^2=a^2$ *[UQ]*

Solution: Similar to above Example.

Example 1.7. *Find the differential equation of all spheres of fixed radius having their centres in the* $xy-plane$. *[UQ]*

Solution: Any point in the $xy-plane$ is of the form $(a,b,0)$. The equation of the sphere with center $(a,b,0)$ and radius r is given by $(x-a)^2+(y-b)^2+(z-0)^2=r^2$. i.e., $(x-a)^2+(y-b)^2+z^2=r^2$. (1)
Here a,b are the two arbitrary constants and r is a given constant.
Differentiating (1) partially w.r.t 'x'

$$2(x-a)+2z\frac{\partial z}{\partial x}=0 \qquad \text{i.e., } x-a+pz=0 \tag{2}$$

Differentiating (1) partially w.r.t 'y'

$$2(y-b)+2z\frac{\partial z}{\partial y}=0\frac{\partial z}{\partial y}=2y(x^2+y^2) \qquad \text{i.e., } y-b+qz=0 \tag{3}$$

From (2) and (3)

$$x-a=-pz \text{ and } y-b=-qz$$

Substituting in (1), we get $z^2(p^2+q^2+1)=r^2$.

Example 1.8. *Form the partial differential equation by eliminating the arbitrary constants* a *and* b *from* $(x-a)^2+(y-b)^2+z^2=1$.

Solution: Similar to above Example.

Example 1.9. *Form the partial differential equation by eliminating the arbitrary constants a and b from $log(az-1) = x + ay + b$.*

Solution: Given $log(az-1) = x + ay + b$. (1)

Differentiating (1) partially w.r.t 'x'

$$\frac{1}{az-1}.a\frac{\partial z}{\partial x} = 1 \qquad \text{i.e., } \frac{ap}{az-1} = 1 \tag{2}$$

Differentiating (1) partially w.r.t 'y'

$$\frac{1}{az-1}.a\frac{\partial z}{\partial y} = a \qquad \text{i.e., } \frac{q}{az-1} = 1 \tag{3}$$

From (2) $ap = az - 1 \Rightarrow a = \dfrac{1}{z-p}$

From (3) $q = az - 1 \Rightarrow \dfrac{z}{z-p} - 1 \left(\because a = \dfrac{1}{z-p}\right)$

$$(z-p)q = z - z + p$$
$$zq = p(1+q)$$

Example 1.10. *Form the partial differential equation by eliminating a and b from $(x-a)^2 + (y-b)^2 = z^2cot^2\alpha$, where is a constant. [UQ]*

Solution: Given $(x-a)^2 + (y-b)^2 = z^2cot^2\alpha$ (1)

Differentiating (1) partially w.r.t 'x'

$$2(x-a) = 2z\frac{\partial z}{\partial x}cot^2\alpha \qquad \text{i.e., } x - a = pzcot^2\alpha \tag{2}$$

Differentiating (1) partially w.r.t 'y'

$$2(y-b) = 2z\frac{\partial z}{\partial y}cot^2\alpha \qquad \text{i.e., } y - b = qzcot^2\alpha \tag{3}$$

Sub. (2) and (3) in (1), we get

$$p^2z^2cot^4\alpha + q^2z^2cot^4\alpha = z^2cot^2\alpha$$
$$cot^2\alpha(p^2+q^2) = 1$$
$$p^2 + q^2 = tan^2\alpha$$

Example 1.11. *Form the partial differential equation by eliminating the arbitrary constants from $2z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$*

Solution: Given $2z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$ (1)

Differentiating (1) partially w.r.t 'x'

$$2\frac{\partial z}{\partial x} = \frac{2x}{a^2} \qquad \text{i.e., } p = \frac{x}{a^2} \tag{2}$$

Differentiating (1) partially w.r.t 'y'

$$2\frac{\partial z}{\partial y} = \frac{2y}{b^2} \qquad \text{i.e., } q = \frac{x}{b^2} \tag{3}$$

From (2) and (3), we get $a^2 = \dfrac{x}{p}$ and $b^2 = \dfrac{y}{q}$

Substituting in (1), we get

$$2z = p^2 + q^2$$

Example 1.12. *Form the partial differential equation by eliminating the arbitrary constants a and b from* $z = axe^y + \frac{1}{2}a^2e^{2y} + b$.

Solution: Given $z = axe^y + \dfrac{1}{2}a^2e^{2y} + b$ (1)

Differentiating (1) partially w.r.t 'x'

$$\frac{\partial z}{\partial x} = ae^y \qquad \text{i.e., } p = ae^y \tag{2}$$

Differentiating (1) partially w.r.t 'y'

$$\frac{\partial z}{\partial y} = axe^y + \frac{1}{2}a^2e^{2y}.2 \qquad \text{i.e., } q = axe^y + a^2e^{2y} \tag{3}$$

Sub. (2) in (3), we get

$$q = px + p^2$$

Example 1.13. *Form the partial differential equation by eliminating the arbitrary constants a and b from* $z = xy + y\sqrt{x^2 + a^2} + b$

Solution: Given $z = xy + y\sqrt{x^2 + a^2} + b$ (1)

Differentiating (1) partially w.r.t 'x'

$$\frac{\partial z}{\partial x} = y + y\frac{2x}{2\sqrt{x^2 + a^2}} \qquad \text{i.e., } p = y + \frac{xy}{\sqrt{x^2 + a^2}} \tag{2}$$

Differentiating (1) partially w.r.t 'y'

$$\frac{\partial z}{\partial y} = x + \sqrt{x^2 + a^2} \qquad \text{i.e., } q = x + \sqrt{x^2 + a^2} \tag{3}$$

From (3), we get $\sqrt{x^2 + a^2} = q - x$

Substituting in (2), we get

$$p = y + \frac{xy}{q - x}$$
$$p = \frac{qy}{q - x}$$
$$\text{i.e., } pq - px = qy$$
$$\text{i.e., } px + qy = pq$$

Example 1.14. *Form the partial differential equation of all spheres whose radii are the same.*

Solution: The equation of all sphere with equal radius can be taken as $(x - a)2 + (y - b)2 + (z - c)2 = r2$ (1)
where a, b, c are arbitrary constants and r is a given constant. Since the number of arbitrary constants is more than the number of independent variables, we will get the p.d.e. of order greater than 1.

Differentiating (1) partially w.r.t 'x'

$$2(x - a) + 2(z - c)\frac{\partial z}{\partial x} = 0 \qquad \text{i.e., } i.e., (x - a) + (z - c)p = 0 \tag{2}$$

Differentiating (1) partially w.r.t 'y'

$$2(y - b) + 2(z - c)\frac{\partial z}{\partial y} = 0 \qquad \text{i.e., } (y - b) + (z - c)q = 0 \tag{3}$$

Differentiating (2) partially w.r.t 'x'

$$1 + (z - c)\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x}.\frac{\partial z}{\partial x} = 0 \qquad \text{i.e., } 1 + (z - c)r + p^2 = 0 \tag{4}$$

Differentiating (2) partially w.r.t 'y'

$$1 + (z - c)\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial y}.\frac{\partial z}{\partial y} = 0 \qquad \text{i.e., } 1 + (z - c)t + q2 = 0 \tag{5}$$

From (4) and (5), we get $z - c = \dfrac{-p^2 - 1}{r}$ and $z - c = \dfrac{-q^2 - 1}{t}$

$$\text{i.e., } z - c = \frac{-p^2 - 1}{r} = \frac{-q^2 - 1}{t}$$
$$t(p^2 + 1) = r(q^2 + 1)$$

Example 1.15. *Obtain the partial differential equation by eliminating a, b, c from $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$*

Solution: Since the number of arbitrary constants is more than the number of independent variables, we will get the p.d.e of order greater than '1'.

Given $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ (1)

Differentiating (1) partially w.r.t 'x'

$$\frac{2x}{a^2} + \frac{2z}{c^2}\frac{\partial z}{\partial x} = 0 \qquad \text{i.e., } \frac{x}{a^2} + \frac{z}{c^2}p = 0 \tag{2}$$

Differentiating (1) partially w.r.t 'y'

$$\frac{2y}{b^2} + \frac{2z}{c^2}\frac{\partial z}{\partial y} = 0 \qquad \text{i.e., } \frac{y}{b^2} + \frac{z}{c^2}q = 0 \tag{3}$$

Differentiating (2) partially w.r.t. 'x'

$$\frac{1}{a^2} + \frac{1}{c^2}\left(z\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x}.\frac{\partial z}{\partial x}\right) = 0 \qquad \text{i.e., } \frac{1}{a^2} + \frac{1}{c^2}\left(zr^2 + p^2\right) = 0 \tag{4}$$

Differentiating (3) partially w.r.t 'y'

$$\frac{1}{b^2} + \frac{1}{c^2}\left(z\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial y}.\frac{\partial z}{\partial y}\right) = 0 \qquad \text{i.e., } \frac{1}{b^2} + \frac{1}{c^2}\left(zt^2 + q^2\right) = 0 \tag{5}$$

Differentiating (2) partially w.r.t 'y'

$$0 + \frac{1}{c^2}\left(z\frac{\partial^2 z}{\partial y \partial x} + \frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\right) = 0 \quad \text{i.e., } \frac{1}{c^2}(zs + pq) = 0 \tag{5}$$

$$\text{i.e., } zs + pq = 0$$

Note : We may also get different partial differential equations. The answer is not unique.

1.3 Formation of partial differential equation by elimination of arbitrary Functions

Note: The elimination of one arbitrary function from a given relation gives a partial differential equation of first order while elimination of two arbitrary function from a given relation gives a second or higher order partial differential equation.

1.3.1 Examples of Formation of P.D.E. by Elimination of Arbitrary Functions

Example 1.16. *Eliminate f from $z = f(x^2 - y^2)$.*

Solution: Given $z = f(x^2 - y^2)$ (1)

Differentiating (1) partially w.r.t 'x'

$$\frac{\partial z}{\partial x} = f'(x^2 - y^2).2x \qquad \text{i.e., } p = 2xf' \tag{2}$$

Differentiating (1) partially w.r.t 'y'

$$\frac{\partial z}{\partial y} = f'(x^2 - y^2)(-2y) \qquad \text{i.e., } q = -2yf' \tag{3}$$

From (2) and (3), we get $f' = \dfrac{p}{2x}$ and $f' = \dfrac{-q}{2y}\dfrac{p}{2x} = \dfrac{-q}{2y}$

$$\text{i.e., } py + qx = 0$$

Example 1.17. *Eliminate f from $z = xy + f(x^2 + y^2)$.*

Solution: Given $z = xy + f(x^2 + y^2)$. (1)

Differentiating (1) partially w.r.t 'x'

$$\frac{\partial z}{\partial x} = y + f'(x^2 + y^2).2x \qquad \text{i.e., } p = y + 2xf' \tag{2}$$

Differentiating (1) partially w.r.t 'y'

$$\frac{\partial z}{\partial y} = x + f'(x^2 + y^2).2y \qquad \text{i.e., } q = x + 2yf' \tag{3}$$

From (2) and (3),we get $f' = \dfrac{p-y}{2x}$ and $f' = \dfrac{q-x}{2y}\dfrac{p-y}{2x} = \dfrac{q-x}{2y}$

$$\text{i.e., } py - qx = y^2 - x^2$$

Example 1.18. *Eliminate f from $z = x + y + f(xy)$.*

Solution: Given $z = x + y + f(xy)$ (1)

Differentiating (1) partially w.r.t 'x'

$$\frac{\partial z}{\partial x} = 1 + f'(xy).y \qquad \text{i.e., } p = 1 + yf' \tag{2}$$

Differentiating (1) partially w.r.t 'y'

$$\frac{\partial z}{\partial y} = 1 + f'(xy).x \qquad \text{i.e., } q = 1 + xf' \tag{3}$$

From (2) and (3), we get $f' = \dfrac{p-1}{y}$ and $f' = \dfrac{q-1}{x}\dfrac{p-1}{y} = \dfrac{q-1}{x}$

$$\text{i.e., } px - qy = x - y$$

Example 1.19. *Obtain PDE by eliminating the arbitrary function f from $xyz = \phi(x + y + z)$.* *[UQ]*

Solution: Given $xyz = \phi(x + y + z)$ (1)

Differentiating (1) partially w.r.t 'x'

$$y\left(x\frac{\partial z}{\partial x} + z\right) = \phi'(x + y + z)\left(1 + \frac{\partial z}{\partial x}\right) \text{ i.e., } y(xp + z) = (1 + p)\phi' \tag{2}$$

Differentiating (1) partially w.r.t 'y'

$$x\left(y\frac{\partial z}{\partial y} + z\right) = \phi'(x + y + z)\left(1 + \frac{\partial z}{\partial y}\right) \text{ i.e., } x(yq + z) = (1 + q)\phi' \tag{3}$$

From (2) and (3), we get

$$\phi' = \frac{y(xp+z)}{1+p} \text{ and } \phi' = \frac{x(yq+z)}{1+q}\frac{y(xp+z)}{1+p} = \frac{x(yq+z)}{1+q}$$
$$\text{i.e., } y(xp+z)(1+q) = x(yq+z)(1+p).$$

Example 1.20. *Eliminate the arbitrary function f from $z = f\left(\frac{y}{x}\right)$.*

Solution: Given $z = f\left(\frac{y}{x}\right)$ (1)

Differentiating (1) partially w.r.t 'x'

$$\frac{\partial z}{\partial x} = f'\left(\frac{y}{x}\right).\left(\frac{-y}{x^2}\right) \qquad \text{i.e., } p = \frac{-y}{x^2}f' \qquad (2)$$

Differentiating (1) partially w.r.t 'y'

$$\frac{\partial z}{\partial y} = f'\left(\frac{y}{x}\right).\frac{1}{x} \qquad \text{i.e., } q = \frac{1}{x}f' \qquad (3)$$

From (2) and (3),we get $f' = \dfrac{-px^2}{y}$ and $f' = qx\dfrac{-px^2}{y} = qx$

$$\text{i.e., } px + qy = 0$$

Example 1.21. *Form a partial differential equation by eliminating f from $z = xf\left(\frac{x}{y}\right)$.* *[UQ]*

Solution: Given $z = xf\left(\frac{x}{y}\right)$ (1)

Differentiating (1) partially w.r.t 'x'

$$\frac{\partial z}{\partial x} = xf'\left(\frac{x}{y}\right).\frac{1}{y} + f\left(\frac{x}{y}\right) \qquad \text{i.e., } p = \frac{x}{y}f' + f \qquad (2)$$

Differentiating (1) partially w.r.t 'y'

$$\frac{\partial z}{\partial y} = xf'\left(\frac{x}{y}\right).\left(\frac{-x}{y^2}\right) \qquad \text{i.e., } q = \frac{-x^2}{y^2}f' \qquad (3)$$

From (1), $f = \dfrac{z}{x}$ From (3), $f' = \dfrac{-qy^2}{x^2}$

Substituting these in (2), we get $p = \dfrac{x}{y}\left(\dfrac{-qy^2}{x^2}\right) + \dfrac{z}{x} \Rightarrow p = -\dfrac{qy}{x} + \dfrac{z}{x}$

$$\text{i.e., } px + qy = z$$

Example 1.22. *Form PDE by eliminating arbitrary function f from $z = f\left(\frac{xy}{z}\right)$.* *[UQ]*

Solution: Given $z = f\left(\frac{xy}{z}\right)$ (1)

Differentiating (1) partially w.r.t 'x'

$$\frac{\partial z}{\partial x} = f'\left(\frac{xy}{z}\right)\left(\frac{zy - xy\frac{\partial z}{\partial x}}{z^2}\right) \qquad \text{i.e., } p = \left(\frac{zy - xyp}{z^2}\right)f' \qquad (2)$$

Differentiating (1) partially w.r.t 'y'

$$\frac{\partial z}{\partial y} = f'\left(\frac{xy}{z}\right)\left(\frac{zx - xy\frac{\partial z}{\partial y}}{z^2}\right) \qquad \text{i.e., } q = \left(\frac{zx - xyq}{z^2}\right)f' \qquad (3)$$

From (2) and (3), we get

$$f' = \frac{pz^2}{yz - xyp} \text{ and } f' = \frac{qz^2}{xz - xyp}\frac{pz^2}{yz - xyp} = \frac{qz^2}{xz - xyp}$$
$$pxz - pqxy = qyz - pqxy$$
$$\text{i.e., } px = qy.$$

Example 1.23. *Form a partial differential equation by eliminating f from $z = x^2 + 2f\left(\frac{1}{y} + logx\right)$.*

Solution: Given $z = x^2 + 2f\left(\frac{1}{y} + logx\right)$ (1)

Differentiating (1) partially w.r.t 'x'

$$\frac{\partial z}{\partial x} = 2x + 2f'\left(\frac{1}{y} + \log x\right).\frac{1}{x} \qquad \text{i.e., } p = 2x + \frac{2}{x}f' \qquad (2)$$

Differentiating (1) partially w.r.t 'y'

$$\frac{\partial z}{\partial y} = 2f'\left(\frac{1}{y} + \log x\right)\left(\frac{-1}{y^2}\right) \qquad \text{i.e., } q = \frac{-2}{y^2}f' \qquad (3)$$

From (2) and (3), we get

$$f' = \frac{x}{2}(p - 2x) \text{ and } f'\frac{-qy^2}{2}\frac{x}{2}(p - 2x) = \frac{-qy^2}{2}$$
$$\text{i.e., } px - 2x^2 = -qy^2.$$
$$\text{i.e., } px + qy^2 = 2x^2.$$

Example 1.24. *Form PDE by eliminating arbitrary function f and g from $z = f(x + ay) + g(x - ay)$.*

Solution: Given $z = f(x + ay) + g(x - ay)$ (1)

Differentiating (1) partially w.r.t 'x'

$$\frac{\partial z}{\partial x} = f'(x + ay) + g'(x - ay) \qquad \text{i.e., } p = f' + g' \qquad (2)$$

Differentiating (1) partially w.r.t 'y'

$$\frac{\partial z}{\partial y} = af'(x+ay) - ag'(x-ay) \qquad \text{i.e., } q = af' - ag' \qquad (3)$$

Differentiating (2) partially w.r.t 'x'

$$\frac{\partial^2 z}{\partial x^2} = f'' + g'' \qquad \text{i.e., } r = f'' + g'' \qquad (4)$$

Differentiating (3) partially w.r.t 'y'

$$\frac{\partial^2 z}{\partial y^2} = a^2 f'' + a^2 g'' \qquad \text{i.e., } t = a^2 \left(f'' + g''\right) \qquad (5)$$

Using (4), we get $t = a^2 r$.

Example 1.25. *Form partial differential equation by eliminating arbitrary function f and g from $z = f(2x+y) + g(3x-y)$.* *[UQ]*

Solution: Given $z = f(2x+y) + g(3x-y)$ (1)

Differentiating (1) partially w.r.t 'x'

$$\frac{\partial z}{\partial x} = 2f'\,(2x+y) + 3g'\,(3x-y) \qquad \text{i.e., } p = 2f' + 3g' \qquad (2)$$

Differentiating (1) partially w.r.t 'y'

$$\frac{\partial z}{\partial y} = f'(2x+y) - g'(3x-y) \qquad \text{i.e., } q = f' - g' \qquad (3)$$

Differentiating (2) partially w.r.t 'x'

$$\frac{\partial^2 z}{\partial x^2} = 4f''(2x+y) + 9g''(3x-y) \qquad \text{i.e., } r = 4f'' + 9g'' \qquad (4)$$

Differentiating (3) partially w.r.t 'y'

$$\frac{\partial^2 z}{\partial y^2} = f''(2x+y) + g''(3x-y) \qquad \text{i.e., } t = f'' + g'' \qquad (5)$$

Differentiating (2) partially w.r.t 'y'

$$\frac{\partial^2 z}{\partial x \partial y} = 2f''(2x+y) - 3g''(3x-y) \qquad \text{i.e., } s = 2f'' - 3g'' \qquad (6)$$

Solving (4) and (5), we get $f'' = \frac{1}{5}(qt - r)$ and $g'' = \frac{1}{5}(r - 4t)$

Sub. the values of f'' and g'' in (6), we get

$$\begin{aligned} s &= \frac{2}{5}(9t - r) - \frac{3}{5}(r - 4t) \\ 5s &= 30t - 5r \\ s &= 6t - r \end{aligned}$$

Example 1.26. *Form partial differential equation by eliminating arbitrary function f and g from $z = f(x+2y) + g(2x-y)$.*

Solution: Given $z = f(x+2y) + g(2x-y)$ (1)

Differentiating (1) partially w.r.t 'x'

$$p = f' + 2g' \tag{2}$$

Differentiating (1) partially w.r.t 'y'

$$q = 2f' - g' \tag{3}$$

Differentiating (2) partially w.r.t 'x' : $r = f'' + 4g''$ (4)

Differentiating (3) partially w.r.t 'y' : $t = 4f'' + g''$ (5)

Differentiating (2) partially w.r.t 'y' : $s = 2f'' - 2g''$ (6)

Solving (4) and (5), we get $f'' = \dfrac{4t-r}{15}, g'' = \dfrac{4r-t}{15}$

Sub. the values of f'' and g'' in (6), we get

$$\begin{aligned} s &= \frac{2}{15}(4t-r) - \frac{2}{15}(4r-t) \\ 15s &= 10t - 10r \\ 3s + 2r &= 2t \end{aligned}$$

Example 1.27. *Eliminate the arbitrary function ϕ and from $z = x\phi(y) + y(x)$ and form the partial differential equation.*

Solution: Given $z = x\phi(y) + y(x)$ (1)

Differentiating (1) partially w.r.t 'x'

$$p = \phi(y) + y\varphi'(x) \tag{2}$$

Differentiating (1) partially w.r.t 'y'

$$q = x\phi'(y) + \varphi(x) \tag{3}$$

Differentiating (2) partially w.r.t 'y'

$$s = \phi'(y) + \varphi'(x) \tag{4}$$

$$\begin{aligned} \text{Now, } px + qy &= x\phi(y) + xy\varphi'(x) + xy\phi'(y) + y\varphi(x) \\ &= x\phi(y) + y\varphi(x) + xy(\phi'(y) + \varphi'(x)) \\ px + qy &= z + xys \qquad \text{[using (1) and (4)]} \end{aligned}$$

Example 1.28. *Form partial differential equation by eliminating arbitrary function f and g from $z = xf(3x+2y) + g(3x+2y)$.*

Solution: Given $z = xf(3x+2y) + g(3x+2y)$ (1)

Differentiating (1) partially w.r.t 'x'

$$p = 3xf' + f + 3g' \tag{2}$$

Differentiating (1) partially w.r.t 'y'

$$q = 2xf' + 2g' \tag{3}$$

Differentiating (2) partially w.r.t 'x'

$$r = 3(x.3f'' + f') + 3f' + 9g'' r = 9xf'' + 9g'' + 6f' \tag{4}$$

Differentiating (3) partially w.r.t 'y' : $\quad t = 4xf'' + 4g''$ (5)

Differentiating (2) partially w.r.t 'y' : $\quad s = 6xf'' + 2f' + 6g''$ (6)

$$\begin{aligned}(4) - 3 \times (6) \Rightarrow r - 3s &= 9xf'' + 9g'' + 6f' - 18xf'' - 6f' - 18g'' \\ &= -9xf'' - 9g'' \\ r - 3s &= -9(xf'' + g'') = -9\left(\frac{t}{4}\right) \qquad [using (5)]\end{aligned}$$

$$4r - 12s + 9t = 0$$

Example 1.29. *Form partial differential equation by eliminating arbitrary function f and g from $z = xf\left(\frac{y}{x}\right) + y\phi(x)$.* *[UQ]*

Solution: Given $z = xf\left(\frac{y}{x}\right) + y\phi(x)$ (1)

Differentiating (1) partially w.r.t 'x'

$$\begin{aligned}\frac{\partial z}{\partial x} &= xf'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) + f\left(\frac{y}{x}\right) + y\phi'(x) \\ \frac{\partial z}{\partial x} &= \frac{-y}{x}f'\left(\frac{y}{x}\right) + f\left(\frac{y}{x}\right) + y\phi'(x) \\ \text{i.e., } p &= \frac{-y}{x}f' + f + y\phi' \end{aligned} \tag{2}$$

Differentiating (1) partially w.r.t 'y'

$$\frac{\partial z}{\partial y} = xf'\left(\frac{y}{x}\right).\frac{1}{x} + \phi(x) \Rightarrow \frac{\partial z}{\partial y} = f'\left(\frac{y}{x}\right) + \phi(x)$$

$$\text{i.e., } q = f' + \phi \tag{3}$$

$$\text{Now, } \frac{\partial^2 z}{\partial x \partial y} = f''\left(\frac{y}{x}\right).\left(\frac{-y}{x^2}\right) + \phi'(x) \qquad \text{i.e., } s = \frac{-y}{x^2}f'' + \phi' \tag{4}$$

$$\frac{\partial^2 z}{\partial y^2} = f''\left(\frac{y}{x}\right).\frac{1}{x} \qquad \text{i.e., } t = \frac{1}{x}f'' \tag{5}$$

$$\therefore px + qy = -yf' + xf + xy\phi' + yf' + y\phi \qquad \text{[using (2) and (3)]}$$
$$px + qy = xf + y\phi + xy\phi'$$

From (1), $\qquad px + qy = z + xy\phi' \qquad (6)$

Using (5) in (4), $s = \dfrac{-y}{x^2}(xt) + \phi' \Rightarrow \phi' = s + \dfrac{yt}{x}$

Sub. the value of ϕ' in (6), we get

$$px + qy = z + xy\left(s + \frac{yt}{x}\right)$$
$$px + qy = z + xys + y^2t$$

Example 1.30. *Form a p.d.e by eliminating arbitrary function f and g from $z = f(x+y)g(x-y)$.*

Solution: Given $z = f(x+y)g(x-y)$ i.e., $z = fg \qquad (1)$

Differentiating (1) partially w.r.t 'x'

$$\frac{\partial z}{\partial x} = fg' + gf' \qquad \text{i.e., } p = fg' + gf' \qquad (2)$$

Differentiating (1) partially w.r.t 'y'

$$\frac{\partial z}{\partial y} = f(-g') + gf' \qquad \text{i.e., } q = gf' - fg' \qquad (3)$$

$$\frac{\partial^2 z}{\partial x^2} = fg'' + g'f' + gf'' + f'g' \qquad \text{i.e., } r = fg'' + gf'' + 2f'g' \qquad (4)$$

$$\frac{\partial^2 z}{\partial y^2} = -[f(-g'') + g'f'] + gf'' + f'(-g')$$
$$t = fg'' + gf'' - 2f'g' \qquad (5)$$

From (2) and (3), we get

$$p^2 - q^2 = f^2g'^2 + g^2f'^2 + 2fgf'g' - (g^2f'^2 + f^2g'^2 - 2fgf'g')$$
$$= 4fgf'g'$$
$$p^2 - q^2 = 4zf'g' \qquad [using(1)] \qquad (6)$$

From (4) and (5), we get

$$r - t = 4f'g' \qquad (7)$$

From (6) and (7), we get

$$p^2 - q^2 = z(r - t)$$

1.3.2 Formation of partial differential equation by elimination of arbitrary function ϕ from ϕ(u, v) $= 0$, where u and v are function of x, y and z.

Let $(u, v) = 0$ (1)

Differentiating (1) partially w.r.t x and y, we get

$$\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x} = 0 \tag{2}$$

$$\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial y} = 0 \tag{3}$$

To eliminate ϕ, it is enough to eliminate $\dfrac{\partial \phi}{\partial u}$ and $\dfrac{\partial \phi}{\partial v}$

From (2) and (3), eliminating $\dfrac{\partial \phi}{\partial u}$ and $\dfrac{\partial \phi}{\partial v}$

From (2) and (3), we get

$$\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{vmatrix} = 0$$

Example 1.31. *Form the partial differential equation by eliminating the arbitrary function ϕ from the relation $\phi(x^2 + y^2 + z^2, xyz) = 0$*

Solution: Given $\phi(x^2 + y^2 + z^2, xyz) = 0$ (1)

Let $u = x^2 + y^2 + z^2, v = xyz$

Equation (1) becomes $\phi(u, v) = 0$ (2)

Eliminating ϕ from (2), we get

$$\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{vmatrix} = 0 \tag{3}$$

Here, $\dfrac{\partial u}{\partial x} = 2x + 2zp, \dfrac{\partial u}{\partial y} = 2y + 2zq$

$\dfrac{\partial v}{\partial x} = xyp + yz, \dfrac{\partial v}{\partial y} = xyq + xz$

From (3), $\begin{vmatrix} 2x + 2zp & xyp + yz \\ 2y + 2zq & xyq + xz \end{vmatrix} = 0$

i.e., $(2x + 2zp)(xyq + xz) - (xyp + yz)(2y + 2zq) = 0$

i.e., $pxz^2 - y^2 - qy(z^2 - y^2) = z(y^2 - x^2)$

Example 1.32. *Form the partial differential equation by eliminating the arbitrary function from $f(x + y + z, xy + z^2) = 0$.* *[UQ]*

Solution: Given $f(x+y+z, xy+z^2) = 0$ (1)

Let $u = x+y+z, v = xy+z^2$

Equation (1) becomes $\phi(u,v) = 0$ (2) Eliminating ϕ from (2), we get

$$\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{vmatrix} = 0 \quad (3)$$

Here, $\dfrac{\partial u}{\partial x} = 1+p, \dfrac{\partial u}{\partial y} = 1+q$

$\dfrac{\partial v}{\partial x} = y+2zp, \dfrac{\partial v}{\partial y} = x+2zq$

From(3),

$$\begin{vmatrix} 1+p & y+2zp \\ 1+q & y+2zq \end{vmatrix} = 0$$

$$\text{i.e., } (1+p)(x+2zq) - (1+q)(y+2zp) = 0$$

$$\text{i.e., } (x-2z)p - (y-2z)q = y-x$$

Example 1.33. *Form the partial differential equation by eliminating the arbitrary function g from the relation* $g\left(\dfrac{y}{x}, x^2+y^2+z^2\right) = 0$

Solution: Given $g\left(\dfrac{y}{x}, x^2+y^2+z^2\right) = 0$ (1)

Let $u = \dfrac{y}{x}, v = x^2+y^2+z^2$

Equation (1) becomes $g(u,v) = 0$ (2)

Eliminating ϕ from (2), we get

$$\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{vmatrix} = 0 \quad (3)$$

$$\frac{\partial u}{\partial x} = \frac{-y}{x^2}, \frac{\partial u}{\partial y} = \frac{1}{x}$$

$$\frac{\partial v}{\partial x} = 2x+2zp \frac{\partial v}{\partial y} = 2y+2zq$$

From (3), $\begin{vmatrix} \dfrac{-y}{x^2} & 2x+2zp \\ \dfrac{1}{x} & \text{2y + 2zq} \end{vmatrix} = 0$

i.e., $\dfrac{-y}{x^2}(2y+2zq) - \dfrac{1}{x}(2x+2zp) = 0$

$$\text{i.e., } xzp + yzq + x^2 + y^2 = 0$$

1.3.3 Exercises & Solutions

Find the partial differential equations by eliminating the arbitrary constants $a, b\&c$ as the case may be:

1. $z = ax + by$ [Ans : $z = px + qy$]
2. $z = a\sin by$ [Ans: $r + t = 0$]
3. $z = axy + b$ [Ans: $px - qy = 0$]
4. $z = ax + a^2y^2 + b$ [Ans: $q = 2yp^2$]
5. $z = ax^4 + by^4$ [Ans: $px + qy = 4z$]
6. $ax + by + cz = 1$ [Ans: $r = 0$ (or) $s = 0$ (or) $t = 0$]
7. $z = ax + (1 - a)y + b$ [Ans: $p + q = 1$]
8. $z = (x - a)^2 + (y - b)^2 + 1$ [Ans: $z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2 + 1$]

Find the partial differential equations by eliminating the arbitrary constants from the following:

1. $z = f(x^2 - y^2)$ [Ans: $py + qx = 0$]
2. $z = f(x + iy) + g(x - iy)$ [Ans: $r + t = 0$]
3. $xy + yz + zx = f(\frac{z}{x+y})$
   [Ans: $p(x + y)(x + 2z) - q(x + y)(y + 2z) = z(x - y)$]
4. $z = xf(x + t) + g(x + t)$ [Ans: $r = 2s - t$]
5. $z = x^2f(y) + y^2g(x)$ [Ans: $xyr = 2(px + qy - 2z)$]
6. $z = f(x - y)$ [Ans: $p + q = 0$]
7. $z = y^2 + 2f(+logy)$ [Ans: $px^2 + qy = 2y^2$]
8. $z = x + yf(x^2 - y^2)$ [Ans: $z = xq + yp$]
9. $f(xy + z^2, x + y + z) = 0$ [Ans: $(2z - x)p - (2z - y)q = x - y$]
10. $\phi = (z^2 - xy, \frac{x}{z}) = 0$ [Ans: $px^2 - q(xy - 2z^2) = zx$]
11. $z = f(ax + by) + g(\alpha x + \beta y)$ [Ans: $b\beta r - (a\beta + b\alpha)s + a\alpha t = 0$]

1.4 Lagrange's Linear Equations

The equation of the form $Pp+Qq = R$ is known as Lagrange's equation, where P, Q and R are functions of x, y and z.

To solve this equation it is enough to solve the subsidiary equations [(or) Lagrange's auxiliary equations]

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{Z} \tag{1}$$

If $u = a$ and $v = b$ are two solutions of (1) then the solution of the given Lagrange's equation is $\phi(u, v) = 0$. Generally, the subsidiary equations can be solved in two ways
1. Method of grouping 2. Method of multipliers

1.4.1 Method of grouping

In the subsidiary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{Z}$$

if the variables can be separated in any pair of equations, then we get a solution of the form $u = a$ and $v = b$.

1.4.2 Method of Multipliers

1. Single Multiplier (l, m, n):

Choose any three multipliers l, m, n may be constants or functions of x, y, z we have

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{ldx + mdy + ndz}{lP + mQ + nR}$$

If it is possible to choose l, m, n such that $lP + mQ + nR = 0$ then

$$ldx + mdy + ndz = 0.$$

If $ldx + mdy + ndz$ is a perfect differential of some function, say $u(x, y, z)$, then $du = 0$

$\therefore u = a$ is a solution.

* **Note :** If $lP + mQ + nR \neq 0$ for any (l, m, n), then go for Double Multipliers as follows

2. Double Multipliers $(l_1, m_1, n_1), (l_2, m_2, n_2)$:

Choose Double Multipliers $(l_1, m_1, n_1), (l_2, m_2, n_2)$ may be constants or functions of x, y, z we have

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l_1dx + m_1dy + n_1dz}{l_1P + m_1Q + n_1R} = \frac{l_2dx + m_2dy + n_2dz}{l_2P + m_2Q + n_2R}$$

Integrating both sides of the equation,

$$\frac{l_1dx + m_1dy + n_1dz}{l_1P + m_1Q + n_1R} = \frac{l_2dx + m_2dy + n_2dz}{l_2P + m_2Q + n_2R}$$

we get $u = 0$

$\therefore u = a$ is a solution.

1.4.3 Working Rule to solve the equation Pp + Qq = R

(i) Form the subsidiary equations $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

(ii) Solve the subsidiary equations by the method of grouping or the method of multipliers or both to get two independent solutions $u = a$ and $v = b$.

(iii) The general solution is $\phi(u, v) = 0$

1.5 Examples of Lagrange's Linear Equations

1.5.1 Examples of Lagrange's Linear Equations by Method of grouping

Example 1.34. *Solve* $px + qy = z$.

Solution: Given $px + qy = z$
This equation is of the form $Pp + Qq = R$, then $P = x, Q = y$ and $R = z$.
The subsidiary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Taking $\frac{dx}{x} = \frac{dy}{y}$
Integrating, we get $logx = logy + loga$

$$\text{i.e., } \frac{x}{y} = a$$

Similarly from $\frac{dy}{y} = \frac{dz}{z}$, we get

$$\frac{y}{z} = b$$

Hence the general solution is $\phi(\frac{x}{y}, \frac{y}{z}) = 0$

Example 1.35. *Find the general solution of $px + qy = 0$.* *[UQ]*

Solution: Given $px + qy = 0$
This equation is of the form $Pp + Qq = R$, then $P = x, Q = y$ and $R = 0$.
The subsidiary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{0}$$

Taking $\frac{dx}{x} = \frac{dy}{y}$
Integrating, we get $logx = logy + loga$

$$\text{i.e., } \frac{x}{y} = a$$

Taking $\frac{dy}{y} = \frac{dz}{0}$, we get

$$dz = 0$$

Integrating, we get $z = b$
Hence the general solution is $\phi(\frac{x}{y}, z) = 0$.

Example 1.36. *Solve $\frac{y^2 z}{x}p + xzq = y^2$.* *[UQ]*

Solution: Given $\frac{y^2z}{x}p + xzq = y^2$

i.e., $y^2zp + x^2zq = xy^2$

The subsidiary equations are

$$\frac{dx}{y^2z} = \frac{dy}{x^2z} = \frac{dz}{xy^2}$$

Taking $\frac{dx}{y^2z} = \frac{dy}{x^2z}$, we get

$$\frac{dx}{y^2} = \frac{dy}{x^2}$$

i.e., $x^2dx = y^2dy$

Integrating, we get $\frac{x^3}{3} = \frac{y^3}{3} + c_1$

i.e., $x^3 - y^3 = a$

Taking $\frac{dx}{y^2z} = \frac{dz}{xy^2}$, we get

$$\frac{dx}{z} = \frac{dz}{x}$$

i.e., $xdx = zdz$

Integrating, we get $\frac{x^2}{2} = \frac{z^2}{2} + c_2$

i.e., $x^2 - z^2 = b$

$\therefore$ The general solution is $\phi(x^3 - y^3, x^2 - z^2) = 0$.

Example 1.37. *Solve $yzp + xzq = xy$.* *[UQ]*

Solution: Given $yzp + xzq = xy$

The subsidiary equations are

$$\frac{dx}{yz} = \frac{dy}{xz} = \frac{dz}{xy}$$

Taking $\frac{dx}{yz} = \frac{dy}{xz}$, we get

$$\frac{dx}{y} = \frac{dy}{x}$$

i.e., $xdx = ydy$

Integrating, we get $\frac{x^2}{2} = \frac{y^2}{2} + c_1$

i.e., $x^2 - y^2 = a$

Taking $\frac{dx}{yz} = \frac{dz}{xy}$, we get

$$xdx = zdz$$

Integrating, we get $\frac{x^2}{2} = \frac{z^2}{2} + c_2$

i.e., $x^2 - z^2 = b$

$\therefore$ The general solution is $\phi(x^2 - y^2, x^2 - z^2) = 0$.

Example 1.38. *Solve $x^2p + y^2q = z^2$.*

Solution: Given $x^2p + y^2q = z^2$

The subsidiary equations are

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z^2}$$

Taking $\frac{dx}{x^2} = \frac{dy}{y^2}$

Integrating, we get $-\frac{1}{x} = -\frac{1}{y} + c_1$

$$\text{i.e., } \frac{1}{x} - \frac{1}{y} = a$$

Taking $\frac{dx}{x^2} = \frac{dz}{z^2}$

Integrating, we get

$$-\frac{1}{x} = -\frac{1}{z} + c_2$$

$$\text{i.e., } \frac{1}{x} - \frac{1}{z} = b$$

$\therefore$ The general solution is $\phi(\frac{1}{x} - \frac{1}{y}, \frac{1}{x} - \frac{1}{z}) = 0$.

Example 1.39. *Solve $pcotx + qcoty = cotz$.*

Solution: Given $pcotx + qcoty = cotz$

The subsidiary equations are

$$\frac{dx}{\cot x} = \frac{dy}{\cot y} = \frac{dz}{\cot z}$$

Taking $\frac{dx}{\cot x} = \frac{dy}{\cot y}$, we have

$\tan x dx = \tan y dy$

Integrating, $logsecx = logsecy + loga$

$$\text{i.e., } \frac{\sec x}{\sec y} = a$$

Similarly taking $\frac{\sec y}{\sec z} = b$

$\therefore$ The general solution is $\phi\left(\frac{\sec x}{\sec y}, \frac{\sec y}{\sec z}\right) = 0$.

1.5.2 Examples of Lagrange's Linear Equations by Method of multipliers

Example 1.40. *Solve* $(y-z)p+(z-x)q=x-y$.

Solution: Given $(y-z)p+(z-x)q=x-y$

The subsidiary equations are $\dfrac{dx}{y-z}=\dfrac{dy}{z-x}=\dfrac{dz}{x-y}$ (1)

Each fraction of (1) $=\dfrac{dx+dy+dz}{y-z+z-x+x-y}=\dfrac{dx+dy+dz}{0}$

$$\Rightarrow dx+dy+dz=0$$

Integrating, $x+y+z=a$

Using x,y,z as multipliers,

each fraction of (1) $=\dfrac{xdx+ydy+zdz}{x\,(y-z)+y\,(z-x)+z\,(x-y)}=\dfrac{xdx+ydy+zdz}{0}$

$$\Rightarrow xdx+ydy+zdz=0$$

$$\text{Integrating, } \frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=c$$

$$\text{i.e., } x^2+y^2+z^2=b$$

$\therefore$ The general solution is $f\left(x+y+z,x^2+y^2+z^2\right)=0$.

Example 1.41. *Solve* $(mz-ny)\dfrac{\partial z}{\partial x}+(nx-lz)\dfrac{\partial z}{\partial y}=ly-mx$. *[UQ]*

Solution: Given $(mz-ny)\,p+(nx-lz)\,q=ly-mx$

The subsidiary equations are $\dfrac{dx}{mz-ny}=\dfrac{dy}{nx-lz}=\dfrac{dz}{ly-mx}$ (1)

Using x,y,z as multipliers, each fraction of (1)=

$$\frac{xdx+ydy+zdz}{x\,(mz-ny)+y\,(nx-lz)+z\,(ly-mx)}=\frac{xdx+ydy+zdz}{0}$$

$$\Rightarrow xdx+ydy+zdz=0$$

Integrating, we get $\dfrac{x^2}{2}+\dfrac{y^2}{2}+\dfrac{z^2}{2}=c_1$

i.e., $x^2+y^2+z^2=a$

Using l,m,n as multipliers, each fraction of (1) =

$$\frac{ldx+mdy+ndz}{l\,(mz-ny)+m\,(nx-lz)+n\,(ly-mx)}=\frac{ldx+mdy+ndz}{0}$$

$$ldx+mdy+ndz=0$$

Integrating, $lx+my+nz=b$

$\therefore$ The general solution is $\phi(x^2+y^2+z^2,lx+my+nz)=0$.

Example 1.42. *Solve* $x\left(y^2-z^2\right)p+y\left(z^2-x^2\right)q=z\left(x^2-y^2\right)$. *[UQ]*

Solution: Given $x\left(y^2-z^2\right)p+y\left(z^2-x^2\right)q=z\left(x^2-y^2\right)$

The subsidiary equations are

$$\frac{dx}{x\left(y^2-z^2\right)}=\frac{dy}{y\left(z^2-x^2\right)}=\frac{dz}{z\left(x^2-y^2\right)}$$

Using x, y, z as multipliers, each fraction of (1)=

$$\frac{xdx+ydy+zdz}{x^2\left(y^2-z^2\right)+y^2\left(z^2-x^2\right)+z^2\left(x^2-y^2\right)}=\frac{xdx+ydy+zdz}{0}$$

$$\Rightarrow xdx+ydy+zdz=0$$

Integrating, we get $\dfrac{x^2}{2}+\dfrac{y^2}{2}+\dfrac{z^2}{2}=c_1$

i.e., $x^2+y^2+z^2=a$

Using $\dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{z}$ as multipliers, each fraction of (1) =

$$\frac{\frac{l}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{\left(y^2-z^2\right)+\left(z^2-x^2\right)+\left(x^2-y^2\right)}=\frac{\frac{l}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}$$

$$\frac{l}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz=0$$

Integrating, $\log x+\log y+\log z=\log b$

i.e., $xyz=b$

$\therefore$ The general solution is $\phi(x^2+y^2+z^2, xyz)=0$.

Example 1.43. *Solve* $(y+z)p+(x+z)q=x+y$. *[UQ]*

Solution: Given $(y+z)p+(x+z)q=x+y$

The subsidiary equations are $\dfrac{dx}{y+z}=\dfrac{dy}{x+z}=\dfrac{dz}{x+y}$ (1)

Each fraction of (1) $=\dfrac{dx+dy+dz}{2\left(x+y+z\right)}=\dfrac{dx-dy}{-\left(x-y\right)}=\dfrac{dy-dz}{-\left(y-z\right)}$

$$\frac{d(x-y)}{(x-y)}=\frac{d(y-z)}{(y-z)}$$

Integrating, we get $\log(x-y)=\log(y-z)+\log a$

i.e., $\dfrac{x-y}{y-z}=a$

Taking $\dfrac{dx+dy+dz}{2\left(x+y+z\right)}=\dfrac{dx-dy}{-\left(x-y\right)}$, we have

$$\frac{1}{2}\frac{d(x+y+z)}{(x+y+z)}=-\frac{d(x-y)}{(x-y)}$$

Integrating, we get $\frac{1}{2}\log(x+y+z) = -\log(x-y) + \log b$

$$\sqrt{(x+y+z)}(x-y) = b$$

The general solution is $\phi\left(\frac{x-y}{y-z}, \sqrt{(x+y+z)}(x-y)\right) = 0.$

Example 1.44. *Solve* $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy.$ *[UQ]*

Solution: The subsidiary equations are

$$\frac{dx}{(x^2 - yz)} = \frac{dy}{(y^2 - zx)} = \frac{dz}{z^2 - xy} \tag{1}$$

Each fraction of (1) =

$$\frac{dx + dy + dz}{(x^2 - yz) + (y^2 - zx) + (z^2 - xy)} = \frac{dx + dy + dz}{x^2 + y^2 + z^2 - xy - yz - zx}$$

Using x, y, z as multipliers, each fraction of (1) =

$$\frac{xdx + ydy + zdz}{x^3 + y^3 + z^3 - 3xyz} = \frac{dx + dy + dz}{x^2 + y^2 + z^2 - xy - yz - zx}$$

$$\frac{xdx + ydy + zdz}{(x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)} = \frac{dx + dy + dz}{x^2 + y^2 + z^2 - xy - yz - zx}$$

$$\frac{xdx + ydy + zdz}{x + y + z} = \frac{dx + dy + dz}{1}$$

$$xdx + ydy + zdz = (x + y + z)d(x + y + z)$$

Integrating, $\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{(x + y + z)^2}{2} + c$

$$xy + yz + zx = a$$

Each fraction of (1) $= \frac{dx - dy}{(x^2 - yz) - (y^2 - zx)} = \frac{dy - dz}{(y^2 - zx) - (z^2 - xy)}$

i.e., $\frac{dx - dy}{(x^2 - y^2) + z(x - y)} = \frac{dy - dz}{(y^2 - z^2) + x(y - z)}$

$$\frac{d(x - y)}{(x - y)(x + y + z)} = \frac{d(y - z)}{(y - z)(x + y + z)}$$

$$\frac{d(x - y)}{(x - y)} = \frac{d(y - z)}{(y - z)}$$

Integrating, $\log(x - y) = \log(y - z) + \log b$

$$\frac{x - y}{y - z} = b$$

$\therefore$ The general solution is $\phi\left(xy + yz + zx, \frac{x - y}{y - z}\right) = 0.$

Example 1.45. *Solve* $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$.

Solution: The subsidiary equations are

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)} \tag{1}$$

Using $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$ as multipliers

$$\text{Each fraction of (1)} = \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{(y-z)+(z-x)+(x-y)} = \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{0}$$

$$\text{i.e., } \frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz = 0$$

Integrating, $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = a$

Using $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers

Each fraction of (1) =

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{x(y-z)+y(z-x)+z(x-y)} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\text{i.e., } \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

Integrating, $\log x + \log y + \log z = \log b$

$$xyz = b$$

$\therefore$ The general solution is $\phi\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}, xyz\right) = 0$.

Example 1.46. *Solve* $y^2p - xyq = x(z-2y)$. *[UQ]*

Solution: The subsidiary equations are

$$p - xyq = x(z-2y) \tag{1}$$

Taking $\frac{dx}{y^2} = \frac{dy}{-xy}$, we have

$$\frac{dx}{y} = \frac{dy}{-x}$$

$$xdx = -ydy$$

Integrating, $\frac{x^2}{2} = -\frac{y^2}{2} + c$

i.e., $x^2 + y^2 = a$

Taking $\frac{dy}{-xy} = \frac{dz}{x(z-2y)}$ we have

$$\frac{dy}{-y} = \frac{dz}{(z-2y)}$$
$$\text{i.e., } (z-2y)dy = -ydz$$
$$\text{i.e., } ydz + zdy - 2ydy = 0$$
$$\text{i.e., } d(yz) - 2ydy = 0$$

Integrating, $yz - y^2 = b$

$\therefore$ The general solution is $\phi\left(x^2+y^2, yz-y^2\right) = 0$.

Example 1.47. *Solve* $(x^2-y^2-z^2)p + 2xyq - 2xz = 0$. *[UQ]*

Solution: The subsidiary equations are $\dfrac{dx}{x^2-y^2-z^2} = \dfrac{dy}{2xy} = \dfrac{dz}{2xz}$ (1)

Taking $\dfrac{dy}{2xy} = \dfrac{dz}{2xz}$, we have

$$\frac{dy}{y} = \frac{dz}{z}$$
$$\log y = \log z + \log a$$
$$\text{i.e., } \frac{y}{z} = a$$

Using x, y, z as multipliers

Each fraction of (1) =

$$\frac{xdx+ydy+zdz}{x\left(x^2-y^2-z^2\right)+2xy^2+2xz^2} = \frac{xdx+ydy+zdz}{x^3+xy^2+xz^2} = \frac{xdx+ydy+zdz}{x\left(x^2+y^2+z^2\right)}$$

$$\text{Taking } \frac{dy}{2xy} = \frac{xdx+ydy+zdz}{x\left(x^2+y^2+z^2\right)}$$
$$\frac{dy}{y} = \frac{2\left(xdx+ydy+zdz\right)}{\left(x^2+y^2+z^2\right)}$$
$$\frac{dy}{y} = \frac{d\left(x^2+y^2+z^2\right)}{\left(x^2+y^2+z^2\right)}$$
$$\log y = \log(x^2+y^2+z^2) + logb$$
$$\text{i.e., } \frac{y}{x^2+y^2+z^2} = b$$

$\therefore$ The general solution is $\phi\left(\dfrac{y}{z}, \dfrac{y}{x^2+y^2+z^2}\right) = 0$.

Example 1.48. *Solve* $(x-2z)p + (2z-y)q = y - x$. *[UQ]*

Solution: The subsidiary equations are $\dfrac{dx}{x-2z}=\dfrac{dy}{2z-y}=\dfrac{dz}{y-x}$ (1)

Each fraction of (1) $=\dfrac{dx+dy+dz}{\mathrm{x-2z+2z-y+y}-x}=\dfrac{dx+dy+dz}{0}$

$$\text{i.e., } dx+dy+dz=0$$
$$\text{i.e., } x+y+z=a$$

Each fraction of (1) $=\dfrac{ydx+xdy}{\mathrm{xy-2yz+2xz-xy}}=\dfrac{ydx+xdy}{\mathrm{2z(x-y)}}$

Taking $\dfrac{ydx+xdy}{\mathrm{2z\,(x-y)}}=\dfrac{dz}{\mathrm{y}-x}$, we have

$$ydx+xdy=-2zdz$$
$$\text{i.e., } d(xy)=-2zdz$$

Integrating, $xy=-z^2+b$

$xy+z^2=b$

The general solution is $\phi\left(x+y+z, xy+z^2\right)=0$.

Example 1.49. *Solve* $(z^2-2yz-y^2)p+(xy+zx)q=xy-zx.$ *[UQ]*

Solution: The subsidiary equations are

$$\frac{dx}{z^2-2yz-y^2}=\frac{dy}{xy+zx}=\frac{dz}{xy-zx} \tag{1}$$

Using x, y, z as multipliers each fraction of (1) $=\dfrac{xdx+ydy+zdz}{x\left(z^2-2yz-y^2\right)+y\left(xy+zx\right)+z\left(xy-zx\right)}=\dfrac{xdx+ydy+zdz}{0}$
we have

$$xdx+ydy+zdz=0$$

Integrating, we get $\dfrac{x^2}{2}+\dfrac{y^2}{2}+\dfrac{z^2}{2}=c_1$

$$x^2+y^2+z^2=a$$

Taking $\dfrac{dy}{xy+zx}=\dfrac{dz}{xy-zx}$, we have

$$\frac{dy}{y+z}=\frac{dz}{y-z}$$
$$\text{i.e., } (y-z)dy=(y+z)dz$$
$$ydy-zdy-ydz-zdz=0$$
$$ydy-(ydz+zdy)-zdz=0$$
$$ydy-d(yz)-zdz=0$$

Integrating, $\dfrac{y^2}{2} - yz - \dfrac{z^2}{2} = c_2$

i.e., $y^2 - z^2 - 2yz = b$

The general solution is $\phi\left(x^2 + y^2 + z^2, y^2 - z^2 - 2yz\right) = 0$.

Example 1.50. *Solve* $pz - qz = z^2 + (x+y)^2$.

Solution: The subsidiary equations are $\dfrac{dx}{z} = \dfrac{dy}{-z} = \dfrac{dz}{z^2 + (x+y)^2}$ (1)

Taking $\dfrac{dx}{z} = \dfrac{dy}{-z}$, we have $dx = -dy$

Integrating, $x + y = a$

Taking $\dfrac{dx}{z} = \dfrac{dz}{z^2 + (x+y)^2}$, we have

$$\frac{dx}{z} = \frac{dz}{z^2 + a^2} \qquad (\because x + y = a)$$

$$\text{i.e., } dx = \frac{zdz}{z^2 + a^2}$$

$$\text{Integrating, } x = \frac{1}{2}\log(z^2 + a^2) + c$$

$$2x = \log(z^2 + a^2) + 2c$$

i.e., $2x - \log\left[z^2 + (x+y)^2\right] = b$

The general solution is $\phi\left\{x + y, 2x - \log\left[z^2 + (x+y)^2\right]\right\} = 0$

Example 1.51. *Solve* $p - q = log(x+y)$.

Solution: The subsidiary equations are $\dfrac{dx}{1} = \dfrac{dy}{-1} = \dfrac{dz}{\log(x+y)}$ (1)

Taking $\dfrac{dx}{1} = \dfrac{dy}{-1}$, we have $dx = -dy$

Integrating, $x + y = a$

Taking $\dfrac{dx}{1} = \dfrac{dz}{\log(x+y)}$, we have

$$dx = \frac{dz}{\log a} \qquad (\because x + y = a)$$

$$\text{i.e., } \log a dx = dz$$

Integrating, $(log a)x = z + b$

i.e., $[\log(x+y)]x = z + b$

$x\log(x+y) - z = b$

The general solution is $\phi(x + y, xlog(x+y) - z) = 0$.

1.5.3 Exercises & Solutions

Solve the following partial differential equations:

1. $z(x-y) = x^2p - y^2q$ $\left[\text{Ans}: \phi\left(\frac{1}{x}+\frac{1}{y}, \frac{z}{x+y}\right) = 0\right]$

2. $(x^2+y^2+yz)p + (x^2+y^2-xz)q = z(x+y)$
$\left[\text{Ans}: \phi(x-y-z, \frac{x^2+y^2}{z^2}) = 0\right]$

3. $(3z-4y)\dfrac{\partial z}{\partial x} + (4x-2z)\dfrac{\partial z}{\partial y} = 2y-3x$
$\left[\text{Ans}: \phi(x^2+y^2+z^2, 2x+3y+4z) = 0\right]$

4. $(y+z)p + (z+x)q = x+y$
$\left[\text{Ans}: \phi\left(\frac{y-z}{x-y}, (x-y)\sqrt{x+y+z}\right) = 0\right]$

5. $(y^2+z^2)p - xyq + xz = 0$ $\left[\text{Ans}: \phi\left(\frac{y}{z}, x^2+y^2+z^2\right) = 0\right]$

6. $(y-z)p + (x-y)q = (z-x)$ $\left[\text{Ans}: (x^2+2yz, x+y+z) = 0\right]$

7. $\dfrac{y-z}{yz}p + \dfrac{z-x}{xy}q = \dfrac{x-y}{xy}$ $\left[\text{Ans}: \phi(x+y+z, xyz) = 0\right]$

8. $p\cot x + q\cot y = \cot z$ $\left[\text{Ans}: \phi\left(\frac{\cos z}{\cos y}, \frac{\cos y}{\cos x}\right) = 0\right]$

9. $(a-x)p + (b-y)q = c-z$ $\left[\text{Ans}: \phi\left(\frac{a-x}{b-y}, \frac{b-y}{c-z}\right) = 0\right]$

1.6 Solution of Partial Differential Equations

A solution or integral of a partial differential equation is a relation between the independent and the dependent variables which satisfies the given differential equation.

Note: There are two distinct types of solution for partial differential equations, one type of solution containing arbitrary constants and the other type of solution containing arbitrary functions. Both these types f solution may be given as solutions of the same partial differential equation.

Any solution of a partial differential equation in which the number of arbitrary constants is equal to the number of independent variables is called the ***complete integral***.

Any solution obtained from the complete integral by giving particular values to the arbitrary constants is called a ***particular integral***.

Any solution obtained from the complete integral by eliminating arbitrary constants is called a ***singular integral***.

Any solution which contains the maximum number of arbitrary functions is called as a ***general integral***.

Let $\phi(x, y, z, a, b) = 0$ be the ***complete integral*** of $f(x, y, z, p, q) = 0$. Then the solution obtained by eliminating a and b from the equations

$$\phi(x, y, z, a, b) = 0 \tag{1}$$

$$\frac{\partial \phi}{\partial a} = 0 \tag{2}$$

$$\frac{\partial \phi}{\partial b} = 0 \tag{3}$$

is called the ***singular integral*** of the partial differential equation.

In (1), we put $b = g(a)$ we get

$$\phi(x, y, z, a, g(a)) = 0 \tag{4}$$

Differentiating (4) w.r.t. 'a', we get

$$\frac{\partial \phi}{\partial a} + \frac{\partial \phi}{\partial g} g'(a) = 0 \tag{5}$$

The obtained by eliminating 'a' from (4) and (5) is called the ***general integral*** of $\phi(x, y, z, a, b) = 0$

1.6.1 Solution of Partial Differential Equations by Direct Integration

A partial differential equation can be solved by successive integration in all cases where the dependent variable occurs only in the partial derivatives. We illustrate this method in the following Examples.

Example 1.52. *Solve* $\dfrac{\partial z}{\partial x} = \cos x$.

Solution: If z is a function of x only then there is no difference between $\dfrac{\partial z}{\partial x}$ and $\dfrac{dz}{dx}$. Therefore to find z we can directly integrate $\dfrac{dz}{dx} = \cos x$ and hence the solution is $z = \sin x + A$, where A is arbitrary constant. But here z is a function of two variables x and y.

$\therefore$ Instead of the arbitrary constant A we can take an arbitrary function of y say $f(y)$.

$\therefore$ The solution is $z = \sin x + f(y)$.

Example 1.53. *Solve* $\dfrac{\partial^2 z}{\partial x \partial y} = sinx.$

Solution: Given $\dfrac{\partial^2 z}{\partial x \partial y} = sinx$. Integrating w.r.t '$x$' $\dfrac{\partial z}{\partial y} = -cosx + f(y)$, where $f(y)$ is an arbitrary function of y. Integrating again w.r.t 'y' $Z = -ycosx + F(y) + g(x)$ where $F(y) = \displaystyle\int f(y)dy$ is an arbitrary function of y and $g(x)$ is an arbitrary function of x.

Example 1.54. *Solve* $\dfrac{\partial^2 z}{\partial x \partial y} = 0.$

Solution : Given $\dfrac{\partial^2 z}{\partial x \partial y} = 0$ Integrating w.r.t 'x' $\dfrac{\partial z}{\partial y} = f(y)$, where $f(y)$ is an arbitrary function of y. Integrating again w.r.t 'y' $z = F(y) + g(x)$, where $F(y)$ is an arbitrary function of y and $g(x)$ is an arbitrary function of x.

Example 1.55. *Solve* $\dfrac{\partial^2 u}{\partial y \partial x} = 4xsin(3xy).$

Solution: *Given* $\dfrac{\partial^2 u}{\partial y \partial x} = 4xsin(3xy)$ *Integrating w.r.t 'y'* $\dfrac{\partial u}{\partial x} = \dfrac{-4x\cos(3xy)}{3x} + f(x) = \dfrac{-4}{3}\cos(3xy) + f(x)$ *Integrating again w.r.t 'x'* $u = \dfrac{-4}{3}\dfrac{\sin(3xy)}{3y} + F(x) + g(y)$ *ie,* $u = \dfrac{-4}{9y}\sin(3xy) + F(x) + g(y)$, *where* $F(x) = \displaystyle\int f(x)dx$ *is an arbitrary function of x and $g(y)$ is an arbitrary function of y.*

Example 1.56. *Solve* $\dfrac{\partial z}{\partial x} = 2x + 3y; \dfrac{\partial z}{\partial y} = 3x + cosy$

Solution: $\dfrac{\partial z}{\partial x} = 2x + 3y$ Integrating w.r.t 'x' $z = x^2 + 3xy + f(y)$ (1)
where $f(y)$ is an arbitrary function of 'y' Differentiating (1) partially w.r.t 'y' $\dfrac{\partial z}{\partial y} = 3x + f'(y)$ By hypothesis, $\dfrac{\partial z}{\partial y} = 3x + cosy$
$\therefore f(y) = siny + c$
$\therefore$ from (1), $z = x^2 + 3xy + siny + c$, where c is an arbitrary constant.

Example 1.57. *Solve* $\dfrac{\partial^2 z}{\partial x \partial y} = sinxsiny$ *for which* $\dfrac{\partial z}{\partial y} = -2siny$, *when* $x = 0$ *and* $z = 0$ *when y is an odd multiple of* $\dfrac{\pi}{2}$.

Solution: $\dfrac{\partial^2 z}{\partial x \partial y} = sinxsiny$
Integrating w.r.t 'x'
$$\frac{\partial z}{\partial y} = -cosxsiny + f(y) \tag{1}$$
Given $\dfrac{\partial z}{\partial y} = -2siny$ when $x = 0$
Hence, $-2siny = -siny + f(y)$
$\therefore f(y) = -siny$
From (1), $\dfrac{\partial z}{\partial y} = -cosxsiny - siny$
Integrating w.r.t 'y'
$$z = cosxcosy + cosy + g(x) \tag{2}$$
Given $z = 0$ when $y = (2n+1)\dfrac{\pi}{2}$ $\left[y \text{is an odd multiple of } \dfrac{\pi}{2}\right]$
Hence $g(x) = 0$
From(2), $z = (cosx + 1)cosy$.

Example 1.58. *Solve* $\dfrac{\partial^2 u}{\partial x \partial t} = e^{-t}\cos x$, *given that* $u = 0$ *when* $t = 0$ *and* $\dfrac{\partial u}{\partial t} = 0$ *when* $x = 0$. *Show also that as* $t \to \infty$, $u \to sinx$.

Solution: Given $\dfrac{\partial^2 u}{\partial x \partial t} = e^{-t}\cos x$
Integrating w.r.t 'x'
$$\frac{\partial u}{\partial t} = e^{-t}\sin x + f(t)$$
When $x = 0$, $\dfrac{\partial u}{\partial t} = 0$
$\therefore f(t) = 0$ Hence $\dfrac{\partial u}{\partial t} = e^{-t}\sin x$
Integrating w.r.t 't'
$u = -e^{-t}\sin x + g(x)$
When $t = 0, u = 0$
$0 = -sinx + g(x)$ i.e., $g(x) = sinx$
$\therefore u = -e^{-t}\sin x + \sin x$
$u = (1 - e^{-t})\sin x$.
Apply $t \to \infty$, we have $e^{-t} \to 0$
$\Rightarrow u = sinx$.

1.6.2 Exercises & Solutions

Solve the following partial differential equation using direct integration

1. $\dfrac{\partial z}{\partial x} = -\dfrac{x}{y}$ $\left[\text{Ans: } z = -\dfrac{x^2}{2y} + \phi(y)\right]$

2. $\dfrac{\partial^2 z}{\partial x^2} = \sin y$ $\left[\text{Ans :} z = \dfrac{x^2}{2}\sin y + xf(y) + \phi(y)\right]$

3. $\dfrac{\partial^2 z}{\partial x \partial y} = x^2 + y^2$ $\left[\text{Ans :} z = \dfrac{x^3 y}{3} + \dfrac{y^3 x}{3} + \phi(y) + f(x)\right]$

4. $\dfrac{\partial z}{\partial x} = 4x - 2y; \dfrac{\partial z}{\partial y} = -2x + 6y$ $\left[\text{Ans :} z = 2x^2 - 2xy + 3y^2 + c\right]$

5. $\dfrac{\partial^2 z}{\partial y^2} - z = 0$ when $y = 0, z = e^x$ and $\dfrac{\partial z}{\partial y} = e^{-x}$
$\left[\text{Ans :} z = e^y \cosh x + e^{-y}\sinh x\right]$

1.7 Solution of standard types of first order partial differential equations

A partial differential equation which involves only the first order partial derivatives p (i.e., $=\dfrac{\partial z}{\partial x}$) and q (i.e., $=\dfrac{\partial z}{\partial x}$) is called a ***first order partial differential equation.***

The general form of first order partial differential equation is $f(x, y, z, p, q) = 0$. We shall see some standard form of such equations and solve them by special methods.

1.7.1 Type I: $f(p, q) = 0$ i.e., the equation contain p and q only.

To find Complete Integral

Given $f(p, q) = 0$ (1)

Let $z = ax + by + c$ be a trial solution of the equation (1). Then

$$p = \frac{\partial z}{\partial x} = a \text{ and } q = \frac{\partial z}{\partial y} = b$$

From (1), we get

$$f(a, b) = 0$$

Hence the complete integral of (1) is

$$z = ax + by + c$$

Solving for b from $f(a, b) = 0$, we get $b = \beta(a)$

$\therefore$ The ***complete integral*** of (1) is

$$z = ax + \beta(a)y + c \qquad (2)$$

since [number of arbitrary constants(a,c)=number of independent variables(x,y)=2]

To find Singular Integral

To obtain the *singular integral* we have to eliminate a and c from the equations $z = ax + \beta(a)y + c$, $\dfrac{\partial z}{\partial a} = 0$ and $\dfrac{\partial z}{\partial c} = 0$

$$\text{i.e., } z = ax + \beta(a)y + c$$
$$x + \beta'(a)y = 0$$
$$1 = 0$$

The last equation being absurd and hence there is ***no singular integral***.

To find General Integral

Put $c = \chi(a)$ in complete integral equation (2), we get

$$z = ax + \beta(a)y + \chi(a) \tag{3}$$

Differentiate (3) w.r.t. 'a', we get

$$z = x + \beta'(a)y + \chi'(a) \tag{4}$$

Eliminating 'a' from (3) and (4), we get ***general integral*** of the given p.d.e.

Note: For the equation of the type of $f(p, q) = 0$ there is no singular integral.

Example 1.59. *Solve $pq = 1$.*

Solution: Given $pq = 1$ (1)

To find Complete Integral :

Let $z = ax + by + c$ be a trial solution of the equation (1). Then

$$p = \frac{\partial z}{\partial x} = a \text{ and } q = \frac{\partial z}{\partial y} = b$$

From (1), we get

$$ab = 1$$
$$\text{i.e., } = \frac{1}{a}$$

Hence the complete integral of (1) is

$$z = ax + \frac{1}{a}y + c \tag{2}$$

since [number of arbitrary constants(a,c)=number of independent variables(x,y)=2]

To find Singular Integral :

To obtain the *singular integral* we have to eliminate 'a' and 'c' from the equations $z = ax + \beta(a)y + c$, $\dfrac{\partial z}{\partial a} = 0$ and $\dfrac{\partial z}{\partial c} = 0$

$$\text{i.e., } z = ax + \frac{1}{a}y + c$$

$$x - \frac{1}{a^2}y = 0$$

$$1 = 0$$

The last equation being absurd and hence there is no singular integral.

To find General Integral :

Put $c = \chi(a)$ in complete integral equation (2), we get

$$z = ax + \frac{1}{a}y + \chi(a) \tag{3}$$

Differentiate (3) w.r.t. 'a', we get

$$z = x - \frac{1}{a^2}y + \chi^{'}(a) \tag{4}$$

Eliminating 'a' from (3) and (4), we get general integral of the given p.d.e.

Example 1.60. *Solve* $p^2 + q^2 = 1$.

Solution: Given $p^2 + q^2 = 1$ (1)

Let $z = ax + by + c$ be a trial solution of (1)

Then $p = \dfrac{\partial z}{\partial x} = a$ and $q = \dfrac{\partial z}{\partial y} = b$

From (1), we get

$$a^2 + b^2 = 1$$

$$\therefore b = \sqrt{1 - a^2}$$

$\therefore$ The complete integral is $z = ax + \sqrt{1 - a^2}y + c$

The singular and general integrals are obtained as first example of this type.

Example 1.61. *Solve* $p + q = pq$.

Solution: Given $p + q = pq$ (1)

Let $z = ax + by + c$ be a trial solution of (1)

Then $p = \dfrac{\partial z}{\partial x} = a$ and $q = \dfrac{\partial z}{\partial y} = b$
From (1), we get

$$a + b = ab$$
$$\text{i.e., } b = \frac{a}{a-1}$$

$\therefore$ The complete integral is $z = ax + \dfrac{a}{a-1}y + c$
The singular and general integrals are obtained as in previous procedure.

Example 1.62. *Solve* $pq + p + q = 0$.

Solution: Solution: Similar to above Example

Example 1.63. *Find the complete integral of* $p = e^q$.

Solution: Given $logp = q$ (1)
Let $z = ax + by + c$ be a trial solution of (1)
Then $p = \dfrac{\partial z}{\partial x} = a$ and $q = \dfrac{\partial z}{\partial y} = b$
From (1), we get

$$\text{log a} = \text{b}$$

$\therefore$ The complete integral is $z = ax + (loga)y + c$
The singular and general integrals are obtained as first example of this type.

Example 1.64. *Solve* $\sqrt{p} + \sqrt{q} = 1$.

Solution: Given $\sqrt{p} + \sqrt{q} = 1$ (1)
Let $z = ax + by + c$ be a trial solution of (1)
Then $p = \dfrac{\partial z}{\partial x} = a$ and $q = \dfrac{\partial z}{\partial y} = b$
From (1), we get

$$\sqrt{a} + \sqrt{b} = 1$$
$$\text{i.e., } b = \left(1 - \sqrt{a}\right)^2$$

$\therefore$ The complete integral is $z = ax + \left(1 - \sqrt{a}\right)^2 y + c$
The singular and general integrals are obtained as first example of this type.

Example 1.65. *Find the complete integral of* $p^2 + q^2 = npq$.

Solution: Given $p^2 + q^2 = npq$ (1)

Let $z = ax + by + c$ be a trial solution of (1)

Then $p = \dfrac{\partial z}{\partial x} = a$ and $q = \dfrac{\partial z}{\partial y} = b$

From (1), we get

$$a^2 + b^2 = n\text{ab}$$
$$\text{i.e., } b^2 - n\text{ab} + \text{a}^2 = 0$$
$$\therefore b = \frac{n\text{a} \pm \sqrt{\text{n}^2a^2 - 4a^2}}{2}$$
$$b = \frac{a}{2}\left[n \pm \sqrt{n^2 - 4}\right]$$

$\therefore$ The complete integral is $z = ax + \dfrac{a}{2}\left[n \pm \sqrt{n^2 - 4}\right] y + c$

1.7.2 Type II: $z = px + qy + f(p, q)$ (Clairaut's Form)

Suppose a partial differential equation of the form

$$z = px + qy + f(p, q) \tag{1}$$

which is said to be of Clairaut's form.

To find complete integral : The complete integral of (1) is by substituting $p = a$ and $q = b$ in (1), i.e.,

$$z = ax + by + f(a, b) \tag{2}$$

where a and b are arbitrary constants.

To find singular integral : Partially differentiating (2) w.r.t 'a' and 'b' and then equating to zero, we get

$$x + \frac{\partial f}{\partial a} = 0 \tag{3}$$
$$y + \frac{\partial f}{\partial b} = 0 \tag{4}$$

By eliminating a and b from (2), (3) and (4), we get the singular integral.

To find General Integral : Put $b = \psi(a)$ in complete integral equation (2), we get

$$z = ax + \beta(a)y + f(a, \beta(a)) \tag{5}$$

Differentiate (3) w.r.t. 'a', we get

$$z = x + \beta^{'}(a)y + f^{'}(a, \beta(a)) \tag{6}$$

Eliminating 'a' from (5) and (6), we get general integral of the given p.d.e.

Example 1.66. *Find the complete and singular integrals of* $z = px + qy + pq$.

Solution: Given $z = px + qy + pq$ (1)

The given partial differential equation is in Clairaut's form.

$\therefore$ The complete integral is $z = ax + by + ab$ (2)

where a and b are arbitrary constants.

To find singular integral : Partially differentiating (2) w.r.t 'a' and 'b' then equating to zero, we get

$$x + b = 0$$
$$y + a = 0$$
$$\therefore a = -y \text{ and } b = -x$$

Substituting the values of 'a' and 'b' in (2), we get

$$z = -yx - xy + (-y)(-x) = -yx - xy + yx$$
$$\therefore z = -xy$$

which is the singular integral.

To find General Integral : Put $b = \beta(a)$ in complete integral equation (2), we get

$$z = ax + \beta(a)y + a\beta(a) \tag{3}$$

Differentiate (3) w.r.t. 'a', we get

$$z = x + \beta^{'}(a)y + a\beta^{'}(a) + \beta(a) \tag{4}$$

Eliminating 'a' from (3) and (4), we get general integral of the given p.d.e.

Example 1.67. *Solve* $z = px + qy + 2\sqrt{pq}$.

Solution: Given $z = px + qy + 2\sqrt{pq}$ (1)

The given partial differential equation is in Clairaut's form

$\therefore$ The complete integral is $z = ax + by + 2\sqrt{ab}$ (2)

where a and b are arbitrary constants.

To find singular integral: Partially differentiating (2) w.r.t a and b then equating to zero, we get

$$x + \sqrt{\frac{b}{a}} = 0 \tag{3}$$

$$y + \sqrt{\frac{a}{b}} = 0 \tag{4}$$

$\therefore x = -\sqrt{\dfrac{b}{a}}$ and $y = -\sqrt{\dfrac{a}{b}}$

From the above two equations we get

$$xy = 1$$

which is the singular integral.

The general integral is obtained as first example of this type.

Example 1.68. *Solve* $z = px + qy + p^2q^2$.

Solution: The given equation is in Clairaut's form.

$\therefore$ The complete integral is $z = ax + by + a^2b^2$ (1)

To find singular integral: Partially differentiating (1) w.r.t a and b then equating to zero, we get

$$x = -2ab^2 \quad (2)$$
$$y = -2a^2b \quad (3)$$
$$xy = 4a^3b^3 \quad (4)$$
$$\text{From (2), } \frac{x}{b} = -2ab$$
$$\text{From (3), } \frac{y}{a} = -2ab$$

$$\begin{aligned}
\text{From (1), } z &= ab\left[\frac{x}{b} + \frac{y}{a} + ab\right] \\
&= ab(-2ab - 2ab + ab) \\
z &= -3a^2b^2 \quad (5) \\
\text{Now, } z^3 &= -27a^6b^6 \\
\text{i.e., } z^3 &= -27(a^3b^3)^2 \\
\text{Using (4), } z^3 &= -27\left(\frac{xy}{4}\right)^2 \\
\therefore z^3 &= \frac{-27}{16}x^2y^2
\end{aligned}$$

which is the required singular integral.

The general integral is obtained as first example of this type.

Example 1.69. *Solve* $z = px + qy + p^2 - q^2$.

Solution: The given partial differential equation is in Clairaut's form

$\therefore$ The complete integral is $z = ax + by + a^2 - b^2$ (1)

To find singular integral: Partially differentiating (2) w.r.t a and b then equating to zero, we get

$$x + 2a = 0$$
$$y - 2b = 0$$

$\therefore a = \dfrac{-x}{2}$ and $b = \dfrac{y}{2}$
Substituting the values of a and b in (1), we get

$$z = \frac{-x^2}{2} + \frac{y^2}{2} + \frac{x^2}{4} - \frac{y^2}{4}$$

ie., $4z = y^2 - x^2$ which is the singular integral.
The general integral is obtained as first example of this type.

Example 1.70. *Solve* $z = px + qy + p^2 + pq + q^2$.

Solution: The given equation is in Clairaut's form.
$\therefore$ The complete integral is $z = ax + by + a^2 + ab + b^2$ (1)
To find singular integral: Partially differentiating (1) w.r.t a and b then equating to zero, we get

$$x + 2a + b = 0 \tag{2}$$
$$y + a + 2b = 0 \tag{3}$$

Solving (2) and (3), we get

$$a = \frac{1}{3}(y - 2x), b = \frac{1}{3}(x - 2y)$$

Substituting the values of a and b in (1), we get

$$3z = xy - x^2 - y^2$$

which is the singular integral.
The general integral is obtained as first example of this type.

Example 1.71. *Solve* $z = px + qy + \sqrt{1 + p^2 + q^2}$.

Solution: The given equation is in Clairaut's form.
$\therefore$ The complete integral is $z = ax + by + \sqrt{1 + a^2 + b^2}$ (1)
To find singular integral: Partially differentiating (1) w.r.t a and b then

equating to zero, we get

$$x + \frac{a}{\sqrt{1+a^2+b^2}} = 0 \Rightarrow x = \frac{-a}{\sqrt{1+a^2+b^2}} \tag{2}$$

$$y + \frac{b}{\sqrt{1+a^2+b^2}} = 0 \Rightarrow y = \frac{-b}{\sqrt{1+a^2+b^2}} \tag{3}$$

From (2) and (3), we get

$$x^2+y^2 = \frac{a^2+b^2}{1+a^2+b^2}$$

$$1-(x^2+y^2) = 1 - \frac{a^2+b^2}{1+a^2+b^2}$$

$$1-x^2-y^2 = \frac{1}{1+a^2+b^2}$$

$$\sqrt{1+a^2+b^2} = \frac{1}{\sqrt{1-x^2-y^2}} \tag{4}$$

From (2) and (3), we get

$$a = \frac{-x}{\sqrt{1-x^2-y^2}} \tag{5}$$

and

$$b = \frac{-y}{\sqrt{1-x^2-y^2}} \tag{6}$$

Substituting (4), (5) and (6) in (1), we get

$$z = \frac{-x^2}{\sqrt{1-x^2-y^2}} - \frac{y^2}{\sqrt{1-x^2-y^2}} + \frac{1}{\sqrt{1-x^2-y^2}}$$

$$z = \sqrt{1-x^2-y^2}$$

$$z^2 = 1-x^2-y^2$$

$$\therefore x^2+y^2+z^2 = 1$$

which is the singular integral.
The general integral is obtained as first example of this type.

Example 1.72. *Find the complete integral of* $pqz = p^2(xq+p^2)+q^2(yp+q^2)$.

Solution: Given $pqz = p^2(xq+p^2)+q^2(yp+q^2)$

$$= p^2xq + p^4 + q^2yp + q^4$$

Dividing by pq we have

$$z = px + qy + \frac{p^4 + q^4}{pq}$$

This is Clairaut's form.
The complete integral is

$$z = ax + by + \frac{a^4 + b^4}{ab},$$

where a and b are arbitrary constants.

1.7.3 Type III: $f(z, p, q) = 0$ ie., equation not containing x and y explicity

Given $f(z, p, q) = 0$ (1)
Let $z = f(x + ay)$ be the solution of (1)
Put $x + ay = u$ Then

$$z = f(u) \text{ and } \frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = a$$

$$p = \frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x} = \frac{dz}{du}$$
$$q = \frac{\partial z}{\partial y} = \frac{dz}{du}\frac{\partial u}{\partial y} = a\frac{dz}{du}$$

Substituting the values of p and q in (1), we get

$$f\left(z, \frac{dz}{du}, a\frac{dz}{du}\right) = 0 \tag{2}$$

This is an ordinary differential equation of first order
Solving for $\frac{dz}{du}$, we obtain

$$\frac{dz}{du} = g(z, a)$$
$$\text{i.e., } \frac{dz}{g(z, a)} = du$$
$$\text{Integrating } \int \frac{dz}{g(z, a)} = u + c$$
$$\phi(z, a) = u + c$$
$$\text{ie., } (z, a) = x + ay + c$$

which is the complete integral of (1).
The *singular integral* can be obtained by the usual methods.

Example 1.73. *Find the complete integral of $z = pq$.*

Solution: Given $z = pq$ (1)
Let $z = f(x + ay)$ be the solution of (1)
Put $u = x + ay$ so that $z = f(u)$

$$p = \frac{dz}{du} \quad \text{and} \quad q = a\frac{dz}{du}$$

Substituting in (1), we get

$$\begin{aligned} z &= a\left(\frac{dz}{du}\right)^2 \\ \frac{dz}{du} &= \sqrt{\frac{z}{a}} \Rightarrow \frac{dz}{\sqrt{z}} = \frac{1}{\sqrt{a}}du \\ \text{Integrating, } 2\sqrt{z} &= \frac{1}{\sqrt{a}}u + c \\ 2\sqrt{az} &= u + \sqrt{a}c \\ 2\sqrt{az} &= u + b, \text{whereb} = \sqrt{\text{a}}\text{c} \\ 4az &= (u + b)^2 \\ 4az &= (x + ay + b)^2 \end{aligned}$$

which is the complete integral.

Example 1.74. *Solve* $z = p^2 + q^2$.

Solution: Given $z = p^2 + q^2$ (1)
Let $z = f(x + ay)$ be the solution of (1)
Put $u = x + ay$ so that $z = f(u)$, then

$$p = \frac{dz}{du} \quad \text{and} \quad q = a\frac{dz}{du}$$

Substituting in (1), we get

$$\begin{aligned} z = \left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2 \quad &\Rightarrow \quad z = (1 + a^2)\left(\frac{dz}{du}\right)^2 \\ \frac{dz}{du} = \sqrt{\frac{z}{1 + a^2}} \quad &\Rightarrow \sqrt{1 + a^2}\frac{dz}{\sqrt{z}} = du \end{aligned}$$

$$\begin{aligned} \text{Integrating, } \sqrt{1 + a^2}2\sqrt{z} &= u + b \\ (1 + a^2)4z &= (u + b)^2 \\ \text{i.e., } 4(1 + a^2)z &= (x + ay + b)^2 \end{aligned}$$

which is the complete integral.

Example 1.75. *Solve* $p(1 + q) = qz$.

Solution: Given $p(1+q)=qz$ (1)
Let $z=f(x+ay)$ be the solution of (1)
Put $u=x+ay$ so that $z=f(u)$, then

$$p=\frac{dz}{du} \quad \text{and} \quad q=a\frac{dz}{du}$$

Substituting in (1), we get

$$\begin{aligned}\frac{dz}{du}\left(1+a\frac{dz}{du}\right)&=a\frac{dz}{du}z &\qquad \Rightarrow 1+a\frac{dz}{du}&=az\\ \frac{dz}{du}&=\frac{az-1}{a} &\qquad \Rightarrow \frac{a}{az-1}dz&=du\end{aligned}$$

$$\text{Integrating, } \log(az-1)=u+b$$

$$\text{i.e., } \log(az-1)=x+ay+b$$

which is the complete integral.

Example 1.76. *Solve* $9\left(p^2z+q^2\right)=4$.

Solution: Given $9\left(p^2z+q^2\right)=4$ (1)
Let $z=f(x+ay)$ be the solution of (1)
Put $u=x+ay$ so that $z=f(u)$, then

$$p=\frac{dz}{du} \quad \text{and} \quad q=a\frac{dz}{du}$$

Substituting in (1), we get

$$9\left[\left(\frac{dz}{du}\right)^2z+a^2\left(\frac{dz}{du}\right)^2\right]=4$$

$$\left(\frac{dz}{du}\right)^2=\frac{4}{9(z+a^2)} \Rightarrow \frac{dz}{du}=\frac{2}{3}\frac{1}{\sqrt{z+a^2}}$$

$$\frac{3}{2}\sqrt{z+a^2}dz=du$$

$$\text{Integrating,we get } \frac{3}{2}\frac{\left(z+a^2\right)^{3/2}}{3/2}=u+b$$

$$\left(z+a^2\right)^{3/2}=x+ay+b$$

which is the complete integral.

Example 1.77. *Solve* $p\left(1+q^2\right)=q(z-a)$.

Solution: Given $p\left(1+q^2\right)=q(z-a)$ (1)
Let $z=f(x+ay)$ be the solution of (1)
Put $u=x+ay$ so that $z=f(u)$, then

$$p = \frac{dz}{du} \quad \text{and} \quad q = a\frac{dz}{du}$$

Substituting in (1), we get

$$\frac{dz}{du}\left[1 + b^2\left(\frac{dz}{du}\right)^2\right] = b\frac{dz}{du}(z-a) \quad \Rightarrow \left[1 + b^2\left(\frac{dz}{du}\right)^2\right] = b(z-a)$$

$$\left(\frac{dz}{du}\right)^2 = \frac{bz - ba - 1}{b^2} \quad \Rightarrow \quad \frac{dz}{du} = \frac{\sqrt{bz - ba - 1}}{b}$$

$$\frac{b}{\sqrt{bz - ba - 1}}dz = du$$

Integrating, $2\sqrt{bz - ba - 1} = u + c$

$$4\left(bz - ba - 1\right)^2 = (u + c)^2$$

$$\text{i.e., } 4\left(bz - ba - 1\right)^2 = (x + by + c)^2$$

which is the complete integral.

Example 1.78. *Solve* $z^2\left(p^2 + q^2 + 1\right) = 1$.

Solution: Given $z^2\left(p^2 + q^2 + 1\right) = 1$ (1)

Let $z = f(x + ay)$ be the solution of (1)

Put $u = x + ay$ so that $z = f(u)$, then

$$p = \frac{dz}{du} \quad \text{and} \quad q = a\frac{dz}{du}$$

Substituting in (1), we get

$$z^2\left[\left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2 + 1\right] = 1$$

$$\left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2 + 1 = \frac{1}{z^2}$$

$$\left(1 + a^2\right)\left(\frac{dz}{du}\right)^2 = \frac{1}{z^2} - 1$$

$$\left(1 + a^2\right)\left(\frac{dz}{du}\right)^2 = \frac{1 - z^2}{z^2}$$

$$\frac{dz}{du} = \frac{\sqrt{1 - z^2}}{z\sqrt{1 + a^2}}$$

$$\frac{z}{\sqrt{1 - z^2}}dz = \frac{1}{\sqrt{1 + a^2}}du$$

Integrating, $\displaystyle\int \frac{z}{\sqrt{1 - z^2}}dz = \frac{1}{\sqrt{1 + a^2}}\int du$ (2)

Put $1 - z^2 = t \Rightarrow -2zdz = dt \Rightarrow zdz = -\dfrac{1}{2}dt$

$$
\begin{aligned}
(2) \Rightarrow -\frac{1}{2}\int \frac{dt}{\sqrt{t}} &= \frac{1}{\sqrt{1+a^2}}\int du \\
-\frac{1}{2}2\sqrt{t} &= \frac{1}{\sqrt{1+a^2}}u + b \\
-\sqrt{t} &= \frac{u}{\sqrt{1+a^2}} + b \\
-\sqrt{t} &= \frac{x+ay}{\sqrt{1+a^2}} + b \\
-\sqrt{1-z^2} &= \frac{x+ay}{\sqrt{1+a^2}} + b
\end{aligned}
$$

which is the complete integral.

Example 1.79. *Solve* $z^2\left(p^2+q^2+1\right) = c^2$.

Solution: Similar to above Example.

Example 1.80. *Solve* $q^2 = z^2p^2\left(1-p^2\right)$.

Solution: Given $q^2 = z^2p^2\left(1-p^2\right)$ (1)
Let $z = f(x+ay)$ be the solution of (1)
Put $u = x+ay$ so that $z = f(u)$, then

$$p = \frac{dz}{du} \quad \text{and} \quad q = a\frac{dz}{du}$$

Substituting in (1), we get

$$
\begin{aligned}
a^2\left(\frac{dz}{du}\right)^2 &= z^2\left(\frac{dz}{du}\right)^2\left[1-\left(\frac{dz}{du}\right)^2\right] \\
\frac{a^2}{z^2} &= 1-\left(\frac{dz}{du}\right)^2 \qquad \Rightarrow \qquad \left(\frac{dz}{du}\right)^2 = \frac{z^2-a^2}{z^2} \\
\frac{dz}{du} &= \frac{\sqrt{z^2-a^2}}{z} \qquad \Rightarrow \frac{z}{\sqrt{z^2-a^2}}dz = du
\end{aligned}
$$

Integrating, $\displaystyle\int \frac{z}{\sqrt{z^2-a^2}}dz = \int du$

$$\text{Put } z^2-a^2 = t \Rightarrow 2zdz = dt \Rightarrow zdz = \frac{1}{2}dt$$

$$
\begin{aligned}
\frac{1}{2}\int \frac{dt}{\sqrt{t}} &= \int du \\
\frac{1}{2}2\sqrt{t} &= u+b \qquad \Rightarrow \qquad \sqrt{t} = u+b \\
\sqrt{z^2-a^2} &= x+ay+b
\end{aligned}
$$

which is the complete integral.

1.7.4 Type IV: $f(x,p) = g(y,q)$ (Separable equations)

A first order partial differential equation is called separable if it can be written as $f(x,p) = g(y,q)$.

Let $f(x,p) = g(y,q) = a$, where a is an arbitrary constant.
$\therefore f(x,p) = a, g(y,q) = a$ Solving for p and q from these two equations we get

$$p = f_1(x,a) \text{ and } q = g_1(y,a)$$

$$\begin{aligned}\text{Since } dz &= \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \\ &= pdx + qdy \\ dz &= f_1(x,a)dx + g_1(x,a)dy\end{aligned}$$

Integrating, we get

$z = \int f_1(x,a)dx + \int g_1(y,a)dy + b$

which gives the complete integral.

Example 1.81. *Solve* $p + q = x + y$

Solution: Given that $p + q = x + y$ (1)

i.e., $p - x = y - q$

Let $p - x = y - q = a$, where a is an arbitrary constant.

Now, $p - x = a \Rightarrow p = a + x$

i.e., $f_1(x,a) = a + x$

$y - q = a \Rightarrow q = y - z$

i.e., $f_2(y,a) = y - a$

$\therefore$ The complete integral is

$$\begin{aligned}z &= \int f_1(x,a)dx + \int f_2(y,a)dy + b \\ z &= \int (a + x)\,dx + \int (y - a)dy + b \\ z &= \frac{(a + x)^2}{2} + \frac{(y - a)^2}{2} + b \\ 2z &= (a + x)^2 + (y - a)^2 + 2b\end{aligned}$$

Example 1.82. *Solve* $p - x^2 = q + y^2$

Solution: Let $p - x^2 = q + y^2 = a$ (1)

Now, $p - x^2 = a \Rightarrow p = a + x^2$

i.e., $f_1(x,a) = a + x^2$

$q + y^2 = a \Rightarrow q = a - y^2$

i.e., $f_2(y, a) = a - y^2$

$\therefore$ The complete integral is

$$z = \int f_1(x, a)dx + \int f_2(y, a)dy + b$$
$$z = \int \left(a + x^2\right)dx + \int \left(a - y^2\right)dy + b$$
$$z = \left(ax + \frac{x^3}{3}\right) + \left(ay - \frac{y^3}{3}\right) + b$$

Example 1.83. *Solve* $p^2 - q^2 = x - y$

Solution: Given $p^2 - x = q^2 - y$

Let $p^2 - x = q^2 - y = a$

Now, $p^2 - x = a \Rightarrow p = \sqrt{a + x}$

i.e., $f_1(x, a) = \sqrt{a + x}$

$q^2 - y = a \Rightarrow q = \sqrt{a + y}$

i.e., $f_2(y, a) = \sqrt{a + y}$

$\therefore$ The complete integral is

$$z = \int f_1(x, a)dx + \int f_2(y, a)dy + b$$
$$z = \int \sqrt{a + x}dx + \int \sqrt{a + y}dy + b$$
$$z = \frac{2}{3}\left(a + x\right)^{3/2} + \frac{2}{3}\left(a + y\right)^{3/2} + b$$

Example 1.84. *Solve* $p^2 + q^2 = x + y$

Solution: Similar to above Example.

Example 1.85. *Solve* $\sqrt{p} + \sqrt{q} = 2x$

Solution: Let $\sqrt{p} - 2x = -\sqrt{q} = a$

$\sqrt{p} - 2x = a \Rightarrow p = a + 2x^2$

i.e., $f_1(x, a) = a + 2x^2$

$-\sqrt{q} = a \Rightarrow q = a^2$

i.e., $f_2(y, a) = a^2$

$\therefore$ The complete integral is

$$z = \int f_1(x, a)dx + \int f_2(y, a)dy + b$$
$$z = \int \left(a + 2x\right)^2 dx + \int a^2 dy + b$$
$$z = \frac{\left(a + 2x\right)^3}{3 \times 2} + a^2 y + b$$

Example 1.86. *Solve* $p^2y\left(1+x^2\right) = qx^2$

Solution: Given $\dfrac{p^2\left(1+x^2\right)}{x^2} = \dfrac{q}{y}$

Let $\dfrac{p^2\left(1+x^2\right)}{x^2} = \dfrac{q}{y} = a$

Now $\dfrac{p^2\left(1+x^2\right)}{x^2} = a \Rightarrow p = \dfrac{\sqrt{a}x}{\sqrt{1+x^2}}$

i.e., $f_1(x,a) = \dfrac{\sqrt{a}x}{\sqrt{1+x^2}}$

$\dfrac{q}{y} = a \Rightarrow q = ay$

i.e., $f_2(y,a) = ay$

$\therefore$ The complete integral is

$$z = \int f_1(x,a)dx + \int f_2(y,a)dy + b$$

$$z = \int \frac{\sqrt{a}x}{\sqrt{1+x^2}}dx + \int aydy + b$$

Put $1+x^2 = t \Rightarrow 2xdx = dt$

$$z = \frac{\sqrt{a}}{2}\int \frac{\mathrm{dt}}{\sqrt{\mathrm{t}}} + a\int ydy + b$$

$$= \frac{\sqrt{a}}{2}2\sqrt{\mathrm{t}} + a\frac{y^2}{2} + bz \qquad = \sqrt{a}\sqrt{1+x^2} + a\frac{y^2}{2} + b$$

Example 1.87. *Solve* $\dfrac{x}{p} + \dfrac{y}{q} = 1$

Solution: Given $\dfrac{x}{p} = 1 - \dfrac{y}{q}$

Let $\dfrac{x}{p} = 1 - \dfrac{y}{q} = a$

Now $\dfrac{x}{p} = a \ \Rightarrow p = \dfrac{x}{a} \qquad \therefore f_1(x,a) = \dfrac{x}{a}$

$1 - \dfrac{y}{2} = a \ \Rightarrow q = \dfrac{y}{a+1} \quad \therefore f_2(y,a) = \dfrac{y}{a+1}$

$\therefore$ The complete integral is

$$z = \int f_1(x,a)dx + \int f_2(y,a)dy + b$$

$$z = \int \frac{x}{a}dx + \int \frac{y}{a+1}dy + b$$

$$z = \frac{x^2}{2a} + \frac{y^2}{2(a+1)} + b$$

Example 1.88. *Solve* $p+q=\sin x+\sin y$

Solution: Given $p-\sin x=q-\sin y$
Let $p-\sin x=q-\sin y=a$

$$\begin{aligned} \text{Now } p-\sin x=a \ &\Rightarrow p=a+\sin x \ \therefore f_1(x,a)= \ a+\sin x \\ q-\sin y=a \ &\Rightarrow q=a+\sin y \ \therefore f_2(y,a)= \ a+\sin y \end{aligned}$$

$\therefore$ The complete integral is

$$\begin{aligned} z &= \int f_1(x,a)dx+\int f_2(y,a)dy+b \\ z &= \int (a+\sin x)\,dx+\int (\text{a}+\text{siny})\,dy+b \\ z &= ax-\cos x+ay-\cos y+b \end{aligned}$$

1.7.5 Exercises & Solutions

Solve the following partial differential equations

Type I : f(p, q)=0

a) $p^2-q^2=4$ $\left[\text{Ans: } z=ax+\sqrt{a^2-4y}+c\right]$

b) $p+sinq=0$ $[\text{Ans : } z=by-xsinb+c]$

c) $p=q$ $[\text{Ans : } z=a(x+y)+c]$

Type II : Clairaut's form:

a) $z=px+qy+\dfrac{p}{q}-p$ $\left[\text{Ans: CI : } z=ax+by+\dfrac{a}{b}-a, \textbf{SI :} \ \dfrac{y}{1-x}\right]$

b) $(1-x)p+(2-y)q=3-z$ $[\text{Ans: CI : } z=ax+by-a-2b+3]$

c) $z=px+qy+logpq$
$[\text{Ans: CI : } z=ax+by+logab, \textbf{SI :} \ z=-2-logxy]$

d) $z=px+qy+3p^{1/3}q^{1/3}$
$\left[\text{Ans: CI : } ax+by+3a^{1/3}b^{1/3}, \textbf{SI :} \ xyz=1\right]$

Type III : f(z,p, q)=0

a) $p^2+pq=z^2$ $\left[\text{Ans: } logz=\dfrac{1}{\sqrt{1+a}}(x+ay)+b\right]$

b) $z^2=p^2+q^2+1$ $\left[\text{Ans: } \cosh^{-1} z=\dfrac{1}{\sqrt{1+a^2}}(x+ay)+b\right]$

c) $p^3+q^3=8z$ $\left[\text{Ans: } (1+a^3)z^2=\dfrac{64}{27}(x+ay+b)^3\right]$

d) $p+q=z$ $\qquad$ [Ans: $x+ay=(1+alogz+b)$]

e) $p^3=qz$ $\qquad$ $\left[\text{Ans: } 4z=a(x+ay-b)^2\right]$

Type IV : f(x,p)=g(y,q)

a) $q^2-p=y-x$ $\qquad$ $\left[\text{Ans: } z=\dfrac{(a+x)^2}{2}+\dfrac{2}{3}(a+y)^{3/2}+b\right]$

b) $yp=2yx+logq$ $\qquad$ $\left[\text{Ans: } z=x2+ax+\dfrac{e^{ay}}{a}\right]$

c) $p^2+q^2=x+y$ $\qquad$ $\left[\text{Ans: } z=\dfrac{2}{3}\left\{(x+a)^{3/2}+(y-a)^{3/2}\right\}+b\right]$

d) $q=xyp^2$ $\qquad$ $\left[\text{Ans: } z=2\sqrt{ax}+\dfrac{ay^2}{2}+b\right]$

e) $x^2p^2=yq^2$ $\qquad$ [Ans: $z=alogx+2a\sqrt{y}+b$]

1.7.6 Equations Reducible to Standard Forms

1.7.7 Type A : $f\left(x^m p, y^n q\right)=0$

An equation of the form $f\left(x^m p, y^n q\right)=0$ where m and n are constants can be transformed into an equation of the first type [i.e., $f(p,q)=0$]

Put $X=x^{1-m}$ and $Y=y^{1-n}$ where $m\neq 1$ and $n\neq 1$.

Then $p=\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial X}\dfrac{dX}{\partial x}=P\left(1-m\right)x^{-m}$ where $P=\dfrac{\partial z}{\partial X}$

$$\therefore x^m p=\left(1-m\right)P$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial Y}\frac{dY}{\partial y}=Q\left(1-n\right)y^{-n} \text{ where } Q=\frac{\partial z}{\partial Y}$$

$$\therefore y^n q=\left(1-n\right)Q$$

Hence the given equation reduces to $f[(1-m)P,(1-n)Q]=0$
which is of the form **$\mathbf{F(P,\ Q)=0}$ (Type I)**

1.7.8 Type B : $f\left(x^m p, y^n q, z\right)=0$

An equation of the form $f\left(x^m p, y^n q, z\right)=0$ can also be transformed to the standard type **$\mathbf{F(P,\ Q,\ z)=0}$ (Type II)** by the substitution $X=x^{1-m}$ and $Y=y^{1-n}$ where $m\neq 1$ and $n\neq 1$.

Note: In the above two types if $m=1$ put $X=\log x$ and if $n=1$, put $Y=\log y$:

$$\therefore p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial X}\frac{dX}{\partial x}=P\frac{1}{x} \text{ where } P=\frac{\partial z}{\partial X}$$

i.e., $xp=P$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial Y}\frac{dY}{\partial x}=Q\frac{1}{y} \text{ where } Q=\frac{\partial z}{\partial Y}$$

i.e., $yq=Q$

$\therefore$ The equations reduce to either $F(P,Q)=0$ or $F(P,Q,z)=0$.

1.7.9 Type C : $f\left(z^k p, z^k q\right)=0$ where k is a constant

An equation of the form $f\left(z^k p, z^k q\right)=0$ can be transformed into an equation of the **first type** [i.e., **$\mathbf{f(p,\ q)=0}$**].

Put $Z=z^{k+1}$ if k $\neq -1$

Then $\dfrac{\partial Z}{\partial x}=\left(k+1\right)z^k p$

$$P=\left(k+1\right)z^k p \text{ where } P=\frac{\partial Z}{\partial x}$$

i.e., $z^k p=\dfrac{1}{k+1}P$

$$Q = (k+1)\, z^k q \text{ where } Q = \frac{\partial Z}{\partial y}$$

i.e., $z^k q = \dfrac{1}{k+1} Q$

Hence the given equation reduces to $f\left(\dfrac{1}{k+1}P, \dfrac{1}{k+1}Q\right) = 0$ which is of the form $\mathbf{F(P,\ Q) = 0}$

Note: If $k = -1$, then

$$\text{Put } Z = \log z$$
$$\frac{\partial Z}{\partial x} = \frac{1}{z} p$$
$$P = z^{-1} p$$

Similarly,

$$P = z^{-1} p$$

Hence the given equation again reduces to the form $F(P, Q) = 0$.

1.7.10 Type D: $f\left(x^m z^k p, y^n z^k q\right) = 0$

This can be transformed to an equation of the form $F(P, Q) = 0$ by the substitution.

$$X = \begin{cases} x^{1-m} \text{ if } m \neq 1 \\ \log x \text{ if } m = 1 \end{cases}$$
$$Y = \begin{cases} y^{1-n} \text{ if } n \neq 1 \\ \log y \text{ if } n = 1 \end{cases}$$
$$Z = \begin{cases} z^{k+1} \text{ if } k \neq -1 \\ \log z \text{ if } k = -1 \end{cases}$$

Example 1.89. *Solve* $xp + yq = 1$

Solution: Given $xp + yq = 1$ (1)

The given differential equation is the form $f\left(x^m p, y^n q\right) = 0$ with $m = 1$, $n = 1$.

Put $X = \log x$ and $Y = \log y$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x} = P\frac{1}{x}, \text{ where } P = \frac{\partial z}{\partial X}$$

$\therefore xp = P$

Similarly, $yq = Q$ where $Q = \dfrac{\partial z}{\partial Y}$

$\therefore$ (1) becomes $P + Q = 1$ (2)

This is of the form $F(P, Q) = 0$ [Type I]

Let $z = aX + bY + c$ be a solution of (2)

$$\frac{\partial z}{\partial X} = a \Rightarrow P = a$$
$$\frac{\partial z}{\partial Y} = b \Rightarrow Q = b$$

Substituting the values of P and Q in (2), we get

$$a + b = 1$$
$$\text{i.e., } b = 1 - a$$

$\therefore$ The complete integral of $P + Q = 1$ is

$$z = aX + (1 - a)Y + b$$

$\therefore$ The required complete integral is

$$z = a \log x + (1 - a) \log y + b$$

Example 1.90. *Solve* $p^2x^2 + q^2y^2 = z^2$

Solution: The given differential equation can be written as
$(xp)^2 + (yq)^2 = z^2$ (1)
This is of the form $f(x^m p, y^n q, z) = 0$ with $m = 1$, $n = 1$.
Put $X = \log x$ and $Y = \log y$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x} = P\frac{1}{x}, \text{whereP} = \frac{\partial z}{\partial X}$$

$\therefore xp = P$
Similarly, $yq = Q$ where $Q = \dfrac{\partial z}{\partial Y}$
$\therefore$ (1) becomes $P^2 + Q^2 = z^2$ (2)
This is of the form $F(P, Q, z) = 0$ [Type II]
Let $z = \phi(X + aY)$ be a solution of (2)
Let $u = X + aY$ so that $z = \phi(u)$
Then $P = \dfrac{dz}{du}$ and $Q = a\dfrac{dz}{du}$

$$\therefore \text{ (2) becomes } \left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2 = z^2$$

$$\left(\frac{dz}{du}\right)^2 \left(1+a^2\right) = z^2$$

$$\frac{dz}{du} = \frac{z}{\sqrt{(1+a^2)}}$$

$$\frac{dz}{z} = \frac{du}{\sqrt{(1+a^2)}}$$

$$\text{Integrating, } \log z = \frac{u}{\sqrt{(1+a^2)}} + b$$

$$\therefore \log z = \frac{X+aY}{\sqrt{\left(1+a^2\right)}} + b$$

$$\log z = \frac{\log x + a\log y}{\sqrt{(1+a^2)}} + b$$

which is the complete integral.

Example 1.91. *Solve* $p^2x^2 + q^2y^2 = z$

Solution: This problem is to similar to above problem.

Example 1.92. *Solve* $z^2 = xypq$

Solution: The given equation can be written as $z = (xp)(yq)$ (1)
This is of the form $f\left(x^m p, y^n q, z\right) = 0$ with $m = 1$, $n = 1$.
Put $X = \log x$ and $Y = \log y$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x} = P\frac{1}{x}, \text{ where } P = \frac{\partial z}{\partial X}$$

$\therefore xp = P$
Similarly, $yq = Q$ where $Q = \dfrac{\partial z}{\partial Y}$
$\therefore$ (1) becomes $z^2 = PQ$ (2)
This is of the form $F(P, Q, z) = 0$ [Type II]
Let $z = \phi(X + aY)$ be a solution of (2)
Let $u = X + aY$ so that $z = \phi(u)$
Then $P = \dfrac{dz}{du}$ and $Q = a\dfrac{dz}{du}$

$\therefore$ (2) becomes $z^2 = a^2\left(\dfrac{dz}{du}\right)^2$

$$\frac{dz}{du} = \frac{z}{\sqrt{a}}$$

$$\frac{dz}{z} = \sqrt{a}du$$

Integrating, $\log z = \sqrt{a}u + b$

i.e., $\log z = \sqrt{a}\,(X + aY) + b$

i.e., $\log z = \sqrt{a}\,(\log x + a\log y) + b$

which is the complete integral.

Example 1.93. *Solve* $z^2\left(p^2x^2 + q^2\right) = 1$

Solution: The given equation can be written as $(px)^2 + q^2 = \dfrac{1}{z^2}$ (1)

This is of the form $f\left(x^m p, q, z\right) = 0$ with $m = 1$.

Put $X = \log x$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x} = P\frac{1}{x},\ \text{where } P = \frac{\partial z}{\partial X}$$

$\therefore xp = P$

$\therefore$ (1) becomes $P^2 + q^2 = \dfrac{1}{z^2}$ (2)

This is of the form $F(P, q, z) = 0$

Let $z = \phi(X + aY)$ be a solution of (2)

Let $u = X + aY$ so that $z = \phi(u)$

Then $P = \dfrac{dz}{du}$ and $Q = a\dfrac{dz}{du}$

$$\therefore \ (2) \text{ becomes } \left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2 = \frac{1}{z^2}$$

$$\left(\frac{dz}{du}\right)^2 (1+a^2) = \frac{1}{z^2}$$

$$\frac{dz}{du} = \frac{1}{z\sqrt{1+a^2}}$$

$$zdz = \frac{1}{\sqrt{1+a^2}}du$$

$$\text{Integrating, } \frac{z^2}{2} = \frac{1}{\sqrt{1+a^2}}u + b$$

$$\frac{z^2}{2} = \frac{1}{\sqrt{1+a^2}}(X+ay) + b$$

$$\frac{z^2}{2} = \frac{1}{\sqrt{1+a^2}}(\log x+ay) + b$$

$$\sqrt{1+a^2}z^2 = 2(\log x + ay) + c, \text{ where } c = 2b$$

which is the complete integral.

Example 1.94. *Solve* $p^2x^4 + y^2zq = 2z^2$

Solution: The given equation can be written as $(x^2p)^2 + (y^2q)z = 2z^2(1)$
This is of the form $f(x^mp, y^nq, z) = 0$ with $m = 2, n = 2$. [Type II]
Put $X = x^{1-m}$ and $Y = y^{1-n}$
i.e., $X = \dfrac{1}{x}$ and $Y = \dfrac{1}{y}$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x} = \frac{\partial z}{\partial X}\left(-\frac{1}{x^2}\right) = -\frac{\mathrm{P}}{\mathrm{x}^2}, \text{whereP} = \frac{\partial z}{\partial X}$$

$\therefore x^2p = -P$
Similarly $y^2q = -Q$, where $Q = \dfrac{\partial z}{\partial Y}$

$$\therefore (1) \text{ becomes } P^2 - Qz = 2z^2 \tag{2}$$

This is of the form $F(P, Q, z) = 0$ [Type II]
Let $z = \phi(X + aY)$ be a solution of (2)
Let $u = X + aY$ so that $z = \phi(u)$
Then $P = \dfrac{dz}{du}$ and $Q = a\dfrac{dz}{du}$

$\therefore$ (2) becomes $\left(\dfrac{dz}{du}\right)^2 - a\left(\dfrac{dz}{du}\right)z = 2z^2$

$$\left(\frac{dz}{du}\right)^2 - az\left(\frac{dz}{du}\right) - 2z^2 = 0$$

$$\frac{dz}{du} = \frac{az \pm \sqrt{a^2z^2 + 8z^2}}{2}$$

$$\frac{dz}{du} = z\left[\frac{a \pm \sqrt{a^2+8}}{2}\right]$$

$$\frac{dz}{z} = \left[\frac{a \pm \sqrt{a^2+8}}{2}\right]du$$

$$\text{Integrating, } \log z = \left[\frac{a \pm \sqrt{a^2+8}}{2}\right]u + b$$

$$\log z = \left[\frac{a \pm \sqrt{a^2+8}}{2}\right](X + aY) + b$$

$$\log z = \left[\frac{a \pm \sqrt{a^2+8}}{2}\right]\left(\frac{1}{x} + \frac{a}{y}\right) + b$$

which is the required complete integral.

Example 1.95. *Solve* $x^4p^2 - yzq = z^2$

Solution: The given equation can be written as $(x^2p)^2 - (yq)z = z^2$ (1)
This is of the form $f\left(x^mp, y^nq, z\right) = 0$ with $m = 2, n = 1$.
Put $X = x^{1-m}$ and $Y = \log y$
i.e., $X = \dfrac{1}{x}$ and $Y = \log y$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x} = \frac{\partial z}{\partial X}\left(-\frac{1}{x^2}\right) = \mathrm{P}\left(-\frac{1}{x^2}\right), \text{whereP} = \frac{\partial z}{\partial X}$$

$\therefore x^2p = -P$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial y} = \frac{\partial z}{\partial Y}\left(\frac{1}{y}\right) = Q\left(\frac{1}{y}\right), \text{whereP} = \frac{\partial z}{\partial Y}$$

i.e., $yq = Q$

$\therefore$ (1) becomes $P^2 - Qz = z^2$ (2)

This is of the form $F(P, Q, z) = 0$ [Type II]
Let $z = \phi(X + aY)$ be a solution of (2)
Let $u = X + aY$ so that $z = \phi(u)$
Then $P = \dfrac{dz}{du}$ and $Q = a\dfrac{dz}{du}$

$\therefore$ (2) becomes $\left(\frac{dz}{du}\right)^2 - a\left(\frac{dz}{du}\right)z = z^2$

$$\left(\frac{dz}{du}\right)^2 - az\left(\frac{dz}{du}\right) - z^2 = 0$$

$$\frac{dz}{du} = \frac{az \pm \sqrt{a^2z^2 + 4z^2}}{2}$$

$$\frac{dz}{du} = z\left(\frac{a \pm \sqrt{a^2+4}}{2}\right)$$

$$\frac{dz}{z} = \left(\frac{a \pm \sqrt{a^2+4}}{2}\right)du$$

$$\text{Integrating, } \log z = \left[\frac{a \pm \sqrt{a^2+4}}{2}\right]u + b$$

$$\log z = \left[\frac{a \pm \sqrt{a^2+4}}{2}\right](X + aY) + b$$

$$\log z = \left[\frac{a \pm \sqrt{a^2+4}}{2}\right]\left(\frac{1}{x} + a\log y\right) + b$$

which is the required complete integral.

Example 1.96. *Solve* $z^2(p^2+q^2) = x^2+y^2$

Solution: The given equation can be written as $(zp)^2 + (zq)^2 = x^2 + y^2$ (1)
Put $Z = z^{k+1}$, where $k = 1$ (Type C)
i.e., $Z = z^2$
$\frac{\partial Z}{\partial x} = 2z\frac{\partial z}{\partial x}$
i.e., $P = 2zp$, where $P = \frac{\partial Z}{\partial x} \Rightarrow zp = \frac{P}{2}$
Similarly, $zq = \frac{Q}{2}$, where $Q = \frac{\partial Z}{\partial y}$

$$\therefore \text{(1)becomes } \frac{P^2}{4} + \frac{Q^2}{4} = x^2 + y^2$$
$$P^2 + Q^2 = 4\left(x^2 + y^2\right)$$
$$P^2 - 4x^2 = 4y^2 - Q^2$$
$$\text{Take } P^2 - 4x^2 = 4y^2 - Q^2 = 4a^2 \text{ (say)} \qquad [Type IV]$$
$$\text{Now } P^2 - 4x^2 = 4a^2$$
$$\Rightarrow P = 2\sqrt{x^2 + a^2}$$
$$\therefore f_1(x, a) = 2\sqrt{x^2 + a^2}$$
$$\text{Now, } 4y^2 - Q^2 = 4a^2$$
$$\Rightarrow Q = 2\sqrt{y^2 - a^2}$$
$$\therefore f_2(y, a) = 2\sqrt{y^2 - a^2}$$
$$\therefore Z = \int f_1(x, a)\, dx + \int f_2(y, a)\, dy + b$$
$$Z = 2\int \sqrt{x^2 + a^2} dx + 2\int \sqrt{y^2 - a^2} dx + b$$
$$Z = 2\left[\frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right)\right] + 2\left[\frac{y}{2}\sqrt{y^2-a^2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{y}{a}\right)\right] + b$$
$$z^2 = x\sqrt{x^2 + a^2} + a^2\sinh^{-1}\left(\frac{x}{a}\right) + y\sqrt{y^2 - a^2} - a^2\cosh^{-1}\left(\frac{y}{a}\right) + b$$

which is the required complete integral.

Example 1.97. *Solve* $p^2 + q^2 = z^2(x^2 + y^2)$

Solution: The given equation can be written as

$$\left(\frac{p}{z}\right)^2 + \left(\frac{q}{z}\right)^2 = x^2 + y^2$$
$$\left(z^{-1}p\right)^2 + \left(z^{-1}q\right)^2 = x^2 + y^2 \tag{1}$$

Put $Z = \log z \qquad [\because k = -1] \qquad (Type C)$

$$\frac{\partial Z}{\partial x} = \frac{1}{z}\frac{\partial z}{\partial x}$$

i.e., $P = z^{-1}p$, where $P = \dfrac{\partial Z}{\partial x}$

Similarly, $Q = z^{-1}q$, where $Q = \dfrac{\partial Z}{\partial y}$

$$\therefore (1)\text{becomes } P^2 + Q^2 = x^2 + y^2$$
$$P^2 - x^2 = y^2 - Q^2$$
$$\text{Take } P^2 - x^2 = y^2 - Q^2 = a^2 \text{ (say)} \qquad [TypeIV]$$
$$P^2 - x^2 = a^2$$
$$\Rightarrow P = \sqrt{x^2 + a^2}$$
$$\therefore f_1(x, a) = \sqrt{x^2 + a^2}$$
$$\text{Now, } y^2 - Q^2 = a^2$$
$$\Rightarrow Q = \sqrt{y^2 - a^2}$$
$$\therefore f_2(y, a) = \sqrt{y^2 - a^2}$$
$$\therefore Z = \int f_1(x, a)\, dx + \int f_2(y, a)\, dy + b$$
$$Z = \int \sqrt{x^2 + a^2} dx + \int \sqrt{y^2 - a^2} dy + b$$
$$Z = \left[\frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right)\right] + \left[\frac{y}{2}\sqrt{y^2-a^2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{y}{a}\right)\right] + b$$
$$\log z = \frac{1}{2}\left[x\sqrt{x^2 + a^2} + a^2\sinh^{-1}\left(\frac{x}{a}\right) + y\sqrt{y^2 - a^2} - a^2\cosh^{-1}\left(\frac{y}{a}\right)\right] + b$$

which is the required complete integral.

1.7.11 Exercises & Solutions

Solve the following partial differential equations:

1. $\dfrac{x^2}{p} + \dfrac{y^2}{q} = z$ $\left[\text{Ans: } \dfrac{3}{2}z^2 = \dfrac{x^3}{a} + \dfrac{y^3}{1-a} + b\right]$

2. $2x^4p^2 - yzq - 3z^2 = 0$ $\left[\text{Ans: } \log z = \dfrac{a \pm \sqrt{a^2+24}}{2}\left(\dfrac{1}{x} + a\log y\right) + y\right]$

3. $xp + yq = 1$ $\left[\text{Ans: } z = \dfrac{1}{1+a}(\log x + a\log y) + b\right]$

4. $\dfrac{p}{x^2} + \dfrac{q}{y^2} = z$ $\left[\text{Ans: } \log z = \dfrac{1}{3(1+a)}\left(x^3 + ay^3\right) + b\right]$

5. $q^2y^2 = z(z - px)$ $\left[\text{Ans: } \log z = \dfrac{-1 \pm \sqrt{1+4a^2}}{2a^2}(\log x + alogy) + b\right]$

1.8 Homogeneous Linear Partial Differential Equation

A linear partial differential equation in which all the partial derivatives are of the same order is called homogeneous linear partial differential equation. Otherwise it is called non homogeneous linear partial equation.

A homogeneous linear partial differential equation of n^{th} order with constant coefficients is of the form

$$a_0\frac{\partial^n z}{\partial x^n} + a_1\frac{\partial^n z}{\partial x^{n-1}\partial y} + \cdots + a_n\frac{\partial^n z}{\partial y^n} = F(x,y) \tag{1}$$

where $a_0, a_1, \ldots a_n$ are constants.

This equation also can be written in the form

$$(a_0D^n + a_1D^{n-1}D' + \cdots + a_nD'^n)z = F(x,y)$$

where $D = \dfrac{\partial}{\partial x}, D' = \dfrac{\partial}{\partial y}$.

i.e., $f(D,D')z = F(x,y)$ (2)

The solution of $f(D,D')z = 0$ is called the complementary function of (2). The particular solution of (2) is called the particular integral function of (2) given symbolically by

$$\text{P.I.} = \frac{F(x,y)}{f(D,D')}$$

Hence the complete solution z = complementary function + particular integral.

i.e., Complete solution z = C.F. + P.I.

1.8.1 To find the complementary function (C.F)

Put $D = m$ and $D' = 1$ in $f(D,D') = 0$, then we get an equation, which is called the auxillary equation of (2).

$\therefore$ the auxillary equation of (2) is $f(m,1) = 0$

i.e., $a_0m^n + a_1m^{n-1} + \cdots + a_n = 0$

Let the roots of this equation be $m_1, m_2, \ldots, m_n$.

Case (i): If the roots are real (or imaginary) and distinct. Then

$C.F. = f_1(y+m_1x) + f_2(y+m_2x) + \cdots + f_n(y+m_nx)$

Case (ii):

(a) If any two roots are equal (ie, $m_1 = m_2 = m$) and others are distinct, Then $C.F. = f_1(y+mx)+xf_2(y+m_2x)+f_3(y+m_3x)+\cdots+f_n(y+m_nx)$.

(b) If any three roots are equal (i.e., $m_1 = m_2 = m_3 = m$)and others are distinct. Then

$$C.F. = f_1(y+mx) + xf_2(y+mx) + x^2f_3(y+mx) + f_4(y+m_4x) + \cdots + f_n(y+m_nx).$$

1.8.2 To find Particular Integral (P.I)

Type 1 : If $F(x,y) = e^{ax+by}$, then

$$\begin{aligned} \text{P.I.} &= \frac{1}{f(D,D')} e^{ax+by} \\ &= \frac{1}{f(a,b)} e^{ax+by} \text{ provided } f(a,b) \neq 0. \end{aligned}$$

If $f(a,b) = 0$, then

$$\text{P.I.} = x.\frac{1}{f'(D,D')} e^{ax+by}$$

where $f'(D,D')$ is the partial derivative of $f(D,D')$ w.r.t D.

Type 2 : If $F(x,y) = sin(ax+by)$, then

$$P.I. = \frac{1}{f(D,D')} \sin(ax+by)$$

Replace D^2 by $-a^2$, D'^2 by $-b^2$ and DD' by $-ab$ in $f(D,D')$ provided the denominator is not equal to zero.
If the denominator is zero, then

$$\text{P.I.} = x\frac{1}{f'(D,D')} \sin(ax+by)$$

where $f'(D,D')$ is the partial derivative of $f(D,D')$ w.r.t D. **Note :** Similar formula holds for $F(x,y) = cos(ax+by)$.

Type 3 : If $F(x,y) = x^m y^n$, then

$$\begin{aligned} \text{P.I.} &= \frac{1}{f(D,D')} x^m y^n \\ &= [f(D,D')]^{-1} x^m y^n \end{aligned}$$

Expand $[f(D,D')]^{-1}$ in ascending powers of D, D' and then operate on $x^m y^n$.

Note : In $x^m y^n$ if $m > n$, then try to write $f(D,D')$ as a function of $\frac{D'}{D}$ and if $m < n$, then try to write $f(D,D')$ as a function of $\frac{D}{D'}$.

Note : $\frac{1}{D}$ denotes integration w.r.t 'x', $\frac{1}{D'}$ denotes integration w.r.t 'y'.

Type 4 : If $F(x,y) = e^{ax+by}\phi(x,y)$, then

$$\begin{aligned}\text{P.I.} &= \frac{1}{f(D,D')}e^{ax+by}\phi(x,y)\\ &= e^{ax+by}\frac{1}{f(D+a,D'+b)}\phi(x,y)\end{aligned}$$

Type 5 : If $F(x,y)$ is any function, resolve $f(D,D')$ into linear factors say $(D-m_1D'),(D-m_2D'),\ldots,(D-m_nD')$, then

$$\text{P.I.} = \frac{1}{(D-m_1D')(D-m_2D')\ldots(D-m_nD')}F(x,y)$$

Note : (1) $\dfrac{1}{D-mD'}F(x,y) = \displaystyle\int F(x,c-mx)dx$ where $y=c-mx$

(2) $\dfrac{1}{D+mD'}F(x,y) = \displaystyle\int F(x,c+mx)dx$ where $y=c+mx$

1.8.3 Examples of Homogeneous Linear P.D.E.

Example 1.98. *Solve* $\left(D^2-5DD'+6D'^2\right)z=0$

Solution: The auxillary equation is

$$\begin{aligned}m^2-5m+6&=0 \qquad [\text{Replace } D \text{ by } m \text{ and } D' \text{ by } 1]\\ (m-2)(m-3)&=0\\ \text{i.e., } m&=2,3\\ \text{C.F.} &= f_1(y+2x)+f_2(y+3x)\end{aligned}$$

$\therefore$ The solution is $z=f_1(y+2x)+f_2(y+3x)$

Example 1.99. *Solve* $\dfrac{\partial^3 z}{\partial x^3}-4\dfrac{\partial^3 z}{\partial x^2\partial y}+4\dfrac{\partial^3 z}{\partial x\partial y^2}=0$

Solution: Given $(D^3-4D^2D'+4DD'2)z=0$
The auxillary equation is

$$\begin{aligned}m^3-4m^2+4m&=0\\ \text{i.e., } m(m-2)^2&=0\\ \text{i.e., } m&=0,2,2\\ \text{C.F.} &= f_1(y+0x)+f_2(y+2x)+xf_3(y+2x)\end{aligned}$$

$\therefore$ The solution is $z=f_1(y+0x)+f_2(y+2x)+xf_3(y+2x)$

Example 1.100. *Solve the equation* $\dfrac{\partial^2 z}{\partial x^2}-3\dfrac{\partial^2 z}{\partial x\partial y}+2\dfrac{\partial^2 z}{\partial y^2}=0$

Solution: Given $(D^2-3DD'+2D'2)z=0$
The auxillary equation is

$$\begin{aligned} m^2 + 3m + 2 &= 0 \\ (m-1)(m-2) &= 0 \\ \text{i.e., } m &= 1, 2 \\ \text{C.F. } &= f_1(y+x) + f_2(y+2x) \end{aligned}$$

$\therefore$ The solution is $z = f_1(y+x) + f_2(y+2x)$

Example 1.101. *Solve the equation* $\left(2D^2 + 5DD' + 2D'^2\right) z = 0$

Solution: The auxillary equation is

$$\begin{aligned} 2m^2 + 5m + 2 &= 0 \\ (2m+1)(m+2) &= 0 \\ \text{i.e., } m &= \frac{-1}{2}, -2 \\ \text{C.F. } &= f_1\left(y - \frac{1}{2}x\right) + f_2(y-2x) \end{aligned}$$

$\therefore$ The solution is $f_1\left(y - \frac{1}{2}x\right) + f_2(y-2x)$

Example 1.102. *Solve* $\left(D^2 - 2DD' + D'^2\right) z = e^{x+2y}$

Solution: The auxillary equation is

$$\begin{aligned} m^2 - 2m + 1 &= 0 \\ (m-1)^2 &= 0 \\ \text{i.e., } m &= 1, 1 \end{aligned}$$

$\therefore$ C.F. is $f_1\left(y-x\right) + xf_2(y-x)$

$$\begin{aligned} \text{P.I. } &= \frac{1}{D^2 - 2DD' + D'^2} e^{x+2y} \\ &= \frac{1}{1-4+4} e^{x+2y} \\ &= e^{x+2y} \end{aligned}$$

$\therefore$ The complete solution $z =$ C.F. + P.I.

$$\text{i.e., } z = f_1(y+x) + xf_2(y+x) + e^{x+2y}.$$

Example 1.103. *Solve* $\left(D^2 - D'^2\right) z = e^{x+2y}$

Solution: The auxillary equation is

$$\begin{aligned} m^2 - 1 &= 0 \\ (m-1)(m+1) &= 0 \\ \text{i.e., } m &= 1, -1 \end{aligned}$$

$\therefore$ C.F. is $f_1\,(y+x) + f_2(y-x)$

$$\begin{aligned}\text{P.I.} &= \frac{1}{D^2 - D'^2} e^{x+2y} \\ &= \frac{1}{1-4} e^{x+2y} \\ &= -\frac{1}{3} e^{x+2y}\end{aligned}$$

$\therefore$ The complete solution $z =$ C.F. + P.I.

$$\text{i.e., } z = f_1(y+x) + f_2(y-x) - \frac{1}{3} e^{x+2y}.$$

Example 1.104. *Solve* $\left(D^2 - 7DD'^2 + 12D'^2\right) z = e^{x-y}$

Solution: The auxillary equation is

$$\begin{aligned} m^2 - 7m + 12 &= 0 \\ (m-3)(m-4) &= 0 \\ \text{i.e., } m &= 3, 4\end{aligned}$$

$\therefore$ C.F. is $f_1(y+3x) + f_2(y+4x)$

$$\begin{aligned}\text{P.I.} &= \frac{1}{D^2 - 7DD' + 12D'^2} e^{x-y} \\ &= \frac{1}{1+7+12} . e^{x-y} \\ &= \frac{1}{20} e^{x-y}\end{aligned}$$

$\therefore$ The complete solution $z =$ C.F. + P.I.

$$\text{i.e., } z = f_1(y+3x) + f_2(y+4x) + \frac{1}{20} e^{x-y}.$$

Example 1.105. *Solve* $\left(6D^2 + 6DD' + D'^2\right) z = \left(e^x + e^{-2y}\right)^2$

Solution: The auxillary equation is

$$\begin{aligned} 6m^2 + 6m + 1 &= 0 \\ \text{i.e., } m &= \frac{-6 \pm \sqrt{36-24}}{12} \\ m &= \frac{-3 \pm \sqrt{3}}{6}\end{aligned}$$

$\therefore$ C.F. is $f_1(y + (\frac{-3+\sqrt{3}}{6})x) + f2(y - (\frac{3+\sqrt{3}}{6})x)$

$$
\begin{aligned}
\text{P.I.} &= \frac{1}{6D^2+6DD'+D'^2}(e^x+e^{-2y})^2 \\
&= \frac{1}{6D^2+6DD'+D'^2}(e^{2x}+e^{-4y}+2e^{x-2y}) \\
&= \frac{e^{2x+0y}}{6D^2+6DD'+D'^2}+\frac{e^{0x-4y}}{6D^2+6DD'+D'^2}+\frac{2e^{x-2y}}{6D^2+6DD'+D'^2} \\
&= \frac{1}{24}e^{2x}+\frac{1}{16}e^{-4y}-e^{x-2y}
\end{aligned}
$$

$\therefore$ The complete solution $z =$ C.F. + P.I.

Example 1.106. *Solve* $(D+D')^2 z = e^{x-y}$

Solution: The auxillary equation is

$$
\begin{aligned}
m^2+2m+1 &= 0 \\
\text{i.e., } m &= -1,-1
\end{aligned}
$$

$\therefore$ C.F. is $f_1(y-x)+xf_2(y-x)$

$$
\begin{aligned}
\text{P.I.} &= \frac{1}{D^2+2DD'-D'^2}e^{x-y} \\
&= \frac{1}{0}e^{x-y} \\
\therefore \text{P.I.} &= x\frac{1}{2D+2D'}e^{x-y} \\
&= x\frac{1}{0}e^{x-y} \\
\therefore \text{P.I.} &= x^2\frac{1}{2}e^{x-y}
\end{aligned}
$$

$\therefore$ The complete solution $z =$ C.F. + P.I.

$$\text{i.e., } z = f_1(y-x)+xf_2(y-x)+\frac{x^2}{2}e^{x-y}$$

Example 1.107. *Solve* $\left(D^4-D'^4\right)^2 z = e^{x+y}$

Solution: The auxillary equation is

$$
\begin{aligned}
m^4-1 &= 0 \\
\text{i.e., } (m^2-1)(m^2+1) &= 0 \\
\text{i.e., } m &= 1,-1,i,-i
\end{aligned}
$$

$\therefore$ C.F. is $f_1(y+x)+f_2(y-x)+f_3(y+ix)+f_4(y-ix)$

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^4 - D'^4} e^{x+y} \\ &= \frac{1}{0}.e^{x+y} \\ \therefore \text{P.I.} &= x\frac{1}{4D^3} e^{x+y} \\ &= \frac{1}{4} x e^{x+y} \end{aligned}$$

$\therefore$ The complete solution $z = \text{C.F.} + \text{P.I.}$
i.e., $z = f_1(y+x) + f_2(y-4x) + f_3(y+ix) + f_4(y-ix) + \frac{1}{4} x e^{x+y}$

Example 1.108. *Solve* $\left(D^3 - 3DD'^2 + 2D'^3\right) z = e^{x+y}$

Solution: The auxillary equation is

$$\begin{aligned} m^3 - 3m + 2 = 0 \\ \text{i.e., } m = 1, 1, -2 \end{aligned}$$

$\therefore$ C.F. is $f_1(y+x) + x f_2(y+x) + f_3(y-2x)$

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^3 - 3DD'^2 + 2D'^3} e^{x+y} \\ &= \frac{1}{0}.e^{x+y} \\ \therefore \text{P.I.} &= x\frac{1}{3D^2 - 3D'^2} e^{x+y} = x\frac{1}{0} e^{x+y} \\ \text{P.I.} &= x^2 \frac{1}{6D} e^{x+y} = x^2 \frac{1}{6} e^{x+y} \end{aligned}$$

$\therefore$ The complete solution $z = \text{C.F.} + \text{P.I.}$
i.e., $z = f_1(y+x) + x f_2(y+x) + f_3(y-2x) + x^2 \frac{1}{6} e^{x+y}$

Example 1.109. *Solve the equation* $\left(D^2 + 3DD' + 2D'^2\right) z = e^x \cosh y$

Solution: The auxillary equation is

$$\begin{aligned} m^2 + 3m + 2 = 0 \\ \text{i.e., } m = -1, -2 \end{aligned}$$

$\therefore$ C.F. is $f_1(y-x) + f_2(y-2x)$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^2+3DD'+2D'^2} e^x \cosh y \\
&= \frac{1}{D^2+3DD'+2D'^2} e^x \left(\frac{e^y+e^{-y}}{2}\right) \\
&= \frac{1}{2}\left[\frac{1}{D^2+3DD'+2D'^2} e^{x+y} + \frac{1}{D^2+3DD'+2D'^2} e^{x-y}\right] \\
&= \frac{1}{2}\left[\frac{1}{6} e^{x+y} + \frac{1}{0} e^{x-y}\right]
\end{aligned}$$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{2}\left[\frac{e^{x+y}}{6} + x\frac{1}{2D+3D'} e^{x-y}\right] \\
&= \frac{1}{2}\left[\frac{e^{x+y}}{6} + x\frac{1}{-1} e^{x-y}\right] \\
\therefore \text{P.I.} &= \frac{1}{2}\left[\frac{e^{x+y}}{6} - x e^{x-y}\right]
\end{aligned}$$

$\therefore$ The complete solution $z = \text{C.F.} + \text{P.I.}$

Example 1.110. *Solve* $\left(D^2+9D'^2\right) z = \cos(2x+3y)$

Solution: The auxillary equation is

$$\begin{aligned}
m^2+9 &= 0 \\
m^2 &= -9 \\
\text{i.e., } m &= \pm 3i
\end{aligned}$$

$\therefore$ C.F. is $f_1(y+3ix) + f_2(y-3ix)$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^2+9D'^2} \cos(2x+3y) \\
&= \frac{1}{-4-81} \cos(2x+3y) \qquad \left[\text{Replacing } D^2 \text{ by } -4, D'^2 \text{ by } -9\right] \\
&= -\frac{1}{85} \cos(2x+3y)
\end{aligned}$$

$\therefore$ The complete solution $z = \text{C.F.} + \text{P.I.}$

i.e., $z = f1(y+3ix) + f2(y-3ix) - \dfrac{1}{85}\cos(2x+3y)$

Example 1.111. *Solve* $\left(D^2-2DD'+D'^2\right) z = \sin(2x+3y)$

Solution: The auxillary equation is

$$\begin{aligned}
m^2-2m+1 &= 0 \\
m &= 1, 1
\end{aligned}$$

$\therefore$ C.F. is $f_1(y+x)+xf_2(y+x)$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^2-2DD'+D'^2}\sin(2x+3y)\\
&= \frac{1}{-4+12-9}\sin(2x+3y)\left[\text{Replacing } D^2\to -4, D'^2\to -9, DD'\to -6\right]\\
&= -sin(2x+3y)
\end{aligned}$$

$\therefore$ The complete solution $z=$ C.F. + P.I.
i.e., $z=f_1(y+x)+xf_2(y+x)-sin(2x+3y)$

Example 1.112. *Solve* $\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial y^2}=\cos 2x\cos 3y$

Solution: Given $\left(D^2+D'^2\right)z=\cos 2x\cos 3y$
The auxillary equation is

$$\begin{aligned}
m^2+1&=0\\
m^2&=-1\\
m&=i,-i
\end{aligned}$$

$\therefore$ C.F. is $f_1(y+ix)+f_2(y-ix)$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^2+D'^2}\cos 2x\cos 3y\\
&= \frac{1}{D^2+D'^2}\frac{1}{2}\left(\cos(2x+3y)+\cos(2x-3y)\right)\\
&= \frac{1}{2}\left(\frac{1}{D^2+D'^2}\cos(2x+3y)+\frac{1}{D^2+D'^2}\cos(2x-3y)\right)\\
&= \frac{1}{2}\left(\frac{1}{-4-9}\cos(2x+3y)+\frac{1}{-4-9}\cos(2x-3y)\right)\\
&= -\frac{1}{26}\left(\cos(2x+3y)+\cos(2x-3y)\right)\\
&= -\frac{1}{26}\left(2\cos\left(\frac{4x}{2}\right)+\cos\left(\frac{6y}{2}\right)\right)\\
&= -\frac{1}{13}\left(\cos 2x cos 3y\right)
\end{aligned}$$

$\therefore$ The complete solution $z=$ C.F. + P.I.
Aliter :

$$
\begin{aligned}
\text{P.I.} &= \frac{1}{f(D^2+D'^2)}\sin ax\sin by \text{ (or) } \cos ax\cos by \\
&= \frac{1}{f(-a^2,-b^2)}\sin ax\sin by \text{ (or) } \cos ax\cos by \text{ provided } f\left(-a^2,-b^2\right)\neq 0 \\
\therefore \text{P.I.} &= \frac{1}{D^2+D'^2}\cos 2x\cos 3y \\
&= \frac{1}{-4-9}\cos 2x\cos 3y \\
\therefore \text{P.I.} &= -\frac{1}{13}\cos 2x\cos 3y
\end{aligned}
$$

Example 1.113. *Solve* $\left(D^2+DD'-6D'^2\right)z=\cos(2x+y)$

Solution: The auxillary equation is

$$
\begin{aligned}
m^2+m-6&=0 \\
\text{i.e., } (m-2)(m+3)&=0 \\
\text{i.e., } m&=2,-3
\end{aligned}
$$

$\therefore$ C.F. is $f_1(y+2x)+f_2(y-3x)$

$$
\begin{aligned}
\text{P.I.} &= \frac{1}{D^2+DD'-6D'^2}\cos(2x+y) \\
&= \frac{1}{-4-2+6}\cos(2x+y)\left[\text{Replacing } D^2\to -4, D'^2\to -1, DD'\to -2\right] \\
&= \frac{1}{0}\cos(2x+y) \\
\therefore \text{P.I.} &= x\frac{1}{2D+D'}\cos(2x+y) \\
\text{P.I.} &= x\frac{2D-D'}{4D^2-D'^2}\cos(2x+y) \\
&= x\frac{2D-D'}{-16+1}\cos(2x+y) \qquad \left[\text{Replacing } D^2 \text{ by } -4, D'^2 \text{ by } -1\right] \\
&= -\frac{x}{15}\left[2D\cos(2x+y)-D'\cos(2x+y)\right] \\
&= -\frac{x}{15}\left[-4\sin(2x+y)+\sin(2x+y)\right] \\
&= \frac{x}{5}\sin(2x+y)
\end{aligned}
$$

$\therefore$ The complete solution $z=$ C.F. + P.I.

Example 1.114. *Write the P.I. of* $\left(D^2+DD'\right)z=sin(x+y)$

Solution:

$$
\begin{aligned}
\text{P.I.} &= \frac{1}{D^2+DD'}\sin(x+y)\\
&= \frac{1}{-1-1}\sin(x+y) \qquad \left[\text{Replacing } D^2 \text{ by } -1, DD' \text{ by } -1\right]\\
\therefore \text{P.I.} &= -\frac{1}{2}\sin(x+y)
\end{aligned}
$$

Example 1.115. *Solve the equation* $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial x \partial y} = \sin x \cos 2y$

Solution: Given $\left(D^2 - DD'\right) z = \sin x \cos 2y$
The auxillary equation is

$$
\begin{aligned}
m^2 - m &= 0\\
\text{i.e., } m &= 0, 1
\end{aligned}
$$

$$
\begin{aligned}
\therefore \text{C.F.} &= f_1(y+0x) + f_2(y+x)\\
&= f_1(y) + f_2(y+x)
\end{aligned}
$$

$$
\begin{aligned}
\text{P.I.} &= \frac{1}{D^2-DD'}\sin x cos 2y\\
&= \frac{1}{D^2-DD'}.\frac{1}{2}\left(\sin(x+2y)+\sin(x-2y)\right)\\
&= \frac{1}{2}\left[\frac{1}{D^2-DD'}\sin(x+2y) + \frac{1}{D^2-DD'}\sin(x-2y)\right]\\
&= \frac{1}{2}\left[\frac{1}{-1+2}\sin(x+2y) + \frac{1}{-1-2}\sin(x-2y)\right]\\
&= \frac{1}{2}\left[sin(x+2y) - \frac{1}{3}\sin(x-2y)\right]
\end{aligned}
$$

$\therefore$ The complete solution $z =$ C.F. + P.I.

Example 1.116. *Solve* $\left(D^2 + 3DD' - 4D'^2\right) z = \cos(2x+y)$

Solution: The auxillary equation is

$$
\begin{aligned}
m^2 + 3m - 4 &= 0\\
\text{i.e., } (m-1)(m+4) &= 0\\
\text{i.e., } m &= 1, -4
\end{aligned}
$$

$$
\therefore \text{ C.F. } = f_1(y+x) + f_2(y-4x)
$$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^2+3DD'-4D'^2}\sin y \\
&= \frac{1}{D^2+3DD'-4D'^2}\sin(0x+y) \\
&= \frac{1}{0+0+4}\sin(0x+y) \quad \left[\text{Replacing } D^2 \text{ by } 0, D'^2 \text{ by } -1, DD' \text{ by } 0\right] \\
\text{P.I.} &= \frac{1}{4}\sin y
\end{aligned}$$

$\therefore$ The complete solution $z =$ C.F. + P.I.

Example 1.117. *Solve* $\left(D^3+D^2D'-DD'^2-D'^3\right)z = 3\sin(x+y)$

Solution: The auxillary equation is

$$m^3+m^2-m-1=0$$
$$\text{i.e., } m = 1,-1,-1$$

$$\begin{aligned}
\therefore\ \text{C.F.} &= f_1(y+x)+f_2(y-x)+xf_3(y-x) \\
\text{P.I.} &= \frac{1}{D^3+D^2D'-DD'^2-D'^3}3\sin(x+y) \\
&= \frac{1}{-D-D'+D+D'}3\sin(x+y) \\
&\quad \left[\text{Replacing } D^2 \text{ by } -1, D'^2 \text{ by } -1, DD' \text{ by } -1\right] \\
&= \frac{1}{0}3\sin(x+y) \\
\therefore \text{P.I.} &= x\frac{1}{3D^2+2DD'-D'^2}3\sin(x+y) \\
&= x\frac{1}{-3-2+1}3\sin(x+y) \\
&\quad \left[\text{Replacing } D^2 \text{ by } -1, D'^2 \text{ by } -1, DD' \text{ by } -1\right] \\
&= -\frac{3}{4}x\sin(x+y)
\end{aligned}$$

$\therefore$ The complete solution $z =$ C.F. + P.I.

Example 1.118. *Solve* $\left(D^3+D^2D'-2DD'^2\right)z = \sin(x-2y)$

Solution: The auxillary equation is

$$m^3+m^2-2m=0$$
$$m(m2+m-2)=0$$
$$\text{i.e., } m = 0,1,-2$$

$$\therefore \text{ C.F. } = f_1(y) + f_2(y+x) + f_3(y-2x)$$

$$\begin{aligned}
\text{P.I. } &= \frac{1}{D^3 + D^2D' - 2DD'^2} sin(x-2y) \\
&= \frac{1}{-D - D' + 8D} \sin(x-2y) \\
&\quad \left[\text{Replacing } D^2 \text{ by } -1, D'^2 \text{ by } -4\right] \\
&= \frac{1}{7D - D'} \sin(x-2y) \\
&= \frac{7D + D'}{-49+4} \sin(x-2y) \\
&\quad \left[\text{Replacing } D^2 \text{ by } -1, D'^2 \text{ by } -4\right] \\
&= -\frac{1}{45}\left[7D\sin(x-2y) + D'\sin(x-2y)\right] \\
&= -\frac{1}{45}\left[7\cos(x-2y) - 2\cos(x-2y)\right] \\
\text{P.I. } &= -\frac{1}{9}\cos(x-2y)
\end{aligned}$$

$\therefore$ The complete solution $z =$ C.F. + P.I.

i.e., $z = f1(y) + f2(y+x) + f3(y-2x) - \dfrac{1}{9}\cos(x-2y)$.

Example 1.119. *Solve* $\left(D^2 - 3DD' + 2D'^2\right) z = \sin x \cos y$

Solution: The auxillary equation is

$$\begin{aligned}
m^2 - 3m + 2 &= 0 \\
(m-1)(m-2) &= 0 \\
\text{i.e., } m &= 1, 2
\end{aligned}$$

$$\therefore \text{ C.F. } = f_1(y+x) + f_2(y+2x)$$

$$\begin{aligned}
\text{P.I. } &= \frac{1}{D^2 - 3DD' + 2D'^2} \sin x \cos y \\
&= \frac{1}{D^2 - 3DD' + 2D'^2} \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right] \\
&= \frac{1}{2}\left[\frac{1}{D^2 - 3DD' + 2D'^2}.\sin(x+y) + \frac{1}{D^2 - 3DD' + 2D'^2}\sin(x-y)\right] \\
&= \frac{1}{2}\left[\frac{1}{-1+3-2}.\sin(x+y) + \frac{1}{-1-3-2}\sin(x-y)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\left[\frac{1}{0}.\sin(x+y) - \frac{1}{6}\sin(x-y)\right] \\
\text{P.I.} &= \frac{1}{2}\left[x\frac{1}{2D-3D'}.\sin(x+y) - \frac{1}{6}\sin(x-y)\right] \\
&= \frac{1}{2}\left[x\frac{2D+3D'}{4D^2-9D'^2}.\sin(x+y) - \frac{1}{6}\sin(x-y)\right] \\
&= \frac{1}{2}\left[x\frac{2D+3D'}{-4+9}.\sin(x+y) - \frac{1}{6}\sin(x-y)\right] \\
&= \frac{1}{2}\left[\frac{x}{5}\left[2D\sin(x+y) + 3D'\sin(x+y)\right] - \frac{1}{6}\sin(x-y)\right] \\
&= \frac{1}{2}\left[\frac{x}{5}\left[2\cos(x+y) + 3\cos(x+y)\right] - \frac{1}{6}\sin(x-y)\right] \\
\text{P.I.} &= \frac{1}{2}\left[x\cos(x+y) - \frac{1}{6}\sin(x-y)\right]
\end{aligned}
$$

$\therefore$ The complete solution $z =$ C.F. + P.I.
i.e., $f_1(y+x) + f_2(y+2x) + \dfrac{1}{2}\left[x\cos(x+y) - \dfrac{1}{6}\sin(x-y)\right].$

Example 1.120. *Solve* $\left(D^2 + 3DD' + 2D'^2\right) z = x^2y^2$

Solution: The auxillary equation is

$$
\begin{aligned}
m^2 + 3m + 2 &= 0 \\
(m+1)(m+2) &= 0 \\
\text{i.e., } m &= -1, -2
\end{aligned}
$$

$$\therefore \text{ C.F. } = f_1(y-x) + f_2(y-2x)$$

$$
\begin{aligned}
\text{P.I.} &= \frac{1}{D^2 + 3DD' + 2D'^2}x^2y^2 \\
&= \frac{1}{D^2\left[1 + \frac{3D'}{D} + \frac{2D'^2}{D^2}\right]}x^2y^2 \\
&= \frac{1}{D^2}\left[1 + \left(\frac{3D'}{D} + \frac{2D'^2}{D^2}\right)\right]^{-1}x^2y^2
\end{aligned}
$$

$$= \frac{1}{D^2}\left[1 - \left(\frac{3D'}{D} + \frac{2D'^2}{D^2}\right) + \left(\frac{3D'}{D} + \frac{2D'^2}{D^2}\right)^2 - \ldots\right] x^2y^2$$
$$= \frac{1}{D^2}\left[1 - \frac{3D'}{D} - \frac{2D'^2}{D^2} + \frac{9D'^2}{D^2}\right] x^2y^2$$
$$= \frac{1}{D^2}\left[1 - \frac{3D'}{D} + \frac{7D'^2}{D^2}\right] x^2y^2$$
$$= \frac{1}{D^2}\left[x^2y^2 - \frac{3}{D}(2x^2y) + \frac{7}{D^2}(2x^2)\right]$$
$$= \frac{1}{D^2}(x^2y^2) - \frac{1}{D^3}(6x^2y) + \frac{1}{D^4}(14x^2)$$
$$= \frac{1}{12}x^4y^2 - \frac{1}{10}x^5y + \frac{7}{180}x^6$$
$$= \frac{1}{2}\left[\frac{x}{5}\left[2\cos(x+y) + 3\cos(x+y)\right] - \frac{1}{6}\sin(x-y)\right]$$
$$\text{P.I. } = \frac{1}{2}\left[x\cos(x+y) - \frac{1}{6}\sin(x-y)\right]$$
$$\left(\because \frac{1}{D^n}x^m = \frac{x^{m+n}}{(m+1)(m+1)\ldots(m+n)}\right)$$

$\therefore$ The complete solution $z =$ C.F. + P.I.

Example 1.121. *Solve* $\left(D^2 - 2DD' + 2D'^2\right) z = x^2y^2$

Solution: The auxillary equation is

$$m^2 - 2m + 2 = 0$$
$$m = \frac{2 \pm \sqrt{4-8}}{2}$$
$$\text{i.e., } m = 1 \pm i$$

$$\therefore \text{ C.F. } = f_1(y + (1+i)x) + f_2(y + (1-i)x)$$

$$\text{P.I. } = \frac{1}{D^2 - 2DD' + 2D'^2}x^3y$$
$$= \frac{1}{D^2\left[1 - \frac{2D'}{D} + \frac{2D'^2}{D^2}\right]}x^3y$$
$$= \frac{1}{D^2}\left[1 - \left(\frac{2D'}{D} - \frac{2D'^2}{D^2}\right)\right]^{-1} x^3y$$
$$= \frac{1}{D^2}\left[1 + \frac{2D'}{D}\right] x^3y = \frac{1}{D^2}\left[x^3y + \frac{2}{D}(x^3)\right]$$
$$= \frac{1}{D^2}(x^3y) + \frac{2}{D^3}(x^3) = \frac{x^5y}{20} + \frac{x^6}{60}$$

$\therefore$ The complete solution $z =$ C.F. + P.I.

Example 1.122. *Solve* $\left(D^2 + 6DD' + 5D'^2\right) z = xy^4$

Solution: The auxillary equation is

$$\begin{aligned} m^2 + 6m + 5 &= 0 \\ (m+1)(m+5) &= 0 \\ \text{i.e., } m &= -1, -5 \end{aligned}$$

$$\therefore \text{ C.F. } = f_1(y-x) + f_2(y-5x)$$

$$\begin{aligned} \text{P.I. } &= \frac{1}{D^2 + 6DD' + 5D'^2} xy^4 = \frac{1}{5D'^2\left[\frac{D^2}{5D'^2} + \frac{6}{5}\frac{D}{D'} + 1\right]} xy^4 \\ &= \frac{1}{5D'^2}\left[1 + \left(\frac{6D}{5D'} + \frac{D^2}{5D'^2}\right)\right]^{-1} xy^4 \\ &= \frac{1}{5D'^2}\left[1 - \left(\frac{6D}{5D'} + \frac{D^2}{5D'^2}\right) + \left(\frac{6D}{5D'} + \frac{D^2}{5D'^2}\right)^2 - \ldots\right] xy^4 \\ &= \frac{1}{5D'^2}\left[1 - \frac{6D}{5D'}\right] xy^4 = \frac{1}{5D'^2}\left[xy^4 - \frac{6y^4}{5D'}\right] \\ &= \frac{1}{5D'^2}(xy^4) - \frac{6}{25D'^3}(y^4) \\ &= \frac{xy^6}{150} - \frac{y^7}{875} \end{aligned}$$

$\therefore$ The complete solution $z =$ C.F. + P.I.

Example 1.123. *Find the general integral of the equation* $\dfrac{\partial^2 z}{\partial x^2} + 3\dfrac{\partial^2 z}{\partial x \partial y} + 2\dfrac{\partial^2 z}{\partial y^2} = x + y$

Solution: Given $\left(D^2 + 3DD' + 2D'^2\right) z = x + y$ The auxillary equation is

$$\begin{aligned} m^2 + 3m + 2 &= 0 \\ (m+1)(m+2) &= 0 \\ \text{i.e., } m &= -1, -2 \end{aligned}$$

$$\therefore \text{ C.F. } = f_1(y-x) + f_2(y-2x)$$

$$
\begin{aligned}
\text{P.I. } &= \frac{1}{D^2+3DD'+2D'^2}(x+y)\\
&= \frac{1}{D^2\left(1+\frac{3D'}{D}+\frac{2D'^2}{D^2}\right)}(x+y)\\
&= \frac{1}{D^2}\left[1+\left(\frac{3D'}{D}+\frac{2D'^2}{D^2}\right)\right]^{-1}(x+y)\\
&= \frac{1}{D^2}\left[1-\left(\frac{3D'}{D}+\frac{2D'^2}{D^2}\right)+\left(\frac{3D'}{D}+\frac{2D'^2}{D^2}\right)^2-\dots\right](x+y)\\
&= \frac{1}{D^2}\left[1-\frac{3D'}{D}\right](x+y) = \frac{1}{D^2}\left[x+y-\frac{3}{D}\right]\\
&= \frac{1}{D^2}(x)+\frac{1}{D^2}(y)-\frac{1}{D^3}(3)\\
&= \frac{x^3}{6}+\frac{x^2y}{2}-\frac{x^3}{2}
\end{aligned}
$$

$\therefore$ The complete solution $z = \text{C.F.} + \text{P.I.}$

Example 1.124. *Solve* $\left(D^2+4DD'-5D'^2\right)z = x+y^2$

Solution: The auxillary equation is

$$
\begin{aligned}
m^2+4m-5+2&=0\\
(m+5)(m-1)&=0\\
\text{i.e., } m&=1,-5
\end{aligned}
$$

$$\therefore \text{ C.F. } = f_1(y+x)+f_2(y-5x)$$

$$
\begin{aligned}
\text{P.I. } &= \frac{1}{D^2+4DD'-5D'^2}(x+y^2) = \frac{1}{-5D'^2\left(\frac{-D^2}{5D'}-\frac{4D}{5D'}+1\right)}(x+y^2)\\
&= -\frac{1}{5D'^2}\left[1-\left(\frac{4D}{5D'}+\frac{D^2}{5D'}\right)\right]^{-1}(x+y^2)\\
&= -\frac{1}{5D'^2}\left[1+\left(\frac{4D}{5D'}+\frac{D^2}{5D'}\right)+\left(\frac{4D}{5D'}+\frac{D^2}{5D'}\right)^2-\dots\right](x+y^2)\\
&= -\frac{1}{5D'^2}\left[1+\frac{4D}{5D'}\right](x+y^2) = -\frac{1}{5D'^2}\left[x+y^2+\frac{4}{5D'}\right]\\
&= -\frac{1}{5}\frac{1}{D'^2}(x)+\frac{1}{D'^2}(y^2)+\frac{4}{5}\frac{1}{D'^3}(1) = -\frac{1}{5}\left(\frac{xy^2}{2}+\frac{y^4}{12}+\frac{2y^3}{15}\right)
\end{aligned}
$$

$\therefore$ The complete solution $z = \text{C.F.} + \text{P.I.}$

i.e., $z = f_1(y+x)+f_2(y-5x)-\dfrac{1}{5}\left(\dfrac{1}{2}xy^2+\dfrac{1}{12}y^4+\dfrac{2}{15}y^3\right)$.

Example 1.125. *Solve* $\left(D^2 - 4DD' + 4D'^2\right) z = e^{x-2y}\cos(2x-y)$

Solution: The auxillary equation is

$$\begin{aligned} m^2 - 4m + 4 &= 0 \\ \text{i.e., } m &= 2, 2 \end{aligned}$$

$$\therefore \text{ C.F. } = f_1(y+2x) + xf_2(y+2x)$$

$$\begin{aligned} \text{P.I. } &= \frac{1}{D^2 - 4DD' + 4D'^2} e^{x-2y}\cos(2x-y) \\ &= e^{x-2y}\frac{1}{(D+1)^2 - 4(D+1)(D'-2) + 4(D'-2)^2}\cos(2x-y) \\ &= e^{x-2y}\frac{1}{D^2 + 10D - 4DD' - 20D' + 4D'^2 + 25}\cos(2x-y) \\ &= e^{x-2y}\frac{1}{-4 + 10D - 8 - 20D' - 4 + 25}\cos(2x-y) \\ &= e^{x-2y}\frac{1}{10D - 20D' + 9}\cos(2x-y) \\ &= e^{x-2y}\frac{((10D - 20D') - 9)\cos(2x-y)}{(10D - 20D')^2 - 81} \\ &= e^{x-2y}\frac{(10D - 20D' - 9)\cos(2x-y)}{100D^2 - 400DD' + 400D'^2 - 81} \\ &= e^{x-2y}\frac{(10D - 20D' - 9)}{-400 - 800 - 400 - 81}\cos(2x-y) \\ &= -\frac{e^{x-2y}}{1681}\left(-20\sin(2x-y) - 20\sin(2x-y) - 9\cos(2x-y)\right) \\ \text{P.I. } &= \frac{1}{1681}e^{x-2y}\left[40\sin(2x-y) + 9\cos(2x-y)\right] \end{aligned}$$

$\therefore$ The complete solution $z =$ C.F. + P.I.

Example 1.126. *Solve* $\left(D^2 - DD' - 2D'^2\right) z = (y-1)e^x$

Solution: The auxillary equation is

$$\begin{aligned} m^2 - m - 2 &= 0 \\ (m-2)(m+1) &= 0 \\ \text{i.e., } m &= 2, -1 \end{aligned}$$

$$\therefore \text{ C.F. } = f_1(y+2x) + f_2(y-x)$$

$$\begin{aligned}
\text{P.I.} &= \frac{(y-1)e^x}{D^2-DD'-2D'^2} \\
&= e^x \frac{1}{(D+1)^2-(D+1)D'-2D'^2}(y-1) \\
&= e^x \frac{1}{D^2+2D+1-DD'-D'-2D'^2}(y-1) \\
&= e^x \left[1+\left(D^2+2D-DD'-D'-2D'^2\right)\right]^{-1}(y-1) \\
&= e^x \left[1-\left(D^2+2D-DD'-D'-2D'^2\right)+\ldots\right](y-1) \\
&= e^x \left[1+DD'+D'\right](y-1) \\
&= e^x \left[y-1+D(1)+1\right] \\
&= e^x y
\end{aligned}$$

$\therefore$ The complete solution $z = \text{C.F.} + \text{P.I.}$
i.e., $z = f_1(y+2x) + f_2(y-x) + ye^x$.

Example 1.127. *Solve* $\left(D^2+DD'-6D'^2\right)z = y\cos x$

Solution: The auxillary equation is

$$\begin{aligned}
m^2+m-6 &= 0 \\
(m+3)(m-2) &= 0 \\
\text{i.e., } m &= 2, -3
\end{aligned}$$

$$\begin{aligned}
\therefore\ \text{C.F.} &= f_1(y+2x) + f_2(y-3x) \\
\text{P.I.} &= \frac{1}{D^2+DD'-6D'^2} y\cos x \\
&= \frac{1}{(D-2D')(D+3D')} y\cos x \\
&= \frac{1}{D-2D'} \int (c+3x)\cos x\,dx \qquad \text{where } y = c+3x \\
&= \frac{1}{D-2D'} \left[(\mathrm{c}+3x)\sin x + 3\cos x\right] \\
&= \frac{1}{D-2D'} \left[(y-3x+3x)\sin x + 3\cos x\right] \qquad \because c = y-3x \\
&= \frac{1}{D-2D'} (y\sin x + 3\cos x) \\
&= \int \left[(c_1-2x)\sin x + 3\cos x\right]dx \qquad \text{where } y = c_1-2x \\
&= \left[(c_1-2x)(-\cos x) - (-2)(-\sin x)\right] + 3\sin x \\
&= -(c_1-2x)\cos x - 2\sin x + 3\sin x \\
&= -y\cos x + \sin x \\
\text{P.I.} &= \sin x - y\cos x
\end{aligned}$$

$\therefore$ The complete solution $z =$ C.F. + P.I.

1.8.4 Exercises & Solutions

Solve the following partial differential equations:

1. $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x \partial y} - 2\dfrac{\partial^2 z}{\partial y^2} = 0$ [Ans: $z = f_1(y+x) + f_2(y-2x)$]

2. $2\dfrac{\partial^2 z}{\partial x^2} + 5\dfrac{\partial^2 z}{\partial x \partial y} + 2\dfrac{\partial^2 z}{\partial y^2} = 0$ [Ans: $z = f_1(y-2x) + f_2(2y-x)$]

3. $\left(D^3 - 2D^2 D'\right) 2e^{2x} + 3x^2 y$
$$\left[\text{Ans: } z = f_1(y) + x f_2(y) + f3(y+2x)\frac{1}{4}e^{2x} + \frac{1}{60}\left(3x^5 y + x^6\right)\right]$$

4. $r + s - 6t = \cos(2x+y)$
$$\left[\text{Ans: } z = f_1(y-3x) + f_2(y+2x) + \frac{1}{5}x\left(\sin 2x + y\right)\right]$$

5. $r + 2s + t = 2(y-x) + \sin(x-y)$
$$\left[\text{Ans: } z = f_1(y-x) + x f_2(y-x) + x^2 y - x^3 + \frac{1}{2}x^2\left(\sin x - y\right)\right]$$

6. $\left(D2 + 2DD + D'^2\right) z = 2\cos y - x \sin y$
[Ans: $z = f_1(y-x) + x f_2(y-x) + x \sin y$]

7. $\left(D^2 + 3DD' + 2D'^2\right) z = x + y$
$$\left[\text{Ans: } z = f_1\left(y-x\right) + f_2\left(y-2x\right) - \frac{x^3}{3} + \frac{x^2 y}{2}\right]$$

8. $\left(2D^2 - 2DD' + D'^2\right) z = 2e^{3y} + e^{x+y} + y^2$
$$\left[\text{Ans: } z = f_1\left(y + \frac{1+i}{2}x\right) + f_2\left(y + \frac{1-i}{2}x\right) + \frac{2}{9}e^{3y} + e^{x+y} + \frac{x^2 y^2}{2}\right]$$

9. $\left(D^2 - DD'\right) z = \cos x \cos 2y$
$$\left[\text{Ans: } z = f_1(y) + f_2(y+x) + \frac{1}{2}\cos\left(x+2y\right) - \frac{1}{6}\cos(x-2y)\right]$$

10. $\left(D^2 - 2DD'\right) z = e^{2x} + x^3 y$
$$\left[\text{Ans: } z = f_1\left(y\right) + f_2\left(y+2x\right)\frac{1}{4}e^{2x} + \frac{x^5 y}{20} + \frac{x^6}{60}\right]$$

1.9 Non-Homogeneous Linear P.D.E.

A linear partial differential equation in which all the partial derivatives are not of the same order is called non-homogeneous linear partial differential equation.

Consider the equation $\qquad f(D, D')z = F(x, y) \qquad (1)$

If $f(D, D')$ is not homogeneous, then (1) is the non-homogeneous linear equation. As in the of homogenous linear equation , the complete solution is

$$z = \text{C.F.} + \text{P.I.}$$

[Finding P.I. of Non-Homo. P.D.E. is same as in Homo. P.D.E.]

1.9.1 To find the complementary function (C.F.)

Case(1): If $(D - m_1 - C_1)\, z = 0$, then C.F. $= e^{C_1 x} f_1\,(y + m_1 x)$

Case(2): If $(D' - m_1 D - C_1)\, z = 0$, then C.F. $= e^{C_1 y} f_1\,(x + m_1 y)$

Case(3): If $(D + m_1 D' + C_1)\, z = 0 \Rightarrow [D - (-m_1)D' - (-C_1)]\, z = 0$,

then C.F. $= e^{-C_1 x}\,(f_1(y - m_1 x))$

Case(4): If $(D' + m_1 D + c_1)\, z = 0 \Rightarrow [D' - (-m_1)D - (-c_1)]\, z = 0$, then

C.F. $= e^{-C_1 y} f_1\,(x - m_1 y)$

Case(5): If $(D - m_1 D' - c_1)\,(D - m_2 D' - c_2)\, z = 0$, then

C.F. $= e^{C_1 x} f_1\,(y + m_1 x) + e^{C_2 x} f_2\,(y + m_2 x)$

Case(6): If $(D - m_1 D' - c_1)^2\, z = 0$, then

C.F. $= e^{C_1 x} f_1\,(y + m_1 x) + x e^{C_1 x} f_2\,(y + m_1 x)$

Case(7): Put $D = h$ and $D' = k$ in $f\,(D, D') = 0$, then we get an

Aux. eqn.

$$f(h, k) = 0 \qquad (2)$$

Solve (2) and find the value of h interm of k or k interm of h.

Let the n values of h be denoted by $f_1(k), f_2(k), f_3(k), \ldots, f_n(k)$.

Then

$$\text{C.F.} = \sum c_1 e^{f_1(k)x+ky} + \sum c_2 e^{f_2(k)x+ky} + \sum c_3 e^{f_3(k)x+ky} + \ldots + \sum c_n e^{f_n(k)x+ky}$$

1.9.2 Examples of Non-Homogeneous Linear P.D.E.

Example 1.128. *Solve* $(D - 2D' - 1)\, z = 0$

Solution: C.F. $= e^{x} f_1(y + 2x)$ $\qquad [\because C_1 = 1, m_1 = 2 \text{ as case(1)}]$

Example 1.129. *Solve* $(D' - 2D - 3)\, z = 0$

Solution: C.F. $= e^{3y} f_1(x + 2y)$ $\qquad [\because C_1 = 3, m_1 = 2 \text{ as case(2)}]$

Example 1.130. *Solve* $(D + 3D' + 1)\, z = 0$

Solution: Given $(D + 3D' + 1)\, z = 0 \Rightarrow [D - (-3)D' - (-1)]$
C.F. $= e^{-x} f_1(y - 3x)$ $\qquad [\because C_1 = 1, m_1 = 3 \text{ as case(3)}]$

Example 1.131. *Solve* $(D' + 5D + 6)\, z = 0$

Solution: Given $(D' + 5D + 6)\, z = 0 \Rightarrow [D' - (-5)D - (-6)]$
C.F. $= e^{-6y} f_1(x - 5y)$ $\qquad [\because C_1 = 6, m_1 = 5 \text{ as case(4)}]$

Example 1.132. *Solve* $(D - 4D' - 1)\,(D' + 5D - 3)\, z = 0$

Solution: Given $(D - 4D' - 1)\,[D' - (-5)D - 3]\, z = 0$
C.F. $= e^{x} f_1(y + 4x) + e^{3y} f_2(x - 5y)$

Example 1.133. *Solve* $(D - 2D' + 2)^3\, z = 0$

Solution: Given $(D - 2D' + 2)^3\, z = 0 \Rightarrow [D - 2D' - (-2)]^3\, z = 0$
C.F. $= e^{-2x} f_1(y + 2x) + x e^{-2x} f_2(y + 2x) + x^2 e^{-2x} f_3(y + 2x)$

Example 1.134. *Solve* $\left(D^2 - DD' + D' - 1\right) z = 0$

Solution:

$$\begin{aligned}
\text{Consider } \left(D^2 - DD' + D' - 1\right) &= \left(D^2 - 1\right) - DD' + D' \\
&= (D - 1)(D + 1) - D'(D - 1) \\
&= (D - 1)(D - D' + 1) \\
&= [D - 0D' - 1][D - D' - (-1)]
\end{aligned}$$

C.F. $= e^{x} f_1(y) + e^{-x} f_2(y + x)$

(OR)

Put $D = h$ and $D' = k$ in

$$f\,(D, D') = 0 \Rightarrow \left(D^2 + DD' + D' - 1\right) z = 0$$

then we get an Aux. eqn.

$$f(h, k) = 0 \Rightarrow \left(h^2 + hk + k - 1\right) = 0 \tag{1}$$

Solve (1) and find the value of h interm of k or k interm of h.

$h^2 + hk + k - 1 = 0$

Now, $h = \dfrac{k \pm \sqrt{k^2 - 4(k-1)}}{2} = \dfrac{k \pm \sqrt{(k-2)^2}}{2} = \dfrac{k \pm (k-2)}{2}$

$i.e., h = k-1, 1$

$$\begin{aligned} \text{C.F. } &= \sum c_1 e^{(k-1)x+ky} + \sum c_2 e^{1x+ky} \\ &= \sum c_1 e^{-x} e^{k(x+y)} + \sum c_2 e^{x+ky} \\ &= e^{-x} f_1(x+y) + e^x f_2(y) \end{aligned}$$

Example 1.135. *Solve* $\left(D^2 - D'^2 - 3D + 3D'\right) z = 0$

Solution: Put $D = h$ and $D' = k$ in

$$f(D, D') = 0 \Rightarrow \left(D^2 - D'^2 - 3D + 3D'\right) z = 0$$

then we get an Aux. eqn.

$$f(h,k) = 0 \Rightarrow \left(h^2 - k^2 - 3h + 3k\right) = 0 \tag{1}$$

Solve (1) and find the value of h interm of k

$$\begin{aligned} \left(h^2 - k^2 - 3(h-k)\right) &= 0 \\ (h-k)(h+k) - 3(h-k) &= 0 \\ (h-k)(h+k-3) &= 0 \\ h &= k, 3-k \end{aligned}$$

$$\begin{aligned} \text{C.F. } &= \sum c_1 e^{kx+ky} + \sum c_2 e^{(3-k)x+ky} = \sum c_1 e^{k(x+y)} + \sum c_2 e^{3x+k(y-x)} \\ &= f_1(x+y) + e^{3x} f_2(y-x) \end{aligned}$$

Example 1.136. *Solve* $\left(2D^2 - DD' - D'^2 + 6D + 3D'\right) z = 0$

Solution: Put $D = h$ and $D' = k$

$$\begin{aligned} \text{A.E. } \left(2h^2 - hk - k^2 + 6h + 3k\right) &= 0 \\ \text{i.e., } \left[2h^2 + h(6-k) + (3k - k^2)\right] &= 0 \end{aligned}$$

$$\begin{aligned} h &= \frac{(k-6) \pm \sqrt{(6-k)^2 - 8(3k-k^2)}}{4} \\ &= \frac{(k-6) \pm \sqrt{k^2 - 12k + 36 - 24k + 8k^2}}{4} = \frac{(k-6) \pm \sqrt{9k^2 - 36k + 36}}{4} \\ &= \frac{(k-6) \pm (3k-6)}{4} = \frac{(k-6)+(3k-6)}{4}, \frac{(k-6)-(3k-6)}{4} \\ &= k-3, -\frac{k}{2} \end{aligned}$$

$$
\begin{aligned}
\text{C.F.} &= \sum c_1 e^{(k-3)x+ky} + \sum c_2 e^{-\frac{k}{2}x+ky} \\
&= \sum c_1 e^{-3x} e^{k(x+y)} + \sum c_2 e^{k\left(y-\frac{x}{2}\right)} \\
&= e^{-3x} f_1(x+y) + f_2\left(y - \frac{x}{2}\right)
\end{aligned}
$$

1.10 Assignment III[Partial Differential Equations]

1. (i) Form a partial differential equation by eliminating a and b from the expression $(x-a)^2 + (y-b)^2 + z^2 = c^2$.
   (ii) Find the partial differential equation of all planes which are at a constant distance a_1 from the origin.

2. (i) Form the partial differential equation by eliminating arbitrary function from $z = xf(2x+y) + g(2x+y)$.
   (ii) Form the PDE by eliminating the arbitrary function ϕ from $\phi\left(x^2+y^2+z^2, ax+by+cz\right) = 0$.

3. Solve the following:
   (i) $p(1+q)z = qz$
   (ii) $x^2p + y^2q = 0$ (iii) $xp + yq = 0$

4. Solve the following:
   (i) $z = px + qy + \log pq$ (ii) $z = px + qy + pq$
   (iii) $z = px + qy + \sqrt{1+p^2+q^2}$ (iv) $z = px + qy + p^2q^2$

5. Solve the following:
   (i) $p(1+q) = qz$ (ii) $x^4p^2 - yzq = z^2$
   (iii) $x^4p^2 + y^2qz = 2z^2$ (iv) $z^2\left(p^2x^2 + q^2\right) = 1$

6. Solve the following:
   (i) $p^2 + q^2 = x^2 + y^2$ (ii) $z^2\left(p^2+q^2\right) = x^2 + y^2$
   (iii) $4z^2q^2 = y + 2zp - x$ (iv) $p^2 + q^2 = z^2\left(x^2+y^2\right)$

7. Solve the following:
   (i) $x^2(y-z)p = y^2(z-x)q = z^2(x-y)$
   (ii) $\left(x^2 - yz\right)p + \left(y^2 - zx\right)q = z^2 - xy$
   (iii) $(y - xz)p + (yz - x)q = (x+y)(x-y)$

(iv) $\left(x^2-y^2-z^2\right)p+2xyq=2zx$
(v) $x\left(z^2-y^2\right)p+y\left(x^2-z^2\right)q=z\left(y^2-x^2\right)$
(vi) $x\left(y^2+z\right)p+y\left(x^2+z\right)q=z\left(x^2-y^2\right)$
(vii) $(3z-4y)p+(4x-2z)q=(2y-3x)$
(viii) $(x-2z)p+(2z-y)q=y-x$
(ix) $x(y-z)p+y(z-x)q=z(x-y)$

8. Solve the following:
(i) $\left(D^3+D^2D'-DD'^2-D'^3\right)z=e^{2x+y}+\cos(x+y)$
(ii) $\dfrac{\partial^3 z}{\partial x^3}-2\dfrac{\partial^3 z}{\partial x^2\partial y}=e^{x+2y}+4\sin(x+y)$
(iii) $\left(D^3-7DD'^2-6D'^3\right)z=\cos(x+2y)+4$
(iv) $\left(D^2+2DD'+D'^2\right)z=\sinh(x+y)+e^{x+2y}$
(v) $\left(D^2+2DD'+D'^2\right)z=x^2y+e^{x-y}$
(vi) $\left(D^2-DD'-2D'^2\right)z=2x+3y+e^{3x+4y}$
(vii) $\left(D^2-D'^2\right)z=e^{x-y}\sin(x+2y)$
(viii) $\left(D^2-5DD'+6D'^2\right)z=y\sin x$
(ix) $r+s-6t=y\cos x$

9. Solve the following:
(i) $\left(D^2-D'^2-3D+3D'\right)z=xy+7$
(ii) $\left(D^2+2DD'-2D-2D'^2\right)z=\sin(x+2y)$
(iii) $\left(2D^2-DD'-D'^2+6D+3D'\right)z=xe^y$
(iv) $\left(D^2+2DD'+D'^2-2D-2D'\right)z=e^{3x+y}+4$
(v) $\left(D^2+D'^2+2DD'+2D+2D'+1\right)z=e^{2x+y}$