Linear DE with constant co-efficients (330 PD)

A DE of the form $\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_n y = X \quad \cdots (1)$

where X is a function of x only and $a_1, a_2 \cdots a_n$ are constants is called a linear differential equation of nth order.

The general solution of (1) is $y = C.F. + P.I.$ where C.F involves n arbitrary constants and P.I doesn't involve any arbitrary constant.

Remarks: It should be remembered that, P.I. appears due to X in (1). Hence if a linear differential equation with constant coefficients is given with $X=0$ then its general solution will not involve P.I. and so for such DE the general solution will be given by $y = C.F.$

Auxilliary equation: Consider the DE (1) with $X=0$. that is

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n) y = 0 \quad \text{or} \quad f(D) y = 0 \quad \cdots (2)$$

assume that $y = e^{mx}$ is a solution of this equation. then

$$y = e^{mx}, \; Dy = me^{mx} \; D^2 y = m^2 e^{mx} \cdots D^n y = m^n e^{mx}$$

so (2) becomes, $(m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots + a_n) e^{mx} = 0$

which will hold if, $m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots + a_n = 0$

or, $f(m) = 0$

This equation is called the auxiliary equation.

Working rule for finding complementary function:

on solving the auxilary equation we shall get n roots. Three cases arise, according as the roots of the auxilary equations are,

i) simple real roots
ii) complex roots
iii) surds in the form $\alpha \pm i\sqrt{\beta}$

case I: First suppose that the auxilary equation has n distinct roots $m_1, m_2 \dots m_n$ then the C.F. is given by, $c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$

where, $c_1, c_2 \dots c_n$ are arbitrary constants.

Again if the auxilary equation has the real root m_K occuring K times and if further the remainging roots of the A.E. are distinct real numbers $m_{K+1}, m_{K+2} \dots m_n$ then C.F. is given by,

$$(c_1 + c_2 x + c_3 x^2 + \dots + c_K x^{K-1}) e^{m_K x} + c_{K+1} e^{m_{K+1} x} + \dots + c_n e^{m_n x}$$

case II: Let $\alpha \pm i\beta$ be a pair of complex roots. Then the corresponding part of the C.F. may be written in the following forms.

$$e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x) / \; c_1 e^{\alpha x} \cos(\beta x + c_2) / c_1 e^{\alpha x} \sin(\beta x + c_2)$$

If however, the auxilary equation has two equal pairs of complex roots $\alpha + i\beta$ and $\alpha - i\beta$ say occur twice, the corresponding part of the C.F. is written as

$$e^{\alpha x} \{ (c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x \}$$

case III: If a pair of the roots of the auxiliary equation involves surds, say $\alpha \pm \sqrt{\beta}$, where β is positive, then the corresponding part of C.F. is one of the following forms.

$$e^{\alpha x} \{ c_1 \cosh(x\sqrt{\beta}) + c_2 \sinh(x\sqrt{\beta}) \}, \; c_1 e^{\alpha x} \cosh(x\sqrt{\beta} + c_2), \; c_1 e^{\alpha x} \sinh(x\sqrt{\beta} \dots$$

It should be noted that the above results are exactly similar to those of case (ii) except that sin and cos have been replaced by sinh and cosh respectively.

Ex-01: $(D^2-3D+2)y=0 \cdots (i)$

Let, $y=e^{mx}$ is a solution of (i)

then we have,

$$D^2e^{mx}-3De^{mx}+2e^{mx}=0$$

$$\Rightarrow (m^2-3m+2)e^{mx}=0$$

A.E. $\Rightarrow m^2-3m+2=0$

$\Rightarrow m=1,2.$

$\therefore$ The general solution is,

$$y=C.F.=c_1e^{x}+c_2e^{2x}$$

Ex-02: $\boxed{(D^3-3D^2+4)y=0}$

Here, A.E.: $m^3-3m^2+4=0$

$$\Rightarrow m^3+m^2-4m^2-4m+4m+4=0$$

$$\Rightarrow m^2(m+1)-4m(m+1)+4(m+1)=0$$

$$\Rightarrow (m+1)(m^2-4m+4)=0.$$

$$\Rightarrow (m+1)(m-2)^2=0$$

$$\Rightarrow m=-1,2,2$$

$\therefore$ G.S.: $y=Ae^{-x}+(B+Cx)e^{2x}$

Ex: 03 $(D^3-5D^2+9D-5)y=0$

Here, A.E.: $m^3-5m^2+9m-5=0$

$$\Rightarrow m^3-m^2-4m^2+4m+5m-5=0$$

$$\Rightarrow m^2(m-1)-4m(m-1)+5(m-1)=0$$

$$\Rightarrow (m-1)(m^2-4m+5)=0$$

$$\Rightarrow m=1,\ \frac{4\pm\sqrt{16-20}}{2}$$

$$\therefore m=1;\ 2\pm i$$

G.S: $y=Ae^{x}+e^{2x}(B\cos x+C\sin x)$

Ex: 04 $(D^4+4)y=0$

Here, A.E.: $m^4+4=0$

$$\Rightarrow m^4-(2i)^2=0$$

$$\Rightarrow (m^2+2i)(m^2-2i)=0$$

$$\Rightarrow m=\pm\sqrt{-2i},\ \pm\sqrt{2i}$$

$$=\pm\sqrt{1^2-2i+i^2}\ \pm\sqrt{1+2i+i^2}$$

$$=\pm\sqrt{(1-i)^2}\ \pm\sqrt{(1+i)^2}$$

$$=1\pm i,\ -1\pm i$$

G.S: $y=e^{-x}(A\cos x+B\sin x)+e^{x}(C\cos x+D\sin x)$

Ex: 05 $(D^2+6D+4)y=0$

Here; A.E.: $m^2+6m+4=0$

$$\Rightarrow m=-3\pm\sqrt{5}$$

$\therefore$ G.S. $y=e^{-3x}(A\cosh x\sqrt{5}+B\sinh x\sqrt{5})$

* $(D^2-6D+9)y=0$

* $(D^2+4)y=0$

* $(D^2-6D+9)y=1+x+x^2$.

Ex: i) $(D^2-4D+13)y=0$ Ans: $y=e^{2x}(A\cos 3x+B\sin 3x)$

ii) $(D^4+2D^2-3)y=0$ Ans: $y=Ae^{x}+Be^{-x}+C\cos\sqrt{3}x+E\sin\sqrt{3}x$

iii) $(D^2+5D+4)y=0$ Ans: $y=Ae^{-x}+Be^{-4x}$

iv) $(D^3-5D^2+7D-3)y=0$ Ans: $y=(A+Bx)e^{x}+Ce^{3x}$

v) $(D^3+2D^2+4D+8)y=0$ Ans: $y=Ae^{-2x}+B\cos 2x+C\sin 2x$

b)

Short methods for finding the particular integral when X is of certain special forms:

a) When X is of the form e^{ax}:

i) $P.I. = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$ where $f(a) \neq 0$

ii) If $f(a) = 0$ then $f(D)$ must possess a factor of the type $(D-a)^r$. In this we apply the following formula

$$\frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{r!} e^{ax}$$

~~$(D^2+4)y$~~

Ex: $(D^2-3D+2)y = e^{5x}$

Here. A.E.: $m^2-3m+2=0$

$\Rightarrow (m-2)(m-1)=0$

$\Rightarrow m = 2, 1$

$\therefore$ C.F.: $y_c = Ae^{2x}+Be^{x}$

P.I.: $y_p = \frac{1}{D^2-3D+2} e^{5x}$

$= \frac{1}{5^2-3.5+2} e^{5x}$

$= \frac{1}{12} e^{5x}$

$y = Ae^{5x}$

$Dy =$

$D^2y =$

$\therefore$ G.S.: $y = y_c + y_p$

$= Ae^{2x} + Be^{x} + \frac{1}{12} e^{5x}$

Ex: $(D^3+3D^2+3D+1)y = e^{-x}$

Here A.E.: $m^3+3m^2+3m+1=0$

$\Rightarrow (m+1)^3 = 0$

$\therefore m = -1, -1, -1$

$\therefore$ C.F.: $y_c = (A+Bx+Cx^2)e^{-x}$

P.I.: $y_p = \frac{1}{D^3+3D^2+3D+1} e^{-x}$

$= \frac{1}{(D+1)^3} e^{-x}$

$= x \frac{1}{3(D+1)^2} e^{-x}$

$= x^2 \frac{1}{6(D+1)} e^{-x}$

$= x^3 \frac{1}{6} e^{-x}$

$= \frac{1}{6} x^3 e^{-x}$

G.S.: $y = y_c + y_p$

$= (A+Bx+Cx^2)e^{-x} + \frac{1}{6} x^3 e^{-x}$

Exercise:

i) $(D^3-9D^2+26D-24)y = e^{x}$

Ans: $Ae^{2x}+Be^{3x}+Ce^{4x} - \frac{1}{6} e^{x}$

ii) $(D^2+2D+1)y = e^{x}+e^{-x}$

Ans: $(A+Bx)e^{-x} + \frac{e^{x}}{4} + \frac{x^2 e^{-x}}{2}$

Decaying oscillation:

Find the solution of IVP:

$$y'' + 2y' + 101y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

Sol:

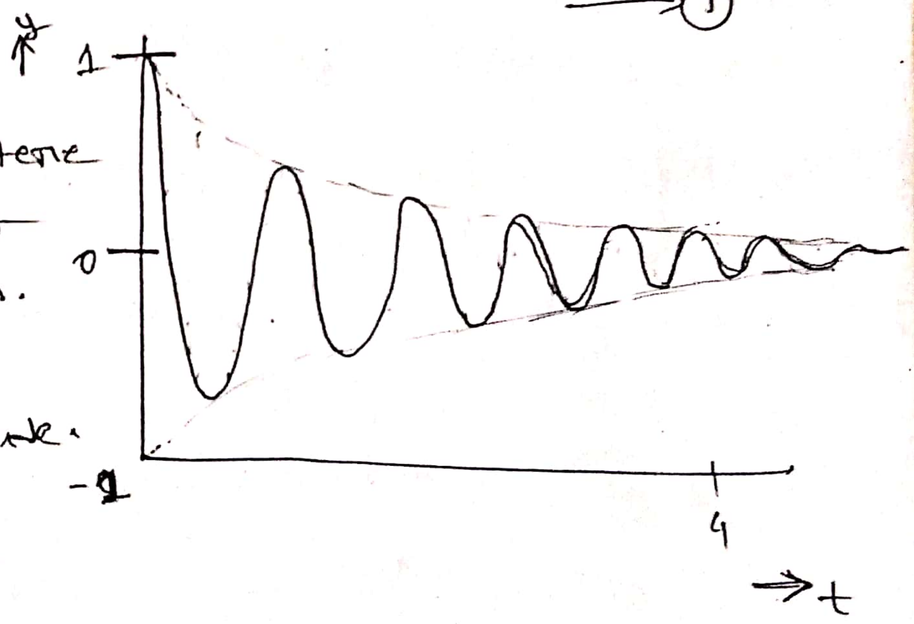
$$m = -1 \pm 10i$$

$$\therefore y(t) = e^{-t}(c_1 \cos 10t + c_2 \sin 10t).$$

$$y(0) = 1 = 1 \cdot (c_1 + 0) \Rightarrow c_1 = 1.$$

$$y'(0) = 0, \quad y'(0) = -1 + 10c_2 \Rightarrow c_2 = \frac{1}{10} = .1.$$

$$\therefore y(t) = e^{-t}(\cos 10t + .1 \sin 10t) \quad (1)$$



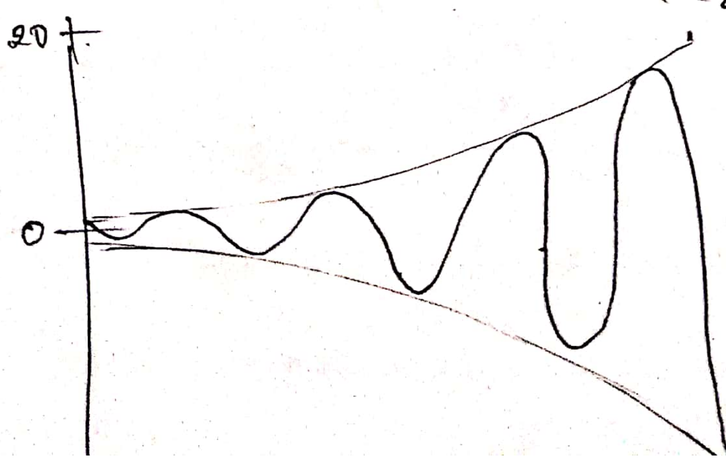
decaying oscillation. Here sine and cos part gives the oscillation. and e^{-t} part gives the decaying amplitude.

$t \to \infty, \; y \to 0.$

Growing oscillation:

$$(D^2 - 4D + 13)y = 0.$$

$$y(t) = e^{2t}(c_1 \cos 3t + c_2 \sin 3t)$$



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H.W.

** $(D^2 - 4D + 13)y = 0$

** $(D^2 + 5D + 4)y = 0$

** $(D^3 + 2D^2 + 4D + 8)y = 0$

** $(D^3 - 9D^2 + 26D - 24)y = e^x$

** $(D^4 - 1)y = e^x \sin x$

** $(D^2 - 4)y = e^x - \cos x$

** $(D^3 - 5D^2 + 7D - 3)y = x^4 + 5x + 2$.

* ~~Say~~ Cauchy - Euler Equation: A linear DE of the form $a_n x^n \dfrac{d^n y}{dx^n} + \cdots\cdots + a_1 x \dfrac{dy}{dx} + a_0 y = g(x)$ where $a_n, a_{n-1}, \cdots a_1, a_0$ are constants is known as a Cauchy Euler equation.

* Method of solution: Let $y = x^m$ be a solution. Then Auxiliary equation $am^2 + (b-a)m + c = 0$.

for 2nd order: $am(m-1) + bm + c = 0$

$am(m-1)(m-2) + bm(m-1) + cm + D = 0$.

Case-I: if m_1 and m_2 are distinct then
$$y = c_1 x^{m_1} + c_2 x^{m_2}$$

Case-II: if $m_1 = m_2$. then $y = c_1 x^{m_1} + c_2 \ln x \, x^{m_1}$

Case-III: if $m = \alpha \pm i\beta$, then
$$y = x^\alpha (c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)).$$

Some problem:

1. Solve $x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} - 4y = 0$.

Sol: Given the DE

$$x^2 y'' - 2xy' - 4y = 0 \longrightarrow (1)$$

Let $y = x^m$ be a solution of (1).

∴ Auxiliary equation,

$$m^2 + (-2-1)m - 4 = 0$$

$$\Rightarrow m^2 - 3m - 4 = 0.$$

$$\therefore m_1 = -1, \ m_2 = 4.$$

∴ Solution of (1) is $y = c_1 \frac{1}{x} + c_2 x^4$.

2. Solve $4x^2 y'' + 8x y' + y = 0$

Sol: Given the DE,

$$4x^2 y'' + 8xy' + y = 0 \longrightarrow (1)$$

Let $y = x^m$ be a solution of (1).

∴ Auxiliary equation of (1),

$$4m^2 + 4m + 1 = 0$$

$$\Rightarrow m = -\frac{1}{2}, -\frac{1}{2}.$$

∴ The general solution of (1) is

$$y = c_1 x^{-\frac{1}{2}} + c_2 \ln x \cdot x^{-\frac{1}{2}}.$$

Solve: $x^3y''' + 5x^2y'' + 7xy' + 8y = 0$.

Sol: Given the equation

$$x^3y''' + 5x^2y'' + 7xy' + 8y = 0 \longrightarrow (1)$$

Let $y = x^m$ be a solution of (1).

$$\therefore \quad m(m-1)(m-2) + 5m(m-1) + 7m + 8 = 0$$

$$\Rightarrow m^3 + 2m^2 + 4m + 8 = 0$$

$$\Rightarrow (m+2)(m^2+4) = 0.$$

$$\therefore \quad m = 2, \pm 2i$$

$$\therefore \quad y = c_1 x^2 + c_2 \cos(2 \ln x) + c_3 \sin(2 \ln x).$$

$\boxed{\text{Exercises 4.7}}$

1. $x^2y'' - 3xy' - 2y = 0$
2. $3x^2y'' + 6xy' + y = 0$
3. $xy'''' + 6y''' = 0$
4. $x^3y''' + xy' - y = 0$
5. $x^2y'' + 3xy' - 4y = 0.$
6. $x^2y'' - 7xy' + 41y = 0.$

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1. when $f(x)$ is polynomial,

$$y_p = Ax^2+Bx+C \,/\, Ax+B \,/\, A$$

2. when $f(x)$ is an exponential,

$$y_p = Ae^x.$$

3. when $f(x)$ is a sine/cosine

$$y_p = A\sin x + B\cos x.$$

* $y''+3y'+2y=0$
* $y''-6y'+9y = 1+x+x^2$
* $y''-4y'+4y = x^2$
* $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 2y = 4+x^2+x^3$
* $y''+2y'+2y = \cos 2x$
* $y''+4y'+4y = \sin 2x$
* $(D+3)^2 y = 4e^{2x}+5e^{-3x}$
* $y''-4y+3y = e^{3x}+xe^{2x}$
* $y''-2y'-3y = e^{2t}+3t^2+4t-5+5\cos 2t$
    * $y_p = -\frac{1}{3}e^{2t} - t^2+1 - \frac{7}{13}\cos 2t - \frac{4}{13}\sin 2t.$
* $y''-2y'-3y = 5e^{3t}$