## Derivatives:

If $f(z)$ is single valued in some region R of the z-plane, the derivative of $f(z)$ is defined as

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

Provided that the limit exists independent of the manner in which $\Delta z \to 0$. In such case we say that $f(z)$ is differentiable at $z$.

## Analytic function:

If the derivative $f'(z)$ exists at all points $z$ of a region $\mathbb{R}$, then $f(z)$ is said to be analytic in $\mathbb{R}$ and is referred to as an analytic function in $\mathbb{R}$.

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problem: Using the definition, find the derivatives of the following functions at the indicated points:

(a) ~~$f(z) = \frac{2z-i}{z+2i}$ at $z=i$~~ $f(z) = 3z^2+4iz-5+i$ at $z=2$

Soln: we know that by definition, the derivative of $f(z)$ at any point $z$ is

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} \quad \cdots \quad (1)$$

Given that,

$$f(z) = 3z^2+4iz-5+i \quad \cdots \quad (2)$$

$$f(z+\Delta z) = 3(z+\Delta z)^2+4i(z+\Delta z)-5+i$$

$$= 3\left[z^2+2z\Delta z+(\Delta z)^2\right]+4iz+4i\Delta z-5+i$$

$$= 3z^2+6z\Delta z+3(\Delta z)^2+4iz+4i\Delta z-5+i$$

$$= \left(3z^2+4iz-5+i\right)+\left[6z+3\Delta z+4i\right]\Delta z$$

[by (2)]

$$= f(z)+\left[6z+3\Delta z+4i\right]\Delta z$$

$$\Rightarrow f(z+\Delta z)-f(z) = \left[6z+3\Delta z+4i\right]\Delta z \quad \cdots \quad (3)$$

Using (3), we get from (1),

$$f'(z) = \lim_{\Delta z \to 0} \frac{\left[6z+3\Delta z+4i\right]\cancel{\Delta z}}{\cancel{\Delta z}}$$

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$$= \lim_{\Delta z \to 0} (6z + 3\Delta z + 4i)$$

$$= 6z + 4i$$

Now at $z = 2$,

$$f'(2) = 6 \cdot 2 + 4i$$
$$= 12 + 4i$$

(ii) $f(z) = 3z^{-2}$ at $z = 1+i$

Soln: we know, by definition the derivative of $f(z)$ at any point $z$ is

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \quad \cdots (1)$$

Given that,

$$f(z) = 3z^{-2}$$
$$= \frac{3}{z^2}$$

$$f(z+\Delta z) = \frac{3}{(z+\Delta z)^2}$$

$$f(z+\Delta z) - f(z) = \frac{3}{(z+\Delta z)^2} - \frac{3}{z^2}$$

$$= 3\left[\frac{z^2 - (z+\Delta z)^2}{z^2(z+\Delta z)^2}\right]$$

$$= 3\left[\frac{z^2 - [z^2 + 2z\Delta z + (\Delta z)^2]}{z^2(z+\Delta z)^2}\right]$$

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$$= 3\left[\frac{\cancel{z^2}-\cancel{z^2}-2z\Delta z-(\Delta z)^2}{z^2(z+\Delta z)^2}\right]$$

$$= 3\left[\frac{-2z\Delta z-(\Delta z)^2}{z^2(z+\Delta z)^2}\right]$$

Now from (1) we get,

$$f'(z) = \lim_{\Delta z\to 0}\frac{3\left[\frac{-2z\Delta z-(\Delta z)^2}{z^2(z+\Delta z)^2}\right]}{\Delta z}$$

$$= \lim_{\Delta z\to 0}\frac{3\cancel{\Delta z}\left[-2z-\Delta z\right]}{\cancel{\Delta z}\; z^2(z+\Delta z)^2}$$

$$= \frac{3[-2z-0]}{z^2(z+0)^2}$$

$$= \frac{-6z}{z^4} = -\frac{6}{z^3}$$

At $z = 1+i$

$$f'(z) = -\frac{6}{(1+i)^3}$$

$$= -\frac{6}{1^3+3\cdot 1^2\cdot i+3\cdot 1\cdot(i)^2+(i)^3}$$

$$= -\frac{6}{1+3i-3-i}$$

$$= -\frac{6}{2i-2}\quad \text{(Ans)}$$

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**Problem:** Show that $\frac{d}{dz}(\bar{z})$ does not exist anywhere.

**Soln:** By definition we have, $f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$

$$\frac{d}{dz}(\bar{z}) = \lim_{\Delta z \to 0} \frac{\overline{z+\Delta z} - \bar{z}}{\Delta z}$$

Here,
$f(z) = \bar{z}$
$f(z+\Delta z) = \overline{z+\Delta z}$

$$= \lim_{\Delta z \to 0} \frac{\bar{z} + \overline{\Delta z} - \bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

Now along the real axis $\Delta x \to 0$, $\Delta y = 0$ so

$$\frac{d}{dz}(\bar{z}) = \lim_{\Delta x \to 0} \frac{\Delta x - 0}{\Delta x + 0}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x}$$

$$= 1$$

Along the imaginary axis $\Delta x = 0$, $\Delta y \to 0$, so

$$\frac{d}{dz}(\bar{z}) = \lim_{\Delta y \to 0} \frac{0 - i\Delta y}{0 + i\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y}$$

$$= -1$$

Since two limits are not equal,

so $\frac{d}{dz}(\bar{z})$ does not exists.

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(11) Prove that $\lim_{z \to 0} \frac{\bar{z}}{z}$ does not exist.

Soln: Let,

$z = x + iy$

$\bar{z} = x - iy$

$\therefore \lim_{z \to 0} \frac{\bar{z}}{z}$

$= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x - iy}{x + iy}$

Taking limit along the real axis $(y = 0, x \to 0)$

$$\lim_{x \to 0} \frac{x - 0}{x + 0} = \lim_{x \to 0} 1 = 1$$

Again taking limit along the imaginary axis $(x = 0, y \to 0)$

$$\lim_{z \to 0} \frac{\bar{z}}{z} = \lim_{y \to 0} \frac{0 - iy}{0 + iy} = -1$$

The above two limits are not equal, the limit depends on manner in which $z \to 0$

Hence $\lim_{z \to 0} \frac{\bar{z}}{z}$ does not exists.

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prove that $f(z)=|z|^2$ is not differentiable except at the origin.

Soln: Given that

$$f(z)=|z|^2$$

By the definition of derivative we have,

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z}$$

Now at $z=0$,

$$f'(0) = \lim_{\Delta z \to 0} \frac{f(\Delta z)-f(0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{|\Delta z|^2-|0|^2}{\Delta z}$$ , since $f(z)=|z|^2$, $f(\Delta z)=|\Delta z|^2$

$$= \lim_{\Delta z \to 0} \frac{\cancel{\Delta z}\,\overline{\Delta z}}{\cancel{\Delta z}}$$

$$= 0$$

Again, if $z \neq 0$, say $z=z_0$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{|z_0+\Delta z|^2-|z_0|^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z_0+\Delta z)\overline{(z_0+\Delta z)}-z_0\overline{z_0}}{\Delta z}$$

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$$= \lim_{\Delta z \to 0} \frac{(z_0+\Delta z)(\overline{z_0+\Delta z}) - z_0\overline{z_0}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z_0\bar{z}_0 + \bar{z}_0\Delta z + z_0\overline{\Delta z} + \Delta z\overline{\Delta z} - \overline{z_0 z_0}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z_0\overline{\Delta z} + \bar{z}_0\Delta z + \Delta z\overline{\Delta z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z_0\overline{\Delta z}}{\Delta z} + \lim_{\Delta z \to 0} \frac{\bar{z}_0\cancel{\Delta z}}{\cancel{\Delta z}} + \lim_{\Delta z \to 0} \frac{\cancel{\Delta z}\overline{\Delta z}}{\cancel{\Delta z}}$$

$$= z_0 \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} + \bar{z}_0 + 0$$

$$= \bar{z}_0 + z_0 \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

Now along real axis, $\Delta x \to 0$, $\Delta y = 0$

$$\therefore f'(z_0) = \bar{z}_0 + z_0 \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x}$$

$$= \bar{z}_0 + z_0 \text{, along real axis}$$

Along imaginary axis $\Delta y \to 0$, $\Delta x = 0$

$$f'(z_0) = \bar{z}_0 + z_0 \lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y}$$

$$= \bar{z}_0 + z_0 \lim_{\Delta y \to 0} (-1)$$

$$= \bar{z}_0 - z_0$$

Since the two limits are not equal, so $f'(z_0)$ does not exist.



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