Method of finding residues:

(a) Residue at a simple pole:

If $f(z)$ has a simple pole at $z=a$,

then $\operatorname{Res}f(a) = \lim\limits_{z\to a} (z-a)f(z)$.

(b) Residue at a pole of order n:

If $f(z)$ has a pole of order n at $z=a$, then

$$\operatorname{Res}(\text{at } z=a) = \frac{1}{(n-1)!} \lim_{z\to 0} \left\{ \frac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n f(z) \right] \right\}$$

Problem: Find the poles and resides of the following functions in the finite plane:

$$f(z) = \frac{2z+1}{z^2-z-2}$$

Soln: Given, $f(z) = \dfrac{2z+1}{z^2-z-2}$

poles of $f(z)$ are obtained from

$$z^2-z-2=0$$

$$\Rightarrow z^2-2z+z-2=0$$

$$\Rightarrow z(z-2)+1(z-2)=0$$

$$\Rightarrow (z-2)(z+1)=0$$

$$\therefore z = 2, -1$$

So $f(z)$ has two simple poles at $z=2$ and $z=-1$.

The residues of $f(z)$ at $z=2$ is

$$\lim_{z\to 2} (z-2) f(z)$$

$$= \lim_{z\to 2} \cancel{(z-2)} \left\{ \frac{2z+1}{\cancel{(z-2)}(z+1)} \right\}$$

$$= \lim_{z\to 2} \frac{2z+1}{z+1}$$

$$= \frac{2\cdot 2+1}{2+1} = \frac{5}{3}$$

The residue of $f(z)$ at $z=-1$ is

$$\lim_{z\to -1} (z+1) f(z)$$

$$= \lim_{z\to -1} \cancel{(z+1)} \left\{ \frac{2z+1}{(z-2)\cancel{(z+1)}} \right\}$$

$$= \lim_{z\to -1} \frac{2z+1}{z-2}$$

$$= \frac{2(-1)+1}{-1-2}$$

$$= \frac{1}{3}$$

(ii) $$f(z) = \frac{z}{(z-1)(z+1)^2}$$

Here $z=1$ is a simple pole and $z=-1$ is a pole of order 2.

$[(z+1)^2=0$

$\Rightarrow (z+1)(z+1)=0$

$\therefore z=-1,-1$

Residue at $z=1$,

$$\lim_{z\to 1} (z-1) \cdot \frac{z}{(z-1)(z+1)^2}$$

$$= \lim_{z\to 1} \frac{z}{(z+1)^2} = \frac{1}{(1+1)^2} = \frac{1}{4}$$

Residue at $z=-1$ is

$$\frac{1}{(2-1)!}\lim_{z\to -1}\frac{d^{2-1}}{dz^{2-1}}\left[(z+1)^2 f(z)\right]$$

$$=\frac{1}{1!}\lim_{z\to -1}\frac{d}{dz}\left[\cancel{(z+1)^2}\frac{z}{(z-1)\cancel{(z+1)^2}}\right]$$

$$=\lim_{z\to -1}\frac{(z-1)\cdot 1-z\cdot 1}{(z-1)^2}$$

$$=\lim_{z\to -1}\frac{-1}{(z-1)^2}$$

$$=\frac{-1}{(-1-1)^2}=\frac{-1}{4}\ \text{(Ans)}$$

(iii) $$f(z)=\frac{z^3}{(2z+1)^3}$$

poles of $f(z)$ are obtained by

$(2z+1)^3=0 \quad \Rightarrow (2z+1)(2z+1)(2z+1)=0$

$\Rightarrow \cancel{2z=-1} \qquad \therefore z=-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}$

$\therefore \cancel{z=}$

So $f(z)$ has a pole of order 3 at $z=-\frac{1}{2}$

Residue of $f(z)$ at $z=-\frac{1}{2}$ is

$$\frac{1}{(3-1)!}\lim_{z\to -\frac{1}{2}}\frac{d^{3-1}}{dz^{3-1}}\left[\left(z+\frac{1}{2}\right)^3 f(z)\right]$$

$$=\frac{1}{2!}\lim_{z\to -\frac{1}{2}}\frac{d^2}{dz^2}\left[\left(z+\frac{1}{2}\right)^3\cdot\frac{z^3}{(2z+1)^3}\right]$$

$$=\frac{1}{2!}\lim_{z\to -\frac{1}{2}}\frac{d^2}{dz^2}\left[\left(\frac{2z+1}{2}\right)^3\cdot\frac{z^3}{(2z+1)^3}\right]$$

$$= \frac{1}{2!} \lim_{z\to -\frac{1}{2}} \cdot \frac{d^2}{dz^2}\left[\frac{z^3}{\cancel{(2z+1)^3}} \cancel{(2z+1)^3} \cdot \frac{z^3}{8}\right]$$

$$= \frac{1}{16} \lim_{z\to -\frac{1}{2}} \frac{d}{dz}(3z^2)$$

$$= \frac{1}{16} \lim_{z\to -\frac{1}{2}} (6z)$$

$$= \frac{1}{16}\left\{\cancel{6}^{3} \times \left(-\frac{1}{\cancel{2}}\right)\right\}$$

$$= -\frac{3}{16} \text{ (Ans)}$$

\# $f(z) = \dfrac{z^2+16}{(z-i)^2(z+3)}$

Soln: Poles of $f(z)$ are obtained by

$(z-i)^2 = 0$ $\quad\quad$ $z+3=0$

$\Rightarrow (z-i)(z-i)=0$ $\quad\quad$ $\Rightarrow z=-3$

$\Rightarrow z = i, i$

$\therefore$ $f(z)$ has a simple pole at $z=-3$

$f(z)$ has a pole of order two at $z=i$

$$\lim_{z\to -3} \cancel{(z+3)} \cdot \frac{z^2+16}{(z-i)^2\cancel{(z+3)}}$$

$$= \frac{(-3)^2+16}{(-3-i)^2}$$

$$= \frac{9+16}{(-3)^2 - 2\cdot(-3)\cdot i + 1} = \frac{25}{8+6i} = \frac{25(8-6i)}{64-36i^2}$$

$$= \frac{25(8-6i)}{64+36}$$

$$= \frac{1}{4}(8-6i)$$

$$= \frac{1}{2}(4-3i)$$