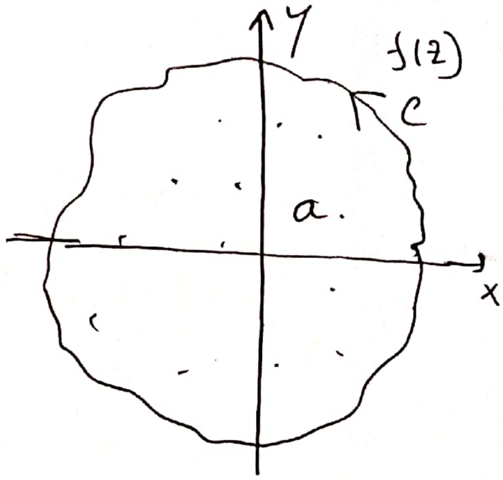
Cauchy Integral formula:

If $f(z)$ is analytic within and on a closed contour C and if 'a' is any point within c then

$$f(a) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{z-a} dz$$

$$\Rightarrow f(a)\, 2\pi i = \oint_c \frac{f(z)}{z-a} dz$$



Cauchy Integral formula for derivatives

$$f'(a) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{(z-a)^2} dz$$

$$f^n(a) = \frac{n!}{2\pi i} \oint_c \frac{f(z)}{(z-a)^{n+1}}$$

Problems: Using cauchy Integral formula evaluate $\int_C \frac{1}{z(z^2+9)} dz$ where C is the circle $|z| = 2$.

Here,

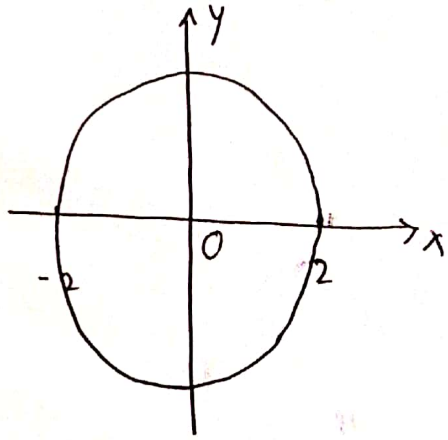
$|z| = 2$

$\Rightarrow |x+iy| = 2$

$\Rightarrow \sqrt{x^2+y^2} = 2$

$\Rightarrow x^2+y^2 = 2^2$

$\therefore |z| = 2$ is a circle with centre $(0,0)$ and radius 2.



Given,

$$\int_C \frac{1}{z(z^2+9)} dz$$

Let, $f(z) = \frac{1}{z^2+9}$

$\therefore z = 0$ lies inside the circle $|z| = 2$.

Now the given integral can be written as

$$\int_C \frac{f(z)}{z-0} dz$$

$= 2\pi i\, f(0)$ [By cauchy's integral formula]

$= 2\pi i \times \frac{1}{9}$ Since $f(z) = \frac{1}{z^2+9}$

$= \frac{2\pi i}{9}$ $f(0) = \frac{1}{9}$

(ii) $\int_c \frac{z}{(9-z^2)(z+i)}$, $|z|=2$

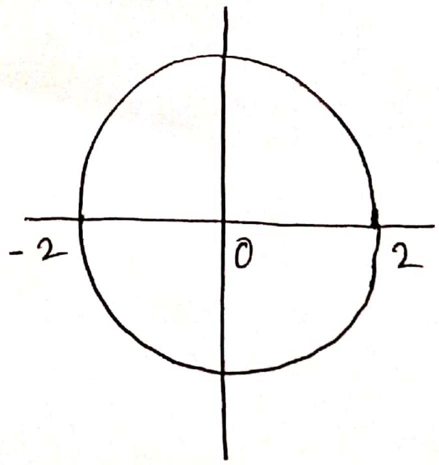
Soln; Here $|z| = 2$

$\Rightarrow |x+iy| = 2$

$\Rightarrow \sqrt{x^2+y^2} = 2$

$\Rightarrow x^2+y^2 = 2^2$

which is a circle with centre (0, 0) and radius 2.



Given.

$\int_c \frac{z}{(9-z^2)(z+i)}$

Let $f(z) = \frac{z}{9-z^2}$

Now $z+i=0$

$\Rightarrow z = -i$

$\Rightarrow |z| = |-i| = 1$ lies inside the Circle $|z|=2$

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the given integral can be written as

$$\int_c \frac{\frac{z}{9-z^2}}{z+i}\,dz$$

$$= \int_c \frac{f(z)}{z-(-i)}\,dz$$

$$= 2\pi i f(-i) \quad \text{[By Cauchy's integral formula]}$$

$$= 2\pi i \times \frac{-i}{10}$$

$$= -\frac{2\pi i^2}{10}$$

$$= \frac{\pi}{5}$$

Since,

$$f(z) = \frac{z}{9-z^2}$$

$$\Rightarrow f(-1) = \frac{-i}{9-(-i)^2}$$

$$= \frac{-i}{9+1}$$

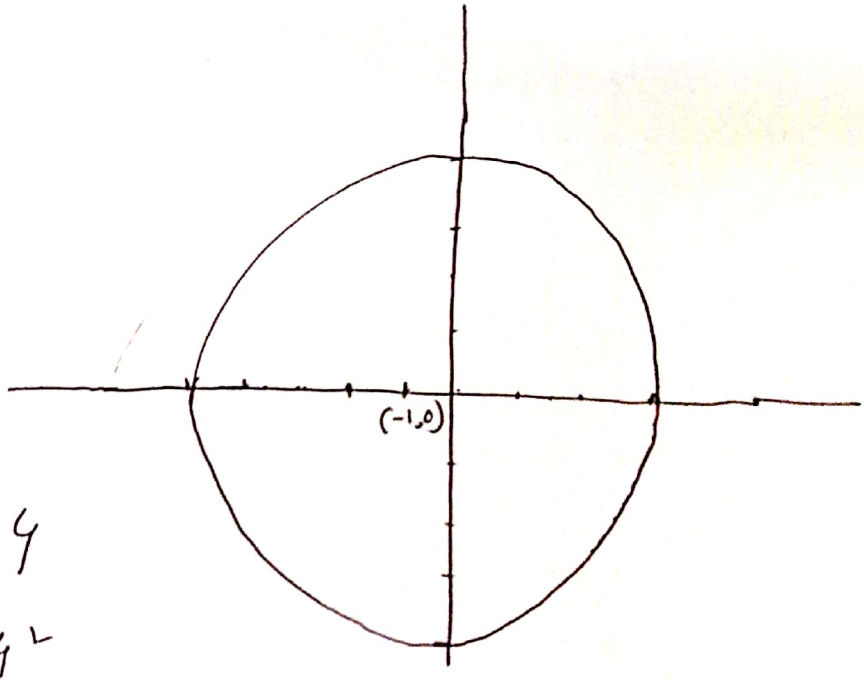
$$= -\frac{i}{10}$$

$$\therefore \int_c \frac{z}{(9-z^2)(z+i)} = \frac{\pi}{5}. \quad \text{(Ans)}$$

(iii) $\int_C \frac{e^{3z}}{z+\pi i} dz$ where C is the circle $|z+1| = 4$

Soln;

Here,



$|z+1| = 4$

$\Rightarrow |x+iy+1| = 4$

$\Rightarrow |(x+1)+iy| = 4$

$\Rightarrow \sqrt{(x+1)^2+y^2} = 4$

$\Rightarrow (x+1)^2 + y^2 = 4^2$

which is a circle with centre $(-1,0)$ and radius 4.

Given $\int_C \frac{e^{3z}}{z+\pi i} dz$

Let $f(z) = e^{3z}$

$z+\pi i = 0$

$\therefore z = -\pi i$

$\Rightarrow |z| = |-\pi i| = 3.1416 < 4$

So ~~$|z|$~~ $z = -\pi i$ lies inside the circle $|z+1| = 4$.

Then by Cauchy's integral formula we get

$$\int_c \frac{e^{3z}}{z+\pi i}\,dz = \int_c \frac{f(z)}{z-(-\pi i)}\,dz$$

$$= 2\pi i\, f(-\pi i)$$

$$= 2\pi i\, e^{-i3\pi}$$

$$= 2\pi i\,(\cos 3\pi - i\sin 3\pi)$$

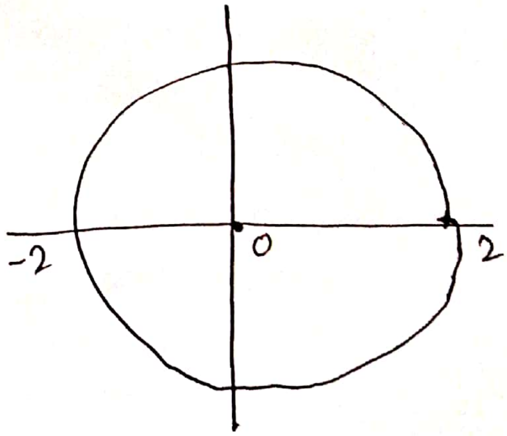
$$= 2\pi i\,(-1-0)$$

$$= -2\pi i \quad \text{(Ans)}$$

[Since $f(z) = e^{3z}$
$f(-\pi i) = e^{-3\pi i}$
And
$e^{-i\theta} = \cos\theta - i\sin\theta$

(iv) Evaluate $\int_c \frac{e^z}{z^2+1}\,dz$ over the circular path $|z| = 2$.

<u>Soln:</u> Given,



$$|z| = 2$$

$$\Rightarrow |x+iy| = 2$$

$$\Rightarrow \sqrt{x^2+y^2} = 2$$

$$\Rightarrow x^2+y^2 = 2^2$$

Which is a circle with centre (0,0) and radius 2.

Now $\mathbb{P}^{\nu}$ Given

$$\int_c \frac{e^z}{z^2+1}\, dz$$

Let $f(z) = e^z$

Now $z^2+1=0$

$\Rightarrow z^2 = -1$

$\therefore z = \pm i$

$|z| = |i| = 1$ lies inside the circle $|z|=2$

$|z| = |-i| = 1$ " " "

Now by cauchy's integral formula we get,

$$\int_c \frac{e^z}{z^2+1}\, dz = \int_c \frac{f(z)}{(z+i)(z-i)}\, dz$$

$$= \frac{1}{2i}\int_c \left[\frac{1}{z-i} - \frac{1}{z+i}\right] f(z)\, dz$$

$$= \frac{1}{2i}\int_c \frac{f(z)}{z-i}\, dz - \frac{1}{2i}\int_c \frac{1}{z+i} f(z)\, dz$$

$$= \frac{1}{2i}\cdot 2\pi i f(i) - \frac{1}{2i} 2\pi i f(-i)$$

[By cauchy's integral formula]

$= \pi e^{i} - \pi e^{-i}$

$= \pi (e^{i} - e^{-i})$

$= \pi (2i \sin 1)$

(Ans)

Since

$f(z) = e^{z}$

$f(i) = e^{i}$

$f(-i) = e^{-i}$

$\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

$\sin 1 = \dfrac{e^{i} - e^{-i}}{2i}$

Problem:

Show that $\oint_c \dfrac{e^{2z}}{(z+1)^4} dz = \dfrac{8\pi i e^{-2}}{3}$

where c is the circle $|z| = 3$

Soln: Given $|z| = 3$

$\therefore |x+iy| = 3$

$\Rightarrow x^2 + y^2 = 3^2$

which is a circle with centre $(0,0)$ and radius 3.

Here, $\oint_c \dfrac{e^{2z}}{(z+1)^4}$

Let $f(z) = e^{2z}$

Now,

$z = -1$ lies inside the circle $|z| = 3$

Then by Cauchy's integral formula for n-th derivative we get

$$\oint_c \frac{e^{2z}}{(z+1)^4}\, dz = \oint_c \frac{f(z)}{\{z-(-1)\}^4}\, dz$$

$$= 2\pi i \, \frac{f'''(-1)}{\lfloor 3} \qquad \cdots (1)$$

Here $f(z) = e^{2z}$

$f'(z) = 2e^{2z}$

$f''(z) = 4e^{2z}$

$f'''(z) = 8e^{2z}$

$f'''(-1) = 8e^{-2}$

Then from (1) we get,

$$\oint_c \frac{e^{2z}}{(z+1)^4}\, dz = 2\pi i \, \frac{8e^{-2}}{6}$$

$$= \frac{8\pi i e^{-2}}{3}.$$