Binomial Distribution

Let p be the probability of occurrence and q be the probability of non-occurrence of a particular event in a single trial so that $p+q=1$. If the experiment is repeated for n independent trials, then the probability of an event may be expressed as

$$P(x) = \binom{n}{x} p^x q^{n-x}; \quad x = 0, 1, 2, \ldots, n$$

where $q = 1-p$ and $\binom{n}{x} = {}^nC_x$;

n = Number of trials;

p = probability of success and $0 \le p \le 1$.

Special case:

If $n=1$; the distribution is known as unit (or point) binomial distribution or Bernoulli distribution.

Underlying conditions of Binomial Distribution:

Underlying conditions of trials

1. There is a fixed number of trials
2. The trials are independent
3. There are only two outcomes for each trial such as success and failure.
4. probability of success remains constant from trial to trial.
5. The number of success x $(x = 0, 1, 2, \ldots, n)$ in n trials is a discrete random variable.

Mean, Variance and standard deviation of a binomial distribution:

Mean: $\mu_1' = \sum_{x=0}^{n} x p(x)$

$= 0 \cdot p(0) + 1 \cdot p(1) + 2 \cdot p(2) + \cdots + n p(n)$

$= 0 + {}^nC_1\, p^1 q^{n-1} + 2 \cdot {}^nC_2\, p^2 q^{n-2} + \cdots + n \cdot {}^nC_n\, p^n q^{n-n}$

$= \frac{n!}{1!\,(n-1)!} p q^{n-1} + 2 \cdot \frac{n!}{2!\,(n-2)!} p^2 q^{n-2} + \cdots + n \cdot \frac{n!}{n!\,(n-n)!} p^n q^0$

$$= \frac{n\cancel{(n-1)!}}{\cancel{(n-1)!}} pq^{n-1} + 2\cdot\frac{n(n-1)\cancel{(n-2)!}}{2!\,\cancel{(n-2)!}} p^2 q^{n-2} + \cdots + np^n$$

$$= npq^{n-1} + \cancel{2}\,\frac{n(n-1)}{\cancel{2}} p^2 q^{n-2} + \cdots + np^n$$

$$= npq^{n-1} + n(n-1)p^2q^{n-2} + \cdots + np^n$$

$$= np\,[q^{n-1} + (n-1)pq^{n-2} + \cdots + p^{n-1}]$$

$$= np(q+p)^{n-1} \qquad [\because (q+p)^n = q^n + {}^nC_1 p^1 q^{n-1} + {}^nC_2 p^2 q^{n-2} + \cdots + p^n]$$

$$= np(1)^{n-1} \qquad [\because q+p=1]$$

$$= np$$

So the mean of a binomial distribution is np.

(*) The variance of a binomial distribution $= npq$

(*) The standard deviation $= \sqrt{npq}$

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* Mean of the binomial distribution is greater than its variance:

Proof: Variance $= npq$

$= np(1-p)$

$= np - np^2$

Variance $=$ Mean $- np^2$

or, Mean $=$ Variance $+ np^2$

$=$ Variance $+$ (+ve quantity)

$\therefore$ Mean $>$ Variance.

Problems

1. The mean and variance of a binomial distribution are 4 and $\frac{4}{3}$ respectively. Find (i) probability function
(ii) $P(x=0)$
(iii) $P(x \geq 1)$ or at least one success

Soln:

n and p are two parameters of a binomial distribution.

We know,

mean $= np$ and variance $= npq$

According to the question

$$np = 4 \quad \dots (1)$$

and $$npq = \frac{4}{3}$$

$$\Rightarrow 4q = \frac{4}{3} \quad [\text{using } (1)]$$

$$\Rightarrow q = \frac{1}{3}$$

$$\therefore p = 1 - q = 1 - \frac{1}{3} = \frac{2}{3}$$

Putting the values of p in (1) we get,

$$n \times \frac{2}{3} = 4$$

$$\Rightarrow n = \frac{12}{2} = 6$$

(i) We know, the probability function of a binomial variate x is

$$P(x) = {}^nC_x \, p^x q^{n-x}; \quad x = 0, 1, 2, \ldots, n$$

$$\therefore P(x) = {}^6C_x \left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{6-x}; \quad x = 0, 1, 2, \ldots$$

(ii) $$P(x=0) = {}^6C_0 \left(\frac{2}{3}\right)^0 \left(\frac{1}{3}\right)^{6-0}$$

$$= 1 \times 1 \times \left(\frac{1}{3}\right)^6$$

$$= 0.0014$$

(iii) $$P(x \geq 1) = 1 - P(x=0)$$

$$= 1 - 0.0014$$

$$= 0.9986$$

2. The mean of a binomial distribution is 40 and standard deviation is 6. Calculate n, p and q.

Soln: Let x be a binomial variate with parameters n and p.

We know, Mean $= np$

Standard deviation $= \sqrt{npq}$

According to the question we have,

$$np = 40 \quad \cdots (1)$$

and $\sqrt{npq} = 6$

$$\Rightarrow npq = 36$$

$$\Rightarrow 40q = 36$$

$$\therefore q = 0.9$$

$$\therefore p = 1 - q = 1 - 0.9$$
$$= 0.1$$

Putting the value of p in (1)

$$n \times 0.1 = 40$$

$$\therefore n = 400$$

3. An unbaised coin is tossed 6 times. Find the probability of getting

(1) Exactly 3 heads
(2) At least 5 heads
(3) At best 3 heads

Soln: Number of trials in the experiment, $n = 6$

~~Let, Number of~~

Let, x be the number of heads

We have, the probability of getting head in a single toss, $p = \frac{1}{2}$
the probability function of binomial variate, x is

$$P(x) = {}^nC_x p^x q^{n-x}; \quad x = 0, 1, 2, \ldots, n$$

$$\therefore P(x) = {}^6C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{6-x};$$

$$x = 0, 1, 2, \ldots, 6$$

(1) probability of getting exactly 3 heads:

$$P(x=3) = {}^6C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^{6-3}$$

$$= {}^6C_3 \left(\frac{1}{2}\right)^6$$

$$= \frac{6 \times 5 \times 4}{1 \times 2 \times 3} \times \frac{1}{64}$$

$$= \frac{5}{16}$$

(2) Probability of getting at least 5 heads

$$P(x \geq 5) = P(x=5) + P(x=6)$$

$$= \binom{6}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^{6-5} + \binom{6}{6}\left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right)^{6-6}$$

$$= \frac{6}{64} + \frac{1}{64} = \frac{7}{64}$$

(3) probability of getting at best 3 heads

$$P(x \leq 3) = P(x=0) + P(x=1) + P(x=2) + P(x=3)$$

$$= \binom{6}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^{6-0} + \binom{6}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^{6-1}$$

$$+ \binom{6}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{6-2} + \binom{6}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^{6-3}$$

$$= \frac{21}{32}.$$

H.W An unbaised coin is tossed 10 times. Find the probability of getting

(1) 3 heads

(2) At least one head

(3) At most one head

Problem: The probability of a new born baby will be boy or girl is equal; then among 5 new born babies, what is the probability that

(i) at least one child is boy
(ii) at best two children are boy

Soln: Since the probability of girl or boy is equal,

$$p = \frac{1}{2}, \quad q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}, \quad n = 5$$

Let, x denotes the number of new born babies who are boy

$$P(x) = {}^nC_x \, p^x \, q^{n-x}; \quad x = 0, 1, 2, \ldots, n$$

$$\Rightarrow P(x) = {}^5C_x \left(\tfrac{1}{2}\right)^x \left(\tfrac{1}{2}\right)^{5-x};$$

$$x = 0, 1, 2, \ldots, 5$$

The probability that at least one child is boy

$$P(x \geq 1) = 1 - P(x = 0)$$

$$= 1 - {}^5C_0 \left(\tfrac{1}{2}\right)^0 \left(\tfrac{1}{2}\right)^{5-0}$$

$$= 1 - 1 \times 1 \times \left(\tfrac{1}{2}\right)^5$$

$$= 1 - 0.03125$$

$$= 0.96875$$

(ii) The probability that at best two children are boy

$$P(x \leq 2) = P(x=0) + P(x=1) + P(x=2)$$

$$= {}^5C_0 \left(\tfrac{1}{2}\right)^0 \left(\tfrac{1}{2}\right)^{5-0} + {}^5C_1 \left(\tfrac{1}{2}\right)^1 \left(\tfrac{1}{2}\right)^{5-1} + {}^5C_2 \left(\tfrac{1}{2}\right)^2 \left(\tfrac{1}{2}\right)^{5-2}$$

$$= 0.5 .$$

<u>problem:</u>

probability that Bangladesh win a cricket test match against pakistan is given to be $\frac{1}{3}$. If Bangladesh and pakistan play three test matches, use binomial distribution to find the probability that

(i) Bangladesh will loss all three test matches

(ii) Bangladesh will win at least one test match.

Soln: Here,

The probability that Bangladesh win test match

$$p = \frac{1}{3}$$

$$q = 1 - p$$
$$= 1 - \frac{1}{3} = \frac{2}{3}$$

$$n = 3$$

Let x denote the number of test matches that Bangladesh wins.

$$p(x) = {}^{n}C_x \, p^x q^{n-x}; \quad x = 0, 1, 2, \ldots, n$$

$$p(x) = {}^{3}C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{n-x}; \quad x = 0, 1, 2, 3$$

(i) The probability that Bangladesh will loss all three test matches

$$P(x=0) = {}^{3}C_0 \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^{3-0}$$
$$= 1 \times 1 \times (0.6667)^3$$
$$= 0.2963$$

(ii) The probability that Bangladesh will win at least one test match:

$$P(x \geq 1) = 1 - P(x=0)$$
$$= 1 - 0.2963$$
$$= 0.7037$$

H.W In a community the probability that a newly child will be a girl is 0.4. Among 4 newly born children in that community, what is the probability that

(i) all the four girls Ans: 0.0256

(ii) at least 2 girls: Ans: 0.5248

(iii) no girl Ans: 0.1296