Rigid Body Examples

Short Course of Mechanics from a Geometric Perspective
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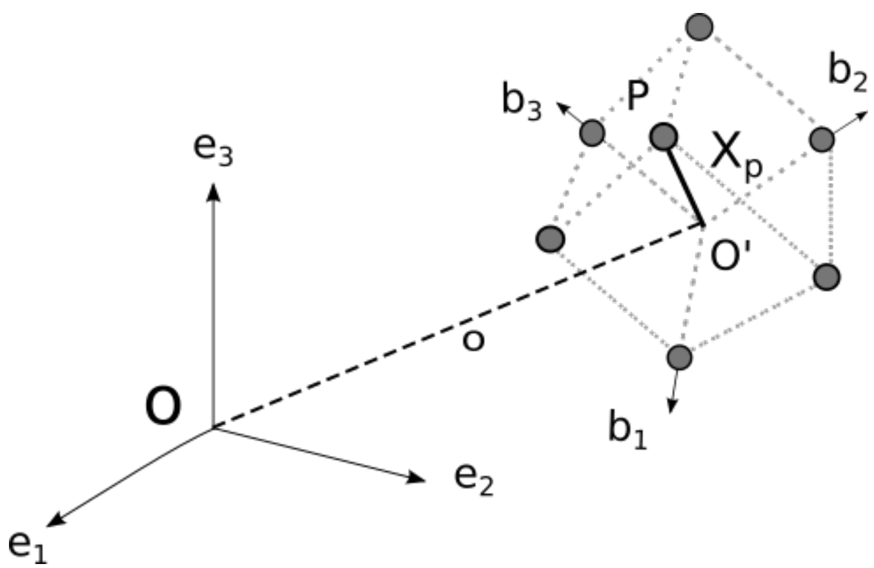
Rigid Body Equations



$$
\begin{aligned}
\pi &= \mathbb{I}_R \omega \\
\dot{R} &= \widehat{\omega} R \\
\dot{p} &= f^e \\
\dot{\pi} &= \tau^e
\end{aligned}
$$

$$
\begin{gathered}
\dot{R} = R\,\widehat{\Omega} \\
\mathbb{I}\dot{\Omega} = \mathbb{I}\Omega \times \Omega + T^e \\
M\ddot{o} = f^e
\end{gathered}
$$

$$
\begin{gathered}
\dot{R} = R\,\widehat{\mathbb{I}^{-1}\Pi} \\
\dot{\Pi} = \Pi \times \mathbb{I}^{-1}\Pi + T^e \\
M\ddot{o} = f^e
\end{gathered}
$$

Rigid Body Equations in Principle Axis Body Frame

$$\boxed{\mathbb{I} \triangleq \sum \mathbb{I}_i = \sum m_i(||X_i||^2 I_{3\times 3} - X_i X_i^T)}$$

$\mathbb{I}$ is symmetric and positive definite $\quad \mathbb{I} = \mathrm{diag}\{\mathbb{I}_1, \mathbb{I}_2, \mathbb{I}_3\}$

$$\boxed{\mathbb{I}\dot{\Omega} = \mathbb{I}\Omega \times \Omega + T^e}$$

$$\boxed{\begin{aligned} \mathbb{I}_1\dot{\Omega}_1 &= (\mathbb{I}_2 - \mathbb{I}_3)\Omega_2\Omega_3 + T_1 \\ \mathbb{I}_2\dot{\Omega}_2 &= (\mathbb{I}_3 - \mathbb{I}_1)\Omega_3\Omega_1 + T_2 \\ \mathbb{I}_3\dot{\Omega}_3 &= (\mathbb{I}_1 - \mathbb{I}_2)\Omega_1\Omega_2 + T_3 \end{aligned}}$$

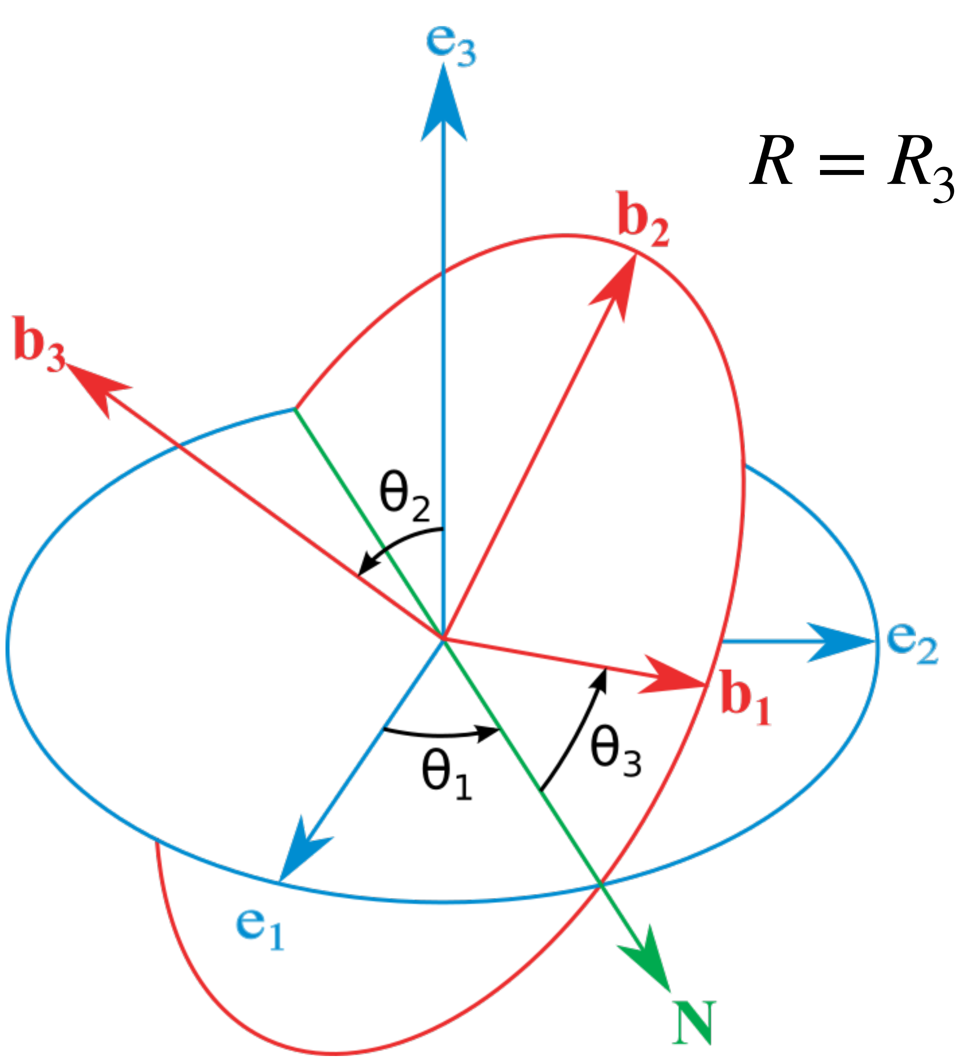
$$\boxed{\dot{\Pi} = \Pi \times \mathbb{I}^{-1}\Pi + T^e}$$

$$\boxed{\begin{aligned} \dot{\Pi}_1 &= \frac{(\mathbb{I}_2 - \mathbb{I}_3)}{\mathbb{I}_2\mathbb{I}_3}\Pi_2\Pi_3 + T_1 \\ \dot{\Pi}_2 &= \frac{(\mathbb{I}_3 - \mathbb{I}_1)}{\mathbb{I}_3\mathbb{I}_1}\Pi_3\Pi_1 + T_2 \\ \dot{\Pi}_3 &= \frac{(\mathbb{I}_1 - \mathbb{I}_2)}{\mathbb{I}_1\mathbb{I}_2}\Pi_1\Pi_2 + T_3 \end{aligned}}$$

Euler Angles

$$R = R_i(\theta_1)R_j(\theta_2)R_k(\theta_3) \in SO(3) \qquad (\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3 \to R \in SO(3)$$

When $i \neq j$ & $j \neq k$ this map is a local isomorphism **i-j-k Euler Angles**



3-1-3 Euler Angles

$$R = R_3(\theta_1)R_1(\theta_2)R_3(\theta_3) = \begin{bmatrix} c_1c_3 - c_2s_1s_3 & c_2c_3s_1 + c_1s_3 & s_1s_2 \\ -c_3s_1 - c_1c_2s_3 & c_1c_2c_3 - s_1s_3 & c_1s_2 \\ s_2s_3 & -c_3s_2 & c_2 \end{bmatrix}$$

$$c_i \triangleq \cos\theta_i \qquad s_i \triangleq \sin\theta_i$$

Singular when $\theta_2 = 0$ or $\theta_2 = \pi$

Gimbal Lock

$$\Omega = \begin{bmatrix} \dot{\theta}_1 \sin\theta_2 \sin\theta_3 + \dot{\theta}_2 \cos\theta_3 \\ \dot{\theta}_1 \sin\theta_2 \cos\theta_3 - \dot{\theta}_2 \sin\theta_3 \\ \dot{\theta}_1 \cos\theta_2 + \dot{\theta}_3 \end{bmatrix}$$

Euler's Rigid Body Equations

$$\Omega = \begin{bmatrix} \dot{\theta}_1 \sin\theta_2 \sin\theta_3 + \dot{\theta}_2 \cos\theta_3 \\ \dot{\theta}_1 \sin\theta_2 \cos\theta_3 - \dot{\theta}_2 \sin\theta_3 \\ \dot{\theta}_1 \cos\theta_2 + \dot{\theta}_3 \end{bmatrix} \qquad \Pi = \mathbb{I}\Omega = \begin{bmatrix} \mathbb{I}_1 \left(\dot{\theta}_1 \sin\theta_2 \sin\theta_3 + \dot{\theta}_2 \cos\theta_3 \right) \\ \mathbb{I}_2 \left(\dot{\theta}_1 \sin\theta_2 \cos\theta_3 - \dot{\theta}_2 \sin\theta_3 \right) \\ \mathbb{I}_3 \left(\dot{\theta}_1 \cos\theta_2 + \dot{\theta}_3 \right) \end{bmatrix}$$

$$\dot{\Pi} = \Pi \times \mathbb{I}^{-1}\Pi + T^e$$

$$\begin{aligned}
\mathbb{I}_1 \sin\theta_2 \sin\theta_3 \ddot{\theta}_1 + \mathbb{I}_1 \cos\theta_3 \ddot{\theta}_2 &= -\mathbb{I}_1 \left(\dot{\theta}_1\dot{\theta}_2 \cos\theta_2 \sin\theta_3 + \dot{\theta}_1\dot{\theta}_3 \sin\theta_2 \cos\theta_3 - \dot{\theta}_2\dot{\theta}_3 \sin\theta_3 \right) + (\mathbb{I}_2 - \mathbb{I}_3)\left(\dot{\theta}_1 \sin\theta_2 \cos\theta_3 - \dot{\theta}_2 \sin\theta_3 \right) \left(\dot{\theta}_1 \cos\theta_2 + \dot{\theta}_3 \right) + T_1^e \\
\mathbb{I}_2 \sin\theta_2 \cos\theta_3 \ddot{\theta}_1 - \mathbb{I}_2 \sin\theta_3 \ddot{\theta}_2 &= -\mathbb{I}_2 \left(\dot{\theta}_1\dot{\theta}_2 \cos\theta_2 \cos\theta_3 - \dot{\theta}_1\dot{\theta}_3 \sin\theta_2 \sin\theta_3 - \dot{\theta}_2\dot{\theta}_3 \cos\theta_3 \right) + (\mathbb{I}_3 - \mathbb{I}_1)\left(\dot{\theta}_1 \cos\theta_2 + \dot{\theta}_3 \right) \left(\dot{\theta}_1 \sin\theta_2 \sin\theta_3 + \dot{\theta}_2 \cos\theta_3 \right) + T_2^e \\
\mathbb{I}_3 \cos\theta_2 \ddot{\theta}_1 + \mathbb{I}_3 \ddot{\theta}_3 &= \mathbb{I}_3 \dot{\theta}_1 \dot{\theta}_2 \sin\theta_2 + (\mathbb{I}_1 - \mathbb{I}_2)\left(\dot{\theta}_1 \sin\theta_2 \sin\theta_3 + \dot{\theta}_2 \cos\theta_3 \right) \left(\dot{\theta}_1 \sin\theta_2 \cos\theta_3 - \dot{\theta}_2 \sin\theta_3 \right) + T_3^e .
\end{aligned}$$

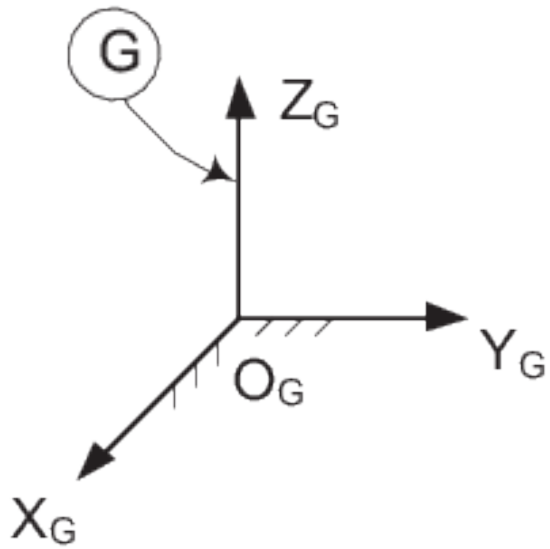
Singular $\qquad \theta_2 = 0 \quad \text{or} \quad \theta_2 = \pi$

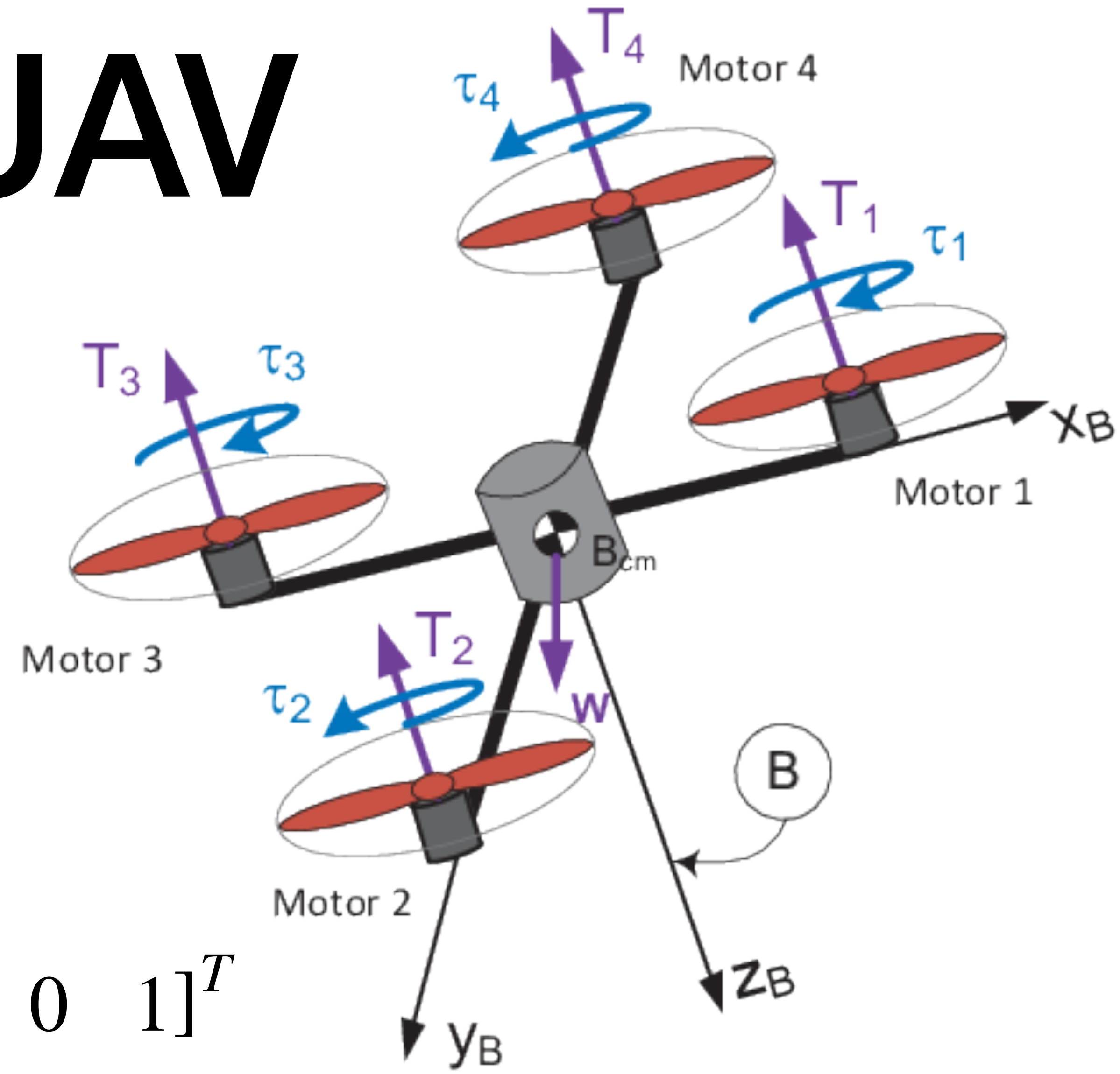
Hopefully you will never use these after this course!!

MRUAV

$$\mathbf{b} = \mathbf{e}R$$

$$O_G B_{cm} = \mathbf{e}o$$





$$\gamma_3 \triangleq [0 \quad 0 \quad 1]^T$$

$$T^u = \begin{bmatrix} 0 & lc_l & -lc_l & 0 \\ -lc_l & 0 & lc_l & 0 \\ -c_d & c_d & -c_d & c_d \end{bmatrix} \begin{bmatrix} \omega_1^2 \\ \omega_2^2 \\ \omega_3^2 \\ \omega_4^2 \end{bmatrix}$$

$$f^u = c_l(\omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2)$$

$$\boxed{\begin{aligned} M\ddot{o} &= f^u R\gamma_3 - mg\gamma_3 \\ \dot{R} &= R\,\widehat{\Omega} \\ \mathbb{I}\dot{\Omega} &= \mathbb{I}\Omega \times \Omega + T^u \end{aligned}} \rightarrow \quad \mathbb{I}\nabla_\Omega \Omega = T^u$$

$$\mathbb{I}\nabla_\Omega \Omega \triangleq \mathbb{I}\dot{\Omega} - \mathbb{I}\Omega \times \Omega$$

Quanser Aero

$$\mathbf{a} = \mathbf{e}R_3(\theta) \qquad \mathbf{b} = \mathbf{a}R_1(\phi)$$

$$R_3(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R_1(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

$$\mathbf{b} = \mathbf{a}R_1(\phi) = \mathbf{e}\,\underbrace{R_3(\theta)R_1(\phi)}_{R}$$

$$R = R_3R_1 = \begin{bmatrix} \cos\theta & -\sin\theta & \sin\theta\sin\phi \\ \sin\theta & \cos\theta\cos\phi & -\cos\theta\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

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$$\gamma_1 = [1 \quad 0 \quad 0]^T \quad \gamma_2 = [0 \quad 1 \quad 0]^T \quad \gamma_3 = [0 \quad 0 \quad 1]^T$$

$$\mathbf{b} = \mathbf{a}R_1(\phi) = \mathbf{e}\,\underbrace{R_3(\theta)R_1(\phi)}_{R}$$

$$\widehat{\Omega} = R^T\dot{R} = \dot{\theta}R_1^T\widehat{\gamma}_3R_1 + \dot{\phi}\,\widehat{\gamma}_1 = \dot{\theta}\widehat{R_1^T\gamma_3} + \dot{\phi}\,\widehat{\gamma}_1 = \dot{\theta}\widehat{R_1^T\gamma_3 + \dot{\phi}\gamma_1}$$

$$\Omega = \dot{\theta}R_1^T\gamma_3 + \dot{\phi}\gamma_1 = \begin{bmatrix} \dot{\phi} \\ \dot{\theta}\sin\phi \\ \dot{\theta}\cos\phi \end{bmatrix} = \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{bmatrix}$$

Quanser Aero

$$\gamma_1 = [1 \quad 0 \quad 0]^T \qquad \gamma_2 = [0 \quad 1 \quad 0]^T \qquad \gamma_3 = [0 \quad 0 \quad 1]^T$$

$$\Omega = \dot{\theta} R_1^T \gamma_3 + \dot{\phi} \gamma_1 = \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \sin \phi \\ \dot{\theta} \cos \phi \end{bmatrix} = \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{bmatrix}$$

$$\mathbf{a} \omega_a = \mathbf{b} \Omega \qquad \rightarrow \qquad \mathbf{a} \omega_a = \mathbf{a} R_1 \Omega$$

$$\mathbf{a} = \mathbf{e} R_3(\theta) \qquad \mathbf{b} = \mathbf{a} R_1(\phi)$$

$$\omega_a = R_1 \Omega = \Omega_3 + \Omega_1 = \begin{bmatrix} \dot{\phi} \\ 0 \\ \dot{\theta} \end{bmatrix}$$

$$\mathbf{b} = \mathbf{a} R_1(\phi) = \mathbf{e} \underbrace{R_3(\theta) R_1(\phi)}_{R}$$

$\gamma_2 \cdot \omega_a = 0 \quad \rightarrow$ **Constraints are automatically satisfied !**

Quanser Aero

$$\gamma_1 = [1 \quad 0 \quad 0]^T \qquad \gamma_2 = [0 \quad 1 \quad 0]^T \qquad \gamma_3 = [0 \quad 0 \quad 1]^T$$

$$\Omega = \dot{\theta} R_1^T \gamma_3 + \dot{\phi}\gamma_1 = \begin{bmatrix} \dot{\phi} \\ \dot{\theta}\sin\phi \\ \dot{\theta}\cos\phi \end{bmatrix} = \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{bmatrix}$$

$$\Pi = \mathbb{I}\Omega = \begin{bmatrix} \mathbb{I}_1\dot{\phi} \\ \mathbb{I}_2\dot{\theta}\sin\phi \\ \mathbb{I}_3\dot{\theta}\cos\phi \end{bmatrix}$$

If the outer gimbal is much lighter than the arm $\quad \rightarrow \quad T = u_1\gamma_1 + u_2\gamma_3 + \lambda\gamma_2$

$$\mathbb{I}_m\dot{\Omega} = \mathbb{I}_m\Omega \times \Omega + T^e \quad \rightarrow \quad \begin{aligned} \mathbb{I}_1\dot{\Omega}_1 &= (\mathbb{I}_2 - \mathbb{I}_3)\Omega_2\Omega_3 + u_1, \\ \mathbb{I}_2\dot{\Omega}_2 &= (\mathbb{I}_3 - \mathbb{I}_1)\Omega_3\Omega_1 + \lambda, \\ \mathbb{I}_3\dot{\Omega}_3 &= (\mathbb{I}_1 - \mathbb{I}_2)\Omega_1\Omega_2 + u_2, \end{aligned}$$

Quanser Aero

$$\gamma_1 = [1 \quad 0 \quad 0]^T \qquad \gamma_2 = [0 \quad 1 \quad 0]^T \qquad \gamma_3 = [0 \quad 0 \quad 1]^T$$

$$\Omega = \dot{\theta} R_1^T \gamma_3 + \dot{\phi} \gamma_1 = \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \sin \phi \\ \dot{\theta} \cos \phi \end{bmatrix} = \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{bmatrix}$$

$$\begin{aligned} \mathbb{I}_1 \dot{\Omega}_1 &= (\mathbb{I}_2 - \mathbb{I}_3)\Omega_2 \Omega_3 + u_1, \\ \mathbb{I}_2 \dot{\Omega}_2 &= (\mathbb{I}_3 - \mathbb{I}_1)\Omega_3 \Omega_1 + \lambda, \\ \mathbb{I}_3 \dot{\Omega}_3 &= (\mathbb{I}_1 - \mathbb{I}_2)\Omega_1 \Omega_2 + u_2, \end{aligned}$$

E.O.M

$$\begin{aligned} \mathbb{I}_1 \ddot{\phi} &= (\mathbb{I}_2 - \mathbb{I}_3)\dot{\theta}^2 \sin \phi \cos \phi + u_1, \\ \mathbb{I}_3 \ddot{\theta} \cos \phi &= \left((\mathbb{I}_1 - \mathbb{I}_2)\cos \phi - \mathbb{I}_3 \sin \phi\right) \dot{\theta}\dot{\phi} + u_2, \end{aligned}$$

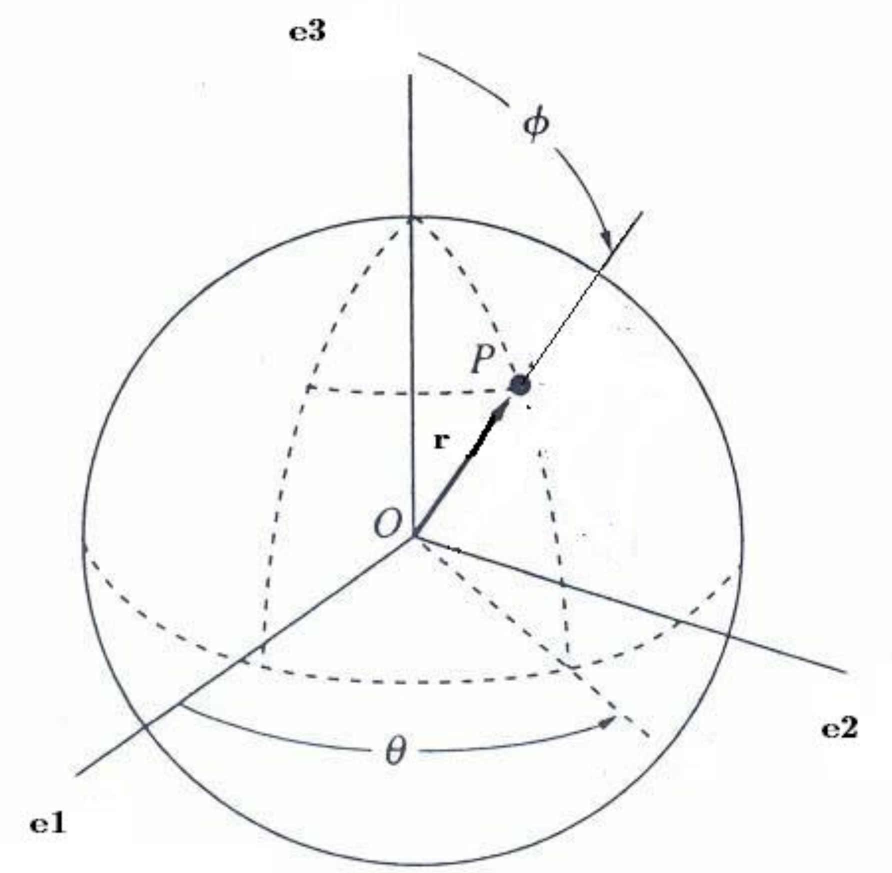
Constraint Moment

$$\lambda = \mathbb{I}_2(\ddot{\theta} \sin \phi + \dot{\theta}\dot{\phi} \cos \phi) - (\mathbb{I}_3 - \mathbb{I}_1)\dot{\theta}\dot{\phi} \cos \phi$$

Spherical Pendulum

$$\mathbf{a} = \mathbf{e}R_3(\theta) \qquad \mathbf{b} = \mathbf{a}R_2(\phi)$$

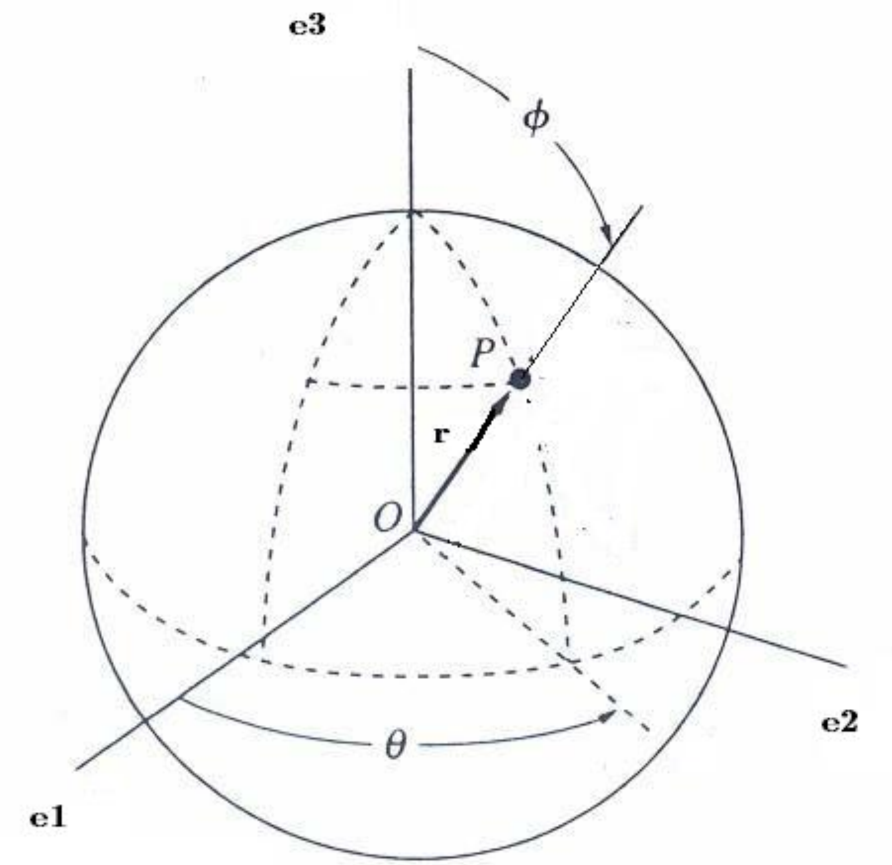
$$R_3(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R_2(\phi) = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}$$



$$\mathbf{b} = \mathbf{a}R_2(\phi) = \mathbf{e}\,\underbrace{R_3(\theta)R_2(\phi)}_{R}$$

$$R = R_3R_2 = \begin{bmatrix} \cos\theta\cos\phi & -\sin\theta & \cos\theta\sin\phi \\ \sin\theta\cos\phi & \cos\theta & \sin\theta\sin\phi \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}$$

Spherical Pendulum



$$\mathbf{a} = \mathbf{e}R_3(\theta) \qquad \mathbf{b} = \mathbf{a}R_2(\phi)$$

$$R_3(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R_2(\phi) = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}$$

$$\mathbf{b} = \mathbf{a}R_2(\phi) = \mathbf{e}\,\underbrace{R_3(\theta)R_2(\phi)}_{R}$$

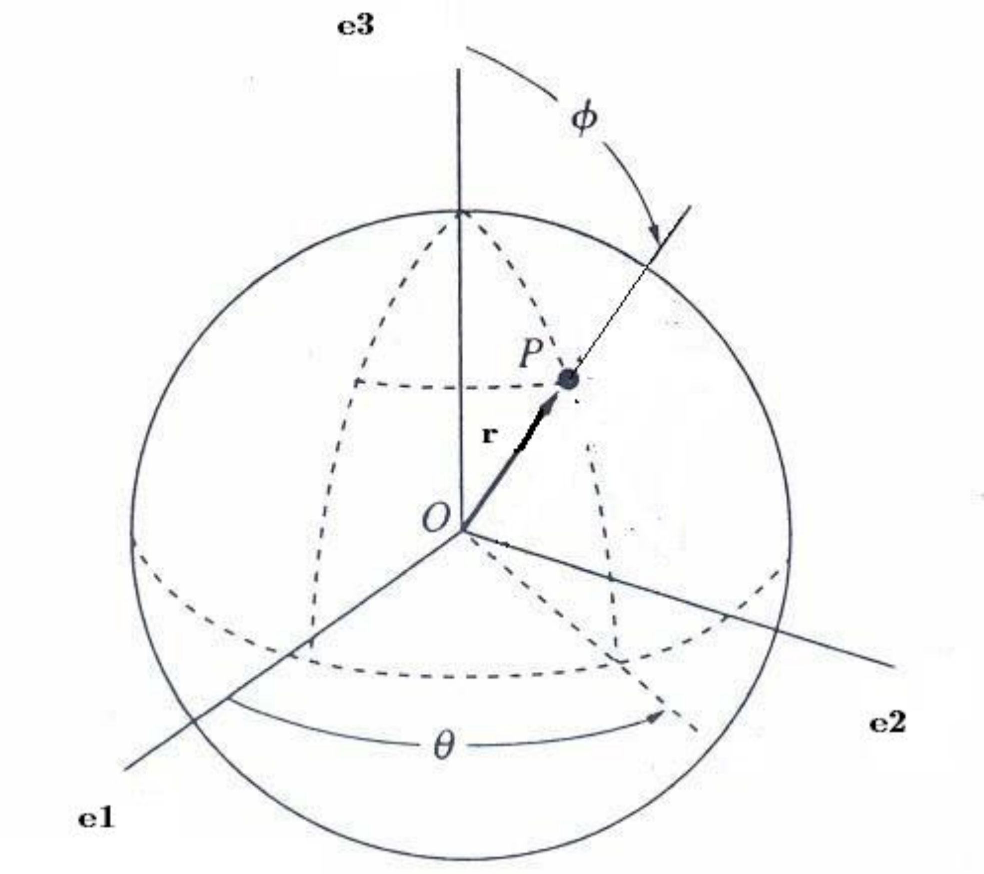
$$\gamma_1 = [1 \quad 0 \quad 0]^T \qquad \gamma_2 = [0 \quad 1 \quad 0]^T \qquad \gamma_3 = [0 \quad 0 \quad 1]^T$$

$$\widehat{\Omega} = R^T\dot{R} = \dot{\theta}R_2^T\widehat{\gamma}_3R_2 + \dot{\phi}\,\widehat{\gamma}_2 = \dot{\theta}\widehat{R_2^T\gamma_3} + \dot{\phi}\,\widehat{\gamma}_2 = \widehat{\dot{\theta}R_2^T\gamma_3 + \dot{\phi}\gamma_2}$$

$$\Omega = \dot{\theta}R_2^T\gamma_3 + \dot{\phi}\gamma_2 = \begin{bmatrix} -\dot{\theta}\sin\phi \\ \dot{\phi} \\ \dot{\theta}\cos\phi \end{bmatrix} = \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{bmatrix}$$

Spherical Pendulum

$$\gamma_1 = [1 \quad 0 \quad 0]^T \qquad \gamma_2 = [0 \quad 1 \quad 0]^T \qquad \gamma_3 = [0 \quad 0 \quad 1]^T$$



$$\Omega = \dot{\theta} R_2^T \gamma_3 + \dot{\phi}\gamma_2 = \begin{bmatrix} -\dot{\theta}\sin\phi \\ \dot{\phi} \\ \dot{\theta}\cos\phi \end{bmatrix} = \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{bmatrix}$$

$$OP = \mathbf{e}x = \mathbf{b}X \qquad X = [0 \quad 0 \quad r]^T$$

$$\dot{x} = R\,\widehat{\Omega}\,X$$
$$\ddot{x} = R(\widehat{\Omega}^2 X + \widehat{\dot{\Omega}}\,X)$$

$$\mathbf{a} = \mathbf{e}R_3(\theta) \qquad \mathbf{b} = \mathbf{a}R_2(\phi)$$

$$\mathbf{b} = \mathbf{a}R_2(\phi) = \mathbf{e}\,\underbrace{R_3(\theta)R_2(\phi)}_{R}$$

$$R^T f^e = F^e = u_1\gamma_1 + u_2\gamma_2 - mgR^T\gamma_3$$

Spherical Pendulum

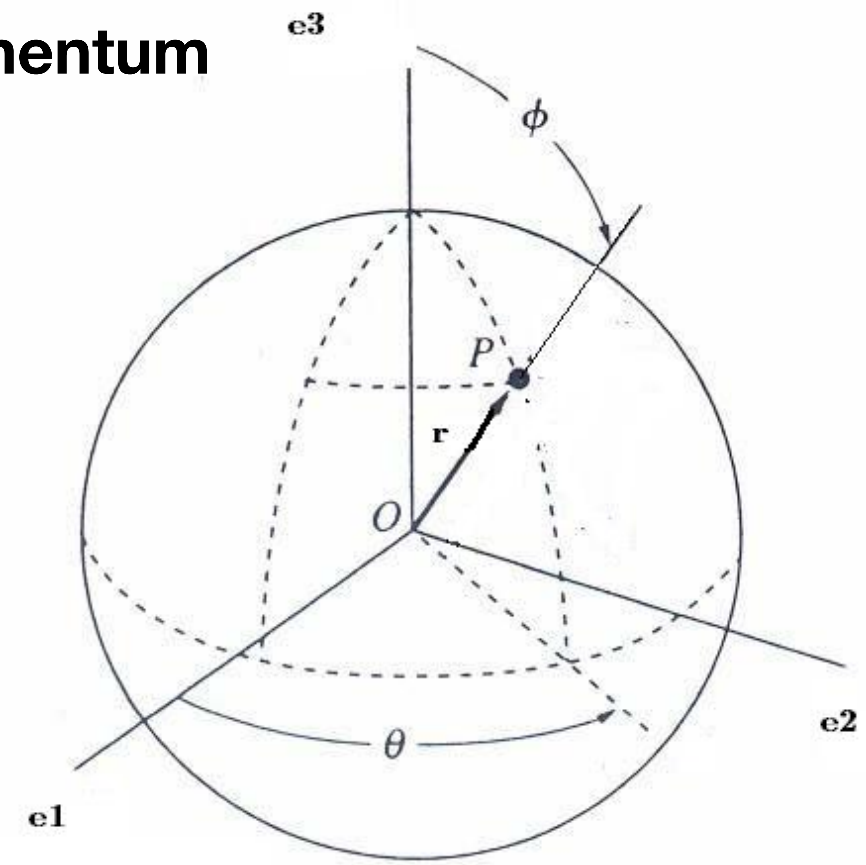
Approach I - Rate of change of Angular Momentum

$$\mathbf{b} = \mathbf{a}R_2(\phi) = \mathbf{e}\,\underbrace{R_3(\theta)R_2(\phi)}_{R}$$

$$OP = \mathbf{e}x = \mathbf{b}X \qquad X = [0 \quad 0 \quad r]^T$$

$$\dot{x} = R\,\widehat{\Omega}\,X$$

$$R^T f^e = F^e = u_1\gamma_1 + u_2\gamma_2 - mgR^T\gamma_3$$

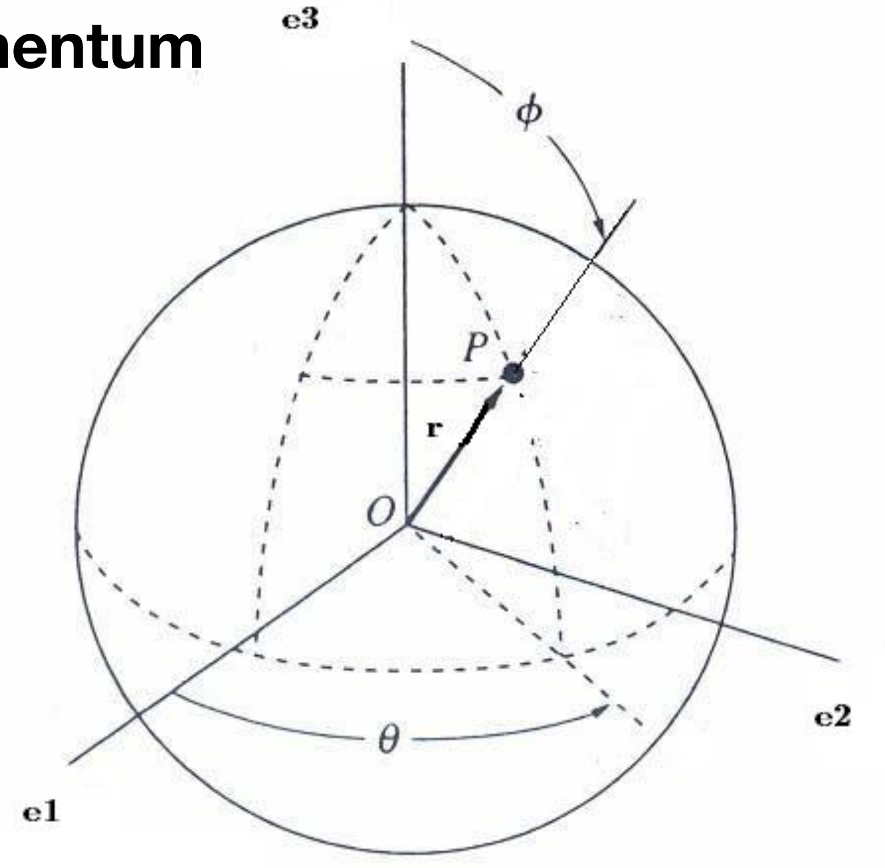


$$\pi = x \times m\dot{x} = (RX) \times (R\,\widehat{\Omega}\,X) = mR(X \times \Omega \times X) = -\,m\,R\,\widehat{X}^{\,2}\Omega = R\mathbb{I}_m\Omega$$

$$\mathbb{I}_m \triangleq -\,m\,\widehat{X}^{\,2} = \begin{bmatrix} mr^2 & 0 & 0 \\ 0 & mr^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Spherical Pendulum

Approach I - Rate of change of Angular Momentum



$$\mathbf{b} = \mathbf{a}R_2(\phi) = \mathbf{e}\,\underbrace{R_3(\theta)R_2(\phi)}_{R}$$

$$OP = \mathbf{e}x = \mathbf{b}X \qquad X = [0 \quad 0 \quad r]^T$$

$$\dot{x} = R\,\widehat{\Omega}\,X$$

$$R^T f^e = F^e = u_1\gamma_1 + u_2\gamma_2 - mgR^T\gamma_3$$

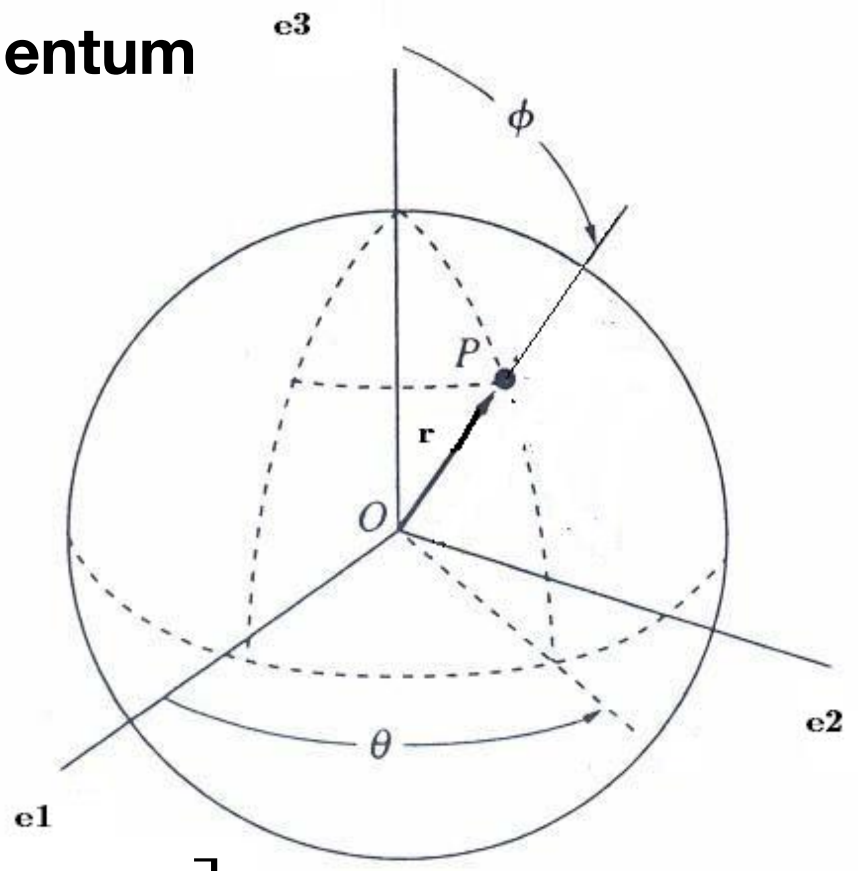
$$\pi = x \times m\dot{x} = (RX) \times (R\,\widehat{\Omega}\,X) = mR(X \times \Omega \times X) = -\,m\,R\,\widehat{X}^{\,2}\Omega = R\mathbb{I}_m\Omega$$

$$x \times f^e = \dot{\pi} = R(\mathbb{I}_m\dot{\Omega} + \Omega \times \mathbb{I}_m\Omega) \quad \rightarrow \quad R^T(x \times f^e) = \mathbb{I}_m\dot{\Omega} - \mathbb{I}_m\Omega \times \Omega$$

$$\boxed{\mathbb{I}_m\dot{\Omega} - \mathbb{I}_m\Omega \times \Omega = X \times F^e = r\gamma_3 \times F^e}$$

Spherical Pendulum

Approach I - Rate of change of Angular Momentum



$$R^T f^e = F^e = u_1\gamma_1 + u_2\gamma_2 - mgR^T\gamma_3$$

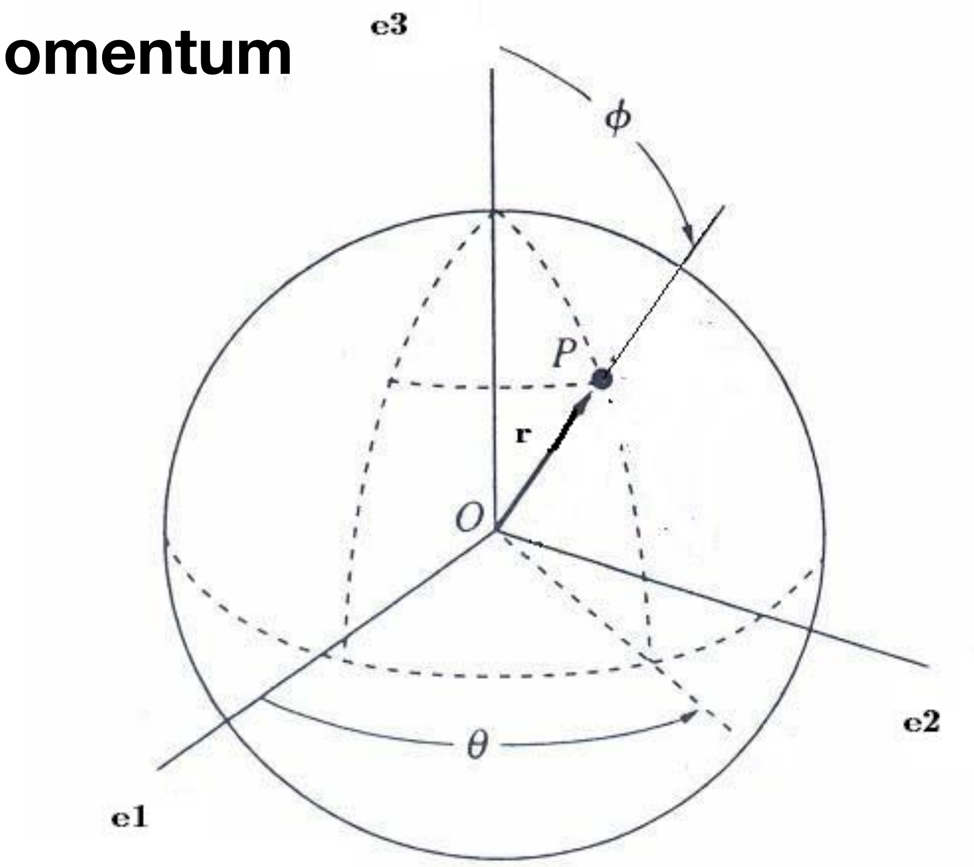
$$\boxed{\mathbb{I}_m\dot{\Omega} - \mathbb{I}_m\Omega \times \Omega = X \times F^e = r\gamma_3 \times F^e}$$

$$\mathbb{I}_m\Omega \times \Omega = \begin{bmatrix} 0 & 0 & mr^2\Omega_2 \\ 0 & 0 & -mr^2\Omega_1 \\ -mr^2\Omega_2 & mr^2\Omega_1 & 0 \end{bmatrix} \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{bmatrix} = \begin{bmatrix} mr^2\Omega_2\Omega_3 \\ -mr^2\Omega_1\Omega_3 \\ 0 \end{bmatrix}$$

$$\gamma_3 \times R_2^T\gamma_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\sin\phi \\ 0 \\ \cos\phi \end{bmatrix} = \begin{bmatrix} 0 \\ -\sin\phi \\ 0 \end{bmatrix}$$

Spherical Pendulum

Approach I - Rate of change of Angular Momentum



$$R^T f^e = F^e = u_1\gamma_1 + u_2\gamma_2 - mgR^T\gamma_3$$

$$\boxed{\mathbb{I}_m\dot{\Omega} - \mathbb{I}_m\Omega\times\Omega = X\times F^e = r\gamma_3\times F^e}$$

$$mr^2\,\dot{\Omega}_1 = mr^2\Omega_2\Omega_3 + u_1$$
$$mr^2\,\dot{\Omega}_2 = mr^2\Omega_1\Omega_3 + mgr\,\sin\phi + u_2$$

$$\Omega = \begin{bmatrix} -\dot{\theta}\sin\phi \\ \dot{\phi} \\ \dot{\theta}\cos\phi \end{bmatrix} = \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{bmatrix} \quad \rightarrow \quad \begin{aligned} mr^2\,\ddot{\theta}\sin\phi &= -\left(2mr^2\dot{\phi}\dot{\theta}\cos\phi + u_1\right) \\ mr^2\,\ddot{\phi} &= -\,mr^2\dot{\theta}^2\sin\phi\cos\phi + mgr\,\sin\phi + u_2 \end{aligned}$$

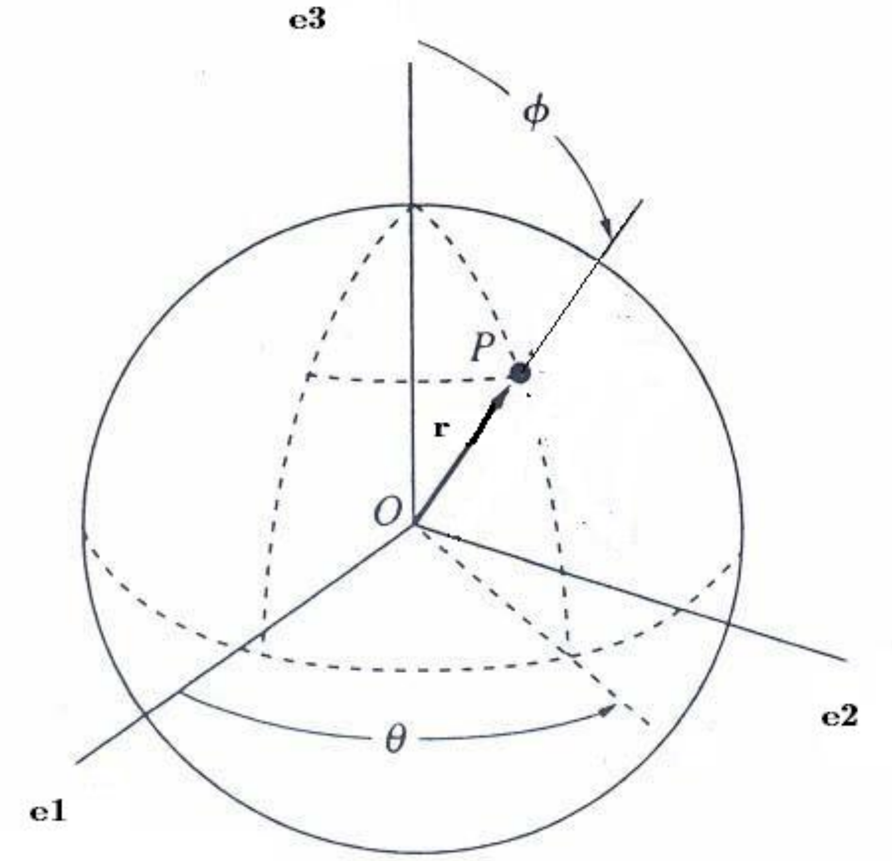
Spherical Pendulum

Approach II - Newton's equations

$$R^T f^e = F^e = u_1\gamma_1 + u_2\gamma_2 - mgR^T\gamma_3$$

$$\dot{x} = R\,\widehat{\Omega}\,X$$

$$\ddot{x} = R\left(\widehat{\Omega}^{\,2}X + \widehat{\dot{\Omega}}\,X\right)$$



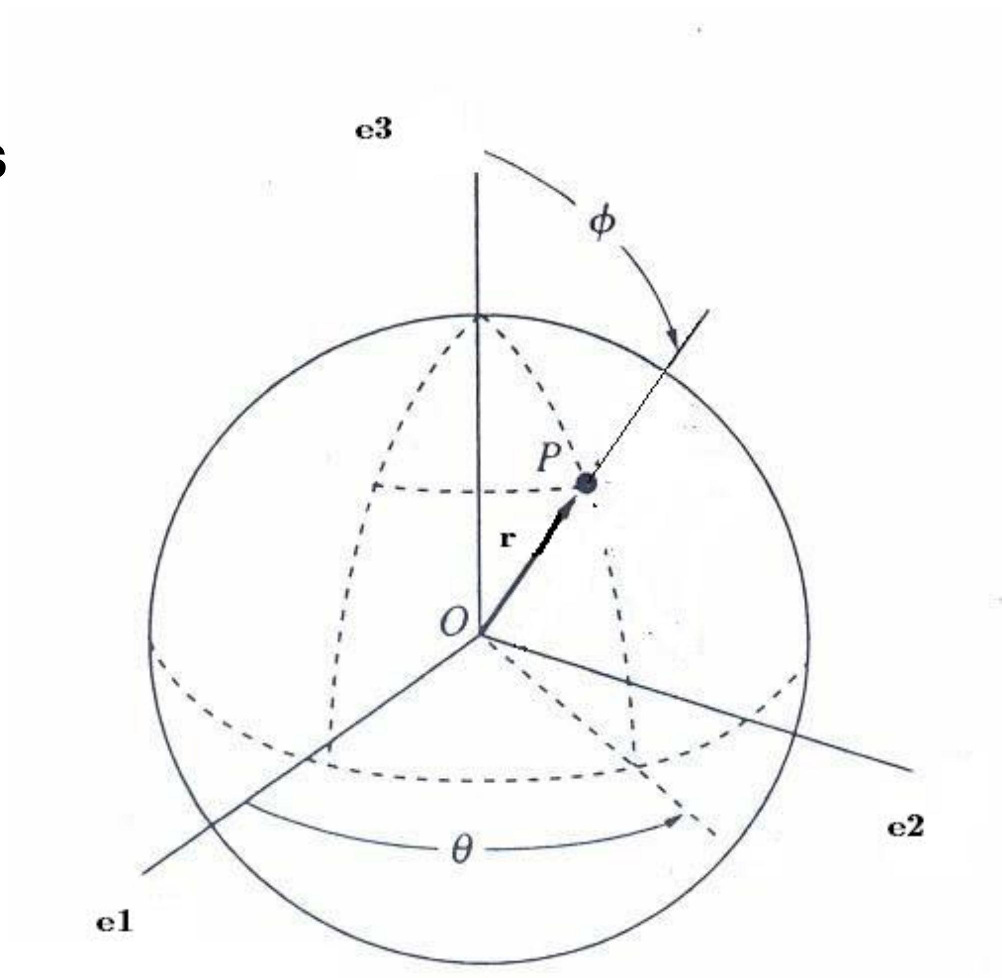
$$f^e = m\ddot{x} = R\left(m\widehat{\Omega}^{\,2}X + m\widehat{\dot{\Omega}}\,X\right) = R\left(m\Omega\times\Omega\times X - mX\times\dot{\Omega}\right)$$

$$m\,\Omega\times\Omega\times X - m\,X\times\dot{\Omega} = R^T f^e = F^e$$

$$m\,X\times\Omega\times\Omega\times X - m\,X\times X\times\dot{\Omega} = X\times F^e$$

Spherical Pendulum

Approach II - Newton's equations



$$R^T f^e = F^e = u_1\gamma_1 + u_2\gamma_2 - mgR^T\gamma_3$$

$$m\, X \times \Omega \times \Omega \times X - m\, X \times X \times \dot{\Omega} = X \times F^e$$

$$\widehat{A}^{\,2} = AA^T - ||A||^2 I_{3\times 3} \quad \rightarrow \quad \begin{aligned} \widehat{X}\,\widehat{\Omega}^{\,2} X &= \widehat{X}\,\Omega\Omega^T X = (X \times \Omega)(\Omega \cdot X) \\ \widehat{\Omega}\,\widehat{X}^{\,2}\Omega &= \widehat{\Omega}\,XX^T\Omega = -\,(X \times \Omega)(\Omega \cdot X) \end{aligned}$$

$$-m\,\widehat{\Omega}\,\widehat{X}^{\,2}\Omega - m\,\widehat{X}^{\,2}\dot{\Omega} = X \times F^e \qquad \rightarrow \qquad \mathbb{I}_m\dot{\Omega} - \mathbb{I}_m\Omega \times \Omega = X \times F^e$$