10.3.5.4.4

EE24BTECH11017 - D.Karthik

Problem Statement : Places A and B are 100 km apart on a highway. One car starts from A and another from B at the same time. If the cars travel in the same direction at different speeds, they meet in 5 hours. If they travel towards each other, they meet in 1 hour. What are the speeds of the two cars?

Solution

Let:

- The speed of the car starting from A be x km/h.
- The speed of the car starting from B be y km/h.

Case 1: Cars Traveling in the Same Direction

- The relative speed is $(x-y)$ km/h. - They meet after 5 hours, so the distance traveled by their relative speed is:

$$5(x-y) = 100$$

$$x - y = 20 \quad \text{(Equation 1)}$$

Case 2: Cars Traveling Towards Each Other

- The relative speed is $(x+y)$ km/h. - They meet after 1 hour, so the total distance covered is:

$$1(x+y) = 100$$

$$x + y = 100 \quad \text{(Equation 2)}$$

Solving the Equations

We now solve the system:

$$x - y = 20$$

$$x + y = 100$$

Adding both equations:

$$(x-y) + (x+y) = 20 + 100$$

$$2x = 120$$

$$x = 60$$

Substituting $x = 60$ into Equation 2:

$$60 + y = 100$$

$$y = 40$$

Conclusion

- The speed of the car starting from A is **60 km/h**.
- The speed of the car starting from B is **40 km/h**.

Given System of Equations

$$x - y = 20 \tag{0.1}$$
$$x + y = 100 \tag{0.2}$$

0.1 Step 1: Matrix Representation

Rewriting the system in matrix form:

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 20 \\ 100 \end{bmatrix} \tag{0.3}$$

Define:

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad B = \begin{bmatrix} 20 \\ 100 \end{bmatrix} \tag{0.4}$$

0.2 Step 2: LU Decomposition

We express A as:

$$A = LU \tag{0.5}$$

where L is a **lower triangular matrix**, and U is an **upper triangular matrix**:

$$L = \begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix} \tag{0.6}$$

Expanding:

$$\begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \tag{0.7}$$

By comparing entries:

$$u_{11} = 1, \quad u_{12} = -1 \tag{0.8}$$
$$l_{21} \cdot u_{11} = 1 \Rightarrow l_{21} = 1 \tag{0.9}$$
$$l_{21} \cdot u_{12} + u_{22} = 1 \Rightarrow (1)(-1) + u_{22} = 1 \Rightarrow u_{22} = 2 \tag{0.10}$$

Thus:

$$L = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \tag{0.11}$$

0.3 *Step 3: Solve LY = B (Forward Substitution)*

We solve:

$$LY = B \tag{0.12}$$

where $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ is an intermediate variable.

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 100 \end{bmatrix} \tag{0.13}$$

Solving:

$$y_1 = 20 \tag{0.14}$$

$$y_1 + y_2 = 100 \Rightarrow 20 + y_2 = 100 \Rightarrow y_2 = 80 \tag{0.15}$$

Thus:

$$Y = \begin{bmatrix} 20 \\ 80 \end{bmatrix} \tag{0.16}$$

Step 3: Solve $UX = Y$ (Backward Substitution)

We solve:

$$UX = Y \tag{0.17}$$

$$\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 20 \\ 80 \end{bmatrix} \tag{0.18}$$

Solving:

$$2y = 80 \Rightarrow y = 40 \tag{0.19}$$

$$x - y = 20 \Rightarrow x - 40 = 20 \Rightarrow x = 60 \tag{0.20}$$

Final Answer

$$\boxed{x = 60, \quad y = 40} \tag{0.21}$$

Conclusion

Thus, the speed of the first car is $60km/h$, and the speed of the second car is $40km/h$.

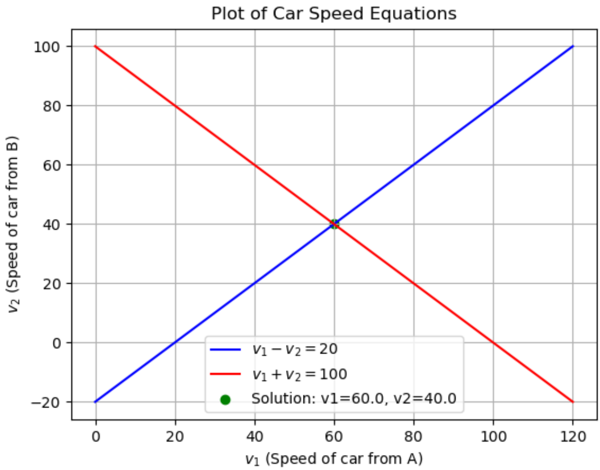

Fig. 0.1: Graphical Representation of the Solution