10.4.4.3

EE24BTECH11017 - D.Karthik

Question:
Is it possible to design a rectangular mango grove whose length is twice its breadth, and the area is $800\,\text{m}^2$? If so, find its length and breadth.
Solution:
Let the breadth of the rectangular mango grove be b meters. Then the length is $2b$ meters (as it is twice the breadth). The area of the rectangle is given as $800\,\text{m}^2$.
The relationship between the length, breadth, and area can be expressed as:

$$\text{Area} = \text{Length} \times \text{Breadth} \tag{0.1}$$
$$800 = 2b \cdot b \tag{0.2}$$

This simplifies to a quadratic equation:

$$2b^2 = 800 \tag{0.3}$$
$$b^2 = 400 \tag{0.4}$$
$$b^2 - 400 = 0 \tag{0.5}$$

Using the Newton-Raphson Method

To solve $B^2 - 400 = 0$ using the Newton-Raphson method:

- Define $f(B) = B^2 - 400$, so $f'(B) = 2B$.
- The iterative formula is:

$$B_{n+1} = B_n - \frac{f(B_n)}{f'(B_n)} = B_n - \frac{B_n^2 - 400}{2B_n} \tag{0.6}$$

Let the initial guess be $B_0 = 10$:

$$B_1 = B_0 - \frac{B_0^2 - 400}{2B_0} = 10 - \frac{10^2 - 400}{2 \times 10} = 25 \tag{0.7}$$
$$B_2 = B_1 - \frac{B_1^2 - 400}{2B_1} = 25 - \frac{25^2 - 400}{2 \times 25} = 20.4 \tag{0.8}$$
$$B_3 = B_2 - \frac{B_2^2 - 400}{2B_2} \approx 20.0001 \tag{0.9}$$

Thus, $B \approx 20\,\text{m}$ and $L = 40\,\text{m}$.

Conclusion

It is **possible** to design such a mango grove with:

$$\text{Length} = 40\,\text{m}, \quad \text{Breadth} = 20\,\text{m}. \tag{0.10}$$

Using Eigenvalue Approach:
For this approach, we can model the problem as solving a quadratic equation using matrix methods. The general quadratic equation is given as:

$$ax^2 + bx + c = 0$$

The corresponding companion matrix for this equation is:

$$\mathbf{C} = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix}$$

For the quadratic equation $2x^2 - 800 = 0$, we have $a = 2, b = 0, c = -800$. The companion matrix becomes:

$$\mathbf{C} = \begin{pmatrix} 0 & 1 \\ 400 & 0 \end{pmatrix}$$

The eigenvalues of the companion matrix correspond to the roots of the quadratic equation. Using the determinant of the matrix, we solve for the eigenvalues:

$$\text{Determinant} = \lambda^2 - 400 = 0 \implies \lambda = \pm 20$$

Hence, the eigenvalues are 20 and -20, representing the breadth and length (ignoring the negative sign for length, we consider the breadth as 20 meters, and length as 40 meters).

Using QR Decomposition Method

To find the eigenvalues of the matrix, we use the QR algorithm:

$$x^2 - 400 = 0 \tag{0.11}$$

the roots of this equation are the eigenvalues of the companion matrix:

$$C = \begin{bmatrix} 0 & 1 \\ 400 & 0 \end{bmatrix} \tag{0.12}$$

Step 2: QR Decomposition
The QR decomposition of a matrix C expresses it as the product of an orthogonal matrix Q and an upper triangular matrix R:

$$C = QR \tag{0.13}$$

1. Extract Column Vectors
The columns of C are:

$$c_1 = \begin{bmatrix} 0 \\ 400 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{0.14}$$

2. Compute Orthonormal Basis for Q
First, compute the first orthonormal vector:

$$q_1 = \frac{c_1}{||c_1||} = \frac{1}{\sqrt{0^2 + 400^2}} \begin{bmatrix} 0 \\ 400 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{0.15}$$

Compute the projection of c_2 onto q_1:

$$\text{proj}_{q_1} c_2 = (q_1^T c_2) q_1 = (0 \cdot 1 + 1 \cdot 0) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{0.16}$$

Compute the second orthonormal vector:

$$q_2 = \frac{c_2 - \text{proj}_{q_1} c_2}{||c_2 - \text{proj}_{q_1} c_2||} = \frac{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}{||\begin{bmatrix} 1 \\ 0 \end{bmatrix}||} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{0.17}$$

Thus, the orthonormal matrix Q is:

$$Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{0.18}$$

3. Compute the Upper Triangular Matrix R

$$R = Q^T C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^T \begin{bmatrix} 0 & 1 \\ 400 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 400 & 0 \end{bmatrix} \tag{0.19}$$

Step 3: Compute Eigenvalues Using the QR Algorithm
The next iterate of the matrix is:

$$C_1 = RQ = \begin{bmatrix} 0 & 1 \\ 400 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 20 & 0 \\ 0 & -20 \end{bmatrix} = \begin{bmatrix} 20 & 0 \\ 0 & -20 \end{bmatrix} \tag{0.20}$$

Since C_1 is diagonal, its diagonal elements are the eigenvalues:

$$\lambda_1 = 20, \quad \lambda_2 = -20 \tag{0.21}$$

1 Conclusion

The eigenvalues of the matrix C are $\lambda_1 = 20$ and $\lambda_2 = -20$. If these eigenvalues represent dimensions, we take the positive root:

$$\text{Breadth} = 20 \text{ m}, \quad \text{Length} = 40 \text{ m} \tag{0.22}$$

Thus, the dimensions of the mango grove are 40×20 meters. **Graphical Representation:**
To visualize this, the rectangle can be plotted with the given dimensions, as shown below:

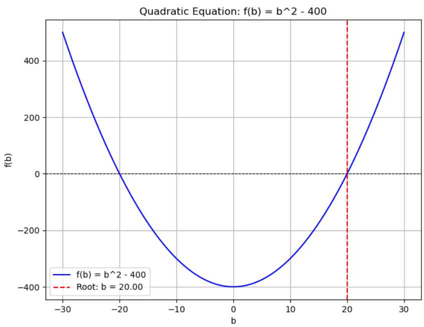

Fig. 0.1: A rectangular mango grove with length 40 m and breadth 20 m.