26. STATISTICS

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① **Measures of Central Tendency** : A measure of central tendency is single value that attempts to describe a set of data by identifying the central position within that set of data As such measure of central tendency are sometimes called measures of central location

Measure of Central Tendency
- Mathematical Averages
  - Mean
  - Median
  - Mode
- Positional Averages
  - Median
  - Quartiles
  - Deciles
  - Percentiles
  - Mode

② **Mean** : Sum of all Values of data divided by no. of values.

i.e. $$\boxed{\bar{x} = \frac{\sum x_i}{n}}$$

Mean
- Arithmetic mean
- Geometric mean
- Harmonic mean

Example If the mean of numbers 27, 31, 89, 107, 156 is 82 then the mean of 130, 126, 68, 50, 1 is

(a) 80
(b) 82
(c) 157
(d) 75 ✓

27	31	89	107	156	$\rightarrow$ 82
130	126	68	50	1	$-$? 75
$\downarrow$ 157	$\downarrow$ 157	$\downarrow$ 157	$\downarrow$ 157	$\downarrow$ 157	$\downarrow$ 157

$$\begin{array}{r} 157 \\ -82 \\ \hline 75 \end{array}$$

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Example The arithmetic mean of $^nC_0, ^nC_1, ---- ^nC_n$ is

(a) $1/n$

(b) $\frac{2^n}{n+1}$ ✓

(c) $2^{n-1}/n$

(d) $\frac{2^n}{n}$

$$\bar{x} = \frac{^nC_0 + ^nC_1 + ^nC_2 + --- + ^nC_n}{n+1}$$

$$= \frac{2^n}{n+1}$$

Example The AM of n numbers of a series is $\bar{x}$. If the sum of first $(n-1)$ terms is K then n^{th} number is

(a) $\bar{x} - K$

(b) $n\bar{x} - K$ ✓

(c) $3\bar{x}$

(d) $\frac{\bar{x}}{3}$

$$\frac{(x_1 + x_2 + --- + x_{n-1}) + x_n}{n} = \bar{x}$$

$$\frac{K + x_n}{n} = \bar{x} \Rightarrow x_n = n\bar{x} - K$$

* Arithmetic mean of Discrete Continuous Series

For Discrete Series. $\boxed{\bar{x} = \frac{\sum f_i x_i}{\sum f_i}}$

Example If the values $1, \frac{1}{2}, \frac{1}{3} --- \frac{1}{n}$ occur at frequencies $1, 2, 3 . . --- n$ in a distribution then mean is

(a) 1

(b) n

(c) $\frac{1}{n}$

(d) $\frac{2}{n+1}$ ✓

x_i	f_i	$f_i x_i$
1	1	1
$\frac{1}{2}$	2	1
$\frac{1}{3}$	3	1
⋮	⋮	⋮
$\frac{1}{n}$	n	1
	$\sum f = \frac{n(n+1)}{2}$	$\sum = n$

$$\bar{x} = \frac{n}{\frac{n(n+1)}{2}}$$

$$= \frac{2}{n+1}$$

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Example: The arithmetic mean of the following frequency distribution :

(a) $2^n/n$

(b) $2^n/n+1$

✓(c) $\frac{n}{2}$

(d) $9/n$

$$\bar{x} = \frac{n(2^{n-1})}{2^n} = \frac{n}{2} \text{ A}$$

x_i	f_i	$f_i x_i$
0	$^nC_0$	0
1	$^nC_1$	$^nC_1$
2	$^nC_2$	$2\,^nC_2$
3	$^nC_3$	$3\,^nC_3$
⋮	⋮	⋮
n	^nC_n	$n\,^nC_n$
	$\Sigma = 2^n$	$n(2^{n-1})$

③ Properties of Arithmetic mean

(1) Mean is dependent on shifting of origin i.e. if $\bar{x}$ is mean of x_i terms then $\bar{x} \pm k$ is a mean of $x_i \pm k$ terms.

(2) Mean is dependent on Change of scale

i.e. If $\bar{x}$ is the mean of x_i terms and y_i be the variables s.t. $y_i = ax_i + b$

then mean of $\boxed{\bar{y} = a(\bar{x}) + b}$

(3) The algebraic sum of the deviations of the values of the variable about mean in a frequency distribution is zero

i.e. $\sum (x_i - \bar{x}) = 0$

e.g

x_i	$x_i - \bar{x}$
1	−2
2	−1
3	0
4	1
5	2
	$\Sigma = 0$

$\bar{x} = 3$

(4) Combined mean: If $\bar{x}_1$ and $\bar{x}_2$ are mean of two groups of sizes n_1 and n_2 resp. Then mean $\bar{x}$ of two groups taken together is

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$$

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Example If the arithmetic mean of the observation $x_1, x_2, x_3 - - - x_n$ is 1 then the mean of $\frac{x_1}{k}, \frac{x_2}{k}, \frac{x_3}{k} - - - - \frac{x_n}{k}$ $(k>0)$ is

(a) Greater than 1 $\quad \bar{x} = 1 \quad$ If $k = \frac{1}{2}$

(b) Less than 1 $\quad$ then $\bar{x}_k = \frac{1}{k} \quad \bar{x}_k = \frac{1}{1/2} = 2$

(c) Equal to 1

✓(d) None of these $\quad$ if $k=7$ then $\bar{x}_k = \frac{1}{7}$

OR all ~~both~~ (a) and (b) and (c)

Example The mean of n items is $\bar{x}$. If first term is increased by 1, second by 2 and so on, then the new mean is

(a) $\bar{x} + n$

(b) $\bar{x} + \frac{n}{2}$

✓(c) $\bar{x} + \frac{n+1}{2}$

(d) N.O.T

$x_1 \quad x_1 + 1$

$x_2 \quad x_2 + 2$

$x_3 \quad x_3 + 3$

$x_n \quad (x_n + n)$

$\bar{x} \quad \bar{x} + \frac{\cancel{n}(n+1)}{2\cancel{n}} = \bar{x} + \frac{n+1}{2}$

mean of $1+2+3 \ldots n = \frac{1+2+3 \ldots n}{n} = \frac{\cancel{n}(n+1)}{2\cancel{n}} = \frac{n+1}{2}$

Example The mean of a set of observations is a. If each observation is multiplied by b and each product is decreased by c then mean of new set is

(a) $\frac{a}{b} + c$

✓(b) $ab - c$

(c) $\frac{a}{b} - c$

(d) $ab + c$

$\bar{x} = a$

new $\bar{x}_1 = ab - c$

Example The average score of boys in an examination of a school is 71 and that of girls is 73. The average score of school in that exam is 71.8. The ratio of the number of boys to the girls is

✓(a) 3:2

(b) 3:4

(c) 1:2

(d) 2:1

b $\quad 71 \quad n_1 \rightarrow 71 n_1$

g $\quad 73 \quad n_2 \rightarrow 73 n_2$

$\frac{71 n_1 + 73 n_2}{n_1 + n_2} = 71.8 \Rightarrow 1.2 n_2 = .8 n_1$

$\frac{n_1}{n_2} = \frac{1.2}{0.8} = 3/2$

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Example A group of 10 items has mean 6. If the mean of 4 of these items is 7.5 then the mean of the remaining items is

(a) 6.5

(b) 5.5 $\qquad 4 \quad 7.5 \longrightarrow 30$

(c) 4.5 $\qquad 6 \quad x \longrightarrow 6x$

✓(d) 5.0 $\quad$ directly → $\frac{60-30}{6} = 5$ $\qquad \frac{30+6x}{10} = 6 \Rightarrow x = 5$

(4) Geometric Mean

$$G.M = (x_1 x_2 x_3 \cdots x_n)^{1/n} = \text{antilog}\left(\frac{\sum \log(x)}{N}\right)$$

As taking log both sides

$$\log GM = \frac{1}{n}\log(x_1 x_2 \cdots x_n) = \frac{1}{n}(\log x_1 + \log x_2 \cdots + \log x_n)$$

$$\log GM = \frac{\sum_1^n \log(x_i)}{N} \Rightarrow GM = \text{antilog}\left(\frac{\sum \log(x)}{N}\right)$$

Example Geometric Mean of $2, 2^2, 2^3 \cdots 2^n$ is

✓(a) $2^{\frac{n+1}{2}}$ $\qquad G.M = (2 \cdot 2^2 \cdot 2^3 \cdots 2^n)^{\frac{1}{n}}$

(b) $2^{\frac{n-1}{2}}$ $\qquad = \left(2^{1+2+3\cdots+n}\right)^{\frac{1}{n}} = 2^{\left(\frac{n(n+1)}{2}\right)\frac{1}{n}}$

(c) $2^{\frac{n}{2}}$

(d) $2^{\frac{2}{n}}$ $\qquad = 2^{\frac{n+1}{2}}$

Examples If G is the G.M of Product of r sets of observations with GM's $G_1, G_2, G_3 \cdots G_n$ resp. then G is equal to (with equal no. of elements)

(a) $\log(G_1) + \log(G_2) + \cdots + \log(G_n)$ $\quad$ So. $G = (G_1^n G_2^n G_3^n)^{\frac{1}{nnn}} = (G_1 G_2 G_3)^{\frac{1}{3}}$

✓(b) $G_1\, G_2 \cdots G_n$ $\quad$ ~~If $G_1 = G_2 = G_3$ then $G = G_1 G_2 G_3$~~

(c) $\log(G_1) \cdot \log(G_2) \log(G_3) \cdots \log(G_n)$ $\quad$ but ~~given is $G_1 \neq G_2 \neq G_3$~~

~~(d)~~ NOT

Given Let $G = \left((x_1 x_2 \cdots x_n) \cdot (y_1 y_2 \cdots y_n) \cdot (z_1 z_2 \cdots z_n)\right)^{\frac{1}{nnn}} = \frac{1}{3n}$

$G_1 = (x_1 x_2 \cdots x_n)^{\frac{1}{n}}$ $\quad G_2 = (y_1 y_2 \cdots y_n)^{\frac{1}{n}}$ $\quad G_3 = (z_1 z_2 \cdots z_n)^{\frac{1}{n}}$

$G_1^n = x_1 x_2 \cdots x_n$ $\quad G_2^n = y_1 y_2 \cdots y_n$ $\quad G_3^n = (z_1 z_2 \cdots z_n)$

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Example If G_1, G_2 are the geometric means of two series of observations and G is the GM of the ratios of the corresponding observations then G is equal to

(a) G_1/G_2 ✓

(b) $\log G_1 - \log G_2$

(c) $\frac{\log G_1}{\log G_2}$

(d) $\log(G_1 G_2)$

$$G_1 = (x_1 x_2 x_3 \cdots x_n)^{1/n}$$

$$G_2 = (y_1 y_2 y_3 \cdots y_n)^{\frac{1}{n}}$$

$$G = \left(\frac{x_1}{y_1}, \frac{x_2}{y_2}, \cdots, \frac{x_n}{y_n}\right)^{\frac{1}{n}}$$

$$G = \frac{G_1}{G_2}$$

(5) Harmonic Mean

Harmonic mean of a non zero observations is the reciprocal of the AM of the reciprocals of the given values

$$HM = \frac{N}{\sum \frac{1}{x_i}} = \frac{N}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots \frac{1}{x_n}} \quad x_i \neq 0$$

In Discrete series $HM = \dfrac{N}{\sum\left(\frac{f}{m}\right)}$ N = Total freq.

Example The Harmonic mean of square of first five natural no. is

(a) $\frac{5}{\sum_{i=1}^{5} i^2}$

(b) $\frac{5^2}{\sum_{i=1}^{5} \frac{1}{i}}$

(c) $\frac{5}{\sum_{i=1}^{5} \frac{1}{i^2}}$ ✓

(d) $\frac{\sum_{i=1}^{5} i^2}{5}$

$$H = \frac{5}{\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2}}$$

$$= \frac{5}{\sum_{i=1}^{5} \frac{1}{i^2}}$$

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Example A boy goes to a school from his home at a speed of x km/hr and come back at a speed of y km/hr then the average speed is given by

(a) AM

(b) GM

(c) HM ✓

(d) NOT

$S = \frac{D}{T}$ $S \propto \frac{1}{T}$ So H.M.

$Avg = \frac{2xy}{x+y}$ i.e. $\frac{2}{\frac{1}{x}+\frac{1}{y}}$

Example A Car completes the first half of its journey with a velocity V_1 and the rest half with a velocity V_2 Then the average velocity of the car for whole journey.

(a) $\frac{2V_1V_2}{V_1+V_2}$ ✓

(b) $\frac{V_1+V_2}{2}$

(c) $\sqrt{V_1V_2}$

(d) NOT

$$Avg = \frac{2}{\frac{1}{V_1}+\frac{1}{V_2}} = \frac{2V_1V_2}{V_1+V_2}$$

(6) **Median :** Middle value of distrubution

* if n is odd, then median $= \left(\frac{n+1}{2}\right)^{th}$ term

* if n is even then median $=$ AM of $\left(\frac{n}{2}\right)^{th}$ & $\left(\frac{n}{2}+1\right)^{th}$ term

Example The daily wages of Ten workers 20, 25, 17, 18, 8, 15, 22, 11, 9, 14 Then median =

(a) 15

(b) 16 ✓

(c) 17

(d) 14

8, 9, 11, 14, 15, 17, 18, 20, 22, 25

$$\text{median} = \frac{\left(\frac{10}{2}\right)^{th} + \left(\frac{10}{2}+1\right)^{th}}{2} = \frac{5^{th}+6^{th}}{2} = \frac{15+17}{2}$$
$$= 16$$

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Example The median of the series 8, 12, 15, 7, x, 19, 22 lies in the interval.

✓(a) [12, 15]
(b) [7, 15]
(c) [15, 17]
(d) [9, 12]

0 7 0 8 0 12 15 19 22 — then median = 12

7, 8 12 15 0 19 0 22 0 — then median = 15

7, 8, 12, x, 15, 19, 22

$12 < x < 15$

Example The median of 21 observation is 18. If two observations 15 and 24 included to the observations then the median of new series is

(a) 15
✓(b) 18
(c) 24
(d) 16

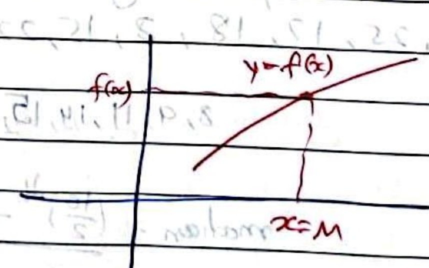
– – – 15 – , 18 , – – 24 –

Example The median can graphically be found from

✓(a) Ogive
(b) Histogram
(c) Frequency Curve
(d) NOT

Example If $y = f(x)$ be monotonically increasing or decreasing function of x and M is the median of variable x then the median of y is

✓(a) $f(M)$
(b) $M/2$
(c) $f^{-1}(M)$
(d) NOT



$y = f(M)$

$M = f^{-1}(y)$

Mode = 3 Median − 2 Mean

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Example. The median for the following distribution

(a) 6
(b) 5
(c) 7
(d) 4

$N = 120$

find $\frac{N}{2} = \frac{120}{2} = 60$

it lies in which cf = 65

So. median = 5

x_i	f_i	cf
1	8	8
2	10	18
3	11	29
4	16	45
5	20	65
6	25	90
7	15	105
8	9	114
9	6	120
	120	

(7) **Mode:** The mode or modal value of a distribution is that value of the variable for which the frequency is maximum.

e.g. Mode of data 10, 20, 50, 20, 30, 20, 50, 30, 10 is 20.

Note: (1) For Frequency Distribution Mean $(\bar{x}) = \frac{\sum f_i x_i}{\sum f_i}$

(2) For Grouped data.

$$\text{Median} = l + \left(\frac{\frac{n}{2} - C_f}{f}\right) \times h$$

l = lower limit of Median class
f = frequency of Median class
C_f = Cumulative Frequency of Proceeding M.C.
h = class size

To find Median class Find N/2 and this value lie in which of cumulative freq. that class is Median class.

(3) For Grouped data

$$\text{Mode} = l + \left(\frac{f_1 - f_0}{2f_1 - f_0 - f_2}\right) \times h$$

l = lower limit of Modal Class
h = class size
f_0 = frequency of proceeding M.C.
f_1 = frequency of Modal class
f_2 = frequency of succeeding M.C.

To find Modal class The class which has the largest frequency is the Modal class.

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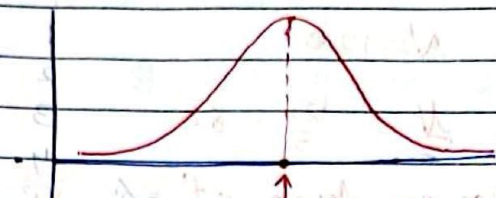
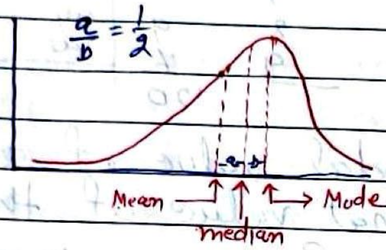
(8) Symmetric Distrubution



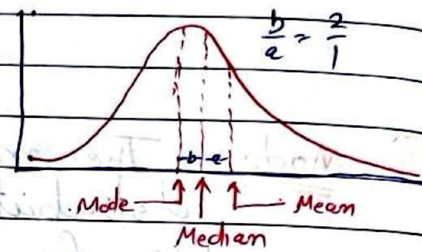
Mode = Mean = Median

SYMMETRIC



SKEWED LEFT
(negatively)

$\bar{x} < M < Z$



SKEWED RIGHT
(Positively)

$Z < M < \bar{x}$

* $\frac{M-\bar{x}}{Z-M} = \frac{a}{b} = \frac{1}{2}$ & $\frac{M-\bar{x}}{Z-\bar{x}} = \frac{a}{a+b} = \frac{1}{3}$

$2M - 2\bar{x} = Z - M$

$\boxed{3M - 2\bar{x} = Z}$

i.e. $\boxed{\text{Mode} = 3\,\text{Median} - 2\,\text{Mean}}$

Example The arithmetic mean and Mode of a data are 24 and 12 respectivly Then the median of the date is

(a) 20 ✓
(b) 18
(c) 21
(d) 22

$Z = 3M - 2\bar{x}$

$12 = 3M - 48$

$\boxed{M = 20}$

or $\frac{\bar{x}-M}{\bar{x}-Z} = \frac{1}{3}$

$M = 24 - \frac{12}{3} = 20$

$M = \bar{x} - \left(\frac{\bar{x}-Z}{3}\right)$

Example If the ratio of mean and median of a certain data is 2:3 then the ratio of its mode and mean is

(a) 4:3
(b) 7:6
(c) 7:8
(d) 5:2 ✓

Mean : Median : Mode

$\underbrace{2 : 3}_{1} \underbrace{\;: 5}_{2}$

Mean < Median

skewed Left.

Mode : Mean = 5 : 2

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Example: If the difference of Mode and median of a date is 24 then the difference of Median and mean is

✓(a) 12
(b) 24
(c) 8
(d) 36

Trick: $\bar{X}\ \underbrace{\ M\ }_{12}\ \overset{24}{\overline{\ Z\ }}$ 1 : 2

$Z\ \underbrace{\ M\ }_{24}\ \overset{12}{\overline{\ X\ }}$ 2 : 1

Example: If in a moderately skewed distribution the value of mode and mean are 6λ and 9λ respectively then value of median is

✓(a) 8λ
(b) 7λ
(c) 6λ
(d) 5λ

Trick: mode 6λ — median 8λ — mean 9λ

2 : 1

Example: Which one of the following measure of marks is the most suitable one of central location for computing intelligence of students

(a) Mode
(b) AM
(c) GM
✓(d) Median

Note:

Measure of Central Tendency	Measure of Dispersion	Moments	Correlation
Mean	Mean Deviation	Skewness	Regression
median	Standard Deviation	Kurtosis	
Mode	Quartile Deviation		
GM	Variance		
HM			
AM			

Question arise :- If we will have all these Averages, then why we need of Dispersion?

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Answer e.g.

Match	Schin Score	OR	Yuvraj Score	OR
1st	120	56	60	57
2nd	10	59	65	58
3rd	110	60	55	60
4th	20	61	70	62
5th	40	64	50	63
	300	300	300	300

Which player is best ?

* Since Averages of both cases is same so these fails to give answer
Thus we need dispersion like

$$\text{Mean Deviation} = \frac{\sum |x_i - \bar{x}|}{n} = \frac{\sum |x_i - M|}{n} = \frac{\sum |x_i - z|}{n}$$

Note. $\sum (x_i - \bar{x}) = 0$ Thus we take modulus

$$Var(x) = \frac{\sum |x_i - \bar{x}|^2}{n} \qquad S.D = \sqrt{\frac{\sum (x_i - \bar{x})^2}{n}}$$

* But in these Cases if these also are same then it will also fails to give answers
Then we move to Symmetric Distribution

Moments

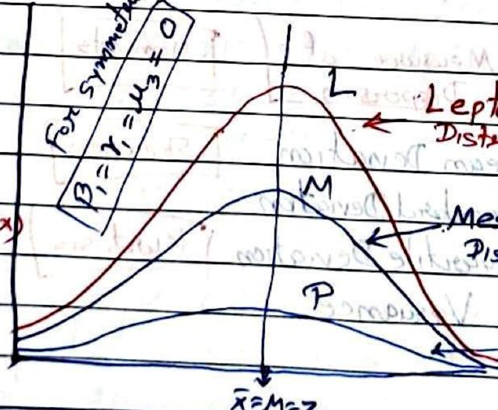
$M_r = \sum (x_i - \bar{x})^r$

$M_1 = \sum (x_i - \bar{x}) = 0$

$M_2 = \sum (x_i - \bar{x})^2 = V(x)$

$M_3 = \sum (x_i - \bar{x})^3$

$M_4 = \sum (x_i - \bar{x})^4$



Leptokurtic Distribution $\boxed{\beta_2 > 3, \ \gamma_2 > 0}$

Normal curve (best)

Mesokurtic Distribution $\boxed{\beta_2 = 3, \ \gamma_2 = 0}$

Platykurtic Distribution $\boxed{\beta_2 < 3, \ \gamma_2 < 0}$

$\boxed{\beta_1 = \frac{(M_3)^2}{(M_2)^3}}$ $\boxed{\beta_2 = \frac{M_4}{(M_2)^2}}$ $\boxed{\gamma_1 = \sqrt{\beta_1}}$ $\boxed{\gamma_2 = \beta_2 - 3}$

It tells Skewness

$\beta_1 = \gamma_1 = 0$

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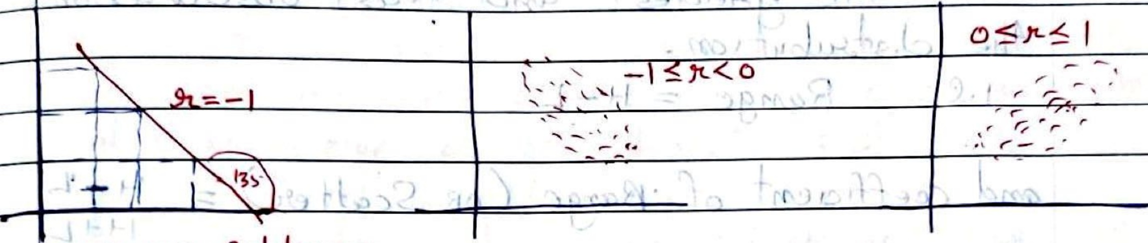
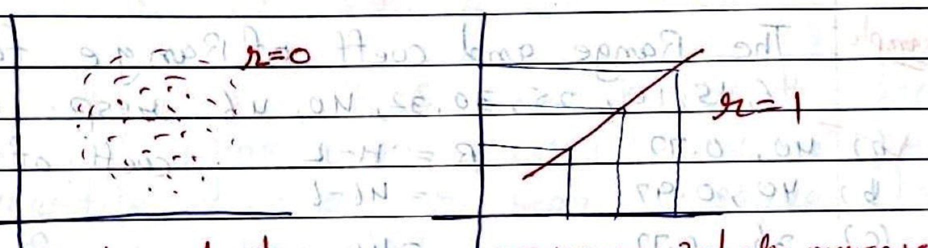
* But if curves are also same then it also fails to give answer. The we will move to the concept Correlation or Regression

Correlation is denoted by 'r' and $\boxed{-1 \le r \le 1}$



one increase 2nd decrease



Independent (No Relation)

one increase 2nd also increase

(9) **Dispersion** : Dispersion is the measure of the variations. It measures the degree of scatteredness of the observations in a distribution around the Central value

Measure of Dispersion

- Range
- Quartile Deviation or the semi-interquartile Range
- Mean Deviation
- Standard Deviation

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Note: Range → Highest value − Lowest value (H−L)

Coefficient of Range = $\frac{H-L}{H+L}$

(10) Range → The Range is the difference b/w the greatest and least observation of the distribution.

i.e. Range = H − L

and coefficient of Range (OR Scatter) = $\frac{H-L}{H+L}$.

Example: The Range and coeff. of Range for 6, 15, 10, 25, 30, 32, 40, 46 resp. are

✓(a) 40, 0.77
(b) 40, 0.97
(c) 36, 0.77
(d) 36, 0.97

$R = H-L$
$= 46-6$
$= 40$

coeff. of $R = \frac{H-L}{H+L}$
$= \frac{46-6}{46+6} = \frac{40}{52}$
$= 0.77$

Example: If the Range of 14, 12, 17, 18, 16, x is 20 and $x > 0$ then the value of x is

(a) 2
(b) 28
✓(c) 32
(d) Cannot be determined

x, 12, 14, 16, 17, 18, x
L H H

$\to 26$

$x - 12 = 20$
$x = 32$

$18 - x = 20$
$\cancel{x = -2}$ as $x > 0$

Example. Which of the following is not a measure of central Tendency

(a) Mean
(b) Mode
(c) Median
✓(d) Range

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Example Range and coeff. of Range for

(a) 40, 0.5
(b) 30, 0.2
(c) 30, 0.5
(d) 40, 0.2

Wages	no. of workers
10-20	53
20-30	35
30-40	20
40-50	12

1st method	2nd Method
Range → 50 - 10 = 40	R = 45 - 15 = 30
Coeff of Range	Coeff. of Range
$= \frac{50-10}{50+10} = \frac{40}{60} = 0.67$	$= \frac{45-15}{45+15} = \frac{30}{60} = 0.5$

Note Range for grouped data is the difference b/w the upper boundary of the highest class and the Lower boundary of the Lowest class

Also

It is calculated by using difference b/w the mid points of the highest and Lowest class.

(ii) Quartile Deviation: If Q_1 and Q_3 be the lower and upper quartiles as introduced then quartile deviation or semi-interquartile range Q is given by

$$Q = \frac{1}{2}(Q_3 - Q_1)$$

$$\text{Coefficient of Quartile deviation} = \frac{(Q_3-Q_1)/2}{(Q_3+Q_1)/2} = \frac{Q_3-Q_1}{Q_3+Q_1}$$

Example If the Quartile deviation of a set of observations is 10 and the third Quartile is 35 then 1st Quartile is

(a) 24
(b) 36
(c) 17
(d) 15

~~It is a Coeff~~

$\cancel{\frac{Q_3-Q_1}{Q_3+Q_1} = 10}$ $\cancel{\Rightarrow 9Q_3 + 11Q_1 = 10}$ $\cancel{\Rightarrow Q_1 = 10 - 9Q_3 = 10-}$

$$Q = \frac{1}{2}(Q_3 - Q_1) \Rightarrow Q_1 = Q_3 - 2Q = 35 - 2\times10 = 15$$

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Example If 25% of the items of a data are less the 35 and 25% of the items are more than 75 The Q.D of data is

(a) 55
(b) 20
(c) 35
(d) 75

$$Q = \frac{Q_3 - Q_1}{2}$$

$$= \frac{75-35}{2} = 20$$

25% 35 75% 75 25%

Q_1 Q_3

Example For the following data calculate Quartile dev. and coeff. of Quartile.

Marks	no. of stu	cf
0-10	16	16
10-20	20	36 cf
20-30	30 f	60
30-40	50	110
40-50	40 f	150
50-60	30	180

(a) 10.25, 0.29.85
(b) 10.625, 0.29.82
(c) 10.5, 0.29.5
(d) 10, 29

$\left(\frac{N}{4}\right)^{th}$ item

$$= \frac{180}{4} = 45$$

$$Q_1 = l + \left(\frac{\frac{N}{4} - cf}{f}\right) \times h$$

$$= 20 + \left(\frac{45-36}{36}\right) \times 10 = 25$$

$$\frac{3N}{4} = 3 \times 45 = 135 \qquad Q_3 = l + \left(\frac{\frac{3n}{4} - cf}{f}\right) \times h$$

$$= 40 + \left(\frac{135-110}{40}\right) \times 10 = 40 + 6.25 = 46.25$$

$$\text{Quartile dev.} = \frac{Q_3 - Q_1}{2} = \frac{46.25-25}{2} = \frac{21.25}{2} = 10.625$$

$$\text{Coeff.} = \frac{Q_3 - Q_1}{Q_3 + Q_1} = \frac{46.25-25}{46.25+25} = \frac{21.25}{71.25} = 0.29$$

(12) Mean deviation about Mean / Median

If $x_1, x_2 - - - x_n$ are n values of a variable x then the mean deviation from a Average A (mean or median) is given by

$$M.D = \frac{1}{n} \sum_{i=1}^{n} |x_i - A|$$

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Example The mean deviation of the data 3, 10, 10, 4, 7, 10, 5 from the mean is

(a) 2
(b) 2.57 ✓
(c) 3
(d) 3.57

x_i	$\lvert x_i - \bar{x} \rvert$
3	4
10	3
10	3
4	3
7	0
10	3
5	2
49	18

$\bar{x} = \frac{49}{7} = 7$

$M.D = \frac{\sum |x_i - \bar{x}|}{n} = \frac{18}{7} = 2.57$

Example The mean deviation from mean of the observation $a, a+d, a+2d, \ldots\ a+2nd$ is

(a) $\frac{n(n+1)d^2}{3}$
(b) $\frac{n(n+1)d^2}{2}$
(c) $a + \frac{n(n+1)d^2}{2}$
(d) $\frac{n(n+1)|d|}{2n+1}$ ✓

x_i	$x_i - \bar{x}$
a	nd
$a+d$	$(n-1)d$
$a+2d$	$(n-2)d$
⋮	⋮
$a+(n-1)d$	d
$\bar{x} = a+nd$	0
$a+(n+1)d$	d
⋮	⋮
$a+2nd$	nd

n terms; n terms

$\bar{x} = a+nd$

Total terms = $2n+1$

$M.D = \frac{2|d|\{1+2+3+\ldots+n\}}{2n+1}$

$= \frac{\cancel{2}|d|\, n(n+1)}{\cancel{2}(2n+1)}$

Note: Mean Deviation About Mean in Discrete & Continuous Series $M.D = \frac{1}{N}\sum f_i |x_i - A|$

Example If the mean deviation about the median of the numbers: $a, 2a, \ldots, 50a$ is 50 then $|a| =$

(a) 2
(b) 3
(c) 4 ✓
(d) 5

x_i	$\lvert x_i - 25.5a \rvert$
a	$24.5\lvert a\rvert$
$2a$	$23.5\lvert a\rvert$
$3a$	$22.5\lvert a\rvert$
⋮	⋮
$48a$	$22.5\lvert a\rvert$
$49a$	$23.5\lvert a\rvert$
$50a$	$24.5\lvert a\rvert$

$n = 50$

$M.D = 50$

$\frac{\sum |x_i - M|}{n} = 50$

$|a|\, 2\,[0.5+1.5+2.5 \ldots + 24.5] = 2500$

$|a|\, \cancel{2}\left[\frac{25}{\cancel{2}}[1+24]\right] = \cancel{2500}\ 100$

$|a| = 4$

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Example The mean deviation of the data is ___ when measured from the median

(a) Largest
✓(b) Least
(c) zero
(d) one

Note. $\sum |x_i - \bar{x}|$ | $\sum (x_i - \bar{x})^2$ — Minimum

$\sum |x_i - M|$ least | $\sum (x_i - M)^2$

$\sum |x_i - z|$ | $\sum (x_i - z)^2$

Note * Mean deviation is minimum about medium
* Variance / standard deviation is minimum about mean

(13) Variance & Standard Deviation

$$\boxed{V(x) = \frac{\sum (x_i - \bar{x})^2}{n}} = \frac{\sum (x_i^2 + \bar{x}^2 - 2x_i\bar{x})}{n}$$

$$= \frac{1}{n}\left[\sum x_i^2 + \bar{x}^2 \sum 1 - 2\bar{x}\left(\sum x_i\right)\right]$$

$$= \frac{1}{n}\left[\sum x_i^2 + \bar{x}^2 n - 2\bar{x}\, n(\bar{x})\right]$$

$$= \frac{1}{n}\left[\sum x_i^2 - n\bar{x}^2\right] = \boxed{\frac{\sum x_i^2}{n} - (\bar{x})^2}$$

Note

$V(ax) = a^2 V(x)$
$V(x+a) = V(x)$
$\Rightarrow \boxed{V(ax+b) = a^2 V(x)}$

$S.D(ax) = |a|\, S.D(x)$
$S.D(x+a) = S.D(x)$
$\Rightarrow SD(ax+b) = |a|\, S.D(x)$

Note. Change of scale

$V(ax) = a^2 V(x)$
$SD(ax) = |a| SD(x)$

Dependent

Shifting of Origin

$V(x+a) = V(x)$
$SD(x+a) = S.D(x)$

Independent

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Example The Variance of the score 2, 4, 6, 8, and 10 is

(a) 2

(b) 4

(c) 6

(d) 8 ✓

$2(\underbrace{1, 2, 3, 4, 5})$

$Var(x) = \frac{(5)^2 - 1}{12} = 2$

$Var(2x) = (2)^2 Var(x) = 2^2 \times 2 = 8$

Note Variance of first n natural number

$1, 2, 3, 4, \ldots n$

$$V(x) = \frac{\sum x_i^2}{n} - \left(\frac{\sum x_i}{n}\right)^2$$

$$= \frac{n(n+1)(2n+1)}{6n} - \left(\frac{\cancel{n}(n+1)}{2\cancel{n}}\right)^2$$

$$= \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2$$

$$= \frac{(n+1)}{2}\left[\frac{2n+1}{3} - \frac{n+1}{2}\right] = \left(\frac{n+1}{2}\right)\left(\frac{n-1}{6}\right)$$

$$\boxed{V(x) = \frac{n^2-1}{12}} \text{ and } S.D = \sqrt{Var(x)}$$

Example. The S.D of Scores 1, 2, 3, 4, 5 is

(a) $\sqrt{2}$ ✓

(b) $\sqrt{3}$

(c) $2/5$

(d) $3/5$

$Var(x) = \frac{n^2-1}{12} = \frac{(5)^2-1}{12} = \frac{24}{12} = 2$

$S.D = \sqrt{Var(x)} = \sqrt{2}$

Example The variance of first n even natural no. are

$2, 4, 6, \ldots 2n$

(a) $\frac{n^2-1}{3}$ ✓

(b) $\frac{n^2-1}{12}$

(c) $\frac{n^2+1}{12}$

(d) $\frac{n-1}{2}$

$V(x) = \frac{n^2-1}{12}$

$V(2x) = (2)^2\left(\frac{n^2-1}{12}\right) = \frac{n^2-1}{3}$

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27. PROBABILITY

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① Set theoretic Representation of statements

(1) Non happening of event A is written as $\bar{A}$

(2) Happening of at least one among events A, B is written as $(A \cup B)$ OR $(A \cap \bar{B}) \cup (\bar{A} \cap B) \cup (A \cap B)$

(3) Happening of both events A & B is written as $(A \cap B)$

(4) Neither A nor B happens is written as
$$(\bar{A} \cap \bar{B}) = (\overline{A \cup B})$$ – D Morgan's Law

② Experiment:

Experiment means an operation which can produce some well-defined outcomes.

e.g. Throwing a die gives six outcomes 1 to 6 number

* **Types of Experiments:**

Experiments
- Deterministic Experiment
- Random Experiment

Deterministic Experiment

Those experiments which when repeated under identical conditions produce the same result or outcome are known as deterministic experiments

eg. Experiments in science or engineering are repeated under identical conditions

Random Experiment

If the result of exp. is not certain but it may be one of the several possible outcomes when repeated under identical conditions is called Random Experiment

e.g. Tossing a coin and throwing a dice

Note. Random Experiment is known as Probabilistic experiment

(3) Sample space & Sample Points

* **Sample space** → The set of all distinct possible outcomes of a random experiment is called the sample space and denoted by S

* **Sample points**: Each outcome of S is called sample point.

For example:

In case of tossing a coin $S = \{H, T\}$ $n(S) = 2$

if k coins are Tossed at a time then $n(S) = 2^k$

(4) Event:

Any subset of the sample S is called an event.

e.g. Throwing a single die gives $S = \{1, 2, 3, 4, 5, 6\}$

Consider $A = \{2, 4, 6\}$, $B = \{1, 3, 5\}$ Since A, B are subsets of S, hence they are events

* **Types of Events**:

Events

- **Simple events**: In case the subset of sample space is a singleton set, then we call it as simple event.
  e.g. occurence of taill while tossing a coin
- **Compound events**: Any event that is a mixture of two or more simple events is called compound event.
  eg. Comming of an even number when rolled a die

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Events

Certain events

An event which is certain to happen is called certain event

e.g. When usual die is thrown getting a no. less than 7. is a certain event

Impossible event

An event which can't happen at all is called impossible event.

eg. When usual die is thrown, getting a number multiple of 13 is an impossible

Events

Mutually Exclusive events

If the happening of any one of them prevents the happening of any of the other events

i.e. A and B are two events s.t. $A \cap B = \phi$

e.g. A die is rolled, $S = \{1, 2, 3, 4, 5, 6\}$

Let A: Appear odd no.

B: Appear even no.

Here A & B are mutually exclusive.

Mutually Exhaustive events

Two or more events are said to be exhaustive if the performance of the experiment always results in the occurrence of at least one of them

i.e. $E_1 \cup E_2 \cup E_3 = S$

e.g. Let A: Getting 2 Head $\{HH\}$

B: Getting 1 Head $\{HT, TH\}$

C: Getting 0 Head $\{TT\}$

$A \cup B \cup C = S$

Events

Independent Evens

If happening of one event is not effected by the other events

e.g. Two dice are thrown together, getting even no. on 1st dice is independent to getting odd on 2nd dice.

Binomial Theorem with Replacement

Dependent events

If happening of one event depends upon the happening of other events

eg. If one dice is thrown appearance 2 and 3 are dependent events

Used in Bayes Theorem, Conditional Probability, without Replacement

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Events

Equally Likely Events

Two or more events said to be equally Likely if there is no reason to expect any one of them to happen in preference to the others

e.g. A one die is thrown appearence numbers on dice have equal chances. i.e 1 to 6.

Complementary events

∴ The complement of any event A is the event [Not A] i.e. the event that A does not occur A and Not A ($\bar{A}$) are complementary events

e.g. S = All types of weathers in a year The event 'Rainy Day' and 'Dry Day' are complementary)

Note. Complementary events are mutually exclusive and exhaustive events

i.e. $A \cap \bar{A} = \phi$ and $A \cup \bar{A} = S$

(5) Three Different Definitions of Probability

(1) Classical OR Mathematical Definition

A random experiment results n simple events which are exhaustive, mutually exclusive and equally likely.

In those n events, m are favourable for the occurrence of the event E then Probability of event E i.e. $\boxed{P(E) = \frac{m}{n}}$

$$\boxed{P(E) = \frac{m}{n} = \frac{\text{no. of favourable outcomes to E}}{\text{no. of all possible outcomes}}}$$

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* Limitations of Classical Definition

It is not Applicable when

(1) Outcomes of random experiment are not equally similar
(2) The random experiment contains infinite many outcomes.

(2) Statistical Definition of Probability

Suppose E is an event of a random experiment which can be repeated any number of times under similar conditions; noting the success or failure of the event in each trial. If $N_r(E)$ denotes the number of successes in first r trials then

$\frac{N_r(E)}{r}$, the relative frequency and denote

by $R_r(E)$ and get Relative frequencies
$R_1(E), R_2(E), R_3(E) --- R_r(E)$

then $$\boxed{P(E) = \lim_{n\to\infty} R_n(E) \text{ if exist}}$$

Since every sequence need not be convergent so there is no guarantee that $\lim_{n\to\infty} R_n(E)$ exists.

* Limitations:
(1) Repeating random experiment infinitely many times is practically impossible
(2) The sequence $R_1(E), R_2(E) --- R_n(E)$ is assumed to tend a limit, which may or may not exist

So keeping above Trouble Axiomatic Definition is formulated by Russian Mathematician KOLMOGOROV

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(3) Axiomatic Definition of Probability

Let S be the sample space of a random experiment. P(S) denote power set of S i.e, all possible subsets S i.e all events. We define Probability function 'p' so that

$$P : \mathcal{P}(S) \to R$$

and

(1) $P(E) \geq 0 \quad \forall E \in \mathcal{P}(S)$ $\quad 0 \leq P(E) \leq 1$
('Axiom of non negativity)

(2) $P(S) = 1$ (Axiom of Certainty

(3) $E_1, E_2 \in P(S)$; $E_1 \cap E_2 = \phi$ then $P(E_1 \cup E_2) = P(E_1) + P(E_2)$
(Axiom of additivity)

(1) Questions on Coins

(1) One coin Tossed : $S = \{H, T\}$
So $P(H) = 1/2 \quad P(T) = 1/2$ $\quad$ 1 1

(2) Two coins Tossed : $S = \{HH, HT, TH, TT\}$ $\quad$ 1 2 1
$P(\text{No Head}) = 1/4$
$P(\text{one head}) = 2/4 \quad P(\text{Two Head}) = 1/4 \quad P(\text{At least one head}) = 3/4$

(3) Three coins Tossed : $P(\text{No head}) = 1/8$
$P(\text{One head}) = 3/8$ $\quad$ 1 3 3 1
$P(\text{Two head}) = 3/8$
$P(\text{Three Head}) = 1/8$

$S = \{$HHH
HHT
HTH
HTT
THH
THT
TTH
TTT$\}$

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(4) Four Coin Tossed

P(No Head) = 1/16
P(one head) = 4/16
P(Two head) = 6/16
P(Three head) = 4/16
P(Four Head) = 1/16

1 4 6 4 1

S = { HHHH, TTTT,
HHHT, TTTH,
HHTH, TTHT,
HHTT, TTHH,
HTHH, THTT,
HTHT, THTH,
HTTH, THHT,
HTTT, THHH }

Note:

No. of Heads	0	1	2	3	4
P(X)	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

For Four Coins

For one coin → 1 1
For Two coin → 1 2 1
For 3 coins → 1 3 3 1
For 4 coins → 1 4 6 4 1
For 5 coins → 1 5 10 10 5 1

No of Head	0	1	2	3
P(X)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Example A Game consists of tossing a coin 3 times and nothing its outcome. A boy wins if all tosses give the same outcome and losses otherwise The probability that the boy losses the game is

(a) $\frac{1}{4}$ (c) $\frac{3}{4}$ ✓

(b) $\frac{2}{4}$ (d) $\frac{1}{3}$

no of Head/Tale	0	1	2	3
P(X)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$
	win	Loss	Loss	win

$P(\text{loss}) = \frac{3}{8} + \frac{3}{8} = \frac{6}{8} = \frac{3}{4}$

Example Four coins are tossed. The chance that there should be two tails is

(a) $\frac{1}{8}$

(b) $1/4$

(c) $\frac{3}{8}$ ✓

(d) $\frac{1}{16}$

no. of H/T	0	1	2	3	4
P(X)	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

$P(x) = \frac{6}{16} = \frac{3}{8}$

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(7) Questions on Dice

(1) One Die Rolled: $S = \{1, 2, 3, 4, 5, 6\}$ $P(\text{each outcome}) = 1/6$

$P(\text{even no.}) = 3/6 = 1/2$ $P(\text{odd no.}) = \frac{3}{6} = \frac{1}{2}$ $P(\text{Prime}) = \frac{3}{6} = \frac{1}{2}$

(2) Two Dice Rolled:

S = { (11) (12) (13) (14) (15) (16)
(21) (22) (23) (24) (25) (26)
(31) (32) (33) (34) (35) (36)
(41) (42) (43) (44) (45) (46)
(51) (52) (53) (54) (55) (56)
(61) (62) (63) (64) (65) (66) }

Sum of digits		Probability
2	-1 →	1/36
3	-1 →	2/36
4	-1 →	3/36
5	-1 →	4/36
6	-1 →	5/36
7	13 →	6/36
8	13 →	5/36
9	13 →	4/36
10	13 →	3/36
11	13 →	2/36
12	13 →	1/36

(3) Three Dice Rolled.

S = { (111) (112) (113) - - -
- - - - - - - (666) }

$\boxed{\text{Total} = 216}$

Sum of digits	fav. cases/216
3	1
4	3
5	6
6	10
7	15
8	21
9	25
10	27
11	27
12	25
13	21
14	15
15	10
16	6
17	3
18	1

		(252)(552)		(363)(633)		
For sum = 12	(156)	(255)	(246)	(336)	(345)	444
no. of case	6	3	6	3	6	1

Total cases = 6 + 3 + 6 + 3 + 6 + 1 = 25

Example Three dices are thrown. The Probability of obtaining a sum of 5 points is

(a) $\frac{1}{6}$ (c) 1/216
✓(b) 1/36 (d) 1/54

(113) | (221)
(311) | (212)
(131) | (122)

sum	
3	1
4	3
5	6

6/216

Example A and B throw 3 dice. If A throws a sum of 16 points, the probability of B throwing a greater sum

(a) $\frac{7}{54}$ ✓(c) $\frac{1}{54}$

(b) $\frac{5}{54}$ (d) $\frac{3}{54}$

Required Probability = P(getting sum 17) + P(getting 18)

$= \frac{3}{216} + \frac{1}{216} = \frac{4}{216} = \frac{1}{54}$

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Tricks: Probability $= \frac{(r-1)(r-2)}{2\times 6^3}$ where $r = 3,4,5,6,7 \& 8$

and Probability $= \frac{(19-r)(20-r)}{2\times 6^3}$ with $r = 17 \& 18$

Example A and B alternately throw a pair of symmetrical dice. A wins if he throws 6 before B throws 7, and B wins if he throws 7 before A throws 6. If A begins the probability of his winning is

(a) $11/36$

(b) $36/61$ ✓

(c) $\frac{31}{61}$

(d) $1/36$

No. of cases to get sum 6 = 5   No. of cases to get sum 7 = 6

$$P(A_w) = P(A_w) + P(A_L B_L A_w) + P(A_L B_L A_L B_L A_w) + \cdots$$

$$= \frac{5}{36} + \frac{31}{36}\times\frac{30}{36}\times\frac{5}{36} + \frac{31}{36}\times\frac{30}{36}\times\frac{31}{36}\times\frac{30}{36}\times\frac{5}{36} + \cdots$$

$$= \frac{5}{36}\Big[1 + \underbrace{\frac{31}{36}\times\frac{30}{36} + \left(\frac{31}{36}\times\frac{30}{36}\right)^2 + \cdots}_{GP}\Big]$$

$a + ar + ar^2 + \cdots$

Sum $= \frac{1}{1-r}$

$$= \frac{5}{36}\times\left(\frac{1}{1-\frac{31}{36}\times\frac{30}{36}}\right)$$

$$= \frac{5}{\cancel{36}}\times\frac{\cancel{36}\times 36}{36\times 36 - 31\times 30} = \frac{5\times 36}{1296 - 930} = \frac{5\times 36}{366} = \frac{30}{61}$$

(8) Questions on Balls.

Example A bag contains 3 black and 4 green balls. What is the probability that 2 balls of diff. colours are drawn with replacement? without replacement.

$B = 3$
$G = 4$

With Replacement

$P(BG) + P(GB)$

Req. Prob. $= \frac{3}{7}\times\frac{4}{7} + \frac{4}{7}\times\frac{3}{7}$

$= \frac{24}{49}$

without replacement

Req. Prob. $= P(BG) + P(GB)$

$= \frac{3}{7}\times\frac{4}{6} + \frac{4}{7}\times\frac{3}{6}$

$= \frac{24}{42} = \frac{4}{7}$

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Example A bag contains 5 black ; 4 white, 3 red balls if a ball is selected randomwise, the probability that it is a black or a red ball

(a) $1/3$

(b) $1/4$

✓(c) $2/3$

(d) $5/12$

$B = 5$, $W = 4$, $R = 3$; 8

$$\text{Req. Prob} = \frac{{}^8C_1}{{}^{12}C_1} = \frac{8}{12} = \frac{2}{3}$$

Example There are 4 red, 3 black and 5 white balls in a bag. The probability of drawing 2 balls of the same colour and one is of diff. colour is

(a) $\frac{5}{44}$

(b) $15/44$

✓(c) $29/44$

(d) $131/195$

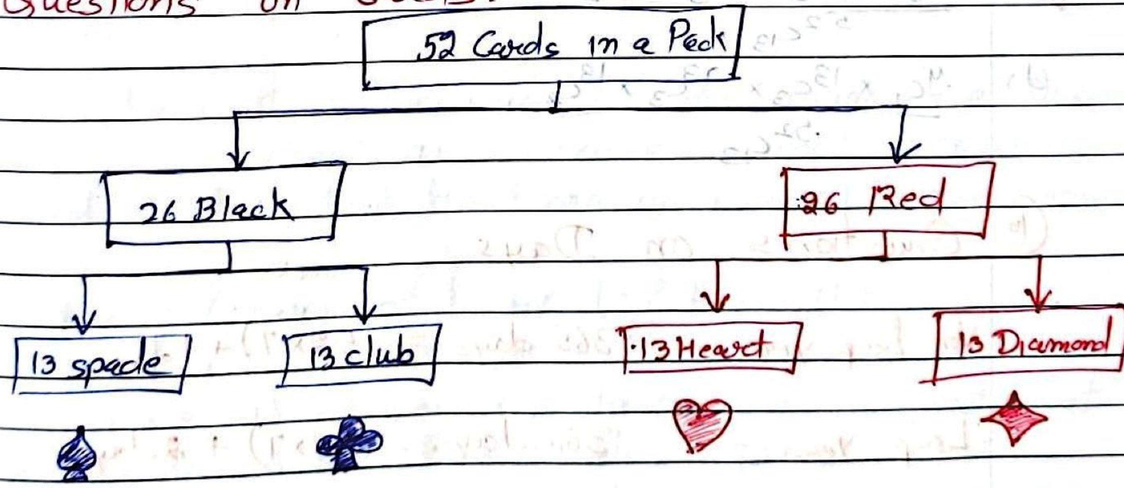
$R - 4$, $B - 3$, $W - 5$

$$P = \frac{{}^4C_2 \times {}^8C_1 + {}^3C_2 \times {}^9C_1 + {}^5C_2 \times {}^7C_1}{{}^{12}C_3}$$

$$= \frac{6 \times 8 + 3 \times 9 + 10 \times 7}{\frac{12}{3} \times \frac{11}{2} \times 10} = \frac{48 + 27 + 70}{220}$$

$$= \frac{145}{220} = \frac{29}{44} \text{ A}$$

⑨ Questions on Cards.



Note Face Cards = 12 (Jack, Queen, King) — Also known as Court Cards

Honour Cards = 16 (Ace, Jack, Queen, King)

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Example The face cards are removed from a well shuffled pack of 52 cards. Out of the remaining cards 4 are drawn at random. The Probability that they belong to different suits and different denominations is

(a) $13^4 / {}^{52}C_4$

(b) $\dfrac{{}^{13}C_4}{{}^{40}C_4}$

✓(c) $\dfrac{{}^{10}P_4}{{}^{40}C_4}$

(d) $\dfrac{13^4}{{}^{40}C_4}$

$P = \dfrac{10 \times 9 \times 8 \times 7}{{}^{40}C_4} = \dfrac{{}^{10}P_4}{{}^{40}C_4}$

Example A pack of cards is distributed among Four hands equally. The probability that 5 spades 3 clubs, 3 hearts and the rest diamonds may be in a particular hand is

(a) ${}^4C_1 \times {}^{13}C_5 \times {}^{13}C_3 \times {}^{13}C_2$

(b) $\dfrac{{}^{13}C_5 \times {}^{26}C_6 \times {}^{13}C_2}{{}^{52}C_{13}}$

✓(c) $\dfrac{{}^{13}C_5 \times {}^{13}C_3 \times {}^{13}C_3 \times {}^{13}C_2}{{}^{52}C_{13}}$

(d) $\dfrac{{}^4C_1 \times {}^{13}C_3 \times {}^{13}C_3 \times {}^{13}C_2}{{}^{52}C_{13}}$

Req. Prob. $= \dfrac{{}^{13}C_5 \times {}^{13}C_3 \times {}^{13}C_3 \times {}^{13}C_2}{{}^{52}C_{13}}$

(10) Questions on Days

Non-leap year — 365 days $= (\underbrace{52 \times 7}_{\text{weeks}}) + 1$ day

Leap Year — 366 days $= (52 \times 7) + 2$ days

Example The Probability that 13th day of the randomly chosen month is a friday is

(a) 1/7

(b) 1/12

✓(c) 1/8

(d) 1/42

Probability of chosen month is $= \dfrac{1}{12}$

Probability of 13th day is friday $= \dfrac{1}{7}$ {There are 7 days in which month can be start so only one way which 13th day is friday}

Req. Prob. $= \dfrac{1}{12} \times \dfrac{1}{7} = \dfrac{1}{84}$

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Example The Probability that the February of leap year will have 5 Saturdays is

(a) $\frac{3}{7}$ ✓(c) $\frac{1}{7}$

(b) $\frac{2}{7}$ (d) $\frac{4}{7}$

In Feb. (leap year)

No. of days $= 29 = \underbrace{28}_{4\text{ saturdays}} + 1$

S|M|T|W|T|F|S

(11) Odds in Favour & Odds Against an Event.

Let $P(A) = \frac{m}{n}$, $m \leq n$

odds in favour of $A = m : (n-m)$ i.e. $\frac{m}{n-m}$

odds against of $A = (n-m) : m$ i.e. $\frac{n-m}{m}$

* **Odds in Favour:** The ratio of the number of ways that an outcome can occur compared to how many ways it cannot occur.

odds in favour of $E = \frac{P(E)}{1-P(E)} = \frac{P(E)}{(P(\bar{E})}$

* **Odds Against:** The ratio of the number of ways that an outcome cannot occur compared to in how many ways it can occur

Odds Against of $E = \frac{1-P(E)}{P(E)} = \frac{P(\bar{E})}{P(E)}$

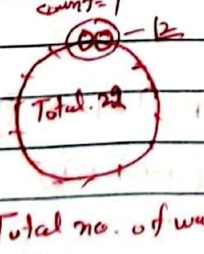
Example A party of 23 persons take their seats at a round table. The odds against two specific person sitting together is

(a) 1 to 11

(b) 11 to 1

✓(c) 10 to 1

(d) 1 to 10



Total 23 persons sit in round Table $= \lfloor 23-1 = \lfloor 22$

$P(A) = \frac{\lfloor 21 \times \lfloor 2}{\lfloor 22} = \frac{2}{22} = \frac{1}{11} \frac{\to m}{\to n}$ $\frac{n-m}{m}$

Total no. of ways to sit 22 person in round $= m = \lfloor 21$

Odds Against $= \frac{11-1}{1} = \frac{10}{1}$

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Example There are three events E_1, E_2 and E_3 one of which must, and only one can happen. The odds are 7 to 4 against E_1 and 5 to 3 against E_2. Find odds against E_3

For one must, and only one happen

(a) 65 : 23
(b) 23 : 65
(c) 23 : 88
(d) 65 : 88

$\boxed{P(E_1)+P(E_2)+P(E_3)=1}$

$P(E_1)=\frac{4}{7+4}=\frac{4}{11}$ $P(E_2)=\frac{3}{5+3}=\frac{3}{8}$

$P(E_1)+P(E_2)+P(E_3)=1$

$\frac{4}{11}+\frac{3}{8}+P(E_3)=1$

$P(E_3)=1-\frac{4}{11}-\frac{3}{8}=\frac{23}{88}$

odds agains $E_3=\frac{88-23}{23}=\frac{65}{23}$

Note. we know $P(A)=\frac{m}{n}$

odds in fav. $=\frac{m}{n-m}$

$\therefore P(n)=\frac{m}{(n-m)+m}$

odds agains $=\frac{n-m}{m}$

$(P(\bar{A}))^T=\frac{(n-m)+m}{m}$

$P(A)=\frac{m}{[illegible]}$

Example From a bag containing 4 white and 5 black balls 3 are drawn at Random. The odds against these all are black.

(a) 5 to 42
(b) 37 to 5
(c) 5 to 37
(d) 42 to 5

$P(B)=\frac{^5C_3}{^9C_3}=\frac{5\times4\times3}{9\times8\times7}=\frac{10}{84}=\frac{5}{42}$

odd against $=\frac{42-5}{5}=\frac{37}{5}$

(12) Addition Theorem on Probability

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

Example The probability that at least one of the events A and B occur is 0.6. If A and B occur simultaneouly with probability 0.2 then $P(\bar{A})+P(\bar{B})=$

(AUB)

(a) 0.4
(b) 0.8
(c) 1.2
(d) 1.6

$P(A\cup B)=0.6$

$P(A\cap B)=0.2$

$P(A\cup B)+P(A\cap B)=P(A)+P(B)$

$\Rightarrow P(A)+P(B)=0.6+0.2=0.8$

$P(\bar{A})+P(\bar{B})=1-P(A)+1-P(B)$

$=2-(P(A)+P(B))=2-0.8=1.2$

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Example From a well shuffled pack of 52 playing cards two cards are drawn at random. The probability that either both are red or both are kings is. $= P(A \cup B)$

2 Red King

(a) $\dfrac{{}^{26}C_2 + {}^{4}C_2}{{}^{52}C_2}$ A: Red card B: King $A \cap B$: 2

✓(b) $\dfrac{{}^{26}C_2 + {}^{4}C_2 - {}^{2}C_2}{{}^{52}C_2}$ $P = \dfrac{{}^{26}C_2 + {}^{4}C_2 - {}^{2}C_2}{{}^{52}C_2}$

(c) $\dfrac{{}^{30}C_2}{{}^{52}C_2}$

(d) $\dfrac{{}^{39}C_2}{{}^{52}C_2}$

Example One number is chosen at random from 1 to 200. The probability that the number is divisible by 4 or 6 is

$6, 12, 18, \ldots, 198$

$192 = 6 + (n-1)6 \Rightarrow n = 3$

$n - 1 = \frac{192}{6} = 32 \Rightarrow$

(a) $\frac{76}{200}$ A: Divisible by 4 — no. divisible by 4 = 50

✓(b) $\frac{67}{200}$ B: Divisible by 6 — " " " 6 = 16+16+1 = 33

$A \cap B$: Divisible by 12 — no. of divis. by 12 = 16

(c) $\frac{66}{200}$

$$\text{Req. Prob} = \frac{n(A) + n(B) - n(A \cap B)}{200} = \frac{50 + 33 - 16}{200}$$

(d) $\frac{87}{200}$ $= \frac{67}{200}$

(13) Questions involving Permutation

Example The Probability that in a random arrangement of the letters of the word UNIVERSITY the 2 I's do not come together is

(a) $\frac{2}{5}$ UNVERSTY (II)

A: Two I's come together

(b) $\frac{3}{5}$ m = favourable cases (ways) $= 9! \times \frac{2!}{2!} = 9!$ Req. Probability $= P(\bar{A}) = 1 - P(A)$

✓(c) $\frac{4}{5}$ n = Total ways $= \frac{10!}{2!}$ $= 1 - \frac{1}{5}$

(d) $\frac{5}{6}$ $P(A) = \frac{m}{n} = \frac{9!}{10!} \times 2! = \frac{2}{10} = \frac{1}{5}$ $= \frac{4}{5}$

Example Six boys and 6 girls to sit in a row at random. The probability that all the girls sit together is

(a) $\frac{\lfloor 6 \lfloor 6}{\lfloor 12}$ (c) $\frac{\lfloor 6 \lfloor 6}{\lfloor 12}$ $(G_1G_2G_3G_4G_5G_6)$ $B_1\ B_2\ B_3\ B_4\ B_5\ B_6$
1 2 3 4 5 6 7

n: Total ways $= \lfloor 12$

✓(b) $\frac{\lfloor 6 \lfloor 7}{\lfloor 12}$ (d) $2 \cdot \frac{\lfloor 6 \lfloor 7}{\lfloor 12}$ m: no. of ways girls sit tog. $= \lfloor 7 \times \lfloor 6$

$P(A) = \frac{m}{n} = \frac{\lfloor 6 \lfloor 7}{\lfloor 12}$.

Example There are Four addressed envelops. The probability that all letters are not placed in the right envelopes is.

(a) $1/4$ A: All letters place in right envelops

(b) $3/4$ m: no. of ways favourable to $A = 1$

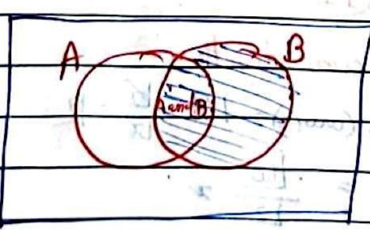
n: Total ways $= \lfloor 4 = 24$

✓(c) $23/24$ $P(A) = \frac{1}{24}$ $\therefore P(\bar{A})$ = All letters not in right

(d) $1/24$ envelops $= 1 - P(A) = 1 - \frac{1}{24} = \frac{23}{24}$

(14) Conditional Probability

- Let A and B be two events associated with a random experiment.
Then the probability of occurrence of event A under the condition that B has already occurred and $P(B) \neq 0$ is called the conditional Probabability and it is denoted by $P(A/B)$ = Prob. of occurrence of A given that B has already occurred.



$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

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Example Given that E and F are events such that $P(E) = 0.5$; $P(F) = 0.4$ and $P(E \cap F) = 0.3$ then what will be the value of $P(F/E)$

(a) 2/5

(b) 3/5

(c) 3/4

(d) 2/4

$$P(F/E) = \frac{P(F \cap E)}{P(E)} = \frac{0.3}{0.5} = \frac{3}{5}$$

Example A pair of dice are rolled. What is the Probability that they sum to 7 given that neither die shows a 2?

(a) 3/25

(b) 2/25

(c) 4/25

(d) 1/25

A: Sum 7 = (16)(25)(34)(43)(52)(61) = 6

B: either die shows 2 = (12)(22)(32)(42)(52)(62) (21)(23)(24)(25)(26) = 11

C: neither die shows 2 = 36 − 11 = 25

Req. Prob $P(A/C) = \frac{P(A \cap C)}{P(C)} = \frac{4/36}{25/36} = \frac{4}{25}$ A

Example A urn contain 7 red and 3 black balls. Two balls are drawn without replacement. Find the probability that the second ball is red if it is known that the first is red.

(a) 2/3

(b) 1/3

(c) 3/4

(d) None.

R − 7

B − 3

$P(\text{both Red}) = \frac{7}{10} \times \frac{6}{9} = P(A \cap B)$

A: 1st ball is Red

B: 2nd Ball is Red

$A \cap B$: both are red

Required $P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{7}{10} \times \frac{6}{9}}{\frac{7}{10}} = \frac{6}{9} = \frac{2}{3}$

short: → $\boxed{\begin{matrix} R-7 \\ B-3 \end{matrix}} \xrightarrow[\text{gone}]{\text{1 Red}} \boxed{\begin{matrix} R-6 \\ B=3 \end{matrix}}$ the $P(R) = 6/9 = 2/3$ A

Example Ten Cards numbered 1 to 10 are placed in a box, mixed up throughly and then one card is drawn randomly. If it is known that the number on the drawn card is more than 3, what is the probability that it is an even number?

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1. (a) $\frac{4}{7}$

(b) $\frac{3}{7}$

(c) $\frac{4}{5}$

(d) None

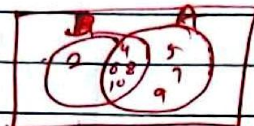
A: no. more than 3 = {4, 5, 6, 7, 8, 9, 10} = 7

B: An even no. = {2, 4, 6, 8, 10} = 5

$A \cap B$ = {4, 6, 8, 10} = 4

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{4/10}{7/10} = \frac{4}{7}$$

short:

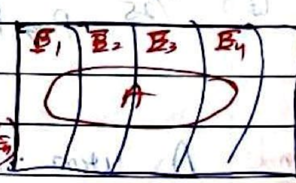


(15) Law of Total Probability:

If an event A can occur with one of the n mutually exclusive and exhaustive events $B_1, B_2, B_3, \ldots, B_n$ and the probabilities $P(A/B_1)$, $P(A/B_2), \ldots, P(A/B_n)$ are known then

$$P(A) = P(A \cap E_1) + P(A \cap E_2) + \ldots + P(A \cap E_n)$$

$$P(A) = P(E_1)P(A/E_1) + P(E_2)P(A/E_2) + \ldots + P(E_n)P(A/E_n)$$



i.e. $$\boxed{P(A) = \sum_{i=1}^{n} P(E_i)\, P(A/E_i)}$$

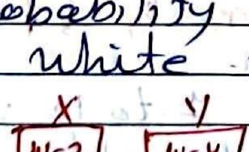
Example A bag X contains 2 white and 3 black balls and another bag Y contains 4 white and 2 black balls. One bag is selected at random and a ball is drawn from it. The probability for the ball chosen to be white is

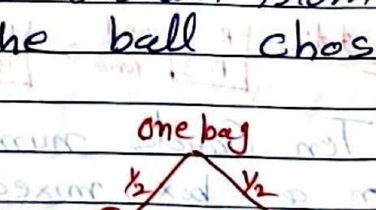
(a) 2/15

(b) 7/15

(c) 8/15 ✓

(d) 14/15





$$\text{Req. } P = \frac{1}{2} \times \frac{2}{5} + \frac{1}{2} \times \frac{4}{6} = \frac{1}{5} + \frac{1}{3} = \frac{8}{15} \text{ A}$$

Events	$P(E_i)$	$P(A/E_i)$	$P(A \cap E_i)$ =	P
E_1	1/2	2/5	1/5	$\frac{1}{5} + \frac{1}{3} = \frac{8}{15}$
E_2	1/2	4/6	1/3	

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Example A purse contains 2 silver and 4 copper coins. A second purse contains 4 silver and 3 copper coins. If a coin is pulled at random from one of the two purses, what is the Probability that it is a silver coin?

(a) $\frac{19}{41}$

(b) $\frac{17}{42}$

✓(c) 19/42

(d) None

S-2	S-4
C-4	C-3

$$\frac{1}{2}\times\frac{2}{6}+\frac{1}{2}\times\frac{4}{7}=\frac{1}{6}+\frac{2}{7}=\frac{19}{42}$$

Example In a set of 10 coins, 2 coins are with heads on both sides. A coin is selected at random from this set and tossed five times. If all the five times, the result was heads, find the probability that the selected coin had heads on both the sides.

E_1: Selected coin has 2 Heads

E_2: Selected coin is normal

A: All 5 times gives Head.

✓(a) 9/40

(b) 8/40

(c) 8/41

(d) 7/41

E_i	$P(E_i)$	$P(A/E_i)$	$P(A\cap E_i)$	Req. Prob
E_1	$\frac{2}{10}$	1	$1/5 = 8/40$	$\frac{8}{40}+\frac{1}{40}=\frac{9}{40}$
E_2	$\frac{8}{10}$	$(\frac{1}{2})^5$	$1/40$	

(16) Multiplication theorem of Probability

Example A bag contains 10 white and 15 black balls. Two balls are drawn in succession without replacement. What is the Probability that first is white and second is black?

✓(a) 1/4

(b) 1/3

(c) 1/5

(d) none

$\boxed{\begin{matrix} W-10 \\ B-25 \end{matrix}}$

$$P(W\cap B) = P(W)\times P(B)$$

$$= \frac{10}{25}\times\frac{15}{24} = \frac{1}{4}$$

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Example A bag contains 19 tickets, numbered from 1 to 19. A ticket is drawn and then another ticket is drawn without replacement. Find the probability that both tickets will show even numbers.

(a) 4/19 ✓

(b) 5/19

(c) 4/18

(d) None

even no. – 2, 4, 6, 8, 10, 12, 14, 16, 18

Req. Prob. $= \frac{\cancel{9}}{19} \times \frac{\cancel{8}^{4}}{\cancel{18}} = \frac{4}{19}$

Multiplication Theorem:

(i) $P(A \cap B) = P(A) \cdot P(B/A)$

(ii) $P(A \cap B \cap C) = P(A)\, P(B/A)\, P(C/A \cap B)$

(iii) $P(A_1 \cap A_2 \cap A_3 \cdots \cap A_n) = P(A_1)\, P(A_2/A_1)\, P(A_3/A_1 \cap A_2) \cdots P(A_n/A_1 \cap A_2 \cap \cdots \cap A_{n-1})$

(17) Baye's Theorem:

Let $E_1, E_2, E_3 \cdots E_n$ be n events and A be any events s.t. A/E_i are known then:

$$P(E_i/A) = \frac{P(E_i)\, P(A/E_i)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2) \cdots + P(E_n)\, P(A/E_n)}$$

Example Three urns have the following composition of balls

Urn 1: 1 white 2 black Urn 2: 2 white, 1 black

Urn 3: 2 white, 2 black. One of Urn is selected at random and a ball is drawn. It turns out to be white. Find Probability that it came from urn III 3.

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Solution

1	2	3
W - 1	W - 2	W - 2
B - 2	B - 1	B - 2

	Choose urn	select white ball	A		
E_1 Urn 1	$1/3$	$\times$ $1/3$	$\frac{1}{9} = \frac{2}{18}$	$\frac{2/18}{9/18} = 2/9$	From urn I
E_2 Urn 2	$1/3$	$\times$ $2/3$	$\frac{2}{9} = \frac{4}{18}$	$\frac{4/18}{9/18} = \frac{4}{9}$	From urn II
E_3 Urn 3	$1/3$	$\times$ $2/4 = \frac{1}{2}$	$\frac{1}{6} = \frac{3}{18}$	$\frac{3/18}{9/18} = \frac{3}{9} = \frac{1}{3}$	From urn III

$\sum P(A_i) P(A/E_i) = \frac{9}{18}$

Example Box A contains 2 black and 3 Red balls While Box B contains 3 black and 4 red balls. Out of these two boxes one is selected at random. And the probability of chosing box A is double that of Box B. If a red ball is drawn from the selected Box then the probability that it comes from Box B is -

Solution

A	B
B - 2	B - 3
R - 3	R - 4

$\frac{P(A)}{P(B)} = \frac{2}{1}$ $P(B) = \frac{1}{3}$

$P(A) = \frac{2}{2+1} = \frac{2}{3}$

Let $P(B) = x = \frac{1}{3}$

$P(A) = 2x = 2/3$

$P(A) + P(B) = 1$

$x + 2x = 1$

$x = 1/3$

A	$2/3$	$\times$ $3/5$	$\cancel{2/5} = \frac{42}{105}$	$\frac{42/105}{62/105} = \frac{42}{62} = \frac{21}{31}$	Comes from Box A.
B	$1/3$	$\times$ $4/7$	$= 4/21 = 20/105$	$\frac{20/105}{62/105} = \frac{20}{62} = \frac{10}{31}$	Comes from Box B

$\sum = 62/105$

Example A man is known to speak the truth 2 out of 3 times. He throws a die and reports that it is a six. Then the probability that it is actually a six is

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$P(\text{Truth}) = \frac{2}{3}$ $P(\text{Lie}) = \frac{1}{3}$

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(a) $2/7$ ✓

(b) $6/7$

(c) $5/7$

(d) None

E_1 : dice shows 6 E_2 : die not shows 6

		Truth		
E_1	$\frac{1}{6} \times \frac{2}{3} = \frac{1}{9} = \frac{2}{18}$		$\frac{2/18}{7/18} = \frac{2}{7}$	For Actually six
E_2	$\frac{5}{6} \times \frac{1}{3} = \frac{5}{18}$	Lie	$\frac{5/18}{7/18} = \frac{5}{7}$	For not Actually six.

$\Sigma = \frac{7}{18}$

(18) Properties of Independent Events

(I) If A, B are independent events $\frac{A}{B} = A$, $B/A = B$

(II) A, B are independent events $\Leftrightarrow P(A \cap B) = P(A)\, P(B)$

(III) If A and B are independent then

$P(\bar{A} \cap B) = P(\bar{A})\, P(B)$ — $\bar{A}$ and B are independent

$P(\bar{A} \cap \bar{B}) = P(\bar{A})\, P(\bar{B})$ — $\bar{A}$ and $\bar{B}$ are independent

A and $\bar{B}$ are independent

Example The Probabilities of solving a problem by three students A, B, C independently are $\frac{1}{3}, \frac{1}{4}, \frac{1}{5}$. The probability that the problem will be solved is —

(a) $\frac{3}{5}$ ✓

(b) $4/5$

(c) $2/5$

(d) None

$P(\text{At least one of } A, B, C \text{ solve}) = 1 - P(\bar{A})P(\bar{B})P(\bar{C})$

$= 1 - \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5}$

$= 1 - \frac{2}{5} = 3/5$

Example Given that the events A & B are such that $P(A) = 1/2$ $P(A \cup B) = 3/5$ & $P(B) = p$ Find p if they are (i) mutually exclusive

(ii) independent.

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(a) $\frac{1}{10}, \frac{1}{5}$ — mutually exclusive (i) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(b) $\frac{1}{5}, \frac{1}{10}$ — Independent $\frac{3}{5} = \frac{1}{2} + P - 0$

$P = \frac{3}{5} - \frac{1}{2} = \frac{6-5}{10} = \frac{1}{10}$

(c) $\frac{1}{10}, \frac{1}{10}$ (ii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(d) None. $= P(A) + P(B) - P(A) \cdot P(B)$

$\frac{3}{5} = \frac{1}{2} + P - \frac{1}{2} \times P$

$\frac{3}{5} - \frac{1}{2} = \frac{1}{2} P \Rightarrow P = \frac{2}{10} = \frac{1}{5}$

Note. $P(A/B) = \frac{P(A \cap B)}{P(B)}$

If A, B are independent then $P(A \cap B) = P(A) \cdot P(B)$

$\Rightarrow P(A/B) = \frac{P(A) \cdot \cancel{P(B)}}{\cancel{P(B)}} = P(A)$

Similarly $P(B/A) = P(B)$

(19) **Random Variable**: Let 'S' be a sample space of a random experiment.

A real valued fn $X : S \to R$ is called Random variable.

e.g. If 'X' is the sum of the numbers on rolling two dice then 'X' is random variable

i.e. $X(1,1) = 2$

$X(2\,3) = 5$

$X(6\,6) = 12$

$X : \underset{\substack{\uparrow \\ \text{Domain}}}{\overset{\text{sample space}}{S}} \to \underset{\text{Range}}{\overset{\text{Real numbers}}{R}}$

$\boxed{\text{Random Variable}}$

$\boxed{\text{Discrete Random Variable}}$	$\boxed{\text{Continuous Random Variable}}$
It is Probability Mass function	It is Probability Density function

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* **Discrete Random Variable**

Let S be the sample space, a random variable $X: S \to R$ is said to be discrete (or) discontinuous if the range of 'X' is countable.

e.g. if 'X' is the number of heads obtaining on tossing 5 coins, then the Range of X is $\{0, 1, 2, 3, 4, 5\}$ and hence X is discrete.

* **Continuous Random Variable**

A random variable $X: S \to R$ is said to be continuous if the range of X is an interval (a, b)

(26) **Probability distribution of the R.V. 'X'**

Let S be the sample space and $X: S \to R$ be a Random variable. The function $F: R \to R$ defined by $F(x) = P(X \leq x)$ is called probability distribution function of X.

Let $F(x)$ be the probability distribution function for the random variable X then

(i) $0 \leq F(x) \leq 1 \quad \forall x \in R$

(ii) $F(x)$ is an increasing function i.e. $x_1, x_2 \in R$ s.t. $x_1 < x_2 \Rightarrow F(x_1) \leq F(x_2)$

(iii) $\lim_{x \to \infty} F(x) = 1$ and $\lim_{x \to -\infty} F(x) = 0$

Note if $F(x) = P(X \leq x)$

$= P(X=0) + P(X=1) + \cdots + P(X=x)$

It functioning Like fi

It is Functioning Like cumulative Frequency

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Example

$$F(8) = P(X \le 8)$$
$$= P(X=1) + P(X=2) + \cdots + P(X=8)$$
$$= \frac{1}{36} + \frac{2}{36} + \cdots + \frac{5}{36}$$
$$= \frac{26}{36}$$

X	P(X)	F(x)
2	1/36	1/36
3	2/36	3/36
4	3/36	6/36
5	4/36	10/36
6	5/36	15/36
7	6/36	21/36
8	5/36	26/36
9	4/36	30/36
10	3/36	33/36
11	2/36	35/36
12	1/36	36/36

1

* Remarks

R1 If $X: S \to \mathbb{R}$ is a discrete Random variable with range $\{x_1, x_2, \ldots\}$ then $\sum_{r=1}^{\infty} P(X = x_r) = 1$

R2 Mean μ of Random variable $= \dfrac{\sum x_r P(X=x_r)}{\sum P(X)}$ Always $\sum P(X) = 1$ (x_i, f_i)

R3 Variance of Random Variable $= \dfrac{\sum (x_r - \mu)^2 P(X = x_r)}{\sum P(X)}$ Always $\to \sum P(X) = 1$

$$= \sum_{r=1}^{\infty} x_r^2 P(X = x_r) - \mu^2$$

Always $\to \sum P(X) = 1$

R4 Standard Deviation - The +ve square root of variance

* Important Note

Random Variable
- Discrete
  - Probability Mass fn
  - $P(X)$
  - $\sum P(X) = 1$
- Continuous
  - Probability Density fn
  - $F(x)$
  - $\int_{-\infty}^{\infty} F(x)\,dx = 1$

$$F(x) = P(X \le x)$$
$$= \sum_{-\infty}^{x} P(X) = \int_{-\infty}^{x} F(x)\,dx$$

Example X is continuous random Variable with Probability density function

$$f(x) = \begin{cases} c/\sqrt{x} & 0 < x < 4 \\ 0 & \text{elsewhere} \end{cases} \quad \text{find } c?$$

Soln $\boxed{\int_{-\infty}^{\infty} f(x)\,dx = 1}$

$$\int_{-\infty}^{0} f(x)\,dx + \int_{0}^{4} f(x)\,dx + \int_{4}^{\infty} f(x)\,dx = 1$$

$$0 + \int_{0}^{4} \frac{c}{\sqrt{x}}\,dx + 0 = 1$$

$$\left[c(2\sqrt{x})\right]_0^4 = 1 \Rightarrow c(4) - 0 = 1$$

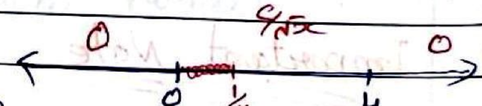
$$\boxed{c = 1/4}$$

Example X is continuous random variable with probability density function

$$f(x) = \begin{cases} c/\sqrt{x} & 0 < x < 4 \\ 0 & \text{elsewhere} \end{cases} \quad \text{Then } P(X < \tfrac{1}{4}) = ? \text{ an } P(X > 1) = ?$$

Soln.

$$\int_0^4 \frac{c}{\sqrt{x}}\,dx = \left[c(2\sqrt{x})\right]_0^4 = 1$$



$$4c = 1 \Rightarrow \boxed{c = 1/4}$$

$$P(X < \tfrac{1}{4}) = \int_0^{1/4} \frac{c}{\sqrt{x}}\,dx = \frac{1}{4}\int_0^{1/4} \frac{1}{\sqrt{x}}\,dx$$

$$= \frac{1}{4}\left[(2\sqrt{x})\right]_0^{1/4}$$

$$= \frac{1}{4} \times \frac{2}{2} = \frac{1}{4}$$

$$P(X > 1) = \int_1^4 \frac{c}{\sqrt{x}}\,dx = \frac{1}{4}\left[2\sqrt{x}\right]_1^4 = \frac{1}{4} \times (4 - 2) = \frac{2}{4} = \frac{1}{2}$$

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Example X is continuous random variable with Probability density function

$$f(x) = \begin{cases} cxe^{-2x} & x>0 \\ 0 & \text{elsewhere} \end{cases}$$

value of $c = ?$ and $P(2 \le x \le 3)$, $P(X>1)$

Soln

$$\int_0^\infty cxe^{-2x}\,dx = 1$$

$$c\int_0^\infty \underset{\text{I}}{x}\,\underset{\text{II}}{e^{-2x}}\,dx = 1 \Rightarrow c\left[\left(\frac{xe^{-2x}}{-2}\right)_0^\infty - \int_0^\infty \frac{e^{-2x}}{-2}\,dx\right] = 1$$

$$\Rightarrow c\left[0 - \left[\frac{e^{-2x}}{4}\right]_0^\infty\right] = 1 \Rightarrow -c\left[0 - \frac{1}{4}\right] = 1$$

$$\Rightarrow \frac{c}{4} = 1 \Rightarrow \boxed{c=4}$$

Now $P(2 \le X \le 3) = \int_2^3 cxe^{-2x}\,dx = 4\int_2^3 \underset{\text{I}}{x}\,\underset{\text{II}}{e^{-2x}}\,dx$

$$= 4\left[\left(\frac{xe^{-2x}}{-2}\right)_2^3 - \int_2^3 \frac{e^{-2x}}{-2}\,dx\right]$$

$$= 4\left[\left(\frac{3e^{-6}}{-2} - \frac{\cancel{2}e^{-4}}{-\cancel{2}}\right) + \frac{1}{2}\left(\frac{e^{-2x}}{-2}\right)_2^3\right]$$

$$= 4\left[\left(-\frac{3}{2}e^{-6} + e^{-4}\right) - \frac{1}{4}e^{-6} + \frac{1}{4}e^{-4}\right]$$

$$= -6e^{-6} + 4e^{-4} - e^{-6} + e^{-4} = 5e^{-4} - 7e^{-6} \text{ A}$$

$$P(X>1) = 4\int_1^\infty xe^{-2x}\,dx = 4\left[\left(\frac{xe^{-2x}}{-2}\right)_1^\infty - \int_1^\infty \frac{e^{-2x}}{-2}\,dx\right]$$

$$= 4\left[0 + \frac{1}{2}e^{-2} + \frac{1}{2}\left(\frac{e^{-2x}}{-2}\right)_1^\infty\right]$$

$$= 4\left[\frac{1}{2}e^{-2} + \frac{1}{4}e^{-2}\right] = 2e^{-2} + e^{-2} = 3e^{-2}$$

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Example The spectrum of R.V - X consists of points 1, 2, 3, ---, n and $P(X=i)$ is proportional to $\frac{1}{i(i+1)}$

Soln

X	$P(X=i)$
1	$K\left[\frac{1}{1}-\frac{1}{2}\right]$
2	$K\left[\frac{1}{2}-\frac{1}{3}\right]$
3	$K\left[\frac{1}{3}-\frac{1}{4}\right]$
4	$K\left[\frac{1}{4}-\frac{1}{5}\right]$
⋮	
n-1	
n	$K\left[\frac{1}{n}-\frac{1}{n+1}\right]$

$$P(x=i) \propto \frac{1}{i(i+1)}$$

$$P(x=i) = \frac{K}{i(i+1)} = K\left[\frac{1}{i}-\frac{1}{i+1}\right]$$

$$\sum P(x=i) = 1$$

$$K\left[1-\frac{1}{n+1}\right] = 1$$

$$K\left[\frac{n}{n+1}\right] = 1 \Rightarrow \boxed{K = \frac{n+1}{n}}$$

(21) **Disturbution Function**

A coin is Tossed 3 times

$X \sim$ no. of Heads

X	P(X)	X. P(X)	F(X)
0	1/8	0	1/8
1	3/8	3/8	$\frac{1}{8}+\frac{3}{8} = 4/8$
2	3/8	3/4	$\frac{4}{8}+\frac{3}{8} = 7/8$
3	1/8	3/8	$\frac{7}{8}+\frac{1}{8} = \frac{8}{8} = 1$

Always $\sum P(x) = 1$ $\sum xP(x) = 12/8$

$$\text{Mean } \bar{x} = \frac{\sum x P(x)}{\sum P(x)} = \frac{12/8}{1} = \frac{12}{8} = \frac{3}{2}$$

Note $\bar{x} = \sum x P(x) = E(x)$ — ~~Expected value of x~~ Mean is called.

$$V(x) = \sum x^2 P(x) - \sum x P(x)$$

Also $\boxed{V(x) = E(x^2) - E(x)}$ — Expected value of x

$$\boxed{\bar{x} = np} \qquad \boxed{V(x) = npq}$$

$$\bar{x} = 3 \times \frac{1}{2} = \frac{3}{2} \qquad V(x) = 3 \times \frac{1}{2} \times \frac{1}{2} = \frac{3}{4}$$

n = no. of coins Tossed
p = Probability for success
q = Probability for failure

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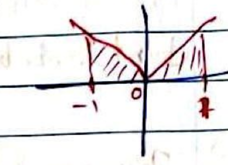
Example $f(x) = \begin{cases} |x| & -1 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$ is $f(x)$ is a p.d.f?

Solution For a $f(x)$ p.d.f $\int_{-1}^{1} f(x)\,dx = 1$

So $\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} |x|\,dx = 2\int_{0}^{1} x\,dx$



$= 2\left(\frac{x^2}{2}\right)_0^1 = 2 \times \frac{1}{2} = 1$

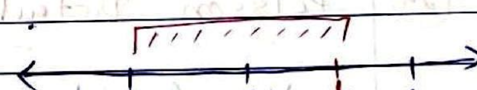
Hence $f(x)$ is a P.d.f.

Example X is continuous random variable with probability density function

$$f(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{2}(1-x^2) & -1 \le x < 0 \\ \frac{1}{2}(1+x^2) & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

Then $P(X < \frac{1}{2}) = ?$

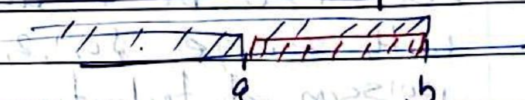
Soln $P(X < \frac{1}{2}) = \int_{-1}^{0} \frac{1}{2}(1+x^2)\,dx + \int_{0}^{1/2} \frac{1}{2}(1+x^2)\,dx$



$= \frac{1}{2}\left(x - \frac{x^3}{3}\right)_{-1}^{0} + \frac{1}{2}\left(x + \frac{x^3}{3}\right)_{0}^{1/2}$

$= \frac{1}{2}\left(-\left(-1+\frac{1}{3}\right)\right) + \frac{1}{2}\left(\frac{1}{2}+\frac{1}{24}\right) = \frac{1}{2}\left(\frac{2}{3}\right) + \frac{1}{2} \times \frac{13}{24} = \frac{1}{3} + \frac{13}{48} = \frac{29}{48}$ A

Note $\boxed{P(a \le x \le b) = P(x \le b) - P(x \le a)}$



$\boxed{F(x) = P(X \le x)}$

$\Rightarrow \boxed{P(a \le x \le b) = F(b) - F(a)}$

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(22) ## Joint Probability Function.

$$\boxed{\sum_x \sum_y f(x,y) = 1}$$ If x, y discrete — Joint Probability Mass function

$$\boxed{\int_x \int_y f(x,y) = 1}$$ If x, y is continuous — Joint Probability Density Function

(23) ## Binomial Distribution:

Let n be +ve integer and p be any real no. s.t. $0 \le p \le 1$. A random variable X with range $\{0, 1, 2, \ldots n\}$ is said binomial distribution or Bernoulli distribution with parameters 'n' and 'p'

$$P(X=r) = {}^nC_r\, q^{n-r} p^r \quad \text{for } r = 0, 1, 2, \ldots n$$

where $p + q = 1$ $\quad \bar{x} = np \quad V(x) = npq$

(24) ## Poisson Distribution:

The poisson distribution is the limiting form of the binomial distribution As $n \to \infty$ and $p \to 0$ so that $np = \lambda$ remains constant.

Note: When the no. of trials is very large, we use poisson distribution.

Let $\lambda > 0$ be a real no., a random variable X with range $\{0, 1, 2, \ldots\}$ is said to follow poisson distribution with parameter λ.

$$\boxed{P(X=r) = \frac{e^{-\lambda} \cdot \lambda^r}{r!}} \quad \text{for } r = 0, 1, 2, \ldots$$

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Note: For the Poisson distribution

$$\boxed{mean = Variance = \lambda}$$

where λ is the parameter of poisson distribution

Example A poisson variable satisfies $P(X=1) = P(X=2)$ find $P(X=5)$.

(a) $\frac{4}{15e^2}$

(b) $\frac{2}{15e^2}$

(c) $\frac{4}{13e^2}$

(d) $\frac{4}{13e^2}$

$P(X=r) = \frac{e^{-\lambda}\lambda^r}{\lfloor r}$

$P(X=1) = P(X=2)$

$\frac{\cancel{e^{-\lambda}}\cancel{\lambda}}{\lfloor 1} = \frac{\cancel{e^{-\lambda}}\lambda^{\cancel{2}}}{\lfloor 2}$

$\boxed{\lambda = 2}$

$P(X=5) = \frac{e^{-2}(2)^5}{\lfloor 5}$

$= \frac{32^4}{\cancel{120} \times e^2}$ 15

$= \frac{4}{15e^2}$

Example The parameter λ of poisson distribution is always

(a) Zero

(b) 1

(c) -1

(d) A finite +ve value.

Example The variance of P.D with parameter λ is

(a) λ (b) $\sqrt{\lambda}$ (c) $1/\lambda$ (d) $1/\sqrt{\lambda}$

Example $f(x) = \begin{cases} cxe^{-2x} & x>0 \\ 0 & \text{elsewhere} \end{cases}$

find $c = ?$ $P(2 \le x \le 3)$ $P(x>1)$

Soln we know $\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow \int_{-\infty}^{0} 0\,dx + \int_{0}^{\infty} cxe^{-2x}dx = 1$

$= 0 + c\left[\left[x\frac{e^{-2x}}{-2} \right]_0^{\infty} - \int_0^{\infty} \frac{e^{-2x}}{-2}dx \right] = 1$

$= c\left[0 - \left[\left(+\frac{1}{4}\right)e^{-2x} \right]_0^{\infty} \right] = 1 \Rightarrow c\left(\frac{1}{4}\right) = 1 \Rightarrow \boxed{c = 1}$

(ii) $P(2 \le x \le 3) = \int_2^3 f(x)\,dx = \int_2^3 4xe^{-2x}\,dx$

$$= 4\left[\left(x\frac{e^{-2x}}{-2}\right)_2^3 - \int_2^3 \frac{e^{-2x}}{-2}\,dx\right]$$

$$= 4\left[\left(-\frac{3}{2}e^{-6} + e^{-4}\right) - \left(\frac{1}{4}e^{-2x}\right)_2^3\right]$$

$$= 4\left[\left(-\frac{3}{2}e^{-6} + e^{-4}\right) - \frac{1}{4}e^{-6} + \frac{1}{4}e^{-4}\right]$$

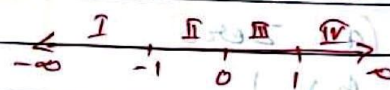
$$= -6e^{-6} + 4e^{-4} - e^{-6} + e^{-4}$$

$$= 5e^{-4} - 7e^{-6} \text{ A}$$

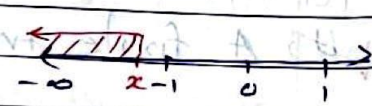
Example $f(x) = \begin{cases} |x| , & -1 < x < 1 \\ 0 & \text{elsewhere} \end{cases} = \begin{cases} -x , & -1 < x < 0 \\ x , & 0 \le x < 1 \\ 0 & \text{elsewhere} \end{cases}$

To find distributive function $F(X)$

Soln. * Case I $\rightarrow$ $x < -1$

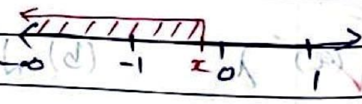


$$F_x(x) = P(X \le x) = \int_{-\infty}^{x} 0\,dx = 0$$



* Case II $\rightarrow$ $-1 < x < 0$

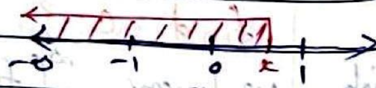
$$F_x(x) = P(X \le x) = \int_{-\infty}^{-1} 0\,dx + \int_{-1}^{x} (-x)\,dx$$



$$= \left[-\frac{x^2}{2}\right]_{-1}^{x} = -\frac{x^2}{2} + \frac{1}{2} = \frac{1}{2}(1 - x^2)$$

* Case-III $\rightarrow$ $0 \le x < 1$

$$F_x(x) = P(X \le x) = \int_{-\infty}^{-1} 0\,dx + \int_{-1}^{0} (-x)\,dx + \int_{0}^{x} x\,dx$$



$$= \left[-\frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2}\right]_0^x = 0 + \frac{1}{2} + \frac{x^2}{2} = \frac{1}{2}\left[1 + x^2\right]$$

* Case IV — $x \geq 1$

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{-1} 0\,dx + \int_{-1}^{0} -x\,dx + \int_{0}^{1} x\,dx + \int_{1}^{x} 0\,dx$$

$$= 0 + \left(\frac{-x^2}{2}\right)_{-1}^{0} + \left(\frac{x^2}{2}\right)_{0}^{1} + 0 = \frac{1}{2} + \frac{1}{2} = 1$$

So.

$$F_X(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{2}(1-x^2) & -1 \leq x < 0 \\ \frac{1}{2}(1+x^2) & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

distributive probability function

Note

$$F'(x) = \begin{cases} 0 & x < -1 \\ -x & -1 \leq x < 0 \\ x & 0 < x < 1 \\ 0 & x \geq 1 \end{cases} = f(x)$$

Now To find $P(X \leq \frac{1}{2})$

* $P(X \leq \frac{1}{2}) = F(\frac{1}{2}) = \frac{1}{2}\left(1 + \left(\frac{1}{2}\right)^2\right) = \frac{5}{8}$ A

* $P(-\frac{1}{2} \leq x \leq \frac{1}{2}) = F(\frac{1}{2}) - F(-\frac{1}{2})$

$$= \frac{5}{8} - \frac{1}{2}\left(1 - \left(\frac{1}{2}\right)^2\right) = \frac{5}{8} - \frac{3}{8} = \frac{2}{8} = \frac{1}{4} \text{ A}$$

* $P(X \geq \frac{1}{2}) = 1 - P(X \leq \frac{1}{2}) = 1 - F(\frac{1}{2}) = 1 - \frac{5}{8} = \frac{3}{8}$ A

Note $\boxed{P(X \geq a) = 1 - P(X \leq a) = 1 - F(a)}$

* Joint probability Function

Example

$$f(x,y) = \begin{cases} k\,x(x-y) & 0 < x < 2,\ -x < y < x \\ 0 & \text{elsewhere} \end{cases}$$

Find k = ?

Soln. we know it is a Joint Prob. Density fn

So $\iint f(x,y)\,dx\,dy = 1$.

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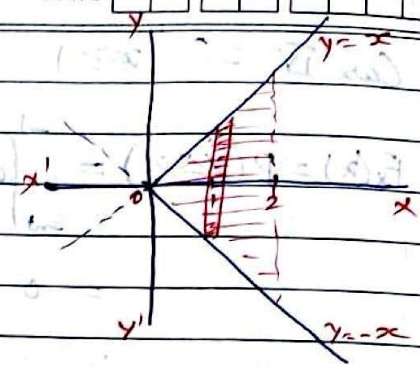
$$\int_{x=0}^{x=2}\left(\int_{y=-x}^{y=x} f(x,y)\,dy\right)dx = 1$$



$$\int_0^2\left(\int_{-x}^{x} k(x-(x-y))\,dy\right)dx = 1$$

$$\int_0^2 kx\left[xy-\frac{y^2}{2}\right]_{-x}^{x} dx = 1$$

$$\int_0^2 kx\left[\left(x^2-\frac{x^2}{2}\right)-\left(-x^2-\frac{x^2}{2}\right)\right]dx = 1$$

$$\int_0^2 kx\left[\frac{x^2}{2}+\frac{3x^2}{2}\right]dx = 1$$

$$\int_0^2 2kx^3\,dx = 1 \Rightarrow \left[2k\frac{x^4}{4}\right]_0^2 = 1$$

$$8k = 1 \Rightarrow \boxed{k = 1/8}$$

$$\boxed{f(x,y) = \frac{1}{8}x(x-y) \quad 0<x<2,\ -x<y<x}$$

Now $f_X(x) = \int_y f(x,y)\,dy = \int_{-x}^{x} \frac{1}{8}x(x-y)\,dy = \frac{x^3}{4}$

$f_Y(y) = \int_x f(x,y)\,dx = \int_{x=y}^{2} \frac{1}{8}x(x-y)\,dx$ or $\int_{x=-y}^{2} \frac{1}{8}x(x-y)\,dx$

$$\Rightarrow \frac{1}{8}\int_y^2 x^2-xy\,dx = \frac{1}{8}\left[\frac{x^3}{3}-\frac{x^2y}{2}\right]_y^2$$

$$= \frac{1}{8}\left[\frac{8}{3}-2y-\frac{y^3}{3}+\frac{y^3}{2}\right] = \frac{1}{8}\left[\frac{8}{3}-2y+\frac{y^3}{6}\right]$$

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New $\int_{-y}^{2} \frac{1}{8}(x^3 - xy) = \frac{1}{8}\left[\frac{x^3}{3} - \frac{x^2 y}{2}\right]_{-y}^{2}$

$= \frac{1}{8}\left[\frac{8}{3} - 2y + \frac{y^3}{3} + \frac{y^3}{2}\right] = \frac{1}{8}\left[\frac{8}{3} - 2y + \frac{5y^3}{6}\right]$

So $f_x(x) = \frac{x^3}{4}$

Marginal Probabilities

$$f_y(y) = \begin{cases} \frac{1}{8}\left[\frac{8}{3} - 2y + \frac{y^3}{6}\right] & 0 < y < 2 \\ \frac{1}{8}\left[\frac{8}{3} - 2y + \frac{5y^3}{6}\right] & -2 < y < 0 \end{cases}$$

(25) **Expected Value**

Probability of $g(x)$

$$E(g(x)) = \int g(x)\, f(x)\, dx = \sum g(x)\, P(x)$$

For [illegible] fn. — for discrete fn.

$E(x) = \int x f(x)\, dx$ — $E(x+y) = \iint (x+y) f(x,y)\, dx\, dy$

$E(x^2) = \int x^2 f(x)\, dx$ — $E(xy) = \iint xy\, f(x,y)\, dx\, dy$

$E(x^r) = \int x^r f(x)\, dx$

Note. For x and y independent $P(x \cap y) = P(x) \cdot P(y)$

Similarly

For x and y independent $E(xy) = E(x)\, E(y)$

* $E(c) = \int c f(x)\, dx = c \underbrace{\int f(x)\, dx}_{\text{Total probability} = 1} = c(1) = c$

* $E(ax+b) = a\, E(x) + b$

Note

Conditional Prob. of x given y: $f_{x/y}(x,y) = \frac{f(x,y)}{f_y(y)}$ — $f_{y/x}(x,y) = \frac{f(x,y)}{f_x(x)}$

Conditional Prob. of Y given X.

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$E(ax) = a\,E(x)$

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$\mu_1' = E(x) = \text{mean} = \bar{x}$ $\quad E(x-A) = \int (x-A) f(x)\,dx$

$\mu_2' = E(x^2)$ $\quad \mu_1 = E(x-\bar{x}) = E(x) - E(\bar{x})$

$\mu_3' = E(x^3)$ $\quad = E(x) - \bar{x}$

$= \bar{x} - \bar{x} = 0$

$\mu_4' = E(x^4)$ $\quad \mu_2 = E(x-\bar{x})^2$ $\quad \mu_3 = E(x-\bar{x})^3$

$$\mu_2 = E(x-\bar{x})^2 = E(x^2 + \bar{x}^2 - 2x\bar{x})$$
$$= E(x^2) + E(\bar{x})^2 - E(2x\bar{x})$$
$$= E(x^2) + \bar{x}^2 - 2\bar{x}\,\underbrace{E(x)}_{\bar{x}}$$
$$= E(x^2) + \bar{x}^2 - 2\bar{x}^2 = \underbrace{E(x^2)}_{\mu_2'} - \underbrace{\bar{x}}_{\mu_1'}$$

$\boxed{\mu_2 = \mu_2' - \mu_1'}$ — Variance.

Similarly $\quad \mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1' - 3\mu_1'^4$

$\mu_1', \mu_2', \mu_3', \mu_4' \longrightarrow$ Moments about origin

$\mu_1\ \mu_2\ \mu_3\ \mu_4 \longrightarrow$ Moments about mean.

Note For Joint Probability functions P_{xy}

Marginal Probabilities
$$\boxed{P(x) = \sum_y P_{xy} \qquad P(y) = \sum_x P_{xy}}$$