№1

Саратовцев Артем 18ПИ-1

3) Несмещенность:

Доказать, что $M[\hat{a}] = a$

$M[\hat{a}] = M[\frac{1}{2n}\sum_{i=1}^{n} X_i]$ = (по св-ву мат. ожидания) = $\frac{1}{2n} M[\sum_{i=1}^{n} X_i]$ = (по св-ву мат. ожидания) =

$= \frac{1}{2n}\sum_{i=1}^{n} M[X_i] = \frac{1}{2\cancel{n}} M[X]\cdot\cancel{n} = \frac{1}{2} M[X]$

$M[X] = \frac{1}{a^2}\int_0^\infty x\cdot x\cdot e^{-x/a} = \frac{1}{a^2}\int_0^\infty (at)^2\cdot e^{t}\cdot a\cdot dt = a\int_0^\infty t^{3-1}\cdot e^{t}\,dt = a\Gamma(3) = 2a$

$x/a = t$
$x = at$
$dx = a\cdot dt$

$M[\hat{a}] = \frac{1}{2}2a = a \Rightarrow$ Несмещенная

4) Состоятельность:

$P\{|\hat{a}-a|<\varepsilon\} \geq 1 - \frac{D[\hat{a}]}{\varepsilon^2}$

$D[\hat{a}] = D[\frac{1}{2n}\sum_{i=1}^{n} X_i] = \frac{1}{4n^2} D[\sum_{i=1}^{n} X_i] = \frac{1}{4n} D[X] = \frac{1}{4n}\cdot 2a^2 = \frac{a^2}{2n}$

$D[X] = M[X^2] - (M[X])^2 = 6a^2 - (2a)^2 = 6a^2 - 4a^2 = 2a^2$

$M[X^2] = \frac{1}{a^2}\int_0^\infty x^2\cdot x\cdot e^{-x/a}dx = \frac{1}{a^2}\int_0^\infty (at)^3\cdot e^{-t}\cdot a\cdot dt = a^2\int_0^\infty t^{4-1}\cdot e^{-t}\cdot dt = 6a^2$

$x/a = t$
$x = at$
$dx = a\cdot dt$

$P\{|\hat{a}-a|<\varepsilon\} = 1 - \frac{a^2}{2n} \underset{n\to\infty}{=} 1 \Rightarrow$ состоятельная

5) Эффективность:

$e = \frac{D[\hat{a}^э]}{D[\hat{a}]} \leq 1$ $\qquad f(x,a) = \frac{1}{a^2}\cdot e^{-x/a}$

$D[\hat{a}^э] = \frac{1}{nJ_a} = \frac{1}{n\cdot 2/a^2} = \frac{a^2}{2n}$ $\qquad \ln(f(x,a)) = -2\ln(a) - x/a$

$J_a = M[\left(\frac{\partial \ln(f(x,a))}{\partial a}\right)^2] = M[C] = M[4/a^2] - M[4x/a^3] + M[x^2/a^4] =$

$= 4/a^2 - 4/a^3\cdot M[X] + 1/a^4\cdot M[X^2] = 4/a^2 - 8/a^2 + 6/a^2 = 2/a^2$

$(\ln(f(x,a)))'_a = -2/a + x/a^2$

$(-2/a + x/a^2)^2 = 4/a^2 - 4x/a^3 + x^2/a^4 = C$

$e = \frac{D[\hat{a}^э]}{D[a]} = \frac{a^2/2n}{a^2/2n} = 1 \Rightarrow \hat{a}$ - эффективная оценка.

№2

Несмещенность:

$$M[\hat a] = M\left[\frac{1}{n}\sum_{i=1}^{n} x_i^2\right] = \frac{1}{n} M\left[\sum_{i=1}^{n} x_i^2\right] = \frac{1}{n}\sum_{i=1}^{n} M[x_i^2] = \frac{1}{n}\cdot n\cdot M[x^2] = M[x^2]$$

$$M[x^2] = \frac{2}{a}\int_0^{\infty} x^2\cdot x\cdot e^{-x^2/a}\,dx = \frac{2}{a}\int_0^{\infty}(at)^{3/2}\cdot\frac{1}{2}\left(\frac{a}{t}\right)^{1/2}\cdot e^{-t}\cdot dt = a\int_0^{\infty} t^{2-1}\cdot e^{-t}\cdot dt = a$$

$x^2/a = t$

$x = \sqrt{at}$

$dx = \frac{1}{2}\sqrt{a/t}\,dt$

$M[\hat a] = a \Rightarrow$ несмещенная

Состоятельность:

$$P\{|\hat a - a| < \varepsilon\} \geq 1 - \frac{D[\hat a]}{\varepsilon^2}$$

$$D[\hat a] = D\left[\frac{1}{n}\sum_{i=1}^{n} x_i^2\right] = \frac{1}{n^2} D\left[\sum_{i=1}^{n} x_i^2\right] = \frac{1}{n} D[x^2] = a^2/n$$

$$D[x^2] = M[x^4] - (M[x^2])^2 = 2a^2 - a^2 = a^2$$

$$M[x^4] = \frac{2}{a}\int_0^{\infty} x^4\cdot x\cdot e^{-x^2/a}\cdot dx = \frac{2}{a}\int_0^{\infty}(at)^{5/2}\cdot\left(\frac{1}{2}\right)\cdot\left(\frac{a}{t}\right)^{1/2}\cdot e^{-t}\cdot dt = a^2\int_0^{\infty} t^{3-1} e^{-t}\,dt = 2a^2$$

$x^2/a = t$

$x = \sqrt{at}$ $\quad dx = \frac{1}{2}\sqrt{a/t}\cdot dt$

$P\{|\hat a - a| < \varepsilon\} \geq 1 - a/n \xrightarrow[n\to\infty]{} 1 \Rightarrow$ состоятельная

Эффективность:

$$e = \frac{D[\hat a^{э}]}{D[\hat a]} \leq 1 \qquad f(x,a) = \frac{2}{a}\cdot e^{-x^2/a}$$

$$D[\hat a^{э}] = \frac{1}{nJ_a} = \frac{1}{n\cdot 1/a^2} = a^2/n \qquad \ln(f(x,a)) = -\ln(a/2) - x^2/a$$

$$J_a = M\left[\left(\frac{\partial \ln(f(x,a))}{\partial a}\right)^2\right] = M[C] = M[1/a^2] - M[2x^2/a^3] + M[x^4/a^4] = \frac{1}{a^2} - \frac{2}{a^3}M[x^2] + \frac{1}{a^4}M[x^4] = \frac{1}{a^2} - \frac{2}{a^2} + \frac{2}{a^2} = 1/a^2$$

$$(\ln(f(x,a)))'_a = \left(-\ln(a/2) - x^2/a\right)'_a = -\frac{1}{a} + \frac{x^2}{a^2}$$

$$\left(-\frac{1}{a} + \frac{x^2}{a^2}\right)^2 = \frac{1}{a^2} - \frac{2x^2}{a^3} + \frac{x^4}{a^4} = C$$

$$e = \frac{D[\hat a^{э}]}{D[a]} = \frac{a^2/n}{a^2/n} = 1 \Rightarrow \hat a \text{ — эффективная оценка}$$