Topic 3: Practice Problems

MATH 249/265

This problem set concerns *sections 1.5 - 1.6*. This includes limits at infinity, continuity, and the Intermediate Value Theorem.

Topic 3: Practice Problems

1. The following questions from *CLP-I Differential Calculus Questions*:
   (a) 1.5: 1 - 15, 17 - 20, 25
   (b) 1.6: 1 - 18, 24

1.5▲ Limits at Infinity

Q[1]: Give a polynomial $f(x)$ with the property that both $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ are (finite) real numbers.

Q[2]: Give a polynomial $f(x)$ that satisfies $\lim_{x\to\infty} f(x) \neq \lim_{x\to-\infty} f(x)$.

Q[3]: Evaluate $\lim_{x\to\infty} 2^{-x}$

Q[4]: Evaluate $\lim_{x\to\infty} 2^{x}$

Q[5]: Evaluate $\lim_{x\to-\infty} 2^{x}$

Q[6]: Evaluate $\lim_{x\to-\infty} \cos x$

Q[7]: Evaluate $\lim_{x\to\infty} x - 3x^5 + 100x^2$.

Q[8]: Evaluate $\lim_{x\to\infty} \dfrac{\sqrt{3x^8+7x^4}+10}{x^4-2x^2+1}$.

Q[9](*): $\lim_{x\to\infty} \left[\sqrt{x^2+5x}-\sqrt{x^2-x}\right]$

Q[10](*): Evaluate $\lim_{x\to-\infty} \dfrac{3x}{\sqrt{4x^2+x}-2x}$.

Q[11](*): Evaluate $\lim_{x\to-\infty} \dfrac{1-x-x^2}{2x^2-7}$.

Q[12](*): Evaluate $\lim_{x\to\infty} \left(\sqrt{x^2+x}-x\right)$

Q[13](*): Evaluate $\lim_{x\to+\infty} \dfrac{5x^2-3x+1}{3x^2+x+7}$.

Q[14](*): Evaluate $\lim_{x\to+\infty} \dfrac{\sqrt{4x+2}}{3x+4}$.

Q[15](*): Evaluate $\lim_{x\to+\infty} \dfrac{4x^3+x}{7x^3+x^2-2}$.

Q[17](*): Evaluate $\lim_{x\to+\infty} \dfrac{5x^2+10}{3x^3+2x^2+x}$.

Q[18]: Evaluate $\lim\limits_{x\to-\infty} \dfrac{x+1}{\sqrt{x^2}}$.

Q[19]: Evaluate $\lim\limits_{x\to\infty} \dfrac{x+1}{\sqrt{x^2}}$

Q[20](*): Find the limit $\lim\limits_{x\to-\infty} \sin\left(\dfrac{\pi}{2}\dfrac{|x|}{x}\right) + \dfrac{1}{x}$.

Q[25]: Give a rational function $f(x)$ with the properties that $\lim\limits_{x\to\infty} f(x) \neq \lim\limits_{x\to-\infty} f(x)$, and both limits are (finite) real numbers.

1.6▲ Continuity

Q[1]: Give an example of a function (you can write a formula, or sketch a graph) that has infinitely many infinite discontinuities.

Q[2]: When I was born, I was less than one meter tall. Now, I am more than one meter tall. What is the conclusion of the Intermediate Value Theorem about my height?

Q[3]: Give an example (by sketch or formula) of a function $f(x)$, defined on the interval $[0,2]$, with $f(0) = 0$, $f(2) = 2$, and $f(x)$ never equal to 1. Why does this not contradict the Intermediate Value Theorem?

Q[4]: Is the following a valid statement?

Suppose f is a continuous function over $[10, 20]$, $f(10) = 13$, and $f(20) = -13$. Then f has a zero between $x = 10$ and $x = 20$.

Q[5]: Is the following a valid statement?

Suppose f is a continuous function over $[10, 20]$, $f(10) = 13$, and $f(20) = -13$. Then $f(15) = 0$.

Q[6]: Is the following a valid statement?

Suppose f is a function over $[10, 20]$, $f(10) = 13$, and $f(20) = -13$, and f takes on every value between -13 and 13. Then f is continuous.

Q[7]: Find a constant k so that the function

$$a(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{when } x \neq 0 \\ k & \text{when } x = 0 \end{cases}$$

is continuous at $x = 0$.

Q[8]: Use the Intermediate Value Theorem to show that the function $f(x) = x^3 + x^2 + x + 1$ takes on the value 12345 at least once in its domain.

Q[9](∗): Describe all points for which the function is continuous: $f(x) = \dfrac{1}{x^2 - 1}$.

Q[10](∗): Describe all points for which this function is continuous: $f(x) = \dfrac{1}{\sqrt{x^2 - 1}}$.

Q[11](∗): Describe all points for which this function is continuous: $\dfrac{1}{\sqrt{1 + \cos(x)}}$.

Q[12](∗): Describe all points for which this function is continuous: $f(x) = \dfrac{1}{\sin x}$.

Q[13](*): Find all values of c such that the following function is continuous at $x = c$:

$$f(x) = \begin{cases} 8 - cx & \text{if} \quad x \leqslant c \\ x^2 & \text{if} \quad x > c \end{cases}$$

Use the definition of continuity to justify your answer.

Q[14](*): Find all values of c such that the following function is continuous everywhere:

$$f(x) = \begin{cases} x^2 + c & x \geqslant 0 \\ \cos cx & x < 0 \end{cases}$$

Use the definition of continuity to justify your answer.

Q[15](*): Find all values of c such that the following function is continuous:

$$f(x) = \begin{cases} x^2 - 4 & \text{if } x < c \\ 3x & \text{if } x \geqslant c\,. \end{cases}$$

Use the definition of continuity to justify your answer.

Q[16](*): Find all values of c such that the following function is continuous:

$$f(x) = \begin{cases} 6 - cx & \text{if} \quad x \leqslant 2c \\ x^2 & \text{if} \quad x > 2c \end{cases}$$

Use the definition of continuity to justify your answer.

Q[17]: Show that there exists at least one real number x satisfying $\sin x = x - 1$

Q[18](*): Show that there exists at least one real number c such that $3^c = c^2$.

Q[24]: Suppose $f(x)$ and $g(x)$ are functions that are continuous over the interval $[a, b]$, with $f(a) \leqslant g(a)$ and $g(b) \leqslant f(b)$. Show that there exists some $c \in [a, b]$ with $f(c) = g(c)$.

Answers to Exercises 1.5 — Jump to TABLE OF CONTENTS

A-1: There are many answers: any constant polynomial has this property. One answer is $f(x) = 1$.

A-2: There are many answers: any odd-degree polynomial has this property. One answer is $f(x) = x$.

A-3: 0

A-4: ∞

A-5: 0

A-6: DNE

A-7: $-\infty$

A-8: $\sqrt{3}$

A-9: 3

A-10: $-\frac{3}{4}$

A-11: $-\dfrac{1}{2}$

A-12: $\frac{1}{2}$

A-13: $\frac{5}{3}$

A-14: 0

A-15: $\frac{4}{7}$

A-17: 0

A-18: -1

A-19: 1

A-20: -1

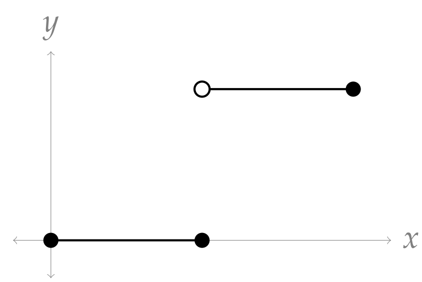
A-25: No such rational function exists.

Answers to Exercises 1.6

A-1: Many answers are possible; the tangent function behaves like this.

A-2: At some time between my birth and now, I was exactly one meter tall.

A-3: One example is $f(x) = \begin{cases} 0 & \text{when } 0 \leqslant x \leqslant 1 \\ 2 & \text{when } 1 < x \leqslant 2 \end{cases}$. The IVT only guarantees $f(c) = 1$ for some c in $[0, 2]$ when f is *continuous* over $[0, 2]$. If f is not continuous, the IVT says nothing.



A-4: Yes

A-5: No

A-6: No

A-7: $k = 0$

A-8: Since f is a polynomial, it is continuous over all real numbers. $f(0) = 0 < 12345$ and $f(12345) = 12345^3 + 12345^2 + 12345 + 1 > 12345$ (since all terms are positive). So by the IVT, $f(c) = 12345$ for some c between 0 and 12345.

A-9: $(-\infty, -1) \cup (-1, 1) \cup (1, +\infty)$

A-10: $(-\infty, -1) \cup (1, +\infty)$

A-11: The function is continuous *except* at $x = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots$.

A-12: $x \neq n\pi$, where n is any integer

A-13: ± 2

A-14: $c = 1$

A-15: $-1, 4$

A-16: $c = 1, c = -1$

A-17: This isn't the kind of equality that we can just solve; we'll need a trick, and that trick is the IVT. The general idea is to show that $\sin x$ is somewhere bigger, and somewhere smaller, than $x - 1$. However, since the IVT can only show us that a function is equal to a constant, we need to slightly adjust our language. Showing $\sin x = x - 1$ is equivalent to showing $\sin x - x + 1 = 0$, so let $f(x) = \sin x - x + 1$, and let's show that it has a real root.

First, we need to note that $f(x)$ is continuous (otherwise we can't use the IVT). Now, we need to find a value of x for which it is positive, and for which it's negative. By checking a few values, we find $f(0)$ is positive, and $f(100)$ is negative. So, by the IVT, there exists a value of x (between 0 and 100) for which $f(x) = 0$. Therefore, there exists a value of x for which $\sin x = x - 1$.

A-18: We let $f(x) = 3^x - x^2$. Then $f(x)$ is a continuous function, since both 3^x and x^2 are continuous for all real numbers.

We want a value a such that $f(a) > 0$. We see that $a = 0$ works since

$$f(0) = 3^0 - 0 = 1 > 0.$$

We want a value b such that $f(b) < 0$. We see that $b = -1$ works since

$$f(-1) = \frac{1}{3} - 1 < 0.$$

So, because $f(x)$ is continuous on $(-\infty, \infty)$ and $f(0) > 0$ while $f(-1) < 0$, then the Intermediate Value Theorem guarantees the existence of a real number $c \in (-1, 0)$ such that $f(c) = 0$.

A-24:

- If $f(a) = g(a)$, or $f(b) = g(b)$, then we simply take $c = a$ or $c = b$.

- Suppose $f(a) \neq g(a)$ and $f(b) \neq g(b)$. Then $f(a) < g(a)$ and $g(b) < f(b)$, so if we define $h(x) = f(x) - g(x)$, then $h(a) < 0$ and $h(b) > 0$. Since h is the difference of two functions that are continuous over $[a, b]$, also h is continuous over $[a, b]$. So, by the Intermediate Value Theorem, there exists some $c \in (a, b)$ with $h(c) = 0$; that is, $f(c) = g(c)$.

- The original .pdf "CLP-1 Differential Calculus Exercises" has full solutions:
  http://www.math.ubc.ca/~CLP/CLP1/problem_book_clp_1.pdf