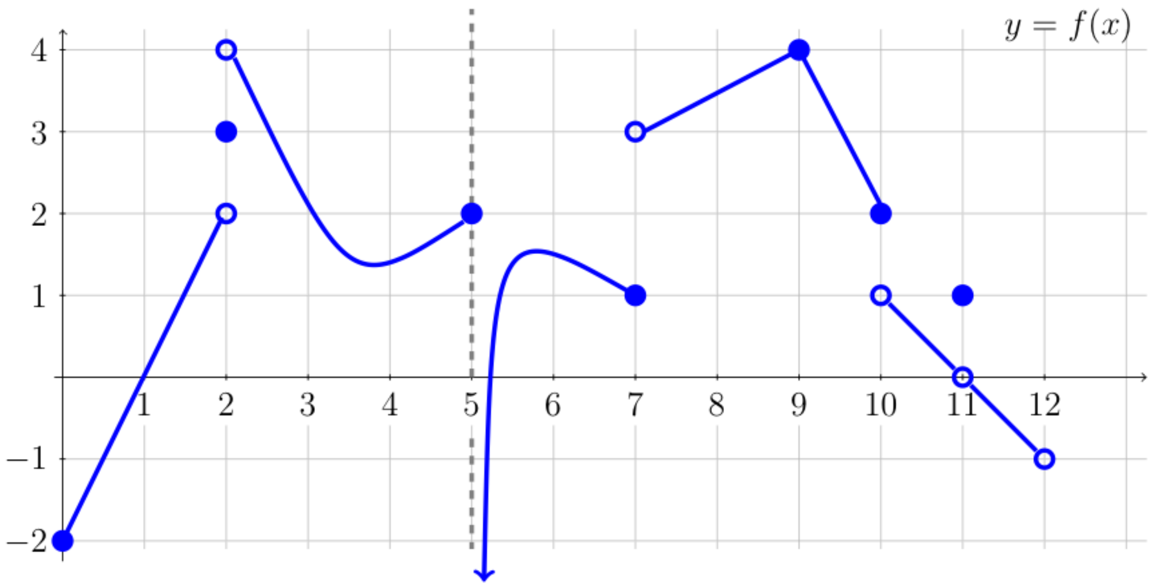
MATH 249 / 265 Term Exam I Written Solutions. F2023

All questions refer to the following graph.



VII-1

Version II

(a) f is not continuous at $x = 2$. Here's why:

Note that $\lim_{x\to 2^-} f(x) = 2$ and $\lim_{x\to 2^+} f(x) = 4$,

since the left and right limits at $x=2$ exist but are not equal, $\lim_{x\to 2} f(x)$ does not exist.

Moreover, since $\lim_{x\to 2^-} f(x) = 2 \neq 4 = \lim_{x\to 2^+}$

we see that f has a jump discontinuity at $x = 2$.

(b) I will evaluate $\lim_{x \to 5^+} \left[\frac{1}{f(x)} \sin\left(\frac{\pi}{x-5}\right) \right]$.

by using the Squeeze Theorem.

First $\quad -1 \leq \sin\left(\frac{\pi}{x-5}\right) \leq 1$

for $x \to 5^+$ (x close to 5 but $x > 5$)

$f(x) < 0$ so dividing the inequality by $f(x)$ we get:

$$\frac{-1}{f(x)} \geq \frac{1}{f(x)} \sin\left(\frac{\pi}{x-5}\right) \geq \frac{1}{f(x)}.$$

Now since $\lim_{x \to 5^+} f(x) = -\infty$, we get that

$$\lim_{x \to 5^+} \frac{-1}{f(x)} = 0 = \lim_{x \to 5^+} \frac{1}{f(x)}.$$

Therefore, by the Squeeze Theorem

$$\lim_{x \to 5^+} \left[\frac{1}{f(x)} \sin\left(\frac{\pi}{x-5}\right) \right] = 0.$$

Version 33

(a) The function f is not continuous at $x=5$.

Indeed, $\lim_{x\to 5^+} f(x) = -\infty$ and $\lim_{x\to 5^-} f(x) = 2$

so $\lim_{x\to 5} f(x)$ does not exist.

Moreover, since $\lim_{x\to 5^+} f(x) = -\infty$, f has an infinite discontinuity at $x=5$.

(b) I'll evaluate $\lim_{x\to 11^+}\left[f(x)\cos\left(\frac{\pi}{11-x}\right)\right]$ by using the Squeeze Theorem.

Recall $-1 \le \cos\left(\frac{\pi}{11-x}\right) \le 1$ and since $f(x) < 0$ for x near 11 and $x > 11$ we multiply by $f(x)$

to get $-f(x) \geq f(x)\cos\left(\frac{\pi}{11-x}\right) \geq f(x)$

Now $\lim_{x\to 11^+} f(x) = 0$ so $\lim_{x\to 11^+}(-f(x)) = -\lim_{x\to 11^+} f(x) = 0$

and by the Squeeze Theorem $\lim_{x\to 11^+}\left[f(x)\cos\left(\frac{\pi}{11-x}\right)\right] = 0$.

Version 55

(a) The function f is not continuous at $x = 7$. Indeed, $\lim_{x \to 7^-} f(x) = 1 = f(7)$, however $\lim_{x \to 7^+} f(x) = 3$. Since $\lim_{x \to 7^-} f(x) = 1$ exists and $\lim_{x \to 7^+} f(x) = 3$ exists, but these limits are not equal, we see that f has a jump discontinuity at $x = 7$.

(b) We know that $-1 \leq \sin(f(x)) \leq 1$ and we can multiply the inequality by $(5-x)^2 \geq 0$ to get

$$-(5-x)^2 \leq (5-x)^2 \sin(f(x)) \leq (5-x)^2$$

Now $\lim_{x \to 5^+} -(5-x)^2 = 0$ and $\lim_{x \to 5^+} (5-x)^2 = 0$.

Therefore by the Squeeze Theorem,

$$\lim_{x \to 5^+} \left[(5-x)^2 \sin(f(x)) \right] = 0.$$

Version 77

(a) The function f is not continuous at $x = 11$.

Indeed, $\lim_{x\to 11} f(x) = 0 \neq 1 = f(11)$

So since $\lim_{x\to 11} f(x)$ exists but is not equal to $f(11)$, we see that f has a removable discontinuity at $x = 11$.

(b) We know that for all $x \neq 1$,

$$-1 \leq \sin\left(\frac{2\pi}{1-x}\right) \leq 1$$

For x near 1 and $x > 1$, $f(x) > 0$ and we can multiply the inequality by $f(x)$ to get

$$-f(x) \leq f(x)\sin\left(\frac{2\pi}{1-x}\right) \leq f(x).$$

Now, $\lim_{x\to 1^+} f(x) = 0$ and $\lim_{x\to 1^+}(-f(x)) = -\lim_{x\to 1^+} f(x) = 0$

Therefore, by the Squeeze Theorem,

$$\lim_{x\to 1^+}\left[f(x)\sin\left(\frac{2\pi}{1-x}\right)\right] = 0.$$