Prime numbers $\Rightarrow$ Divisible by only 1 and number itself.

$\Rightarrow$ No of factors = 2

Eg : Is 1 a prime number ?

Prime: 2, 3, 5, 7, 11, 13 . . . .

Q Given n, tell if it is a prime number.

Approach 1:

$n \to 1$

$n \to n$

$\xrightarrow[\text{have factors}]{\text{doesn't}}$ $[2, N-1]$

$\Rightarrow$ Traverse from 2 to n-1
check if there is
a factor.

Approach 2: traversse from $[2, \sqrt{N}]$

12

1 x 12
2 x 6
3 x 4
4 x 3
6 x 2
12 x 1

$\boxed{i}$ $\boxed{n/i}$

$i \leq n/i$

$i^2 \leq n$

$\boxed{i < \sqrt{n}}$

TC: $O(\sqrt{N})$

SC: $O(1)$

A prime number does not have factors between 2 and $\sqrt{N}$

Q Find all the primes [1, N]

Ex 1 : $n = 10$ return 2, 3, 5, 7

$$\boxed{\begin{array}{c}\text{Using earlier approach} \\ TC : n\sqrt{n}\end{array}}$$

SIEVE OF ERATOSTHENES

~~1~~ 2 3 ~~4~~ 5 ~~6~~ 7 ~~8~~

~~9~~ ~~10~~ 11 ~~12~~ 13 ~~14~~ ~~15~~ ~~16~~

17 ~~18~~ 19 ~~20~~ ~~21~~ ~~22~~ 23 ~~24~~

~~25~~ ~~26~~ ~~27~~ ~~28~~ 29 ~~30~~ 31 ~~32~~

~~33~~ ~~34~~ ~~35~~ ~~36~~ 37 ~~38~~ ~~39~~ ~~40~~

41 ~~42~~ 43 ~~44~~ ~~45~~ ~~46~~ 47 ~~48~~

~~49~~ ~~50~~

7×2
7×3
7×4
7×5
7×6
7×7
7×8

$[2, \sqrt{n}]$

Observation: While marking the multiples for a prime i, you can start from $i \times i$ and move forward.

Observation2: We need to traverse from $[2, \sqrt{n}]$

Pseudo Code

```
bool primes[n+1] => T
primes[0] = primes[1] = F;

for (int i = 2 ; i <= √n ; i++) {
    if (primes[i]) {
        for (int j = i×i ; j <= N ; j += i) {
            primes[j] = F;
        }
    }
}

for (int i = 2 ; i <= N ; i++) {
    if (primes[i])
        print(i);
}
```

TC : (Iterations by primes) × (Iterations by no prime)

+ $O(\sqrt{n})$

$$TC : \frac{n}{2} + \frac{n}{3} + \frac{n}{5} + \frac{n}{7} \cdots\cdots + \frac{n}{x}$$

$$n\left\{\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} \cdots \frac{1}{x}\right\} + O(\sqrt{n})$$

$\Downarrow$

Sum of reciprocals of primes till n $\Rightarrow$ $\log(\log n)$

$\Rightarrow$ $n \log \log n \simeq \boxed{O(n)}$

$2^{10} \simeq 10^3$

$$TC : O(n \log \log n)$$

$n = 10^6$ $\quad \approx \log \log 10^6$

$n = 2^{20}$ $\quad \log \log 2^{20} \Rightarrow \log 20 \simeq \boxed{4}$

Q1 Given N, find smallest prime factors of all numbers from 2 – N.

Ex = $\boxed{n = 10}$

2	3	4	5	6	7	8	9	10
2	3	2	5	2	7	2	3	2

⇒ if (primes [i] == i)
↳ prime

Ex $\boxed{n = 30}$

1 2 3 ~~4~~ 2 5 ~~6~~ 2 7 ~~8~~ 2

~~9~~ 3 ~~10~~ 2 11 ~~12~~ 2 13 ~~14~~ 2 ~~15~~ 3 ~~16~~ 2

17 ~~18~~ 2 19 ~~20~~ 2 ~~21~~ 3 ~~22~~ 2 23 ~~24~~ 2

25 ~~26~~ 2 ~~27~~ 3 ~~28~~ 2 29 ~~30~~ 2

Pseudo Code:

spf ⇒ smallest prime factor.

```
spf [n+1];
for(int i=1; i<=n; i++){
    spf [i] = i
}
for (int i=2; i<=√n; i++) {
    if (spf [i] == i) {
        for (int j = i*i; j<=N; j+=i) {
            if ( spf [j] == j)
                spf[j] = i;
        }
    }
}
```

Tc: O (n log log n)
Sc: O (n)

Q3 Count no of divisors. Given Q Queries

Ex1 : $n = 72$

1×72

2×36

3×24

4×18

6×12

8×9

$$n \Rightarrow p_1^{x} \times p_2^{y}$$

no of divisors of n $\Rightarrow \boxed{(x+1) \times (y+1)}$

$\boxed{\text{Prime Factorisation}}$

2	72
2	36
2	18
3	9
3	3
	1

$$\boxed{72 \Rightarrow 2^3 \times 3^2}$$

$\Downarrow$

$(3+1) \times (2+1)$

$\Rightarrow 4 \times 3 \Rightarrow 12$

$72 \Rightarrow 2^3 \times 3^2$

$$\begin{bmatrix} 2^0 \\ 2^1 \\ 2^2 \\ 2^3 \end{bmatrix} \quad \begin{bmatrix} 3^0 \\ 3^1 \\ 3^2 \end{bmatrix}$$

4×3

$\Rightarrow 12$

$2^0 \times 3^0 \Rightarrow 1$

$2^0 \times 3^1 \Rightarrow 3$

$2^0 \times 3^2 \Rightarrow 9$

$2^1 \times 3^0 \Rightarrow 2$

$2^1 \times 3^1 \Rightarrow 6$

$2^1 \times 3^2 \Rightarrow 18$

$n \Rightarrow p_1^x \, p_2^y \, p_3^z$

no of divisors $\Rightarrow (x+1) \times (y+1) \times (z+1)$

Pseudo Code

Step 1: Create the spf array.

Step 2: Given N, find number of divisors.

```
int total => 1 .

while (N > 1) {
    int p => spf[N];
    int c => 0
    while (N%p == 0) {
        c++
        N /= p;
    }
    total => total x (1+c);
}
```

N = 72, p = 2, c => 0

N => 36, c => 1

N => 18, c => 2

N => 9, c => 3

Total = 4.

$\frac{N}{2}, N/4, N/8, N/16 \ldots 1$

$\log n$

$TC : (a \log n + n)$

Sieve

Q Are prime numbers finite ?

$\boxed{\text{No}}$

$$[P_1, P_2, P_3 \; . \; - - - \; P_n]$$

Co-primes

$$\boxed{\gcd(a, b) = 1}$$

3, 4 9, 7
3, 7 9, 2

∴ Adjacent numbers are always co-prime.

$$\boxed{\gcd(X, X+1) = 1}$$

Let's say we have finite prime numbers.

$$\Rightarrow [P_1, P_2 \ldots \text{—} \text{—} \text{—} \quad P_n]$$

$$X \Rightarrow P_1 \times P_2 \times P_3 \ldots \quad P_n$$

$$\gcd(X, X+1) \Rightarrow \underline{1}$$

$$X+1 \Rightarrow P_x^{a} \times P_y^{b}$$