Optimal Control WS20/21: Homework 1

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December 6, 2020

a) Discretize cost functional:

$$J \approx x_N^T Q x_N + \sum_{k=0}^{N-1} x_k^T Q x_k + u_k^T R u_k \tag{1}$$

b) Matrix representation of discretized linear dynamics:

We know

$$x(t) = e^{At} x(0) + \int_0^t e^{A(t-\tau)} B u(\tau) d\tau \tag{2}$$

Discretizing with time step size h

$$x_k \overset{\text{def}}{=} x(kh) \overset{\text{def}}{=} x(t_k)$$

and inserting it into (2) yields therefore for the state x_{k+1} the expression

$$\begin{aligned} x_{k+1} &= e^{Ah(k+1)} x_0 + \int_0^{(k+1)h} e^{A((k+1)h-\tau)} B u(\tau) d\tau & (3) \\ &= e^{Ah} \left[e^{Akh} x_0 + \int_0^{kh} e^{A(kh-\tau)} B u(\tau) d\tau \right] + \int_{kh}^{(k+1)h} e^{A(hT+h-\tau)} B u(\tau) d\tau. & (4) \end{aligned}$$

Where we assume that the steering signal u is constant between the discretization time samples t_k. We simplify this expression by substituting with $v(\tau) = kh + h - \tau$ and obtain

$$\begin{aligned} x_{k+1} &= e^{Ah} x_k - \left(\int_{v(kh)}^{v((k+1)h)} e^{Av} dv \right) B u_k & (5) \\ &= e^{Ah} x_k - \left(\int_h^0 e^{Av} dv B \right) u_k. & (6) \\ &= \underbrace{e^{Ah}}_{=:A_d} x_k + \underbrace{\left(\int_0^h e^{Av} dv B \right)}_{=:B_d} u_k. & (7) \end{aligned}$$

This verifies the given Matrix representation.
Euler Approximation:

$$\dot{x}(t_k) \approx \frac{x(t_{k+1}) - x(t_k)}{h}. \tag{8}$$

Rearranging (8) yields

$$\begin{aligned} x(t_{k+1}) - x(t_k) &\approx h\dot{x}(t_k) & (9) \\ &= hAx(t_k) + hBu(t_k) & (10) \\ \iff \quad x(t_{k+1}) &\approx (I + Ah)x(t_k) + hBu(t_k) & (11) \\ x_{k+1} &\approx \underbrace{(I + Ah)}_{=:A_{\text{eul}}} x_k + hBu_k. & (12) \end{aligned}$$

Relation to (7):
By the definition of the matrix exponential, we have

$$A_d = e^{Ah} = \sum_{k=0}^{\infty} \frac{A^k h^k}{k!}. \tag{13}$$

Neglecting all quadratic and higher terms yields with A_d from **b)**

$$A_d \approx I + Ah = A_{\text{eul}}$$

which is exactly the matrix obtained by the euler approximation.

c) Reformulate the discrete OC problem

$$\begin{aligned} \min_{x,u}. \quad & \sum_{k=ß}^{N-1} g(x,u) = x_N^T Q x_N + \sum_{k=0}^{N-1} x_k^T Q x_k + u_k^T R u_k \\ \text{subject to} \quad & x_{k+1} = A_d x_k + B_d u_k \\ & x_0 = \bar{x} \end{aligned} \tag{14}$$

We begin by defining the optimization variables vector y. It consists of all inputs and all states that occur from the initial time to the finite time horizon. So we define

$$u := \begin{pmatrix} u_0 \\ \vdots \\ u_{N-1} \end{pmatrix}; \quad x := \begin{pmatrix} x_0 \\ \vdots \\ x_N \end{pmatrix}; \quad y := \begin{pmatrix} x \\ u \end{pmatrix}. \tag{15}$$

For the objective function, we obtain $g(x,u) = \frac{1}{2} y^T H y + 0^T y + 0$ with the block diagonal matrix

$$H = \text{diag}(\underbrace{Q, \ldots, Q}_{N\text{blocks}}, \underbrace{R, \ldots, R}_{N-1\text{blocks}}).$$

Further, we implement the given system dynamics as linear equality constraints. Rearranging the difference equation yields

$$A_d x_k - x_{k+1} + B u_k = 0 \qquad \text{for all } k = 1, \ldots, N-1.$$

By stacking these equations for all $k \in \{1, \ldots, N-1\}$, we can formulate them as $A_{\text{eq}} y = b_{\text{eq}}$ with $A_{\text{eq}} = \begin{pmatrix} A_{\text{eq}}^x & A_{\text{eq}}^u \end{pmatrix}$,
and more detailed,

$$A_{\text{eq}}^x = \begin{pmatrix} I_{n\times n} & 0 & \cdots & \cdots & 0 \\ A_d & -I_{n\times n} & \cdots & \cdots & \vdots \\ 0 & \ddots & \ddots & & 0 \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & \cdots & & A_d & -I_{n\times n} \end{pmatrix},$$

$$A_{\text{eq}}^u = \begin{pmatrix} 0 & \cdots & 0 \\ B_d & \ddots & \vdots \\ \vdots & \ddots & 0 \\ 0 & \cdots & B_d \end{pmatrix}.$$

$$b_{\text{eq}} = \begin{pmatrix} \bar{x} \\ \vdots \\ 0 \end{pmatrix}.$$

The first row of blocks implements the initial condition $x_0 = \bar{x}$.