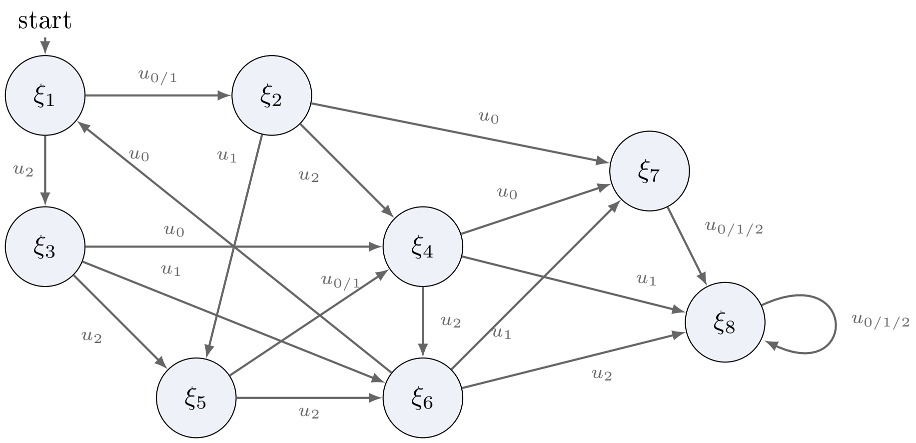
Homework exercise 2

Value function iteration and MPC

Submission deadline: February 5, 2021, 2pm

Problem 1:

Consider the following graph



where the labels along the paths denote the inputs that are required to do the transition from one state to another. Let $\mathcal{X} = \{\xi_1, \ldots, \xi_n\}$ denote the state space where $n = 8$ and let $\mathcal{U} = \{u_0, u_1, u_2\} = \{0, 1, 2\}$ denote the control space. We aim to minimize the cost along a path of infinite length where the cost is defined as

$$\sum_{k=0}^{\infty} 0.9^k f_0(x_k, v_k) \qquad x_k \in \mathcal{X}, v_k \in \mathcal{U}$$

and where the transition costs are given by

$$\begin{aligned}
&f_0(\xi_1, u_{0/1}) = 1 && f_0(\xi_1, u_2) = 1 && \\
&f_0(\xi_2, u_0) = 3 && f_0(\xi_2, u_1) = 6 && f_0(\xi_2, u_2) = 2\\
&f_0(\xi_3, u_0) = 1 && f_0(\xi_3, u_1) = 2 && f_0(\xi_3, u_2) = 3\\
&f_0(\xi_4, u_0) = 2 && f_0(\xi_4, u_1) = 6 && f_0(\xi_4, u_2) = 3\\
&f_0(\xi_5, u_{0/1}) = 0 && f_0(\xi_5, u_2) = 1 && \\
&f_0(\xi_6, u_0) = 5 && f_0(\xi_6, u_1) = 1 && f_0(\xi_6, u_2) = 1\\
&f_0(\xi_7, u_{0/1/2}) = 2 && && \\
&f_0(\xi_8, u_{0/1/2}) = 0. && &&
\end{aligned}$$

a) Read the dynamics from the graph and write the problem as a discrete-time infinite-horizon optimal control problem. Provide the value function iteration for this problem.

b) Implement the value function iteration in Matlab to compute the value function. Also compute the corresponding optimal feedback as well as the optimal input sequence when starting in the state ξ_1.

c) Suppose that we start the value function iteration with V^0 such that

$$V^0(\xi_i) \leq TV^0(\xi_i) \qquad \forall \xi_i \in \mathcal{X},$$

which generates a sequence of vectors $V^k = \left[V^k(\xi_1), \ldots, V^k(\xi_n)\right]^\top$. Show that $V^k \to V^*$ and that $V^0(\xi_i) \leq V^*(\xi_i)$ for all $\xi_i \in \mathcal{X}$, where V^* denotes the value function.
Hint: Use properties of the dynamic programming operator.

d) Deduce from c) that $V^* = [V^*(\xi_1), \ldots, V^*(\xi_n)]^\top$ is also the solution to the problem

$$\begin{aligned} \max_{V(\xi_1),\ldots,V(\xi_n)} \quad & \sum_{i=1}^{n} V(\xi_i) \\ \text{s.t.} \quad & V(\xi_i) \leq TV(\xi_i) \qquad i = 1, \ldots, n. \end{aligned} \tag{1}$$

e) Show that we can transform problem (1) into the linear program (LP) where

$$\begin{aligned} V^* = \arg\max_{V(\xi_1),\ldots,V(\xi_n)} \quad & \sum_{i=1}^{n} V(\xi_i) \\ \text{s.t.} \quad & V(\xi_i) \leq f_0(\xi_i, u) + \alpha V(f(\xi_i, u)) \\ & \qquad\qquad \forall u \in \mathcal{U}(\xi_i), \ \ i = 1, \ldots, n \end{aligned}$$

then write it in the form

$$\begin{aligned} V^* = \arg\min_{V} \quad & c^\top V \\ \text{s.t.} \quad & AV \leq b. \end{aligned} \tag{2}$$

f) Compute the value function in MATLAB by solving the LP in (2) using the command `linprog`. How can the corresponding optimal feedback be computed using this linear programming approach?

Problem 2:

Consider the discrete-time system

$$x_{k+1} = Ax_k + Bu_k,$$

where

$$A = \begin{bmatrix} 1 & 3 \\ -0.5 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 0.6 \\ -0.7 \end{bmatrix}.$$

a) Compute the equilibrium of the unforced system, i.e. for $u_k = 0, k = 0, 1, 2, \ldots.$ Is it stable, i.e. is $|\lambda_i(A)| < 1, i = 1, 2$?

b) For the given discrete-time system formulate the optimization problem of the MPC scheme at time step k with actual state $\bar{x}$ with

- an input constraint $|u_j| \leq 1$,
- a quadratic stage cost $f_0(x_j, u_j) = x_j^\top x_j + u_j^2$,
- a time horizon $N = 3$,
- a quadratic terminal cost

  $$\phi(x_3) = {x_3}^\top P x_3 \qquad \text{where } P = \begin{bmatrix} 4.2 & 7 \\ 7 & 36.1 \end{bmatrix},$$

- a terminal region $\mathcal{X}_f = \{x \in \mathbb{R}^2 : x^\top P x \leq c\}$ with $c = \frac{\lambda_{\min}(P)}{|K|^2} = 1.3325$ and $K = \begin{bmatrix} -0.3 & 1.4 \end{bmatrix}$ based on a local feedback $u = -Kx$.

c) Show that the MPC scheme from b) gives a stabilizing controller, i.e. show that there exists a feasible u such that

1. the control u is feasible for all $x \in \mathcal{X}_f$,
2. for all $x \in \mathcal{X}_f$ it is $\phi(x_{j+1}) - \phi(x_j) \leq -f_0(x_j, u_j)$ and
3. the terminal region $\mathcal{X}_f$ is invariant.

Hint 1: Consider the controller $u_j = -Kx_j$.
Hint 2: For 3., first show that 2. holds for all $x \in \mathbb{R}^2$ (you can use MATLAB to do the necessary calculations!) and use these results to show 3.

d) Define a suitable optimization variable z and write the optimal control problem from b) as a quadratically constrained quadratic program, i.e. write it in the form

$$\begin{aligned} \min_z \quad & z^\top H z \\ \text{s.t.} \quad & A_{eq} z = B_{eq} \\ & A_{ineq} z \leq B_{ineq} \\ & z^\top T z \leq d. \end{aligned}$$

Is the problem convex?

e) Implement the MPC algorithm in MATLAB and simulate the system for $k = 0, 1, \ldots, 30$. Use the MATLAB command `fmincon()` to solve the optimization problem from d). Provide plots of the resulting evolution of the state as well as of the determined input.

f) What happens if you consider another initial condition, e.g. $x_0 = \begin{bmatrix} 1 & -0.9 \end{bmatrix}^\top$? How can you explain this behavior?

g) Compute the solution of the LQ problem

$$\begin{aligned} \min \quad & \sum_{k=0}^{\infty} x_k^\top x_k + u_k^2 \\ \text{s.t.} \quad & x_{k+1} = Ax_k + Bu_k \end{aligned}$$

in MATLAB using the command `dlqr` and simulate the closed loop with the optimal input. Provide plots of the state as well as of the input and compare your results with those obtained in e). Does the optimal input satisfy the input constraints defined in b)?