Optimal Control WS20/21: Homework 2

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January 17, 2021

Problem 1

a) Formulating the problem as an discrete-time, infinite-horizon o. c. problem yields

$$\begin{aligned}
\min_{u_0,u_1,\dots} \quad & \sum_{k=0}^{\infty} f_0(x_k,u_k) = 0.9^k f_0(x_k,u_k) \\
\text{subject to} \quad & x_{k+1} = f(x_k,u_k), \\
& x_k \in \mathcal{X} = \{\xi_1,\dots,\xi_8\}, \\
& u_k \in \mathcal{U} = \{0,1,2\}, \\
& x_0 = \xi_1,
\end{aligned} \tag{1}$$

where the dynamics $f : \mathcal{X} \times \mathcal{U} \longrightarrow \mathcal{X}$ are defined by the arrows in the graph.

The Value function iteration is defined by the Bellman operator

$$V_{k+1}(x) = TV_k(x) = \min_{u\in\mathcal{U}}\{f_0(x,u) + \alpha V_k(x)\} \quad \text{with} \quad \alpha = 0.9. \tag{2}$$

Evaluated for this particular problem, this yields

$$\begin{aligned}
V_{k+1}(\xi_1) &= \min_{u\in\mathcal{U}}\{1+0.9V_k(\xi_2), 1+0.9V_k(\xi_2)\}, \\
V_{k+1}(\xi_2) &= \min_{u\in\mathcal{U}}\{3+0.9V_k(\xi_7), 6+0.9V_k(\xi_5), 3+0.9V_k(\xi_4)\}, \\
V_{k+1}(\xi_3) &= \min_{u\in\mathcal{U}}\{1+0.9V_k(\xi_4), 2+0.9V_k(\xi_6), 3+0.9V_k(\xi_5)\}, \\
V_{k+1}(\xi_4) &= \min_{u\in\mathcal{U}}\{2+0.9V_k(\xi_7), 6+0.9V_k(\xi_8), 3+0.9V_k(\xi_6)\}, \\
V_{k+1}(\xi_5) &= \min_{u\in\mathcal{U}}\{0+0.9V_k(\xi_4), 0+0.9V_k(\xi_4), 1+0.9V_k(\xi_6)\}, \\
V_{k+1}(\xi_6) &= \min_{u\in\mathcal{U}}\{5+0.9V_k(\xi_1), 1+0.9V_k(\xi_7), 1+0.9V_k(\xi_8)\}, \\
V_{k+1}(\xi_7) &= \min_{u\in\mathcal{U}}\{2+0.9V_k(\xi_8), 2+0.9V_k(\xi_8), 2+0.9V_k(\xi_8)\}, \\
V_{k+1}(\xi_8) &= \min_{u\in\mathcal{U}}\{0+0.9V_k(\xi_8), 0+0.9V_k(\xi_8), 0+0.9V_k(\xi_8)\},
\end{aligned} \tag{3}$$

starting with an arbitrary inital value function $V^0 : \mathcal{X} \longrightarrow \mathbb{R}$.

b) Matlab Results TODO!

c) We suppose

$$V^0(\xi) \le TV^0(\xi) \quad \forall_{\xi\in\mathcal{X}}, \tag{4}$$

for an initial function V^0 for a value function iteration with the Bellman Operator, defined in (2). From the lectures, we have the following properties.

$$\|TV^1 - TV^2\|_\infty \le \alpha \, \|V^1 - V^2\|_\infty \tag*{(Contraction) (5)}$$

$$V^1(\xi) \le V^2(\xi) \quad \forall_{\xi\in\mathcal{X}} \Longrightarrow TV^1(\xi) \le TV^2(\xi) \quad \forall_{\xi\in\mathcal{X}} \tag*{(Monotonicity) (6)}$$

V^* solves the Bellman Equation V^* is the Value function, since we have discounted costs. (7)

First we show, that the iteration converges to the value function of the problem. Since (5) holds and $\alpha \in (0,1)$,we deduce the convergence of $\{V_k\}_{k\in\mathbb{N}}$ by the Banach-Fixed-Point-Theorem. Since we have such a limit $\lim\limits_{k\to\infty} V_k \longrightarrow V^*$, it has to satisfy the fixed point property $TV^* = V^*$. Hence it satisfies the Bellman equation and (7) verifies V^* as the value function of our problem.

Second, it remains to show that $V^0(\xi) \leq V^*(\xi) \forall_{\xi\in\mathcal{X}}$ holds.
By assumption (4) and the monotonicity property (6), we infer inductively, that $V_k(\xi) \leq V_{k+1}(\xi)$ for all $\xi \in \mathcal{X}$. Thus, the sequence $\{V_k(\xi)\}_{k\in\mathbb{N}_0}$ is non-decreasing for any $\xi \in \mathcal{X}$. This proves the claim $V^0(\xi) \leq V^*(\xi) \forall_{\xi\in\mathcal{X}}$.

Problem 2

a)

b)