

# Optimal Control WS20/21: Homework 2

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## Problem 1

a) Formulating the problem as an discrete-time, infinite-horizon o. c. problem yields

$$\begin{aligned} \min_{u_0, u_1, \dots} \quad & \sum_{k=0}^{\infty} \alpha^k f_0(x_k, u_k) = \sum_{k=0}^{\infty} 0.9^k f_0(x_k, u_k) \\ \text{subject to} \quad & x_{k+1} = f(x_k, u_k), \\ & x_k \in \mathcal{X} = \{\xi_1, \dots, \xi_8\}, \\ & u_k \in \mathcal{U} = \{0, 1, 2\}, \\ & x_0 = \xi_1, \end{aligned} \tag{1}$$

where the dynamics  $f : \mathcal{X} \times \mathcal{U} \rightarrow \mathcal{X}$  are defined by the arrows in the graph.

The Value function iteration is defined by the Bellman operator

$$V_{k+1}(x) = TV_k(x) = \min_{u \in \mathcal{U}} \{f_0(x, u) + \alpha V_k(x)\} \quad \text{with } \alpha = 0.9. \tag{2}$$

Evaluated for this particular problem, this yields

$$\begin{aligned} V_{k+1}(\xi_1) &= \min_{u \in \mathcal{U}} \{1 + 0.9V_k(\xi_2), 1 + 0.9V_k(\xi_2)\}, \\ V_{k+1}(\xi_2) &= \min_{u \in \mathcal{U}} \{3 + 0.9V_k(\xi_7), 6 + 0.9V_k(\xi_5), 3 + 0.9V_k(\xi_4)\}, \\ V_{k+1}(\xi_3) &= \min_{u \in \mathcal{U}} \{1 + 0.9V_k(\xi_4), 2 + 0.9V_k(\xi_6), 3 + 0.9V_k(\xi_5)\}, \\ V_{k+1}(\xi_4) &= \min_{u \in \mathcal{U}} \{2 + 0.9V_k(\xi_7), 6 + 0.9V_k(\xi_8), 3 + 0.9V_k(\xi_6)\}, \\ V_{k+1}(\xi_5) &= \min_{u \in \mathcal{U}} \{0 + 0.9V_k(\xi_4), 0 + 0.9V_k(\xi_4), 1 + 0.9V_k(\xi_6)\}, \\ V_{k+1}(\xi_6) &= \min_{u \in \mathcal{U}} \{5 + 0.9V_k(\xi_1), 1 + 0.9V_k(\xi_7), 1 + 0.9V_k(\xi_8)\}, \\ V_{k+1}(\xi_7) &= \min_{u \in \mathcal{U}} \{2 + 0.9V_k(\xi_8), 2 + 0.9V_k(\xi_8), 2 + 0.9V_k(\xi_8)\}, \\ V_{k+1}(\xi_8) &= \min_{u \in \mathcal{U}} \{0 + 0.9V_k(\xi_8), 0 + 0.9V_k(\xi_8), 0 + 0.9V_k(\xi_8)\}, \end{aligned} \tag{3}$$

starting with an arbitrary initial value function  $V^0 : \mathcal{X} \rightarrow \mathbb{R}$ .

b) Value function  $V : \mathbb{R}^8 \rightarrow \mathbb{R}$ , obtained after 1000 value function iterations:

$$\begin{aligned} V(\xi_1) &= 3.61 \\ V(\xi_2) &= 4.80 \\ V(\xi_3) &= 2.90 \\ V(\xi_4) &= 3.80 \\ V(\xi_5) &= 1.90 \\ V(\xi_6) &= 1.00 \\ V(\xi_7) &= 2.00 \\ V(\xi_8) &= 0.00. \end{aligned} \tag{4}$$

The values in (4) induce the following state feedback:

$$\begin{aligned}
k(\xi_1) &= u_2 \\
k(\xi_2) &= u_1 \\
k(\xi_3) &= u_1 \\
k(\xi_4) &= u_1 \\
k(\xi_5) &= u_2 \\
k(\xi_6) &= u_2 \\
k(\xi_7) &= u_0 \\
k(\xi_8) &= u_0.
\end{aligned} \tag{5}$$

For the initial state  $x_0 = \xi_1$ , the state-feedback given in (5) yields the optimal input sequence

$$u^* = (2 \quad 1 \quad 2),$$

that steers the state to the terminal state  $\xi_8$  via the route  $\xi_1, \xi_3, \xi_6, \xi_8$ .

c) We suppose

$$V^0(\xi) \leq TV^0(\xi) \quad \forall \xi \in \mathcal{X}, \tag{6}$$

for an initial function  $V^0$  for a value function iteration with the Bellman Operator, defined in (2). From the lectures, we have the following properties.

$$\|TV^1 - TV^2\|_\infty \leq \alpha \|V^1 - V^2\|_\infty \tag{Contraction} \tag{7}$$

$$V^1(\xi) \leq V^2(\xi) \quad \forall \xi \in \mathcal{X} \implies TV^1(\xi) \leq TV^2(\xi) \quad \forall \xi \in \mathcal{X} \tag{Monotonicity} \tag{8}$$

$V^*$  solves the Bellman Equation  $V^*$  is the Value function, since we have discounted costs.  $(9)$

First we show, that the iteration converges to the value function of the problem. Since (7) holds and  $\alpha \in (0, 1)$ , we deduce the convergence of  $\{V_k\}_{k \in \mathbb{N}}$  by the Banach-Fixed-Point-Theorem. Since we have such a limit  $\lim_{k \rightarrow \infty} V_k \longrightarrow V^*$ , it has to satisfy the fixed point property  $TV^* = V^*$ . Hence it satisfies the Bellman equation and (9) verifies  $V^*$  as the value function of our problem.

Second, it remains to show that  $V^0(\xi) \leq V^*(\xi) \forall \xi \in \mathcal{X}$  holds.

By assumption (6) and the monotonicity property (8), we infer inductively, that  $V_k(\xi) \leq V_{k+1}(\xi)$  for all  $\xi \in \mathcal{X}$ . Thus, the sequence  $\{V_k(\xi)\}_{k \in \mathbb{N}_0}$  is non-decreasing for any  $\xi \in \mathcal{X}$ . This proves the claim  $V^0(\xi) \leq V^*(\xi) \forall \xi \in \mathcal{X}$ .

d) Maximization Problem:

$$\begin{aligned}
&\max_{V(\xi_1), \dots, V(\xi_n)} \sum_{k=0}^n V(\xi_k) \\
&\text{subject to} \quad V(\xi_i) \leq TV(\xi_i) \quad \forall i=1, \dots, n.
\end{aligned} \tag{10}$$

To show:  $V^*$  is a solution to the maximization problem given above. Proof by contradiction. Suppose, there exists an admissible Function  $\tilde{V} : \mathbb{R}^n \longrightarrow \mathbb{R}$  with

$$\tilde{V}(\xi_i) > V^*(\xi_i) \tag{11}$$

for at least one  $i \in \{1, \dots, n\}$ . Since  $\tilde{V}$  is admissible for the given max. problem, we know that  $\tilde{V}(\xi_i) \leq T\tilde{V}(\xi_i) \quad \forall i \in \{1, \dots, n\}$ . From exercise c) we know, that hence  $\tilde{V}(\xi_i) \leq T^k \tilde{V}(\xi_i) \leq V^*(\xi_i)$  holds for all  $i \in \{1, \dots, n\}$  and for all  $k \in \mathbb{N}$ . This clearly contradicts (11). Thus,  $V^*$  maximizes the objective of the given problem.

e) Transform (10) into

$$\begin{aligned} V^* = \arg \max_{V(\xi_1), \dots, V(\xi_n)} & \sum_{k=0}^n V(\xi_k) \\ \text{subject to} & V(\xi_i) \leq f_0(\xi_i, u) + \alpha V(f(\xi, u)) \\ & \forall u \in \mathcal{U}(\xi_i) \forall i=1, \dots, n. \end{aligned} \quad (12)$$

As shown in d),  $V^*$  maximizes the objective of problem (10). It remains to show, that the constraints of problems (10) and (14) are equivalent. The operator  $T$  is defined by the Bellman Equation, as written in (2). We have

$$\begin{aligned} V(\xi_i) \leq TV(\xi_i) &= \min_{u \in \mathcal{U}} \{f_0(\xi_i, u) + \alpha V_k(\xi_i)\} \\ &\leq f_0(\xi_i, \tilde{u}) + \alpha V_k(\xi_i) \end{aligned} \quad (13)$$

for any input signal  $\tilde{u} \in \mathcal{U}$ . This arises directly from the definition of the minimum. Thus, (10) and (14) denote the same problem.

It remains to formulate this as a linear program

$$\begin{aligned} V^* = \arg \max_V & c^T V \\ \text{subject to} & AV \leq b. \end{aligned} \quad (14)$$

With  $V = (V(\xi_1), \dots, V(\xi_n))^T$ , the objective  $\max \sum_{k=0}^n V(\xi_k)$  is the same as  $\min c^T V$  with

$$c = -(1, \dots, 1)^T \in \mathbb{R}^n.$$

Now we rewrite the constraints as a linear inequality  $AV = b$ .

For a state  $\xi_i \in \mathcal{X}$ , the inequality

$$\begin{aligned} V(\xi_i) &\leq f_0(\xi_i, u) + \alpha V(f(\xi_i, u)) \\ \iff V(\xi_i) - \alpha V(f(\xi_i, u)) &\leq f_0(\xi_i, u) \end{aligned} \quad (15)$$

has to hold for each possible input  $u \in \mathcal{U}$ , which leads to three scalar inequalities per state. Such an inequality can be represented with a line in the matrix  $A$ , with an 1 in the  $i$ -th column and  $-\alpha$  in the  $f(\xi_i, u)$ -th column. The corresponding entry of the vector  $b$  is  $f_0(\xi_i, u)$ . The resulting matrices  $A, b$  for the particular problem can be found in the solutions-sheet or can be generated by the matlab script.

One can determine the optimal input  $u^*$  in any state  $\xi_i$  by using the input that makes inequality (13) an equality. This minimizes the  $V$ -value of the next state that will be reached.

## Problem 2

a) System

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \quad \text{with} \\ A &= \begin{pmatrix} 1 & 3 \\ -0.5 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad x_0 = \begin{pmatrix} 0.6 \\ -0.7 \end{pmatrix}. \end{aligned} \quad (16)$$

The origin  $x^e = 0 \in \mathbb{R}^2$  is a equilibrium point of (16), since  $x^e = Ax^e$ . We determine the eigenvalues by computing roots of the characteristic polynomial of  $A$ .

$$\det(A - \lambda I) = \det \begin{pmatrix} 1 - \lambda & 3 \\ 3 & 1 - \lambda \end{pmatrix} = \lambda^2 - 4\lambda + 1.$$

hence, roots of the char. pol. are

$$\lambda_{1,2} = 1 \pm i \frac{\sqrt{6}}{2} \quad \text{with } i \text{ the imaginary unit.}$$

Since  $|\lambda_i| > 1$  for  $i = 1, 2$ ,  $x^e$  is an unstable equilibrium.

b) MPC optimization problem at time  $k$  with given state  $x_k = \bar{x}$ :

$$\begin{aligned}
& \min_{u_k, \dots, u_{k+N}} \sum_{j=k}^{k+N-1} f_0(x_j, u_j) + \phi(x_{k+N}) \\
& = \sum_{j=k}^{k+N-1} = x_j^T Q x_j + u_j^T R u_j + x_{k+N}^T P x_{k+N} \\
& \text{subject to} \quad x_{j+1} = A x_j + B u_j \text{ for all } j = k, \dots, k+N \quad (\text{dynamic constraints}) \\
& \quad u_j \leq 1, \text{ for all } j = k, \dots, k+N \quad (\text{input constraints}) \\
& \quad u_j \leq 1, \text{ for all } j = k, \dots, k+N \\
& \quad x_{k+N}^T P x_{k+N} \leq c, \quad (\text{terminal constraint}) \\
& \quad x_k = \bar{x}, \quad (\text{initial state constraint})
\end{aligned} \tag{17}$$

with

$$Q = I_2, R = 1, N = 3, c = \frac{\lambda_{\min}(P)}{|K|^2}, K = (0.3, 1.4).$$