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Optimal Control

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Homework exercise 1

Solution of finite horizon open loop optimal control problems via nonlinear programming

Submission deadline: January 8, 2021

Consider the following optimal control problem

$$\min \quad J = \int_0^{t_f} x(t)^\top Q\, x(t) + u(t)^\top R\, u(t)\, dt \tag{1a}$$

$$\text{s.t.} \quad \dot{x}(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}}_{=:A} x(t) + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{=:B} u(t) \tag{1b}$$

with Q and R symmetric positive definite and $x(0) = x_0 \in \mathbb{R}^2$. We want to obtain an approximate solution of this problem by transforming the infinite dimensional optimal control problem into a finite dimensional nonlinear program via discretization. In the following we consider the time discretization $t_k := kh$, where $h = \frac{t_f}{N}$, and $N > 0$, and assume that the control input $u(t)$ is piecewise constant on the equidistant time intervals $[t_k, t_{k+1})$.

a) Discretize the cost functional (1a) by using an appropriate approximation method for the integral, e.g. the rectangle method.

b) Utilize the general representation of the solution of a linear differential equation given by

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} B u(\tau) d\tau$$

in order to verify that the discretization of (1b) is given by the discrete-time system

$$x_{k+1} = e^{Ah}\, x_k + \left(\int_0^h e^{A\tau} B d\tau \right) u_k$$

with $x_k = x(t_k)$ and $u_k = u(t_k)$.

c) Utilize the Euler approximation of the derivative

$$\dot{x}(t_k) \approx \frac{x(t_{k+1}) - x(t_k)}{h}$$

in order to derive another discretization of (1b). How are the discretizations from b) and c) related?

In the following we consider the discretization of the optimal control problem (1) given by

$$\begin{aligned} \min \quad & \sum_{k=0}^{N-1} g(x_k, u_k) \\ \text{s.t.} \quad & x_{k+1} = A_D x_k + B_D u_k \end{aligned} \tag{2}$$

where g, A_D and B_D were determined in a) – c).

d) Define a suitable optimization variable y to write the discretized problem (2) as a quadratic program of the form

$$\begin{aligned} \min_{y} \quad & \frac{1}{2} y^\top H y + f^\top y + d \\ \text{s.t.} \quad & A_{\text{ineq}} y \le b_{\text{ineq}} \\ & A_{\text{eq}} y = b_{\text{eq}}. \end{aligned} \tag{3}$$

e) Derive the necessary conditions at an optimum of (3) and argue whether or not they are also sufficient. Give the optimal solution y^* in terms of H, f, d, A_{eq}, b_{eq}, A_{ineq}, b_{ineq}.

f) Let $t_f = 5$, $Q = I_2$, $R = 1$, $N = 50$ and $x(0) = \begin{bmatrix} -2 & 3 \end{bmatrix}^\top$. Solve the quadratic program (3) numerically with the help of MATLAB by solving the KKT-conditions from e) as well as by using the function `quadprog`. Compare the results of both discretization methods from b) and c) and provide plots of x_k and u_k for both cases.

g) Let $Q = \alpha I_2$ with $\alpha > 0$. How does a change in the parameter α affect the solution and how can you explain this behavior? Provide plots of x_k and u_k for $\alpha = 0.1$, $\alpha = 1$ and $\alpha = 10$.

h) Simulate the continuous-time system (1b) applying the piecewise continuous optimal input obtained in f) when using the exact discretization. How are the closed loop solutions of the continuous-time system and the discrete-time system related? Provide a plot of the error between the two solutions at the discretization time points t_k.