Optimal Control WS20/21: Homework 1

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a) Discretize cost functional:

$$J \approx x_N^T Q x_N + \sum_{k=0}^{N-1} x_k^T Q x_k + u_k^T R u_k \tag{1}$$

b) Matrix representation of discretized linear dynamics:

We know

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \tag{2}$$

Discretizing with time step size h

$$x_k \stackrel{\text{def}}{=} x(kh) \stackrel{\text{def}}{=} x(t_k)$$

and inserting it into (2) yields therefore for the state x_{k+1} the expression

$$\begin{aligned} x_{k+1} &= e^{Ah(k+1)}x_0 + \int_0^{(k+1)h} e^{A((k+1)h-\tau)}Bu(\tau)d\tau & (3)\\ &= e^{Ah}\left[e^{Akh}x_0 + \int_0^{kh} e^{A(kh-\tau)}Bu(\tau)d\tau\right] + \int_{kh}^{(k+1)h} e^{A(hT+h-\tau)}Bu(\tau)d\tau. & (4) \end{aligned}$$

Where we assume that the steering signal u is constant between the discretization time samples t_k. We simplify this expression by substituting with $v(\tau) = kh + h - \tau$ and obtain

$$\begin{aligned} x_{k+1} &= e^{Ah}x_k - \left(\int_{v(kh)}^{v((k+1)h)} e^{Av}dv\right)Bu_k & (5)\\ &= e^{Ah}x_k - \left(\int_h^0 e^{Av}dvB\right)u_k. & (6)\\ &= \underbrace{e^{Ah}}_{=:A_d}x_k + \underbrace{\left(\int_0^h e^{Av}dvB\right)}_{=:B_d}u_k. & (7) \end{aligned}$$

This verifies the given Matrix representation.

c) Euler Approximation:

$$\dot{x}(t_k) \approx \frac{x(t_{k+1}) - x(t_k)}{h}. \tag{8}$$

Rearranging (8) yields

$$\begin{aligned} x(t_{k+1}) - x(t_k) &\approx h\dot{x}(t_k) & (9)\\ &= hAx(t_k) + hBu(t_k) & (10)\\ \iff \quad x(t_{k+1}) &\approx (I + Ah)x(t_k) + hBu(t_k) & (11)\\ x_{k+1} &\approx \underbrace{(I + Ah)}_{=:A_{\text{eul}}}x_k + hBu_k. & (12) \end{aligned}$$

Relation to (7):
By the definition of the matrix exponential, we have

$$A_d = e^{Ah} = \sum_{k=0}^{\infty} \frac{A^k h^k}{k!}. \tag{13}$$

Neglecting all quadratic and higher terms yields with A_d from **b)**

$$A_d \approx I + Ah = A_{\text{eul}}$$

which is exactly the matrix obtained by the euler approximation.