

Optimal Control WS20/21: Homework 2

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Problem 1

a) Formulating the problem as an discrete-time, infinite-horizon o. c. problem yields

$$\begin{aligned} \min_{u_0, u_1, \dots} \quad & \sum_{k=0}^{\infty} \alpha^k f_0(x_k, u_k) = \sum_{k=0}^{\infty} 0.9^k f_0(x_k, u_k) \\ \text{subject to} \quad & x_{k+1} = f(x_k, u_k), \\ & x_k \in \mathcal{X} = \{\xi_1, \dots, \xi_8\}, \\ & u_k \in \mathcal{U} = \{0, 1, 2\}, \\ & x_0 = \xi_1, \end{aligned} \tag{1}$$

where the dynamics $f : \mathcal{X} \times \mathcal{U} \rightarrow \mathcal{X}$ are defined by the arrows in the graph.

The Value function iteration is defined by the Bellman operator

$$V_{k+1}(x) = TV_k(x) = \min_{u \in \mathcal{U}} \{f_0(x, u) + \alpha V_k(x)\} \quad \text{with } \alpha = 0.9. \tag{2}$$

Evaluated for this particular problem, this yields

$$\begin{aligned} V_{k+1}(\xi_1) &= \min_{u \in \mathcal{U}} \{1 + 0.9V_k(\xi_2), 1 + 0.9V_k(\xi_3)\}, \\ V_{k+1}(\xi_2) &= \min_{u \in \mathcal{U}} \{3 + 0.9V_k(\xi_7), 6 + 0.9V_k(\xi_5), 3 + 0.9V_k(\xi_4)\}, \\ V_{k+1}(\xi_3) &= \min_{u \in \mathcal{U}} \{1 + 0.9V_k(\xi_4), 2 + 0.9V_k(\xi_6), 3 + 0.9V_k(\xi_5)\}, \\ V_{k+1}(\xi_4) &= \min_{u \in \mathcal{U}} \{2 + 0.9V_k(\xi_7), 6 + 0.9V_k(\xi_8), 3 + 0.9V_k(\xi_6)\}, \\ V_{k+1}(\xi_5) &= \min_{u \in \mathcal{U}} \{0 + 0.9V_k(\xi_4), 0 + 0.9V_k(\xi_4), 1 + 0.9V_k(\xi_6)\}, \\ V_{k+1}(\xi_6) &= \min_{u \in \mathcal{U}} \{5 + 0.9V_k(\xi_1), 1 + 0.9V_k(\xi_7), 1 + 0.9V_k(\xi_8)\}, \\ V_{k+1}(\xi_7) &= \min_{u \in \mathcal{U}} \{2 + 0.9V_k(\xi_8), 2 + 0.9V_k(\xi_8), 2 + 0.9V_k(\xi_8)\}, \\ V_{k+1}(\xi_8) &= \min_{u \in \mathcal{U}} \{0 + 0.9V_k(\xi_8), 0 + 0.9V_k(\xi_8), 0 + 0.9V_k(\xi_8)\}, \end{aligned} \tag{3}$$

starting with an arbitrary initial value function $V^0 : \mathcal{X} \rightarrow \mathbb{R}$.

b) Value function $V : \mathbb{R}^8 \rightarrow \mathbb{R}$, obtained after 1000 value function iterations:

$$\begin{aligned} V(\xi_1) &= 3.61 \\ V(\xi_2) &= 4.80 \\ V(\xi_3) &= 2.90 \\ V(\xi_4) &= 3.80 \\ V(\xi_5) &= 1.90 \\ V(\xi_6) &= 1.00 \\ V(\xi_7) &= 2.00 \\ V(\xi_8) &= 0.00. \end{aligned} \tag{4}$$

The values in (4) induce the following state feedback:

$$\begin{aligned}
k(\xi_1) &= u_2 \\
k(\xi_2) &= u_1 \\
k(\xi_3) &= u_1 \\
k(\xi_4) &= u_1 \\
k(\xi_5) &= u_2 \\
k(\xi_6) &= u_2 \\
k(\xi_7) &= u_0 \\
k(\xi_8) &= u_0.
\end{aligned} \tag{5}$$

For the initial state $x_0 = \xi_1$, the state-feedback given in (5) yields the optimal input sequence

$$u^* = (2 \quad 1 \quad 2),$$

that steers the state to the terminal state ξ_8 via the route $\xi_1, \xi_3, \xi_6, \xi_8$.

c) We suppose

$$V^0(\xi) \leq TV^0(\xi) \quad \forall \xi \in \mathcal{X}, \tag{6}$$

for an initial function V^0 for a value function iteration with the Bellman Operator, defined in (2). From the lectures, we have the following properties.

$$\|TV^1 - TV^2\|_\infty \leq \alpha \|V^1 - V^2\|_\infty \tag{Contraction} \tag{7}$$

$$V^1(\xi) \leq V^2(\xi) \quad \forall \xi \in \mathcal{X} \implies TV^1(\xi) \leq TV^2(\xi) \quad \forall \xi \in \mathcal{X} \tag{Monotonicity} \tag{8}$$

V^* solves the Bellman Equation V^* is the Value function, since we have discounted costs. (9)

First we show, that the iteration converges to the value function of the problem. Since (7) holds and $\alpha \in (0, 1)$, we deduce the convergence of $\{V_k\}_{k \in \mathbb{N}}$ by the Banach-Fixed-Point-Theorem. Since we have such a limit $\lim_{k \rightarrow \infty} V_k \longrightarrow V^*$, it has to satisfy the fixed point property $TV^* = V^*$. Hence it satisfies the Bellman equation and (9) verifies V^* as the value function of our problem.

Second, it remains to show that $V^0(\xi) \leq V^*(\xi) \forall \xi \in \mathcal{X}$ holds.

By assumption (6) and the monotonicity property (8), we infer inductively, that $V_k(\xi) \leq V_{k+1}(\xi)$ for all $\xi \in \mathcal{X}$. Thus, the sequence $\{V_k(\xi)\}_{k \in \mathbb{N}_0}$ is non-decreasing for any $\xi \in \mathcal{X}$. This proves the claim $V^0(\xi) \leq V^*(\xi) \forall \xi \in \mathcal{X}$.

d) Maximization Problem:

$$\begin{aligned}
&\max_{V(\xi_1), \dots, V(\xi_n)} \sum_{k=0}^n V(\xi_k) \\
&\text{subject to} \quad V(\xi_i) \leq TV(\xi_i) \quad \forall i=1, \dots, n.
\end{aligned} \tag{10}$$

To show: V^* is a solution to the maximization problem given above. Proof by contradiction. Suppose, there exists an admissible Function $\tilde{V} : \mathbb{R}^n \longrightarrow \mathbb{R}$ with

$$\tilde{V}(\xi_i) > V^*(\xi_i) \tag{11}$$

for at least one $i \in \{1, \dots, n\}$. Since \tilde{V} is admissible for the given max. problem, we know that $\tilde{V}(\xi_i) \leq T\tilde{V}(\xi_i) \quad \forall i \in \{1, \dots, n\}$. From exercise c) we know, that hence $\tilde{V}(\xi_i) \leq T^k \tilde{V}(\xi_i) \leq V^*(\xi_i)$ holds for all $i \in \{1, \dots, n\}$ and for all $k \in \mathbb{N}$. This clearly contradicts (11). Thus, V^* maximizes the objective of the given problem.

e) Transform (10) into

$$\begin{aligned} V^* = \arg \max_{V(\xi_1), \dots, V(\xi_n)} & \sum_{k=0}^n V(\xi_k) \\ \text{subject to} & V(\xi_i) \leq f_0(\xi_i, u) + \alpha V(f(\xi, u)) \\ & \forall u \in \mathcal{U}(\xi_i) \forall i=1, \dots, n. \end{aligned} \quad (12)$$

As shown in d), V^* maximizes the objective of problem (10). It remains to show, that the constraints of problems (10) and (14) are equivalent. The operator T is defined by the Bellman Equation, as written in (2). We have

$$\begin{aligned} V(\xi_i) \leq TV(\xi_i) &= \min_{u \in \mathcal{U}} \{f_0(\xi_i, u) + \alpha V_k(\xi_i)\} \\ &\leq f_0(\xi_i, \tilde{u}) + \alpha V_k(\xi_i) \end{aligned} \quad (13)$$

for any input signal $\tilde{u} \in \mathcal{U}$. This arises directly from the definition of the minimum. Thus, (10) and (14) denote the same problem.

It remains to formulate this as a linear program

$$\begin{aligned} V^* = \arg \max_V & c^T V \\ \text{subject to} & AV \leq b. \end{aligned} \quad (14)$$

With $V = (V(\xi_1), \dots, V(\xi_n))^T$, the objective $\max. \sum_{k=0}^n V(\xi_k)$ is the same as $\min. c^T V$ with

$$c = -(1, \dots, 1)^T \in \mathbb{R}^n.$$

Now we rewrite the constraints as a linear inequality $AV = b$.

For a state $\xi_i \in \mathcal{X}$, the inequality

$$\begin{aligned} V(\xi_i) &\leq f_0(\xi_i, u) + \alpha V(f(\xi_i, u)) \\ \iff V(\xi_i) - \alpha V(f(\xi_i, u)) &\leq f_0(\xi_i, u) \end{aligned} \quad (15)$$

has to hold for each possible input $u \in \mathcal{U}$, which leads to three scalar inequalities per state. Such an inequality can be represented with a line in the matrix A , with an 1 in the i -th column and $-\alpha$ in the $f(\xi_i, u)$ -th column. The corresponding entry of the vector b is $f_0(\xi_i, u)$. The resulting matrices A, b for the particular problem can be found in the solutions-sheet or can be generated by the matlab script.

One can determine the optimal input u^* in any state ξ_i by using the input that makes inequality (13) an equality. This minimizes the V -value of the next state that will be reached.

Problem 2

a) System

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \quad \text{with} \\ A &= \begin{pmatrix} 1 & 3 \\ -0.5 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad x_0 = \begin{pmatrix} 0.6 \\ -0.7 \end{pmatrix}. \end{aligned} \quad (16)$$

The origin $x^e = 0 \in \mathbb{R}^2$ is a equilibrium point of (16), since $x^e = Ax^e$. We determine the eigenvalues by computing roots of the characteristic polynomial of A .

$$\det(A - \lambda I) = \det \begin{pmatrix} 1 - \lambda & 3 \\ 3 & 1 - \lambda \end{pmatrix} = \lambda^2 - 4\lambda + 1.$$

hence, roots of the char. pol. are

$$\lambda_{1,2} = 1 \pm i \frac{\sqrt{6}}{2} \quad \text{with } i \text{ the imaginary unit.}$$

Since $|\lambda_i| > 1$ for $i = 1, 2$, x^e is an unstable equilibrium.

b) MPC optimization problem at time k with given state $x_k = \bar{x}$:

$$\begin{aligned}
& \min_{u_k, \dots, u_{k+N}} \sum_{j=k}^{k+N-1} f_0(x_j, u_j) + \phi(x_{k+N}) \\
& = \sum_{j=k}^{k+N-1} x_j^T Q x_j + u_j^T R u_j + x_{k+N}^T P x_{k+N} \\
& \text{subject to } x_{j+1} = A x_j + B u_j \text{ for all } j = k, \dots, k+N \quad (\text{dynamic constraints}) \\
& \quad u_j \leq 1, \text{ for all } j = k, \dots, k+N, \quad (\text{input constraints}) \\
& \quad -u_j \leq 1, \text{ for all } j = k, \dots, k+N, \\
& \quad x_{k+N}^T P x_{k+N} \leq c, \quad (\text{terminal constraint}) \\
& \quad x_k = \bar{x}, \quad (\text{initial state constraint})
\end{aligned} \tag{17}$$

with

$$Q = I_2, R = 1, N = 3, c = \frac{\lambda_{\min}(P)}{|K|^2}, K = (0.3, 1.4).$$

c) For the given controller $u = Kx$ for \mathcal{X}_f , we show

- For all states $x \in \mathcal{X}_f$, $u = Kx$ is a feasible input signal.
From linear algebra, we know

$$\lambda_{\min}(P) \|x\|^2 \leq x^T P x \iff \|x\|^2 \leq \frac{x^T P x}{\lambda_{\min}(P)}, \tag{18}$$

where $\|\cdot\|$ denotes the euclidean norm. We can use that to estimate

$$\|u\| = \|Kx\| = x^T K K^T x = \|K\|^2 x^T x = \|K\|^2 \|x\|^2 \tag{19}$$

$$\stackrel{(18)}{\leq} \|K\|^2 \frac{x^T P x}{\lambda_{\min}(P)} \leq \|K\|^2 \frac{c}{\lambda_{\min}(P)} = 1. \tag{20}$$

Hence, for $x \in \mathcal{X}_f$, satisfying the terminal constraint from (17), u is a feasible input.

- The condition $\phi(x_{j+1}) - \phi(x_j) \leq f_0(x_j, u_j)$ is satisfied for all states x_j .
Since we have linear system dynamics and quadratic costs as a quadratic expression in x_j .
Plugging in our particular system, yields

$$\phi(x_{j+1}) - \phi(x_j) \leq f_0(x_j, u_j) \tag{21}$$

$$\iff x_{j+1}^T P x_{j+1} - x_j^T P x_j \leq -x_j^T Q x_j - u_j^T u_j \tag{22}$$

$$\iff (A x_j + B u_j)^T P (A x_j + B u_j) - x_j^T P x_j \leq -x_j^T Q x_j - u_j^T R u_j. \tag{23}$$

With the state feedback $u_j = -Kx_j$, we get

$$((A - BK)x_j)^T P ((A - BK)x_j) - x_j^T P x_j \leq -x_j^T Q x_j - (-Kx_j)^T R (-Kx_j) \tag{24}$$

$$\iff x_j^T ((A - BK)^T P (A - BK) - P + Q + K^T R K) x_j \leq 0 \tag{25}$$

Hence with $M := (A - BK)^T P (A - BK) - P + Q + K^T R K$, we have

$$\phi(x_{j+1}) - \phi(x_j) \leq f_0(x_j, u_j) \iff x_j^T M x_j \leq 0 \iff M \preceq 0.$$

For our example, we have $M = \begin{pmatrix} -0.266 & 1.068 \\ 1.068 & -6.364 \end{pmatrix} \preceq 0$, thus the condition holds true.

- The terminal region \mathcal{X}_f is positively invariant
We show the invariance of the terminal region by proving $x_j^T P x_j < c$ for all times. Since f_0 is positive definite, we have with (21)

$$\phi(x_{j+1}) - \phi(x_j) \leq f_0(x_j, u_j) \leq 0.$$

Hence $x_j^T P x_j$ is decreasing over time and remaining smaller than c and hence $x_j \in \mathcal{X}_f$.

d) Write as quadratic optimization problem with quadratic constraints.

The optimization variable has to contain the input signals as well as the resulting states

$$z = (x_k \dots x_{k+N} u_k \dots u_{k+N-1})^T \in \mathbb{R}^{(N+1)n+Nm}, \quad (26)$$

where n is the state space dimension and u the dimension of the input signal. Impömentation of the objective function then is achieved by setting

$$H = \text{diag}(Q, \dots, Q, P, R, \dots, R) \in \mathbb{R}^{(N+1)n+Nm \times (N+1)n+Nm} \quad (27)$$

with $N+1$ blocks of Q and N blocks of R . The system dynamics are implemented by the equality constraints

$$x_{j+1} = Ax_j + Bu_j \iff Ax_j - x_{j+1} + Bu_j = 0 \quad (28)$$

for each time sample $j = k, \dots, k+N$. Further, we take the initial condition $x_k = \bar{x}$ into account. This yields the linear equation system $A_{\text{eq}} z = (A_{\text{eq}}^x \ A_{\text{eq}}^u) = b_{\text{eq}}$ with

$$A_{\text{eq}}^x = \begin{pmatrix} I & 0 & \dots & \dots & 0 & 0 \\ A & -I & 0 & \dots & \dots & 0 \\ 0 & A & -I & 0 & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \dots & 0 \\ 0 & 0 & \dots & A & -I & 0 \\ 0 & 0 & 0 & 0 & A & -I \end{pmatrix} \quad (29)$$

$$A_{\text{eq}}^u = \begin{pmatrix} 0 & 0 & \dots & 0 \\ B & 0 & \dots & 0 \\ 0 & B & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & \dots & 0 & B \end{pmatrix} \quad (30)$$

$$b_{\text{eq}} = (x(k) \dots 0)^T, \quad (31)$$

where the first lines of $A_{\text{eq}}^x, A_{\text{eq}}^u, b_{\text{eq}}$ represent the initial condition. The input constraints are represented within the inequality constraints of the problem. Therefore

$$A_{\text{in}} = \begin{pmatrix} 0 & I \\ 0 & -I \end{pmatrix} \quad (32)$$

$$b_{\text{in}} = (1 \ \dots \ 1)^T. \quad (33)$$

Lastly, the terminal constraints $x_N^T P x_N \leq c$ are implemented by $z^T T z$ with a blockdiagonalmatrix T that consists of zeros exept at the position that meets x_N , this block is assigned with P . By definiteness of P, Q, R , the matrices T, H in the quadratic terms are positive (semi-)definite, while all other constraints are affine. This means, that the optimization problem is convex.