Optimal Control WS20/21: Homework 2

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Problem 1

a) Formulating the problem as an discrete-time, infinite-horizon o. c. problem yields

$$\begin{aligned}
\min_{u_0,u_1,\dots} \quad & \sum_{k=0}^{\infty} \alpha^k f_0(x_k,u_k) = \sum_{k=0}^{\infty} 0.9^k f_0(x_k,u_k) \\
\text{subject to} \quad & x_{k+1} = f(x_k,u_k), \\
& x_k \in \mathcal{X} = \{\xi_1,\dots,\xi_8\}, \\
& u_k \in \mathcal{U} = \{0,1,2\}, \\
& x_0 = \xi_1,
\end{aligned} \tag{1}$$

where the dynamics $f : \mathcal{X} \times \mathcal{U} \longrightarrow \mathcal{X}$ are defined by the arrows in the graph.

The Value function iteration is defined by the Bellman operator

$$V_{k+1}(x) = TV_k(x) = \min_{u\in\mathcal{U}}\{f_0(x,u) + \alpha V_k(x)\} \quad \text{with} \quad \alpha = 0.9. \tag{2}$$

Evaluated for this particular problem, this yields

$$\begin{aligned}
V_{k+1}(\xi_1) &= \min_{u\in\mathcal{U}}\{1 + 0.9V_k(\xi_2), 1 + 0.9V_k(\xi_3)\}, \\
V_{k+1}(\xi_2) &= \min_{u\in\mathcal{U}}\{3 + 0.9V_k(\xi_7), 6 + 0.9V_k(\xi_5), 3 + 0.9V_k(\xi_4)\}, \\
V_{k+1}(\xi_3) &= \min_{u\in\mathcal{U}}\{1 + 0.9V_k(\xi_4), 2 + 0.9V_k(\xi_6), 3 + 0.9V_k(\xi_5)\}, \\
V_{k+1}(\xi_4) &= \min_{u\in\mathcal{U}}\{2 + 0.9V_k(\xi_7), 6 + 0.9V_k(\xi_8), 3 + 0.9V_k(\xi_6)\}, \\
V_{k+1}(\xi_5) &= \min_{u\in\mathcal{U}}\{0 + 0.9V_k(\xi_4), 0 + 0.9V_k(\xi_4), 1 + 0.9V_k(\xi_6)\}, \\
V_{k+1}(\xi_6) &= \min_{u\in\mathcal{U}}\{5 + 0.9V_k(\xi_1), 1 + 0.9V_k(\xi_7), 1 + 0.9V_k(\xi_8)\}, \\
V_{k+1}(\xi_7) &= \min_{u\in\mathcal{U}}\{2 + 0.9V_k(\xi_8), 2 + 0.9V_k(\xi_8), 2 + 0.9V_k(\xi_8)\}, \\
V_{k+1}(\xi_8) &= \min_{u\in\mathcal{U}}\{0 + 0.9V_k(\xi_8), 0 + 0.9V_k(\xi_8), 0 + 0.9V_k(\xi_8)\},
\end{aligned} \tag{3}$$

starting with an arbitrary inital value function $V^0 : \mathcal{X} \longrightarrow \mathbb{R}$.

b) Value function $V : \mathbb{R}^8 \longrightarrow \mathbb{R}$, obtained after 1000 value function iterations:

$$\begin{aligned}
V(\xi_1) &= 3.61 \\
V(\xi_2) &= 4.80 \\
V(\xi_3) &= 2.90 \\
V(\xi_4) &= 3.80 \\
V(\xi_5) &= 1.90 \\
V(\xi_6) &= 1.00 \\
V(\xi_7) &= 2.00 \\
V(\xi_8) &= 0.00.
\end{aligned} \tag{4}$$

The values in (4) induce the following state feedback:

$$\begin{aligned} k(\xi_1) &= u_2 \\ k(\xi_2) &= u_1 \\ k(\xi_3) &= u_1 \\ k(\xi_4) &= u_1 \\ k(\xi_5) &= u_2 \\ k(\xi_6) &= u_2 \\ k(\xi_7) &= u_0 \\ k(\xi_8) &= u_0. \end{aligned} \tag{5}$$

For the initial state $x_0 = \xi_1$, the state-feedback given in (5) yields the optimal input sequence

$$u^* = \begin{pmatrix} 2 & 1 & 2 \end{pmatrix},$$

that steers the state to the terminal state ξ_8 via the route $\xi_1, \xi_3, \xi_6, \xi_8$.

c) We suppose

$$V^0(\xi) \leq TV^0(\xi) \quad \forall_{\xi \in \mathcal{X}}, \tag{6}$$

for an initial function V^0 for a value function iteration with the Bellman Operator, defined in (2). From the lectures, we have the following properties.

$$\|TV^1 - TV^2\|_\infty \leq \alpha \, \|V^1 - V^2\|_\infty \qquad \text{(Contraction)} \tag{7}$$

$$V^1(\xi) \leq V^2(\xi) \quad \forall_{\xi \in \mathcal{X}} \Longrightarrow TV^1(\xi) \leq TV^2(\xi) \quad \forall_{\xi \in \mathcal{X}} \qquad \text{(Monotonicity)} \tag{8}$$

V^* solves the Bellman Equation V^* is the Value function, since we have discounted costs. (9)

First we show, that the iteration converges to the value function of the problem. Since (7) holds and $\alpha \in (0,1)$,we deduce the convergence of $\{V_k\}_{k \in \mathbb{N}}$ by the Banach-Fixed-Point-Theorem. Since we have such a limit $\lim\limits_{k \to \infty} V_k \longrightarrow V^*$, it has to satisfy the fixed point property $TV^* = V^*$. Hence it satisfies the Bellman equation and (9) verifies V^* as the value function of our problem.

Second, it remains to show that $V^0(\xi) \leq V^*(\xi) \forall_{\xi \in \mathcal{X}}$ holds.
By assumption (6) and the monotonicity property (8), we infer inductively, that $V_k(\xi) \leq V_{k+1}(\xi)$ for all $\xi \in \mathcal{X}$. Thus, the sequence $\{V_k(\xi)\}_{k \in \mathbb{N}_0}$ is non-decreasing for any $\xi \in \mathcal{X}$. This proves the claim $V^0(\xi) \leq V^*(\xi) \forall_{\xi \in \mathcal{X}}$.

d) Maximization Problem:

$$\begin{aligned} \max_{V(\xi_1), \dots, V(\xi_n)} \quad & \sum_{k=0}^{n} V(\xi_i) \\ \text{subject to} \quad & V(\xi_i) \leq TV(\xi_i) \quad \forall_{i=1,\dots,n}. \end{aligned} \tag{10}$$

To show: V^* is a solution to the maximization problem given above. Proof by contradiction. Suppose, there exists an admissible Function $\tilde{V} : \mathbb{R}^n \longrightarrow \mathbb{R}$ with

$$\tilde{V}(\xi_i) > V^*(\xi_i) \tag{11}$$

for at least one $i \in \{1, \dots, n\}$. Since $\tilde{V}$ is admissible for the given max. problem, we know that $\tilde{V}(\xi_i) \leq T\tilde{V}(\xi_i) \quad \forall_{i \in \{1, \dots, n\}}$. From excercise c) we know, that hence $\tilde{V}(\xi_i) \leq T^k \tilde{V}(\xi_i) \leq V^*(\xi_i)$ holds for all $i \in \{1, \dots, n\}$ and for all $k \in \mathbb{N}$. This clearly contradicts (11). Thus, V^* maximizes the objective of the given problem.

e) Transform (10) into

$$V^* = \underset{V(\xi_1),\ldots,V(\xi_n)}{\text{arg max.}} \quad \sum_{k=0}^{n} V(\xi_i)$$
$$\text{subject to} \quad V(\xi_i) \leq f_0(\xi_i, u) + \alpha V(f(\xi, u)) \quad \forall_{u\in\mathcal{U}(\xi_i)} \forall_{i=1,\ldots,n}. \tag{12}$$

As shown in d), V^* maximizes the objective of problem (10). It remains to show, that the constraints of problems (10) and (14) are equivalent. The operator T is defined by the Bellman Equation, as written in (2). We have

$$\begin{aligned} V(\xi_i) \leq TV(\xi_i) &= \min_{u\in\mathcal{U}}\{f_0(\xi_i, u) + \alpha V_k(\xi_i)\} \\ &\leq f_0(\xi_i, \tilde{u}) + \alpha V_k(\xi_i) \end{aligned} \tag{13}$$

for any input signal $\tilde{u} \in \mathcal{U}$. This arises directly from the definition of the minimum. Thus, (10) and (14) denote the same problem.

It remains to formulate this as a linear program

$$\begin{aligned} V^* = \underset{V}{\text{arg max.}} \quad & c^T V \\ \text{subject to} \quad & AV \leq b. \end{aligned} \tag{14}$$

With $V = (V(\xi_1), \ldots, V\xi_n)^T$, the objective max. $\sum_{k=0}^{n} V(\xi_i)$ is the same as min. $c^T V$ with

$$c = -(1, \ldots, 1)^T \in \mathbb{R}^n.$$

Now we rewrite the constraints as a linear inequality $AV = b$.

For a state $\xi_i \in \mathcal{X}$, the inequality

$$\begin{aligned} & V(\xi_i) \leq f_0(\xi, u) + \alpha V(f(\xi_i, u)) \\ \iff & V(\xi_i) - \alpha V(f(\xi_i, u)) \leq f_0(\xi_i, u) \end{aligned} \tag{15}$$

has to hold for each possible input $u \in \mathcal{U}$, which leads to three scalar inequalities per state. Such an inequality can be represented with a line in the matrix A, with an 1 in the i-th column and $-\alpha$ in the $f(\xi_i, u)$-th column. The corresponding entry of the vector b is $f_0(\xi_i, u)$. The resutling matrices A, b for the particular problem can be found in the solutions-sheet or can be generated by the matlab script.

One can determine the optimal input u^* in any state ξ_i by using the input that makes inequality (13) an equality. This minimizes the V-value of the next state that will be reached.

Problem 2

a) System

$$\begin{aligned} x_{k+1} = Ax_k + Bu_k \qquad & \text{with} \\ A = \begin{pmatrix} 1 & 3 \\ -0.5 & 1 \end{pmatrix}, \ B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ & x_0 = \begin{pmatrix} 0.6 \\ -0.7 \end{pmatrix}. \end{aligned} \tag{16}$$

The origin $x^{\mathrm{e}} = 0 \in \mathbb{R}^2$ is a equlibrium point of (16), since $x^{\mathrm{e}} = Ax^{\mathrm{e}}$. We determine the eigenvalues by computing roots of the characteristic polynomial of A.

$$\det(A - \lambda I) = det \begin{pmatrix} 1-\lambda & -0.5 \\ 3 & 1-\lambda \end{pmatrix} = \lambda^2 - 4\lambda + 1.$$

hence, roots of the char. pol. are

$$\lambda_{1,2} = 1 \pm i\frac{\sqrt{6}}{2} \qquad \text{with } i \text{ the imaginary unit.}$$

Since $|\lambda_i| > 1$ for $i = 1, 2$, x^{e} is an unstable equilibrium.

b) MPC optimization problem at time k with given state $x_k = \bar{x}$:

$$
\begin{aligned}
\min_{u_k,\ldots,u_{k+N}} \quad & \sum_{j=k}^{k+N-1} f_0(x_j, u_j) + \phi(x_{k+N}) \\
= & \sum_{j=k}^{k+N-1} = x_j^T Q x_j + u_j^T R u_j + x_{k+N}^T P x_{k+N} \\
\text{subject to} \quad & x_{j+1} = Ax_j + Bu_j \text{ for all } j = k, \ldots, k+N \qquad \text{(dynamic constraints)} \\
& u_j \leq 1, \text{ for all } j = k, \ldots, k+N, \qquad \text{(input constraints)} \\
& -u_j \leq 1, \text{ for all } j = k, \ldots, k+N, \\
& x_{k+N}^T P x_{k+N} \leq c, \qquad \text{(terminal constraint)} \\
& x_k = \bar{x}, \qquad \text{(initial state constraint)}
\end{aligned}
\tag{17}
$$

with

$$Q = I_2, R = 1, N = 3, c = \frac{\lambda_{\min}(P)}{|K|^2}, K = (0.3, 1.4).$$

c) For the given controller $u = Kx$ for $\mathcal{X}_f$, we show

- For all states $x \in \mathcal{X}_f$, $u = Kx$ is a feasible input signal.
  From linear algebra, we know

$$\lambda_{\min}(P)\ \|x\|^2 \leq x^T P x \iff \|x\|^2 \leq \frac{x^T P x}{\lambda_{\min}(P)}, \tag{18}$$

  where $\|\cdot\|$ denotes the euclidean norm. We can use that to estimate

$$\|u\| = \|Kx\| = x^T K K^T x = \|K\|^2\ x^T x = \|K\|^2 \|x\|^2 \tag{19}$$

$$\overset{(18)}{\leq} \|K\|^2\ \frac{x^T P x}{\lambda_{\min}(P)} \leq \|K\|^2\ \frac{c}{\lambda_{\min}(P)} = 1. \tag{20}$$

  Hence, for $x \in \mathcal{X}_f$, satisfying the terminal constraint from (17), u is a feasible input.

- The condition $\phi(x_{j+1}) - \phi(x_j) \leq f_0(x_j, u_j)$ is satisfied for all states x_j.
  Since we have linear system dynamics and quadratic costs as a quadratic expression in x_j. Plugging in our particular system, yields

$$
\begin{aligned}
& \phi(x_{j+1}) - \phi(x_j) \leq f_0(x_j, u_j) && (21) \\
\iff \quad & x_{j+1}^T P x_{j+1} - x_j^T P x_j \leq -x_j^T Q x_j - u_j^T u_j && (22) \\
\iff \quad & (Ax_j + Bu_j)^T P (Ax_j + Bu_j) - x_j^T P x_j \leq -x_j^T Q x_j - u_j^T R u_j. && (23)
\end{aligned}
$$

  With the state feedback $u_j = -Kx_j$, we get

$$
\begin{aligned}
& ((A - BK)x_j)^T P ((A - BK)x_j) - x_j^T P x_j \leq -x_j^T Q x_j - (-Kx_j)^T R (-Kx_j) && (24) \\
\iff & x_j^T ((A - BK)^T P (A - BK) - P + Q + K^T R K) x_j \leq 0 && (25)
\end{aligned}
$$

  Hence with $M := (A - BK)^T P (A - BK) - P + Q + K^T R K$, we have

$$\phi(x_{j+1}) - \phi(x_j) \leq f_0(x_j, u_j) \iff x_j^T M x_j \leq 0 \iff M \preccurlyeq 0.$$

  For our example, we have $M = \begin{pmatrix} -0.266 & 1.068 \\ 1.068 & -6.364 \end{pmatrix} \preccurlyeq 0$, thus the condition holds true.

- The terminal region $\mathcal{X}_f$ is positively invariant
  We show the invariance of the terminal region by proving $x_j^T P x_j < c$ for all times. Since f_0 is positive definite, we have with (21)

$$\phi(x_{j+1}) - \phi(x_j) \leq f_0(x_j, u_j) \leq 0.$$

  Hence $x_j^T P x_j$ is decreasing over time and remaining smaller than c and hence $x_j \in \mathcal{X}_f$.

d) Write as quadratic optimization problem with quadratic constraints.
The optimization variable has to contain the input signals as well as the resulting states

$$z = (x_k \ \dots \ x_{k+N} \ u_k \ \dots \ u_{k+N-1})^T. \in \mathbb{R}^{(N+1)n+Nm}, \tag{26}$$

where n is the state space dimension and u the dimension of the input signal. Impölementation of the objective function then is achieved by setting

$$H = \mathrm{diag}(Q, \dots, Q, P, R, \dots, R) \in \mathbb{R}^{(N+1)n+Nm \times (N+1)n+Nm} \tag{27}$$

with $N+1$ blocks of Q and N blocks of R. The system dynamics are implemented by the equality constraints

$$x_{j+1} = Ax_j + Bu_j \iff Ax_j - x_{j+1} + Bu_j = 0 \tag{28}$$

for each time sample $j = k, \dots k+N$. Further, we take the initial condition $x_k = \bar{x}$ into account. This yields the linear equation system $A_{\text{eq}} z = \begin{pmatrix} A_{\text{eq}}^x & A_{\text{eq}}^u \end{pmatrix} = b_{\text{eq}}$ with

$$A_{\text{eq}}^x = \begin{pmatrix} A & -I & 0 & \cdots & \cdots & 0 \\ 0 & A & -I & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \cdots & 0 \\ 0 & 0 & \cdots & A & -I & 0 \\ 0 & 0 & 0 & 0 & A & -I \\ I & 0 & \cdots & \cdots & 0 & 0 \end{pmatrix} \tag{29}$$

$$A_{\text{eq}}^u = \begin{pmatrix} B & 0 & \cdots & 0 \\ 0 & B & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & \cdots & 0 & B \\ 0 & 0 & \cdots & 0 \end{pmatrix} \tag{30}$$

$$b_{\text{eq}} = \begin{pmatrix} 0 \cdots x(k) \end{pmatrix}^T, \tag{31}$$

where the last lines of $A_{\text{eq}}^x, A_{\text{eq}}^u, b_{\text{eq}}$ represent the initial condition. The input constraints are represented within the inequality constraints of the problem. Therefore

$$A_{\text{in}} = \begin{pmatrix} 0 & I \\ 0 & -I \end{pmatrix} \tag{32}$$

$$b_{\text{in}} = \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix}^T. \tag{33}$$

Lastly, the terminal constraints $x_N^T P x_N \leq c$ are implemented by $z^T T z$ with a blockdiagonalmatrix T that consists of zeros exept at the position that meets x_N, this block is assigned with P.
By definiteness of P, Q, R, the matrices T, H in the quadratic terms are positive (semi-)definite, while all other constraints are affine. This means, that the optimization problem is convex.