Optimal Control WS20/21: Homework 2

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Problem 1

a) Formulating the problem as an discrete-time, infinite-horizon o. c. problem yields

$$
\begin{aligned}
\min_{u_0, u_1, \dots} \quad & \sum_{k=0}^{\infty} \alpha^k f_0(x_k, u_k) = \sum_{k=0}^{\infty} 0.9^k f_0(x_k, u_k) \\
\text{subject to} \quad & x_{k+1} = f(x_k, u_k), \\
& x_k \in \mathcal{X} = \{\xi_1, \dots, \xi_8\}, \\
& u_k \in \mathcal{U} = \{0, 1, 2\}, \\
& x_0 = \xi_1,
\end{aligned}
\tag{1}
$$

where the dynamics $f : \mathcal{X} \times \mathcal{U} \longrightarrow \mathcal{X}$ are defined by the arrows in the graph.

The Value function iteration is defined by the Bellman operator

$$
V_{k+1}(x) = TV_k(x) = \min_{u \in \mathcal{U}} \{f_0(x, u) + \alpha V_k(x)\} \quad \text{with} \quad \alpha = 0.9.
\tag{2}
$$

Evaluated for this particular problem, this yields

$$
\begin{aligned}
V_{k+1}(\xi_1) &= \min_{u \in \mathcal{U}} \{1 + 0.9 V_k(\xi_2), 1 + 0.9 V_k(\xi_3)\}, \\
V_{k+1}(\xi_2) &= \min_{u \in \mathcal{U}} \{3 + 0.9 V_k(\xi_7), 6 + 0.9 V_k(\xi_5), 3 + 0.9 V_k(\xi_4)\}, \\
V_{k+1}(\xi_3) &= \min_{u \in \mathcal{U}} \{1 + 0.9 V_k(\xi_4), 2 + 0.9 V_k(\xi_6), 3 + 0.9 V_k(\xi_5)\}, \\
V_{k+1}(\xi_4) &= \min_{u \in \mathcal{U}} \{2 + 0.9 V_k(\xi_7), 6 + 0.9 V_k(\xi_8), 3 + 0.9 V_k(\xi_6)\}, \\
V_{k+1}(\xi_5) &= \min_{u \in \mathcal{U}} \{0 + 0.9 V_k(\xi_4), 0 + 0.9 V_k(\xi_4), 1 + 0.9 V_k(\xi_6)\}, \\
V_{k+1}(\xi_6) &= \min_{u \in \mathcal{U}} \{5 + 0.9 V_k(\xi_1), 1 + 0.9 V_k(\xi_7), 1 + 0.9 V_k(\xi_8)\}, \\
V_{k+1}(\xi_7) &= \min_{u \in \mathcal{U}} \{2 + 0.9 V_k(\xi_8), 2 + 0.9 V_k(\xi_8), 2 + 0.9 V_k(\xi_8)\}, \\
V_{k+1}(\xi_8) &= \min_{u \in \mathcal{U}} \{0 + 0.9 V_k(\xi_8), 0 + 0.9 V_k(\xi_8), 0 + 0.9 V_k(\xi_8)\},
\end{aligned}
\tag{3}
$$

starting with an arbitrary inital value function $V^0 : \mathcal{X} \longrightarrow \mathbb{R}$.

b) Value function $V : \mathbb{R}^8 \longrightarrow \mathbb{R}$, obtained after 1000 value function iterations:

$$
\begin{aligned}
V(\xi_1) &= 3.61 \\
V(\xi_2) &= 4.80 \\
V(\xi_3) &= 2.90 \\
V(\xi_4) &= 3.80 \\
V(\xi_5) &= 1.90 \\
V(\xi_6) &= 1.00 \\
V(\xi_7) &= 2.00 \\
V(\xi_8) &= 0.00.
\end{aligned}
\tag{4}
$$

The values in (4) induce the following state feedback:

$$
\begin{aligned}
k(\xi_1) &= u_2 \\
k(\xi_2) &= u_1 \\
k(\xi_3) &= u_1 \\
k(\xi_4) &= u_1 \\
k(\xi_5) &= u_2 \\
k(\xi_6) &= u_2 \\
k(\xi_7) &= u_0 \\
k(\xi_8) &= u_0.
\end{aligned}
\tag{5}
$$

For the initial state $x_0 = \xi_1$, the state-feedback given in (5) yields the optimal input sequence

$$
u^* = \begin{pmatrix} 2 & 1 & 2 \end{pmatrix},
$$

that steers the state to the terminal state ξ_8 via the route $\xi_1, \xi_3, \xi_6, \xi_8$.

c) We suppose

$$
V^0(\xi) \leq TV^0(\xi) \quad \forall_{\xi \in \mathcal{X}},
\tag{6}
$$

for an initial function V^0 for a value function iteration with the Bellman Operator, defined in (2). From the lectures, we have the following properties.

$$
\|TV^1 - TV^2\|_\infty \leq \alpha \|V^1 - V^2\|_\infty \tag{Contraction}
$$
$$
\tag{7}
$$

$$
V^1(\xi) \leq V^2(\xi) \quad \forall_{\xi \in \mathcal{X}} \implies TV^1(\xi) \leq TV^2(\xi) \quad \forall_{\xi \in \mathcal{X}} \tag{Monotonicity}
$$
$$
\tag{8}
$$

V^* solves the Bellman Equation V^* is the Value function, since we have discounted costs. (9)

First we show, that the iteration converges to the value function of the problem. Since (7) holds and $\alpha \in (0,1)$, we deduce the convergence of $\{V_k\}_{k \in \mathbb{N}}$ by the Banach-Fixed-Point-Theorem. Since we have such a limit $\lim_{k \to \infty} V_k \longrightarrow V^*$, it has to satisfy the fixed point property $TV^* = V^*$. Hence it satisfies the Bellman equation and (9) verifies V^* as the value function of our problem.

Second, it remains to show that $V^0(\xi) \leq V^*(\xi) \forall_{\xi \in \mathcal{X}}$ holds.
By assumption (6) and the monotonicity property (8), we infer inductively, that $V_k(\xi) \leq V_{k+1}(\xi)$ for all $\xi \in \mathcal{X}$. Thus, the sequence $\{V_k(\xi)\}_{k \in \mathbb{N}_0}$ is non-decreasing for any $\xi \in \mathcal{X}$. This proves the claim $V^0(\xi) \leq V^*(\xi) \forall_{\xi \in \mathcal{X}}$.

d) Maximization Problem:

$$
\begin{aligned}
\max_{V(\xi_1),\ldots,V(\xi_n)} \quad & \sum_{k=0}^{n} V(\xi_i) \\
\text{subject to} \quad & V(\xi_i) \leq TV(\xi_i) \quad \forall_{i=1,\ldots,n}.
\end{aligned}
\tag{10}
$$

To show: V^* is a solution to the maximization problem given above. Proof by contradiction. Suppose, there exists an admissible Function $\tilde{V} : \mathbb{R}^n \longrightarrow \mathbb{R}$ with

$$
\tilde{V}(\xi_i) > V^*(\xi_i)
\tag{11}
$$

for at least one $i \in \{1, \ldots, n\}$. Since $\tilde{V}$ is admissible for the given max. problem, we know that $\tilde{V}(\xi_i) \leq T\tilde{V}(\xi_i) \quad \forall_{i \in \{1,\ldots,n\}}$. From excercise c) we know, that hence $\tilde{V}(\xi_i) \leq T^k\tilde{V}(\xi_i) \leq V^*(\xi_i)$ holds for all $i \in \{1, \ldots, n\}$ and for all $k \in \mathbb{N}$. This clearly contradicts (11). Thus, V^* maximizes the objective of the given problem.

e) Transform (10) into

$$V^* = \underset{V(\xi_1),\dots,V(\xi_n)}{\arg \max.} \sum_{k=0}^{n} V(\xi_i)$$

$$\text{subject to} \quad V(\xi_i) \le f_0(\xi_i, u) + \alpha V(f(\xi, u)) \tag{12}$$

$$\forall_{u \in \mathcal{U}(\xi_i)} \forall_{i=1,\dots,n}.$$

As shown in d), V^* maximizes the objective of problem (10). It remains to show, that the constraints of problems (10) and (14) are equivalent. The operator T is defined by the Bellman Equation, as written in (2). We have

$$V(\xi_i) \le TV(\xi_i) = \min_{u \in \mathcal{U}} \{ f_0(\xi_i, u) + \alpha V_k(\xi_i) \}$$

$$\le f_0(\xi_i, \tilde{u}) + \alpha V_k(\xi_i) \tag{13}$$

for any input signal $\tilde{u} \in \mathcal{U}$. This arises directly from the definition of the minimum. Thus, (10) and (14) denote the same problem.

It remains to formulate this as a linear program

$$V^* = \underset{V}{\arg \max.} \quad c^T V$$

$$\text{subject to} \quad AV \le b. \tag{14}$$

With $V = (V(\xi_1),\dots,V\xi_n)^T$, the objective max. $\sum_{k=0}^{n} V(\xi_i)$ is the same as min. $c^T V$ with

$$c = -(1,\dots,1)^T \in \mathbb{R}^n.$$

Now we rewrite the constraints as a linear inequality $AV = b$.

For a state $\xi_i \in \mathcal{X}$, the inequality

$$V(\xi_i) \le f_0(\xi, u) + \alpha V(f(\xi_i, u))$$

$$\iff V(\xi_i) - \alpha V(f(\xi_i, u)) \le f_0(\xi_i, u) \tag{15}$$

has to hold for each possible input $u \in \mathcal{U}$, which leads to three scalar inequalities per state. Such an inequality can be represented with a line in the matrix A, with an 1 in the i-th column and $-\alpha$ in the $f(\xi_i, u)$-th column. The corresponding entry of the vector b is $f_0(\xi_i, u)$. The resutling matrices A, b for the particular problem can be found in the solutions-sheet or can be generated by the matlab script.

One can determine the optimal input u^* in any state ξ_i by using the input that makes inequality (13) an equality. This minimizes the V-value of the next state that will be reached.

Problem 2

a) System

$$x_{k+1} = Ax_k + Bu_k \quad \text{with}$$

$$A = \begin{pmatrix} 1 & 3 \\ -0.5 & 1 \end{pmatrix}, \ B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ x_0 = \begin{pmatrix} 0.6 \\ -0.7 \end{pmatrix}. \tag{16}$$

The origin $x^e = 0 \in \mathbb{R}^2$ is a equlibrium point of (16), since $x^e = Ax^e$. We determine the eigenvalues by computing roots of the characteristic polynomial of A.

$$\det(A - \lambda I) = \det \begin{pmatrix} 1 - \lambda & -0.5 \\ 3 & 1 - \lambda \end{pmatrix} = \lambda^2 - 4\lambda + 1.$$

hence, roots of the char. pol. are

$$\lambda_{1,2} = 1 \pm i \frac{\sqrt{6}}{2} \quad \text{with } i \text{ the imaginary unit.}$$

Since $|\lambda_i| > 1$ for $i = 1, 2$, x^e is an unstable equilibrium.

b) MPC optimization problem at time k with given state $x_k = \bar{x}$:

$$\min_{u_k, \ldots, u_{k+N}} \sum_{j=k}^{k+N-1} f_0(x_j, u_j) + \phi(x_{k+N})$$

$$= \sum_{j=k}^{k+N-1} = x_j^T Q x_j + u_j^T R u_j + x_{k+N}^T P x_{k+N}$$

subject to $x_{j+1} = A x_j + B u_j$ for all $j = k, \ldots, k+N$ (dynamic constraints)

$\qquad\qquad\quad u_j \leq 1,$ for all $j = k, \ldots, k+N$ (input constraints)

$\qquad\qquad\quad u_j \leq 1,$ for all $j = k, \ldots, k+N$

$\qquad\qquad\quad x_{k+N}^T P x_{k+N} \leq c,$ (terminal constraint)

$\qquad\qquad\quad x_k = \bar{x},$ (initial state constraint)

(17)

with

$$Q = I_2, R = 1, N = 3, c = \frac{\lambda_{\min}(P)}{|K|^2}, K = (0.3, 1.4).$$