Lecture 6: Recursion

Marvin Zhang
06/28/2016

Announcements

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- Quiz 2 is *this Thursday*
  - Topics covered may include environment diagrams and higher–order functions

Roadmap

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Functions

Data

Mutability

Objects

Interpretation

Paradigms

Applications

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  - To study this idea through:
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    - recursion (today and tomorrow!)
    - orders of growth

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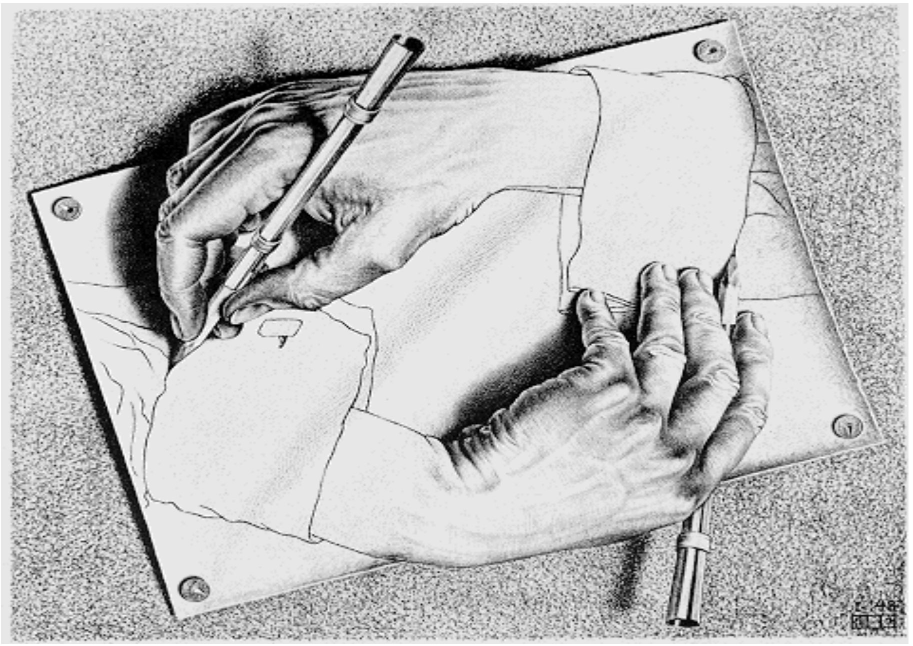
- How is this possible? We’ll see some examples next.



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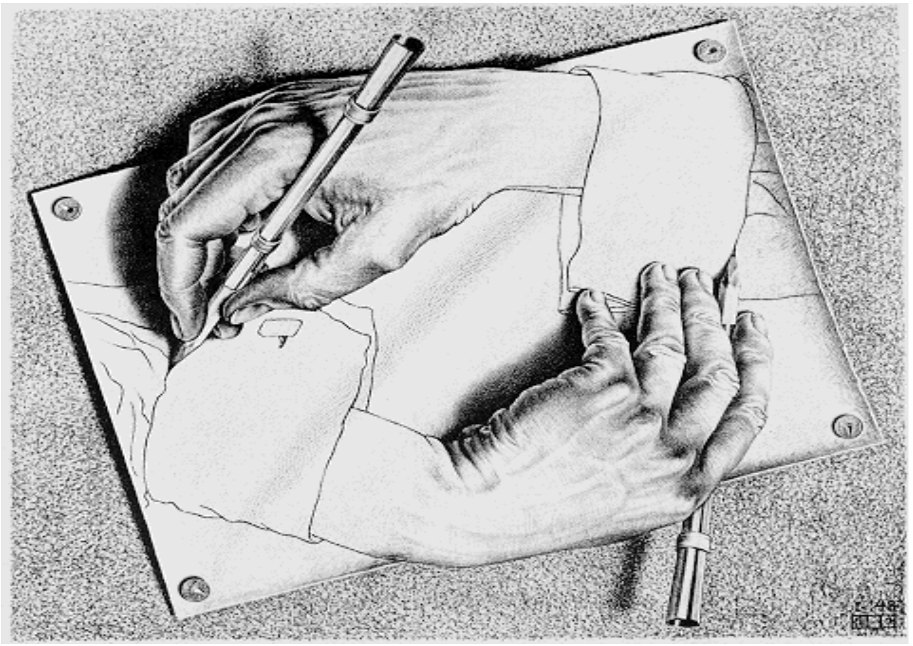

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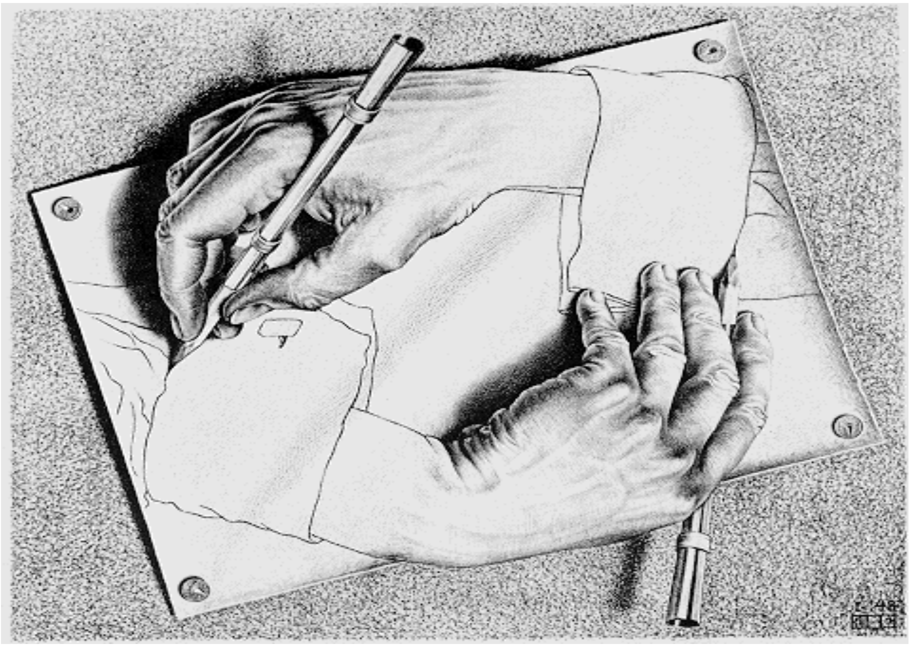

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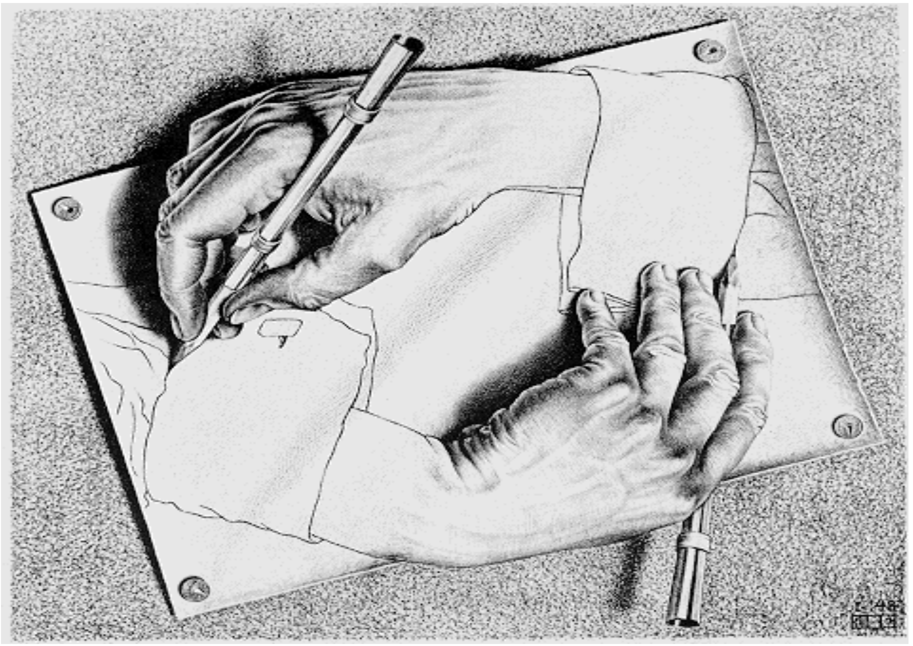

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Recursion (demo)

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Anatomy of a Recursive Function

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def factorial(n):
    """Return the factorial of n."""
    if n == 0:
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Verifying Correctness

The easy way, and the right way

Recursion in Environment Diagrams

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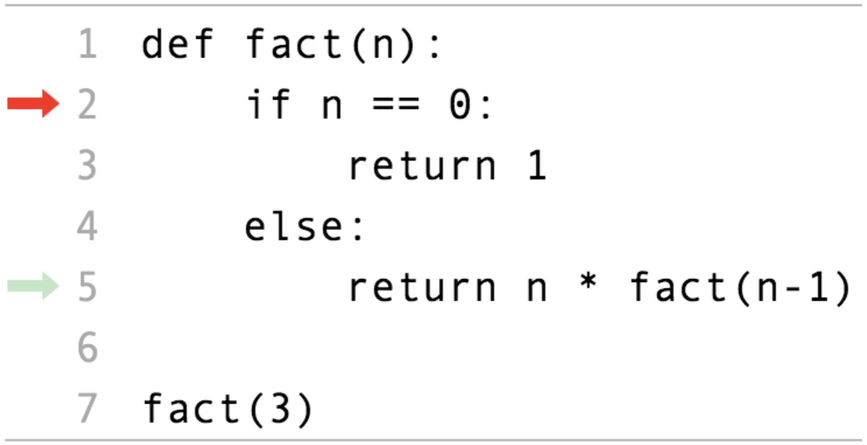

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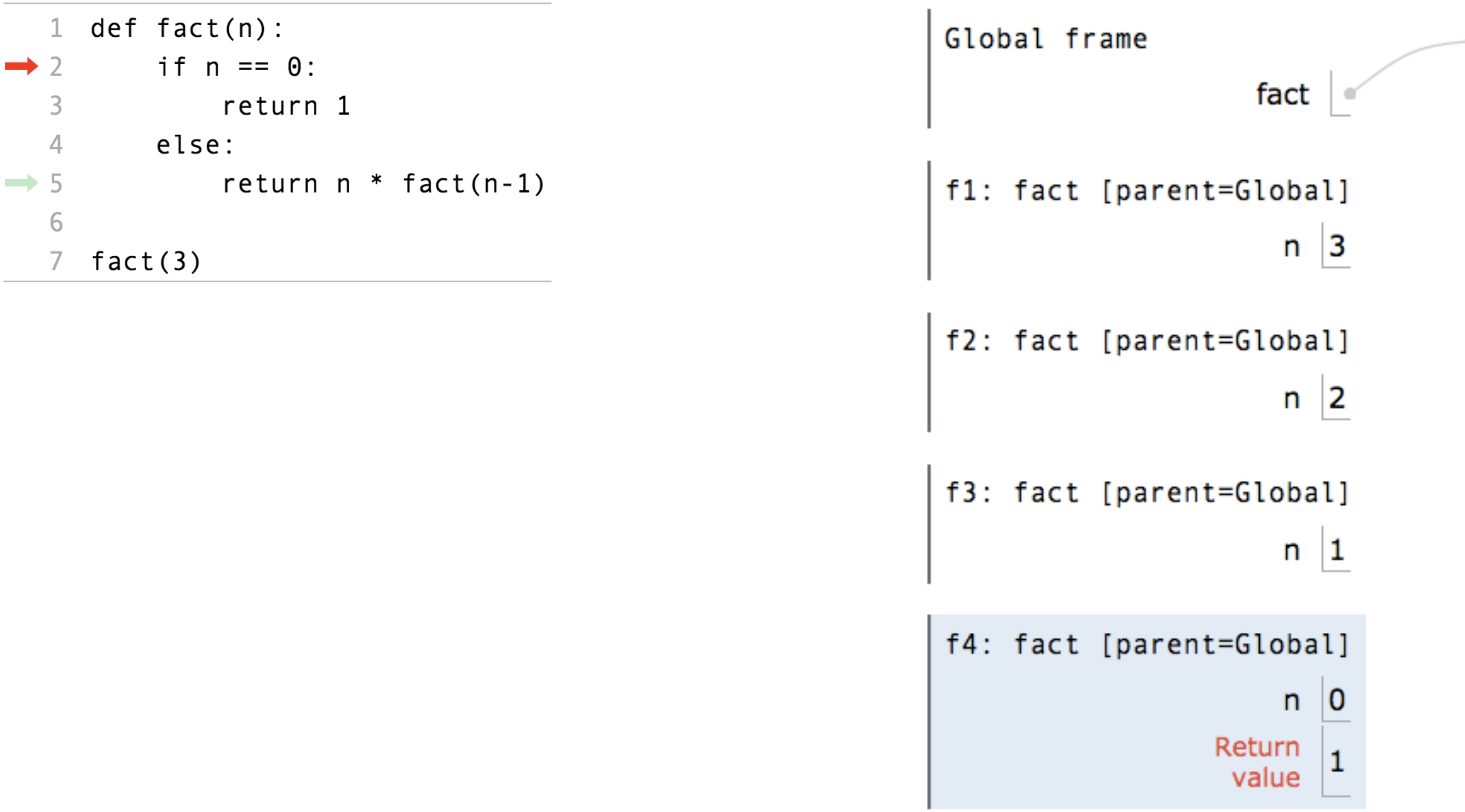

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```
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                  fact  ●

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                     n  3

f2: fact [parent=Global]
                     n  2

f3: fact [parent=Global]
                     n  1

f4: fact [parent=Global]
                     n  0
          Return value  1
```

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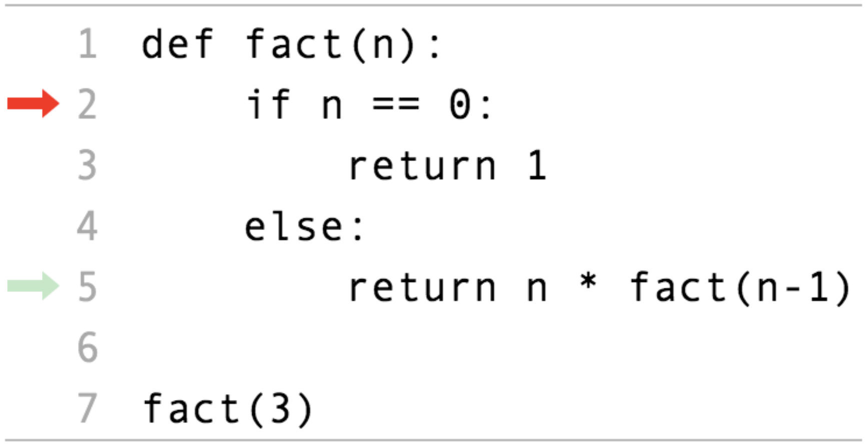

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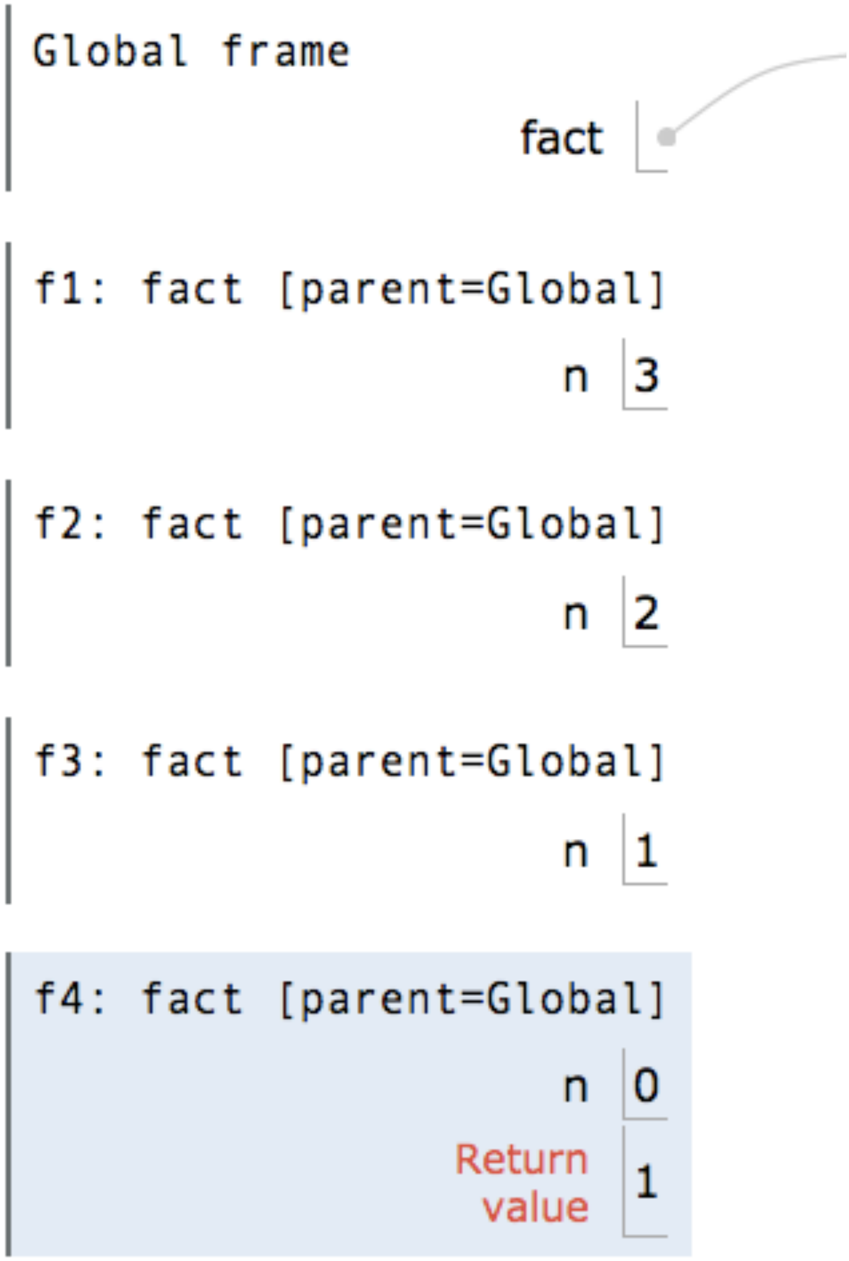

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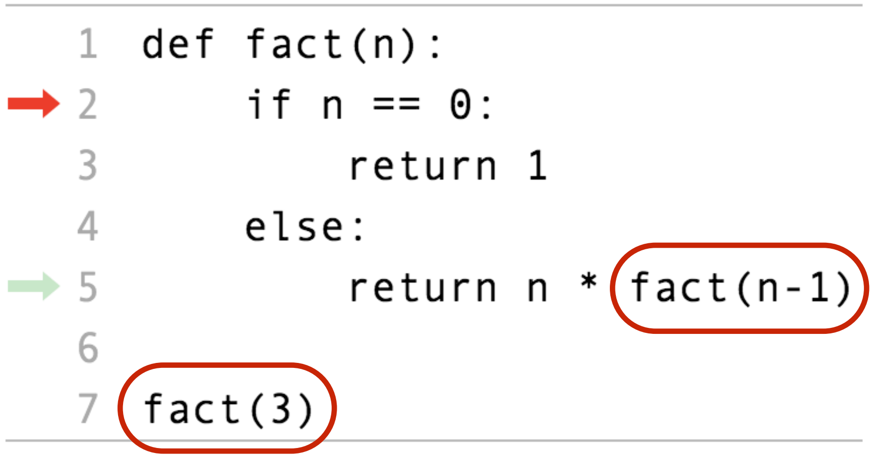

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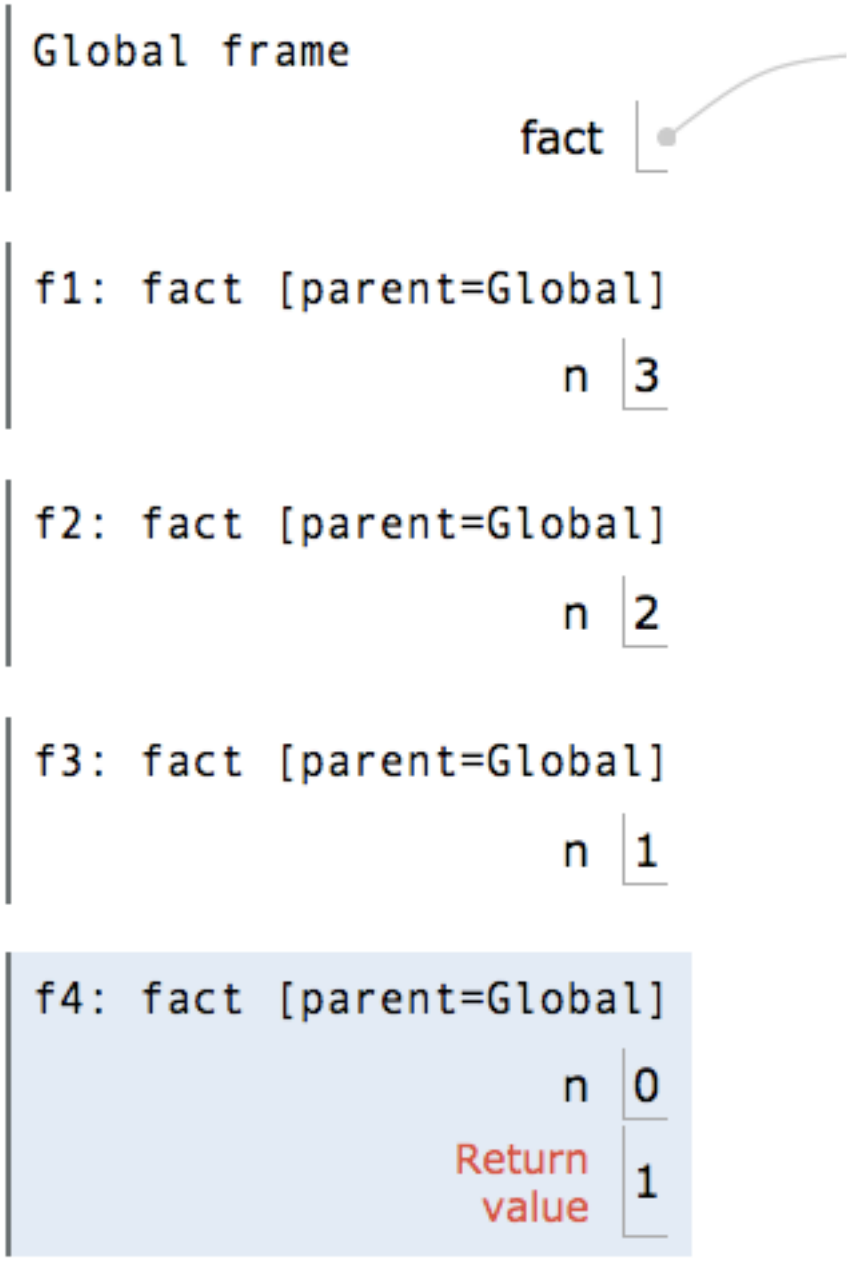

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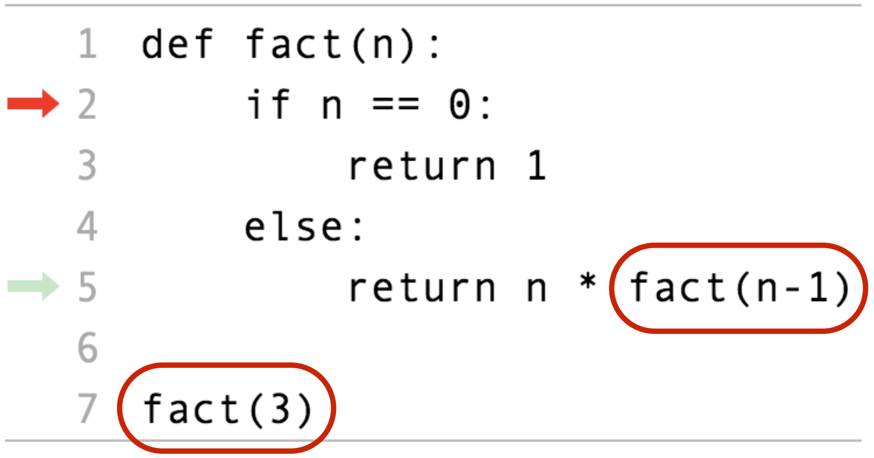

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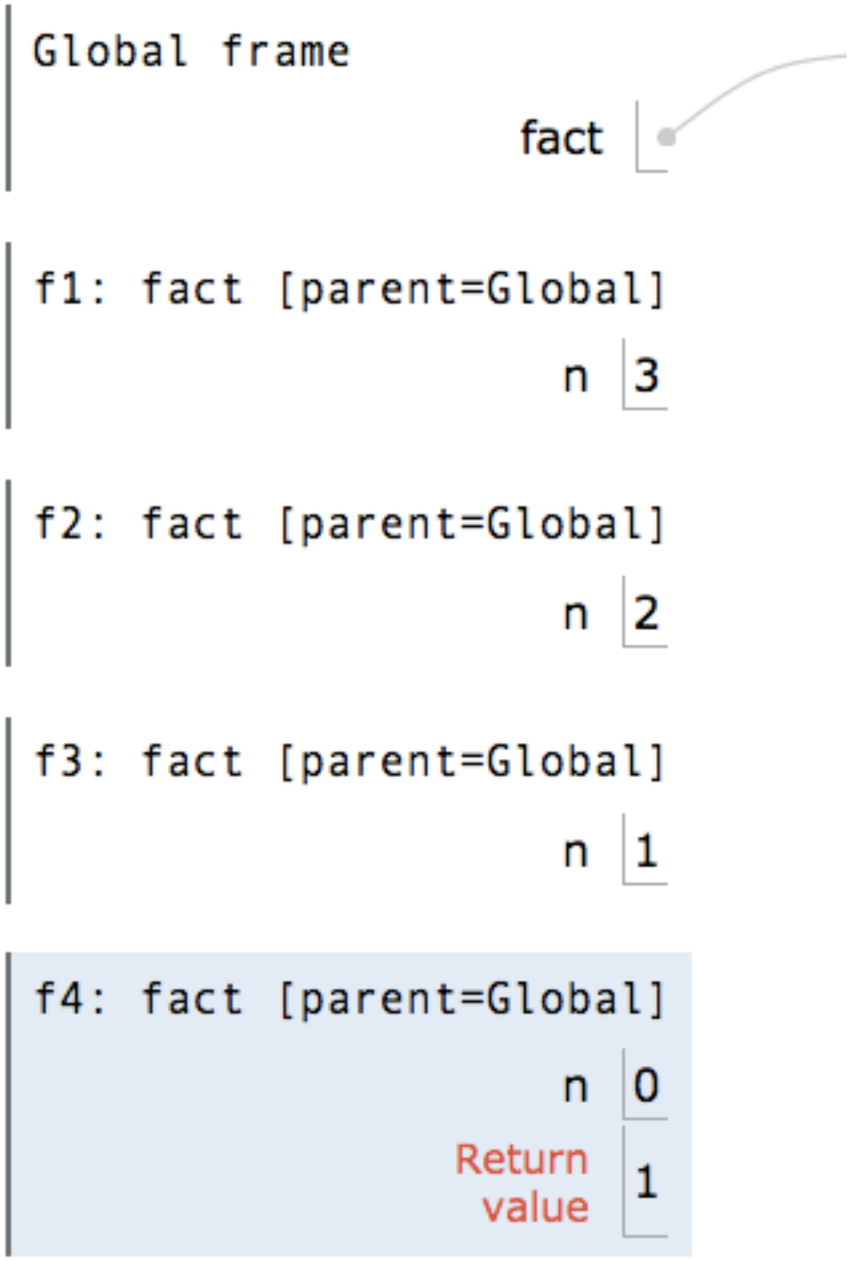

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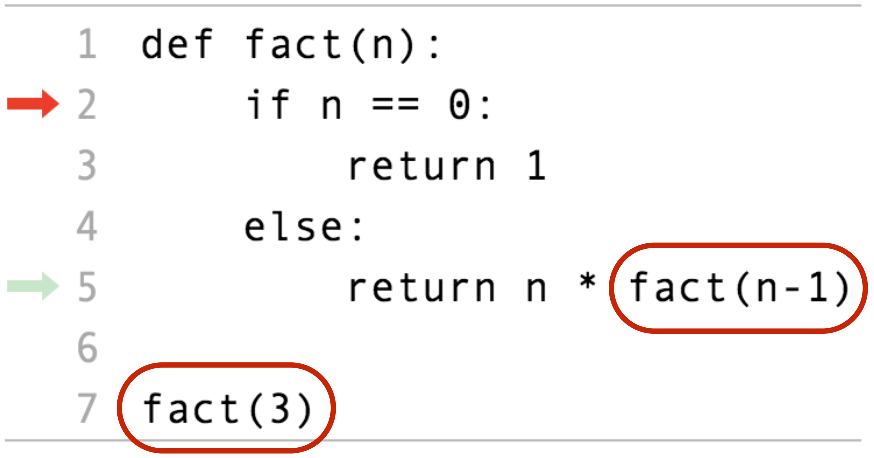

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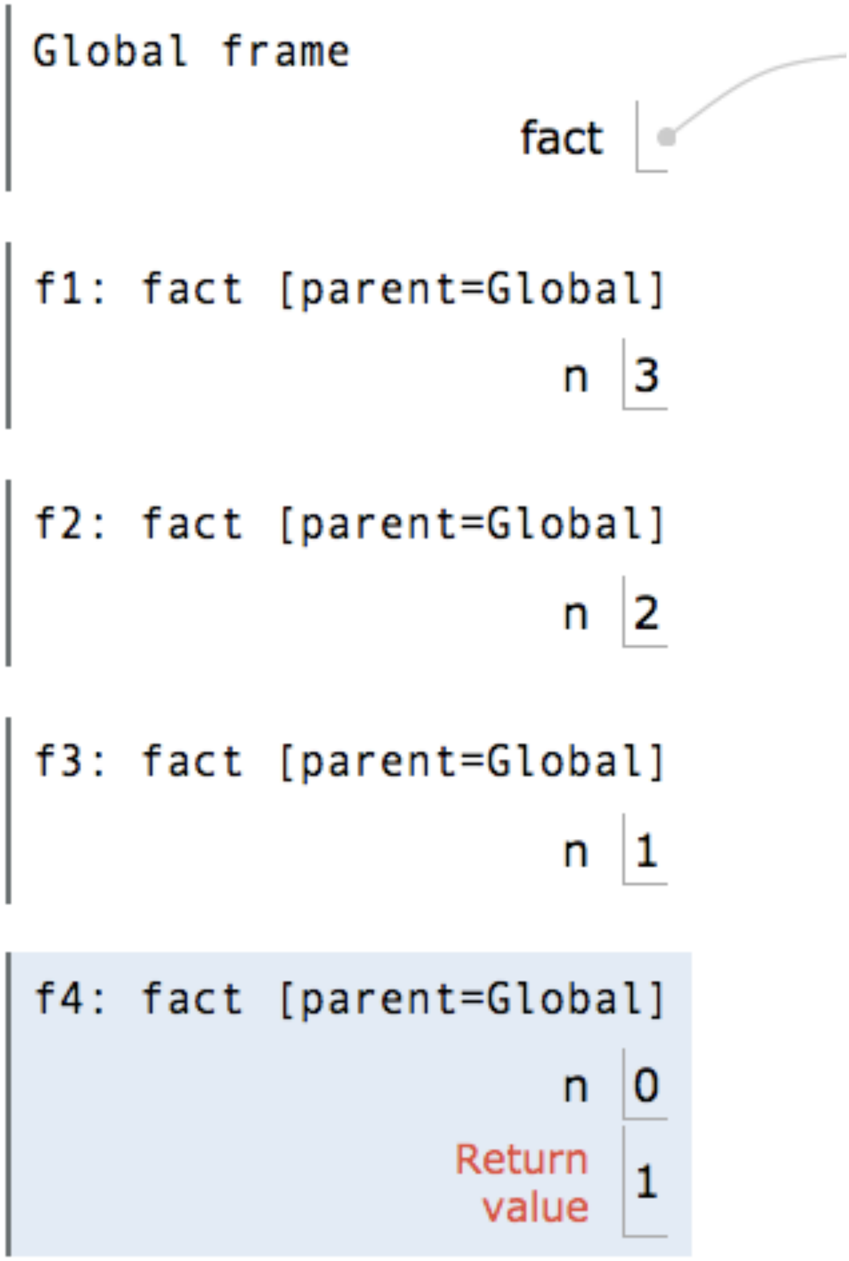

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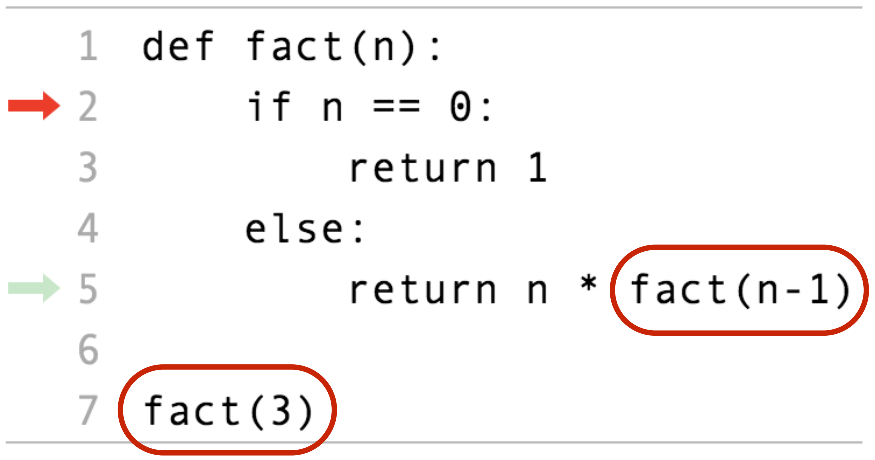

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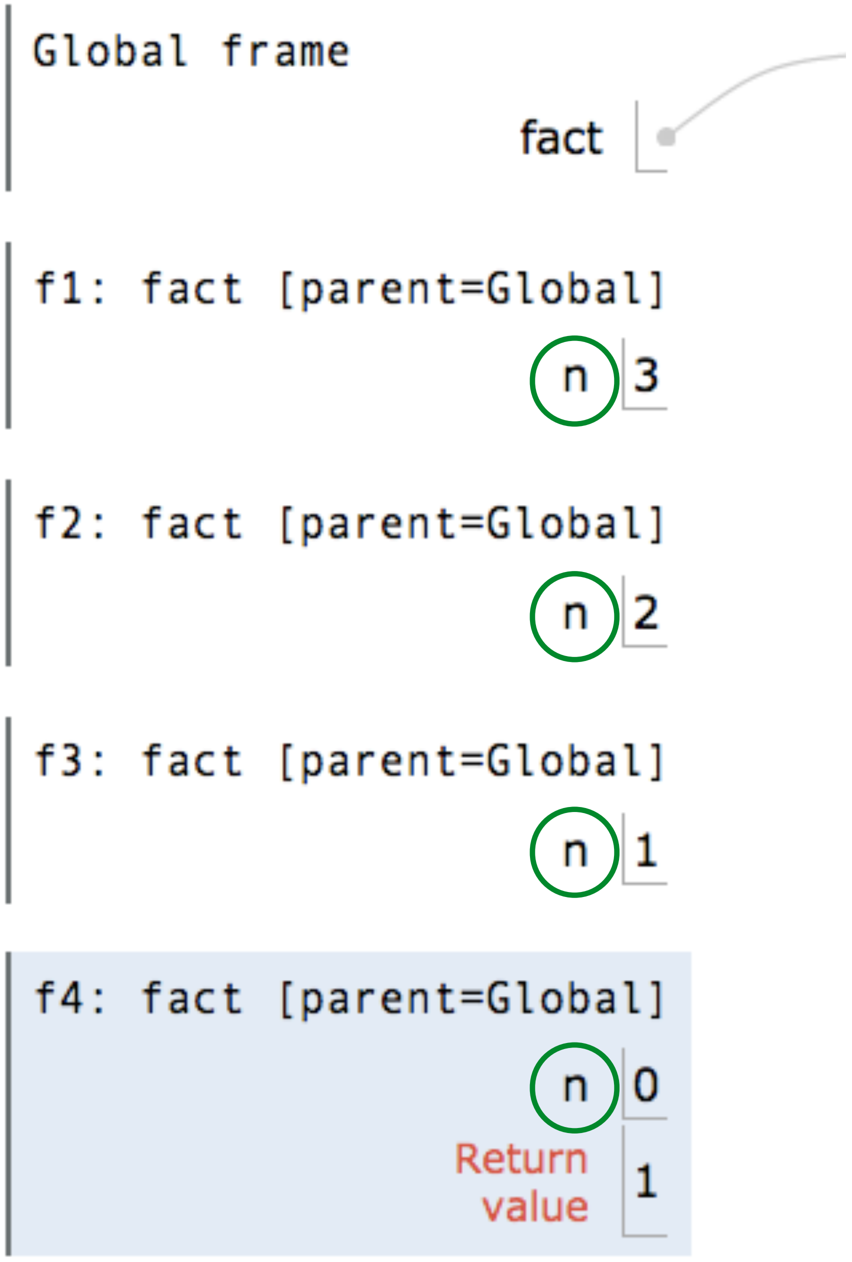

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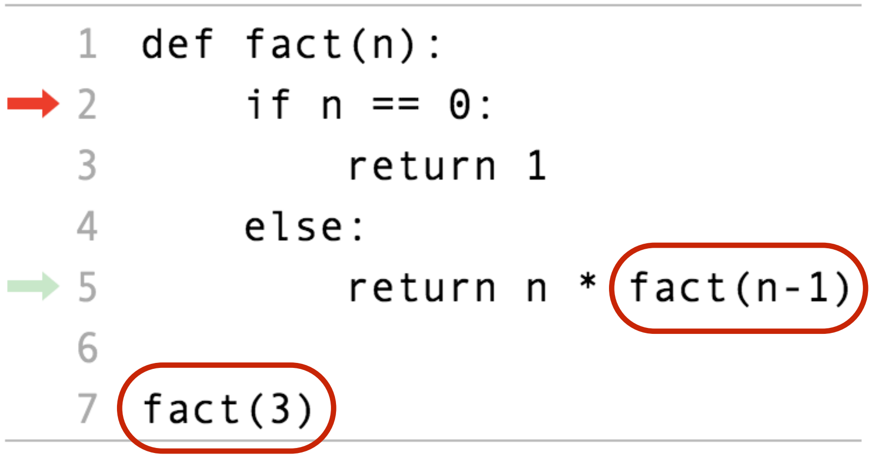

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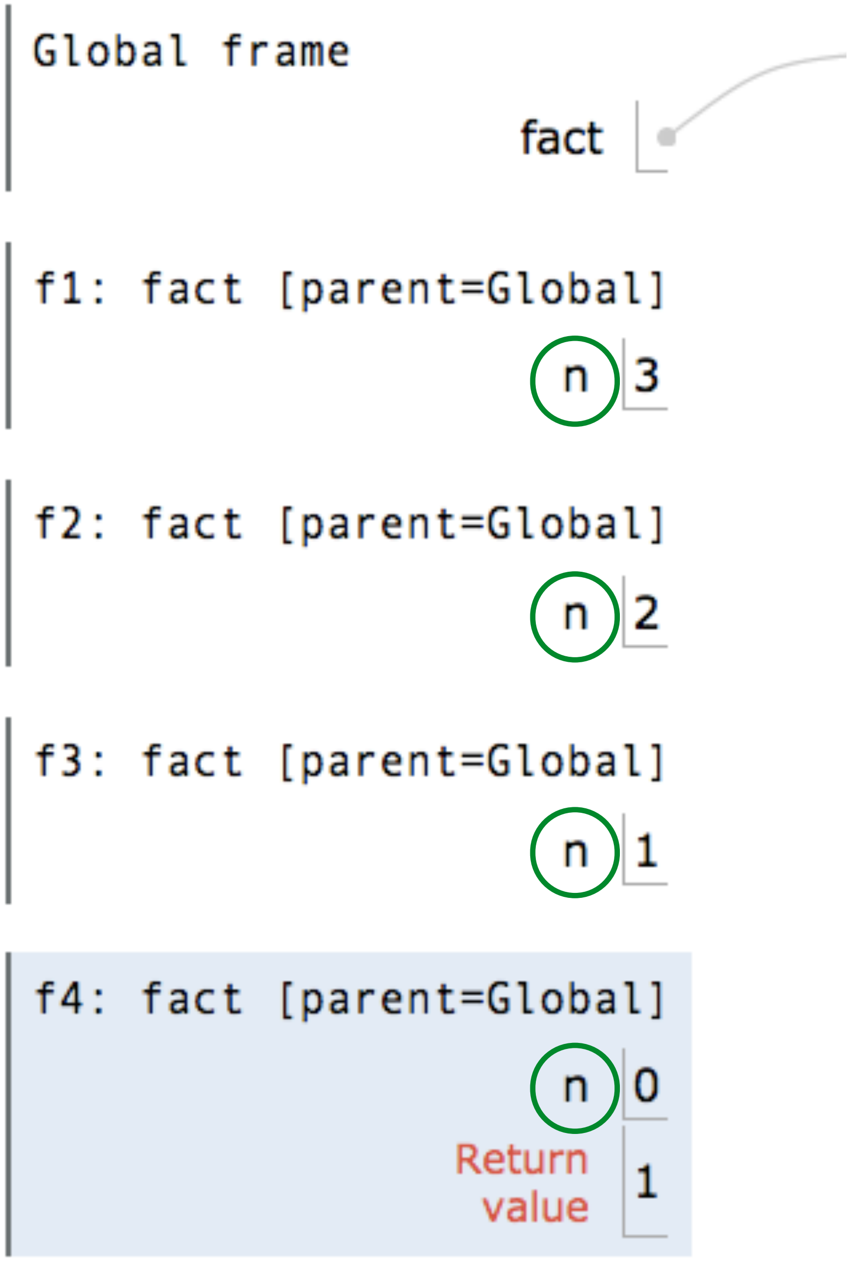
- Each call to `fact` solves a simpler problem than the last: smaller `n`



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3. Verify that `factorial(n)` is correct.

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if n < 10:
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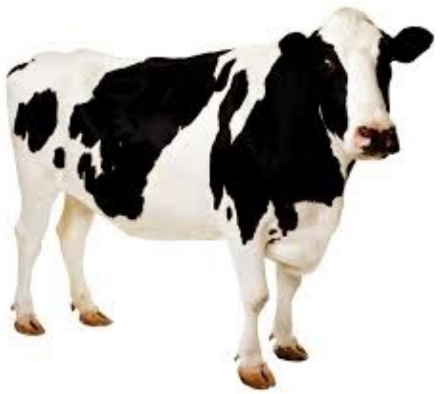

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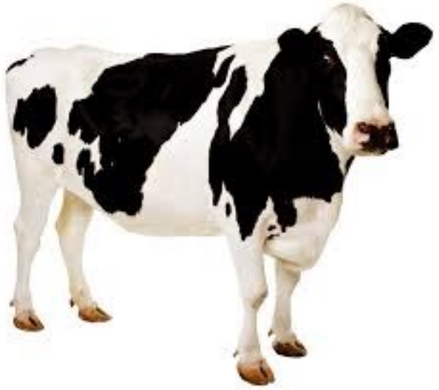

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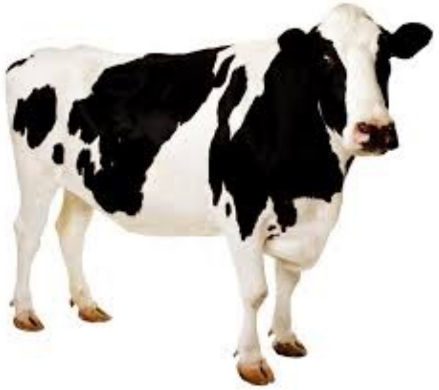

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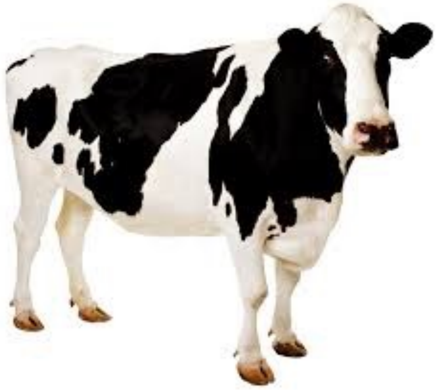

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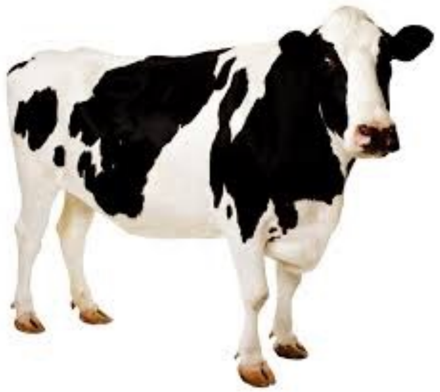

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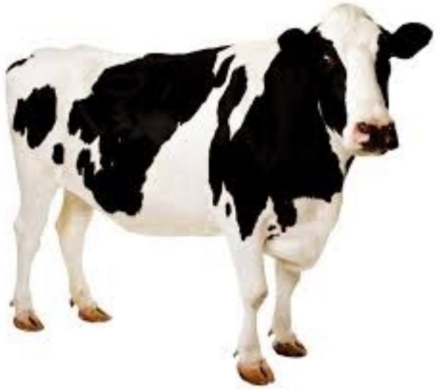

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Iteration vs Recursion

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Using iteration:

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def fact(n):
    if n == 0:
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Math:

Iteration vs Recursion (demo)

- Iteration is a special case of recursion
- Converting iteration to recursion is formulaic, but converting recursion to iteration can be more tricky

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```python
def reverse(word):
    """Return the reverse of the string word."""
    if len(word) < 2:
        return word
    else:
        return reverse(word[1:]) + word[0]
```

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