Enumerating m-Length Walks in Directed Graphs with Constant Delay

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Motivation

- There are many settings in which we wish to enumerate every object within a class of combinatorial objects.
- As such classes are often of exceptional size relative to the description of the class.
- One such example is **regular languages**, where a deterministic, finite automaton can describe a class of possibly unbounded size.
    - We are motivated particularly by the class of *prefix closed langagues* avoiding a set of forbidden substrings, via a problem originating in chemistry.

Our Results

Theorem 1. There is an algorithm which, given a number m, enumerates, without repetitions, all walks of length m in a (directed) graph $G = (V, E)$ with a worst-case delay of $O(1)$, after preprocessing taking $O(|E|)$ time.

Overview

Given the number of walks in a graph (up to n^n), there are two main objectives with the output of our algorihtm:

- **Reconstructability**: The output must allow the most recently enumerated walk to be reconstructed "efficiently".
  - Lower Bound: $O(m)$ (for an m-length walk).
  - Upper Bound $O(n^n)$.
- **Efficient**: The algorithm needs to minimise the delay between outputs.
  - Lower Bound $O(1)$.
  - Upper Bound $O(n^n)$.

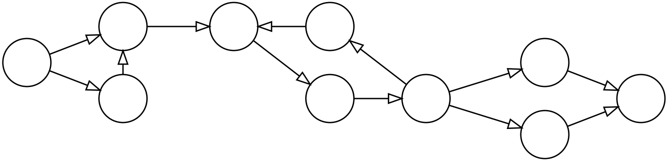
We match the lower bound for both conditions by introducing the data structures of **default edges**, **default paths**, and the **default graph**.

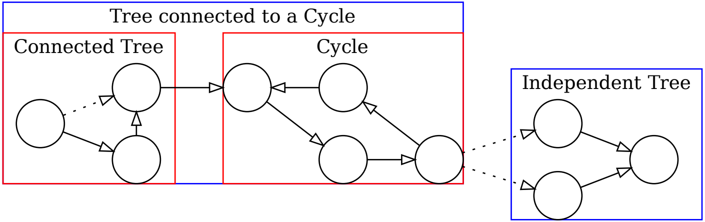
Default Edges, Paths, and Graphs

Definition

Given a graph $G = (V, E)$ and vertex $v \in V$, the **Default Edge** from v is the edge from v in the longest walk starting at v in $\mathcal{G}$. The **Default Path** from v is the path formed by following default edges from each state starting at v. The **Default Graph** of $\mathcal{G}$, $D(\mathcal{G})$, is the graph induced by the default edges.

Default Graph Example





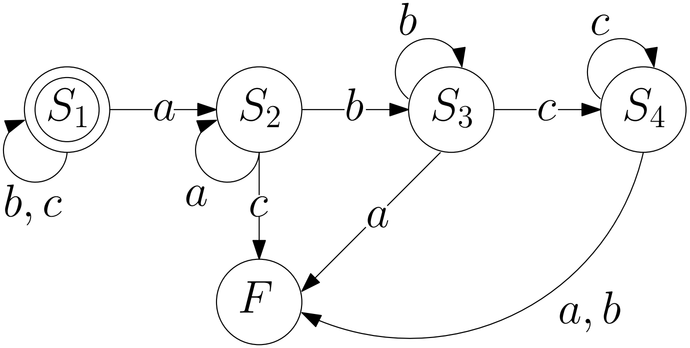
Using the Defualt Graph to represent walks

Key Idea. We represent each walk as a list of vertex-length pairs, each representing a sub walk. The tuple (v, ℓ) represents the walk on the delault path from v for ℓ steps.

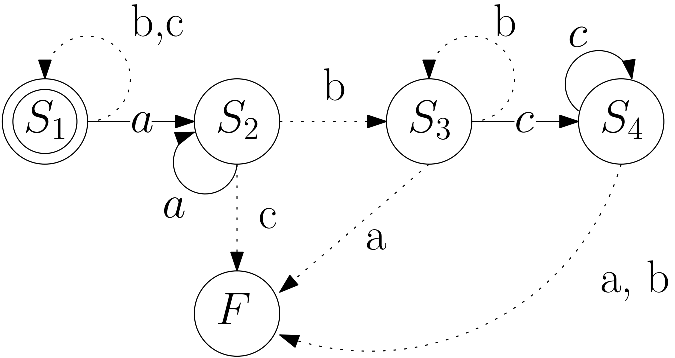
For **strings** accepted by a given automaton $\mathcal{A}$, each walk represents a string, and the tuples in the represetnation represent *substrings*

.

Representation example



Representation example

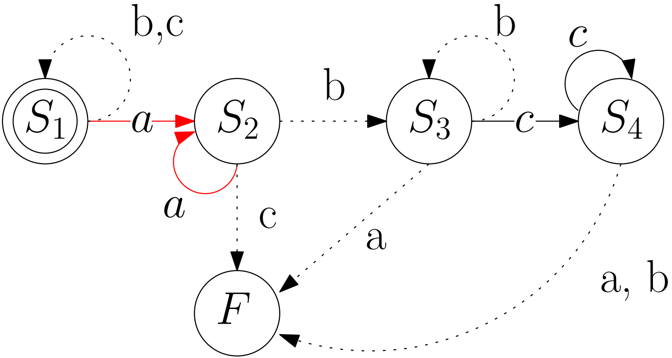


String	default path representation
aaaabbbbcccc	$[(S_1, 4), (S_3, 0), (S_3, 0), (S_3, 0), (S_3, 0), (S_3, 4)]$
aaaaaabccccc	$[(S_1, 6), (S_3, 5)]$
abcccccccccc	$[(S_1, 1)(S_3, 10)]$

Output

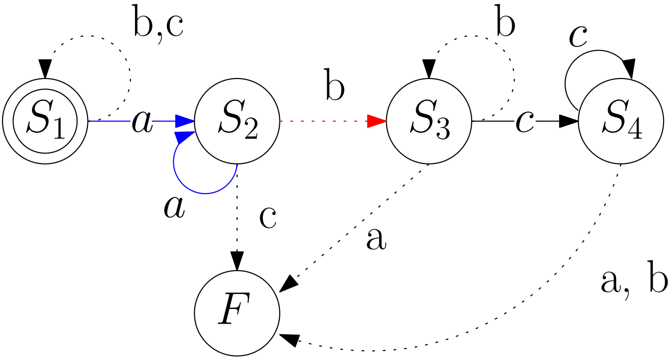
- At each step, we assume that we have a stack of state-length pairs, allowing the full word to be reconstructed in $O(n)$ time.
- Our goal now is to **edit** the stack with a constant number of operations in order to represent the next string.
- At each step, the output is a representation of the edit operations, allowing the strings to be explicitly output on demand.

Representation example



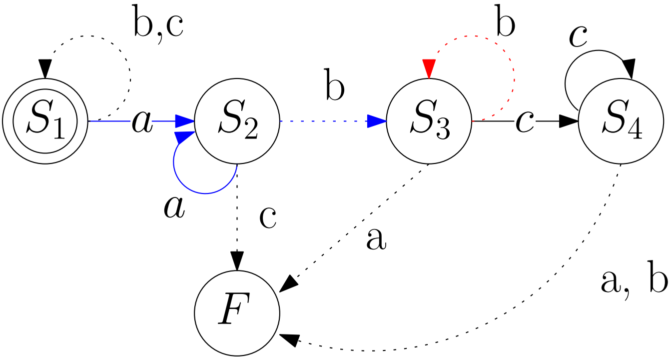
$(S_1, 4) \mapsto aaaa$

Representation example



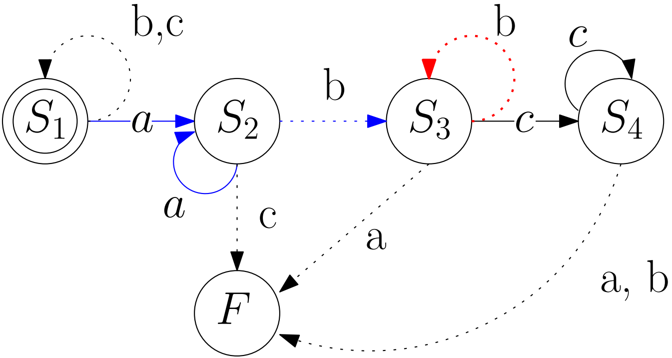
$(S_1, 4)(S_3, 0) \mapsto aaaab$

Representation example



$(S_1, 4)(S_3, 0)(S_3, 0) \mapsto aaaabb$

Representation example



$(S_1, 4)(S_3, 0)(S_3, 0)(S_3, 0) \mapsto aaaabbb$

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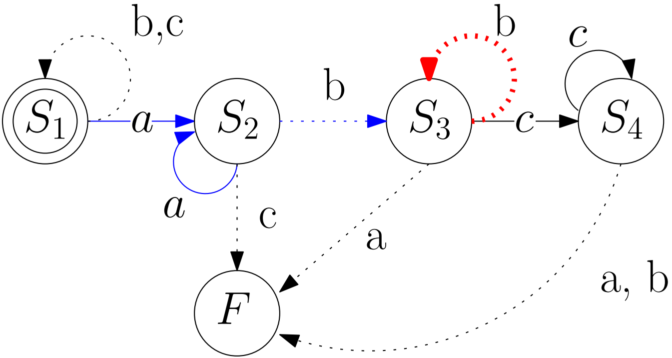

$(S_1, 4)(S_3, 0)(S_3, 0)(S_3, 0)(S_3, 0) \mapsto aaaabbbb$

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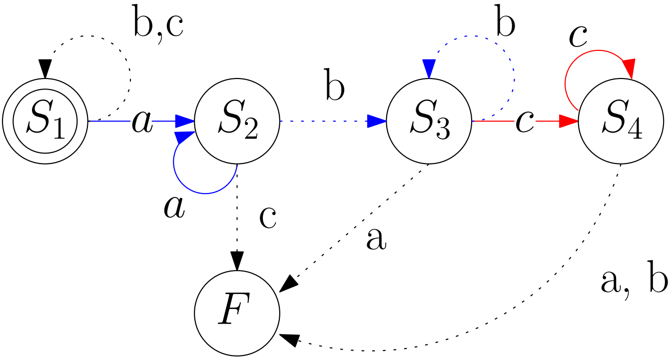


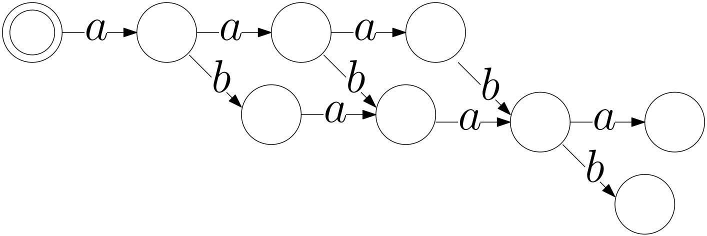
$(S_1, 4)(S_3, 0)(S_3, 0)(S_3, 0)(S_3, 0)(S_3, 4) \mapsto aaaabbbbcccc$

High Level Idea

Our algorithm operates as follows:

- Starting with a walk W (which we assume has been output) we find the the possible ways of *branching* from this walk.
- Each node on the path is weighted by the depth of the node in the default path, and the length of the **second** longest walk from the node.
- Using these weights, we do a **range maximum query** on the graph, and choose the best state on the walk to branch from.

Example



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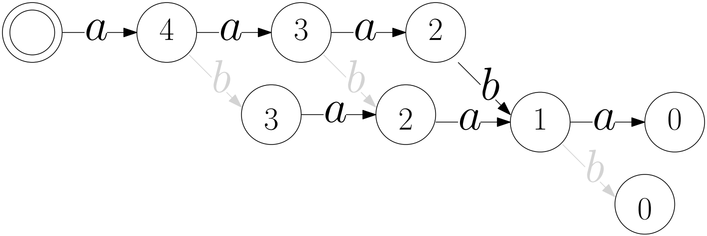

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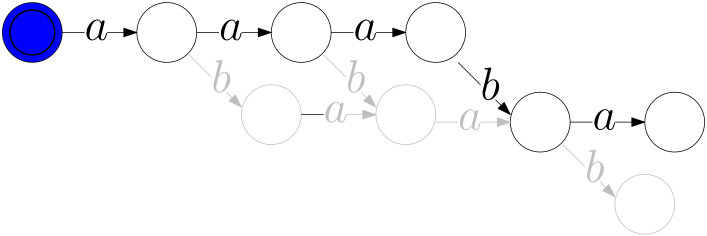

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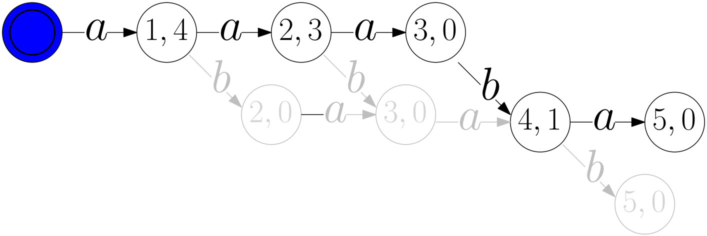

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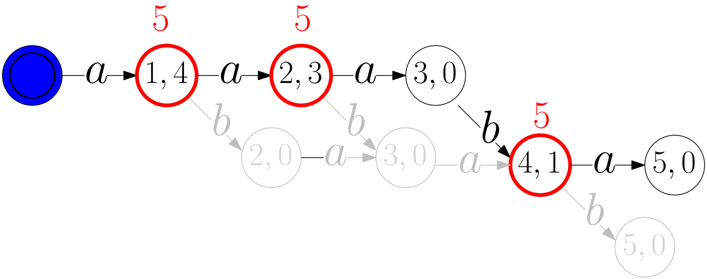

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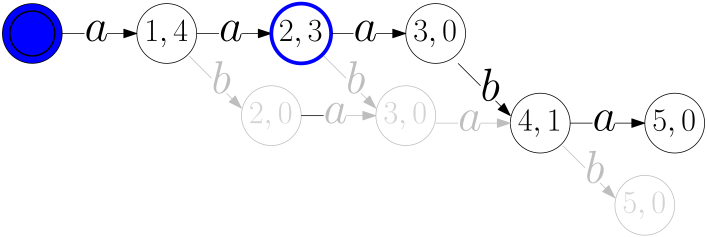

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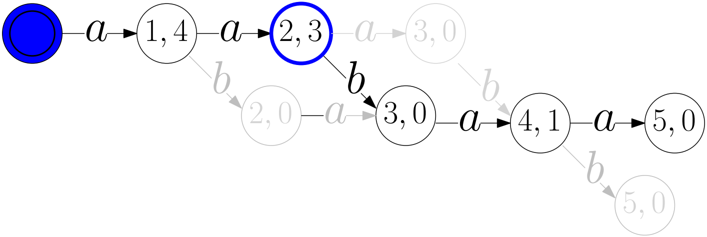

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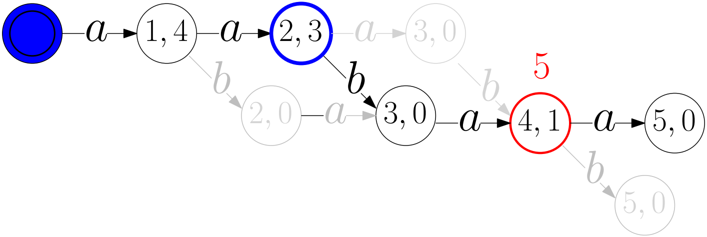

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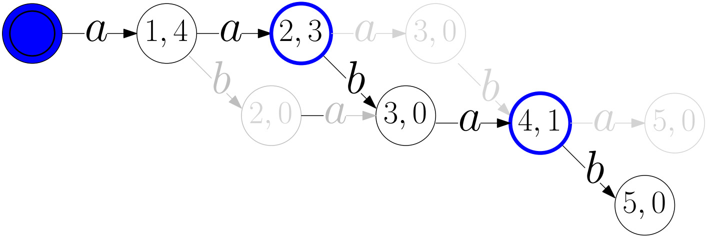

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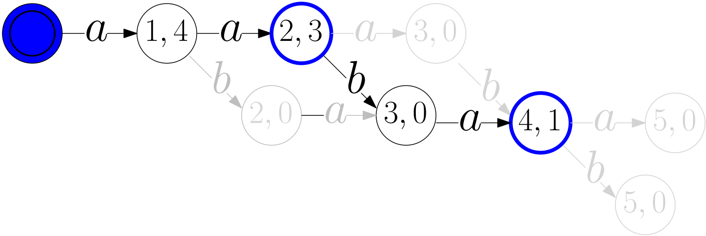

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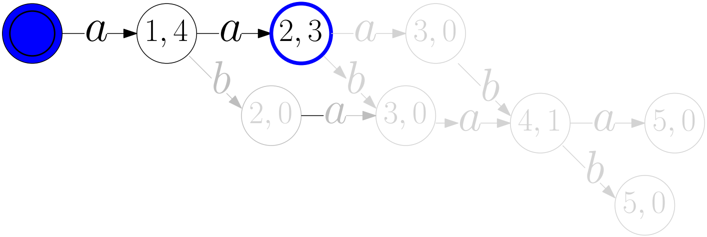

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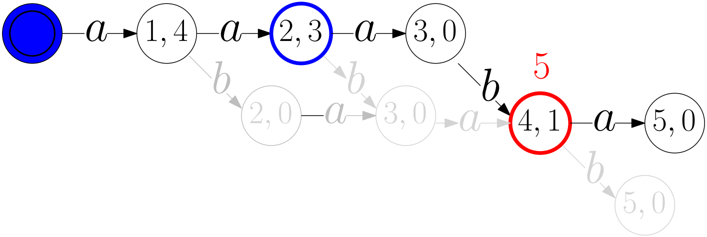

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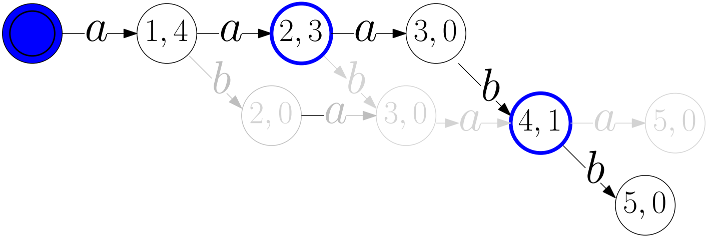

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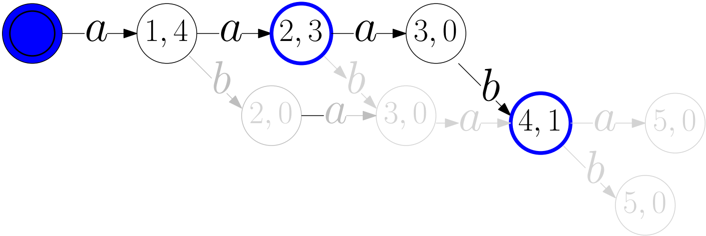

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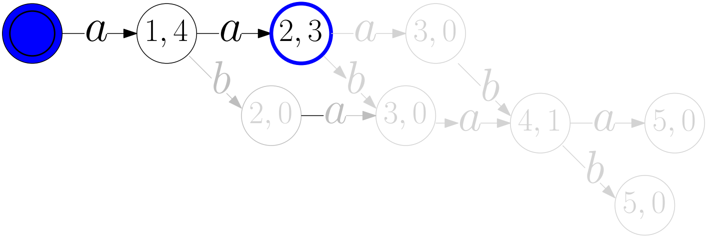

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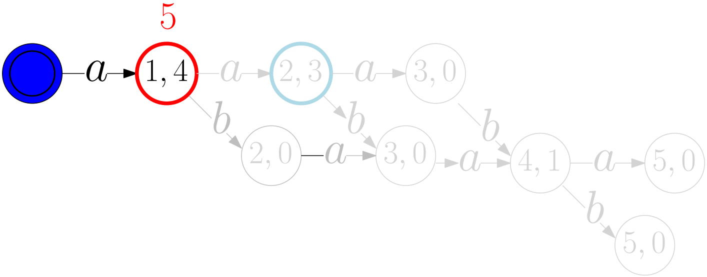

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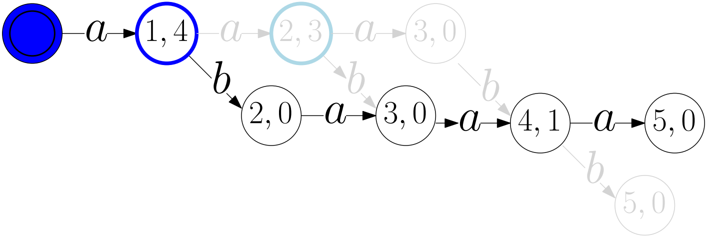

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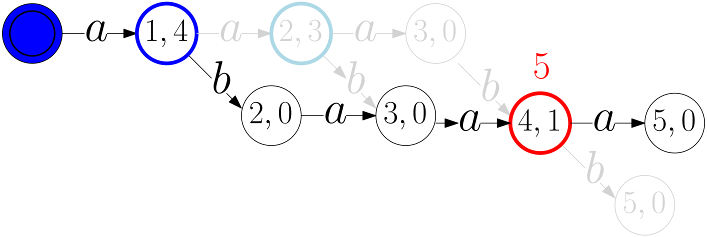

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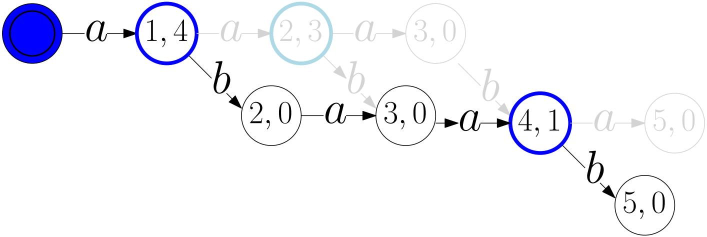

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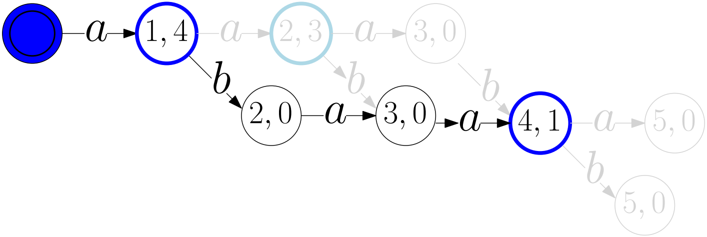

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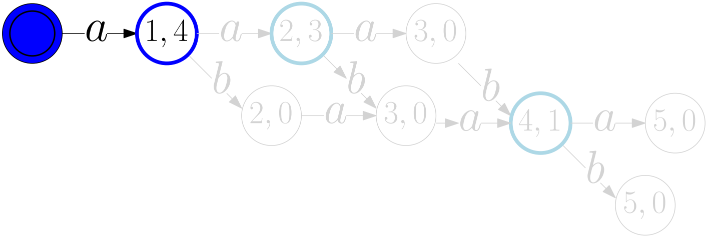

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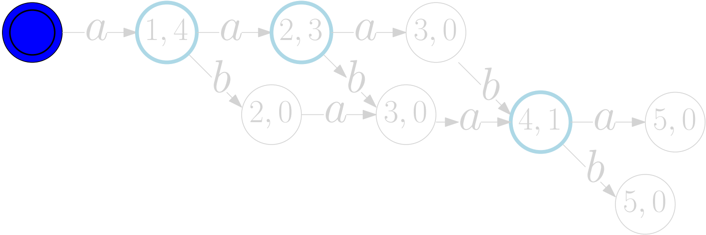

Avoiding unnecessary backtracking

- To avoid unnecessary backtracking in the execution, we use **tail recursion**, to ensure that we can skip any backtracking to a node that is **exhausted**.
- A node is **exhausted** if every walk from the node has already been enumerated (from the current prefix).

Additional Results

- Along with the enumeration results, we give auxiliary results for **ranking** and **unranking** walks in the order output by the enumeration of the algorithm.
- The **rank** of a walk is the integer corresponding to the number of walks output before it.
- The **unranking** problem takes an integer i and asks for the walk with a rank of i.

Ranking and Unranking results (informal)

The **rank** of an m-length walk in the enumeration output order of the graph $G = (V, E)$ with m vertices can be computed in $O(|E| + mn^{\omega} + m^2\Delta)$ time, where ω is the exponent for matrix multiplication and Δ is the largest degree of any vertex in G.

Given an integer i, the i^{th} m-lengthwalk in the enumeration output order of the graph G with n vertices can be computed in $O(|E| + mn^{\omega} + m^2\Delta)$ time, where ω is the exponent for matrix multiplication and Δ is the largest degree of any vertex in G.

Summary

Results:

- An enumeration algorithm with constant delay for walks of length m in the graph G
- Polynomial time algorithms for **ranking** and **unranking** in enumeration order.

Open Problems:

- Can we enumerate **regular languages** with constant delay?
- Can we rank and unrank in linear (Or $O(\Delta n^{\omega})$) time?