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Harmonious Colourings of Temporal Matchings

Duncan Adamson

With thanks to Viktor Zamaraev.

(Temporal) Graphs

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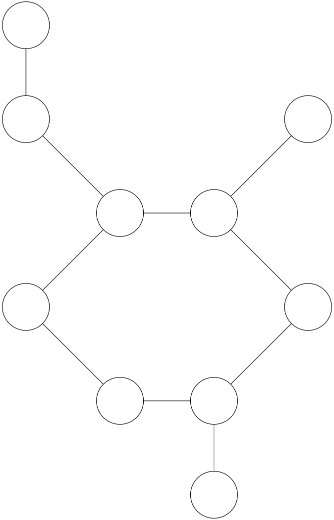

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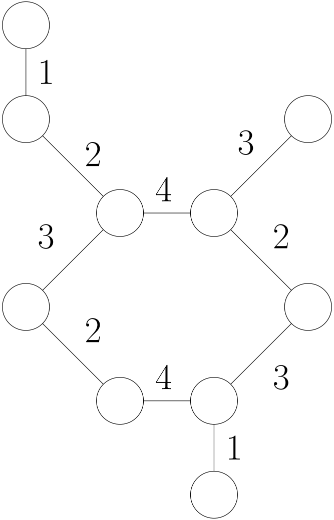

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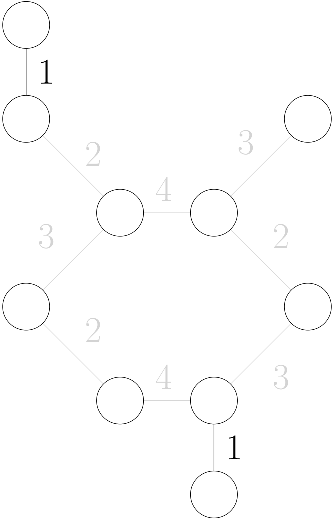

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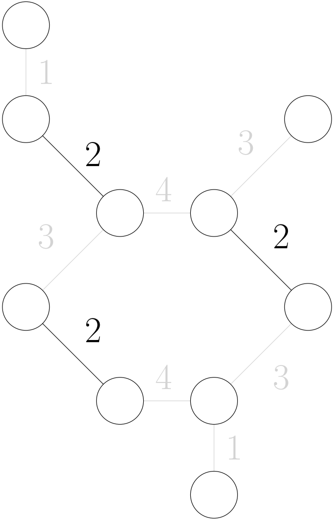

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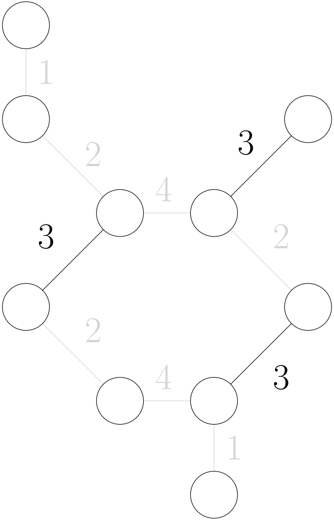

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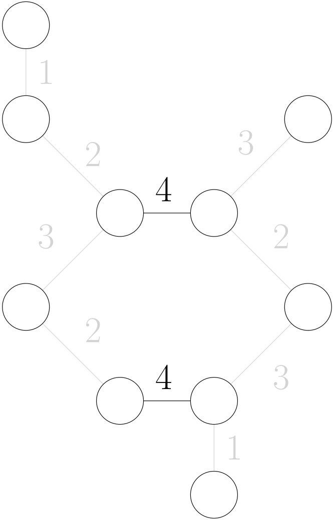

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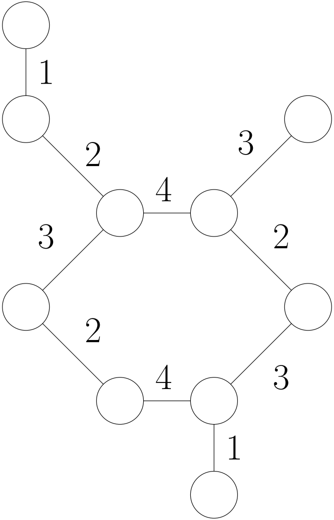
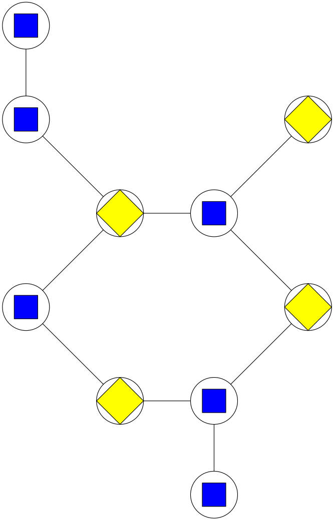
- A **Temporal Graph** corresponds to a sequence of graphs, sharing the same vertex set, but with distinct edge sets. The number of graphs is the **lifetime** of the graph.
- The **Underlying graph** of a given temporal graph $\mathcal{G} = (G_1, G_2, \ldots, G_T) = ((V, E_1), (V, E_2), \ldots, V, E_T)$, is the graph $G' = (V, E_1 \cup E_2 \cup \cdots \cup E_T)$.



Colourings



- A **Colouring** of a graph is an assignment of some set of labels to the set of vertices.
- A colouring is **valid** if no two **adjacent** vertices (vertices sharing an edge) share the same colour.
- A colouring is a **c-colouring** if the mapping contains (at most) c-colours.
- We define a c colouring by a mapping $\psi : V \mapsto 1, 2, \ldots, c$.
- The **chromatic number** of a graph is the smallest value c such that there is a c-colouring of the graph.

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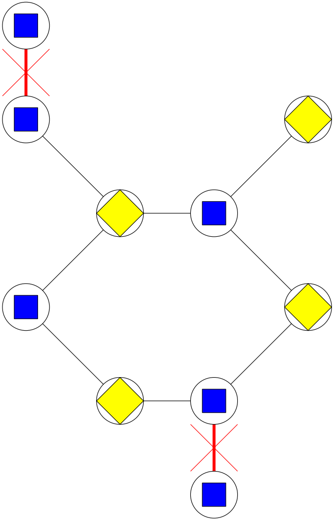

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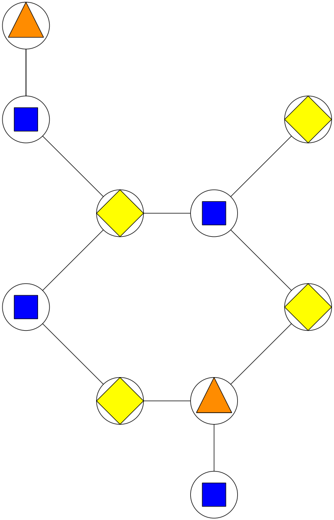

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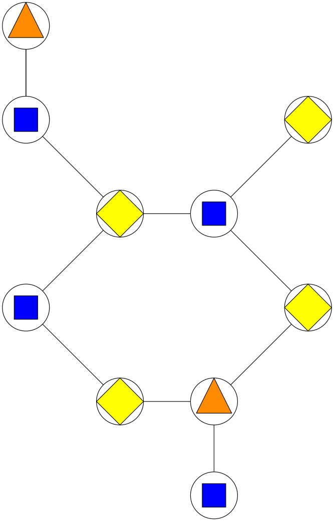
Harmonious Colourings

- A colouring is **harmonious** if the pair of colours assigned to each edge is **unique**.
- The *harmonious chromatic number* of a graph is the smallest number c such that there is a c-colouring of the graph that is harmonious.



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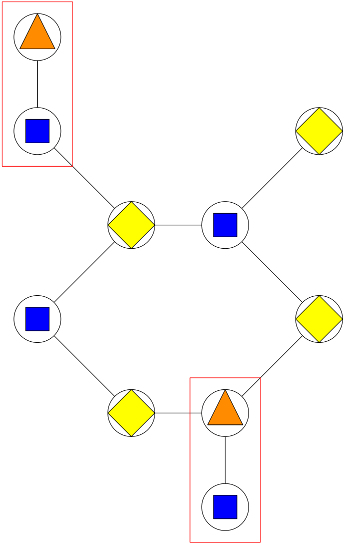

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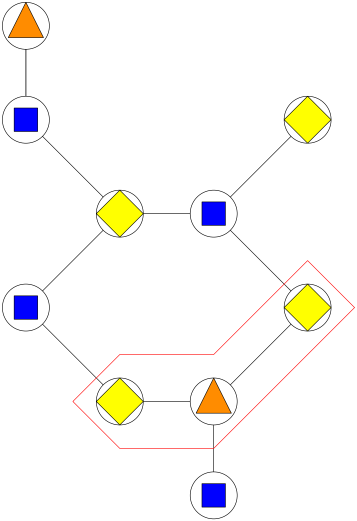

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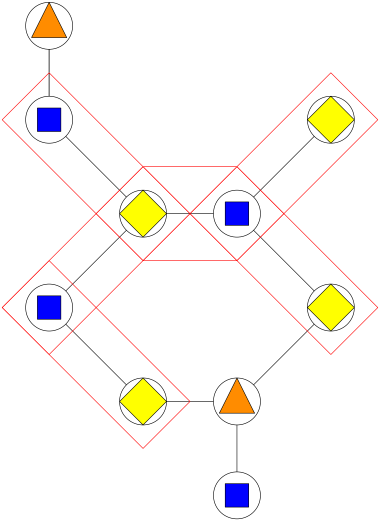

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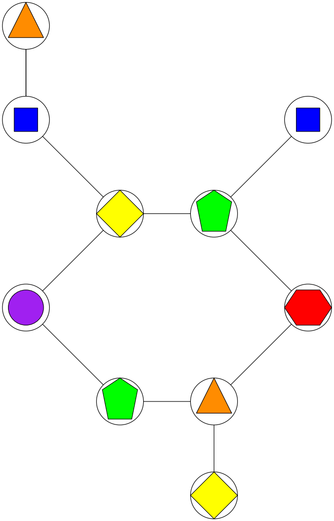
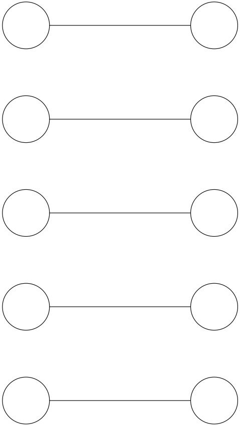


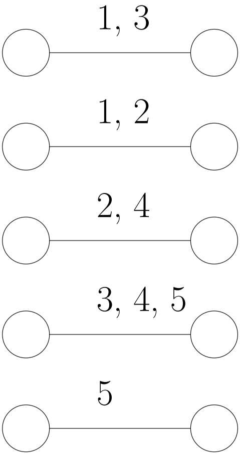
Matching Graph



- A graph is a **matching** if every vertex has (at most) 1 edge.
- A temporal graph is a **temporal matching** if the underlying graph is a matching.

(k, t)-Temporal Matchings

- A temporal matching is a (k, t)-temporal matching if each timestep contains at most k edges, and each edge is active for at most t time steps.



Temporal Harmonious Colouring

Definition

A colouring $\psi : V \mapsto 1, 2, \ldots, c$ of a temporal graph $\mathcal{G} = (G_1, G_2, \ldots, G_T)$ is a **temporal harmonious c-colouring** if ψ is a harmonious colouring for every $G_i \in \mathcal{G}$ (i.e., at each timestep the graph is still harmoniously coloured). Note the the underlying graph is not necessarily harmoniously coloured by ψ.

Problem

Given a temporal graph $\mathcal{G}$ and integer $c \in \mathbb{N}$, does there exist a temporal harmonious c-colouring of $\mathcal{G}$.

Existing Hardness Results

- It is NP-hard to find a valid colouring if the underlying graph is unrestricted, even if only one edge is active each timestep.
- It is NP-hard to find a harmonious colouring on trees, so our problem is hard if at least one of the time steps is allowed to be an arbitrary tree.
- However, for a matching, a harmonious c-colouring can be found (if one exists) in $O(n)$ time via a greedy approach.

Our Results

(k, t)	Result
$k \geq 2, t \geq 4$	NP-hard
$k \geq 3, t \geq 2$	NP-hard
$k \in \mathbb{N}, t \geq 2$	NP-hard to approximate within $O(n^{(1-\epsilon)/2})$
$k = 1, t \in \mathbb{N}$	Temporal Harmonious Chromatic number in $O(n)$ time
$k \leq 2, t \leq 3$	Temporal Harmonious Chromatic number in $O(n)$ time
$k, t \in \mathbb{N}$	Lower bound of $\sqrt{2k}$
$k, t \in \mathbb{N}$	Upper bound $t(k-1)+2$

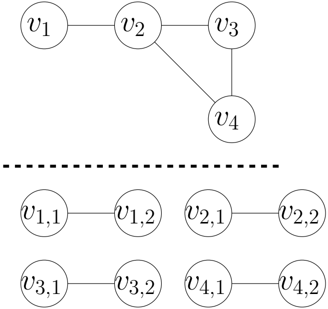
Encoding static graphs as $(2, t)$-temporal matchings

Idea: Encode a static graph G as a (k, t)-temporal matching $\mathcal{G}'$ s.t. the temporal harmonious chromatic number of $\mathcal{G}'$ can be used to tightly bound the chromatic number of G.

Construction

Given $G = (V, E)$, we construct $\mathcal{G}'$ where:

1. For each vertex $v_i \in V$, we create a pair of vertices $v_{i,1}, v_{i,2}$, and edge $(v_{i,1}, v_{i,2})$.



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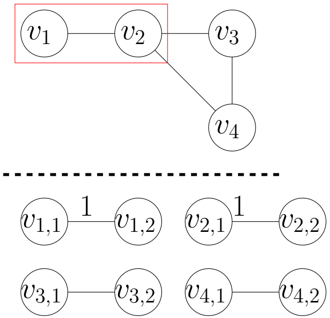

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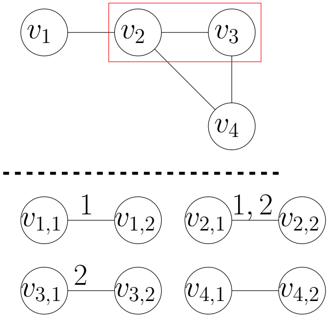

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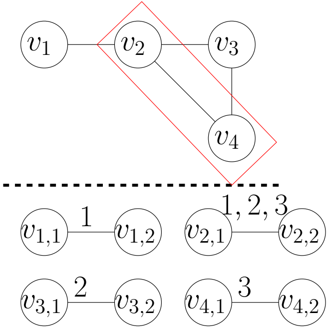

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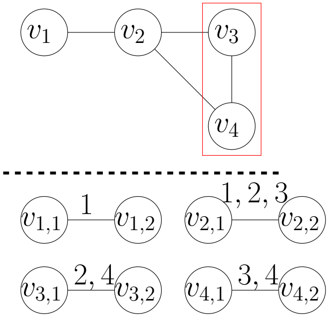
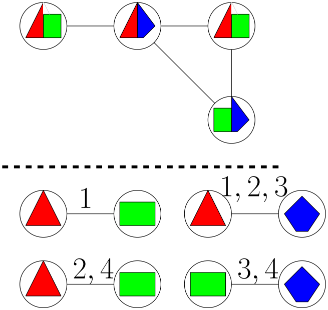


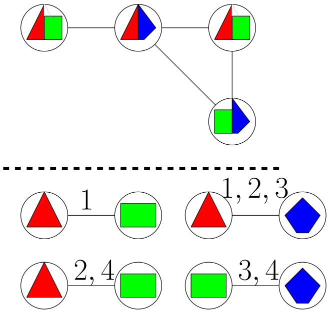
Combining the Chromatic Numbers $\mathcal{G}' \rightarrow G$

- A temporal harmonious colouring ψ of $\mathcal{G}'$ can be used to construct a colouring ψ' of G by assigning to v_i the colour $(\psi(v_{i,1}), \psi(v_{i,2}))$, with the tentative assumption that $\psi(v_{i,1}) < \psi(v_{i,2})$ for some ordering of the colours.
- This mapping transforms a temporal harmonious c-colouring of $\mathcal{G}'$ into a $c(c-1)/2$-colouring of G.



Combining the Chromatic Numbers $G \to \mathcal{G}'$

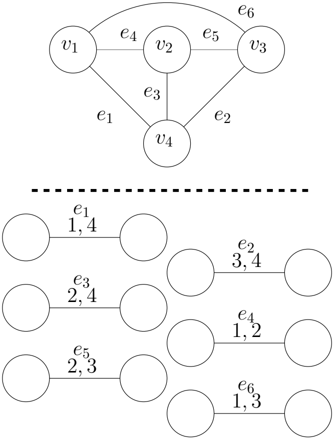
- In the other direction, given a c-colouring ϕ of G, we make a temporal harmonious colouring ϕ' of $\mathcal{G}'$ by matching each colour in $1, 2, \ldots, c$ to a pair in $1, 2, \ldots, c' \times 1, 2, \ldots, c'$, where c' is the smallest value satisfying $c'(c'-1)/2 \geq c$.



Hardness Results for $(2, 4)$-Temporal Matchings

- Using the above construction, we can use the hardness of 3-edge colouring cubic graphs, equivalent to 3-colouring the adjacency graph, to derive hardness for temporal harmonious colouring.
- If we can get a temporal harmonious 3-colouring of $\mathcal{G}'$, then we can get a $3(3-1)/2 = 3$-colouring of the graph G.
- If we can get a 3-colouring of G then we can get a 3-colouring of $\mathcal{G}'$.
- As cubic graphs have a maximum degree of 4, $\mathcal{G}'$ is a $(2, 4)$-matching, giving hardness for this case.
- Our hardness of approximation result follows the same outline.

Hardness Results for $(3, 2)$-Temporal Matchings



- We use the same idea as above to encode the c-edge colouring problem as a $(3, 2)$ temporal matching, by having each vertex pair in the matching represent an edge, with a timestep for each vertex, with every adjacent edge active.

$(2,3)$-matchings

- Using the reverse construction from our hardness results, we can transform a $(2,3)$ temporal matching $\mathcal{G}$ into a cubic graph G.
- If G has a chromatic number of:
  - 0, then $\mathcal{G}$ has either no vertices (and therefore a temporal chromatic number of 0), or only vertices with degree 0 (and therefore a temporal chromatic number of 1).
  - 1, then no two pairs are active at the same time, so $\mathcal{G}$ has a temporal chromatic number of 2.
  - 2 or 3, then $\mathcal{G}$ has a temporal chromatic number of 3.
  - 4, then $\mathcal{G}$ has a temporal chromatic number of 4.

$(2,3)$-matchings

- We can determine the chromatic number of G in $O(n)$ time as either:
  - G is empty, and has a chromatic number of 0.
  - G is a set of disconnected vertices, and has a chromatic number of 1.
  - G is bipartite (determinable in $O(n)$) time, and has a chromatic number of 2.
  - G does not have a chromatic number of $0, 1, 2$ or 4.
  - G is the complete graph of size 4, and has a chromatic number of 4 (decidable in $O(n)$ time).

Lower bound

- If $\mathcal{G}$ is a (k, t)-temporal matching (and not a $(k-1, t)$-temporal matching), then there is a least 1 time step with k edges.
- Therefore, the minimum harmonious chromatic number of this timestep is the value c such that

$$\begin{aligned} c(c-1)/2 &\geq k \\ c(c-1) &\geq 2k \\ c^2 &\geq 2k \\ c &\geq \sqrt{2k} \end{aligned}$$

giving our bound.

Upper Bound

We use a greed algorithm to get an upper bound of $t(k-1)+2$ on the temporal chromatic number of a (k,t)-temporal **paths** and **matchings**.

```
col(1) ← 1
col(2) ← 2.
for i ∈ V \ {1, 2} do
    Colours ← {1, 2, . . . c} \ {col(i − 1)}
    for k ∈ ActiveTimesteps((i − 1, i)) do
        for (j − 1, j) ∈ E_k do
            if ((j, j + 1), (i − 1, i)) ∈ E_k then
                if col(j) = col(i − 1) then
                    Colours ← Colours \ {col(j + 1)}
                if col(j + 1) = col(i − 1) then
                    Colours ← Colours \ {col(j)}
    col(i) ← min(Colours)
```

Conclusion

Results

- Finding Harmonious Colourings of (k, t)-temporal matchings is hard for most values of k and t.
- However, we can find the temporal harmonious chromatic number for $(2, 3)$ matchings in $O(n)$ time.

Open Problems:

- Can we show that the upper bound of $t(k-1)+2$ is tight (for paths)?
- Can we improve the lower bound?

Thank you for your attention.
Any Questions?