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Word-Representable Temporal Graphs

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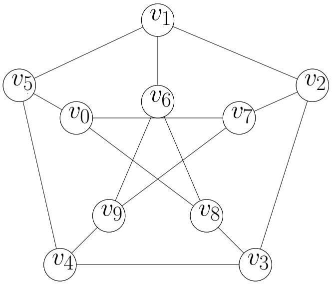
Word-Representable Graphs

- **Word-representable graphs** are a class of graphs that can be represented by a word.
- These generalise several fundamental classes of graphs, including circle graphs, three-colourable graphs, and comparability graphs.
- Every word can be used to represent a graph, but not every graph can be represented by a word.
- A graph may have multiple word representations.

Word-Representable Graphs

- Given a word $w = w_1 w_2 \dots w_n$, we make the graph $G(w) = (V, E)$ based on the **alphabet** and **subsequences** of w.
- We create the vertex set V as $V = \{v_x \mid x \in \mathrm{alph}(w)\}$
- Let $\pi_{x,y}(w)$ denote the subsequence of w containing every instance of the symbols x and y.
- $(v_x, v_y) \in E$ iff $\pi_{x,y}(w) \in \{(xy)^k, (yx)^k, y(xy)^k, x(yx)^k\}$.

$w = 13872960749354128307685019456 2$



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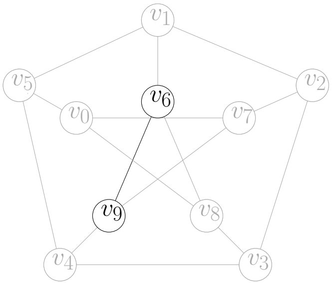
$w = 13872960749354128307685019456 2$

$\pi(w)_{6,9} = 96 \quad 9 \qquad 6 \quad 9 \; 6$



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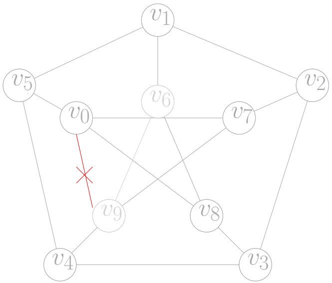


Temporal Graphs

- A **temporal graph** is a generalisation of (static) graphs when, rather than one single set of edges, there is an **ordered sequence** of edge sets.
- By convention:

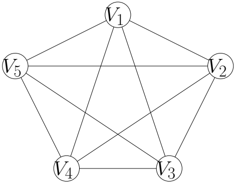
$$\mathcal{G} = (V, E_1, E_2, \ldots, E_T)$$

  where:
  - V is the set of vertices,
  - $E_i \subseteq V \times V$ is a set of edges.
- We call each set of edges a **time step**, and the number of sets as the **lifetime** of the graph.
- An edge e is **active** in timestep i iff $e \in E_i$.
- The **underlying graph** is the static graph

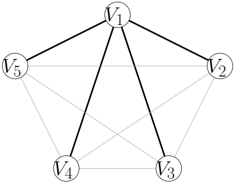
$$U(\mathcal{G}) = \left(V, \bigcup_{i \in [1,T]} E_i \right).$$

Temporal Graph Example

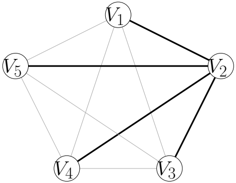
$$\mathcal{G} = (\{v_1, v_2, v_3, v_4, v_5\}, E_1, E_2, E_3, E_4, E_5)$$



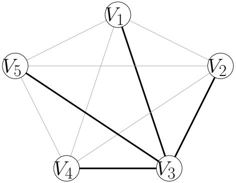
Underlying Graph



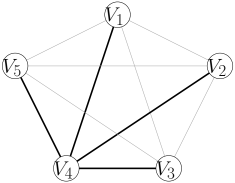
$E_1 = (v_1, v_2), (v_1, v_3), (v_1, v_4), (v_1, v_5)$



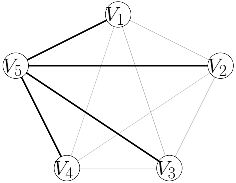
$E_2 = (v_1, v_2), (v_2, v_3), (v_2, v_4), (v_2, v_5)$



$E_3 = (v_1, v_3), (v_2, v_3), (v_3, v_4), (v_3, v_5)$



$E_4 = (v_1, v_4), (v_2, v_4), (v_3, v_4), (v_4, v_5)$



$E_5 = (v_1, v_5), (v_2, v_5), (v_3, v_5), (v_4, v_5)$

Word-Representable Temporal Graphs

- In order to generalise word-representable graphs to temporal graphs, we need to represent different timesteps in a "reasonable" way.
- This means, being able represent a wide class of graphs in each timestep, while still being a compact representation of the temporal graph.

Word-Representable Temporal Graphs

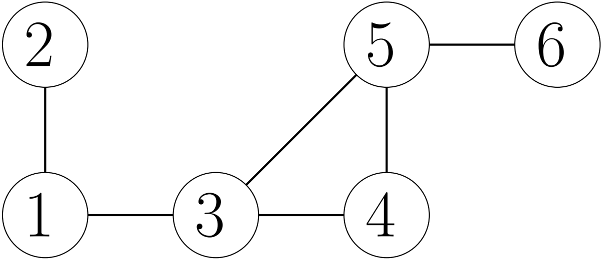
- Given a word $w = w_1 w_2 \dots w_n$, we construct the temporal graph $\mathcal{G}(w) = (V, E_1, E_2, \dots, E_T)$ with the underlying graph $G(w)$ (i.e., the static graph represented by w).
- To determine the timesteps, we split w into a set of factors (**timestep words**) $u_1, u_2, \dots u_T$ such that $w = u_1 u_2 \dots u_T$.
- u_1 is defined as $w[1, i_1]$, where i_1 satisfies $|\operatorname{alph}(w[1, i_1])| = i_1$ and $w[i_1 + 1] \in \operatorname{alph}(w[1, i_1])$.
  - Informally, $i_1 + 1$ is the index of the first repeated symbol in w.
- Now, let $i_{t-1} = |u_1 u_2 \dots u_{t-1}|$. The index i_t is the value such that $|\operatorname{alph}(w[i_{t-1} + 1, i_t])| = i_t - i_{t-1}$ and $w[i_t + 1] \in \operatorname{alph}(w[i_{t-1} + 1, i_t])$.
  - Informally, $i_t + 1$ is the index of the first repeated symbol in the suffix of w, $w[i_{t-1} + 1, n]$.
- u_t is then set to $w[i_{t-1} + 1, i_t]$.

Word-Representable Temporal Graphs

- Let $w = u_1 u_2 \dots u_T$ as in the previous slide.
- Let $G(w) = (V, E)$ be the static graph represented by w, and $\mathcal{G}(w) = (V, E_1, E_2, \dots, E_T)$ the temporal graph represented by w.
- We set the edge set E_t to $\{(v_i, v_j) \in E \mid i \in \text{alph}(u_t) \text{ or } j \in \text{alph}(u_t)\}$.
    - Informally, E_t contains every edge incident to some vertex in the factor u_t.

Example Graph

$$1\,2\,3\,1\,4\,5\,6\,5\,3\,4\,6\,5$$



Example Graph

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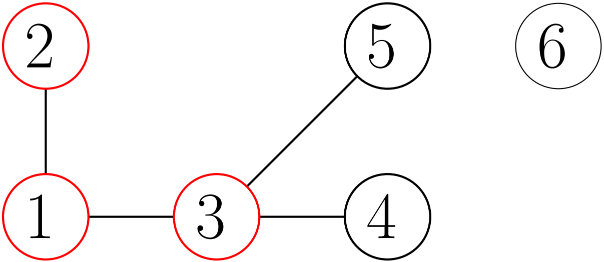
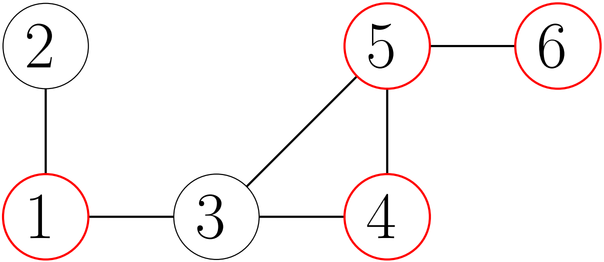


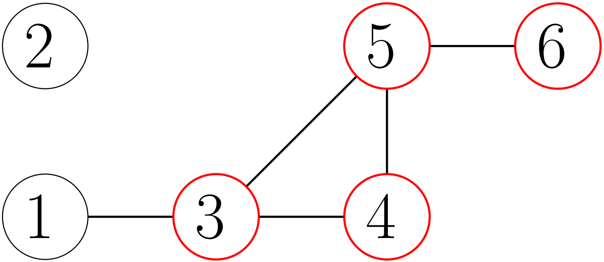
Example Graph

1 2 3 1 4 5 6 5 3 4 6 5



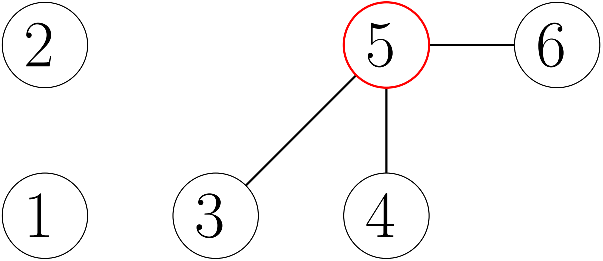
Example Graph

1 2 3 1 4 5 6 5 3 4 6 5



Example Graph

$$1\,2\,3\,1\,4\,5\,6\,5\,3\,4\,6\,5$$



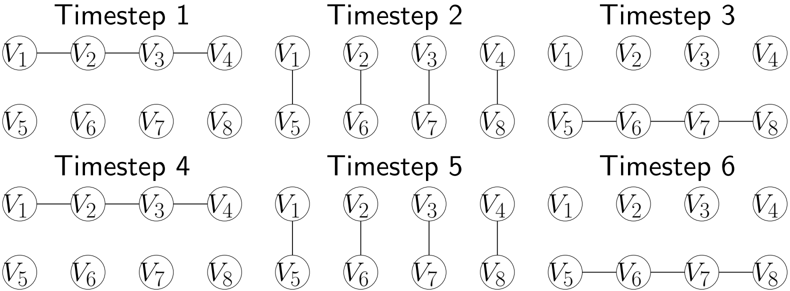
Why are we interested in these graphs?

- In many networks, a node is either **active** or **inactive** (a phone is on or off, a server is connected or disconnected).
- When active, we can assume, in a simple model, that all connections are available.
- Word-representable temporal graphs provide a subclass of such graphs that can be represented in a **compact manner**.

Temporal Walks

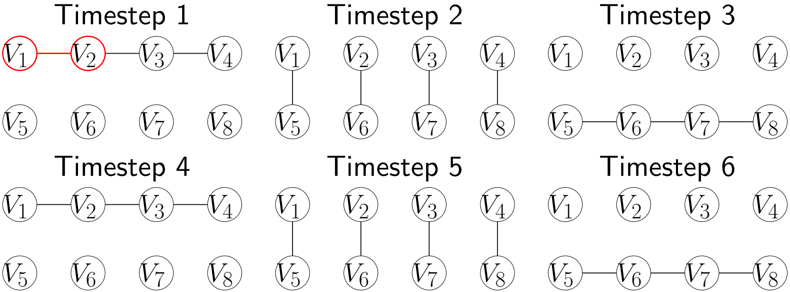
- A **temporal walk** is the temporal analogue of a walk in a static graph.
- A temporal walk on the temporal graph $\mathcal{G}$ is an ordered sequence of edges, $e_1, e_2, \ldots, e_k$, and timesteps $t_1, t_2, \ldots, t_k$ such that:
  - $e_1, e_2, \ldots, e_k$ form a walk in $U(\mathcal{G})$, and,
  - e_i is active in timestep t_i,
  - $1 \leq t_1 < t_2 < \cdots < t_k \leq T$
- Given an agent following a temporal walk, if there is some timestep t that does not appear in the sequence of timesteps, we say the agent is **waiting**.

Temporal Walks Example



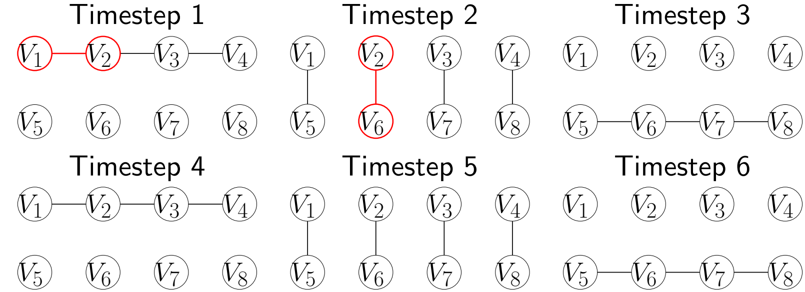
Temporal Walks Example

$((v_1, v_2), 1)$



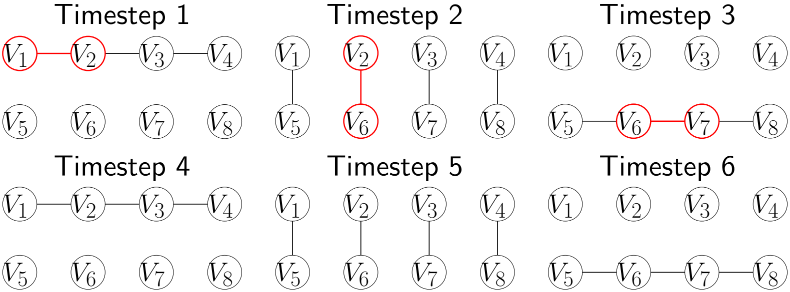
Temporal Walks Example

$((v_1, v_2), 1), ((v_2, v_6), 2)$



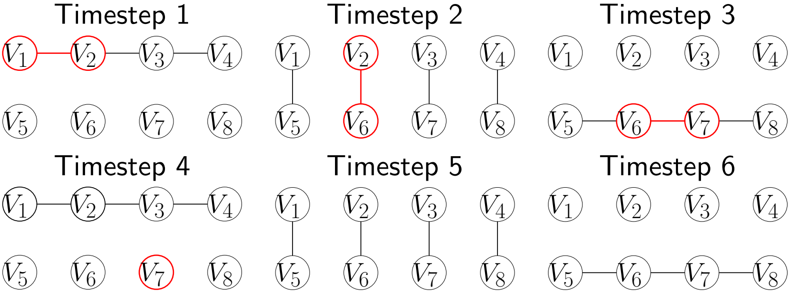
Temporal Walks Example

$((v_1, v_2), 1), ((v_2, v_6), 2), ((v_6, v_7), 3)$



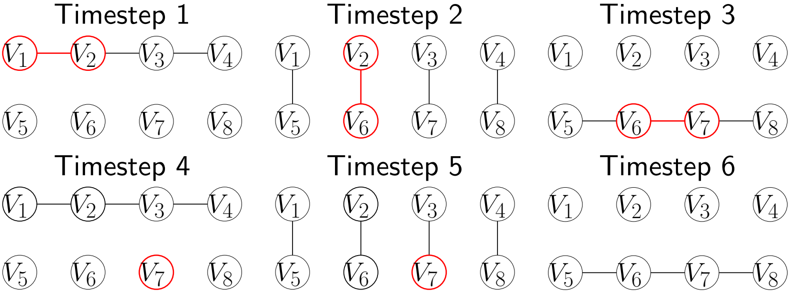
Temporal Walks Example

$((v_1, v_2), 1), ((v_2, v_6), 2), ((v_6, v_7), 3)$



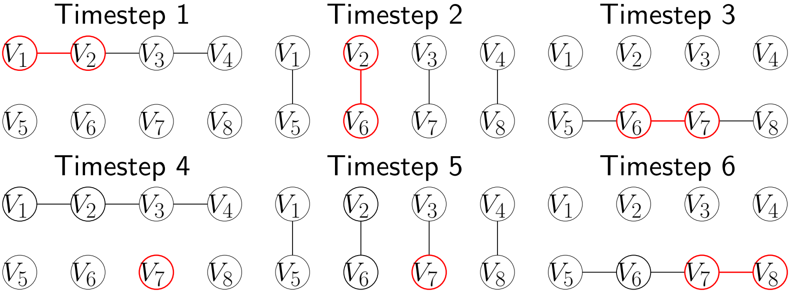
Temporal Walks Example

$((v_1, v_2), 1), ((v_2, v_6), 2), ((v_6, v_7), 3)$



Temporal Walks Example

$((v_1, v_2), 1), ((v_2, v_6), 2), ((v_6, v_7), 3), ((v_7, v_8), 6)$



Problems on Temporal Graphs

- Most algorithmic work on temporal graphs focuses on questions of **reachability** and **exploration**.
- **Reachability:** Given a temporal graph $\mathcal{G}$ and pair of vertices v_i, v_j, does there exist a temporal walk from v_i to v_j?
  - Optimisation Variant: What is the earliest an agent starting at v_i can reach v_j?
- **Exploration:** Given a temporal graph $\mathcal{G}$, does there exist a temporal walk visiting every vertex at least once?
  - Optimisation Variant: What is the earliest an agent starting at v_i can visit every vertex in the graph?

Always Connected Graphs

- A temporal graph is *Always Connected* if every timestep contains only a single connected-component.
- A word-representable temporal graph is always connected if the corresponding temporal graph is always connected.
- It is known that any always connected temporal graph with n vertices and a lifespan of at least n^2 can be explored, and that this bound is tight.

Exploring Always Connected Temporal Graphs

Theorem

Let $w \in \Sigma^*$ be a word representing an always connected temporal graph $\mathcal{G}(w) = (V, E_1, E_2, \ldots, E_T)$ with a minimum degree of δ and $T \geq 2\delta\sigma$. Then, there exists a temporal walk exploring $\mathcal{G}$.

Outline

High Level Idea:

- We find a walk exploring the **underlying graph**.
- A **Eulerian tour** of a **spanning tree** gives such a walk of length at most $2n - 1$.

Problem:

- Transforming this into a temporal walk requires these edges to be active in order.

Lemma (Informal)

Given any $t \in [1, T - \delta]$, $E = \bigcup_{d \in [\delta]} E_{t+d}$.

Sketch

- Let $w = u_1 u_2 \dots u_T$, $\mathcal{G}(w) = (V, E_1, E_2, \dots, E_T)$, and $G(w) = (V, E)$.
- To be always connected, there needs to be, for every vertex $v_i \in V$ and $t \in [1, T]$, some $v_j \in V$ s.t. $(v_i, v_j) \in E_t$.
  - By extension, $\{j \in [1, \sigma] \mid i = j \text{ or } (v_i, v_j) \in E\} \cap \text{alph}(u_t) \neq \emptyset$,
- As $(v_i, v_j) \in E$ iff i and j alternate, then, for every $t \in [1, T - deg(v_i) - 1]$, $i \in \text{alph}(u_t u_{t+1} \dots u_{t+deg(v_i)+1})$.
  - Informally, the letter i has to appear at least once every $deg(v_i)$ timesteps.

Sketch (part 2)

- Now, if $(v_i, v_j) \in E$, then, if i appears at least once every $deg(v_i)$ timesteps, then j must also appear at least once every $deg(v_i)$ consecutive timesteps words.
- Similarly, if j appears at least once every $deg(v_i)$ timesteps, then i must also appear at least once every $deg(v_j)$ consecutive timestep words.
- So, i appears at least once every $\min(deg(v_i), deg(v_j))$ timestep words.
- In general, i appears at least once every

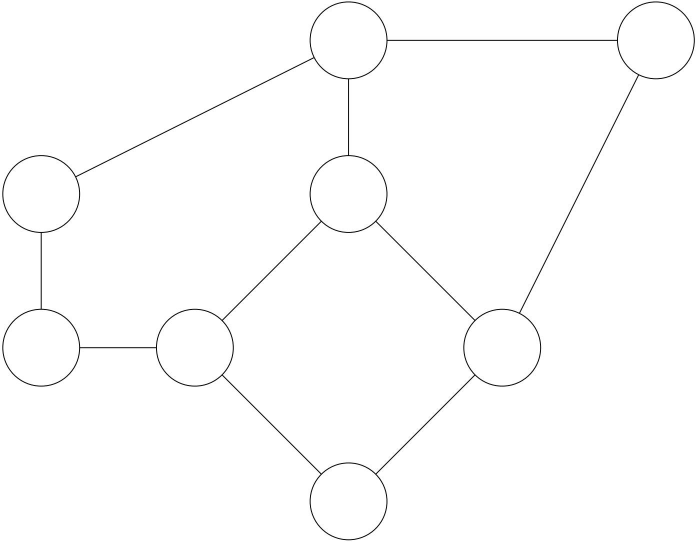
$$\min\left(deg(v_i), \min_{v_j \in N(v_i)}(deg(v_j))\right)$$

consecutive timestep words.

Sketch (part 3)

- Extrapolating out, if $(v_i, v_j) \in E$, and j appears in every d consecutive timestep words, then i appears (at least once) in every d consecutive timestep words.
- So, if there exists a path between v_i and v_k, and k appears in every d' consecutive timestep words, i appears (at least once) in every d' consecutive timestep words.
- Hence, every symbol is active at least once every $\delta + 1$ timesteps.
- By extension, each edge is active at least once every δ timesteps.

Example



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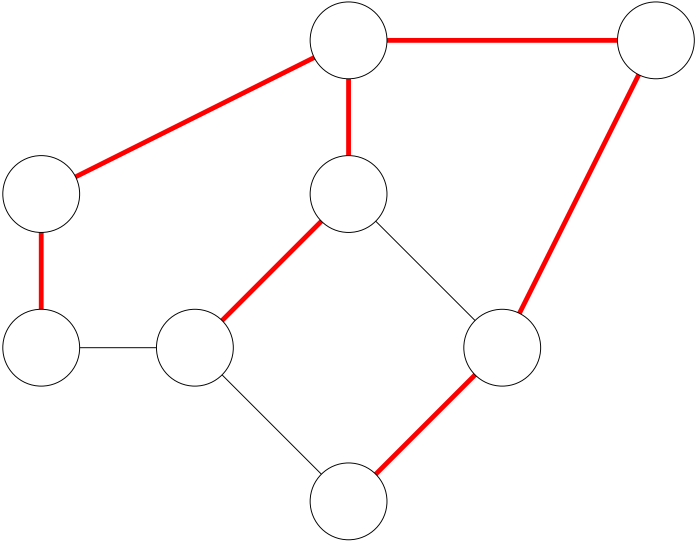

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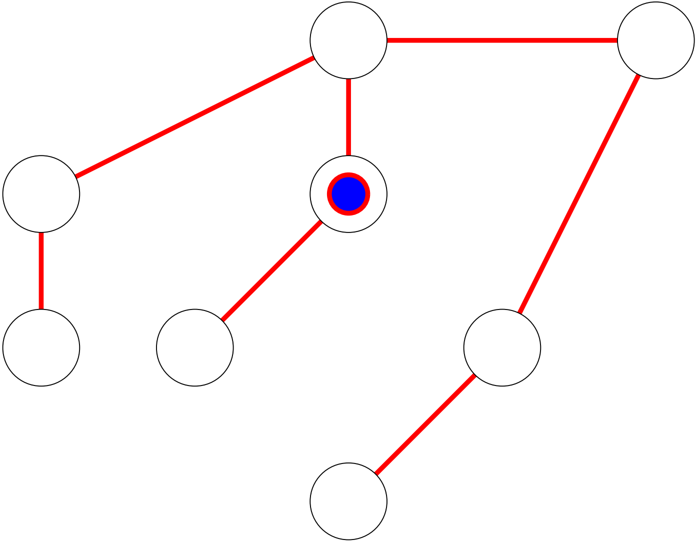

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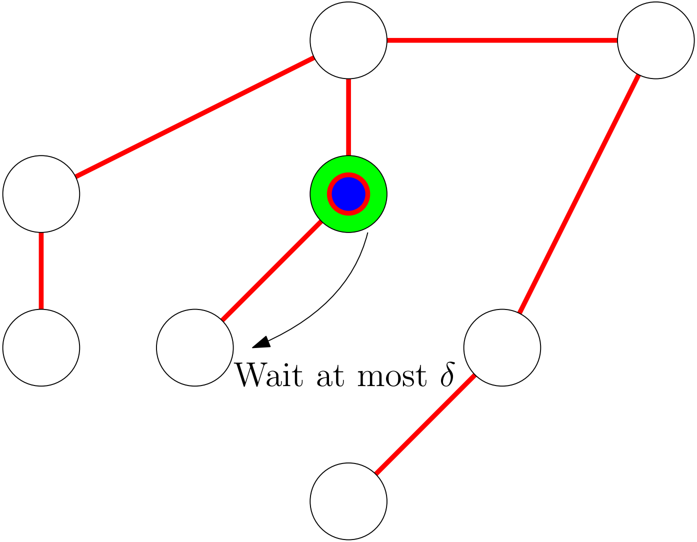

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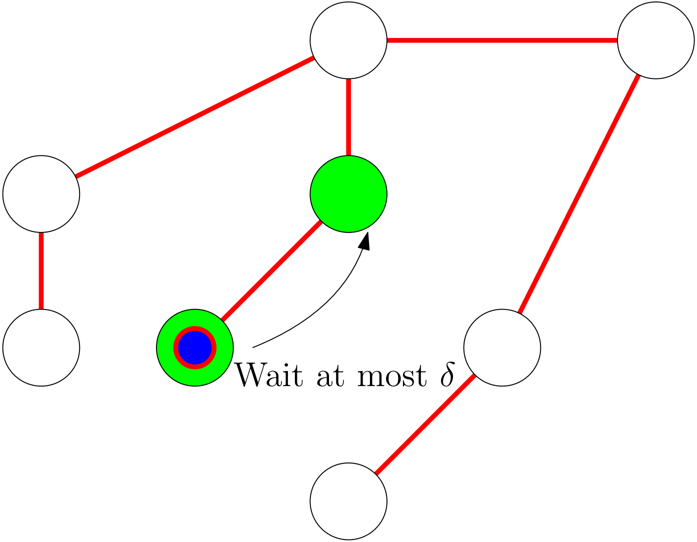

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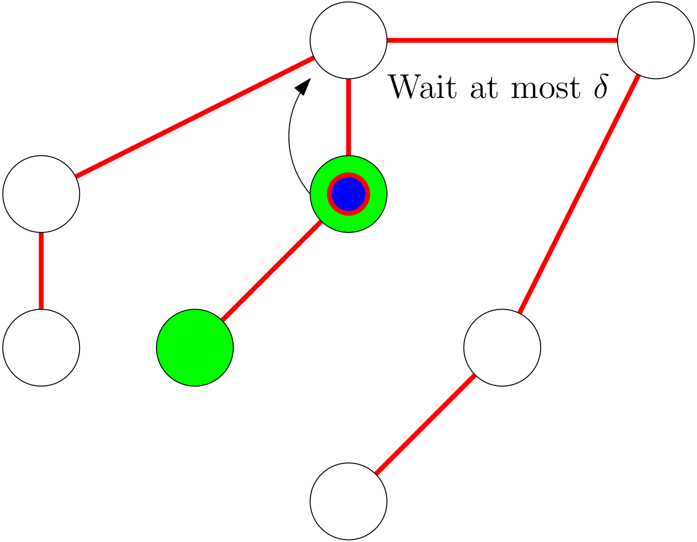

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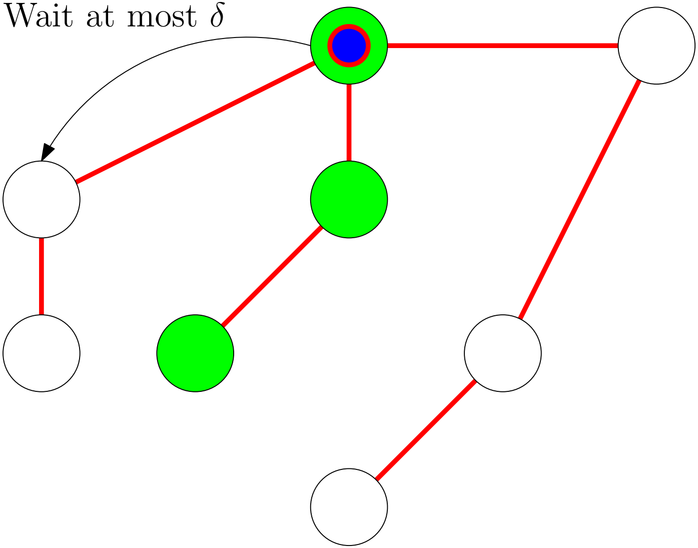

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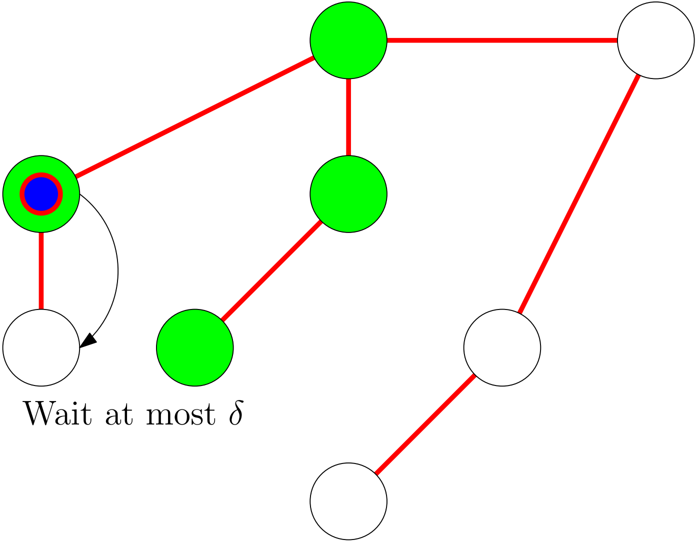
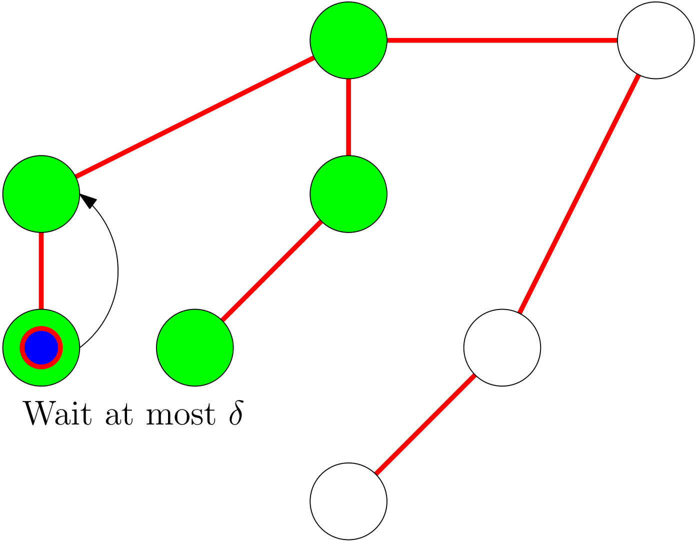


Wait at most δ

Example



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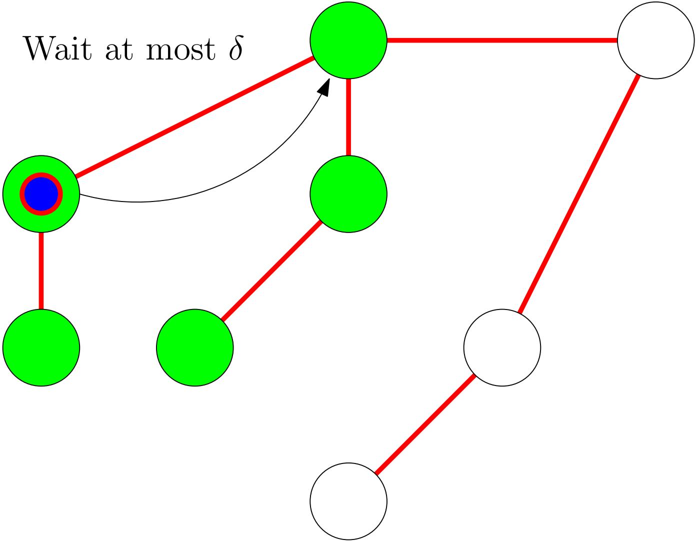

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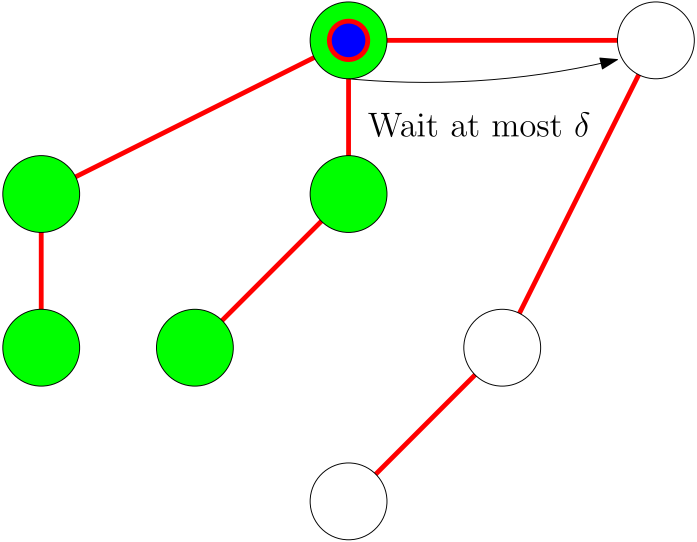

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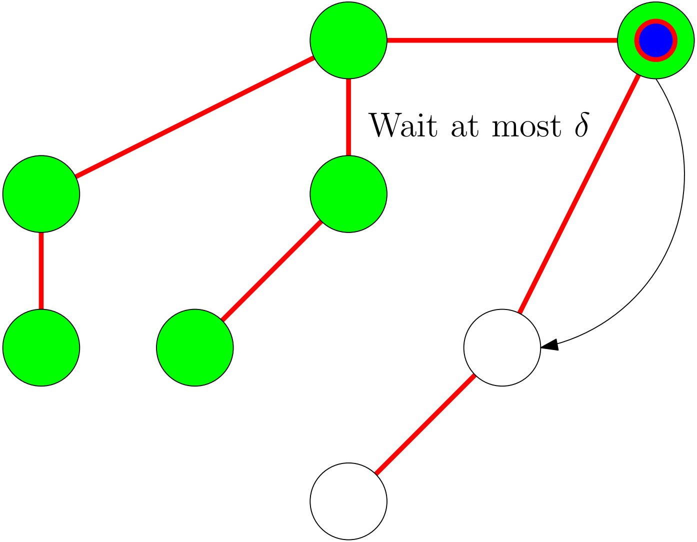

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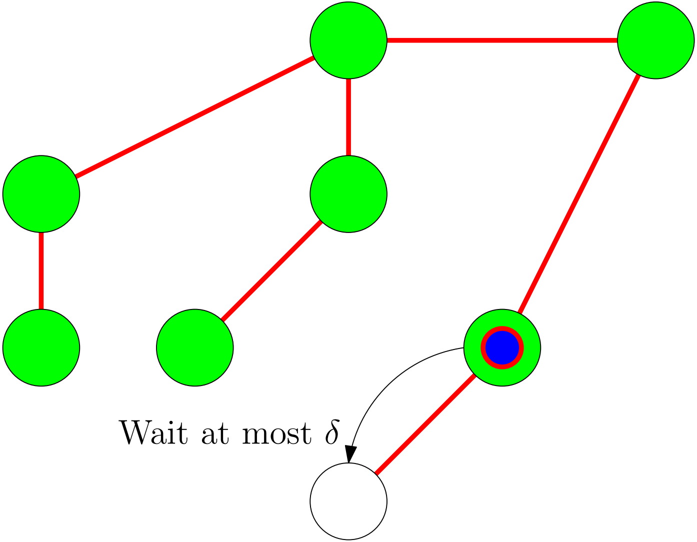

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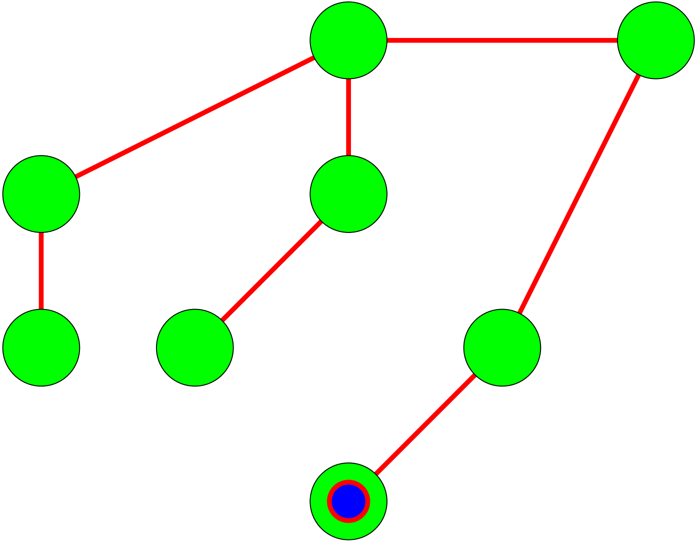

Connected Graphs

Theorem

Let $w \in \Sigma^*$ be a word representing a connected temporal graph $\mathcal{G}(w) = (V, E_1, E_2, \ldots, E_T)$ with a diameter of d and $T \geq 2d\sigma$. Then, there exists a temporal walk exploring $\mathcal{G}$.

Theorem

Given some number of vertices σ and diameter $d < \sigma$, there exists a word w representing a connected graph requiring $d\sigma/2$ timesteps to explore.

Next Steps

- Is the $O(\delta\sigma)$ algorithm asymptotically optimal for exploring word-representable always connected temporal graphs?
- Is the $2d\sigma$ bound for exploring word-representable connected temporal graph tight?
- What kind of graphs can be represented at each timestep?
- Thank you for listening!