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Riemann hypothesis

In mathematics, the **Riemann hypothesis** is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. Many consider it to be the most important unsolved problem in pure mathematics (Bombieri 2000). It is of great interest in number theory because it implies results about the distribution of prime numbers. It was proposed by Bernhard Riemann (1859), after whom it is named.

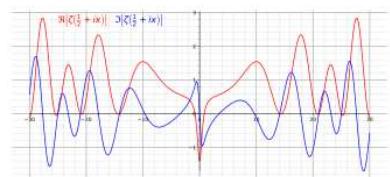
The Riemann hypothesis and some of its generalizations, along with Goldbach's conjecture and the twin prime conjecture, comprise Hilbert's eighth problem in David Hilbert's list of 23 unsolved problems; it is also one of the Clay Mathematics Institute's Millennium Prize Problems. The name is also used for some closely related analogues, such as the Riemann hypothesis for curves over finite fields.

The Riemann zeta function $\zeta(s)$ is a function whose argument s may be any complex number other than 1, and whose values are also complex. It has zeros at the negative even integers; that is, $\zeta(s) = 0$ when s is one of $-2, -4, -6, \dots$. These are called its *trivial zeros*. However, the negative even integers are not the only values for which the zeta function is zero. The other ones are called *nontrivial zeros*. The Riemann hypothesis is concerned with the locations of these nontrivial zeros, and states that:

The real part of every nontrivial zero of the Riemann zeta function is $\frac{1}{2}$.

Thus, if the hypothesis is correct, all the nontrivial zeros lie on the critical line consisting of the complex numbers $\frac{1}{2} + it$, where t is a real number and i is the imaginary unit.

There are several nontechnical books on the Riemann hypothesis, such as Derbyshire (2003), Rockmore (2005), (Sabbagh 2003a, 2003b), du Sautoy (2003), and Watkins (2015). The books Edwards (1974), Patterson (1988), Borwein et al. (2008), Mazur & Stein (2015) and Broughan (2017) give mathematical introductions, while Titchmarsh (1986), Ivić (1985) and Karatsuba & Voronin (1992) are advanced monographs.



The real part (red) and imaginary part (blue) of the Riemann zeta function along the critical line $\text{Re}(s) = 1/2$. The first nontrivial zeros can be seen at $\text{Im}(s) = \pm 14.135, \pm 21.022$ and ± 25.011 .

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Riemann zeta function

The [Riemann zeta function](#) is defined for complex s with real part greater than 1 by the [absolutely convergent infinite series](#)

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \dots$$

Leonhard Euler already considered this series in the 1730s for real values of s , in conjunction with his solution to the [Basel problem](#). He also proved that it equals the [Euler product](#)

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} = \frac{1}{1 - 2^{-s}} \cdot \frac{1}{1 - 3^{-s}} \cdot \frac{1}{1 - 5^{-s}} \cdot \frac{1}{1 - 7^{-s}} \cdot \frac{1}{1 - 11^{-s}} \dots$$

where the infinite product extends over all prime numbers p .^[1]

The Riemann hypothesis discusses zeros outside the region of convergence of this series and Euler product. To make sense of the hypothesis, it is necessary to [analytically continue](#) the function to obtain a form that is valid for all complex s . This is permissible because the zeta function is [meromorphic](#), so its analytic continuation is guaranteed to be unique and functional forms equivalent over their [domains](#). One begins by showing that the zeta function and the [Dirichlet eta function](#) satisfy the relation

$$\left(1 - \frac{2}{2^s}\right) \zeta(s) = \eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \dots$$

But the series on the right converges not just when the real part of s is greater than one, but more generally whenever s has positive real part. Thus, this alternative series extends the zeta function from $\operatorname{Re}(s) > 1$ to the larger domain $\operatorname{Re}(s) > 0$, excluding the zeros $s = 1 + 2\pi i n / \ln(2)$ of $1 - 2/2^s$ where n is any nonzero integer (see [Dirichlet eta function](#)). The zeta function can be extended to these values too by taking limits, giving a finite value for all values of s with positive real part except for the [simple pole](#) at $s = 1$.

In the strip $0 < \operatorname{Re}(s) < 1$ the zeta function satisfies the [functional equation](#)

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s).$$

One may then define $\zeta(s)$ for all remaining nonzero complex numbers s ($\operatorname{Re}(s) \leq 0$ and $s \neq 0$) by applying this equation outside the strip, and letting $\zeta(s)$ equal the right-hand side of the equation whenever s has non-positive real part (and $s \neq 0$).

If s is a negative even integer then $\zeta(s) = 0$ because the factor $\sin(\pi s/2)$ vanishes; these are the [trivial zeros](#) of the zeta function. (If s is a positive even integer this argument does not apply because the zeros of the [sine](#) function are cancelled by the poles of the [gamma function](#) as it takes negative integer arguments.)

The value $\zeta(0) = -1/2$ is not determined by the functional equation, but is the limiting value of $\zeta(s)$ as s approaches zero. The functional equation also implies that the zeta function has no zeros with negative real part other than the trivial zeros, so all non-trivial zeros lie in the critical strip where s has real part between 0 and 1.

Origin

...es ist sehr wahrscheinlich, dass alle Wurzeln reell sind. Hiervon wäre allerdings ein strenger Beweis zu wünschen; ich habe indess die Aufsuchung desselben nach einigen flüchtigen vergeblichen Versuchen vorläufig bei Seite gelassen, da er für den nächsten Zweck meiner Untersuchung entbehrlich schien.

...it is very probable that all roots are real. Of course one would wish for a rigorous proof here; I have for the time being, after some fleeting vain attempts, provisionally put aside the search for this, as it appears dispensable for the immediate objective of my investigation.

— Riemann's statement of the Riemann hypothesis, from [\(Riemann 1859\)](#). (He was discussing a version of the zeta function, modified so that its roots (zeros) are real rather than on the critical line.)

Riemann's original motivation for studying the zeta function and its zeros was their occurrence in his [explicit formula](#) for the [number of primes](#) $\pi(x)$ less than or equal to a given number x , which he published in his 1859 paper "[On the Number of Primes Less Than a Given Magnitude](#)". His formula was given in terms of the related function

$$\Pi(x) = \pi(x) + \frac{1}{2}\pi(x^{\frac{1}{2}}) + \frac{1}{3}\pi(x^{\frac{1}{3}}) + \frac{1}{4}\pi(x^{\frac{1}{4}}) + \frac{1}{5}\pi(x^{\frac{1}{5}}) + \frac{1}{6}\pi(x^{\frac{1}{6}}) + \dots$$

which counts the primes and prime powers up to x , counting a prime power p^n as $\frac{1}{n}$. The number of primes can be recovered from this function by using the [Möbius inversion formula](#),

$$\begin{aligned} \pi(x) &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \Pi(x^{\frac{1}{n}}) \\ &= \Pi(x) - \frac{1}{2}\Pi(x^{\frac{1}{2}}) - \frac{1}{3}\Pi(x^{\frac{1}{3}}) - \frac{1}{5}\Pi(x^{\frac{1}{5}}) + \frac{1}{6}\Pi(x^{\frac{1}{6}}) - \dots, \end{aligned}$$

where μ is the [Möbius function](#). Riemann's formula is then

$$\Pi_0(x) = \text{li}(x) - \sum_{\rho} \text{li}(x^{\rho}) - \log(2) + \int_x^{\infty} \frac{dt}{t(t^2 - 1) \log(t)}$$

where the sum is over the nontrivial zeros of the zeta function and where Π_0 is a slightly modified version of Π that replaces its value at its points of discontinuity by the average of its upper and lower limits:

$$\Pi_0(x) = \lim_{\varepsilon \rightarrow 0} \frac{\Pi(x - \varepsilon) + \Pi(x + \varepsilon)}{2}.$$

The summation in Riemann's formula is not absolutely convergent, but may be evaluated by taking the zeros ρ in order of the absolute value of their imaginary part. The function li occurring in the first term is the (unoffset) logarithmic integral function given by the Cauchy principal value of the divergent integral

$$\text{li}(x) = \int_0^x \frac{dt}{\log(t)}.$$

The terms $\text{li}(x^{\rho})$ involving the zeros of the zeta function need some care in their definition as li has branch points at 0 and 1, and are defined (for $x > 1$) by analytic continuation in the complex variable ρ in the region $\text{Re}(\rho) > 0$, i.e. they should be considered as $\text{Ei}(\rho \ln x)$. The other terms also correspond to zeros: the dominant term $\text{li}(x)$ comes from the pole at $s = 1$, considered as a zero of multiplicity -1 , and the remaining small terms come from the trivial zeros. For some graphs of the sums of the first few terms of this series see Riesel & Göhl (1970) or Zagier (1977).

This formula says that the zeros of the Riemann zeta function control the oscillations of primes around their "expected" positions. Riemann knew that the non-trivial zeros of the zeta function were symmetrically distributed about the line $s = 1/2 + it$, and he knew that all of its non-trivial zeros must lie in the range $0 \leq \text{Re}(s) \leq 1$. He checked that a few of the zeros lay on the critical line with real part $1/2$ and suggested that they all do; this is the Riemann hypothesis.

The result has caught the imagination of most mathematicians because it is so unexpected, connecting two seemingly unrelated areas in mathematics; namely, number theory, which is the study of the discrete, and complex analysis, which deals with continuous processes. (Burton 2006, p. 376)

Consequences

The practical uses of the Riemann hypothesis include many propositions known true under the Riemann hypothesis, and some that can be shown to be equivalent to the Riemann hypothesis.

Distribution of prime numbers

Riemann's explicit formula for the number of primes less than a given number in terms of a sum over the zeros of the Riemann zeta function says that the magnitude of the oscillations of primes around their expected position is controlled by the real parts of the zeros of the zeta function. In particular the error term in the prime number theorem is closely related to the position of the zeros. For example, if β is the upper bound of the real parts of the zeros, then (Ingham 1932):^{Theorem 30, p.83}, (Montgomery & Vaughan 2007):^{p. 430}

$$\pi(x) - \text{li}(x) = O(x^{\beta} \log x).$$

It is already known that $1/2 \leq \beta \leq 1$ (Ingham 1932).^{:p. 82}

Von Koch (1901) proved that the Riemann hypothesis implies the "best possible" bound for the error of the prime number theorem. A precise version of Koch's result, due to Schoenfeld (1976), says that the Riemann hypothesis implies

$$|\pi(x) - \text{li}(x)| < \frac{1}{8\pi} \sqrt{x \log(x)}, \quad \text{for all } x \geq 2657,$$

where $\pi(x)$ is the prime-counting function, and $\log(x)$ is the natural logarithm of x .

Schoenfeld (1976) also showed that the Riemann hypothesis implies

$$|\psi(x) - x| < \frac{1}{8\pi} \sqrt{x} \log^2(x), \quad \text{for all } x \geq 73.2,$$

where $\psi(x)$ is Chebyshev's second function.

Dudek (2014) proved that the Riemann hypothesis implies that for all $x \geq 2$ there is a prime p satisfying

$$x - \frac{4}{\pi} \sqrt{x} \log x < p \leq x.$$

This is an explicit version of a theorem of Cramér.

Growth of arithmetic functions

The Riemann hypothesis implies strong bounds on the growth of many other arithmetic functions, in addition to the primes counting function above.

One example involves the Möbius function μ . The statement that the equation

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

is valid for every s with real part greater than $1/2$, with the sum on the right hand side converging, is equivalent to the Riemann hypothesis. From this we can also conclude that if the Mertens function is defined by

$$M(x) = \sum_{n \leq x} \mu(n)$$

then the claim that

$$M(x) = O\left(x^{\frac{1}{2} + \varepsilon}\right)$$

for every positive ε is equivalent to the Riemann hypothesis (J.E. Littlewood, 1912; see for instance: paragraph 14.25 in Titchmarsh (1986)). (For the meaning of these symbols, see Big O notation.) The determinant of the order n Redheffer matrix is equal to $M(n)$, so the Riemann hypothesis can also be stated as a condition on the growth of these determinants. The Riemann hypothesis puts a rather tight bound on the growth of M , since Odlyzko & te Riele (1985) disproved the slightly stronger Mertens conjecture

$$|M(x)| \leq \sqrt{x}.$$

The Riemann hypothesis is equivalent to many other conjectures about the rate of growth of other arithmetic functions aside from $\mu(n)$. A typical example is Robin's theorem (Robin 1984), which states that if $\sigma(n)$ is the divisor function, given by

$$\sigma(n) = \sum_{d|n} d$$

then

$$\sigma(n) < e^\gamma n \log \log n$$

for all $n > 5040$ if and only if the Riemann hypothesis is true, where γ is the Euler–Mascheroni constant.

Another example was found by Jérôme Franel, and extended by Landau (see Franel & Landau (1924)). The Riemann hypothesis is equivalent to several statements showing that the terms of the Farey sequence are fairly regular. One such equivalence is as follows: if F_n is the Farey sequence of order n , beginning with $1/n$ and up to $1/1$, then the claim that for all $\varepsilon > 0$

$$\sum_{i=1}^m |F_n(i) - \frac{i}{m}| = O(n^{\frac{1}{2} + \varepsilon})$$

is equivalent to the Riemann hypothesis. Here

$$m = \sum_{i=1}^n \phi(i)$$

is the number of terms in the Farey sequence of order n .

For an example from group theory, if $g(n)$ is Landau's function given by the maximal order of elements of the symmetric group S_n of degree n , then Massias, Nicolas & Robin (1988) showed that the Riemann hypothesis is equivalent to the bound

$$\log g(n) < \sqrt{\text{Li}^{-1}(n)}$$

for all sufficiently large n .

Lindelöf hypothesis and growth of the zeta function

The Riemann hypothesis has various weaker consequences as well; one is the Lindelöf hypothesis on the rate of growth of the zeta function on the critical line, which says that, for any $\varepsilon > 0$,

$$\zeta\left(\frac{1}{2} + it\right) = O(t^\varepsilon),$$

as $t \rightarrow \infty$.

The Riemann hypothesis also implies quite sharp bounds for the growth rate of the zeta function in other regions of the critical strip. For example, it implies that

$$\begin{aligned} e^\gamma &\leq \limsup_{t \rightarrow +\infty} \frac{|\zeta(1+it)|}{\log \log t} \leq 2e^\gamma \\ \frac{6}{\pi^2} e^\gamma &\leq \limsup_{t \rightarrow +\infty} \frac{1/|\zeta(1+it)|}{\log \log t} \leq \frac{12}{\pi^2} e^\gamma \end{aligned}$$

so the growth rate of $\zeta(1+it)$ and its inverse would be known up to a factor of 2 (Titchmarsh 1986).

Large prime gap conjecture

The prime number theorem implies that on average, the gap between the prime p and its successor is $\log p$. However, some gaps between primes may be much larger than the average. Cramér proved that, assuming the Riemann hypothesis, every gap is $O(\sqrt{p} \log p)$. This is a case in which even the best bound that can be proved using the Riemann hypothesis is far weaker than what seems true: Cramér's conjecture implies that every gap is $O((\log p)^2)$, which, while larger than the average gap, is far smaller than the bound implied by the Riemann hypothesis. Numerical evidence supports Cramér's conjecture (Nicely 1999).

Analytic criteria equivalent to the Riemann hypothesis

Many statements equivalent to the Riemann hypothesis have been found, though so far none of them have led to much progress in proving (or disproving) it. Some typical examples are as follows. (Others involve the divisor function $\sigma(n)$.)

The Riesz criterion was given by Riesz (1916), to the effect that the bound

$$-\sum_{k=1}^{\infty} \frac{(-x)^k}{(k-1)!\zeta(2k)} = O\left(x^{\frac{1}{4}+\epsilon}\right)$$

holds for all $\epsilon > 0$ if and only if the Riemann hypothesis holds.

Nyman (1950) proved that the Riemann hypothesis is true if and only if the space of functions of the form

$$f(x) = \sum_{\nu=1}^n c_{\nu} \rho\left(\frac{\theta_{\nu}}{x}\right)$$

where $\rho(z)$ is the fractional part of z , $0 \leq \theta_{\nu} \leq 1$, and

$$\sum_{\nu=1}^n c_{\nu} \theta_{\nu} = 0,$$

is dense in the Hilbert space $L^2(0,1)$ of square-integrable functions on the unit interval. Beurling (1955) extended this by showing that the zeta function has no zeros with real part greater than $1/p$ if and only if this function space is dense in $L^p(0,1)$

Salem (1953) showed that the Riemann hypothesis is true if and only if the integral equation

$$\int_0^\infty \frac{z^{-\sigma-1} \phi(z)}{e^{x/z} + 1} dz = 0$$

has no non-trivial bounded solutions ϕ for $1/2 < \sigma < 1$.

Weil's criterion is the statement that the positivity of a certain function is equivalent to the Riemann hypothesis. Related is Li's criterion, a statement that the positivity of a certain sequence of numbers is equivalent to the Riemann hypothesis.

Speiser (1934) proved that the Riemann hypothesis is equivalent to the statement that $\zeta'(s)$, the derivative of $\zeta(s)$, has no zeros in the strip

$$0 < \Re(s) < \frac{1}{2}.$$

That $\zeta(s)$ has only simple zeros on the critical line is equivalent to its derivative having no zeros on the critical line.

The Farey sequence provides two equivalences, due to Jerome Franel and Edmund Landau in 1924.

Consequences of the generalized Riemann hypothesis

Several applications use the generalized Riemann hypothesis for Dirichlet L-series or zeta functions of number fields rather than just the Riemann hypothesis. Many basic properties of the Riemann zeta function can easily be generalized to all Dirichlet L-series, so it is plausible that a method that proves the Riemann hypothesis for the Riemann zeta function would also work for the generalized Riemann hypothesis for Dirichlet L-functions. Several results first proved using the generalized Riemann hypothesis were later given unconditional proofs without using it, though these were usually much harder. Many of the consequences on the following list are taken from Conrad (2010).

- In 1913, Grönwall showed that the generalized Riemann hypothesis implies that Gauss's list of imaginary quadratic fields with class number 1 is complete, though Baker, Stark and Heegner later gave unconditional proofs of this without using the generalized Riemann hypothesis.

- In 1917, Hardy and Littlewood showed that the generalized Riemann hypothesis implies a conjecture of Chebyshev that

$$\lim_{x \rightarrow 1^-} \sum_{p>2} (-1)^{(p+1)/2} p^x = +\infty,$$

which says that primes $3 \pmod{4}$ are more common than primes $1 \pmod{4}$ in some sense. (For related results, see [Prime number theorem](#) § [Prime number race](#).)

- In 1923 Hardy and Littlewood showed that the generalized Riemann hypothesis implies a weak form of the Goldbach conjecture for odd numbers: that every sufficiently large odd number is the sum of three primes, though in 1937 Vinogradov gave an unconditional proof. In 1997 Deshouillers, Effinger, te Riele, and Zinoviev showed that the generalized Riemann hypothesis implies that every odd number greater than 5 is the sum of three primes. In 2013 Harald Helfgott proved the ternary Goldbach conjecture without the GRH dependence, subject to some extensive calculations completed with the help of David J. Platt.
- In 1934, Chowla showed that the generalized Riemann hypothesis implies that the first prime in the arithmetic progression $a \pmod{m}$ is at most $Km^2 \log(m)^2$ for some fixed constant K .
- In 1967, Hooley showed that the generalized Riemann hypothesis implies Artin's conjecture on primitive roots.
- In 1973, Weinberger showed that the generalized Riemann hypothesis implies that Euler's list of idoneal numbers is complete.
- [Weinberger \(1973\)](#) showed that the generalized Riemann hypothesis for the zeta functions of all algebraic number fields implies that any number field with class number 1 is either Euclidean or an imaginary quadratic number field of discriminant $-19, -43, -67$, or -163 .
- In 1976, G. Miller showed that the generalized Riemann hypothesis implies that one can test if a number is prime in polynomial time via the [Miller test](#). In 2002, Manindra Agrawal, Neeraj Kayal and Nitin Saxena proved this result unconditionally using the [AKS primality test](#).
- [Odlyzko \(1990\)](#) discussed how the generalized Riemann hypothesis can be used to give sharper estimates for discriminants and class numbers of number fields.
- [Ono & Soundararajan \(1997\)](#) showed that the generalized Riemann hypothesis implies that [Ramanujan's integral quadratic form](#) $x^2 + y^2 + 10z^2$ represents all integers that it represents locally, with exactly 18 exceptions.

Excluded middle

Some consequences of the RH are also consequences of its negation, and are thus theorems. In their discussion of the Hecke, Deuring, Mordell, Heilbronn theorem, ([Ireland & Rosen 1990](#), p. 359) say

The method of proof here is truly amazing. If the generalized Riemann hypothesis is true, then the theorem is true. If the generalized Riemann hypothesis is false, then the theorem is true. Thus, the theorem is true!! (punctuation in original)

Care should be taken to understand what is meant by saying the generalized Riemann hypothesis is false: one should specify exactly which class of Dirichlet series has a counterexample.

Littlewood's theorem

This concerns the sign of the error in the prime number theorem. It has been computed that $\pi(x) < \text{li}(x)$ for all $x \leq 10^{25}$ (see [this table](#)), and no value of x is known for which $\pi(x) > \text{li}(x)$.

In 1914 Littlewood proved that there are arbitrarily large values of x for which

$$\pi(x) > \text{li}(x) + \frac{1}{3} \frac{\sqrt{x}}{\log x} \log \log \log x,$$

and that there are also arbitrarily large values of x for which

$$\pi(x) < \text{li}(x) - \frac{1}{3} \frac{\sqrt{x}}{\log x} \log \log \log x.$$

Thus the difference $\pi(x) - \text{li}(x)$ changes sign infinitely many times. Skewes' number is an estimate of the value of x corresponding to the first sign change.

Littlewood's proof is divided into two cases: the RH is assumed false (about half a page of Ingham 1932, Chapt. V), and the RH is assumed true (about a dozen pages). Stanisław Knapowski followed this up and published a paper on the number of times $\Delta(n)$ changes sign in the interval $\Delta(n)$.^[2]

Gauss's class number conjecture

This is the conjecture (first stated in article 303 of Gauss's *Disquisitiones Arithmeticae*) that there are only finitely many imaginary quadratic fields with a given class number. One way to prove it would be to show that as the discriminant $D \rightarrow -\infty$ the class number $h(D) \rightarrow \infty$.

The following sequence of theorems involving the Riemann hypothesis is described in Ireland & Rosen 1990, pp. 358–361:

Theorem (Hecke; 1918). Let $D < 0$ be the discriminant of an imaginary quadratic number field K . Assume the generalized Riemann hypothesis for L -functions of all imaginary quadratic Dirichlet characters. Then there is an absolute constant C such that

$$h(D) > C \frac{\sqrt{|D|}}{\log |D|}.$$

Theorem (Deuring; 1933). If the RH is false then $h(D) > 1$ if $|D|$ is sufficiently large.

Theorem (Mordell; 1934). If the RH is false then $h(D) \rightarrow \infty$ as $D \rightarrow -\infty$.

Theorem (Heilbronn; 1934). If the generalized RH is false for the L -function of some imaginary quadratic Dirichlet character then $h(D) \rightarrow \infty$ as $D \rightarrow -\infty$.

(In the work of Hecke and Heilbronn, the only L -functions that occur are those attached to imaginary quadratic characters, and it is only for those L -functions that *GRH is true* or *GRH is false* is intended; a failure of GRH for the L -function of a cubic Dirichlet character would, strictly speaking, mean GRH is false, but that was not the kind of failure of GRH that Heilbronn had in mind, so his assumption was more restricted than simply *GRH is false*.)

In 1935, Carl Siegel later strengthened the result without using RH or GRH in any way.

Growth of Euler's totient

In 1983 J. L. Nicolas proved (Ribenboim 1996, p. 320) that

$$\varphi(n) < e^{-\gamma} \frac{n}{\log \log n}$$

for infinitely many n , where $\varphi(n)$ is Euler's totient function and γ is Euler's constant.

Ribenboim remarks that:

The method of proof is interesting, in that the inequality is shown first under the assumption that the Riemann hypothesis is true, secondly under the contrary assumption.

Generalizations and analogs

Dirichlet L-series and other number fields

The Riemann hypothesis can be generalized by replacing the Riemann zeta function by the formally similar, but much more general, global L-functions. In this broader setting, one expects the non-trivial zeros of the global L-functions to have real part 1/2. It is these conjectures, rather than the classical Riemann hypothesis only for the single Riemann zeta function, which account for the true importance of the Riemann hypothesis in mathematics.

The generalized Riemann hypothesis extends the Riemann hypothesis to all Dirichlet L-functions. In particular it implies the conjecture that Siegel zeros (zeros of L-functions between 1/2 and 1) do not exist.

The extended Riemann hypothesis extends the Riemann hypothesis to all Dedekind zeta functions of algebraic number fields. The extended Riemann hypothesis for abelian extension of the rationals is equivalent to the generalized Riemann hypothesis. The Riemann hypothesis can also be extended to the L-functions of Hecke characters of number fields.

The grand Riemann hypothesis extends it to all automorphic zeta functions, such as Mellin transforms of Hecke eigenforms.

Function fields and zeta functions of varieties over finite fields

Artin (1924) introduced global zeta functions of (quadratic) function fields and conjectured an analogue of the Riemann hypothesis for them, which has been proved by Hasse in the genus 1 case and by Weil (1948) in general. For instance, the fact that the Gauss sum, of the quadratic character of a finite field of size q (with q odd), has absolute value \sqrt{q} is actually an instance of the Riemann hypothesis in the function field setting. This led Weil (1949) to conjecture a similar statement for all algebraic varieties; the resulting Weil conjectures were proved by Pierre Deligne (1974, 1980).

Arithmetic zeta functions of arithmetic schemes and their L-factors

Arithmetic zeta functions generalise the Riemann and Dedekind zeta functions as well as the zeta functions of varieties over finite fields to every arithmetic scheme or a scheme of finite type over integers. The arithmetic zeta function of a regular connected equidimensional arithmetic scheme of Kronecker dimension n can be factorized into the product of appropriately defined L-factors and an auxiliary factor Jean-Pierre Serre (1969–1970). Assuming a functional equation and meromorphic continuation, the generalized Riemann hypothesis for the L-factor states that its zeros inside the critical strip $\Re(s) \in (0, n)$ lie on the central line. Correspondingly, the generalized Riemann hypothesis for the arithmetic zeta function of a regular connected equidimensional arithmetic scheme states that its zeros inside the critical strip lie on vertical lines $\Re(s) = 1/2, 3/2, \dots, n - 1/2$ and its poles inside the critical strip lie on vertical lines $\Re(s) = 1, 2, \dots, n - 1$. This is known for schemes in positive characteristic and follows from Pierre Deligne (1974, 1980), but remains entirely unknown in characteristic zero.

Selberg zeta functions

Selberg (1956) introduced the Selberg zeta function of a Riemann surface. These are similar to the Riemann zeta function: they have a functional equation, and an infinite product similar to the Euler product but taken over closed geodesics rather than primes. The Selberg trace formula is the analogue for these functions of the explicit formulas in prime number theory. Selberg proved that the Selberg zeta functions satisfy the analogue of the Riemann hypothesis, with the imaginary parts of their zeros related to the eigenvalues of the Laplacian operator of the Riemann surface.

Ihara zeta functions

The Ihara zeta function of a finite graph is an analogue of the Selberg zeta function, which was first introduced by Yasutaka Ihara in the context of discrete subgroups of the two-by-two p-adic special linear group. A regular finite graph is a Ramanujan graph, a mathematical model of efficient communication networks, if and only if its Ihara zeta function satisfies the analogue of the Riemann hypothesis as was pointed out by T. Sunada.

Montgomery's pair correlation conjecture

Montgomery (1973) suggested the pair correlation conjecture that the correlation functions of the (suitably normalized) zeros of the zeta function should be the same as those of the eigenvalues of a random hermitian matrix. Odlyzko (1987) showed that this is supported by large-scale numerical calculations of these correlation functions.

Montgomery showed that (assuming the Riemann hypothesis) at least $2/3$ of all zeros are simple, and a related conjecture is that all zeros of the zeta function are simple (or more generally have no non-trivial integer linear relations between their imaginary parts). Dedekind zeta functions of algebraic number fields, which generalize the Riemann zeta function, often do have multiple complex zeros (Radziejewski 2007). This is because the Dedekind zeta functions factorize as a product of powers of Artin L-functions, so zeros of Artin L-functions sometimes give rise to multiple zeros of Dedekind zeta functions. Other examples of zeta functions with multiple zeros are the L-functions of some elliptic curves: these can have multiple zeros at the real point of their critical line; the Birch-Swinnerton-Dyer conjecture predicts that the multiplicity of this zero is the rank of the elliptic curve.

Other zeta functions

There are many other examples of zeta functions with analogues of the Riemann hypothesis, some of which have been proved. Goss zeta functions of function fields have a Riemann hypothesis, proved by Sheats (1998). The main conjecture of Iwasawa theory, proved by Barry Mazur and Andrew Wiles for cyclotomic fields, and Wiles for totally real fields, identifies the zeros of a p -adic L-function with the eigenvalues of an operator, so can be thought of as an analogue of the Hilbert–Pólya conjecture for p -adic L-functions (Wiles 2000).

Attempted proofs

Several mathematicians have addressed the Riemann hypothesis, but none of their attempts has yet been accepted as a proof. Watkins (2007) lists some incorrect solutions, and more are frequently announced (https://arxiv.org/find/grp_math/1/AND+ti:+AND+Riemann+hypothesis+subj:+AND+General+mathematics/0/1/0/all/0/1).

Operator theory

Hilbert and Pólya suggested that one way to derive the Riemann hypothesis would be to find a self-adjoint operator, from the existence of which the statement on the real parts of the zeros of $\zeta(s)$ would follow when one applies the criterion on real eigenvalues. Some support for this idea comes from several analogues of the Riemann zeta functions whose zeros correspond to eigenvalues of some operator: the zeros of a zeta function of a variety over a finite field correspond to eigenvalues of a Frobenius element on an étale cohomology group, the zeros of a Selberg zeta function are eigenvalues of a Laplacian operator of a Riemann surface, and the zeros of a p-adic zeta function correspond to eigenvectors of a Galois action on ideal class groups.

Odlyzko (1987) showed that the distribution of the zeros of the Riemann zeta function shares some statistical properties with the eigenvalues of random matrices drawn from the Gaussian unitary ensemble. This gives some support to the Hilbert–Pólya conjecture.

In 1999, Michael Berry and Jonathan Keating conjectured that there is some unknown quantization \hat{H} of the classical Hamiltonian $H = xp$ so that

$$\zeta(1/2 + i\hat{H}) = 0$$

and even more strongly, that the Riemann zeros coincide with the spectrum of the operator $1/2 + i\hat{H}$. This is in contrast to canonical quantization, which leads to the Heisenberg uncertainty principle $[x, p] = 1/2$ and the natural numbers as spectrum of the quantum harmonic oscillator. The crucial point is that the Hamiltonian should be a self-adjoint operator so that the quantization would be a realization of the Hilbert–Pólya program. In a connection with this quantum mechanical problem Berry and Connes had proposed that the inverse of the potential of the Hamiltonian is connected to the half-derivative of the function

$$N(s) = \frac{1}{\pi} \operatorname{Arg} \xi(1/2 + i\sqrt{s})$$

then, in Berry–Connes approach

$$V^{-1}(x) = \sqrt{4\pi} \frac{d^{1/2} N(x)}{dx^{1/2}}$$

(Connes 1999). This yields a Hamiltonian whose eigenvalues are the square of the imaginary part of the Riemann zeros, and also that the functional determinant of this Hamiltonian operator is just the Riemann Xi function. In fact the Riemann Xi function would be proportional to the functional determinant (Hadamard product)

$$\det(H + 1/4 + s(s - 1))$$

as proved by Connes and others, in this approach

$$\frac{\xi(s)}{\xi(0)} = \frac{\det(H + s(s - 1) + 1/4)}{\det(H + 1/4)}.$$

The analogy with the Riemann hypothesis over finite fields suggests that the Hilbert space containing eigenvectors corresponding to the zeros might be some sort of first cohomology group of the spectrum $\operatorname{Spec}(\mathbb{Z})$ of the integers. Deninger (1998) described some of the attempts to find such a cohomology theory (Leichtnam 2005).

Zagier (1981) constructed a natural space of invariant functions on the upper half plane that has eigenvalues under the Laplacian operator that correspond to zeros of the Riemann zeta function—and remarked that in the unlikely event that one could show the existence of a suitable positive definite inner product on this space, the Riemann hypothesis would follow. Cartier (1982) discussed a related example, where due to a bizarre bug a computer program listed zeros of the Riemann zeta function as eigenvalues of the same Laplacian operator.

Schumayer & Hutchinson (2011) surveyed some of the attempts to construct a suitable physical model related to the Riemann zeta function.

Lee–Yang theorem

The Lee–Yang theorem states that the zeros of certain partition functions in statistical mechanics all lie on a "critical line" with their real part equals to 0, and this has led to some speculation about a relationship with the Riemann hypothesis (Knauf 1999).

Turán's result

Pál Turán (1948) showed that if the functions

$$\sum_{n=1}^N n^{-s}$$

have no zeros when the real part of s is greater than one then

$$T(x) = \sum_{n \leq x} \frac{\lambda(n)}{n} \geq 0 \text{ for } x > 0,$$

where $\lambda(n)$ is the Liouville function given by $(-1)^r$ if n has r prime factors. He showed that this in turn would imply that the Riemann hypothesis is true. But Haselgrove (1958) proved that $T(x)$ is negative for infinitely many x (and also disproved the closely related Pólya conjecture), and Borwein, Ferguson & Mossinghoff (2008) showed that the smallest such x is 72 185 376 951 205. Spira (1968) showed by numerical calculation that the finite Dirichlet series above for $N=19$ has a zero with real part greater than 1. Turán also showed that a somewhat weaker assumption, the nonexistence of zeros with real part greater than $1+N^{-1/2+\epsilon}$ for large N in the finite Dirichlet series above, would also imply the Riemann hypothesis, but Montgomery (1983) showed that for all sufficiently large N these series have zeros with real part greater than $1 + (\log \log N)/(4 \log N)$. Therefore, Turán's result is vacuously true and cannot help prove the Riemann hypothesis.

Noncommutative geometry

Connes (1999, 2000) has described a relationship between the Riemann hypothesis and noncommutative geometry, and shows that a suitable analog of the Selberg trace formula for the action of the idèle class group on the adèle class space would imply the Riemann hypothesis. Some of these ideas are elaborated in Lapidus (2008).

Hilbert spaces of entire functions

Louis de Branges (1992) showed that the Riemann hypothesis would follow from a positivity condition on a certain Hilbert space of entire functions. However Conrey & Li (2000) showed that the necessary positivity conditions are not satisfied. Despite this obstacle, de Branges has continued to work on an attempted proof of the Riemann hypothesis along the same lines, but this has not been widely accepted by other mathematicians(Sarnak 2005).

Quasicrystals

The Riemann hypothesis implies that the zeros of the zeta function form a quasicrystal, a distribution with discrete support whose Fourier transform also has discrete support. Dyson (2009) suggested trying to prove the Riemann hypothesis by classifying, or at least studying, 1-dimensional quasicrystals.

Arithmetic zeta functions of models of elliptic curves over number fields

When one goes from geometric dimension one, e.g. an algebraic number field, to geometric dimension two, e.g. a regular model of an elliptic curve over a number field, the two-dimensional part of the generalized Riemann hypothesis for the arithmetic zeta function of the model deals with the poles of the zeta function. In dimension one the study of the zeta integral in Tate's thesis does not lead to new important information on the Riemann hypothesis. Contrary to this, in dimension two work of Ivan Fesenko on two-dimensional generalisation of Tate's thesis includes an integral representation of a zeta integral closely related to the zeta function. In this new situation, not possible in dimension one, the poles of the zeta function can be studied via the zeta integral and associated adele groups. Related conjecture of Fesenko (2010) on the positivity of the fourth derivative of a boundary function associated to the zeta integral essentially implies the pole part of the generalized Riemann hypothesis. Suzuki (2011) proved that the latter, together with some technical assumptions, implies Fesenko's conjecture.

Multiple zeta functions

Deligne's proof of the Riemann hypothesis over finite fields used the zeta functions of product varieties, whose zeros and poles correspond to sums of zeros and poles of the original zeta function, in order to bound the real parts of the zeros of the original zeta function. By analogy, Kurokawa (1992) introduced multiple zeta functions whose zeros and poles correspond to sums of zeros and poles of the Riemann zeta function. To make the series converge he restricted to sums of zeros or poles all with non-negative imaginary part. So far, the known bounds on the zeros and poles of the multiple zeta functions are not strong enough to give useful estimates for the zeros of the Riemann zeta function.

Location of the zeros

Number of zeros

The functional equation combined with the [argument principle](#) implies that the number of zeros of the zeta function with imaginary part between 0 and T is given by

$$N(T) = \frac{1}{\pi} \operatorname{Arg}(\xi(s)) = \frac{1}{\pi} \operatorname{Arg}\left(\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s)s(s-1)/2\right)$$

for $s=1/2+iT$, where the argument is defined by varying it continuously along the line with $\operatorname{Im}(s)=T$, starting with argument 0 at $\infty+iT$. This is the sum of a large but well understood term

$$\frac{1}{\pi} \operatorname{Arg}\left(\Gamma\left(\frac{s}{2}\right)\pi^{-s/2}s(s-1)/2\right) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + 7/8 + O(1/T)$$

and a small but rather mysterious term

$$S(T) = \frac{1}{\pi} \operatorname{Arg}(\zeta(1/2+iT)) = O(\log(T)).$$

So the density of zeros with imaginary part near T is about $\log(T)/2\pi$, and the function S describes the small deviations from this. The function $S(t)$ jumps by 1 at each zero of the zeta function, and for $t \geq 8$ it decreases monotonically between zeros with derivative close to $-\log t$.

[Karatsuba \(1996\)](#) proved that every interval $(T, T+H]$ for $H \geq T^{\frac{27}{82}+\varepsilon}$ contains at least

$$H(\ln T)^{\frac{1}{3}} e^{-c\sqrt{\ln \ln T}}$$

points where the function $S(t)$ changes sign.

[Selberg \(1946\)](#) showed that the average moments of even powers of S are given by

$$\int_0^T |S(t)|^{2k} dt = \frac{(2k)!}{k!(2\pi)^{2k}} T(\log \log T)^k + O(T(\log \log T)^{k-1/2}).$$

This suggests that $S(T)/(\log \log T)^{1/2}$ resembles a Gaussian random variable with mean 0 and variance $2\pi^2$ ([Ghosh \(1983\)](#) proved this fact). In particular $|S(T)|$ is usually somewhere around $(\log \log T)^{1/2}$, but occasionally much larger. The exact order of growth of $S(T)$ is not known. There has been no unconditional improvement to Riemann's original bound $S(T)=O(\log T)$, though the Riemann hypothesis implies the slightly smaller bound $S(T)=O(\log T/\log \log T)$ ([Titchmarsh 1986](#)). The true order of magnitude may be somewhat less than this, as random functions with the same distribution as $S(T)$ tend to have growth of order about $\log(T)^{1/2}$. In the other direction it cannot be too small: [Selberg \(1946\)](#) showed that $S(T) \neq o((\log T)^{1/3}/(\log \log T)^{7/3})$, and assuming the Riemann hypothesis [Montgomery](#) showed that $S(T) \neq o((\log T)^{1/2}/(\log \log T)^{1/2})$.

Numerical calculations confirm that S grows very slowly: $|S(T)| < 1$ for $T < 280$, $|S(T)| < 2$ for $T < 6800$ 000, and the largest value of $|S(T)|$ found so far is not much larger than 3 ([Odlyzko 2002](#)).

Riemann's estimate $S(T) = O(\log T)$ implies that the gaps between zeros are bounded, and Littlewood improved this slightly, showing that the gaps between their imaginary parts tends to 0.

Theorem of Hadamard and de la Vallée-Poussin

[Hadamard \(1896\)](#) and [de la Vallée-Poussin \(1896\)](#) independently proved that no zeros could lie on the line $\operatorname{Re}(s) = 1$. Together with the functional equation and the fact that there are no zeros with real part greater than 1, this showed that all non-trivial zeros must lie in the interior of the critical strip $0 < \operatorname{Re}(s) < 1$. This was a key step in their first proofs of

the prime number theorem.

Both the original proofs that the zeta function has no zeros with real part 1 are similar, and depend on showing that if $\zeta(1+it)$ vanishes, then $\zeta(1+2it)$ is singular, which is not possible. One way of doing this is by using the inequality

$$|\zeta(\sigma)^3 \zeta(\sigma + it)^4 \zeta(\sigma + 2it)| \geq 1$$

for $\sigma > 1$, t real, and looking at the limit as $\sigma \rightarrow 1$. This inequality follows by taking the real part of the log of the Euler product to see that

$$|\zeta(\sigma + it)| = \exp \Re \sum_{p^n} \frac{p^{-n(\sigma+it)}}{n} = \exp \sum_{p^n} \frac{p^{-n\sigma} \cos(t \log p^n)}{n},$$

where the sum is over all prime powers p^n , so that

$$|\zeta(\sigma)^3 \zeta(\sigma + it)^4 \zeta(\sigma + 2it)| = \exp \sum_{p^n} p^{-n\sigma} \frac{3 + 4 \cos(t \log p^n) + \cos(2t \log p^n)}{n}$$

which is at least 1 because all the terms in the sum are positive, due to the inequality

$$3 + 4 \cos(\theta) + \cos(2\theta) = 2(1 + \cos(\theta))^2 \geq 0.$$

Zero-free regions

De la Vallée-Poussin (1899–1900) proved that if $\sigma + it$ is a zero of the Riemann zeta function, then $1 - \sigma \geq \frac{C}{\log(t)}$ for some positive constant C . In other words, zeros cannot be too close to the line $\sigma = 1$: there is a zero-free region close to this line. This zero-free region has been enlarged by several authors using methods such as Vinogradov's mean-value theorem. Ford (2002) gave a version with explicit numerical constants: $\zeta(\sigma + it) \neq 0$ whenever $|t| \geq 3$ and

$$\sigma \geq 1 - \frac{1}{57.54(\log |t|)^{2/3} (\log \log |t|)^{1/3}}.$$

Zeros on the critical line

Hardy (1914) and Hardy & Littlewood (1921) showed there are infinitely many zeros on the critical line, by considering moments of certain functions related to the zeta function. Selberg (1942) proved that at least a (small) positive proportion of zeros lie on the line. Levinson (1974) improved this to one-third of the zeros by relating the zeros of the zeta function to those of its derivative, and Conrey (1989) improved this further to two-fifths.

Most zeros lie close to the critical line. More precisely, Bohr & Landau (1914) showed that for any positive ε , all but an infinitely small proportion of zeros lie within a distance ε of the critical line. Ivić (1985) gives several more precise versions of this result, called *zero density estimates*, which bound the number of zeros in regions with imaginary part at most T and real part at least $1/2 + \varepsilon$.

Hardy–Littlewood conjectures

In 1914 Godfrey Harold Hardy proved that $\zeta(\frac{1}{2} + it)$ has infinitely many real zeros.

The next two conjectures of Hardy and John Edensor Littlewood on the distance between real zeros of $\zeta(\frac{1}{2} + it)$ and on the density of zeros of $\zeta(\frac{1}{2} + it)$ on the interval $(T, T + H]$ for sufficiently large $T > 0$, and $H = T^{a+\varepsilon}$ and with as small as possible value of $a > 0$, where $\varepsilon > 0$ is an arbitrarily small number, open two new directions in the investigation of the Riemann zeta function:

1. For any $\varepsilon > 0$ there exists a lower bound $T_0 = T_0(\varepsilon) > 0$ such that for $T \geq T_0$ and $H = T^{\frac{1}{4}+\varepsilon}$ the interval $(T, T+H]$ contains a zero of odd order of the function $\zeta\left(\frac{1}{2} + it\right)$.

Let $N(T)$ be the total number of real zeros, and $N_0(T)$ be the total number of zeros of odd order of the function $\zeta\left(\frac{1}{2} + it\right)$ lying on the interval $(0, T]$.

2. For any $\varepsilon > 0$ there exists $T_0 = T_0(\varepsilon) > 0$ and some $c = c(\varepsilon) > 0$, such that for $T \geq T_0$ and $H = T^{\frac{1}{2}+\varepsilon}$ the inequality $N_0(T+H) - N_0(T) \geq cH$ is true.

Selberg's zeta function conjecture

Atle Selberg (1942) investigated the problem of Hardy–Littlewood 2 and proved that for any $\varepsilon > 0$ there exists such $T_0 = T_0(\varepsilon) > 0$ and $c = c(\varepsilon) > 0$, such that for $T \geq T_0$ and $H = T^{0.5+\varepsilon}$ the inequality $N(T+H) - N(T) \geq cH \log T$ is true. Selberg conjectured that this could be tightened to $H = T^{0.5}$. A. A. Karatsuba (1984a, 1984b, 1985) proved that for a fixed ε satisfying the condition $0 < \varepsilon < 0.001$, a sufficiently large T and $H = T^{a+\varepsilon}$, $a = \frac{27}{82} = \frac{1}{3} - \frac{1}{246}$, the interval $(T, T+H)$ contains at least $cH \ln(T)$ real zeros of the Riemann zeta function $\zeta\left(\frac{1}{2} + it\right)$ and therefore confirmed the Selberg conjecture. The estimates of Selberg and Karatsuba can not be improved in respect of the order of growth as $T \rightarrow \infty$.

Karatsuba (1992) proved that an analog of the Selberg conjecture holds for almost all intervals $(T, T+H]$, $H = T^\varepsilon$, where ε is an arbitrarily small fixed positive number. The Karatsuba method permits to investigate zeros of the Riemann zeta-function on "supershorts" intervals of the critical line, that is, on the intervals $(T, T+H]$, the length H of which grows slower than any, even arbitrarily small degree T . In particular, he proved that for any given numbers $\varepsilon, \varepsilon_1$ satisfying the conditions $0 < \varepsilon, \varepsilon_1 < 1$ almost all intervals $(T, T+H]$ for $H \geq \exp\{(\ln T)^\varepsilon\}$ contain at least $H(\ln T)^{1-\varepsilon_1}$ zeros of the function $\zeta\left(\frac{1}{2} + it\right)$. This estimate is quite close to the one that follows from the Riemann hypothesis.

Numerical calculations

The function

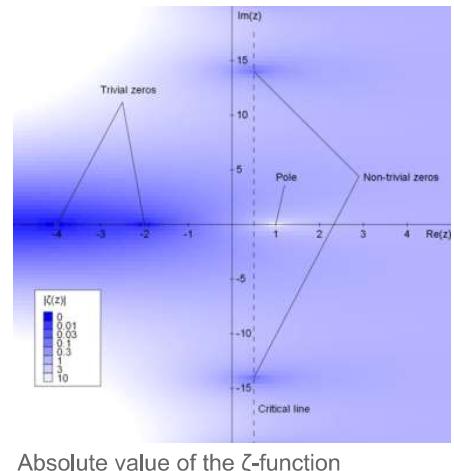
$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

has the same zeros as the zeta function in the critical strip, and is real on the critical line because of the functional equation, so one can prove the existence of zeros exactly on the real line between two points by checking numerically that the function has opposite signs at these points. Usually one writes

$$\zeta\left(\frac{1}{2} + it\right) = Z(t)e^{-i\theta(t)}$$

where Hardy's function Z and the Riemann–Siegel theta function θ are uniquely defined by this and the condition that they are smooth real functions with $\theta(0)=0$. By finding many intervals where the function Z changes sign one can show that there are many zeros on the critical line. To verify the Riemann hypothesis up to a given imaginary part T of the zeros, one also has to check that there are no further zeros off the line in this region. This can be done by calculating the total number of zeros in the region using Turing's method and checking that it is the same as the number of zeros found on the line. This allows one to verify the Riemann hypothesis computationally up to any desired value of T (provided all the zeros of the zeta function in this region are simple and on the critical line).

Some calculations of zeros of the zeta function are listed below. So far all zeros that have been checked are on the critical line and are simple. (A multiple zero would cause problems for the zero finding algorithms, which depend on finding sign changes between zeros.) For tables of the zeros, see Haselgrove & Miller (1960) or Odlyzko.



Year	Number of zeros	Author
1859?	3	B. Riemann used the Riemann–Siegel formula (unpublished, but reported in Siegel 1932).
1903	15	J. P. Gram (1903) used Euler–Maclaurin summation and discovered Gram's law . He showed that all 10 zeros with imaginary part at most 50 range lie on the critical line with real part $1/2$ by computing the sum of the inverse 10th powers of the roots he found.
1914	$79 (\gamma_n \leq 200)$	R. J. Backlund (1914) introduced a better method of checking all the zeros up to that point are on the line, by studying the argument $S(T)$ of the zeta function.
1925	$138 (\gamma_n \leq 300)$	J. I. Hutchinson (1925) found the first failure of Gram's law, at the Gram point g_{126} .
1935	195	E. C. Titchmarsh (1935) used the recently rediscovered Riemann–Siegel formula , which is much faster than Euler–Maclaurin summation. It takes about $O(T^{3/2+\epsilon})$ steps to check zeros with imaginary part less than T , while the Euler–Maclaurin method takes about $O(T^{2+\epsilon})$ steps.
1936	1041	E. C. Titchmarsh (1936) and L. J. Comrie were the last to find zeros by hand.
1953	1104	A. M. Turing (1953) found a more efficient way to check that all zeros up to some point are accounted for by the zeros on the line, by checking that Z has the correct sign at several consecutive Gram points and using the fact that $S(T)$ has average value 0. This requires almost no extra work because the sign of Z at Gram points is already known from finding the zeros, and is still the usual method used. This was the first use of a digital computer to calculate the zeros.
1956	15 000	D. H. Lehmer (1956) discovered a few cases where the zeta function has zeros that are "only just" on the line: two zeros of the zeta function are so close together that it is unusually difficult to find a sign change between them. This is called "Lehmer's phenomenon", and first occurs at the zeros with imaginary parts 7005.063 and 7005.101, which differ by only .04 while the average gap between other zeros near this point is about 1.
1956	25 000	D. H. Lehmer
1958	35 337	N. A. Meller
1966	250 000	R. S. Lehman
1968	3 500 000	Rosser, Yohe & Schoenfeld (1969) stated Rosser's rule (described below).
1977	40 000 000	R. P. Brent
1979	81 000 001	R. P. Brent
1982	200 000 001	R. P. Brent, J. van de Lune , H. J. J. te Riele , D. T. Winter
1983	300 000 001	J. van de Lune, H. J. J. te Riele
1986	1 500 000 001	van de Lune, te Riele & Winter (1986) gave some statistical data about the zeros and give several graphs of Z at places where it has unusual behavior.
1987	A few of large ($\sim 10^{12}$) height	A. M. Odlyzko (1987) computed smaller numbers of zeros of much larger height, around 10^{12} , to high precision to check Montgomery's pair correlation conjecture .
1992	A few of large ($\sim 10^{20}$) height	A. M. Odlyzko (1992) computed a 175 million zeros of heights around 10^{20} and a few more of heights around 2×10^{20} , and gave an extensive discussion of the results.
1998	10000 of large ($\sim 10^{21}$) height	A. M. Odlyzko (1998) computed some zeros of height about 10^{21}
2001	10 000 000 000	J. van de Lune (unpublished)
2004	$\sim 900 000 000 000$ ^[3]	S. Wedeniwski (ZetaGrid distributed computing)
2004	10 000 000 000 000 and a few of large (up to $\sim 10^{24}$) heights	X. Gourdon (2004) and Patrick Demichel used the Odlyzko–Schönhage algorithm . They also checked two billion zeros around heights $10^{13}, 10^{14}, \dots, 10^{24}$.
2020	12 363 153 437 138 up to height 3 000 175 332 800	Platt & Trudgian (2020). They also verified the work of Gourdon (2004) and others.

Gram points

A Gram point is a point on the critical line $1/2 + it$ where the zeta function is real and non-zero. Using the expression for the zeta function on the critical line, $\zeta(1/2 + it) = Z(t)e^{-i\theta(t)}$, where Hardy's function, Z , is real for real t , and θ is the Riemann–Siegel theta function, we see that zeta is real when $\sin(\theta(t)) = 0$. This implies that $\theta(t)$ is an integer multiple of π , which allows for the location of Gram points to be calculated fairly easily by inverting the formula for θ . They are usually numbered as g_n for $n = 0, 1, \dots$, where g_n is the unique solution of $\theta(t) = n\pi$.

Gram observed that there was often exactly one zero of the zeta function between any two Gram points; Hutchinson called this observation **Gram's law**. There are several other closely related statements that are also sometimes called Gram's law: for example, $(-1)^n Z(g_n)$ is usually positive, or $Z(t)$ usually has opposite sign at consecutive Gram points. The imaginary parts y_n of the first few zeros (in blue) and the first few Gram points g_n are given in the following table

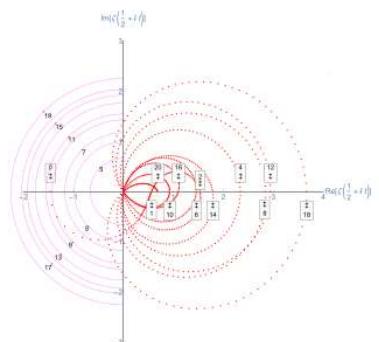
		g_{-1}	y_1	g_0	y_2	g_1	y_3	g_2	y_4	g_3	y_5	g_4	y_6	g_5
0.000	3.436	9.667	14.135	17.846	21.022	23.170	25.011	27.670	30.425	31.718	32.935	35.467	37.586	38.999

The first failure of Gram's law occurs at the 127th zero and the Gram point g_{126} , which are in the "wrong" order.

g_{124}	y_{126}	g_{125}	y_{126}	g_{127}	y_{128}	g_{127}	y_{129}	g_{128}
279.148	279.229	280.802	282.455	282.465	283.211	284.104	284.836	285.752

A Gram point t is called good if the zeta function is positive at $1/2 + it$. The indices of the "bad" Gram points where Z has the "wrong" sign are 126, 134, 195, 211, ... (sequence A114856 in the OEIS). A *Gram block* is an interval bounded by two good Gram points such that all the Gram points between them are bad. A refinement of Gram's law called Rosser's rule due to [Rosser, Yohe & Schoenfeld \(1969\)](#) says that Gram blocks often have the expected number of zeros in them (the same as the number of Gram intervals), even though some of the individual Gram intervals in the block may not have exactly one zero in them. For example, the interval bounded by g_{125} and g_{127} is a Gram block containing a unique bad Gram point g_{126} , and contains the expected number 2 of zeros although neither of its two Gram intervals contains a unique zero. Rosser et al. checked that there were no exceptions to Rosser's rule in the first 3 million zeros, although there are infinitely many exceptions to Rosser's rule over the entire zeta function.

Gram's rule and Rosser's rule both say that in some sense zeros do not stray too far from their expected positions. The distance of a zero from its expected position is controlled by the function S defined above, which grows extremely slowly: its average value is of the order of $(\log \log T)^{1/2}$, which only reaches 2 for T around 10^{24} . This means that both rules hold most of the time for small T but eventually break down often. Indeed, [Trudgian \(2011\)](#) showed that both Gram's law and Rosser's rule fail in a positive proportion of cases. To be specific, it is expected that in about 73% one zero is enclosed by two successive Gram points, but in 14% no zero and in 13% two zeros are in such a Gram-interval on the long run.



This is a polar plot of the first 20 non-trivial Riemann zeta function zeros (including Gram points) along the critical line $\zeta(1/2 + it)$ for real values of t running from 0 to 50. The consecutively labeled zeros have 50 red plot points between each, with zeros identified by concentric magenta rings scaled to show the relative distance between their values of t . Gram's law states that the curve usually crosses the real axis once between zeros.

Arguments for and against the Riemann hypothesis

Mathematical papers about the Riemann hypothesis tend to be cautiously noncommittal about its truth. Of authors who express an opinion, most of them, such as [Riemann \(1859\)](#) and [Bombieri \(2000\)](#), imply that they expect (or at least hope) that it is true. The few authors who express serious doubt about it include [Ivić \(2008\)](#), who lists some reasons for skepticism, and [Littlewood \(1962\)](#), who flatly states that he believes it false, that there is no evidence for it and no imaginable reason it would be true. The consensus of the survey articles ([Bombieri 2000](#), [Conrey 2003](#), and [Sarnak 2005](#)) is that the evidence for it is strong but not overwhelming, so that while it is probably true there is reasonable doubt.

Some of the arguments for and against the Riemann hypothesis are listed by [Sarnak \(2005\)](#), [Conrey \(2003\)](#), and [Ivić \(2008\)](#), and include the following:

- Several analogues of the Riemann hypothesis have already been proved. The proof of the Riemann hypothesis for varieties over finite fields by Deligne (1974) is possibly the single strongest theoretical reason in favor of the Riemann hypothesis. This provides some evidence for the more general conjecture that all zeta functions associated with automorphic forms satisfy a Riemann hypothesis, which includes the classical Riemann hypothesis as a special case. Similarly Selberg zeta functions satisfy the analogue of the Riemann hypothesis, and are in some ways similar to the Riemann zeta function, having a functional equation and an infinite product expansion analogous to the Euler product expansion. But there are also some major differences; for example, they are not given by Dirichlet series. The Riemann hypothesis for the Goss zeta function was proved by Sheats (1998). In contrast to these positive examples, some Epstein zeta functions do not satisfy the Riemann hypothesis even though they have an infinite number of zeros on the critical line (Titchmarsh 1986). These functions are quite similar to the Riemann zeta function, and have a Dirichlet series expansion and a functional equation, but the ones known to fail the Riemann hypothesis do not have an Euler product and are not directly related to automorphic representations.
- At first, the numerical verification that many zeros lie on the line seems strong evidence for it. But analytic number theory has had many conjectures supported by substantial numerical evidence that turned out to be false. See Skewes number for a notorious example, where the first exception to a plausible conjecture related to the Riemann hypothesis probably occurs around 10^{316} ; a counterexample to the Riemann hypothesis with imaginary part this size would be far beyond anything that can currently be computed using a direct approach. The problem is that the behavior is often influenced by very slowly increasing functions such as $\log \log T$, that tend to infinity, but do so so slowly that this cannot be detected by computation. Such functions occur in the theory of the zeta function controlling the behavior of its zeros; for example the function $S(T)$ above has average size around $(\log \log T)^{1/2}$. As $S(T)$ jumps by at least 2 at any counterexample to the Riemann hypothesis, one might expect any counterexamples to the Riemann hypothesis to start appearing only when $S(T)$ becomes large. It is never much more than 3 as far as it has been calculated, but is known to be unbounded, suggesting that calculations may not have yet reached the region of typical behavior of the zeta function.
- Denjoy's probabilistic argument for the Riemann hypothesis (Edwards 1974) is based on the observation that if $\mu(x)$ is a random sequence of "1"s and "-1"s then, for every $\varepsilon > 0$, the partial sums

$$M(x) = \sum_{n \leq x} \mu(n)$$

(the values of which are positions in a simple random walk) satisfy the bound

$$M(x) = O(x^{1/2+\varepsilon})$$

with probability 1. The Riemann hypothesis is equivalent to this bound for the Möbius function μ and the Mertens function M derived in the same way from it. In other words, the Riemann hypothesis is in some sense equivalent to saying that $\mu(x)$ behaves like a random sequence of coin tosses. When $\mu(x)$ is nonzero its sign gives the parity of the number of prime factors of x , so informally the Riemann hypothesis says that the parity of the number of prime factors of an integer behaves randomly. Such probabilistic arguments in number theory often give the right answer, but tend to be very hard to make rigorous, and occasionally give the wrong answer for some results, such as Maier's theorem.

- The calculations in Odlyzko (1987) show that the zeros of the zeta function behave very much like the eigenvalues of a random Hermitian matrix, suggesting that they are the eigenvalues of some self-adjoint operator, which would imply the Riemann hypothesis. All attempts to find such an operator have failed.
- There are several theorems, such as Goldbach's weak conjecture for sufficiently large odd numbers, that were first proved using the generalized Riemann hypothesis, and later shown to be true unconditionally. This could be considered as weak evidence for the generalized Riemann hypothesis, as several of its "predictions" are true.
- Lehmer's phenomenon (Lehmer 1956), where two zeros are sometimes very close, is sometimes given as a reason to disbelieve the Riemann hypothesis. But one would expect this to happen occasionally by chance even if the Riemann hypothesis is true, and Odlyzko's calculations suggest that nearby pairs of zeros occur just as often as predicted by Montgomery's conjecture.
- Patterson (1988) suggests that the most compelling reason for the Riemann hypothesis for most mathematicians is the hope that primes are distributed as regularly as possible.^[4]

Notes

1. Leonhard Euler. Variae observationes circa series infinitas. (<http://eulerarchive.maa.org/docs/originals/E072.pdf>) Commentarii academiae scientiarum Petropolitanae 9, 1744, pp. 160–188, Theorems 7 and 8. In Theorem 7 Euler

- proves the formula in the special case $s = 1$, and in Theorem 8 he proves it more generally. In the first corollary to his Theorem 7 he notes that $\zeta(1) = \log \infty$, and makes use of this latter result in his Theorem 19, in order to show that the sum of the inverses of the prime numbers is $\log \log \infty$.
2. Knapowski, Stanisław (1962). "On sign-changes of the difference $\pi(x)-\text{li}(x)$ " (<https://doi.org/10.4064%2Faa-7-2-107-119>). *Acta Arithmetica*. 7 (2): 107–119. doi:10.4064/aa-7-2-107-119 (<https://doi.org/10.4064%2Faa-7-2-107-119>). ISSN 0065-1036 (<https://www.worldcat.org/issn/0065-1036>).
 3. Weisstein, Eric W. "Riemann Zeta Function Zeros" (<https://mathworld.wolfram.com/RiemannZetaFunctionZeros.html>). *mathworld.wolfram.com*. Retrieved 28 April 2020. "ZetaGrid is a distributed computing project attempting to calculate as many zeros as possible. It had reached 1029.9 billion zeros as of Feb. 18, 2005."
 4. p. 75: "One should probably add to this list the 'Platonic' reason that one expects the natural numbers to be the most perfect idea conceivable, and that this is only compatible with the primes being distributed in the most regular fashion possible..."

References

- Artin, Emil (1924), "Quadratische Körper im Gebiete der höheren Kongruenzen. II. Analytischer Teil", *Mathematische Zeitschrift*, 19 (1): 207–246, doi:10.1007/BF01181075 (<https://doi.org/10.1007%2FBF01181075>), S2CID 117936362 (<https://api.semanticscholar.org/CorpusID:117936362>)
- Backlund, R. J. (1914), "Sur les Zéros de la Fonction $\zeta(s)$ de Riemann" (<http://gallica.bnf.fr/ark:/12148/bpt6k3111d/f1983.image>), *C. R. Acad. Sci. Paris*, 158: 1979–1981
- Beurling, Arne (1955), "A closure problem related to the Riemann zeta-function", *Proceedings of the National Academy of Sciences of the United States of America*, 41 (5): 312–314, Bibcode:1955PNAS...41..312B (<https://ui.adsabs.harvard.edu/abs/1955PNAS...41..312B>), doi:10.1073/pnas.41.5.312 (<https://doi.org/10.1073%2Fpnas.41.5.312>), MR 0070655 (<https://www.ams.org/mathscinet-getitem?mr=0070655>), PMC 528084 (<https://www.ncbi.nlm.nih.gov/pmc/articles/PMC528084>), PMID 16589670 (<https://pubmed.ncbi.nlm.nih.gov/16589670>)
- Bohr, H.; Landau, E. (1914), "Ein Satz über Dirichletsche Reihen mit Anwendung auf die ζ -Funktion und die L -Funktionen", *Rendiconti del Circolo Matematico di Palermo*, 37 (1): 269–272, doi:10.1007/BF03014823 (<https://doi.org/10.1007%2FBF03014823>), S2CID 121145912 (<https://api.semanticscholar.org/CorpusID:121145912>)
- Bombieri, Enrico (2000), *The Riemann Hypothesis – official problem description* (http://www.claymath.org/sites/default/files/official_problem_description.pdf) (PDF), Clay Mathematics Institute, retrieved 2008-10-25
Reprinted in (Borwein et al. 2008).
- Borwein, Peter; Choi, Stephen; Rooney, Brendan; Weirathmueller, Andrea, eds. (2008), *The Riemann Hypothesis: A Resource for the Afficionado and Virtuoso Alike*, CMS Books in Mathematics, New York: Springer, doi:10.1007/978-0-387-72126-2 (<https://doi.org/10.1007%2F978-0-387-72126-2>), ISBN 978-0-387-72125-5
- Borwein, Peter; Ferguson, Ron; Mossinghoff, Michael J. (2008), "Sign changes in sums of the Liouville function", *Mathematics of Computation*, 77 (263): 1681–1694, Bibcode:2008MaCom..77.1681B (<https://ui.adsabs.harvard.edu/abs/2008MaCom..77.1681B>), doi:10.1090/S0025-5718-08-02036-X (<https://doi.org/10.1090/S0025-5718-08-02036-X>), MR 2398787 (<https://www.ams.org/mathscinet-getitem?mr=2398787>)
- de Branges, Louis (1992), "The convergence of Euler products", *Journal of Functional Analysis*, 107 (1): 122–210, doi:10.1016/0022-1236(92)90103-P (<https://doi.org/10.1016/0022-1236%2892%2990103-P>), MR 1165869 (<https://www.ams.org/mathscinet-getitem?mr=1165869>)
- Broughan, Kevin (2017), *Equivalents of the Riemann Hypothesis*, Cambridge University Press, ISBN 978-1108290784
- Burton, David M. (2006), *Elementary Number Theory* (<https://books.google.com/books?id=XMQjuoTqqRMC>), Tata McGraw-Hill Publishing Company Limited, ISBN 978-0-07-061607-3
- Cartier, P. (1982), "Comment l'hypothèse de Riemann ne fut pas prouvée", *Seminar on Number Theory, Paris 1980–81 (Paris, 1980/1981)*, Progr. Math., 22, Boston, MA: Birkhäuser Boston, pp. 35–48, MR 0693308 (<https://www.ams.org/mathscinet-getitem?mr=0693308>)
- Connes, Alain (1999), "Trace formula in noncommutative geometry and the zeros of the Riemann zeta function", *Selecta Mathematica. New Series*, 5 (1): 29–106, arXiv:math/9811068 (<https://arxiv.org/abs/math/9811068>), doi:10.1007/s000290050042 (<https://doi.org/10.1007%2Fs000290050042>), MR 1694895 (<https://www.ams.org/mathscinet-getitem?mr=1694895>), S2CID 55820659 (<https://api.semanticscholar.org/CorpusID:55820659>)
- Connes, Alain (2000), "Noncommutative geometry and the Riemann zeta function", *Mathematics: frontiers and perspectives*, Providence, R.I.: American Mathematical Society, pp. 35–54, MR 1754766 (<https://www.ams.org/mathscinet-getitem?mr=1754766>)

- **Connes, Alain** (2016), "An Essay on the Riemann Hypothesis", in **Nash, J. F.; Rassias, Michael** (eds.), *Open Problems in Mathematics*, New York: Springer, pp. 225–257, arXiv:1509.05576 (<https://arxiv.org/abs/1509.05576>), doi:10.1007/978-3-319-32162-2_5 (https://doi.org/10.1007%2F978-3-319-32162-2_5)
- **Conrey, J. B.** (1989), "More than two fifths of the zeros of the Riemann zeta function are on the critical line" (<http://www.digizeitschriften.de/resolveppn/GDZPPN002206781>), *J. Reine Angew. Math.*, **1989** (399): 1–16, doi:10.1515/crll.1989.399.1 (<https://doi.org/10.1515%2Fcrl.1989.399.1>), MR 1004130 (<https://www.ams.org/mathscinet-getitem?mr=1004130>), S2CID 115910600 (<https://api.semanticscholar.org/CorpusID:115910600>)
- **Conrey, J. Brian** (2003), "The Riemann Hypothesis" (<http://www.ams.org/notices/200303/fea-conrey-web.pdf>) (PDF), *Notices of the American Mathematical Society*: 341–353 Reprinted in (**Borwein et al.** 2008).
- **Conrey, J. B.; Li, Xian-Jin** (2000), "A note on some positivity conditions related to zeta and L-functions", *International Mathematics Research Notices*, **2000** (18): 929–940, arXiv:math/9812166 (<https://arxiv.org/abs/math/9812166>), doi:10.1155/S1073792800000489 (<https://doi.org/10.1155%2FS1073792800000489>), MR 1792282 (<https://www.ams.org/mathscinet-getitem?mr=1792282>), S2CID 14678312 (<https://api.semanticscholar.org/CorpusID:14678312>)
- **Deligne, Pierre** (1974), "La conjecture de Weil. I" (http://www.numdam.org/item?id=PMIHES_1974_43_2_73_0), *Publications Mathématiques de l'IHÉS*, **43**: 273–307, doi:10.1007/BF02684373 (<https://doi.org/10.1007%2FBF02684373>), MR 0340258 (<https://www.ams.org/mathscinet-getitem?mr=0340258>), S2CID 123139343 (<https://api.semanticscholar.org/CorpusID:123139343>)
- **Deligne, Pierre** (1980), "La conjecture de Weil : II" (http://www.numdam.org/item?id=PMIHES_1980_52_1_37_0), *Publications Mathématiques de l'IHÉS*, **52**: 137–252, doi:10.1007/BF02684780 (<https://doi.org/10.1007%2FBF02684780>), S2CID 189769469 (<https://api.semanticscholar.org/CorpusID:189769469>)
- **Deninger, Christopher** (1998), "Some analogies between number theory and dynamical systems on foliated spaces" (<http://www.mathematik.uni-bielefeld.de/documenta/xvol-icm/00/Deninger.MAN.html>), *Proceedings of the International Congress of Mathematicians, Vol. I* (Berlin, 1998), Documenta Mathematica, pp. 163–186, MR 1648030 (<https://www.ams.org/mathscinet-getitem?mr=1648030>)
- **Dudek, Adrian W.** (2014-08-21), "On the Riemann hypothesis and the difference between primes", *International Journal of Number Theory*, **11** (3): 771–778, arXiv:1402.6417 (<https://arxiv.org/abs/1402.6417>), Bibcode:2014arXiv1402.6417D (<https://ui.adsabs.harvard.edu/abs/2014arXiv1402.6417D>), doi:10.1142/S1793042115500426 (<https://doi.org/10.1142%2FS1793042115500426>), ISSN 1793-0421 (<https://www.worldcat.org/issn/1793-0421>), S2CID 119321107 (<https://api.semanticscholar.org/CorpusID:119321107>)
- **Dyson, Freeman** (2009), "Birds and frogs" (<http://www.ams.org/notices/200902/rtx090200212p.pdf>) (PDF), *Notices of the American Mathematical Society*, **56** (2): 212–223, MR 2483565 (<https://www.ams.org/mathscinet-getitem?mr=2483565>)
- **Edwards, H. M.** (1974), *Riemann's Zeta Function*, New York: Dover Publications, ISBN 978-0-486-41740-0, MR 0466039 (<https://www.ams.org/mathscinet-getitem?mr=0466039>)
- **Fesenko, Ivan** (2010), "Analysis on arithmetic schemes. II", *Journal of K-theory*, **5** (3): 437–557, doi:10.1017/is010004028jkt103 (<https://doi.org/10.1017%2Fis010004028jkt103>)
- **Ford, Kevin** (2002), "Vinogradov's integral and bounds for the Riemann zeta function", *Proceedings of the London Mathematical Society*, Third Series, **85** (3): 565–633, arXiv:1910.08209 (<https://arxiv.org/abs/1910.08209>), doi:10.1112/S0024611502013655 (<https://doi.org/10.1112%2FS0024611502013655>), MR 1936814 (<https://www.ams.org/mathscinet-getitem?mr=1936814>), S2CID 121144007 (<https://api.semanticscholar.org/CorpusID:121144007>)
- **Franel, J.; Landau, E.** (1924), "Les suites de Farey et le problème des nombres premiers" (Franel, 198–201); "Bemerkungen zu der vorstehenden Abhandlung von Herrn Franel (Landau, 202–206)", *Göttinger Nachrichten*: 198–206
- **Ghosh, Amit** (1983), "On the Riemann zeta function—mean value theorems and the distribution of $|S(T)|$ ", *J. Number Theory*, **17**: 93–102, doi:10.1016/0022-314X(83)90010-0 ([https://doi.org/10.1016%2F0022-314X\(83\)90010-0](https://doi.org/10.1016%2F0022-314X(83)90010-0))
- **Gourdon, Xavier** (2004), *The 10^{13} first zeros of the Riemann Zeta function, and zeros computation at very large height* (<http://numbers.computation.free.fr/Constants/Miscellaneous/zetazeros1e13-1e24.pdf>) (PDF)
- **Gram, J. P.** (1903), "Note sur les zéros de la fonction $\zeta(s)$ de Riemann" (<https://zenodo.org/record/1930945>), *Acta Mathematica*, **27**: 289–304, doi:10.1007/BF02421310 (<https://doi.org/10.1007%2FBF02421310>), S2CID 115327214 (<https://api.semanticscholar.org/CorpusID:115327214>)
- **Hadamard, Jacques** (1896), "Sur la distribution des zéros de la fonction $\zeta(s)$ et ses conséquences arithmétiques", *Bulletin de la Société Mathématique de France*, **14**: 199–220, doi:10.24033/bsmf.545 (<https://doi.org/10.24033%2Fbsmf.545>) Reprinted in (**Borwein et al.** 2008).
- **Hardy, G. H.** (1914), "Sur les Zéros de la Fonction $\zeta(s)$ de Riemann" (<http://gallica.bnf.fr/ark:/12148/bpt6k3111d.image.f1014.langEN>), *C. R. Acad. Sci. Paris*, **158**: 1012–1014, JFM 45.0716.04 (<https://zbmath.org/?form=at=complete&q=an:45.0716.04>) Reprinted in (**Borwein et al.** 2008).

- Hardy, G. H.; Littlewood, J. E. (1921), "The zeros of Riemann's zeta-function on the critical line" (<https://zenodo.org/record/1447415>), *Math. Z.*, **10** (3–4): 283–317, doi:10.1007/BF01211614 (<https://doi.org/10.1007%2FBF01211614>), S2CID 126338046 (<https://api.semanticscholar.org/CorpusID:126338046>)
- Haselgrove, C. B. (1958), "A disproof of a conjecture of Pólya", *Mathematika*, **5** (2): 141–145, doi:10.1112/S0025579300001480 (<https://doi.org/10.1112/S0025579300001480>), ISSN 0025-5793 (<http://www.worldcat.org/issn/0025-5793>), MR 0104638 (<https://www.ams.org/mathscinet-getitem?mr=0104638>), Zbl 0085.27102 (<https://zbmath.org/?format=complete&q=an:0085.27102>) Reprinted in (Borwein et al. 2008).
- Haselgrove, C. B.; Miller, J. C. P. (1960), *Tables of the Riemann zeta function*, Royal Society Mathematical Tables, Vol. 6, Cambridge University Press, ISBN 978-0-521-06152-0, MR 0117905 (<https://www.ams.org/mathscinet-getitem?mr=0117905>) Review (<https://www.jstor.org/stable/2003098>)
- Hutchinson, J. I. (1925), "On the Roots of the Riemann Zeta-Function", *Transactions of the American Mathematical Society*, **27** (1): 49–60, doi:10.2307/1989163 (<https://doi.org/10.2307%2F1989163>), JSTOR 1989163 (<https://www.jstor.org/stable/1989163>)
- Ingham, A.E. (1932), *The Distribution of Prime Numbers*, Cambridge Tracts in Mathematics and Mathematical Physics, **30**, Cambridge University Press. Reprinted 1990, ISBN 978-0-521-39789-6, MR1074573 (<https://mathscinet.ams.org/mathscinet-getitem?mr=1074573>)
- Ireland, Kenneth; Rosen, Michael (1990), *A Classical Introduction to Modern Number Theory (Second edition)*, New York: Springer, ISBN 0-387-97329-X
- Ivić, A. (1985), *The Riemann Zeta Function*, New York: John Wiley & Sons, ISBN 978-0-471-80634-9, MR 0792089 (<https://www.ams.org/mathscinet-getitem?mr=0792089>) (Reprinted by Dover 2003)
- Ivić, Aleksandar (2008), "On some reasons for doubting the Riemann hypothesis", in Borwein, Peter; Choi, Stephen; Rooney, Brendan; Weirathmueller, Andrea (eds.), *The Riemann Hypothesis: A Resource for the Afficionado and Virtuoso Alike*, CMS Books in Mathematics, New York: Springer, pp. 131–160, arXiv:math.NT/0311162 (<https://arxiv.org/abs/math.NT/0311162>), ISBN 978-0-387-72125-5
- Karatsuba, A. A. (1984a), "Zeros of the function $\zeta(s)$ on short intervals of the critical line", *Izv. Akad. Nauk SSSR, Ser. Mat.* (in Russian), **48** (3): 569–584, MR 0747251 (<https://www.ams.org/mathscinet-getitem?mr=0747251>)
- Karatsuba, A. A. (1984b), "Distribution of zeros of the function $\zeta(1/2 + it)$ ", *Izv. Akad. Nauk SSSR, Ser. Mat.* (in Russian), **48** (6): 1214–1224, MR 0772113 (<https://www.ams.org/mathscinet-getitem?mr=0772113>)
- Karatsuba, A. A. (1985), "Zeros of the Riemann zeta-function on the critical line", *Trudy Mat. Inst. Steklov.* (in Russian) (167): 167–178, MR 0804073 (<https://www.ams.org/mathscinet-getitem?mr=0804073>)
- Karatsuba, A. A. (1992), "On the number of zeros of the Riemann zeta-function lying in almost all short intervals of the critical line", *Izv. Ross. Akad. Nauk, Ser. Mat.* (in Russian), **56** (2): 372–397, Bibcode:1993IzMat..40..353K (<https://ui.adsabs.harvard.edu/abs/1993IzMat..40..353K>), doi:10.1070/IM1993v04n02ABEH002168 (<https://doi.org/10.1070%2FIM1993v04n02ABEH002168>), MR 1180378 (<https://www.ams.org/mathscinet-getitem?mr=1180378>)
- Karatsuba, A. A.; Voronin, S. M. (1992), *The Riemann zeta-function*, de Gruyter Expositions in Mathematics, **5**, Berlin: Walter de Gruyter & Co., doi:10.1515/9783110886146 (<https://doi.org/10.1515%2F9783110886146>), ISBN 978-3-11-013170-3, MR 1183467 (<https://www.ams.org/mathscinet-getitem?mr=1183467>)
- Keating, Jonathan P.; Snaith, N. C. (2000), "Random matrix theory and $\zeta(1/2 + it)$ ", *Communications in Mathematical Physics*, **214** (1): 57–89, Bibcode:2000CMaPh.214...57K (<https://ui.adsabs.harvard.edu/abs/2000CMaPh.214...57K>), doi:10.1007/s002200000261 (<https://doi.org/10.1007%2Fs002200000261>), MR 1794265 (<https://www.ams.org/mathscinet-getitem?mr=1794265>), S2CID 11095649 (<https://api.semanticscholar.org/CorpusID:11095649>)
- Knauf, Andreas (1999), "Number theory, dynamical systems and statistical mechanics", *Reviews in Mathematical Physics. A Journal for Both Review and Original Research Papers in the Field of Mathematical Physics*, **11** (8): 1027–1060, Bibcode:1999RvMaP..11.1027K (<https://ui.adsabs.harvard.edu/abs/1999RvMaP..11.1027K>), doi:10.1142/S0129055X99000325 (<https://doi.org/10.1142/S0129055X99000325>), MR 1714352 (<https://www.ams.org/mathscinet-getitem?mr=1714352>)
- von Koch, Niels Helge (1901), "Sur la distribution des nombres premiers" (<https://zenodo.org/record/2347595>), *Acta Mathematica*, **24**: 159–182, doi:10.1007/BF02403071 (<https://doi.org/10.1007%2BF02403071>), S2CID 119914826 (<https://api.semanticscholar.org/CorpusID:119914826>)
- Kurokawa, Nobushige (1992), "Multiple zeta functions: an example", *Zeta functions in geometry (Tokyo, 1990)*, Adv. Stud. Pure Math., **21**, Tokyo: Kinokuniya, pp. 219–226, MR 1210791 (<https://www.ams.org/mathscinet-getitem?mr=1210791>)
- Lapidus, Michel L. (2008), *In search of the Riemann zeros*, Providence, R.I.: American Mathematical Society, doi:10.1090/mkb/051 (<https://doi.org/10.1090/mkb/051>), ISBN 978-0-8218-4222-5, MR 2375028 (<https://www.ams.org/mathscinet-getitem?mr=2375028>)
- Lavrik, A. F. (2001) [1994], "Zeta-function" (<https://www.encyclopediaofmath.org/index.php?title=Zeta-function>), *Encyclopedia of Mathematics*, EMS Press

- **Lehmer, D. H.** (1956), "Extended computation of the Riemann zeta-function", *Mathematika. A Journal of Pure and Applied Mathematics*, **3** (2): 102–108, doi:10.1112/S0025579300001753 (<https://doi.org/10.1112/S0025579300001753>), MR 0086083 (<https://www.ams.org/mathscinet-getitem?mr=0086083>)
- **Leichtnam, Eric** (2005), "An invitation to Deninger's work on arithmetic zeta functions", *Geometry, spectral theory, groups, and dynamics*, Contemp. Math., **387**, Providence, RI: Amer. Math. Soc., pp. 201–236, doi:10.1090/conm/387/07243 (<https://doi.org/10.1090/conm/387/07243>), MR 2180209 (<https://www.ams.org/mathscinet-getitem?mr=2180209>).
- **Levinson, N.** (1974), "More than one-third of the zeros of Riemann's zeta function are on $\sigma = 1/2$ ", *Adv. Math.*, **13** (4): 383–436, doi:10.1016/0001-8708(74)90074-7 ([https://doi.org/10.1016/0001-8708\(74\)90074-7](https://doi.org/10.1016/0001-8708(74)90074-7)), MR 0564081 (<https://www.ams.org/mathscinet-getitem?mr=0564081>)
- **Littlewood, J. E.** (1962), "The Riemann hypothesis", *The scientist speculates: an anthology of partly baked idea*, New York: Basic books
- van de Lune, J.; **te Riele, H. J. J.**; Winter, D. T. (1986), "On the zeros of the Riemann zeta function in the critical strip. IV", *Mathematics of Computation*, **46** (174): 667–681, doi:10.2307/2008005 (<https://doi.org/10.2307/2008005>), JSTOR 2008005 (<https://www.jstor.org/stable/2008005>), MR 0829637 (<https://www.ams.org/mathscinet-getitem?mr=0829637>)
- Massias, J.-P.; Nicolas, Jean-Louis; Robin, G. (1988), "Évaluation asymptotique de l'ordre maximum d'un élément du groupe symétrique" (<http://matwbn.icm.edu.pl/tresc.php?wyd=6&tom=50&jez=>), *Acta Arithmetica*, **50** (3): 221–242, doi:10.4064/aa-50-3-221-242 (<https://doi.org/10.4064/2Faa-50-3-221-242>), MR 0960551 (<https://www.ams.org/mathscinet-getitem?mr=0960551>)
- Mazur, Barry; Stein, William (2015), *Prime Numbers and the Riemann Hypothesis* (<http://wstein.org/rh/>)
- **Montgomery, Hugh L.** (1973), "The pair correlation of zeros of the zeta function", *Analytic number theory*, Proc. Sympos. Pure Math., **XXIV**, Providence, R.I.: American Mathematical Society, pp. 181–193, MR 0337821 (<https://www.ams.org/mathscinet-getitem?mr=0337821>) Reprinted in (Borwein et al. 2008).
- **Montgomery, Hugh L.** (1983), "Zeros of approximations to the zeta function", in Erdős, Paul (ed.), *Studies in pure mathematics. To the memory of Paul Turán*, Basel, Boston, Berlin: Birkhäuser, pp. 497–506, ISBN 978-3-7643-1288-6, MR 0820245 (<https://www.ams.org/mathscinet-getitem?mr=0820245>)
- **Montgomery, Hugh L.**; Vaughan, Robert C. (2007), *Multiplicative Number Theory I. Classical Theory*, Cambridge studies in advanced mathematics, **97**, Cambridge University Press, ISBN 978-0-521-84903-6
- Nicely, Thomas R. (1999), "New maximal prime gaps and first occurrences" (<http://www.trnicely.net/gaps/gaps.html>), *Mathematics of Computation*, **68** (227): 1311–1315, Bibcode:1999MaCom..68.1311N (<https://ui.adsabs.harvard.edu/abs/1999MaCom..68.1311N>), doi:10.1090/S0025-5718-99-01065-0 (<https://doi.org/10.1090/S0025-5718-99-01065-0>), MR 1627813 (<https://www.ams.org/mathscinet-getitem?mr=1627813>).
- Nyman, Bertil (1950), *On the One-Dimensional Translation Group and Semi-Group in Certain Function Spaces*, PhD Thesis, University of Uppsala: University of Uppsala, MR 0036444 (<https://www.ams.org/mathscinet-getitem?mr=0036444>)
- Odlyzko, A. M.; te Riele, H. J. J. (1985), "Disproof of the Mertens conjecture" (https://archive.today/2012071011237/http://gdz.sub.uni-goettingen.de/no_cache/dms/load/img/?IDDOC=262633), *Journal für die reine und angewandte Mathematik*, **1985** (357): 138–160, doi:10.1515/crll.1985.357.138 (<https://doi.org/10.1515%2Fcrl.1985.357.138>), MR 0783538 (<https://www.ams.org/mathscinet-getitem?mr=0783538>), S2CID 13016831 (<https://api.semanticscholar.org/CorpusID:13016831>), archived from the original (http://gdz.sub.uni-goettingen.de/no_cache/dms/load/img/?IDDOC=262633) on 2012-07-11
- Odlyzko, A. M. (1987), "On the distribution of spacings between zeros of the zeta function", *Mathematics of Computation*, **48** (177): 273–308, doi:10.2307/2007890 (<https://doi.org/10.2307%2F2007890>), JSTOR 2007890 (<https://www.jstor.org/stable/2007890>), MR 0866115 (<https://www.ams.org/mathscinet-getitem?mr=0866115>)
- Odlyzko, A. M. (1990), "Bounds for discriminants and related estimates for class numbers, regulators and zeros of zeta functions: a survey of recent results" (http://www.numdam.org/item?id=JTNB_1990__2_1_119_0), *Séminaire de Théorie des Nombres de Bordeaux*, Série 2, **2** (1): 119–141, doi:10.5802/jtnb.22 (<https://doi.org/10.5802%2Fjtnb.22>), MR 1061762 (<https://www.ams.org/mathscinet-getitem?mr=1061762>)
- Odlyzko, A. M. (1992), *The 10^{20} -th zero of the Riemann zeta function and 175 million of its neighbors* (<http://www.dtc.umn.edu/~odlyzko/unpublished/zeta.10to20.1992.pdf>) (PDF) This unpublished book describes the implementation of the algorithm and discusses the results in detail.
- Odlyzko, A. M. (1998), *The 10^{21} -st zero of the Riemann zeta function* (<http://www.dtc.umn.edu/~odlyzko/unpublished/zeta.10to21.pdf>) (PDF)
- Ono, Ken; Soundararajan, K. (1997), "Ramanujan's ternary quadratic form", *Inventiones Mathematicae*, **130** (3): 415–454, Bibcode:1997InMat.130..415O (<https://ui.adsabs.harvard.edu/abs/1997InMat.130..415O>), doi:10.1007/s002220050191 (<https://doi.org/10.1007%2Fs002220050191>), S2CID 122314044 (<https://api.semanticscholar.org/CorpusID:122314044>)

- Patterson, S. J. (1988), *An introduction to the theory of the Riemann zeta-function*, Cambridge Studies in Advanced Mathematics, 14, Cambridge University Press, doi:10.1017/CBO9780511623707 (<https://doi.org/10.1017/CBO9780511623707>), ISBN 978-0-521-33535-5, MR 0933558 (<https://www.ams.org/mathscinet-getitem?mr=0933558>)
- Platt, David; Trudgian, Tim (2020), *The Riemann hypothesis is true up to $3 \cdot 10^{12}$* , arXiv:2004.09765v1 (<https://arxiv.org/abs/2004.09765v1>)
- Radziejewski, Maciej (2007), "Independence of Hecke zeta functions of finite order over normal fields", *Transactions of the American Mathematical Society*, 359 (5): 2383–2394, doi:10.1090/S0002-9947-06-04078-5 (<https://doi.org/10.1090/S0002-9947-06-04078-5>), MR 2276625 (<https://www.ams.org/mathscinet-getitem?mr=2276625>), "There are infinitely many nonisomorphic algebraic number fields whose Dedekind zeta functions have infinitely many nontrivial multiple zeros."
- Ribenboim, Paulo (1996), *The New Book of Prime Number Records*, New York: Springer, ISBN 0-387-94457-5
- Riemann, Bernhard (1859), "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse" (<http://www.maths.tcd.ie/pub/HistMath/People/Riemann/Zeta/>), *Monatsberichte der Berliner Akademie*. In *Gesammelte Werke*, Teubner, Leipzig (1892), Reprinted by Dover, New York (1953). Original manuscript (<http://www.claymath.org/publications/riemanns-1859-manuscript/>) (with English translation). Reprinted in (Borwein et al. 2008) and (Edwards 1974)
- Riesel, Hans; Göhl, Gunnar (1970), "Some calculations related to Riemann's prime number formula", *Mathematics of Computation*, 24 (112): 969–983, doi:10.2307/2004630 (<https://doi.org/10.2307/2004630>), JSTOR 2004630 (<https://www.jstor.org/stable/2004630>), MR 0277489 (<https://www.ams.org/mathscinet-getitem?mr=0277489>)
- Riesz, M. (1916), "Sur l'hypothèse de Riemann", *Acta Mathematica*, 40: 185–190, doi:10.1007/BF02418544 (<https://doi.org/10.1007/BF02418544>)
- Robin, G. (1984), "Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann", *Journal de Mathématiques Pures et Appliquées*, Neuvième Série, 63 (2): 187–213, MR 0774171 (<http://www.ams.org/mathscinet-getitem?mr=0774171>)
- Rosser, J. Barkley; Yohe, J. M.; Schoenfeld, Lowell (1969), "Rigorous computation and the zeros of the Riemann zeta-function. (With discussion)", *Information Processing 68 (Proc. IFIP Congress, Edinburgh, 1968)*, Vol. 1: Mathematics, Software, Amsterdam: North-Holland, pp. 70–76, MR 0258245 (<https://www.ams.org/mathscinet-getitem?mr=0258245>)
- Rudin, Walter (1973), *Functional Analysis*, 1st edition (January 1973), New York: McGraw-Hill, ISBN 0-070-54225-2
- Salem, Raphaël (1953), "Sur une proposition équivalente à l'hypothèse de Riemann", *Les Comptes rendus de l'Académie des sciences*, 236: 1127–1128, MR 0053148 (<https://www.ams.org/mathscinet-getitem?mr=0053148>)
- Sarnak, Peter (2005), *Problems of the Millennium: The Riemann Hypothesis* (2004) (http://www.claymath.org/sites/default/files/sarnak_rh_0.pdf) (PDF), Clay Mathematics Institute, retrieved 2015-07-28 Reprinted in (Borwein et al. 2008).
- Schoenfeld, Lowell (1976), "Sharper bounds for the Chebyshev functions $\theta(x)$ and $\psi(x)$. II", *Mathematics of Computation*, 30 (134): 337–360, doi:10.2307/2005976 (<https://doi.org/10.2307/2005976>), JSTOR 2005976 (<https://www.jstor.org/stable/2005976>), MR 0457374 (<https://www.ams.org/mathscinet-getitem?mr=0457374>)
- Schumayer, Daniel; Hutchinson, David A. W. (2011), "Physics of the Riemann Hypothesis", *Reviews of Modern Physics*, 83 (2): 307–330, arXiv:1101.3116 (<https://arxiv.org/abs/1101.3116>), Bibcode:2011RvMP...83..307S (<https://ui.adsabs.harvard.edu/abs/2011RvMP...83..307S>), doi:10.1103/RevModPhys.83.307 (<https://doi.org/10.1103/RevModPhys.83.307>), S2CID 119290777 (<https://api.semanticscholar.org/CorpusID:119290777>)
- Selberg, Atle (1942), "On the zeros of Riemann's zeta-function", *SKR. Norske Vid. Akad. Oslo I.*, 10: 59 pp, MR 0010712 (<https://www.ams.org/mathscinet-getitem?mr=0010712>)
- Selberg, Atle (1946), "Contributions to the theory of the Riemann zeta-function", *Arch. Math. Naturvid.*, 48 (5): 89–155, MR 0020594 (<https://www.ams.org/mathscinet-getitem?mr=0020594>)
- Selberg, Atle (1956), "Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series", *J. Indian Math. Soc. (N.S.)*, 20: 47–87, MR 0088511 (<https://www.ams.org/mathscinet-getitem?mr=0088511>)
- Serre, Jean-Pierre (1969–1970), "Facteurs locaux des fonctions zeta des variétés algébriques (définitions et conjectures)" (<https://eudml.org/doc/110758>), *Séminaire Delange-Pisot-Poitou*, 19
- Sheats, Jeffrey T. (1998), "The Riemann hypothesis for the Goss zeta function for $F_q[T]$ ", *Journal of Number Theory*, 71 (1): 121–157, arXiv:math/9801158 (<https://arxiv.org/abs/math/9801158>), doi:10.1006/jnth.1998.2232 (<https://doi.org/10.1006/jnth.1998.2232>), MR 1630979 (<https://www.ams.org/mathscinet-getitem?mr=1630979>), S2CID 119703557 (<https://api.semanticscholar.org/CorpusID:119703557>)
- Siegel, C. L. (1932), "Über Riemanns Nachlaß zur analytischen Zahlentheorie", *Quellen Studien zur Geschichte der Math. Astron. Und Phys. Abt. B: Studien 2*: 45–80 Reprinted in *Gesammelte Abhandlungen*, Vol. 1. Berlin: Springer-Verlag, 1966.

- Speiser, Andreas (1934), "Geometrisches zur Riemannschen Zetafunktion" (https://web.archive.org/web/20150627115412/http://gdz.sub.uni-goettingen.de/index.php?id=11&PPN=PPN235181684_0110&DMDID=DMDLOG_0032&L=1), *Mathematische Annalen*, **110**: 514–521, doi:10.1007/BF01448042 (<https://doi.org/10.1007%2FBF01448042>), JFM 60.0272.04 (<https://zbmath.org/?format=complete&q=an:60.0272.04>), S2CID 119413347 (<https://api.semanticscholar.org/CorpusID:119413347>), archived from the original (http://gdz.sub.uni-goettingen.de/index.php?id=11&PPN=PPN235181684_0110&DMDID=DMDLOG_0032&L=1) on 2015-06-27
- Spira, Robert (1968), "Zeros of sections of the zeta function. II", *Mathematics of Computation*, **22** (101): 163–173, doi:10.2307/2004774 (<https://doi.org/10.2307%2F2004774>), JSTOR 2004774 (<https://www.jstor.org/stable/2004774>), MR 0228456 (<https://www.ams.org/mathscinet-getitem?mr=0228456>)
- Stein, William; Mazur, Barry (2007), *What is Riemann's Hypothesis?* (<https://web.archive.org/web/20090327181331/http://modular.math.washington.edu/edu/2007/simuw07/notes/rh.pdf>) (PDF), archived from the original (<http://modular.math.washington.edu/edu/2007/simuw07/notes/rh.pdf>) (PDF) on 2009-03-27
- Suzuki, Masatoshi (2011), "Positivity of certain functions associated with analysis on elliptic surfaces", *Journal of Number Theory*, **131** (10): 1770–1796, doi:10.1016/j.jnt.2011.03.007 (<https://doi.org/10.1016%2Fj.jnt.2011.03.007>)
- Titchmarsh, Edward Charles (1935), "The Zeros of the Riemann Zeta-Function", *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, The Royal Society, **151** (873): 234–255, Bibcode:1935RSPSA.151..234T (<https://ui.adsabs.harvard.edu/abs/1935RSPSA.151..234T>), doi:10.1098/rspa.1935.0146 (<https://doi.org/10.1098%2Frspa.1935.0146>), JSTOR 96545 (<https://www.jstor.org/stable/96545>)
- Titchmarsh, Edward Charles (1936), "The Zeros of the Riemann Zeta-Function", *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, The Royal Society, **157** (891): 261–263, Bibcode:1936RSPSA.157..261T (<https://ui.adsabs.harvard.edu/abs/1936RSPSA.157..261T>), doi:10.1098/rspa.1936.0192 (<https://doi.org/10.1098%2Frspa.1936.0192>), JSTOR 96692 (<https://www.jstor.org/stable/96692>)
- Titchmarsh, Edward Charles (1986), *The theory of the Riemann zeta-function* (2nd ed.), The Clarendon Press Oxford University Press, ISBN 978-0-19-853369-6, MR 0882550 (<https://www.ams.org/mathscinet-getitem?mr=0882550>)
- Trudgian, Timothy (2011), "On the success and failure of Gram's Law and the Rosser Rule", *Acta Arithmetica*, **125** (3): 225–256, doi:10.4064/aa148-3-2 (<https://doi.org/10.4064%2Faa148-3-2>)
- Turán, Paul (1948), "On some approximative Dirichlet-polynomials in the theory of the zeta-function of Riemann", *Danske Vid. Selsk. Mat.-Fys. Medd.*, **24** (17): 36, MR 0027305 (<https://www.ams.org/mathscinet-getitem?mr=0027305>) Reprinted in (Borwein et al. 2008).
- Turing, Alan M. (1953), "Some calculations of the Riemann zeta-function", *Proceedings of the London Mathematical Society*, Third Series, **3**: 99–117, doi:10.1112/plms/s3-3.1.99 (<https://doi.org/10.1112%2Fplms%2Fs3-3.1.99>), MR 0055785 (<https://www.ams.org/mathscinet-getitem?mr=0055785>)
- de la Vallée-Poussin, Ch.J. (1896), "Recherches analytiques sur la théorie des nombres premiers", *Ann. Soc. Sci. Bruxelles*, **20**: 183–256
- de la Vallée-Poussin, Ch.J. (1899–1900), "Sur la fonction $\zeta(s)$ de Riemann et la nombre des nombres premiers inférieurs à une limite donnée", *Mem. Couronnes Acad. Sci. Belg.*, **59** (1) Reprinted in (Borwein et al. 2008).
- Weil, André (1948), *Sur les courbes algébriques et les variétés qui s'en déduisent*, Actualités Sci. Ind., no. 1041 = Publ. Inst. Math. Univ. Strasbourg 7 (1945), Hermann et Cie., Paris, MR 0027151 (<https://www.ams.org/mathscinet-getitem?mr=0027151>)
- Weil, André (1949), "Numbers of solutions of equations in finite fields", *Bulletin of the American Mathematical Society*, **55** (5): 497–508, doi:10.1090/S0002-9904-1949-09219-4 (<https://doi.org/10.1090%2FS0002-9904-1949-09219-4>), MR 0029393 (<https://www.ams.org/mathscinet-getitem?mr=0029393>) Reprinted in Oeuvres Scientifiques/Collected Papers by Andre Weil ISBN 0-387-90330-5
- Weinberger, Peter J. (1973), "On Euclidean rings of algebraic integers", *Analytic number theory (St. Louis Univ., 1972)*, Proc. Sympos. Pure Math., **24**, Providence, R.I.: Amer. Math. Soc., pp. 321–332, MR 0337902 (<https://www.ams.org/mathscinet-getitem?mr=0337902>)
- Wiles, Andrew (2000), "Twenty years of number theory", *Mathematics: frontiers and perspectives*, Providence, R.I.: American Mathematical Society, pp. 329–342, ISBN 978-0-8218-2697-3, MR 1754786 (<https://www.ams.org/mathscinet-getitem?mr=1754786>)
- Zagier, Don (1977), "The first 50 million prime numbers" (https://web.archive.org/web/20090327181245/http://modular.math.washington.edu/edu/2007/simuw07/misc/zagier-the_first_50_million_prime_numbers.pdf) (PDF), *Math. Intelligencer*, Springer, **0**: 7–19, doi:10.1007/BF03039306 (<https://doi.org/10.1007%2BF03039306>), MR 0643810 (<https://www.ams.org/mathscinet-getitem?mr=0643810>), S2CID 189886510 (<https://api.semanticscholar.org/CorpusID:189886510>), archived from the original (http://modular.math.washington.edu/edu/2007/simuw07/misc/zagier-the_first_50_million_prime_numbers.pdf) (PDF) on 2009-03-27

- Zagier, Don (1981), "Eisenstein series and the Riemann zeta function", *Automorphic forms, representation theory and arithmetic (Bombay, 1979)*, Tata Inst. Fund. Res. Studies in Math., **10**, Tata Inst. Fundamental Res., Bombay, pp. 275–301, MR 0633666 (<https://www.ams.org/mathscinet-getitem?mr=0633666>)

Popular expositions

- Sabbagh, Karl (2003a), *The greatest unsolved problem in mathematics* (<https://archive.org/details/riemannhypothesis00sabb>), Farrar, Straus and Giroux, New York, ISBN 978-0-374-25007-2, MR 1979664 (<https://www.ams.org/mathscinet-getitem?mr=1979664>)
- Sabbagh, Karl (2003b), *Dr. Riemann's zeros* (<https://books.google.com/books?id=JesSAQAAQAAJ>), Atlantic Books, London, ISBN 978-1-843-54101-1
- du Sautoy, Marcus (2003), *The music of the primes* (<https://archive.org/details/musicoftimessea00dusa>), HarperCollins Publishers, ISBN 978-0-06-621070-4, MR 2060134 (<https://www.ams.org/mathscinet-getitem?mr=2060134>)
- Rockmore, Dan (2005), *Stalking the Riemann hypothesis* (<https://archive.org/details/stalkingriemannh00danr>), Pantheon Books, ISBN 978-0-375-42136-5, MR 2269393 (<https://www.ams.org/mathscinet-getitem?mr=2269393>)
- Derbyshire, John (2003), *Prime Obsession*, Joseph Henry Press, Washington, DC, ISBN 978-0-309-08549-6, MR 1968857 (<https://www.ams.org/mathscinet-getitem?mr=1968857>)
- Watkins, Matthew (2015), *Mystery of the Prime Numbers*, Liberalis Books, ISBN 978-1782797814, MR 0000000 (<https://www.ams.org/mathscinet-getitem?mr=0000000>)
- Frenkel, Edward (2014), *The Riemann Hypothesis* (https://www.google.com/search?sxsrf=ACYBGNTYki6txmHIRlwZbkChP9TyCDuFA%3A1568142650416&source=hp&ei=OvV3Xe6hF8aE-gTn6qu4BA&q=numberphile+riemann+hypothesis&oq=numberphile+riemann&gs_l=psy-ab.1.0.0j0i22i30l2.2619.9717..11617...1.0..0.573.3549.0j16j3j5-1.....0....1..gws-wiz.....0i131j35i39j0110j0i22i10i30.tM-qtL3ED8) Numberphile, Mar 11, 2014 (video)

External links

- American institute of mathematics, *Riemann hypothesis* (<http://www.aimath.org/WWN/rh/>)
- Zeroes database (<https://www.lmfdb.org/zeros/zeta/>), 103 800 788 359 zeroes
- *The Key to the Riemann Hypothesis* - Numberphile (<https://www.youtube.com/watch?v=VTveQ1ndH1c>), a YouTube video about the Riemann hypothesis by Numberphile
- Apostol, Tom, *Where are the zeros of zeta of s?* (<http://www.math.wisc.edu/~robbin/funnysongs.html#Zeta>) Poem about the Riemann hypothesis, sung (<http://www.olimu.com/RIEMANN/Song.htm>) by John Derbyshire.
- Borwein, Peter, *The Riemann Hypothesis* (<https://web.archive.org/web/20090327181245/http://oldweb.cecm.sfu.ca/~pborwein/COURSE/MATH08/LECTURE.pdf>) (PDF), archived from the original (<http://oldweb.cecm.sfu.ca/~pborwein/COURSE/MATH08/LECTURE.pdf>) on 2009-03-27 (Slides for a lecture)
- Conrad, K. (2010), *Consequences of the Riemann hypothesis* (<https://mathoverflow.net/q/17232>)
- Conrey, J. Brian; Farmer, David W, *Equivalences to the Riemann hypothesis* (<https://web.archive.org/web/20100316235054/http://aimath.org/pl/rhequivalences>), archived from the original (<http://aimath.org/pl/rhequivalences>) on 2010-03-16
- Gourdon, Xavier; Sebah, Pascal (2004), *Computation of zeros of the Zeta function* (<http://numbers.computation.free.fr/Constants/Miscellaneous/zetazeroscompute.html>) (Reviews the GUE hypothesis, provides an extensive bibliography as well).
- Odlyzko, Andrew, *Home page* (<http://www.dtc.umn.edu/~odlyzko/>) including papers on the zeros of the zeta function (<http://www.dtc.umn.edu/~odlyzko/doc/zeta.html>) and tables of the zeros of the zeta function (http://www.dtc.umn.edu/~odlyzko/zeta_tables/index.html)
- Odlyzko, Andrew (2002), *Zeros of the Riemann zeta function: Conjectures and computations* (<http://www.dtc.umn.edu/~odlyzko/talks/riemann-conjectures.pdf>) (PDF) Slides of a talk
- Pegg, Ed (2004), *Ten Trillion Zeta Zeros* (http://www.maa.org/editorial/mathgames/mathgames_10_18_04.html), Math Games website. A discussion of Xavier Gourdon's calculation of the first ten trillion non-trivial zeros
- Pugh, Glen, *Java applet for plotting Z(t)* (<http://web.viu.ca/pugh/RiemannZeta/RiemannZetaLong.html>)

- Rubinstein, Michael, *algorithm for generating the zeros* (https://web.archive.org/web/20070427221654/http://pmmac03.math.uwaterloo.ca/~mrubinst/l_function_public/L.html), archived from the original (http://pmmac03.math.uwaterloo.ca/~mrubinst/l_function_public/L.html) on 2007-04-27.
- du Sautoy, Marcus (2006), *Prime Numbers Get Hitched* (https://web.archive.org/web/20170922145127/http://seedmagazine.com/content/article/prime_numbers_get_hitched/), Seed Magazine, archived from the original (http://www.seedmagazine.com/news/2006/03/prime_numbers_get_hitched.php) on 2017-09-22, retrieved 2006-03-27
- Stein, William A., *What is Riemann's hypothesis* (<https://web.archive.org/web/20090104104251/http://modular.math.washington.edu/edu/2007/simuw07/index.html>), archived from the original (<http://modular.math.washington.edu/edu/2007/simuw07/index.html>) on 2009-01-04
- de Vries, Andreas (2004), *The Graph of the Riemann Zeta function $\zeta(s)$* (<http://math-it.org/Mathematik/Riemann/RiemannApplet.html>), a simple animated Java applet.
- Watkins, Matthew R. (2007-07-18), *Proposed proofs of the Riemann Hypothesis* (<http://secamlocal.ex.ac.uk/~mwatkins/zeta/RHproofs.htm>)
- *Zetagrid* (<https://web.archive.org/web/20131005173705/http://www.zetagrid.net/>) (2002) A distributed computing project that attempted to disprove Riemann's hypothesis; closed in November 2005

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