# General 2D Bin Packing Problem

A difficult but interesting problem

# Who are we?

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# Introduction

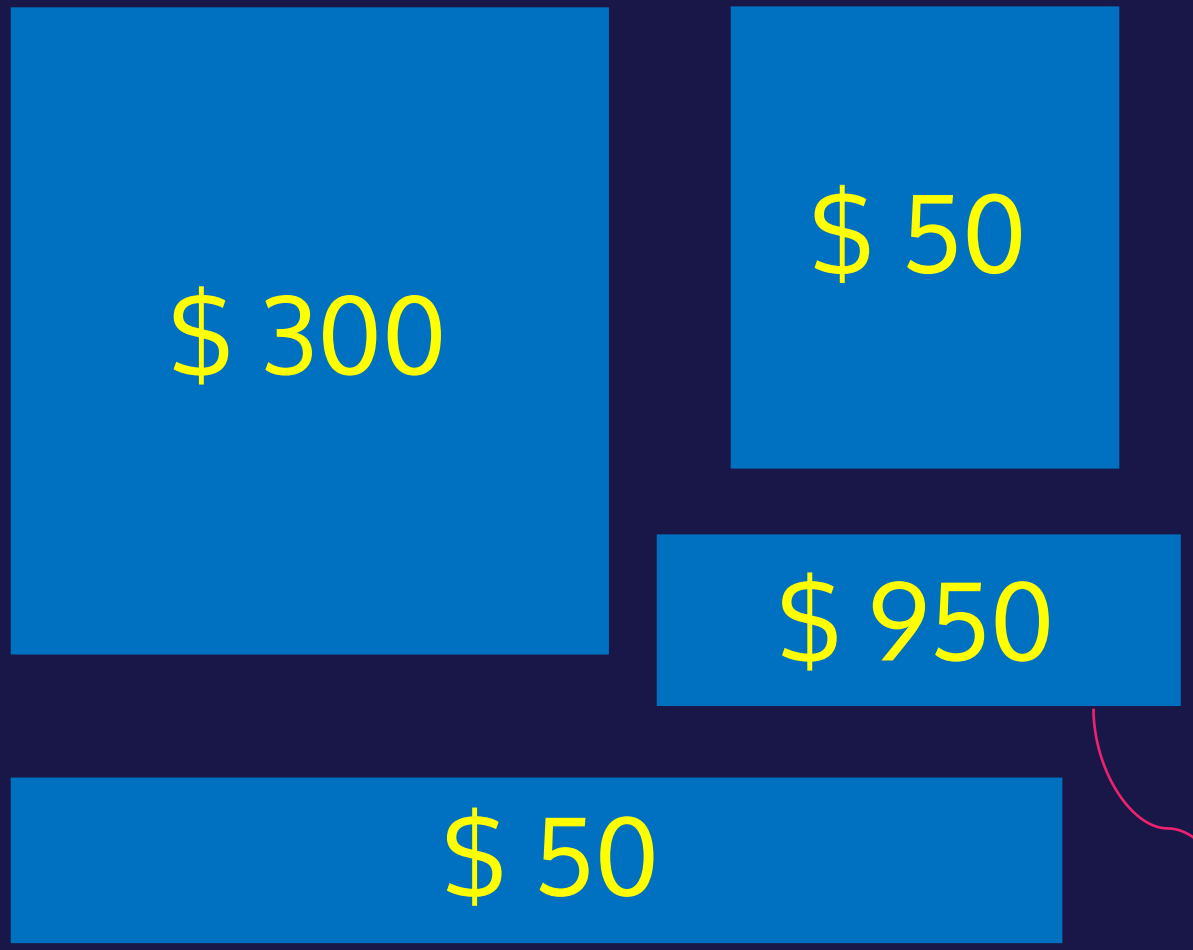
## General 2D Bin Packing Problem

Ok, I got “2D”, but what is “Bin Packing”?

# Items

A lot of “rectangle” items,
generally in different sizes.
We need to transport them all.

Don’t disappoint our customers.

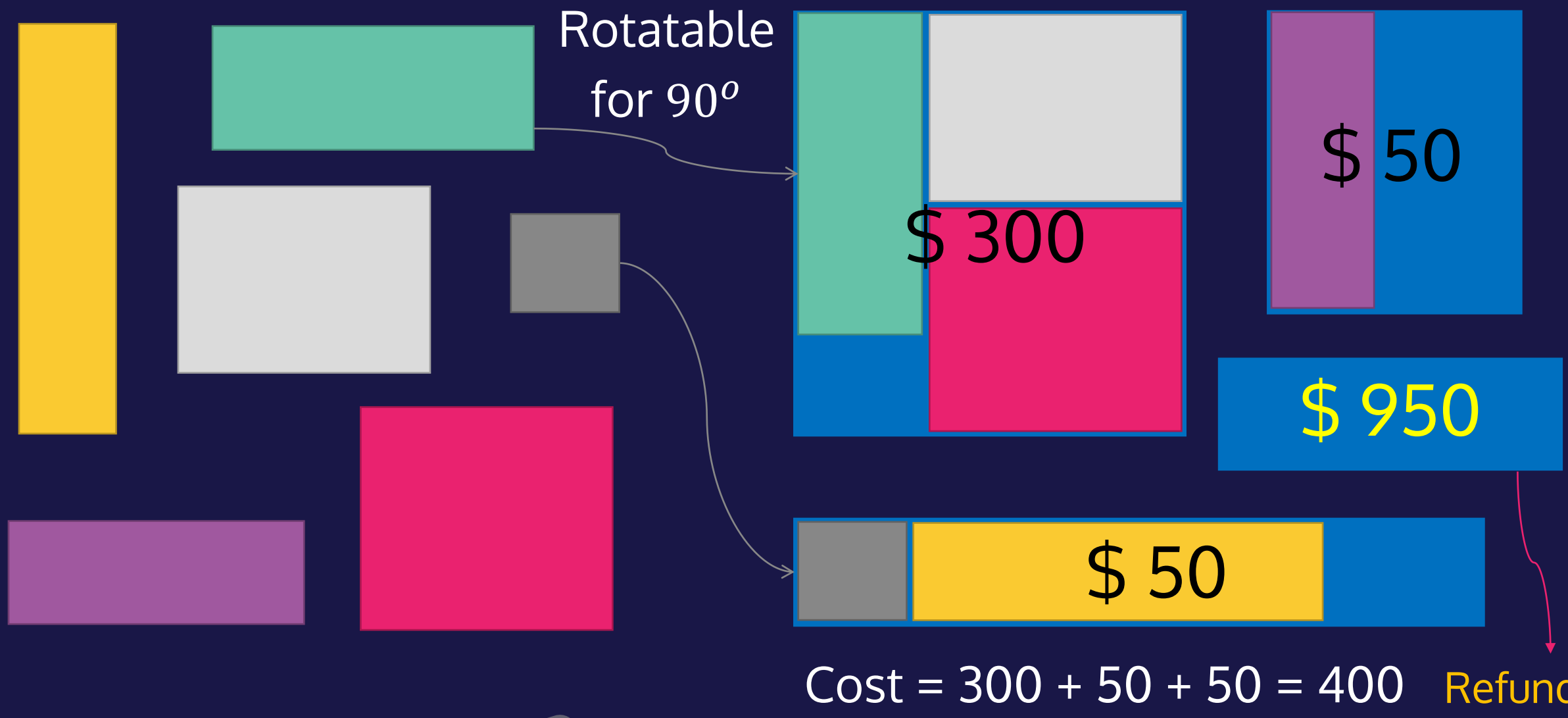


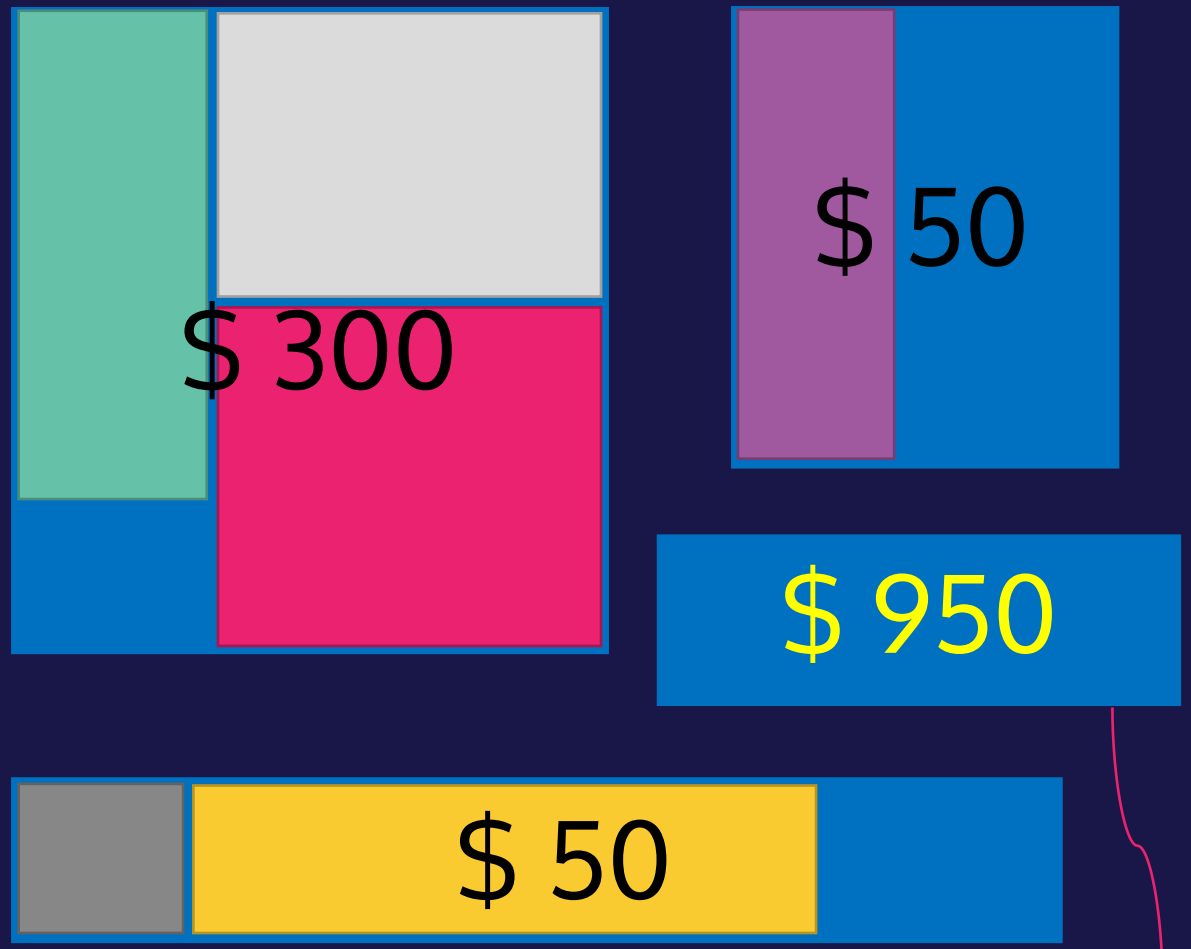

# Cars

We have no car, so we rent some.

We can refund if we don't use a car.

In general, the cars have different sizes and costs.


Rotatable
for 90º
$ 300
$ 50
$ 950
$ 50
Cost = 300 + 50 + 50 = 400
Refund!




# Cost and Objective

The total cost

= sum of rental fees of used cars

We want to minimize this cost.

Cost = 300 + 50 + 50 = 400

# Remark

- Each item has a size
- Each car has a size capacity and a cost
- Each item must be in one car
- All items in a car must somehow fit it orthogonally, in which items are rotatable for $90^o$
- Cost is the sum of fees of all used cars, minimize it

# Fun facts! Why “General 2D Bin Packing”?

## “Bin”

In articles and papers, instead of “car”, they wrote “bin”.

## “2D”

The original version of this problem is one-dimensional (1D), where each item has a “weight” instead of “size”. It is already an NP-hard problem.

## “General”

There are variations like:

- The cost is the number of cars used (rental fee = 1)
- Items are not rotatable

Which are much simpler.

# Input specification

# _sample_data.txt

```
7 3
2 3
1 5
2 2
3 2
1 4
3 2
4 1
3 5 100
4 5 150
5 6 200
```

**n** = 7: Number of rectangles

**m** = 3: Number of cars

Side lengths of 7 rectangles

$w_6 = 3$: Width of the 6th rectangle

$h_6 = 2$: Height of the 6th rectangle

Side lengths and costs of 3 cars

$W_3 = 5, H_3 = 6$: Width, height of the 3rd car

$c_3 = 200$: Cost of the 3rd car

# Data generation

And properties of data

# Define the number of rectangles

There will be 75 different input files, each of them has the number of rectangles (**n**) as follow:

- 50 easy and medium files: From 5 to 54 with step 1, i.e. 5, 6, 7, …, 54
- 10 hard files: From 60 to 330 with step 30, i.e. 60, 90, 120, …, 330
- 14 very hard and severe files: From 350 to 1000 with step 50, i.e. 350, 400, 450, …, 1000
- 1 sample file ☺

# The generating algorithm for each file

- Randomize rectangles
- Randomly put rectangles in cars
- Shrink sizes of cars
- Randomly generate a few more cars and randomly assign costs for all cars

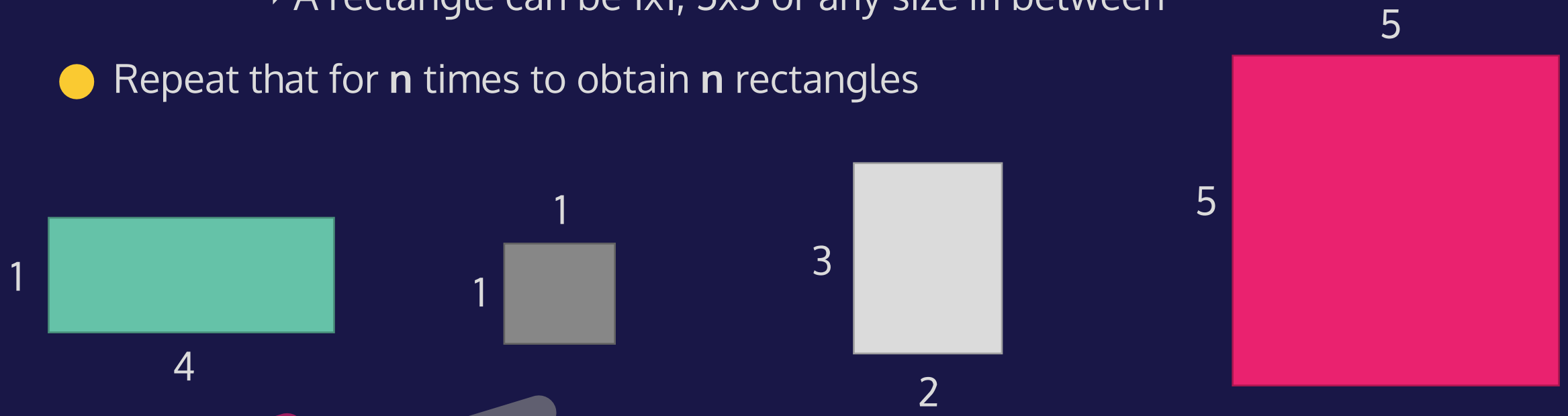
Note that the probability distribution of all random variables in this algorithm is uniform.

# Randomize rectangles

- Randomize a rectangle: each side is a random integer from 1 to 5

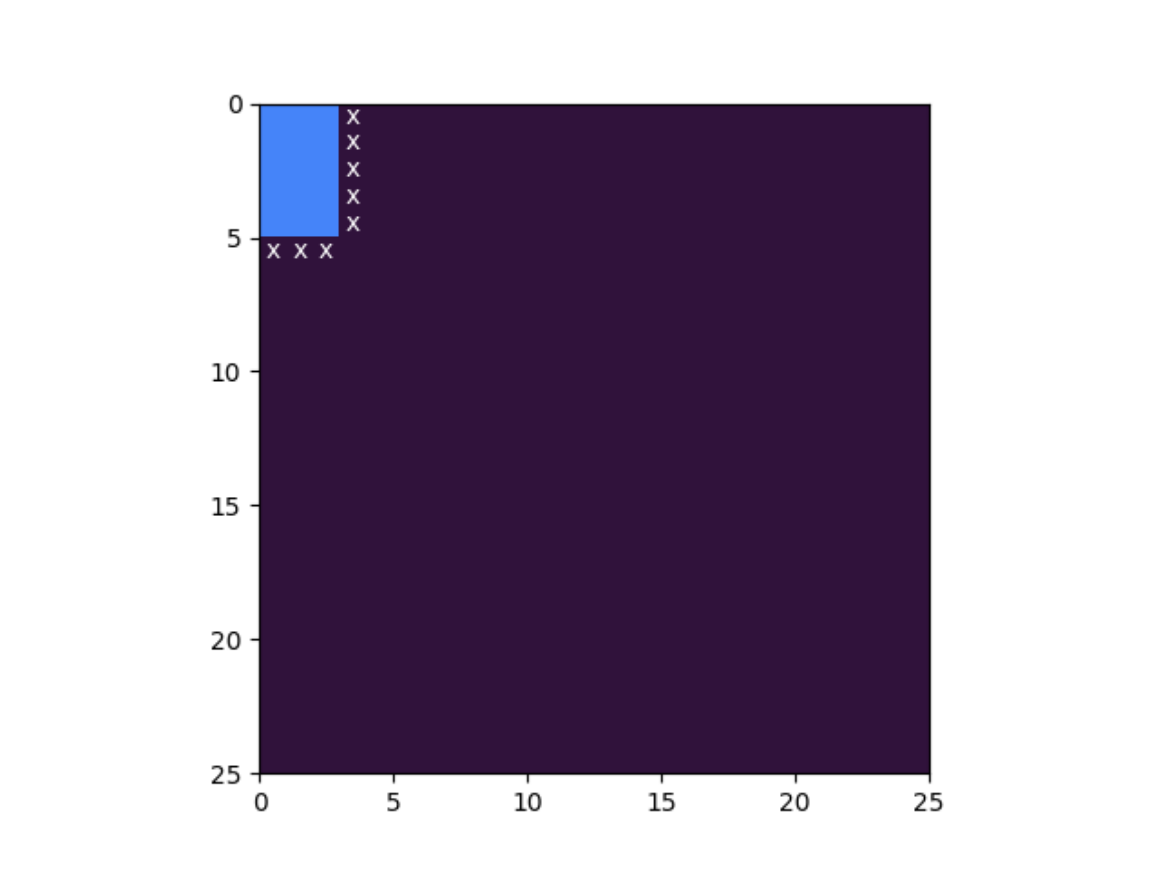
  → A rectangle can be 1x1, 5x5 or any size in between

- Repeat that for **n** times to obtain **n** rectangles



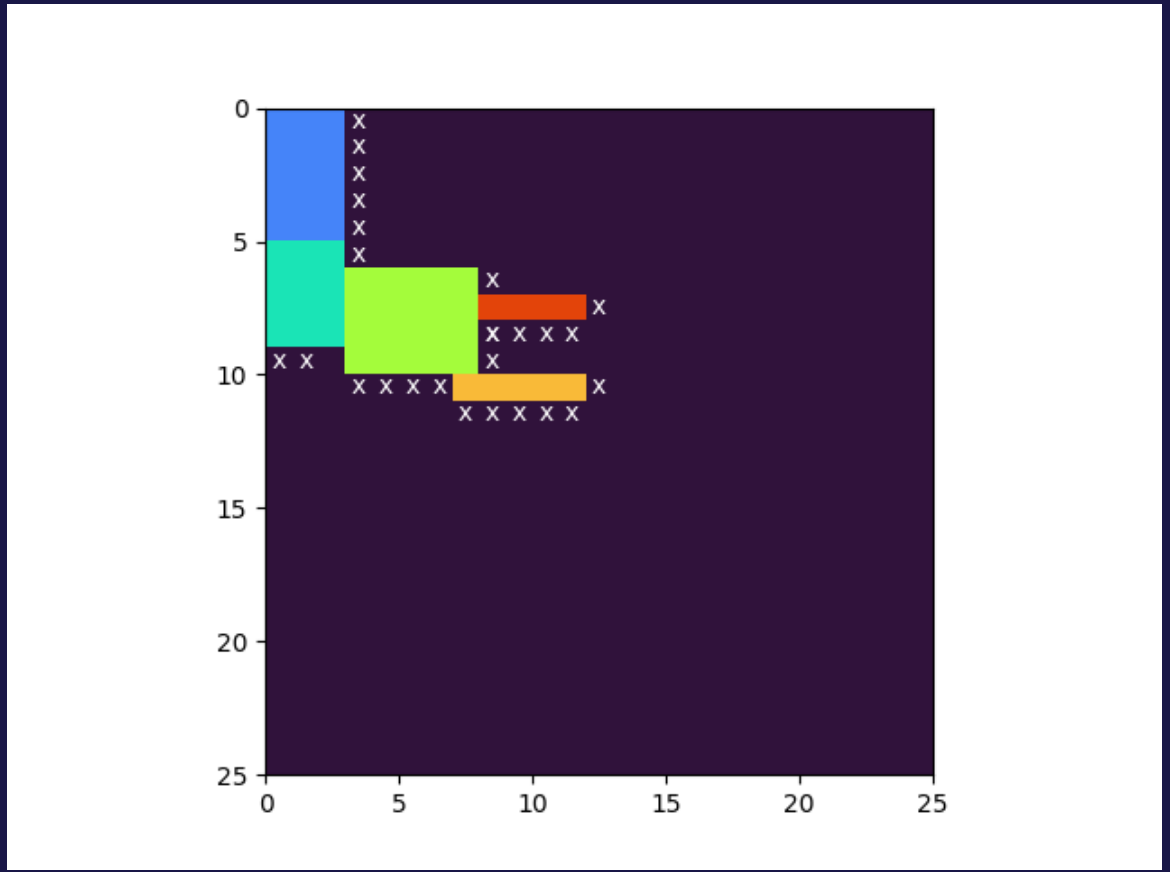
# Randomly put rectangles in cars

- Out of **n** rectangles, randomly pick from 2 to 5 rectangles
- Initialize a 25x25 car (there is a mathematical reason behind it)
- Put the first rectangle in the upper-left corner
- Mark any place that is next to the right or lower bound of the rectangle

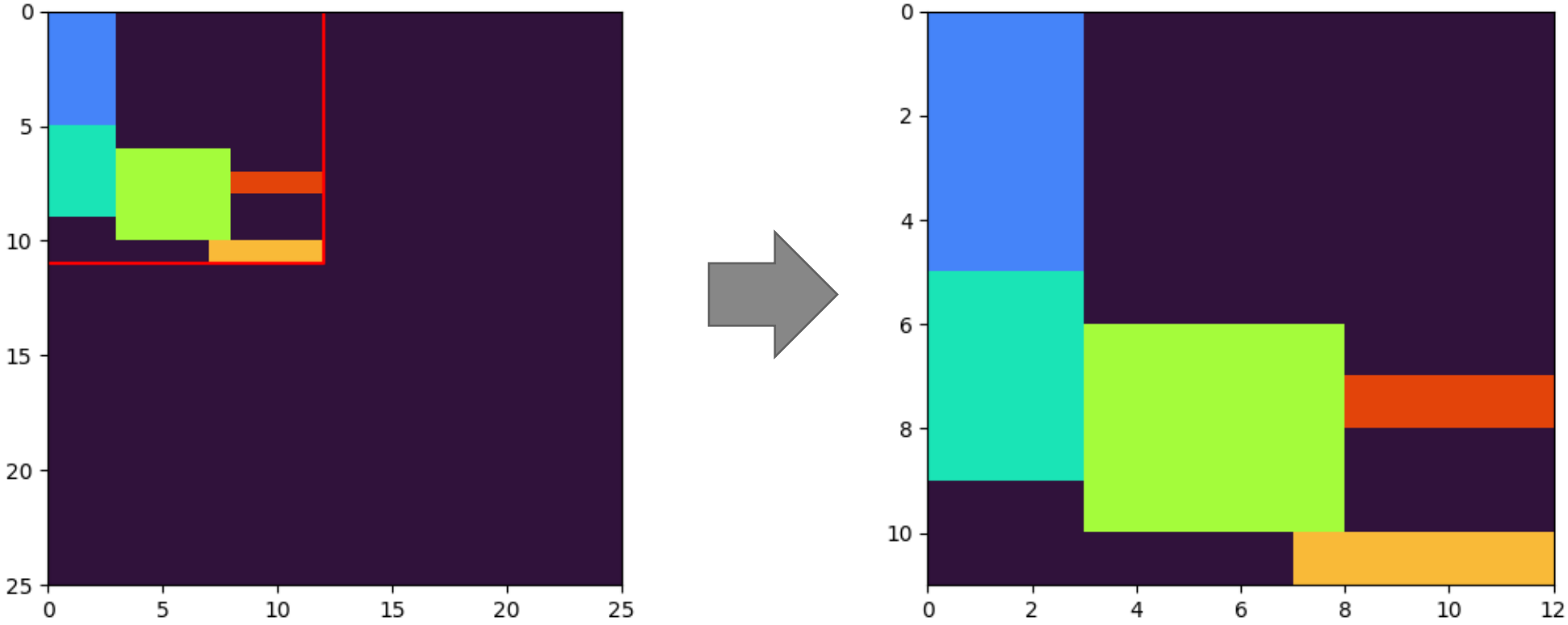


# Randomly put rectangles in cars

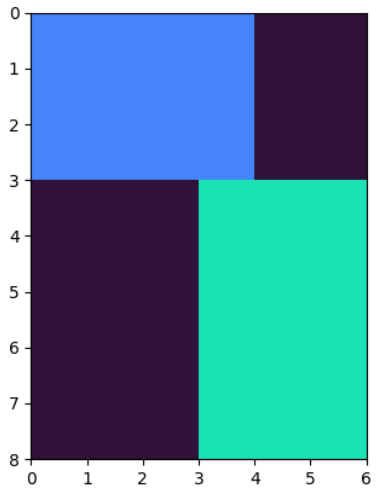
- Out of the rectangles left, choose one and randomly put it in any marked place
- Repeat the step above until there is no rectangle left (from 1 to 4 times of repetition)

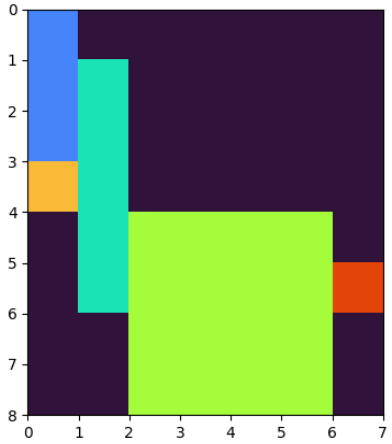


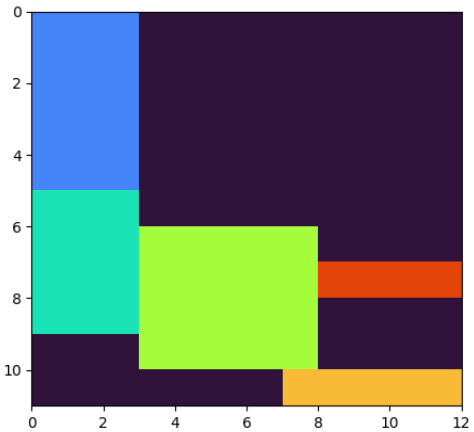
# Shrink sizes of cars

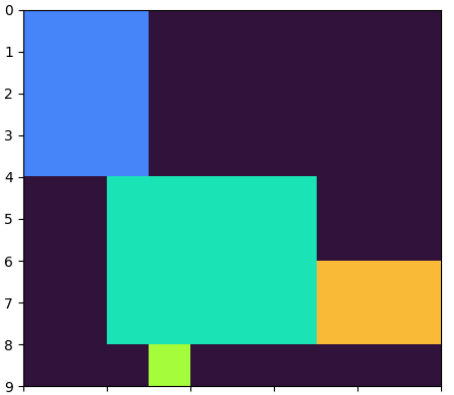


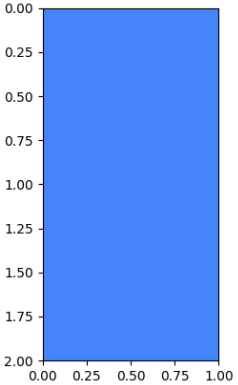
# Some cars of 0030.txt

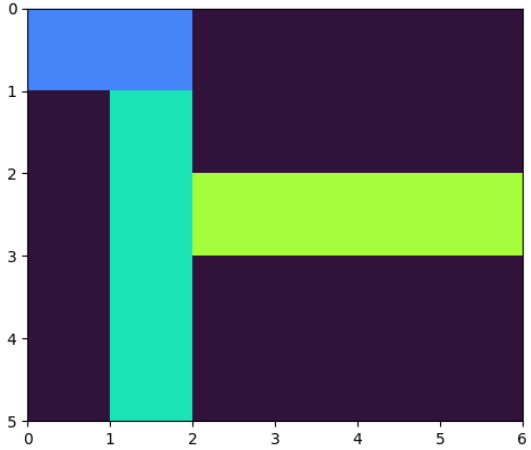


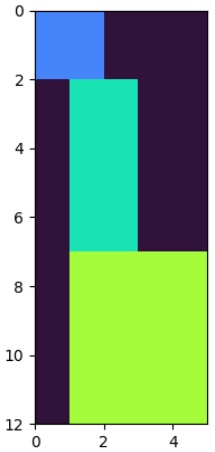



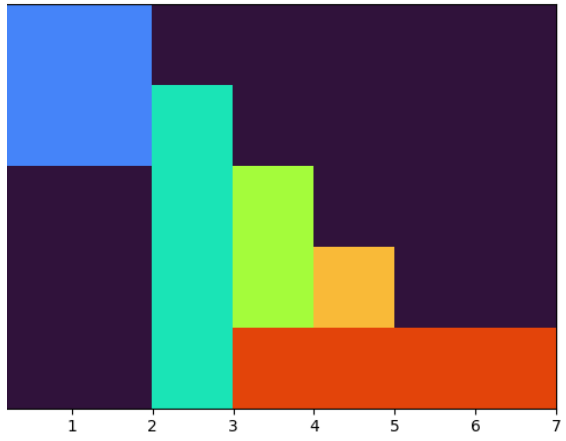



# Randomly generate a few more cars and randomly assign costs for all cars

- Randomly generate a few more cars:

  Random each side of each car: From 1 to 25

  → A car can be 1x1, 25x25 or any size in between

  "A few" = Ceil of 1/5 number of the used cars

- Randomly assign a cost to every car so far:

  From 100 to 1000 with step 50, i.e. 100, 150, 200, …, 1000

# Data properties

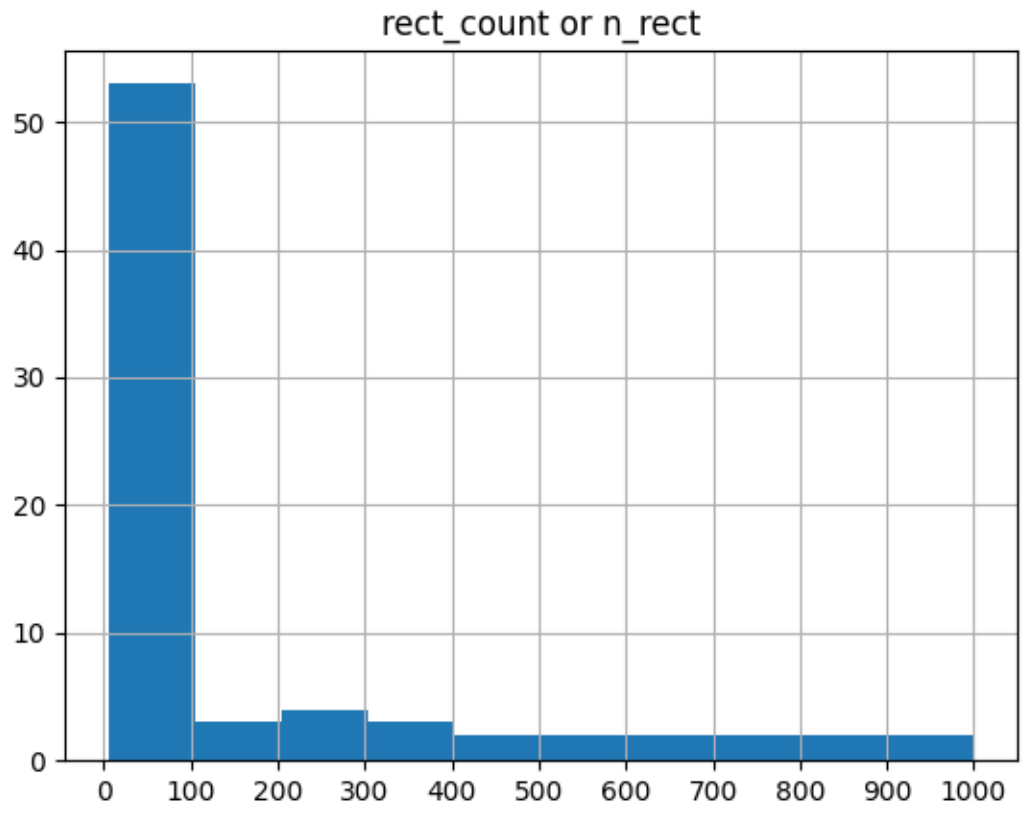
- There are a lot more cars than needed
- Theoretically, the maximum number of rectangles that can fit into one car is $25 \times 25 = 625$

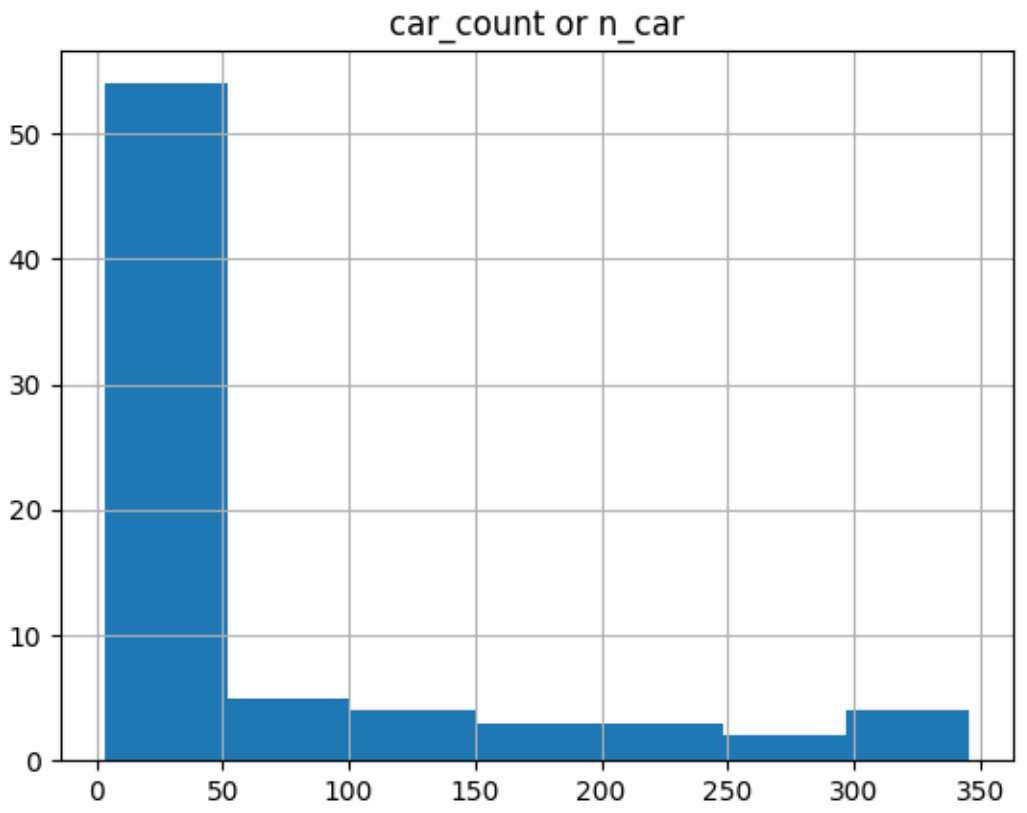
→ Backtracking, Brute force, Dynamic programming, or Branch and bound is impossible due to complexity (if implemented, the mean branching factor is 60, the mean depth is 172, so there are about $60^{172} \cong 7 \times 10^{305}$ nodes, for each node, its legitimation is also a very costly backtracking algorithm)

- Multiple "kinds" of rectangle, only with 5 different side-lengths:

  Small: 1x1, Big: 5x5, Normal: 3x2, "Annoying": 1x5, …

# Data properties: Size distribution visualized



# Problem Formulation

Building CP and MIP model

# Denotation

- $R = \{1, \ldots, n\}$ is the set of given rectangles

  Item i has width $w_i$ and height $h_i$

- $B = \{1, \ldots, m\}$ is the set of available bins

  Bin k has width $W_k$, height $H_k$ and cost $c_k$

# Variables

- $o_i \in \{0, 1\}$ represents the orientation of rectangle i
- $l_i$ , $r_i$, $t_i$ , $b_i$ are left, right, top and bottom coordinates of item i
- Binary variable $u_k$ is 1 iff bin k is used
- Binary variable $p_{ik}$ is 1 iff item i is placed in bin k
- $y_i = k \in \{1, \ldots , m\}$, i.e. item i is placed in bin k

# CP model - Constraints

- $o_i = 0 \Rightarrow r_i = l_i + w_i \wedge t_i = b_i + h_i$ $\quad \forall i \in R$ (1)
- $o_i = 1 \Rightarrow r_i = l_i + h_i \wedge t_i = b_i + w_i$ $\quad \forall i \in R$ (2)
- $y_i = y_j \Rightarrow r_i \leq l_i \vee r_j \leq l_i \vee t_i \leq b_j \vee t_j \leq b_i$ $\quad \forall i, j \in R, i < j$ (3)
- $y_i = k \Rightarrow r_i \leq W_i \wedge t_i \leq H_i$ $\quad \forall i, j \in R, k \in B$ (4)
- $p_{ik} = 1 \Rightarrow y_i = k$ $\quad \forall i \in R, k \in B$ (5)
- $\sum_{i=1}^{n} p_{ik} \geq 1 \Rightarrow u_k = 1$ $\quad \forall k \in B$ (6)

# Constraints

- (1) If an item doesn't rotate, its right = its left + its width and its top = its bottom + its height
- (2) If the item rotates then its right = its left + its height and its top = its bottom + its width
- (3) If two items are placed in the same bin, they can't overlap each other
- (4) If one item is place in a bin then its right and top coordinates can't exceed the bin
- (5) Item **i** is placed in bin **k**
- (6) Bin **k** is used when at least one item is placed in it

# MIP model - Constraints

- $o_i = 0 \Rightarrow r_i = l_i + w_i \wedge t_i = b_i + h_i$ (1); $o_i = 1 \Rightarrow r_i = l_i + h_i \wedge t_i = b_i + w_i$ (2)

  To MIP: $r_i = l_i + w_i * (1 - o_i) + h_i * o_i$ ; $t_i = b_i + h_i * (1 - o_i) + w_i * o_i$

- $y_i = y_j \Rightarrow r_i \leq l_i \vee r_j \leq l_i \vee t_i \leq b_j \vee t_j \leq b_i$ (3)

  To MIP: $r_i \leq M * (1 - x_1) + l_j$; $r_j \leq M * (1 - x_2) + l_i$;

  $t_i \leq M * (1 - x_3) + b_j$; $t_j \leq M * (1 - x_4) + b_i$;

  $x_1 + x_2 + x_3 + x_4 + (1 - x_0) * M \geq 1$;

  $x_1 + x_2 + x_3 + x_4 \leq x_0 * M$; $b = 1 - x_{00}$;

  $y_j + x_{00} \leq y_i + M * x_{01}$; $y_i + x_{00} \leq y_j + M * x_{02}$; $x_{01} + x_{02} \leq x_{00}$;

# MIP model - Constraints

- $y_i = k \Rightarrow r_i \leq W_i \wedge t_i \leq H_i$ (4)

  To MIP: $k + x_{00} \leq y_i + M * x_{01};\ y_i + x_{00} \leq k + M * x_{02}; x_{01} + x_{02} = x_{00};$

  $l_i \leq w_k + x_{00} * M; r_i \leq w_k + x_{00} * M; t_i \leq h_k + x_{00} * M; b_i \leq h_k + x_{00} * M;$

- $p_{ik} = 1 \Rightarrow y_i = k$; To MIP: $y_i = k * p_{ik}$ (5)

- $\sum_{i=1}^{n} p_{ik} \geq 1 \Rightarrow u_k = 1$ (6)

  To MIP: $S = \sum_{i=1}^{n} p_{ik}$

  $S \leq (1 - e) * M; S + e * M \geq 1; u_k = 1 - e;$

# Heuristic

The quickness of heuristic + The completeness of backtracking

# The heuristic algorithm

## Sort

- Cars by "economic efficiency" (cost/area), in ascending order.
- Rectangles by area, in descending order.

Note that the **`For`** loops in the next step iterate through those sorted lists.

## Best-fit (but sounds like first-fit)

- Initialize all cars, each contains no rectangle.
- **`For`** each rectangle, **`for`** each car, try to contain the rectangle in the car, if the rectangle fits, contain it in the car and **`break`** the car loop (to the next rectangle).

## "Try to contain"

In a car, try to contain the new rectangle along with the old rectangle(s) that are contained.

→ Since there is no certain way to fix the places of all the rectangles, a backtracking algorithm must be run to check.

# But why?

## Sort

- The best cars will be at the top of the car list.
- The largest rectangles will be at the top of the rectangle list.

## Best-fit (but sounds like first-fit)

- The largest rectangles will fit in some best cars.
- When continue to iterate through next rectangles, some smaller one could fit in best cars too.

→ Try to use the best cars as much as possible.

## Why sort rectangles?

If we sort rectangles, there will be **fewer** rectangles to fit in the best cars, since: (some **big** + some *small* rectangles) < a lot of *small* rectangles

→ A certain way to reduce the overall running time of the backtracking algorithm.

# Analysis of algorithms

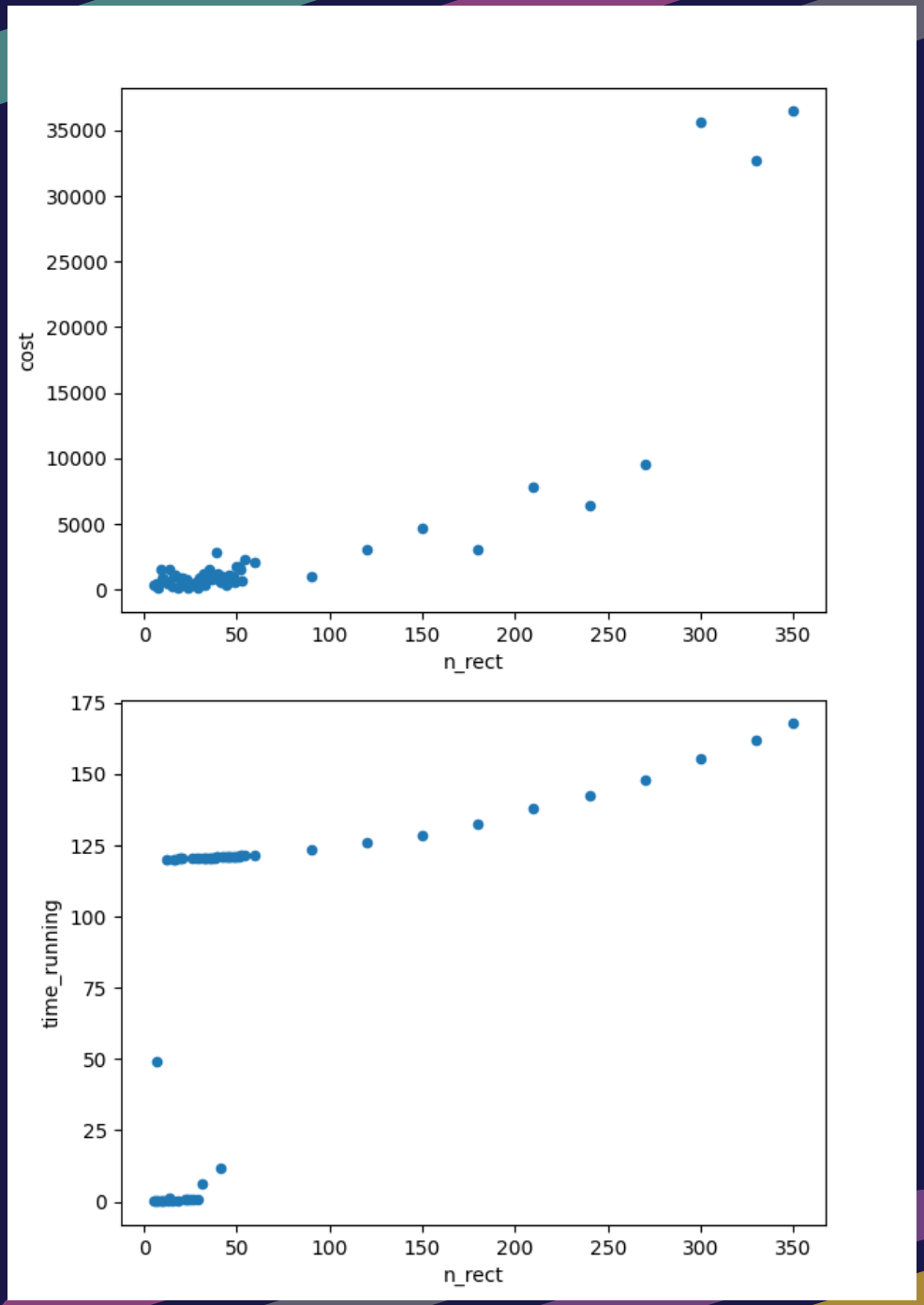
Practically analyze their performance

# CP model

- Out of 75 tests, the model can solve 62, with a 120-second time limitation
- The objective values (costs) of solved instances are excellent in general
- About half of the tests are hard enough for the algorithm to reach the time limit

Cost

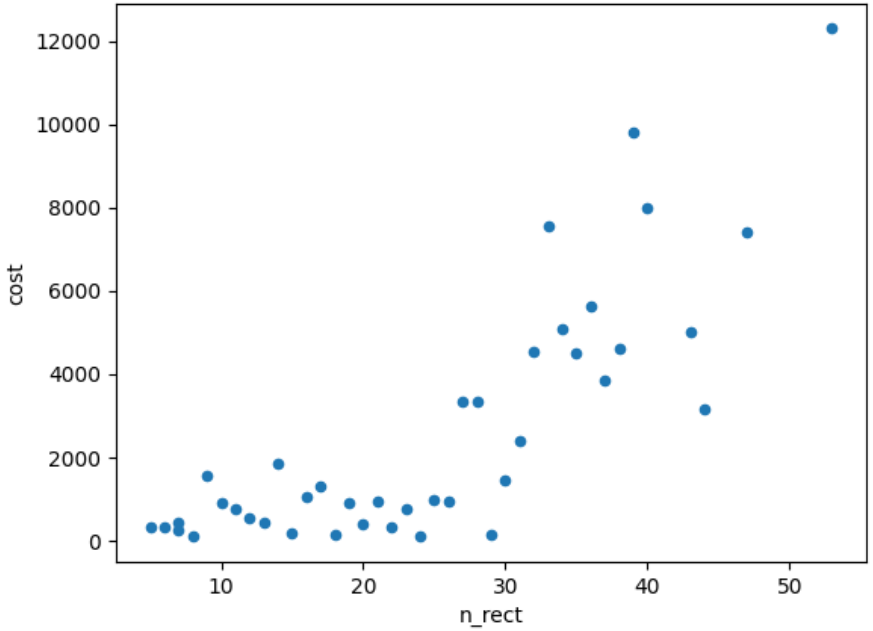
Running time



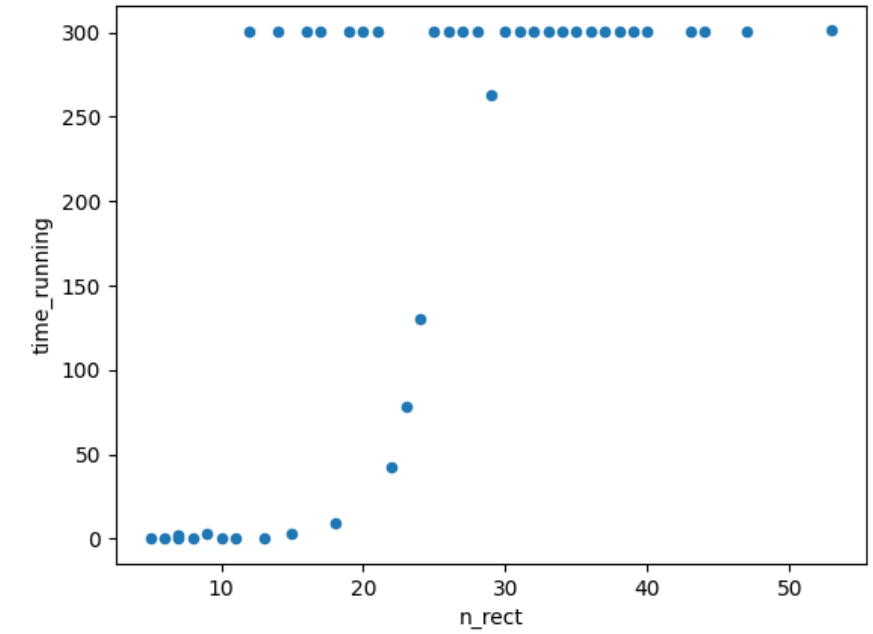
# MIP model

- Out of 75 tests, the model can solve 41, with a 300-second time limitation
- The objective values (costs) of solved instances are excellent in general
- About 2/3 of the tests are hard enough for the algorithm to reach the time limit

Cost



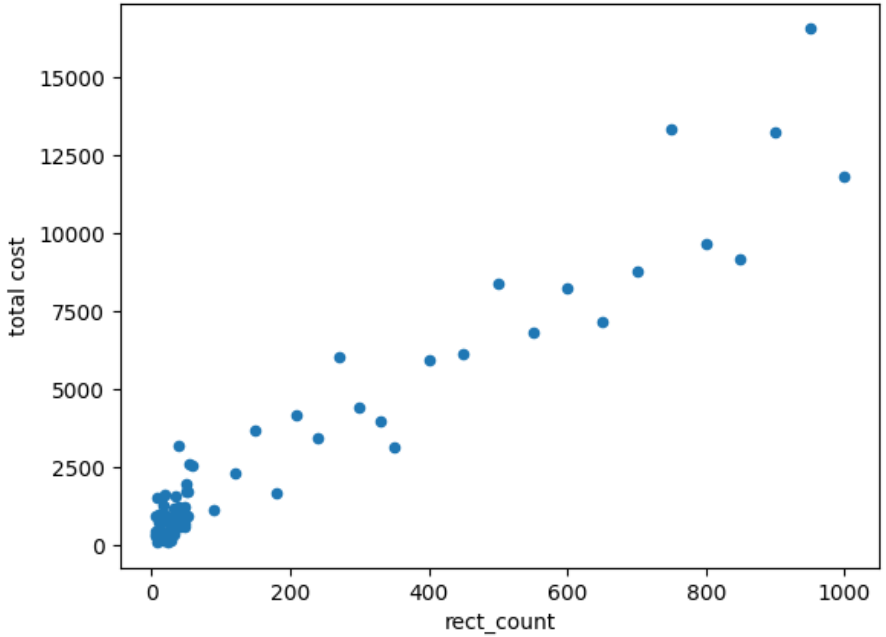

Running time



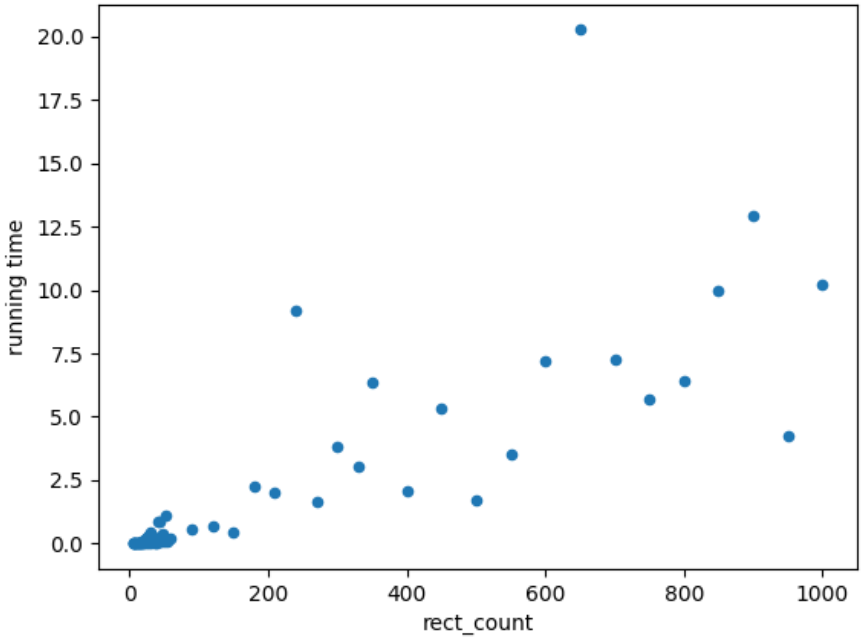
# Heuristic

- It can solve all 75 tests without any time limitation
- The objective values (costs) of instances are good
- The running time form an exponential curve with the number of rectangles (test size)
- Fun fact! The test sizes here are small enough to consider that the running time is linear without big differences.

Cost




Running time



# General 2D Bin Packing Problem

You can find interesting figures, data analysis, and even things that we tried to do in https://github.com/htnminh/optimization-project

Thanks for your attention!