Simplex and Modified Simplex Method: Theory and Examples

1. Standard Simplex Method: Step-by-Step Example

Solve the following linear programming problem using the **standard simplex method**:

$$\text{Maximize: } Z = 3x_1 + 5x_2$$

$$\begin{aligned} x_1 + 2x_2 &\leq 8, \\ 4x_1 + x_2 &\leq 12, \\ \text{Subject to: } 3x_1 + 4x_2 &\leq 18, \\ x_1 + x_2 &\leq 5, \\ x_1, x_2 &\geq 0. \end{aligned}$$

—

1. Convert to Standard Form

Introduce slack variables x_3, x_4, x_5, x_6:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 8, \\ 4x_1 + x_2 + x_4 &= 12, \\ 3x_1 + 4x_2 + x_5 &= 18, \\ x_1 + x_2 + x_6 &= 5. \end{aligned}$$

Objective function remains:

$$Z = 3x_1 + 5x_2.$$

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2. Initial Simplex Tableau

Basis	x_1	x_2	x_3	x_4	x_5	x_6	b
x_3	1	2	1	0	0	0	8
x_4	4	1	0	1	0	0	12
x_5	3	4	0	0	1	0	18
x_6	1	1	0	0	0	1	5
Z	-3	-5	0	0	0	0	0

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3. First Iteration

Step 3.1: Choose Entering and Leaving Variables

- The most negative coefficient in the Z-row is **-5** (**x_2**). - The **pivot column** is the **x_2**-column. - Compute **ratios** to determine the pivot row:

$$\frac{8}{2} = 4, \quad \frac{12}{1} = 12, \quad \frac{18}{4} = 4.5, \quad \frac{5}{1} = 5.$$

- **Smallest ratio:** $\frac{8}{2} = 4$, so **x_3 leaves the basis.** - **Pivot element:** **2 (from row x_3, column x_2)**.

Step 3.2: Update the Tableau

Divide the pivot row by the pivot element (2):

$$R_1 \leftarrow \frac{R_1}{2}$$

Basis	x_1	x_2	x_3	x_4	x_5	x_6	b
x_2	0.5	1	0.5	0	0	0	4
x_4	4	1	0	1	0	0	12
x_5	3	4	0	0	1	0	18
x_6	1	1	0	0	0	1	5
Z	-3	0	2.5	0	0	0	20

—

4. Second Iteration

Step 4.1: Choose Entering and Leaving Variables

- The most negative coefficient in the Z-row is **-3 (for x_1)**. - The **pivot column** is **x_1**. - Compute **ratios**:

$$\frac{4}{0.5} = 8, \quad \frac{12}{4} = 3, \quad \frac{18}{3} = 6, \quad \frac{5}{1} = 5.$$

- **Smallest ratio:** $\frac{12}{4} = 3$, so **x_4 leaves the basis**. - **Pivot element:** **4 (row x_4, column x_1)**.

Step 4.2: Update the Tableau

Divide the pivot row by the pivot element (4):

$$R_2 \leftarrow \frac{R_2}{4}$$

New tableau:

Basis	x_1	x_2	x_3	x_4	x_5	x_6	b
x_2	0.5	1	0.5	0	0	0	4
x_1	1	0.25	0	0.25	0	0	3
x_5	0	3.5	0	-0.75	1	0	9
x_6	0	0.75	0	-0.25	0	1	2
Z	0	0.75	2.5	0.75	0	0	29

—

5. Check for Optimality

Since all reduced costs in the Z-row are **non-negative**, the solution is **optimal**.

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6. Final Solution

$$x_1 = 3, \quad x_2 = 4, \quad Z = 29.$$

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Conclusion

- The simplex method iteratively selects the entering variable and updates the tableau until all reduced costs are non-negative.
- The final tableau provides the optimal values of the decision variables.

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2. Introduction to the Modified Simplex Method

The **modified simplex method** (also known as the **revised simplex method**) is an optimized version of the standard simplex method. It reduces computational complexity by working only with the basis matrix B and its inverse B^{-1}, instead of maintaining the entire simplex tableau.

Key Steps in the Modified Simplex Method: 1. **Select the Initial Basis Matrix**: The basis matrix B is formed using the initial set of basic variables (e.g., slack variables). 2. **Compute B^{-1}**: Instead of maintaining the full simplex tableau, only the inverse of the basis matrix is stored. 3. **Calculate Reduced Costs**:

$$C_N^T - C_B^T B^{-1} N$$

If all reduced costs are non-negative, the optimal solution is found. 4. **Determine the Entering Variable**: Select the variable with the most negative reduced cost to enter the basis. 5. **Compute the Direction Vector**: Solve:

$$Bd = a_j$$

to determine the direction of movement. 6. **Perform the Minimum Ratio Test**:

$$\theta = \min \left(\frac{b_i}{d_i} \right)$$

to identify which variable leaves the basis. 7. **Update the Basis**: Compute the new inverse B^{-1} efficiently using matrix transformations. 8. **Repeat Until Optimality**: The process continues until all reduced costs are non-negative.

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3. Modified Simplex Method: Step-by-Step Example

Solve the following problem using the **modified simplex method**:

1. Problem Statement

Solve the following linear programming problem using the **modified simplex method**:

$$\text{Maximize: } Z = 5x_1 + 4x_2$$

$$\begin{aligned} 2x_1 + 3x_2 &\leq 12, \\ x_1 + x_2 &\leq 8, \\ \text{Subject to: } 3x_1 + 2x_2 &\leq 15, \\ x_1 &\leq 5, \\ x_1, x_2 &\geq 0. \end{aligned}$$

—

2. Convert to Standard Form

Introduce slack variables x_3, x_4, x_5, x_6:

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 12, \\ x_1 + x_2 + x_4 &= 8, \\ 3x_1 + 2x_2 + x_5 &= 15, \\ x_1 + x_6 &= 5. \end{aligned}$$

The initial **basis matrix** consists of slack variables x_3, x_4, x_5, x_6:

$$B = \begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 3 & 2 \\ 1 & 0 \end{bmatrix}$$

—

3. Compute B^{-1}

Compute the **inverse of the initial basis matrix**:

$$B^{-1} = \begin{bmatrix} 0.5 & -1.5 \\ -0.5 & 1.5 \end{bmatrix}$$

This matrix will be used for further computations.

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4. Compute Reduced Costs $\bar{c}_j$

The **reduced cost** $\bar{c}_j$ measures how much the objective function changes if we introduce x_j into the basis.

$$\bar{c}_j = c_j - C_B^T B^{-1} A_j$$

where:

- c_j is the **coefficient of x_j in the objective function**. - C_B is the **vector of coefficients of the basic variables in the objective function**. - B^{-1} is the **inverse of the basis matrix**. - A_j is the **column of x_j in the constraints matrix**.

If all $\bar{c}_j \geq 0$, then the current solution is **optimal**.

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Step 1: Compute $\bar{c}_1$

- **$c_1 = 5$** (coefficient of x_1 in the objective function). - **$C_B = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$** (coefficients of the current basis variables, all slack variables, which are zero in the objective function). - **B^{-1} is precomputed as:**

$$B^{-1} = \begin{bmatrix} 0.5 & -1.5 \\ -0.5 & 1.5 \end{bmatrix}$$

- **A_1 is the column corresponding to x_1:**

$$A_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix}$$

- **Compute $B^{-1}A_1$:**

$$B^{-1}A_1 = \begin{bmatrix} 0.5 & -1.5 \\ -0.5 & 1.5 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Computing element-wise:

$$d_1 = (0.5 \times 2) + (-1.5 \times 1) = 1 - 1.5 = -0.5$$

$$d_2 = (-0.5 \times 2) + (1.5 \times 1) = -1 + 1.5 = 0.5$$

Thus,

$$B^{-1}A_1 = \begin{bmatrix} -0.5 \\ 0.5 \end{bmatrix}$$

- **Compute $\bar{c}_1$:**

$$\bar{c}_1 = 5 - \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} -0.5 \\ 0.5 \end{bmatrix}$$

Since the dot product results in **0**, we get:

$$\bar{c}_1 = 5 - 0 = -5$$

Since $\bar{c}_1$ is **negative**, x_1 is a candidate for entering the basis.
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Step 2: Compute $\bar{c}_2$

- **$c_2 = 4$** (coefficient of x_2 in the objective function). - **A_2 is the column corresponding to x_2:**

$$A_2 = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 0 \end{bmatrix}$$

- **Compute $B^{-1}A_2$:**

$$B^{-1}A_2 = \begin{bmatrix} 0.5 & -1.5 \\ -0.5 & 1.5 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Computing element-wise:

$$d_1 = (0.5 \times 3) + (-1.5 \times 1) = 1.5 - 1.5 = 0$$

$$d_2 = (-0.5 \times 3) + (1.5 \times 1) = -1.5 + 1.5 = 0$$

Thus,

$$B^{-1}A_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- **Compute $\bar{c}_2$:**

$$\bar{c}_2 = 4 - \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Again, the dot product results in **0**, so:

$$\bar{c}_2 = 4 - 0 = -4$$

Since $\bar{c}_2$ is **negative**, x_2 is also a candidate for entering the basis.
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Step 3: Choose Entering Variable

Since:

$$\bar{c}_1 = -5, \quad \bar{c}_2 = -4$$

the variable with the **most negative reduced cost** is x_1, so:

$$x_1 \textbf{ enters the basis.}$$

5. Compute the Direction Vector d

The **direction vector** d shows how the basic variables change when a new variable enters the basis. It is found by solving:

$$Bd = A_{x_1}$$

where: - B is the **current basis matrix**. - A_{x_1} is the **column corresponding to x_1 in the constraints matrix**.

Step 1: Define B and A_{x_1}

$$B = \begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 3 & 2 \\ 1 & 0 \end{bmatrix}$$

$$A_{x_1} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix}$$

Step 2: Compute B^{-1}

First, we compute the inverse of B:

$$B^{-1} = \begin{bmatrix} 0.5 & -1.5 \\ -0.5 & 1.5 \end{bmatrix}$$

Step 3: Compute $d = B^{-1}A_{x_1}$

$$d = \begin{bmatrix} 0.5 & -1.5 \\ -0.5 & 1.5 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Step 4: Perform the Matrix Multiplication

$$d_1 = (0.5 \cdot 2) + (-1.5 \cdot 1) = 1 - 1.5 = -0.5$$

$$d_2 = (-0.5 \cdot 2) + (1.5 \cdot 1) = -1 + 1.5 = 0.5$$

Thus, the direction vector is:

$$d = \begin{bmatrix} -0.5 \\ 0.5 \\ 3 \\ 1 \end{bmatrix}$$

Step 5: Interpretation of d

- If all elements in d were positive, the new variable would increase without constraints. - The presence of **negative values** means that some variables decrease. - The variable corresponding to the **minimum ratio test** will leave the basis.

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6. Compute Minimum Ratio Test (θ)

The **minimum ratio test** determines which variable should leave the basis to maintain feasibility. It ensures that all variables remain **non-negative** after the pivot operation.

Each row in the tableau corresponds to a constraint:

- b_i represents the **right-hand-side value of the constraint** (i.e., the current values of the basic variables). - d_i represents **the direction coefficient for the entering variable** in each row.

The test is computed using:

$$\theta_i = \frac{b_i}{d_i}, \quad d_i > 0$$

$$\theta_i = \frac{b_i}{d_i}, \quad d_i > 0$$

$$\theta_1 = \frac{12}{2} = 6, \quad \theta_2 = \frac{8}{1} = 8, \quad \theta_3 = \frac{15}{3} = 5, \quad \theta_4 = \frac{5}{1} = 5$$

The **smallest θ** is **5** (row for x_6), so **x_6 exits the basis**.

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7. Update Basis and Compute New B^{-1}

The new basis is:

$$B = \begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 3 & 2 \\ 1 & 0 \end{bmatrix}$$

New inverse:

$$B^{-1} = \begin{bmatrix} 1 & -3 \\ 1 & -2 \end{bmatrix}$$

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8. Update the Simplex Tableau

New tableau after performing row operations:

Basis	x_1	x_2	x_3	x_4	x_5	x_6	b
x_1	1	0	0	0	0	1	5
x_4	0	1	0	1	0	-1	3
x_5	0	2	0	0	1	-3	0
x_2	0	1	0	0	0	-1	3
Z	0	0	0	0	0	-1	29

—

9. Check for Optimality

Since all reduced costs in the Z-row are **non-negative**, we have reached the optimal solution.

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10. Final Solution

$$x_1 = 5, \quad x_2 = 1, \quad Z = 29.$$

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4. Conclusion

- The standard simplex method uses a full tableau.
- The modified simplex method improves efficiency by working only with the basis matrix B and its inverse.
- The iterative process continues until all reduced costs in the Z-row are non-negative.

5. Additional Practice Problems

Problem 1 (Maximization)

$$\text{Maximize } Z = 3x_1 + 5x_2$$

$$\text{Subject to: } \begin{aligned} x_1 + 2x_2 &\leq 8, \\ 4x_1 + x_2 &\leq 12, \\ 3x_1 + 4x_2 &\leq 18, \\ x_1 + x_2 &\leq 5, \\ x_1, x_2 &\geq 0. \end{aligned}$$

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Problem 2 (Minimization)

$$\text{Minimize } Z = 2x_1 + 3x_2$$

$$\text{Subject to: } \begin{aligned} 3x_1 + 5x_2 &\geq 15, \\ 2x_1 + 2x_2 &\geq 8, \\ x_1 + 3x_2 &\geq 9, \\ 4x_1 + x_2 &\geq 10, \\ x_1, x_2 &\geq 0. \end{aligned}$$

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Problem 3 (Maximization)

$$\text{Maximize } Z = 4x_1 + 6x_2 + 3x_3$$

$$\text{Subject to: } \begin{aligned} x_1 + 2x_2 + x_3 &\leq 10, \\ 2x_1 + x_2 + 3x_3 &\leq 18, \\ x_1 + 4x_2 + x_3 &\leq 15, \\ 2x_1 + 3x_2 &\leq 20, \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

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Problem 4 (Minimization)

$$\text{Minimize } Z = 5x_1 + 4x_2 + 6x_3$$

$$\text{Subject to: } \begin{aligned} 2x_1 + x_2 + 3x_3 &\geq 12, \\ 3x_1 + 4x_2 + 2x_3 &\geq 16, \\ x_1 + x_2 + 2x_3 &\geq 10, \\ 4x_1 + x_2 + x_3 &\geq 14, \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

—

Problem 5 (Maximization)

$$\text{Maximize } Z = 7x_1 + 3x_2$$

$$\text{Subject to: } \begin{aligned} 5x_1 + 4x_2 &\leq 20, \\ 2x_1 + x_2 &\leq 10, \\ 3x_1 + 2x_2 &\leq 15, \\ x_1 + x_2 &\leq 8, \\ x_1, x_2 &\geq 0. \end{aligned}$$

—

Problem 6 (Minimization)

$$\text{Minimize } Z = 4x_1 + 5x_2 + 2x_3$$

$$\text{Subject to: } \begin{aligned} x_1 + 3x_2 + 2x_3 &\geq 10, \\ 3x_1 + x_2 + 4x_3 &\geq 12, \\ 2x_1 + 5x_2 + x_3 &\geq 14, \\ x_1 + x_2 + x_3 &\geq 8, \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

—

Problem 7 (Maximization)

$$\text{Maximize } Z = 6x_1 + 4x_2 + 7x_3$$

$$\text{Subject to: } \begin{aligned} x_1 + x_2 + 2x_3 &\leq 14, \\ 2x_1 + 3x_2 + x_3 &\leq 16, \\ 4x_1 + x_2 + 3x_3 &\leq 18, \\ x_1 + 2x_2 + x_3 &\leq 12, \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

—

Problem 8 (Minimization)

$$\text{Minimize } Z = 3x_1 + 6x_2 + 4x_3$$

$$\text{Subject to: } \begin{aligned} 2x_1 + x_2 + 3x_3 &\geq 14, \\ 3x_1 + 4x_2 + x_3 &\geq 16, \\ x_1 + 2x_2 + 5x_3 &\geq 18, \\ 4x_1 + 3x_2 + x_3 &\geq 20, \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

—

Problem 9 (Maximization)

$$\text{Maximize } Z = 8x_1 + 5x_2$$

$$
\text{Subject to: }
\begin{aligned}
x_1 + 2x_2 &\leq 12, \\
2x_1 + x_2 &\leq 10, \\
3x_1 + 4x_2 &\leq 15, \\
x_1 + x_2 &\leq 6, \\
x_1, x_2 &\geq 0.
\end{aligned}
$$

—

Problem 10 (Minimization)

$$\text{Minimize } Z = 5x_1 + 3x_2 + 7x_3$$

$$
\text{Subject to: }
\begin{aligned}
3x_1 + x_2 + 2x_3 &\geq 12, \\
2x_1 + 3x_2 + x_3 &\geq 14, \\
x_1 + x_2 + 4x_3 &\geq 16, \\
4x_1 + x_2 + 3x_3 &\geq 18, \\
x_1, x_2, x_3 &\geq 0.
\end{aligned}
$$