Solving a Linear Programming Problem Using the Simplex Method

Problem Statement

Solve the following problem:

$$\text{Maximize: } Z = 3x_1 + 5x_2$$

$$\text{Subject to: } \begin{aligned} x_1 + 2x_2 &\leq 6, \\ 3x_1 + 2x_2 &\leq 12, \\ x_1 &\geq 0, \\ x_2 &\geq 0. \end{aligned}$$

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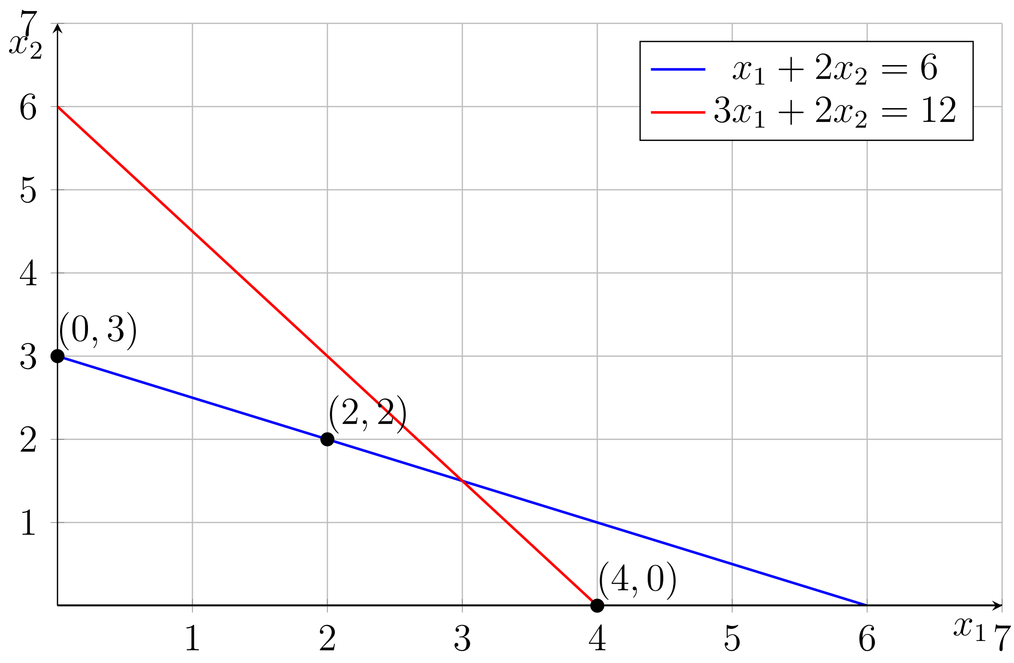
Graphical Representation of Constraints and Feasible Region



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Direction of Maximization

The objective function is $Z = 3x_1 + 5x_2$. The direction of maximization is determined as follows:

1. **Define the Line of Constant Z**: - For any constant $Z = k$, the function can be written as:

$$3x_1 + 5x_2 = k$$

- This represents a straight line with slope $-\frac{3}{5}$.

2. **Determine the Gradient Vector**: - The gradient vector of Z, $\nabla Z = (3, 5)$, points in the direction of increasing Z. - This vector is orthogonal to the lines of constant Z.

3. **Move Parallel Lines**: - To maximize Z, move the line $3x_1+5x_2 = k$ in the direction of the gradient vector ∇Z until it touches the last point in the feasible region.

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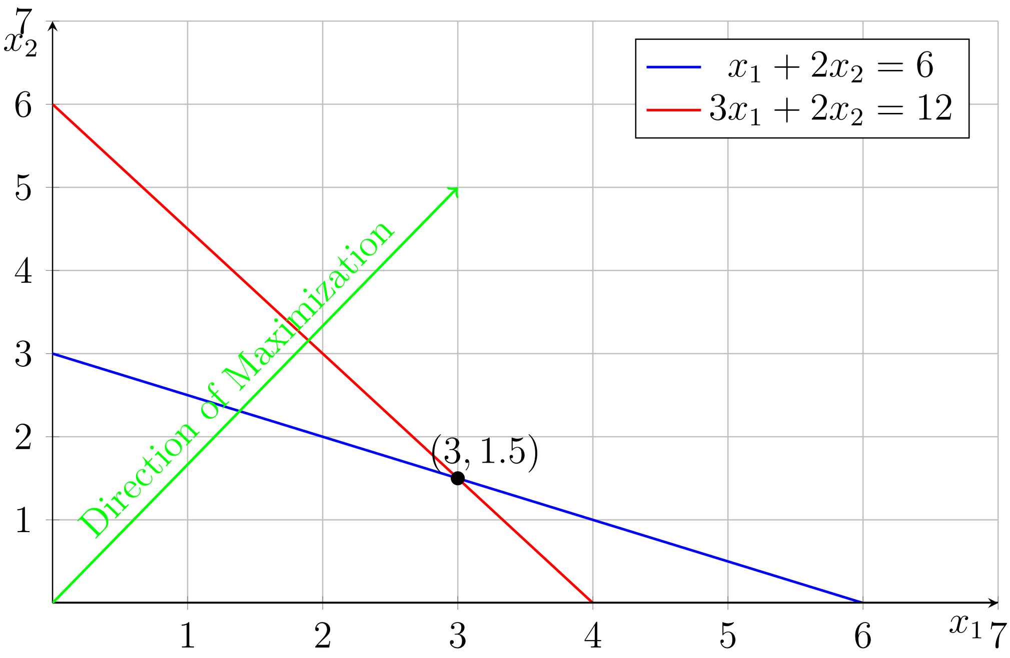
Graphical Representation



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Simplex Method Solution

Canonical Form

Introducing slack variables x_3 and x_4, the constraints become:

$$x_1 + 2x_2 + x_3 = 6,$$
$$3x_1 + 2x_2 + x_4 = 12,$$
$$x_1, x_2, x_3, x_4 \geq 0.$$

The objective function is rewritten as:

$$Z = 3x_1 + 5x_2.$$

Initial Simplex Tableau

Basis	x_1	x_2	x_3	x_4	b
x_3	1	2	1	0	6
x_4	3	2	0	1	12
Z	-3	-5	0	0	0

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Step 1: Selecting the Pivot Element

1. **Pivot Column**: The most negative coefficient in the Z-row is -5 (x_2).
2. **Pivot Row**: Perform the ratio test b_i/a_{i2}:

$$\frac{6}{2} = 3, \quad \frac{12}{2} = 6.$$

The smallest ratio is 3, so x_3 leaves the basis.

Pivot Element: 2.

Step 2: Update the Tableau

1. Divide the pivot row by the pivot element 2:

$$\text{New Row 1: } \frac{[1, 2, 1, 0, 6]}{2} = [0.5, 1, 0.5, 0, 3].$$

2. Update the other rows:

$$\text{Row 2: } [3, 2, 0, 1, 12] - 2 \cdot [0.5, 1, 0.5, 0, 3] = [2, 0, -1, 1, 6].$$

$$Z\text{-row: } [-3, -5, 0, 0, 0] + 5 \cdot [0.5, 1, 0.5, 0, 3] = [-0.5, 0, 2.5, 0, 15].$$

Updated Tableau:

Basis	x_1	x_2	x_3	x_4	b
x_2	0.5	1	0.5	0	3
x_4	2	0	-1	1	6
Z	-0.5	0	2.5	0	15

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Step 3: Second Iteration

1. **Pivot Column**: The most negative coefficient in the Z-row is -0.5 (x_1). 2. **Pivot Row**: Perform the ratio test b_i/a_{i1}:

$$\frac{3}{0.5} = 6, \quad \frac{6}{2} = 3.$$

The smallest ratio is 3, so x_4 leaves the basis.
Pivot Element: 2.

Step 4: Final Tableau

1. Divide the pivot row by the pivot element 2:

$$\text{New Row 2: } \frac{[2, 0, -1, 1, 6]}{2} = [1, 0, -0.5, 0.5, 3].$$

2. Update the other rows:

$$\text{Row 1: } [0.5, 1, 0.5, 0, 3] - 0.5 \cdot [1, 0, -0.5, 0.5, 3] = [0, 1, 0.75, -0.25, 1.5].$$

$$Z\text{-row: } [-0.5, 0, 2.5, 0, 15] + 0.5 \cdot [1, 0, -0.5, 0.5, 3] = [0, 0, 2.25, 0.25, 16.5].$$

Final Tableau:

Basis	x_1	x_2	x_3	x_4	b
x_2	0	1	0.75	-0.25	1.5
x_1	1	0	-0.5	0.5	3
Z	0	0	2.25	0.25	16.5

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Answer

From the final tableau: - The optimal solution is:

$$x_1 = 3, \quad x_2 = 1.5, \quad x_3 = 0, \quad x_4 = 0.$$

- The maximum value of the objective function is:

$$Z = 3(3) + 5(1.5) = 16.5.$$

Interpretation: - $x_1 = 3$: This means that in the optimal solution, the first decision variable has a value of 3. - $x_2 = 1.5$: This means that the second decision variable has a value of 1.5. - $x_3 = 0$, $x_4 = 0$: The slack variables are zero, indicating that the constraints $x_1 + 2x_2 \leq 6$ and $3x_1 + 2x_2 \leq 12$ are binding.

Thus, the optimal solution satisfies all constraints, and Z reaches its maximum value of 16.5.

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Additional Problem for Practice

Solve the following linear programming problem:

$$\text{Maximize: } Z = 4x_1 + 6x_2$$

$$\text{Subject to: } \begin{aligned} 2x_1 + 3x_2 &\leq 12, \\ x_1 + x_2 &\leq 5, \\ x_1 &\geq 0, \\ x_2 &\geq 0. \end{aligned}$$

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