An interestring idea and thinking for the quesiton

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1. Core idea

1. Initial question

Firstly, we need to understand our question of the “Level 5 / 7 : Rewriting”.

Objects

P Q R S: Prop

Assumptions

h1 : $R \leftrightarrow S$

h2 : $\neg((P \to Q \vee \neg S) \wedge (S \vee P \to \neg Q) \to S \to Q) \leftrightarrow P \wedge Q \wedge \neg S$

Goal :

$\neg((P \to Q \vee \neg R) \wedge (R \vee P \to \neg Q) \to R \to Q) \leftrightarrow P \wedge Q \wedge \neg R$

2. The transformation of equivalent form

Denote the h2 as “hypothesis”:

$$\text{hypothesis} : \neg((P \to Q \vee \neg S) \wedge (S \vee P \to \neg Q) \to S \to Q) \leftrightarrow P \wedge Q \wedge \neg S$$

For short, we use “LHS, RHS” to see it’s structure

$$\text{hypothesis} : \text{LHS(P, Q, S)} \leftrightarrow \text{RHS(P, Q, S)}$$

$$\text{LHS(S)} \leftrightarrow \text{RHS(S)}$$

What we want is to prove

$$\text{equation} : \text{LHS(R)} \leftrightarrow \text{RHS(R)}$$

Use the above notation, we have

Objects

P Q R S: Prop

Assumptions

equivalence : $R \leftrightarrow S$

hypothesis : LHS(S) $\leftrightarrow$ RHS(S)

Goal :

equation : LHS(R) $\leftrightarrow$ RHS(R)

3. What we can do?

We can use the “apply” tactic to change our target, i.e.

$$|\text{target} : B$$

$$|h : A \to B$$

$$\text{apply F} \Rightarrow |\text{target*} : A$$

As a result, if we can prove the existence of some “F” in Lambda Term, such that

$$F : \text{hypothesis} \to \text{euqation},$$

$$(\text{LHS(S)} \leftrightarrow \text{RHS(S)}) \mapsto (\text{LHS(R)} \leftrightarrow \text{RHS(R)})$$

then, we can use “apply F” to get our new target $\text{LHS(S)} \leftrightarrow \text{RHS(S)}$, who is just the hypothesis we have. As a result, we can use “assumption” to kill it.

2. Build the “F” in L-term

I want to build and prove a theorem that for any Propositional Logic structure, we can substitute any term S with logical equivalent term R. But I found I can’t use the **pattern match** in the question. So, maybe it’s nearly impossible to prove the “substitution theorem” for general logic equation (At least I can’t make it at the moment). So, we can try to make it partially. I mean, after the “apply Iff.intro”, what we need is just one side of “F”. We denote the start with “*”, it looks like

$$\begin{tikzcd}
\text{target:} & \text{LHS(R)} \arrow[r] \arrow[d, dashed, bend right, "F_{\text{left}}^{-1}"'] & \text{RHS(R)} \arrow[d, dashed, bend right, "F_{\text{right}}^{-1}"'] \\
\text{Hypothesis:} & \text{LHS(S)} \arrow[r] \arrow[u, dashed, bend right, "F_{\text{left}}"'] & \text{RHS(S)} \arrow[u, dashed, bend right, "F_{\text{right}}"']
\end{tikzcd}$$

After the tactic “apply F_right”, our new target will be $\text{LHS(R)} \to \text{RHS(S)}$:

$$\begin{tikzcd}
 & \text{LHS(R)} \arrow[dr, bend left, "\text{targer after 'apply'}"'] & \text{RHS(R)} \\
\text{Hypothesis:} & \text{LHS(S)} \arrow[r] & \text{RHS(S)} \arrow[u, dashed, "\text{F_right}"']
\end{tikzcd}$$

As we can use lambda term with type annotation like fun x : P ∧ Q ∧ ¬S ⇒ ..., I give a construction for the “F_right”:

```
// apply F_right:  P ∧ Q ∧ ¬S =>  P ∧ Q ∧ ¬R
apply (fun Right_S : P ∧ Q ∧ ¬S =>
    And.intro
      (And.left Right_S)
```

```
    (And.intro
      (And.left $ And.right Right_S)
      (
        (fun hrs
        => fun hns
        => fun a
        => hns (mp a)) mp (And.right $ And.right Right_S)
      )
    )
  )
```

The key is to catch a term with type annotation, then use deconstructor to cut, rearrange the pieces. The next step is using “apply Hypothesis”; then we need to prove:

$$\text{target: LHS(R)} \to \text{LHS(S)}$$

which the “inverse” of part of “F” in previous graph:

$$\begin{tikzcd}
\text{target:} & \text{LHS(R)} \arrow[r] \arrow[d, dashed, bend right, "F_{\text{left}}^{-1}"'] & \text{RHS(R)} \arrow[d, dashed, bend right, "F_{\text{right}}^{-1}"'] \\
\text{Hypothesis:} & \text{LHS(S)} \arrow[r] \arrow[u, dashed, bend right, "F_{\text{left}}"'] & \text{RHS(S)} \arrow[u, dashed, bend right, "F_{\text{right}}"']
\end{tikzcd}$$

And here, I believe it’s a little bit difficult to but such F_{left}^{-1} only in L-term. So I just write this document for fun.