Published on *STAT 510* (https://onlinecourses.science.psu.edu/stat510)

Home > 1.2 Sample ACF and Properties of AR(1) Model

1.2 Sample ACF and Properties of AR(1) Model

This lesson defines the sample autocorrelation function (ACF) in general and derives the pattern of the ACF for an AR(1) model. Recall from Lesson 1.1 for this week that an AR(1) model is a linear model that predicts the present value of a time series using the immediately prior value in time.

Stationary Series

As a preliminary, we define an important concept, that of a stationary series. For an ACF to make sense, the series must be a *weakly stationary* series. This means that the autocorrelation for any particular lag is the same regardless of where we are in time.

Definition: A series x_t is said to be **(weakly) stationary** if it satisfies the following properties:

- The mean $E(x_t)$ is the same for all t.
- The variance of x_t is the same for all t.
- The covariance (and also correlation) between x_t and x_{t-h} is the same for all t.

Definition: Let x_t denote the value of a time series at time t. The ACF of the series gives correlations between x_t and x_{t-h} for h = 1, 2, 3, etc. Theoretically, the autocorrelation between x_t and x_{t-h} equals

$$\frac{\text{Covariance}(x_t, x_{t-h})}{\text{Std.Dev.}(x_t)\text{Std.Dev.}(x_{t-h})} = \frac{\text{Covariance}(x_t, x_{t-h})}{\text{Variance}(x_t)}$$

The denominator in the second formula occurs because the standard deviation of a stationary series is the same at all times.

The last property of a weakly stationary series says that the theoretical value of an autocorrelation of particular lag is the same across the whole series. An interesting property of a stationary series is that theoretically it has the same structure forwards as it does backwards.

Many stationary series have recognizable ACF patterns. Most series that we encounter in practice, however, are not stationary. A continual upward trend, for example, is a violation of the requirement that the mean is the same for all t. Distinct seasonal patterns also violate that requirement. The strategies for dealing with nonstationary series will unfold during the first three weeks of the semester.

The First-order Autoregression Model

We'll now look at theoretical properties of the AR(1) model. Recall from Lesson 1.1, that the 1st-order autoregression model is denoted as AR(1). In this model, the value of x at time t is a linear

Loading [MathJax]/extensions/MathZoom.js

function of the value of *x* at time *t–1*. The algebraic expression of the model is as follows:

$$x_t = \delta + \phi_1 \, x_{t-1} + w_t$$

Assumptions:

- $w_t \overset{iid}{\sim} N(0, \sigma_w^2)$, meaning that the errors are independently distributed with a normal distribution that has mean 0 and constant variance.
- Properties of the errors w_t are independent of x_t.
- The series x_1, x_2, ... is (weakly) stationary. A requirement for a stationary AR(1) is that $|\phi_1| < 1$. We'll see why below.

Properties of the AR(1):

Formulas for the mean, variance, and ACF for a time series process with an AR(1) model follow.

- The (theoretical) mean of x_t is

$$\mu = \frac{\delta}{1 - \phi_1}$$

- The variance of is

$$\mathrm{Var}(x_t) = \frac{\sigma_w^2}{1 - \phi_1^2}$$

- The correlation between observations *h* time periods apart is

$$\rho_h = \phi_1^h$$

This defines the theoretical ACF for a time series variable with an AR(1) model. (Note: φ_1 is the slope in the AR(1) model and we now see that it also is the lag 1 autocorrelation.)

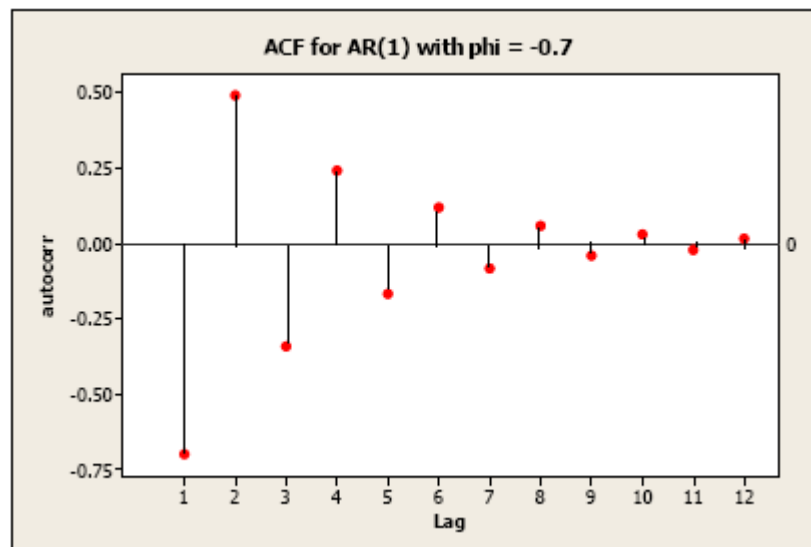Details of the derivations of these properties are in the Appendix to this lesson for interested students.

Pattern of ACF for AR(1) Model

The ACF property defines a distinct pattern for the autocorrelations. For a positive value of φ_1, the ACF exponentially decreases to 0 as the lag *h* increases. For negative φ_1, the ACF also exponentially decays to 0 as the lag increases, but the algebraic signs for the autocorrelations alternate between positive and negative.

Following is the ACF of an AR(1) with φ_1= 0.6, for the first 12 lags. Note the tapering pattern.

Loading [MathJax]/extensions/MathZoom.js

ACF for AR(1) with Phi = .6
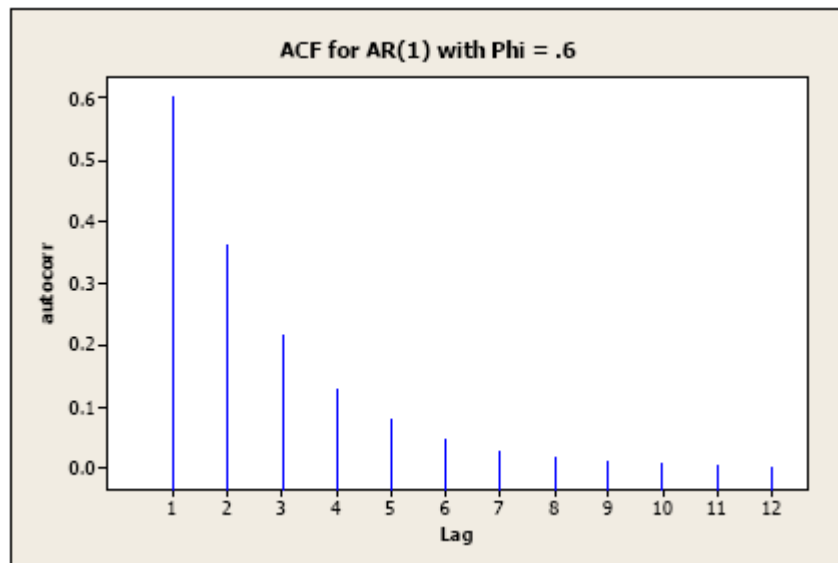
The ACF of an AR(1) with $\varphi_1 = -0.7$ follows. Note the alternating and tapering pattern.

ACF for AR(1) with phi = -0.7

Example 1: In Example 1 of Lesson 1.1, we used an AR(1) model for annual earthquakes in the world with seismic magnitude greater than 7. Here's the sample ACF of the series:

Loading [MathJax]/extensions/MathZoom.js
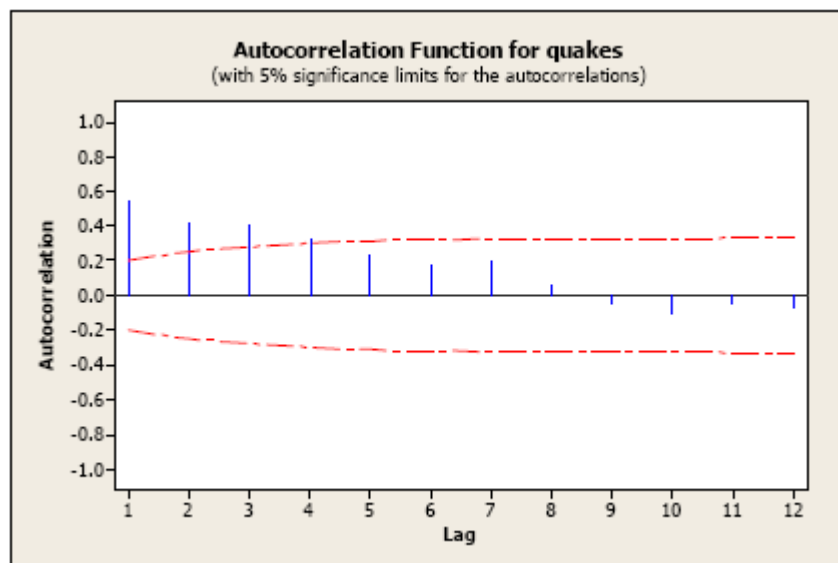
Lag. `ACF`

1. `0.541733`
2. `0.418884`
3. `0.397955`
4. `0.324047`
5. `0.237164`
6. `0.171794`
7. `0.190228`
8. `0.061202`
9. `-0.048505`
10. `-0.106730`
11. `-0.043271`
12. `-0.072305`

The sample autocorrelations taper, although not as fast as they should for an AR(1). For instance, theoretically the lag 2 autocorrelation for an AR(1) = squared value of lag 1 autocorrelation. Here, the observed lag 2 autocorrelation = .418884. That's somewhat greater than the squared value of the first lag autocorrelation ($.541733^2$= 0.293). But, we managed to do okay (in Lesson 1.1) with an AR(1) model for the data. For instance, the residuals looked okay. This brings up an important point – the sample ACF will rarely fit a perfect theoretical pattern. A lot of the time you just have to try a few models to see what fits.

We'll study the ACF patterns of other ARIMA models during the next three weeks. Each model has a different pattern for its ACF, but in practice the interpretation of a sample ACF is not always so clear-cut.

A reminder: Residuals usually are theoretically assumed to have an ACF that has correlation = 0 for all lags.

Example 2:

Here's a time series of the daily cardiovascular mortality rate in Los Angeles County, 1970-1979

Loading [MathJax]/extensions/MathZoom.js
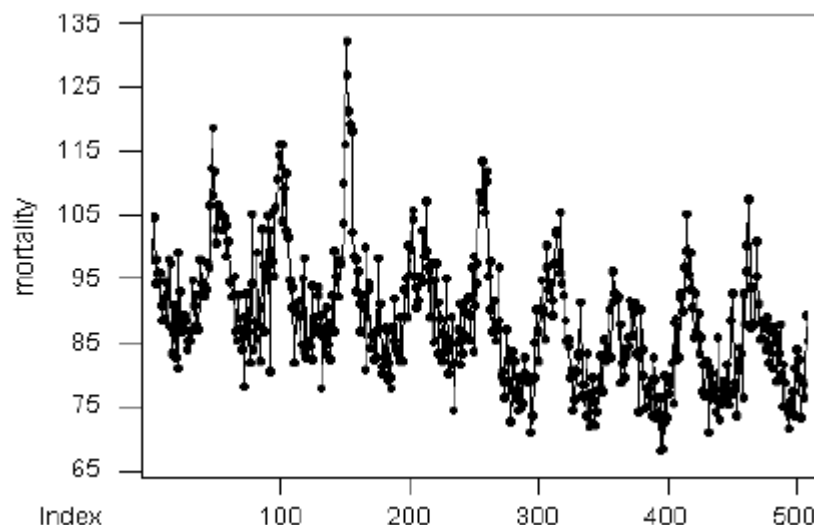
There is a slight downward trend, so the series may not be stationary. To create a (possibly) stationary series, we'll examine the **first differences** $y_t = x_t - x_{t-1}$. This is a common time series method for creating a de-trended series and thus potentially a stationary series. Think about a straight line – there are constant differences in average y for each change of 1-unit in x.

The time series plot of the first differences is the following:
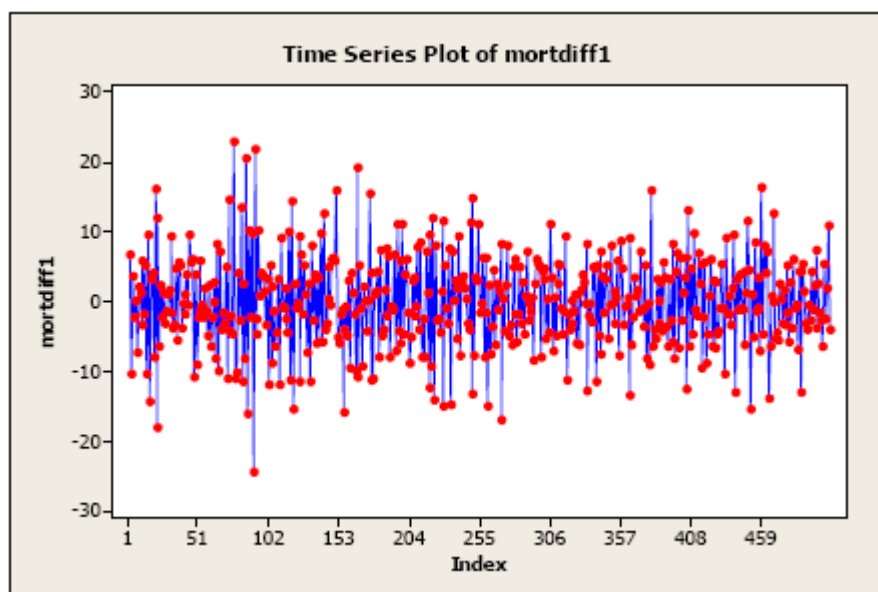
The following plot is the sample estimate of the autocorrelation function of 1st differences:

Loading [MathJax]/extensions/MathZoom.js
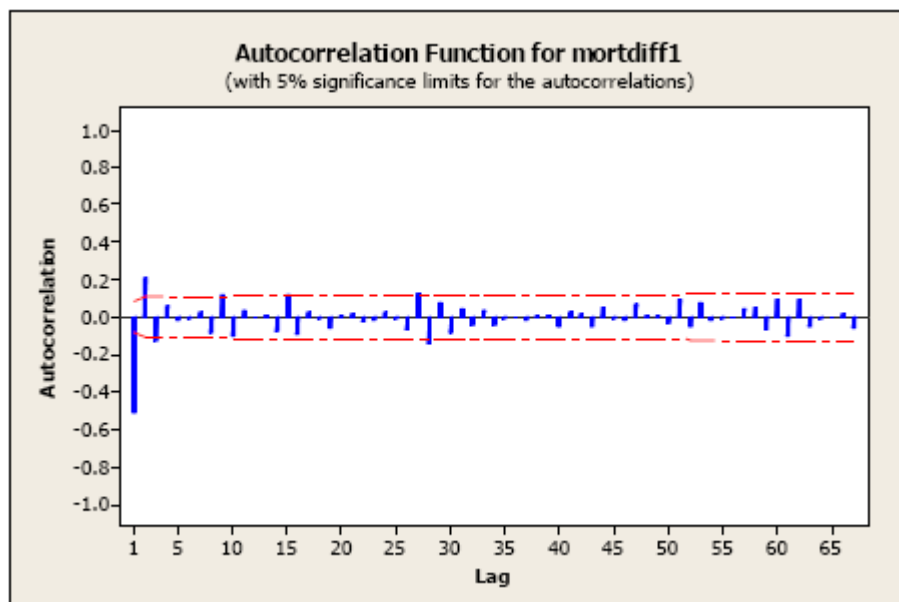
Lag. `ACF`

1. `-0.506029`
2. `0.205100`
3. `-0.126110`
4. `0.062476`
5. `-0.015190`

This looks like the pattern of an AR(1) with a negative lag 1 autocorrelation. Note that the lag 2 correlation is roughly equal to the squared value of the lag 1 correlation. The lag 3 correlation is nearly exactly equal to the cubed value of the lag 1 correlation, and the lag 4 correlation nearly equals the fourth power of the lag 1 correlation. Thus an AR(1) model may be a suitable model for the first differences $y_t = x_t - x_{t-1}$.

Let y_t denote the first differences, so that $y_t = x_t - x_{t-1}$ and $y_{t-1} = x_{t-1} - x_{t-2}$. We can write this AR(1) model as

$$y_t = \delta + \phi_1 y_{t-1} + w_t$$

Using R, we found that the estimated model for the *first differences* is

$$\hat{y}_t = -0.04627 - 0.50636 y_{t-1}$$

Some R code for this example will be given in Lesson 1.3 for this week.

Appendix Derivations of Properties of AR(1)

Generally you won't be responsible for reproducing theoretical derivations, but interested students may want to see the derivations for the theoretical properties of an AR(1).

The algebraic expression of the model is as follows:

Loading [MathJax]/extensions/MathZoom.js $x_t = \delta + \phi_1 x_{t-1} + w_t$

Assumptions:

- $w_t \overset{iid}{\sim} N(0, \sigma_w^2)$, meaning that the errors are independently distributed with a normal distribution that has mean 0 and constant variance.
- Properties of the errors w_t are independent of x_t.
- The series $x_1, \ x_2, \ ...$ is (weakly) stationary. A requirement for a stationary AR(1) is that $|\phi_1| < 1$. We'll see why below.

Mean:

$$E(x_t) = E(\delta + \phi_1 x_{t-1} + w_t) = E(\delta) + E(\phi_1 x_{t-1}) + E(w_t) = \delta + \phi_1 E(x_{t-1}) + 0$$

With the stationary assumption, $E(x_t) = E(x_{t-1})$. Let μ µ denote this common mean. Thus $\mu = \delta + \phi_1 \mu$. Solve for µ to get

$$\mu = \frac{\delta}{1 - \phi_1}$$

Variance:

By independence of errors and values of x,

$$\begin{aligned} \operatorname{Var}(x_t) &= \operatorname{Var}(\delta) + \operatorname{Var}(\phi_1 x_{t-1}) + \operatorname{Var}(w_t) \\ &= \phi_1^2 \operatorname{Var}(x_{t-1}) + \sigma_w^2 \end{aligned}$$

By the stationary assumption, $\operatorname{Var}(x_t) = \operatorname{Var}(x_{t-1})$. Substitute $\operatorname{Var}(x_t)$ for $\operatorname{Var}(x_{t-1})$ and then solve for $\operatorname{Var}(x_t)$. Because $\operatorname{Var}(x_t) > 0$, it follows that $(1 - \phi_1^2) > 0$ and therefore $|\phi_1| < 1$.

Autocorrelation Function (ACF)

To start, assume the data have mean 0, which happens when δ=0, and $x_t = \varphi_1 x_{t-1} + w_t$. In practice this isn't necessary, but it simplifies matters. Values of variances, covariances and correlations are not affected by the specific value of the mean.

Let γ$_h$ = E($x_t x_{t+h}$) = E($x_t x_{t-h}$), the covariance observations *h* time periods apart (when the mean = 0). Let ρ_h = correlation between observations that are *h* time periods apart.

Covariance and correlation between observations one time period apart

$$\gamma_1 = \operatorname{E}(x_t x_{t+1}) = \operatorname{E}(x_t(\phi_1 x_t + w_{t+1})) = \operatorname{E}(\phi_1 x_t^2 + x_t w_{t+1}) = \phi_1 \operatorname{Var}(x_t)$$

$$\rho_1 = \frac{\operatorname{Cov}(x_t, x_{t+1})}{\operatorname{Var}(x_t)} = \frac{\phi_1 \operatorname{Var}(x_t)}{\operatorname{Var}(x_t)} = \phi_1$$

Covariance and correlation between observations h time periods apart

To find the covariance γ$_h$, multiply each side of the model for x_t by x_{t-h}, then take expectations.

Loading [MathJax]/extensions/MathZoom.js $x_t = \phi_1 x_{t-1} + w_t$

$$x_{t-h}x_t = \phi_1 x_{t-h}x_{t-1} + x_{t-h}w_t$$

$$E(x_{t-h}x_t) = E(\phi_1 x_{t-h}x_{t-1}) + E(x_{t-h}w_t)$$

$$\gamma_h = \phi_1 \gamma_{h-1}$$

If we start at γ_1, and move recursively forward we get $\gamma_h = \phi_1^h \gamma_0$. By definition, $\gamma_0 = \text{Var}(x_t)$, so this is $\gamma_h = \phi_1^h \text{Var}(x_t)$. The correlation

$$\rho_h = \frac{\gamma_h}{\text{Var}(x_t)} = \frac{\phi_1^h \text{Var}(x_t)}{\text{Var}(x_t)} = \phi_1^h$$

Source URL: https://onlinecourses.science.psu.edu/stat510/node/60

Loading [MathJax]/extensions/MathZoom.js