THE UNIVERSITY OF TEXAS AT AUSTIN
McCOMBS SCHOOL OF BUSINESS

Diagnostics & Transformations 1

Lecture 17

STA 371G

Predicting the fuel economy (miles per gallon) for different car models of the 70s.

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Regression assumptions:

- Linearity
- Independent errors
- Normally distributed errors
- Equal Variance (Homoscedasticity)

Predicting MPG from Horsepower

```
model <- lm(MPG ~ HP, data=auto_mpg)
summary(model)
```

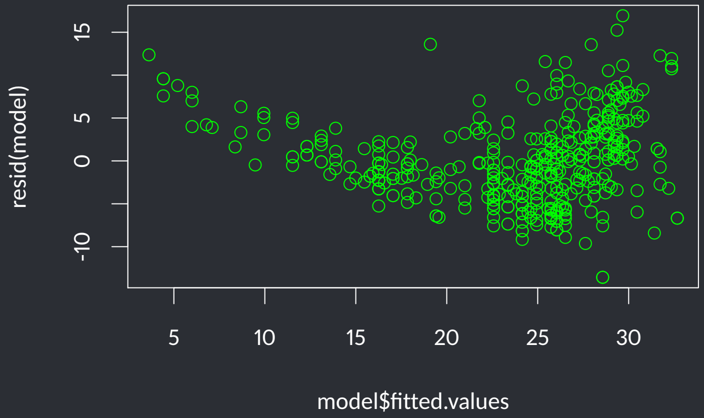
```
Call:
lm(formula = MPG ~ HP, data = auto_mpg)

Residuals:
     Min       1Q   Median       3Q      Max 
-13.5710  -3.2592  -0.3435   2.7630  16.9240 

Coefficients:
             Estimate Std. Error t value Pr(>|t|)    
(Intercept) 39.935861   0.717499   55.66   <2e-16 ***
HP          -0.157845   0.006446  -24.49   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 4.906 on 390 degrees of freedom
Multiple R-squared:  0.6059,Adjusted R-squared:  0.6049 
F-statistic: 599.7 on 1 and 390 DF,  p-value: < 2.2e-16
```

```r
plot(model$fitted.values, resid(model), col='green')
```



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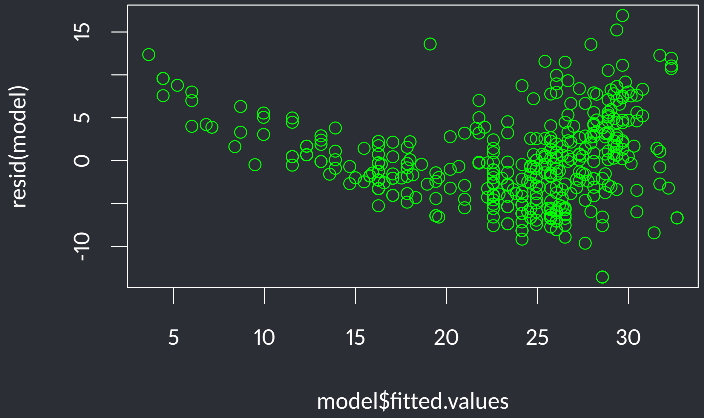

What's wrong here?

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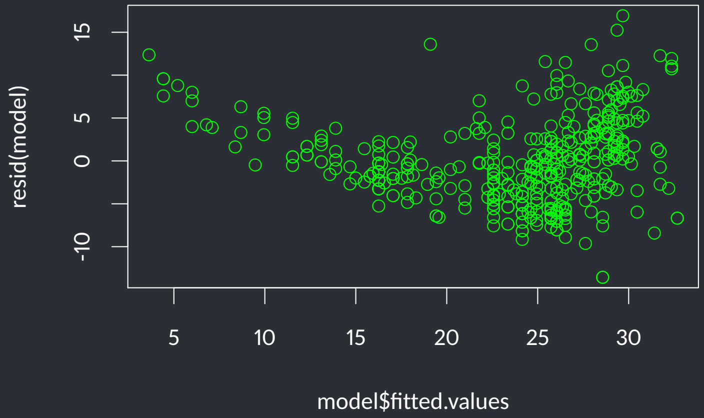


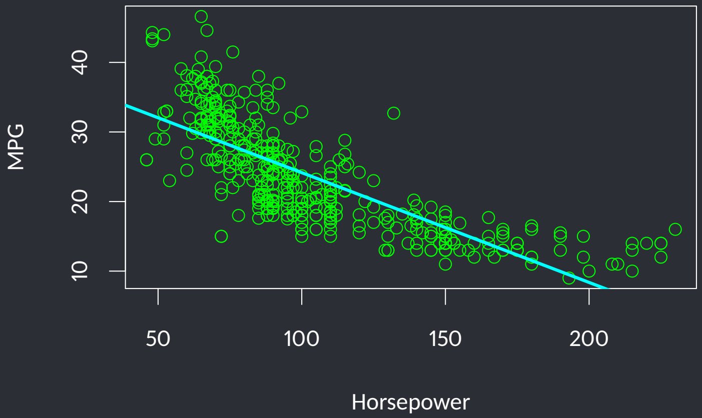
What’s wrong here?

- The trend in the residuals implies linearity issues.
- The “funnel” implies equal variance issues.

Addressing the linearity issue

The relation between MPG and horsepower does not seem to be linear.

```
plot(auto_mpg$HP, auto_mpg$MPG, col='green',
     xlab='Horsepower', ylab='MPG')
abline(model, col='cyan', lwd=3)
```



Addressing the linearity issue

If we could horizontally shift the data on the far right towards the left, the plot would look more linear.

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So let's predict the MPG of a car not from the horsepower, but from a "transformation" of the horsepower.

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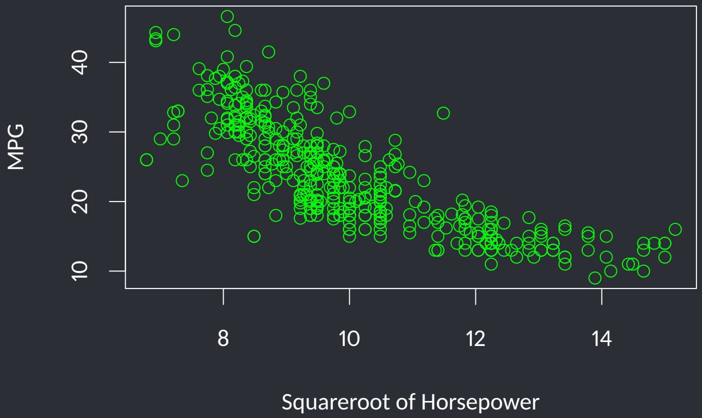
So let's predict the MPG of a car not from the horsepower, but from a "transformation" of the horsepower.

For example, the relation between MPG and HP is not linear, but the one between MPG and $\sqrt{HP}$ could be!

Addressing the linearity issue

```
auto_mpg$HP_sqrt <- sqrt(auto_mpg$HP)
plot(auto_mpg$HP_sqrt, auto_mpg$MPG, col='green',
     xlab='Squareroot of Horsepower', ylab='MPG')
```



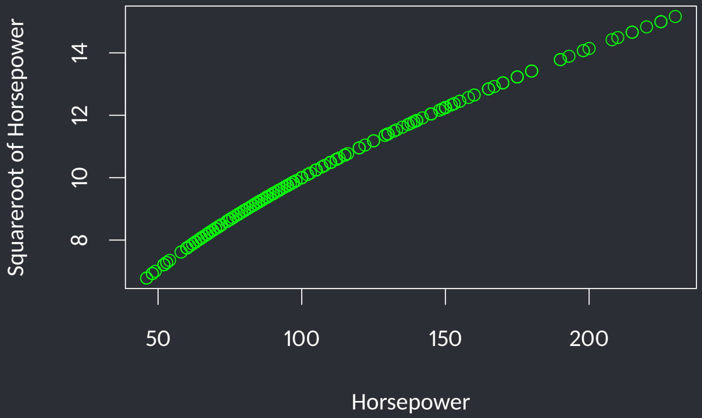
Addressing the linearity issue

It indeed seems a bit better. Notice the change in the range of the horizontal axis.

Addressing the linearity issue

It indeed seems a bit better. Notice the change in the range of the horizontal axis. It has changed from [49,225] to [7,15]. The shift is larger for the data on the far right.

```
plot(auto_mpg$HP, auto_mpg$HP_sqrt, col='green',
    xlab='Horsepower', ylab='Squareroot of Horsepower')
```

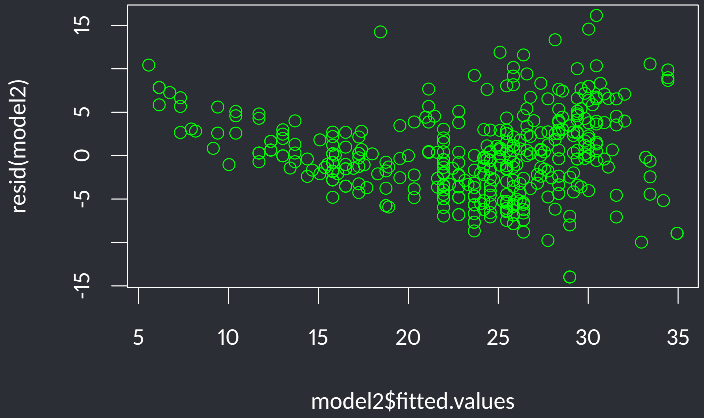


Addressing the linearity issue

```
model2 <- lm(MPG ~ HP_sqrt, data=auto_mpg)
summary(model2)
```

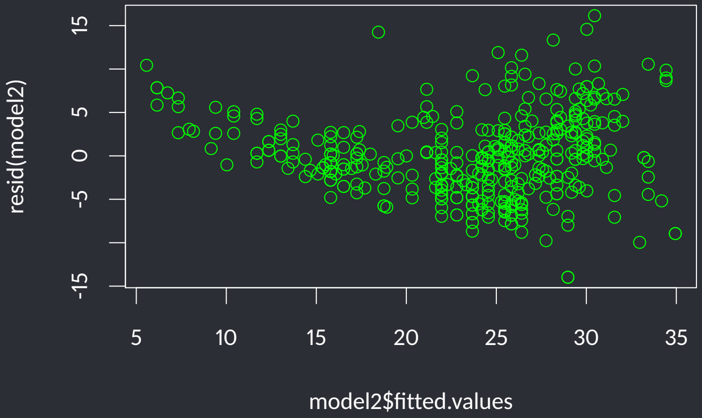
```
Call:
lm(formula = MPG ~ HP_sqrt, data = auto_mpg)

Residuals:
     Min       1Q   Median       3Q      Max 
-13.9768  -3.2239  -0.2252   2.6881  16.1411 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)   58.705      1.349   43.52   <2e-16 ***
HP_sqrt       -3.503      0.132  -26.54   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 4.665 on 390 degrees of freedom
Multiple R-squared:  0.6437,Adjusted R-squared:  0.6428 
F-statistic: 704.6 on 1 and 390 DF,  p-value: < 2.2e-16
```

```r
plot(model2$fitted.values, resid(model2), col='green')
```



The trend flattened a bit.

```r
plot(model2$fitted.values, resid(model2), col='green')
```



The trend flattened a bit.

Can we do better? Let’s try some other transformation.

Logarithmic transformation

One of the most common transformations is the logarithmic transformation with base e (natural logarithm).

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$$e = 2.7182818284\ldots$$

$$e^2 = 7.389$$

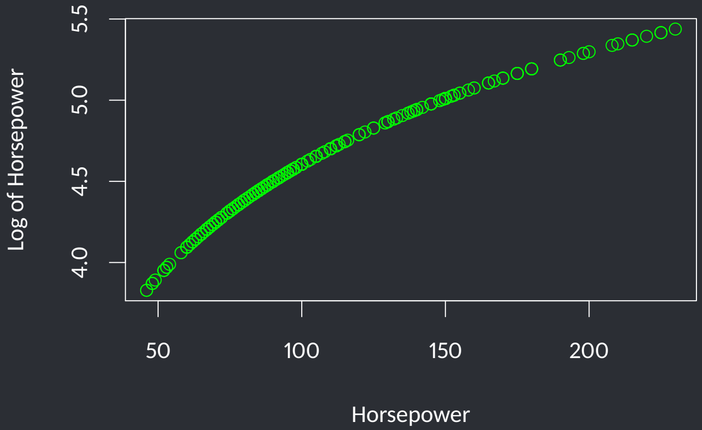
$$\log(e^2) = 2$$

In general:

$$y = e^x \qquad \longleftrightarrow \qquad \log y = x.$$

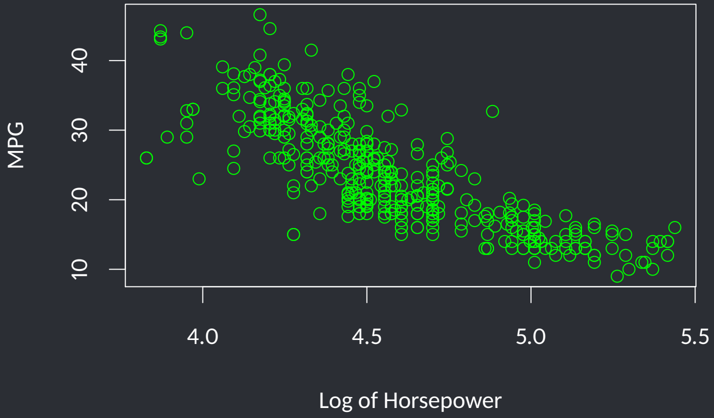
Logarithmic transformation

```
auto_mpg$HP_ln <- log(auto_mpg$HP)
plot(auto_mpg$HP, auto_mpg$HP_ln, col='green',
    xlab='Horsepower', ylab='Log of Horsepower')
```



Logarithmic transformation

```
plot(auto_mpg$HP_ln, auto_mpg$MPG, col='green',
     xlab='Log of Horsepower', ylab='MPG')
```



```r
model3 <- lm(MPG ~ HP_ln, data=auto_mpg)
summary(model3)
```

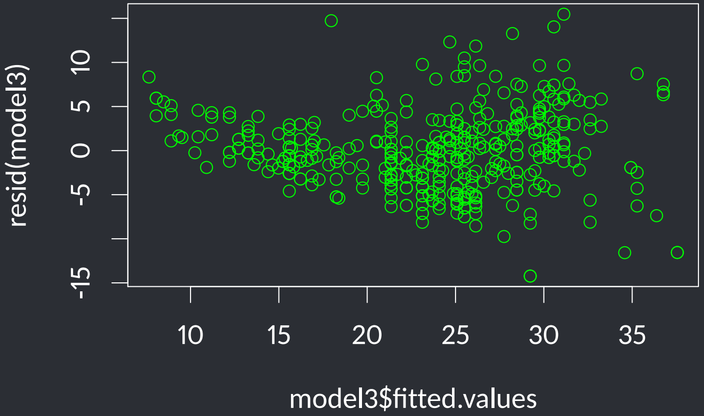
```
Call:
lm(formula = MPG ~ HP_ln, data = auto_mpg)

Residuals:
     Min       1Q   Median       3Q      Max
-14.2299  -2.7818  -0.2322   2.6661  15.4695

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 108.6997     3.0496   35.64   <2e-16 ***
HP_ln       -18.5822     0.6629  -28.03   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 4.501 on 390 degrees of freedom
Multiple R-squared:  0.6683,Adjusted R-squared:  0.6675
F-statistic: 785.9 on 1 and 390 DF,  p-value: < 2.2e-16
```

```r
plot(model3$fitted.values, resid(model3), col='green')
```



The trend flattened even more.

Transforming a Predictor

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It is equivalent to "cutting the distribution of *X* into vertical slices and changing the spacing of the slices."

It does not affect the vertical locations of the data (MPG did not change!).

When the nonlinearity is the biggest issue in the model, transforming the predictor is a good start.

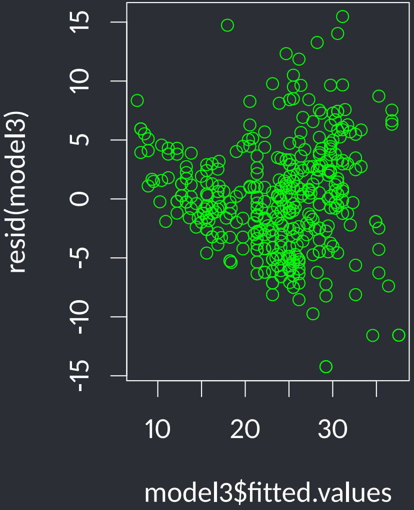
Finding the right transformation is a bit of art, field knowledge and trial and error.

Transforming a Predictor

x	$\sqrt{x}$	$\log x$
1	1	0
10	3.16	2.3
100	10	4.61
1000	31.62	6.91
10000	100	9.21
100000	316.23	11.51

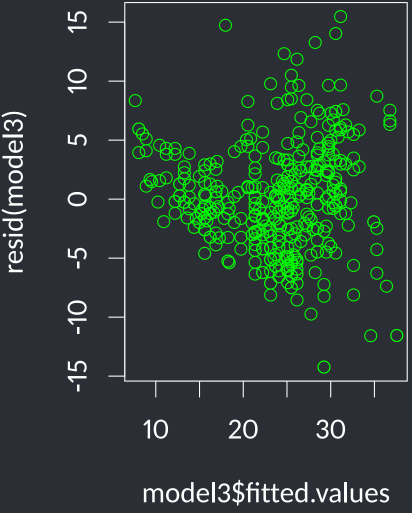
Addressing the equal variance issue

- The (unexplained) variance in the response is higher in some regions.



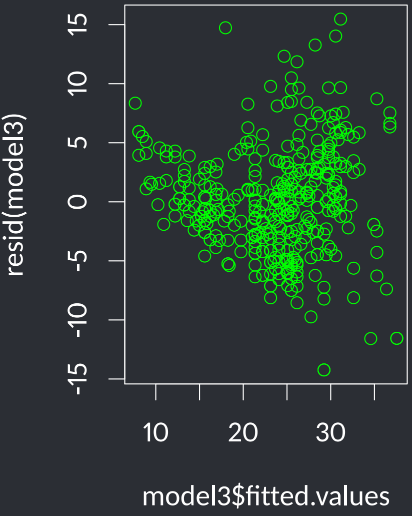
Addressing the equal variance issue

- The (unexplained) variance in the response is higher in some regions.
- Remember how log-transformation shrinks larger numbers more than it shrinks smaller numbers.



Addressing the equal variance issue

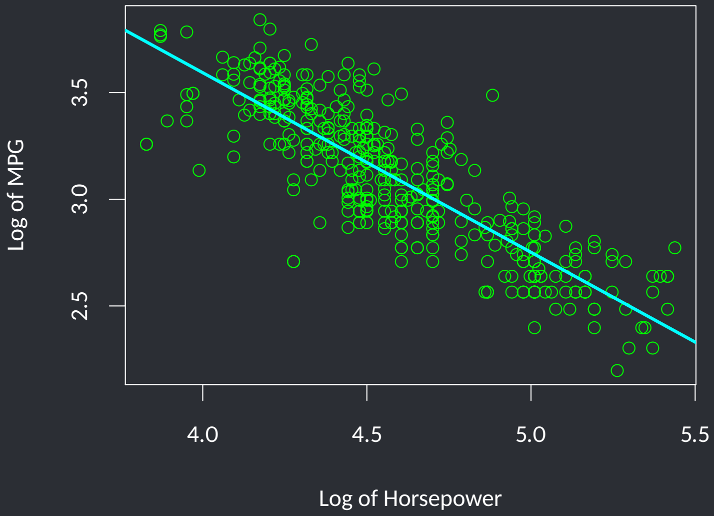
- The (unexplained) variance in the response is higher in some regions.
- Remember how log-transformation shrinks larger numbers more than it shrinks smaller numbers.
- Log-transformation of the response often helps with fixing heteroscedasticity (and non-normality)!



```r
auto_mpg$MPG_ln <- log(auto_mpg$MPG)
model4 <- lm(MPG_ln ~ HP_ln, data=auto_mpg)
summary(model4)
```

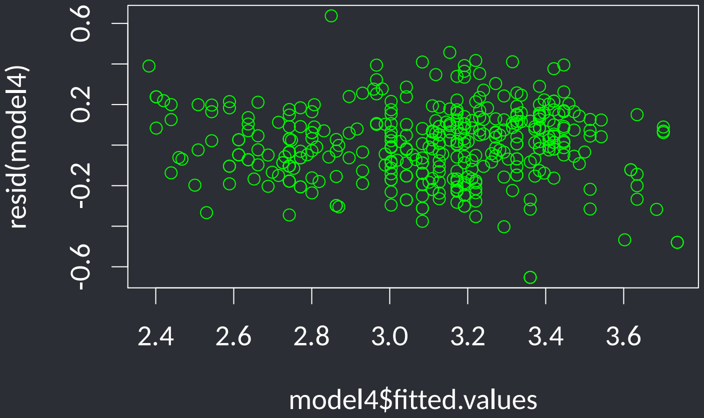
```
Call:
lm(formula = MPG_ln ~ HP_ln, data = auto_mpg)

Residuals:
    Min      1Q  Median      3Q     Max 
-0.6523 -0.1218  0.0079  0.1163  0.6373 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)   6.9606     0.1215    57.3   <2e-16 ***
HP_ln        -0.8418     0.0264   -31.9   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.18 on 390 degrees of freedom
Multiple R-squared:  0.723,Adjusted R-squared:  0.722 
F-statistic: 1.02e+03 on 1 and 390 DF,  p-value: <2e-16
```

```r
plot(auto_mpg$HP_ln, auto_mpg$MPG_ln, col='green',
     xlab='Log of Horsepower', ylab='Log of MPG')
abline(model4, col='cyan', lwd=3)
```



Beautiful!

```r
plot(model4$fitted.values, resid(model4), col='green')
```



Look how awesome our models are getting

Predictor (X)	Response (Y)	R^2	Residual SE
HP	MPG	0.61	4.91
$\sqrt{\mathrm{HP}}$	MPG	0.64	4.66
$\log \mathrm{HP}$	MPG	0.67	4.5
$\log \mathrm{HP}$	$\log \mathrm{MPG}$	0.72	0.18

Interpretation of β values

The model before the transformations was:

$$\widehat{\text{MPG}} = 39.94 - 0.16 \cdot \text{HP}$$

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The interpretation of -0.16 is “for each unit of increase in the horsepower, the MPG estimate reduces by 0.16”.

After transforming the predictor or the response, this interpretation does not hold!

Interpretation of β values

When the square root of HP is used:

$$\widehat{\text{MPG}} = 58.71 - 3.5 \cdot \sqrt{\text{HP}}$$

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When the square root of HP is used:

$$\widehat{\text{MPG}} = 58.71 - 3.5 \cdot \sqrt{\text{HP}}$$

The interpretation of -3.5 is "for each unit of increase in the square root of the horsepower, the MPG estimate reduces by 3.5".

Interpretation of β values

Similarly, in the following model:

$$\widehat{\text{MPG}} = 108.7 - 18.58 \cdot \log \text{HP}$$

Interpretation of β values

Similarly, in the following model:

$$\widehat{\text{MPG}} = 108.7 - 18.58 \cdot \log \text{HP}$$

The interpretation of -18.58 is “for each unit of increase in the natural logarithm of the horsepower, the MPG estimate reduces by 18.58”.

Transformation strategy

- If the model has two or three of the equal variance, normality and linearity issues, try transforming Y.

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- There is no rule for which transformations will work in all cases; trial and error may be required.

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- After transforming the response, if the nonlinearity is not fixed, try transforming the predictor(s) as well.
- There is no rule for which transformations will work in all cases; trial and error may be required.
- Remember, the interpretations of the coefficients will change after you transform one or more variables!