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Conjugate network calculus: A dual approach applying the Legendre transform ☆

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Abstract

Network calculus is a theory of deterministic queuing systems that has successfully been applied to derive performance bounds for communication networks. Founded on min–plus convolution and de-convolution, network calculus obeys a strong analogy to system theory. Yet, system theory has been extended beyond the time domain applying the Fourier transform thereby allowing for an efficient analysis in the frequency domain. A corresponding dual domain for network calculus has not been elaborated, so far.

In this paper we show that in analogy to system theory such a dual domain for network calculus is given by convex/concave conjugates referred to also as the Legendre transform. We provide solutions for dual operations and show that min–plus convolution and de-convolution become simple addition and subtraction in the Legendre domain. Additionally, we derive expressions for the Legendre domain to determine upper bounds on backlog and delay at a service element and provide representative examples for the application of conjugate network calculus.

Keywords: Network calculus; Legendre transform; Fenchel duality theorem

1. Introduction

Network calculus [3,11] is a theory of deterministic queuing systems that allows analyzing various fields in computer networking. Being a powerful and elegant theory, network calculus obeys a number of analogies to classical system theory, however, under a min–plus algebra, where addition becomes computation of the minimum and multiplication becomes addition [1,3,11].

System theory applies a characterization of systems by their response to the Dirac impulse, which constitutes the neutral element. A system is considered to be linear, if a constantly scaled input signal results in a corresponding scaling of the output signal and if the sum of two input signals results in the sum of the two output signals that correspond to the individual input signals. The output of a system can be efficiently computed by convolution of the input signal and the system's impulse response and the concatenation of separate systems can be expressed by convolution of the individual impulse responses.

Network calculus relates very much to the above properties of system theory, while being based on

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the calculus for network delay presented in [5,6] and on Generalized Processor Sharing in [17,18]. The neutral element of network calculus is the burst function and systems are described by their burst response. Min–plus linearity is fulfilled, if the addition of a constant to the input signal results in an addition of the same constant to the output signal and if the minimum of two input signals results in the minimum of the two output signals that correspond to the individual input signals. The output of a system is given as the min–plus convolution respective de-convolution of the input signal and the system's burst response and the concatenation of separate systems can be described by min–plus convolution of the individual burst responses. Extensions and a comprehensive overview on current network calculus are given in [3,11].

Yet, system theory provides another practical domain for analysis applying the Fourier transform, which is particularly convenient due to its clearness and because the convolution integral becomes a simple multiplication in the Fourier domain. Besides, fast algorithms for Fourier transformation exist.

A corresponding domain in network calculus has, however, not been elaborated in depth so far. In [15] it is shown that the backlog bound at a constant rate server is equal to the Legendre, respectively Fenchel, transform [1,21] of the input. A similar concept is used in [10,16], where the output of a network element is computed in the Legendre domain. Related theories are, however, far more developed in the field of morphological signal processing [7,14] where the slope transform has been successfully applied. Yet, it can be shown that the Legendre transform provides the basis for a new and comprehensive theory, which constitutes a dual approach to network calculus that can be efficiently applied to a variety of problems [8,20]. As for the Fourier transform, fast algorithms for the Legendre transform exist [13].

The remainder of this paper is organized as follows: first, Section 2 briefly summarizes some essential elements of network calculus, including arrival curves for input and output as well as service curves. Then, Section 3 introduces the Legendre transform convex and concave conjugates and related properties, especially min–plus convolution and de-convolution. In Section 4 the derived theory is applied to introduce the *conjugate network calculus* comprised of the transformed elements of network calculus described in Section 2. Section 5 provides two sample applications of conjugate network calculus and finally Section 6 concludes the paper.

2. Elements of network calculus

The foundations of network calculus are min–plus convolution and de-convolution, where min–plus operations can be derived from the corresponding classical operations by replacing addition by computation of the minimum and multiplication by addition. Consequently the algebraic structure that is used by network calculus is the commutative dioid $(\mathbb{R} \cup \infty, \min, +)$ [11].

Definition 1 (*Min–plus convolution and min–plus de-convolution*). The min–plus convolution $\otimes$ of two functions $f(t)$ and $g(t)$ is defined as

$$(f \otimes g)(t) = \inf_u [f(t-u) + g(u)]$$

and min–plus de-convolution $\oslash$ as

$$(f \oslash g)(t) = \sup_u [f(t+u) - g(u)].$$

In the context of network calculus $t \geqslant u \geqslant 0$ respectively $t \geqslant 0$ and $u \geqslant 0$ is often applied to min–plus convolution respectively min–plus de-convolution.

The algebraic structure $(\mathscr{F}, \min, \otimes)$ is again a commutative dioid, where $\mathscr{F}$ is the set of wide-sense increasing functions with $f(s) \leqslant f(t)$ for all $s \leqslant t$ and $f(t) = 0$ for $t < 0$ [11]. Note however, that min–plus de-convolution does not generally fulfill these properties. In particular it is not commutative.

Furthermore, in the context of network calculus, arrival and service curves play an important role. Arrival curves constitute upper bounds on the input and output of network elements, while service curves represent lower bounds on the service offered by network elements. Networks are usually comprised of more than one network element and a network service is generally composed of a concatenation of the services of the individual network elements. This section provides the formal definitions for arrival and service curves, and the computation rules for concatenated service curves and single server output bounds, which will be used in the sequel. The corresponding proofs can be found, for example, in [3,11].

2.1. Arrival curves

Flows or aggregates of flows can be described by arrival functions $F(t)$ that are given as the cumulated number of bits seen in an interval $[0, t]$. Arrival curves $\alpha(t)$ are defined to give upper bounds on the arrival functions.

Definition 2 (*Arrival curve*). An arrival function $F(t)$ conforms to an arrival curve $\alpha(t)$, if for all $t \geqslant 0$ and all $s \in [0, t]$

$$\alpha(t-s) \geqslant F(t) - F(s).$$

A typical constraint for incoming flows is given by the leaky-bucket algorithm, which allows for bursts of size b and a defined sustainable rate r.

Definition 3 (*Leaky-bucket arrival curve*). The arrival curve that is enforced by a leaky bucket is given by

$$\alpha(t) = \begin{cases} 0, & t = 0, \\ b + r \cdot t, & t > 0. \end{cases}$$

2.2. Service curves

The service that is offered by the scheduler on an outgoing link can be characterized by a minimum service curve, denoted by $\beta(t)$.

Definition 4 (*Service curve*). A lossless network element with input arrival function $F(t)$ and output arrival function $F'(t)$ is said to offer a service curve $\beta(t)$ if for any $t \geqslant 0$ there exists at least one $s \in [0, t]$ such that

$$F'(t) - F(s) \geqslant \beta(t-s).$$

A characteristic service curve is the rate-latency type with rate R and latency T.

Definition 5 (*Rate-latency service curve*). The rate-latency service curve is defined as

$$\beta(t) = R \cdot [t - T]^+,$$

where $[\cdots]^+$ is zero if the argument is negative.

2.3. Concatenation

Networks are usually comprised of more than one network element and a network service is generally composed of a series of individual network element services. The service curve of a concatenation of service elements can be efficiently described by min–plus convolution of the individual service curves.

Theorem 1 (Concatenation). *The service curve $\beta(t)$ of the concatenation of n service elements with service curves $\beta_i(t)$ becomes*

$$\beta(t) = \bigotimes_{i=1}^{n} \beta_i(t).$$

2.4. Output bounds

Bounds on the output from a service element can be derived to be the min–plus de-convolution of the bound on the input and the corresponding service curve.

Theorem 2 (Output bound). *Consider a service element $\beta(t)$ with input that is bounded by $\alpha(t)$. A bound on the output $\alpha'(t)$ is given by*

$$\alpha'(t) = \alpha(t) \oslash \beta(t).$$

2.5. Performance bounds

The bounds on the performance delivered by a particular service element can be immediately determined from the arrival curve of traffic traversing the element and from the pertaining service curve.

Theorem 3 (Server performance thresholds). *Consider a server with service curve $\beta(t)$. Let $Q(t)$ be the backlog at the server at time t and let $D(t)$ be the virtual delay of the last packet that arrives at time t for traffic with an arrival curve $\alpha(t)$. Then the backlog is upper bounded by*

$$Q \leqslant \sup_{u \geqslant 0} \{\alpha(u) - \beta(u)\} = (\alpha \oslash \beta)(0)$$

and the maximum delay in case of FIFO scheduling is bounded by

$$D \leqslant \inf\{d \geqslant 0 : \alpha(u) \leqslant \beta(u+d)\ \forall u \geqslant 0\}.$$

The maximum backlog and maximum delay can be determined from the maximum vertical and horizontal deviations of the arrival and service curve, respectively.

3. The Legendre transform

In this section we show the existence of eigenfunctions in classical and in particular in min–plus system theory. The corresponding eigenvalues immediately yield the Fourier respective Legendre transform. Following the definition of convex and concave conjugates, the two major operations of network calculus, min–plus convolution and de-convolution, are derived in the Legendre domain and finally a list of properties of the Legendre transform is provided.

3.1. Eigenfunctions and eigenvalues

Let us recall the definition of eigenfunctions and eigenvalues in classical and in min–plus algebra.

Definition 6 (*Eigenfunctions and eigenvalues*). Consider a linear operator $\mathscr{A}$ on a function space. The function $f(t)$ is an eigenfunction for $\mathscr{A}$ with associated eigenvalue λ, if

$$\mathscr{A}[f(t)] = f(t) \cdot \lambda.$$

Accordingly in min–plus algebra eigenfunctions and eigenvalues are defined by

$$\mathscr{A}[f(t)] = f(t) + \lambda.$$

The output $h(t)$ of a linear time-invariant system with impulse response $g(t)$ and input $f(t)$ is given by the convolution integral

$$h(t) = f(t) * g(t) = \int_{-\infty}^{+\infty} f(t-u)g(u)\,\mathrm{d}u.$$

The functions $f(t) = \mathrm{e}^{\mathrm{j}2\pi st}$ are known to be eigenfunctions for the convolution integral as shown by

$$\mathrm{e}^{\mathrm{j}2\pi st} * g(t) = \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{j}2\pi s(t-u)} g(u)\,\mathrm{d}u = \mathrm{e}^{\mathrm{j}2\pi st} \cdot G(s),$$

where the eigenvalue

$$G(s) = \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{j}2\pi su} g(u)\,\mathrm{d}u$$

is equivalent to the Fourier transform of $g(t)$.

In analogy, network calculus applies min–plus convolution to derive lower bounds on the output of network elements, respectively min–plus de-convolution to derive upper bounds. For explanatory reasons the following derivation is made applying min–plus de-convolution according to Definition 1, which provides an upper bound on the output of a network element with burst response $g(t)$ and upper bounded input $f(t)$. Eigenfunctions with regard to min–plus de-convolution are the affine functions $b + s \cdot t$ as established by

$$\begin{aligned}(b + s \cdot t) \oslash g(t) &= \sup_u [b + s \cdot (t+u) - g(u)] \\ &= b + s \cdot t + G(s),\end{aligned}$$

where the eigenvalue

$$G(s) = \sup_u [s \cdot u - g(u)]$$

is the Fenchel conjugate that applies for convex functions $g(t)$. The Fenchel conjugate of a differentiable function is generally denoted as the Legendre transform. Under the assumption of differentiability we use the terms Legendre transform and conjugate interchangeably.

3.2. Convex and concave conjugates

Before providing further details on the Legendre transform, convexity and concavity are defined.

Definition 7 (*Convexity and concavity*). A function $f(t)$ is convex, if for all $u \in [0,1]$

$$f(u \cdot s + (1-u) \cdot t) \leqslant u \cdot f(s) + (1-u) \cdot f(t).$$

A function $g(t)$ is concave, if for all $u \in [0,1]$

$$g(u \cdot s + (1-u) \cdot t) \geqslant u \cdot g(s) + (1-u) \cdot g(t).$$

Among others the following properties are of particular interest: if $f(t) = -g(t)$ is convex then $g(t)$ is concave. The sum of two convex or two concave functions is convex, respectively concave. If the domain of a convex or concave function is smaller than $\mathbb{R}$, the function can be extended to $\mathbb{R}$ while retaining convexity respectively concavity by setting it to $+\infty$, respectively $-\infty$, where it is undefined [21].

The Legendre transform is defined independently for convex and concave functions. Details can be found in [21]. Further on, set-valued extensions exist that allow transforming arbitrary functions [7,14] which, however, are not used here. Let $\mathscr{L}$ denote the Legendre transform in general, where we for clarity distinguish between convex conjugates $\overline{\mathscr{L}}$ and concave conjugates $\underline{\mathscr{L}}$.

Definition 8 (*Convex and concave conjugates*). The convex Fenchel[1] conjugate is defined as

$$F(s) = \overline{\mathscr{L}}(f(t))(s) = \sup_t [s \cdot t - f(t)]$$

and the concave conjugate as

$$G(s) = \underline{\mathscr{L}}(g(t))(s) = \inf_t [s \cdot t - g(t)].$$

If $f(t) = -g(t)$ is convex then $\underline{\mathscr{L}}(g(t))(s) = -\overline{\mathscr{L}}(f(t))(-s)$ holds.

[1] Convex and concave conjugates can also be derived by means of the Fenchel duality theorem, which will be discussed in Section 4.5.

3.3. Min–plus convolution and de-convolution

The foundation of network calculus are min–plus convolution and de-convolution for which corresponding operations in the Legendre domain are derived here.

Theorem 4 (Min–plus convolution in the Legendre domain). *The min–plus convolution of two convex functions $f(t)$ and $g(t)$ in the time domain becomes an addition in the Legendre domain.*

Theorem 4 has already been reported in [1,21].

Proof. The min–plus convolution of two convex functions is convex [11]. Thus, it is meaningful to apply the convex conjugate which becomes

$$\begin{aligned}\overline{\mathscr{L}}((f\otimes g)(t))(s) &= \sup_t[s\cdot t - \inf_u[f(t-u)+g(u)]]\\ &= \sup_t[s\cdot t + \sup_u[-f(t-u)-g(u)]]\\ &= \sup_u[\sup_t[s\cdot(t-u)-f(t-u)]\\ &\quad + s\cdot u - g(u)]\\ &= \sup_u[\overline{\mathscr{L}}(f(t))(s)+s\cdot u-g(u)]\\ &= \overline{\mathscr{L}}(f(t))(s)+\sup_u[s\cdot u-g(u)]\\ &= \overline{\mathscr{L}}(f(t))(s)+\overline{\mathscr{L}}(g(t))(s). \qquad \square\end{aligned}$$

Lemma 1 (Concavity of min–plus de-convolution). *The min–plus de-convolution of a concave function $f(t)$ and a convex function $g(t)$ is concave.*

Proof. The proof is a variation of a proof provided in [11], where it is shown that the min–plus convolution of two convex functions is convex. Define $\mathscr{S}_{-f(t)}$ and $\mathscr{S}_{g(-t)}$ to be the epigraphy of $-f(t)$ and $g(-t)$ according to

$$\begin{aligned}\mathscr{S}_{-f(t)} &= \{(t,\eta)\in\mathbb{R}^2 \mid -f(t)\leqslant\eta\},\\ \mathscr{S}_{g(-t)} &= \{(t,\vartheta)\in\mathbb{R}^2 \mid g(-t)\leqslant\vartheta\}.\end{aligned}$$

Since $-f(t)$ and $g(-t)$ are both convex, the corresponding epigraphy are also convex [11,21] as well as the sum $\mathscr{S}=\mathscr{S}_{-f(t)}+\mathscr{S}_{g(-t)}$ that is

$$\begin{aligned}\mathscr{S} = \{&(r+s,\eta+\vartheta) \mid (r,\eta)\in\mathbb{R}^2,\ (s,\vartheta)\in\mathbb{R}^2,\\ &-f(r)\leqslant\eta,\ g(-s)\leqslant\vartheta\}.\end{aligned}$$

Substitution of $r+s$ by t, s by $-u$, and $\eta+\vartheta$ by ξ yields

$$\begin{aligned}\mathscr{S} = \{&(t,\xi)\in\mathbb{R}^2 \mid (-u,\vartheta)\in\mathbb{R}^2,\ -f(t+u)\\ &\leqslant \xi-\vartheta,\ g(u)\leqslant\vartheta\}.\end{aligned}$$

Since $\mathscr{S}$ is convex, $h(t)=\inf\{\xi\in\mathbb{R} \mid (t,\xi)\in\mathscr{S}\}$ is also convex [21]:

$$\begin{aligned}h(t) &= \inf\{\xi\in\mathbb{R} \mid (-u,\vartheta)\in\mathbb{R}^2,\\ &\qquad -f(t+u)\leqslant\xi-\vartheta,\ g(u)\leqslant\vartheta\}\\ &= \inf\{\xi\in\mathbb{R} \mid (-u)\in\mathbb{R},\ -f(t+u)\\ &\qquad +g(u)\leqslant\xi\} = \inf_u\{-f(t+u)+g(u)\}.\end{aligned}$$

It follows that $(f \oslash g)(t)=\sup_u[f(t+u)-g(u)]$ is concave. $\square$

Theorem 5 (Min–plus de-convolution in the Legendre domain). *The min–plus de-convolution of a concave function $f(t)$ and a convex function $g(t)$ in the time domain becomes a subtraction in the Legendre domain.*

Proof. With Lemma 1 we find

$$\begin{aligned}\underline{\mathscr{L}}((f\oslash g)(t))(s) &= \inf_t[s\cdot t-\sup_u[f(t+u)-g(u)]]\\ &= \inf_t[s\cdot t+\inf_u[-f(t+u)+g(u)]]\\ &= \inf_u[\inf_t[s\cdot(t+u)-f(t+u)]\\ &\quad +g(u)-s\cdot u]\\ &= \inf_u[\underline{\mathscr{L}}(f(t))(s)+g(u)-s\cdot u]\\ &= \underline{\mathscr{L}}(f(t))(s)-\sup_u[s\cdot u-g(u)]\\ &= \underline{\mathscr{L}}(f(t))(s)-\overline{\mathscr{L}}(g(t))(s). \qquad \square\end{aligned}$$

3.4. Properties of the Legendre transform

The Legendre transform exhibits a number of useful properties of which Table 1 lists the most relevant ones. More details can be found in [21].

The Legendre transform is self-dual, that is it is its own inverse. More precisely $\mathscr{L}(\mathscr{L}(f))(t)=(\mathrm{cl}f)(t)$, where $(\mathrm{cl}f)(t)$ is the closure of $f(t)$ that is defined to be $(\mathrm{cl}f)(t)=\liminf_{s\to t}f(s)$ for convex functions and $(\mathrm{cl}f)(t)=\limsup_{s\to t}f(s)$ for concave functions. Thus, if $f(t)$ is convex then $(\mathrm{cl}f)(t)\leqslant f(t)$ and if $f(t)$ is concave then $(\mathrm{cl}f)(t)\geqslant f(t)$. If $(\mathrm{cl}f)(t)=f(t)$ then $f(t)$ is said to be closed. The Legendre transform of a convex function is a closed convex function, respectively the Legendre transform of a concave function is a closed concave function.

Now consider an arbitrary function $f(t)$. The convex conjugate becomes $\overline{\mathscr{L}}(f(t))(s)=\overline{\mathscr{L}}(\mathrm{cl}(\mathrm{conv}f)(t))(s)$, where the convex hull $(\mathrm{cl}(\mathrm{conv}f))(t)$ of $f(t)$ is the greatest closed convex function majorized by

Table 1
Properties of the Legendre transform

Time domain	Legendre domain
$f(t)$ $f(t) = \mathscr{L}(F(s))(t)$	$F(s) = \mathscr{L}(f(t))(s)$ $F(s)$
$f(t)$ f convex	$F(s) = \overline{\mathscr{L}}(f(t))(s) = \sup_t[s \cdot t - f(t)]$ F convex
$f(t)$ f concave	$F(s) = \underline{\mathscr{L}}(f(t))(s) = \inf_t[s \cdot t - f(t)]$ F concave
$f(t) + c$ $f(t) \cdot c$ $f(t) + t \cdot c$ $f(t + c)$ $f(t \cdot c)$	$F(s) - c$ $F(s/c) \cdot c$ $F(s - c)$ $F(s) - s \cdot c$ $F(s/c)$
$f(t) = g(t) \otimes h(t)$, g convex, h convex	$F(s) = G(s) + H(s)$, G convex, H convex
$f(t) = g(t) \oslash h(t)$, g concave, h convex	$F(s) = G(s) - H(s)$, G concave, H convex

$f(t)$. It can be seen as the pointwise supremum on all affine functions majorized by $f(t)$ such that $(\mathrm{cl}(\mathrm{conv} f))(t) = \sup_{b,r}\{b + r \cdot t : (\forall s : b + r \cdot s \leqslant f(s))\}$ and consequently $(\mathrm{cl}(\mathrm{conv} f))(t) \leqslant f(t)$ holds. For the concave conjugate $\underline{\mathscr{L}}(f(t))(s) = \underline{\mathscr{L}}(\mathrm{cl}(\mathrm{conc} f)(t))(s)$ holds, where the concave hull follows as $(\mathrm{cl}(\mathrm{conc} f))(t) = \inf_{b,r}\{b + r \cdot t : (\forall s : b + r \cdot s \geqslant f(s))\}$ and $(\mathrm{cl}(\mathrm{conc} f))(t) \geqslant f(t)$.

4. Conjugate network calculus

After the introduction of convex and concave conjugates and of the dual min–plus operations in the Legendre domain we can derive the dual operations to the network calculus concatenation theorem and output theorem. However, the prerequisite for application of the dual operations is a transformation of arrival and service curves into this domain, which will be presented first. The set of dual elements in the Legendre domain shall be denoted by the term *conjugate network calculus*. These dual operations are complemented by a dual approach to determine performance bounds in the Legendre domain. Each of the following sub-sections presents the dual element in the Legendre domain that corresponds to the element of network calculus presented in the pertaining sub-section of Section 2.

4.1. Arrival curves

According to Definition 3 arrival curves of leaky-bucket type are concave and defined for $t \geqslant 0$. We apply the concave extension described in Section 3.2 by setting $\alpha(t) = -\infty$ for $t < 0$. Strictly, the extended arrival curve does not belong to the set $\mathscr{F}$, which is usually applied by network calculus in the time domain, where $f(t) \in \mathscr{F}$ implies $f(t) = 0$ for $t < 0$. The concave extension is, however, meaningful in this context. Thus, we can derive the concave conjugate of a leaky-bucket arrival curve according to Corollary 1.

Corollary 1 (Conjugate leaky-bucket arrival curve). *The concave conjugate of the leaky-bucket constraint given in Definition* 3 *can be computed according to Definition* 8 *and is given as*

$$A(s) = \inf_t[s \cdot t - \alpha(t)] = \begin{cases} -\infty, & s < r, \\ -b, & s \geqslant r. \end{cases}$$

In [12] the burstiness curve is defined to be the maximal backlog at a constant rate server with rate s and input $\alpha(t)$, whereby it has been pointed out in [15] that the burstiness curve is actually the Legendre transform $-\underline{\mathscr{L}}(\alpha(t))(s)$. Thus, we obtain a very clear interpretation of $A(s)$.

Generally the concave hull of an arbitrary arrival curve is a valid arrival curve, since $\mathrm{cl}(\mathrm{conc}\,\alpha)(t) \geqslant \alpha(t)$ for all t. Note that the hull must not be derived explicitly in the time domain, since it follows immediately from $\underline{\mathscr{L}}(\mathrm{cl}(\mathrm{conc}\,\alpha)(t))(s) = \underline{\mathscr{L}}(\alpha(t))(s)$.

4.2. Service curves

The rate-latency service curve according to Definition 5 is convex and defined for $t \geqslant 0$. The convex extension described in Section 3.2 allows setting the curve to $+\infty$ for $t < 0$. However, in this context it is more meaningful to set the service curve to zero for $t < 0$ which results in a convex function that belongs to $\mathscr{F}$ where $f(t) \in \mathscr{F}$ implies $f(t) = 0$ for $t < 0$.

Corollary 2 (Conjugate rate-latency service curve). *The convex conjugate of the rate-latency service curve given in Definition* 5 *can be computed based on Definition* 8 *and follows immediately according to*

$$B(s) = \sup_t[s \cdot t - \beta(t)] = \begin{cases} +\infty, & s < 0, \\ s \cdot T, & s \geqslant 0,\ s \leqslant R, \\ +\infty, & s > R. \end{cases}$$

The conjugate $B(s)$ of the service curve $\beta(t)$ can be interpreted as the backlog bound that holds, if a constant bit rate stream with rate s is input to the respective network element.

Generally the convex hull of an arbitrary service curve is a valid service curve, since $\mathrm{cl}(\mathrm{conv}\,\beta)(t) \leqslant \beta(t)$ for all t. Note that the hull must not be derived explicitly in the time domain, since it follows immediately from $\overline{\mathscr{L}}(\mathrm{cl}(\mathrm{conv}\,\beta)(t))(s) = \overline{\mathscr{L}}(\beta(t))(s)$.

4.3. Concatenation

The concatenation of service elements, can be represented by min–plus convolution of the individual service curves according to Theorem 1. With Theorem 4 we can immediately formulate the following corollary.

Corollary 3 (Conjugate concatenation). *The conjugate service curve $B(s)$ of the concatenation of n service elements is given as the sum of the individual conjugate service curves $B_i(s)$ according to*

$$B(s) = \sum_{i=1}^{n} B_i(s).$$

Since it is known that $\overline{\mathscr{L}}(\overline{\mathscr{L}}(\beta))(t) = \overline{\mathscr{L}}(B(s))(t) = (\mathrm{cl}(\mathrm{conv}\,\beta))(t) \leqslant \beta(t)$ we find that $\overline{\mathscr{L}}(B(s))(t)$ is generally a valid service curve.

Here, we provide an example for the concatenation of rate-latency service elements. Consider n service elements in series with service curves $\beta_i(t) = R_i \cdot [t - T_i]^+$ for all t. The corresponding conjugates are $B_i(s) = s \cdot T_i$ for $0 \leqslant s \leqslant R_i$ and $+\infty$ elsewhere. The sum is $B(s) = s \cdot \sum_i T_i$ for $0 \leqslant s \leqslant \min_i[R_i]$ and $+\infty$ elsewhere. An example for $n = 2$ is shown in Fig. 1.

The result is convex and deriving the convex conjugate of $B(s)$ we find $(\mathrm{cl}(\mathrm{conv}\,\beta))(t) = \overline{\mathscr{L}}(B(s))(t) = \min_i[R_i] \cdot [t - \sum_i T_i]^+$. The result is exact since $(\mathrm{cl}(\mathrm{conv}\,\beta))(t) = \beta(t)$, where $\beta(t) = \otimes_{i=1}^{n} \beta_i(t)$. The service curve that is derived here by Legendre transform is exactly the same as the one that follows by min–plus convolution in the time domain.

4.4. Output bounds

For the output bound defined in Theorem 2 we can formulate the following corollary by applying Theorem 5.

Corollary 4 (Conjugate output bound). *The conjugate output bound $A'(s)$ of a service element with conjugate service curve $B(s)$ and constrained input with conjugate input bound $A(s)$ is provided by*

$$A'(s) = A(s) - B(s).$$

As stated before $\underline{\mathscr{L}}(\underline{\mathscr{L}}(\alpha'))(t) = \underline{\mathscr{L}}(A'(s))(t) = (\mathrm{cl}(\mathrm{conc}\,\alpha'))(t) \geqslant \alpha(t)$ holds and we find that $\underline{\mathscr{L}}(A'(s))(t)$ is generally a valid output arrival curve.

As an example consider the output bound that can be derived for a rate-latency service element with service curve $\beta(t) = R \cdot [t - T]^+$ for all t and leaky-bucket constrained input with arrival curve $\alpha(t) = b + r \cdot t$ for $t \geqslant 0$, zero for $t = 0$ and $-\infty$ for $t < 0$. The respective conjugates are $B(s) = s \cdot T$ for $0 \leqslant s \leqslant R$ and $+\infty$ else and $A(s) = -b$ for $s \geqslant r$ and $-\infty$ else. The difference is $A'(s) = -b - s \cdot T$ for $r \leqslant s \leqslant R$ and $-\infty$ else. The example is shown in Fig. 2.

The result is concave according to Lemma 1 and the concave conjugate of $A'(s)$ becomes $(\mathrm{cl}(\mathrm{conc}\,\alpha'))(t) = \underline{\mathscr{L}}(A'(s))(t) = b + r \cdot (t + T)$ for $t \geqslant -T$ and $(\mathrm{cl}(\mathrm{conc}\,\alpha'))(t) = \underline{\mathscr{L}}(A'(s))(t) = b + R \cdot (t + T)$ for $t < -T$. Again our solution is exact since $(\mathrm{cl}(\mathrm{conc}\,\alpha'))(t) = \alpha'(t)$, where $\alpha'(t) = \alpha(t) \oslash \beta(t)$. The result is well known for $t \geqslant 0$ from min–plus de-convolution in the time domain. Further

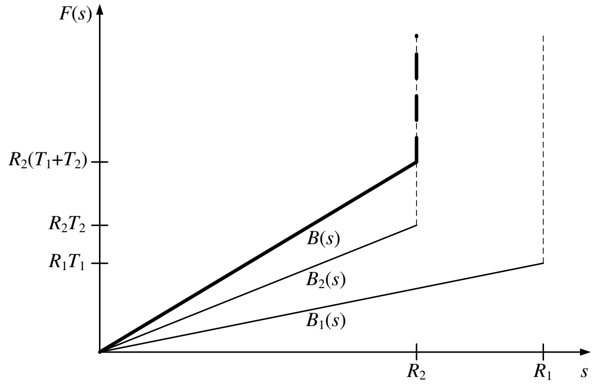

Fig. 1. Conjugate concatenated service curve.

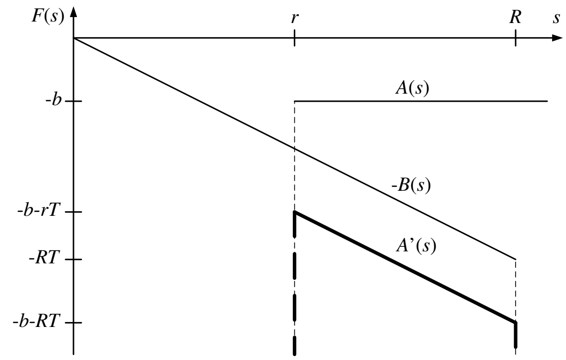

Fig. 2. Conjugate output arrival curve.

on, it is shown in [11] that the result of min–plus de-convolution is also valid for $t < 0$. Note that min–plus de-convolution in the time domain as well as in the Legendre domain is not closed in $\mathscr{F}$, where $f(t) \in \mathscr{F}$ implies $f(t) = 0$ for $t < 0$. In contrast, the output arrival curve $\alpha'(t)$ is strictly positive for $0 > t > -T - b/R$. The usual approach applied by network calculus in the time domain is to truncate functions for $t < 0$ implicitly by allowing only values $t \geqslant u \geqslant 0$ in min–plus convolution, respective $t \geqslant 0$ and $u \geqslant 0$ in min–plus de-convolution. Here, we require an explicit truncation of the output arrival curve $\alpha'(t)$ for $t < 0$ before interpreting the result of the Legendre transform $\underline{\mathscr{L}}(\alpha'(t))(s)$ as a meaningful backlog bound for a virtual subsequent constant rate network node.

4.5. Performance bounds

According to Theorem 3 the maximum backlog is given as the maximum vertical deviation between arrival and service curve. The derivation of this maximum deviation can be mapped to the problem considered in the Fenchel duality theorem, which deals with the problem of finding the minimum distance between two functions f_1 and f_2, where in our case we define $f_1 : \mathbb{R} \to (-\infty, \infty]$ is a convex and $f_2 : \mathbb{R} \to [-\infty, \infty)$ a concave extended real-valued function.

Theorem 6 (Fenchel duality theorem). *Let the functions f_1 and $-f_2$ be convex with $f_1, -f_2 : \mathbb{R} \to (-\infty, \infty]$. Then*

$$\inf_{x\in\mathbb{R}} \{f_1(x) - f_2(x)\} = \sup_{s\in\mathbb{R}} \{F_2(s) - F_1(s)\}$$

holds, with $F_1(s)$ and $F_2(s)$ being the convex and concave conjugate of $f_1(x)$ and $f_2(x)$, respectively.

Proof. We will partly reproduce the proof given in [2], but limited to functions on $\mathbb{R}$ instead of functions on $\mathbb{R}^n$. First, we want to apply the Lagrange approach for optimization with equality constraints. Therefore we restate the problem

$$\begin{aligned} &\text{minimize} \quad f_1(x) - f_2(x) \\ &\text{subject to} \quad x \in \mathbb{R} \end{aligned}$$

to the problem

$$\begin{aligned} &\text{minimize} \quad f_1(y) - f_2(z) \\ &\text{subject to} \quad z - y = 0 \end{aligned}$$

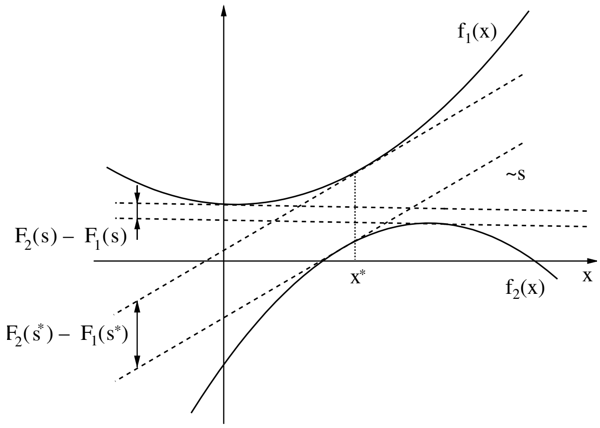

Fig. 3. Illustration of Fenchel's duality theory.

with $y \in \mathrm{dom}(f_1)$ and $z \in \mathrm{dom}(f_2)$. With the Lagrangian given by $L(y, z, s) = f_1(y) - f_2(z) + (z - y)s$ we obtain the dual function:

$$\begin{aligned} q(s) &= \inf_{y\in\mathbb{R}, z\in\mathbb{R}} \{f_1(y) - f_2(z) + (z - y)s\} \\ &= \inf_{z\in\mathbb{R}} \{sz - f_2(z)\} + \inf_{y\in\mathbb{R}} \{f_1(y) - sy\} \\ &= \inf_{z\in\mathbb{R}} \{sz - f_2(z)\} - \sup_{y\in\mathbb{R}} \{sy - f_1(y)\} \\ &= F_2(s) - F_1(s), \end{aligned}$$

where $F_1(s) = \sup_{y\in\mathbb{R}}\{sy - f_1(y)\}$ and $F_2(s) = \inf_{z\in\mathbb{R}}\{sz - f_2(z)\}$ denote the convex respective concave Fenchel conjugates[2] of a convex function $f_1(t)$ respective a concave function $f_2(t)$ as previously defined in Definition 8.

Now, with the functions $F_1 : \mathbb{R} \to (-\infty, \infty]$ and $F_2 : \mathbb{R} \to [-\infty, \infty)$ and according to duality theory in optimization [2], we can derive the dual problem

$$\begin{aligned} &\text{maximize} \quad q(s) = F_2(s) - F_1(s) \\ &\text{subject to} \quad s \in \mathbb{R}, \end{aligned}$$

which corresponds to $\sup_{s\in\mathbb{R}}\{F_2(s) - F_1(s)\}$. □

Fig. 3 illustrates a graphical interpretation of the Fenchel conjugates and of the Fenchel duality theorem. The convex conjugate $F_1(s)$ for a particular s can be constructed by drawing the lower tangent to f_1 with a slope of s. As $F_1(s) = \sup_{x\in\mathbb{R}}\{sx - f_1(x)\} = -\inf_{x\in\mathbb{R}}\{f_1(x) - sx\}$ holds, the ordinate of this tangent equals $-F_1(s)$. Accordingly, the

[2] As f_1 and $-f_2$ are convex functions and $\mathbb{R}$ is a convex set with a linear constraint $z - y = 0$ there is no duality gap between the primal function $\inf_{x,y\in\mathbb{R}, z=y}\{f_1(y) - f_2(z)\}$ and $\inf_{y\in\mathbb{R}, z\in\mathbb{R}}\{f_1(y) - f_2(z) + (z - y)s\}$.

negative concave conjugate $-F_2(s)$ is given where the upper tangent to f_2 with slope s intersects with the vertical axis.

Fenchel's duality theorem says that the minimum vertical distance between a convex and concave curve is equivalent to the maximum difference of the concave conjugate and the convex conjugate. The first order necessary condition for an extremum of $f_1(x) - f_2(x)$ yields $\mathrm{d}f_1(x)/\mathrm{d}x = \mathrm{d}f_2(x)/\mathrm{d}x$, i.e., at the minimum distance the tangents to both functions must be parallel. Consequently, the maximum vertical distance of parallel tangents to $f_1(x)$ and $f_2(x)$ at one x^* corresponds to the minimum distance of the two functions.

By applying the Fenchel duality theorem we can derive the backlog bound in the Legendre domain.

Theorem 7 (Backlog bound in the Legendre domain). *The maximum backlog at the server considered in Theorem 3 is given as*

$$Q = -\sup_{s\in\mathbb{R}}\{A(s) - B(s)\}.$$

Proof. The analogy to Fenchel's duality theorem becomes apparent, when f_1 corresponds to the convex minimum service curve and f_2 corresponds to the concave arrival curve. Generally the concave arrival curve α resides above the convex function β on a subset $\mathscr{U} \subset \mathbb{R}$. This implies a negative difference $\beta(t) - \alpha(t)\ \forall t \in \mathscr{U}$ such that the minimization yields the maximum vertical distance between arrival and minimum service curve. Therefore we can immediately apply Fenchel's duality theorem and obtain

$$\inf_{t\in\mathbb{R}}\{\beta(t) - \alpha(t)\} = \sup_{s\in\mathbb{R}}\{A(s) - B(s)\}. \qquad \square$$

Theorem 8 (Delay bound in the Legendre domain). *An upper bound on the delay at the server considered in Theorem 3 in case of FIFO scheduling is indicated by the difference of the slopes of the tangents to $B(s)$ and $A(s)$ in the same s^*, which intersect the vertical axis in the same point.*

Proof. We cannot easily apply Fenchel's duality theorem for the horizontal deviation of arrival and minimum service curve unless we want to express the deviation as vertical distance of the inverse functions of α and β. Instead, we assume that α and β are closed functions and make use of their convexity and concavity properties. As a consequence, for their biconjugates $\underline{\mathscr{L}}(\underline{\mathscr{L}}(\alpha))(t) = \alpha(t)$ and $\overline{\mathscr{L}}(\overline{\mathscr{L}}(\beta)) = \beta(t)$ holds. Fig. 4 depicts the conjugate transforms of a concave arrival curve and a convex minimum service curve. The value of the biconjugate for example of $\beta(t)$ is given as the intersection of the tangent to $B(s)$ with slope t with the vertical axis.

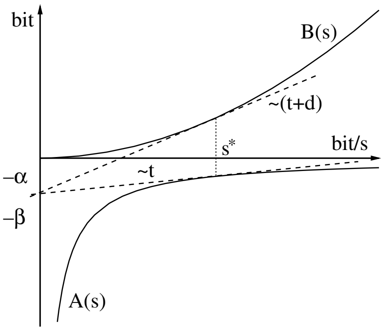

Fig. 4. Illustration of maximum delay in the conjugate transform domain.

The problem of determining the maximum horizontal deviation in the time domain can be stated as follows:

$$\begin{aligned}&\text{maximize} \quad d\\ &\text{subject to} \quad \alpha(t) = \beta(t+d), \quad t, d \in \mathbb{R}.\end{aligned}$$

The Lagrangian for this problem is $L(t,s) = d + s(\alpha(t) - \beta(t+d))$ and the first order necessary condition $\partial L/\partial t = 0$ and $\partial L/\partial s = 0$ yields

$$\begin{aligned}&\partial\alpha(t)/\partial t = \partial\beta(t+d)/\partial t,\\ &\alpha(t) = \beta(t+d).\end{aligned}$$

The equivalent condition to $\alpha(t) = \beta(t+d)$ in the Legendre domain means that the tangents to the conjugate functions must intersect with the vertical axis at the same point, as illustrated in Fig. 4. Condition $\partial\alpha(t)/\partial t = \partial\beta(t+d)/\partial t$ translates to requiring tangents to the conjugate functions at the same s. Consequently, the points on $\alpha(t)$ and $\beta(t)$ with maximum horizontal deviation in the time domain correspond to the points on $A(s)$ and $B(s)$ in the Legendre domain, where the tangents to A and B at the same s intersect in the same point on the vertical axis. Due to the characteristics of the Fenchel conjugates the difference of the slopes of these tangents indicates the maximum horizontal deviation in the time domain. $\square$

We do not claim that the computation of performance bounds in the conjugate transform domain is simpler. But even if the complexity is the same in both domains, the benefit is that one can determine performance bounds once all arrival and service curves have been transformed. In summary, as major driver for conjugate network calculus we see the simplification of convolution and de-convolution operations, while determining the performance bounds in the conjugate transform domain additionally alleviates the need to re-transform into the time domain.

5. Example applications

The application of network calculus and performance analysis in the Legendre domain will be demonstrated using the example of calculating the maximum delay and maximum backlog of *Dual Leaky Bucket* (DLB) constrained traffic at a rate-latency server. Furthermore, the example of service-curve-based routing following the proposal presented in [19] intends to illustrate the advantage of conjugate network calculus in terms of simplifying calculations which are more complex in the time domain.

5.1. Rate-latency performance bounds

The example of DLB-constrained traffic traversing a rate-latency server originates from the Integrated Services Model [22] and considers a variable bit rate flow, which is upper bounded by an arrival curve $\alpha(t) = \min[pt + M, \rho t + \sigma]$.

Theorem 9 (DLB—rate-latency bounds in the Legendre domain). *Consider a variable bit rate flow, which is upper bounded by a DLB arrival curve, served at a rate-latency server that guarantees a minimum service curve* $\beta = r[t-d]^+$, *where* $p \geqslant r \geqslant \rho$ *shall hold. Then, the maximum backlog can be computed by*

$$Q = \sigma + \rho d + \left(\frac{\sigma - M}{p - \rho} - d\right)^+ (\rho - r)$$

and the maximum packet delay, when being served in FIFO order, can be calculated from

$$D = \frac{M}{r} + \frac{\sigma - M}{p - \rho}\frac{p - r}{r} + d.$$

These performance bounds have been derived in [11], for example, and correspond to the maximum horizontal and vertical deviations between α and β, respectively (cf. Fig. 5).

We will provide an alternative proof for Theorem 9 serving as sample application of Theorems 7 and 8.

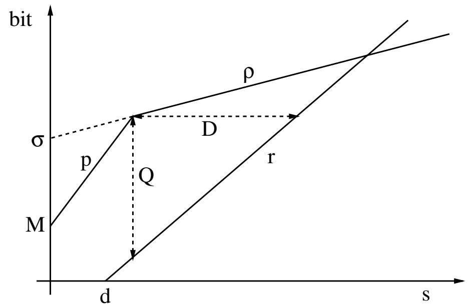

Fig. 5. DLB-LR performance bounds in the time domain.

Proof of Theorem 9. In order to derive the maximum backlog bound we have to determine the minimum vertical distance between $B(s)$ and $A(s)$.

Note that $A(s)$ for a concave $\alpha(t) \in \mathscr{F}$ is generally negative and so is $A(s) - B(s)$. The concave conjugate of a DLB arrival curve $\alpha(t) = \min[pt + M, \rho t + \sigma]$ and the convex conjugate of a latency-rate service curve $\beta(t) = r[t-d]^+$ are denoted by

$$A(s) = \begin{cases} -\infty, & s < \rho, \\ \dfrac{\sigma - M}{p - \rho}(s - \rho) - \sigma, & \rho \leqslant s \leqslant p, \\ -M, & s > p, \end{cases}$$

$$B(s) = \begin{cases} \infty, & s < 0, \\ sd, & 0 \leqslant s \leqslant r, \\ \infty, & s > r. \end{cases}$$

As depicted in Fig. 6 the minimum vertical distance between $B(s)$ and $A(s)$ is either at $s = \rho$ in case that $d \geqslant (\sigma - M)/(p - \rho)$ holds, or at $s = r$ otherwise. For $s = \rho$ this distance is $\sigma + \rho d$, which corresponds to the maximum backlog of Theorem 9 for $(\sigma - M)/(p - \rho) - d \leqslant 0$.

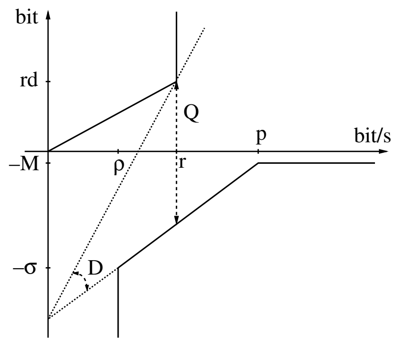

Fig. 6. DLB-LR performance bounds in the Legendre domain.

If $s = r$ applies, the vertical distance is denoted by

$$Q = rd + \sigma - \frac{\sigma - M}{p - \rho}(r - \rho) = \sigma + rd + \frac{\sigma - M}{p - \rho}(\rho - r),$$

which corresponds to the maximum backlog given in Theorem 9 with

$$\left[\frac{\sigma - M}{p - \rho} - d\right]^{+}(\rho - r) = (\rho - r)\frac{\sigma - M}{p - \rho} - \rho d + rd.$$

Thus we have derived the backlog bounds from the minimum vertical distance of the conjugate arrival and service curves. In order to derive the delay bound we need to determine the tangents to the conjugates at the same s^* which intersect in the same ordinate. As illustrated in Fig. 6 such intersecting tangents can only be at $s^* = r$. The tangent to $A(s)$ has a slope of $(\sigma - M)/(p - \rho)$, the tangent to $B(s)$ a slope of $d + \sigma/r + \rho(\sigma - M)/[r(p - \rho)]$. Now the maximum delay bound can be calculated from the difference of the slopes of these tangents, where

$$D = d + \frac{\sigma}{r} + \frac{\sigma - M}{p - \rho}\left(\frac{\rho}{r} - 1\right) = d + \frac{\sigma(p - r) - M(\rho - r)}{r(p - \rho)} = d + \frac{M}{r} + \frac{\sigma - M}{p - \rho}\frac{p - r}{r}$$

corresponds to D in Theorem 9. $\square$

5.2. Service-curve-based routing in the Legendre domain

We consider the problem of finding a feasible path in a certain network for traffic with a particular arrival curve $\alpha(t)$ and delay demand $d_{e2e,\max}$. In this context the feasibility of a path is defined as follows:

Definition 9. For traffic which is upper bounded by $\alpha(t)$ and for a given delay demand $d_{xy,\max}$ between two network nodes x and y the necessary condition for a series of network elements to become a feasible path for that demand is

$$\inf\{\tau \geqslant 0 : \alpha(t) \leqslant \beta_{xy}(t + \tau)\ \forall t \geqslant 0\} \leqslant d_{xy,\max}$$

where according to Theorem 1 the end-to-end service curve $\beta_{xy}(t)$ is computed by

$$\beta_{xy}(t) = \bigotimes_{i=x}^{y-1} \beta_{i,i+1}(t)$$

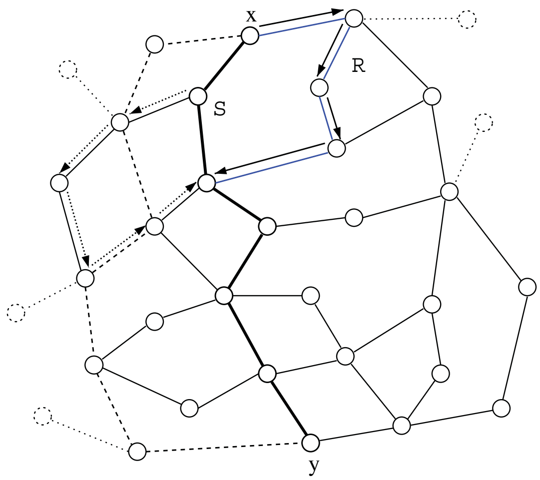

Fig. 7. Local search in the neighborhood of a minimum weight path.

with $\beta_{i,i+1}(t)$ denoting the service curve assigned to the link between node i and its link peer $i + 1$.

Initially, one path shall be given between source x and destination y, which, for example, may result from a minimum hop computation. Fig. 7 depicts an excerpt of an example graph representing the network with a given (minimum hop) path.

In case that an initially computed path is not feasible, [19] proposes to deploy *tabu-search* as local search heuristic in order to exploit the neighborhood of the initial path to find a feasible solution. The neighborhood of a path is given as the number of nodes adjacent to each of the nodes along the current path. A *move* of the tabu-search procedure means replacing one or more links of the current path by another sequence of links. After selecting the starting node of an alternative path a minimum hop path to one downstream node of the original path shall constitute a substitute path. For example, the link from the source node to the next downstream node can be replaced by choosing the link to the right hand neighbor. The substitute path is marked by solid arrows in Fig. 7 and is denoted by $\mathscr{R}$, while the substituted path is denoted by $\mathscr{S}$. The service curve of the substitute path is given by $\beta_{\mathscr{R}} = \bigotimes_{i \in \mathscr{R}} \beta_i$ and the resulting end-to-end service curve is computed from

$$\beta_{xy,\text{subs}} = \beta_{\mathscr{R}} \otimes \bigotimes_{j \in \{x,\ldots,y-1\} \setminus \mathscr{S}} \beta_{j,j+1} = \bigotimes_{i \in \mathscr{R}} \beta_{i,i+1} \otimes \bigotimes_{j \in \{x,\ldots,y-1\} \setminus \mathscr{S}} \beta_{j,j+1}.$$

Procedure *COMPUTEROUTE* $(p[\cdot], \alpha, d_{xy,\max})$:

1 $p_{\text{best}}[\cdot] \leftarrow p[\cdot]$; $\beta_{\text{best}} \leftarrow \beta_{xy}$
2 **while** $\inf_{\tau \geq 0} \{\alpha(t-\tau) \leq \beta_{\text{best}}(t)\} > d_{xy,\max}$ **do**
3 $\quad \mathcal{N}_p[\cdot,\cdot] \leftarrow$ `ADJACENCIES` $(p_{\text{best}}[\cdot])$
4 $\quad$ **forall** $i \in \mathcal{I}$ **do**
5 $\quad\quad$ **forall** $j \in \mathcal{J}_i$ **do**
6 $\quad\quad\quad (\mathcal{S}[\cdot], \mathcal{R}[\cdot], \beta_{\mathcal{R}}) \leftarrow$ `FIND-SUBSTITUTE` $(p_{\text{best}}[i], \mathcal{N}_p[i,j], \alpha, d_{xy,\max})$
7 $\quad\quad\quad$ **if** `IS-TABU` $(\mathcal{R}, p_{\text{best}}[\cdot]) = true$ **then**
\# `Exclude the neighborhood`
\# `node leading to tabu-solution`
8 $\quad\quad\quad\quad \mathcal{N}_p[i,\cdot] \leftarrow \mathcal{N}_p[i,\cdot] - N_p[i,j]$
9 $\quad\quad\quad\quad (\mathcal{S}[\cdot], \mathcal{R}[\cdot], \beta_{\mathcal{R}}) \leftarrow$ `FIND-SUBSTITUTE` $(p_{\text{best}}[i], \mathcal{N}_p[i,j], \alpha, d_{xy,\max})$
10 $\quad\quad\quad \beta_{xy,\text{subs}} \leftarrow \beta_{\mathcal{R}} \otimes \left(\bigotimes_{k \in \{x,\ldots,y-1\} \setminus \mathcal{S}} \beta_{k,k+1}\right)$
11 $\quad\quad\quad$ **if** $\beta_{xy,\text{subs}} > \beta_{\text{best}}$ **then**
12 $\quad\quad\quad\quad p_{\text{best}}[\cdot] \leftarrow p[\cdot]$; $\beta_{\text{best}} \leftarrow \beta_{xy,\text{subs}}$
13 $\quad\quad\quad\quad$ `TABU-MOVE` $(p_{\text{best}}[\cdot], \mathcal{R})$
14 **return** $(p_{\text{best}}[\cdot], \beta_{\text{best}})$

Fig. 8. Algorithm to find a new feasible path.

A procedure `COMPUTEROUTE` to find a feasible path is depicted in Fig. 8 using *pseudo-code* notation [4]. The parameter $p[\cdot]$ is a vector denoting the sequence of nodes traversed by the initially configured path, the parameters α and $d_{xy,\max}$ represent the arrival curve and delay demand of the traffic to be routed, respectively. The two-dimensional array $\mathcal{N}_p[\cdot,\cdot]$ describes the neighborhood of the path $p[\cdot]$, where for each node $p[i]$ the array element $\mathcal{N}_p[i,j]$ with $j \in \mathcal{J}$ indicates one direct neighbor[3] and $\mathcal{J}$ denotes the number of neighbors. This neighborhood is always adapted to the currently configured path (cf. line 3). The procedure checks all substitute paths, which are determined by a sub procedure `FIND-SUBSTITUTE`, beginning at node $p[i]$ for all nodes $i \in I$ along $p[\cdot]$. If changing the route to include the substitute path $\mathscr{R}$ does not lead to a *tabu-move* and if the service curve $\beta_{xy,\text{subs}}$ along the new path results in a lower delay performance for traffic constrained by α, then the new path is set as best currently found path $p_{\text{best}}[\cdot]$ and the pertaining service curve is stored as β_{best} (lines 4–13). A tabu-move is a move which is stored in the so called *tabu-list*. The tabu-list stores the recent history of moves and comes into play when attempting to leave a local extremum. The identified next moves in terms of the substitute path are stored in the tabu-list (cf. line 13). For further details on tabu-search we refer to [9]. The computational complexity of this procedure mainly stems from the frequently invoked min–plus convolution in line 10 and from the infimum operation in the while condition in line 2.

In the Legendre domain the min–plus convolution is transformed to a simple addition of the convex conjugates $B_{k,k+1}(s)$ such that line 10 of the procedure transforms to

$$B_{xy,\text{subs}}(s) \leftarrow B_{\mathscr{R}}(s) + \sum_{k \in \{x,\ldots,y-1\} \setminus \mathscr{S}} B_{k,k+1}(s)$$

with $B_{\mathscr{R}}(s) = \sum_{l \in \mathscr{R}} B_{l,l+1}(s)$, where the pointwise addition of the convex conjugates can be considered significantly less complex than the min–plus convolution operation, which is mainly comprised of an infimum operation.

The feasibility criterion of Definition 9 transforms to requiring that the maximum difference between the slopes of the tangents to $A(s)$ and $B_{xy}(s)$, which intersect in one ordinate, be less than $d_{xy,\max}$. However, if we consider a particular arrival curve, for example a leaky-bucket constrained arrival process, then we can derive more convenient feasibility criteria. If α corresponds to a leaky bucket, then according to Corollary 1 the concave conjugate is given by $A(s) = -b \ \forall s \geqslant r$ and $A(s) = -\infty \ \forall s < r$. In this case the tangent to $A(s)$ intersects with the y-axis in $-b$ and the feasibility criterion is that the function $T(s) = s d_{xy,\max} - b$ must reside above

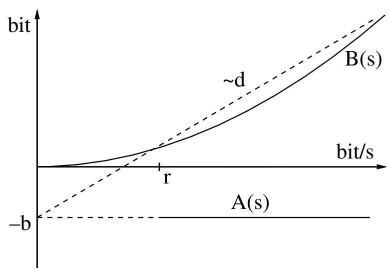

Fig. 9. Path feasibility criterion in Legendre domain for leaky-bucket constrained traffic.

[3] A node with direct link to $p[i]$.

$B_{xy}(s)$ or be a tangent to $B_{xy}(s)$ for one $s^* > r$. Fig. 9 illustrates a graphical explanation of this criterion.

With a horizontal tangent to $A(s)$ the slope of the tangent to $B(s)$ which intersects in the same ordinate must not exceed $d_{xy,\max}$. Consequently, all feasible service curve conjugates $B(s)$ must have a tangent with a smaller slope, i.e., either $B(s)$ resides below the line $T(s) = sd_{xy,\max} - b$ or $T(s)$ is tangent to $B(s)$. The condition "for at least one $s^* \geqslant r$" ensures that the common s^*, where both tangents touch $A(s)$ and $B(s)$ respectively (cf. Theorem 8), is in a subset of $\mathbb{R}$, where $A(s) \neq -\infty$.

6. Conclusions

In this paper we have shown that the Legendre transform provides a dual domain for analysis of data networks applying network calculus. Our work is a significant extension of the known analogy of system theory and network calculus where the Legendre transform corresponds to the Fourier transform in system theory. In particular, we have proven that min–plus convolution and min–plus de-convolution correspond to addition respective subtraction in the Legendre domain, which allows for an efficient analysis and fast computation. Furthermore, we have derived bounds on backlog and delay in the Legendre domain using the conjugate arrival and service curves. Example applications of this conjugate network calculus demonstrate the characteristic properties of network calculus and performance analysis in the Legendre domain.

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