CHARACTERIZATION OF A SIMPLE COMMUNICATION NETWORK USING LEGENDRE TRANSFORM

Takashi Hisakado and Kohshi Okumura

Department of Electrical Engineering
Kyoto University
Kyoto, Japan
{hisakado, kohshi}@kuee.kyoto-u.ac.jp

Vladimir Vukadinović and Ljiljana Trajković

School of Engineering Science
Simon Fraser University
Vancouver, Canada
{vladimir, ljilja}@cs.sfu.ca

ABSTRACT

We describe an application of the Legendre transform to communication networks. The Legendre transform applied to max-plus algebra linear systems corresponds to the Fourier transform applied to conventional linear systems. Hence, it is a powerful tool that can be applied to max-plus linear systems and their identification. Linear max-plus algebra has been already used to describe simple data communication networks. We first extend the Legendre transform as the slope transform to non-concave/non-convex functions. We then use it to analyze a simple communication network. We also propose an identification method for its transfer characteristic, and we confirm the results using the ns-2 network simulator.

1. INTRODUCTION

The majority of communication network systems are nonlinear. Their analysis is rather complex because of this inherent nonlinearity. In this paper, we present an approach to the problem by using the max-plus algebra and the Legendre transform.

The max-plus and min-plus algebra are powerful tools for analyzing discrete event systems [1, 2]. Discrete event systems can be described using linear max-plus or min-plus equations, even though they are nonlinear in the sense of the conventional system theory. The max-plus and min-plus algebra are based on idempotency [4]. Applications of the max-plus algebra [3, 5] and the min-plus algebra [2, 6] have been proposed for the analysis of communication networks. For example, the window flow control of Transport Control Protocol (TCP) has been reported to be max-plus linear [3]. Furthermore, the fractal scaling of TCP traffic was also observed [5]. The network calculus has been proposed for a number of problems in networking, such as window flow control, multimedia smoothing, and establishing bounds for packet loss rates [2, 6].

The Legendre transform is important in max-plus algebra linear systems because it corresponds to the Fourier transform in conventional linear system theory. In addition, as in the case of fast Fourier transform, fast Legendre transform was also introduced [7, 8, 9]. However, the Legendre transform is usually applied only to convex or concave functions [1]. Nevertheless, in the area of morphological signal processing, the extended Legendre transform, called the slope transform, is extended to non-convex/non-concave functions [10, 11]. Because the morphology corresponds to the convolution in the max-plus algebra, the Legendre transform proved to be a powerful tool in morphological signal processing.

Our approach employs the extended Legendre transform that can be applied to non-convex/non-concave functions. We apply it to communication networks and we propose a method for analysis and identification of certain performance parameters of simple packet data networks.

2. LEGENDRE TRANSFORM

The Legendre transform $\mathcal{L}[x(u)](s)$ of the function $x(u)$ that is concave or convex [12] and has an invertible derivative is defined as:

$$X(s) := \mathcal{L}[x(u)](s) = x(u^*) - su^* \text{ , where } s = \frac{\mathrm{d}x}{\mathrm{d}u}(u^*). \quad (1)$$

Hence, if the tangent at a point $(u_0, x(u_0))$ has slope s_0, then the corresponding Legendre transform is the segment $x(u_0) - s_0 u_0$ on the y-axis of the tangent $s_0(u - u_0) + x(u_0)$. An example of the Legendre transform is shown in Figure 1. The maximum of $x(u)$ corresponds to the intercepts of $X(s)$, and the minimum of $X(s)$ corresponds to the intercepts of $x(u)$.

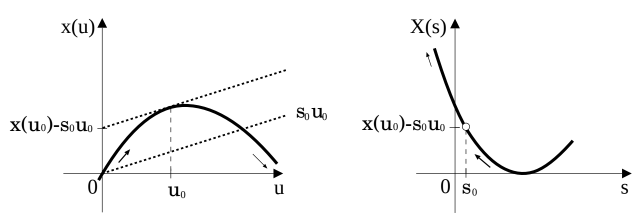

Figure 1: The Legendre transform of a smooth concave function.

In order to extend the transform to non-smooth functions, we consider the interval-valued derivative:

$$\frac{\mathrm{d}x}{\mathrm{d}u}(u_0) = \left[\lim_{x \uparrow u_0} \frac{\mathrm{d}x}{\mathrm{d}u}, \lim_{x \downarrow u_0} \frac{\mathrm{d}x}{\mathrm{d}u} \right]. \quad (2)$$

In this case, the equation $\dfrac{\mathrm{d}x(u)}{\mathrm{d}u} = s$ implies that the interval contains s. The notion of the interval-valued derivative is used in convex analysis, where it is called a subdifferential [10, 13]. An example of the Legendre transform of a non-smooth function is shown in Figure 2.

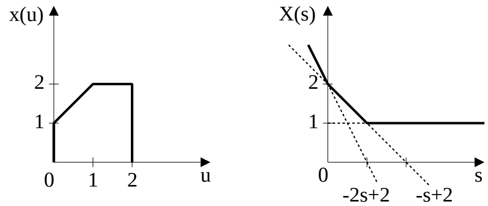

Figure 2: The Legendre transform of a non-smooth concave function.

In order to apply the transform to a non-convex/non-concave functions, we use the extended Legendre transform, also called the slope transform in morphological signal processing [10, 11]. It is redefined as:

$$X(s) := \mathcal{L}[x(u)](s) = \left\{ x(u^*) - su^* \;\middle|\; s = \frac{\mathrm{d}x}{\mathrm{d}u}(u^*) \right\}. \quad (3)$$

In this case, the Legendre transform $X(s)$ is set-valued. Using the extended Legendre transform, we can consider arbitrary function $x(u)$. The extended inverse Legendre transform is defined as:

$$x(u) := \mathcal{L}^{-1}[X(s)](u) = \left\{ X(s^*) + us^* \;\middle|\; \frac{\mathrm{d}X}{\mathrm{d}s}(s^*) = u \right\}. \quad (4)$$

In our approach, we use both the extended Legendre transform and the extended inverse Legendre transform. Figure 3 shows an example of the Legendre transform of a non-convex/non-concave function.

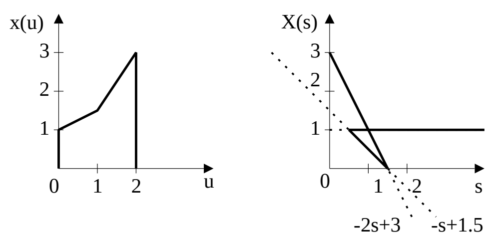

Figure 3: The Legendre transform of a non-convex/non-concave function.

3. SIMPLE NETWORK SYSTEM DESCRIBED BY MAX-PLUS ALGEBRA

We assume that in max-plus algebra communication networks can be described as linear time-invariant systems. In max-plus algebra, the addition is replaced by max (denoted $\oplus$) and the multiplication by plus (denoted $\otimes$) [1]. Numerous discrete event systems can be described via max-plus linear equations. For example, it was shown that Transport Communication Protocol (TCP) is also max-plus linear [3].

We consider a single input/single output system. The input of the system is denoted by the time instance $x(k)$ when the k-th packet is sent from a source. The output of the system is the time instance $y(k)$ when the k-th packet reaches the destination. In this case, both the input and the output are non-decreasing functions. Assuming that the network under consideration does not experience losses, the output is described by the convolution of the response characteristic $h(k)$ and the input $x(k)$:

$$\begin{aligned} y(k) &= \bigoplus_{i=-\infty}^{k} h(k-i) \otimes x(i) \\ &= \max_{0 \le i \le k} \left\{ h(k-i) + x(i) \right\}, \end{aligned} \quad (5)$$

where $x(k) = h(k) = -\infty$ for $k < 0$. The response characteristic $h(k)$ is protocol and network dependent. It also depends on the previous state of the network. The Legendre transform of $y(k)$ in Equation (5) is

$$\begin{aligned} \mathcal{L}[y](s) &= \mathcal{L}[h](s) + \mathcal{L}[x](s) \\ &= H(s) + X(s), \end{aligned} \quad (6)$$

where $H(s)$ and $X(s)$ denote the Legendre transform of the set $\{h(k)\}$ and $\{x(k)\}$, respectively. Equation (6) shows that in the Legendre domain the convolution becomes addition [10]. However, because $H(s)$ is set-valued, it is in general difficult to evaluate the addition. We now consider a network with a simple transfer characteristic:

$$H(s) := \mathcal{L}[h](s) = \left\{ \begin{array}{lll} d & \text{if} & s \ge 1/w \\ \infty & \text{if} & s < 1/w \end{array} \right. , \quad (7)$$

where constants d and w correspond to the minimum packet delay and the maximum throughput in the network, respectively. Using this transfer characteristic we can approximately calculate output.

We describe here a simplified method to obtain the output $\bar{Y}(s) = \mathcal{L}[\bar{y}(u)]$, where $\bar{y}(u)$ denotes the piecewise linear interpolation of the set $\{y(k)\}$. First, we find the piecewise linear interpolation of the input set $\{x(k)\}$, denoted by $\bar{x}(u)$. Let $\bar{X}(s)$, $\dot{X}_k(s)$, and $\dot{Y}_k(s)$ denote the Legendre transform of $\bar{x}(u)$, and the k-th input $x(k)$ and output $y(k)$, respectively. We assume that $x(0) = 0$, $\dot{Y}_0(s) = H(\infty)$, and $s_{-1} = \infty$, and that $\dot{Y}_{k-1}(s)$ is known. We then calculate $\dot{Y}_k(s)$ using the following algorithm:

1. Calculate $\dot{Y}_k'(s) = \dot{X}_k(s) + H(s)$.
2. Find the intersection of $\dot{Y}_k'(s)$ and $\dot{Y}_{k-1}(s)$. The value s of the intersection of $\dot{Y}_k'(s)$ and $\dot{Y}_{k-1}(s)$ is denoted by s_{k-1}.
3. Obtain $\dot{Y}_k(s)$ that is parallel to $\dot{X}_k(s)$ and passes through the intersection point.

Hence, we obtain $\dot{Y}_k(s)$ from $\dot{Y}_{k-1}(s)$. $\bar{Y}(s)$ is the union of those $\dot{Y}_k$(s). Its domain is bounded by $[s_{k-1}, s_k]$. The output $\bar{y}(u)$ is calculated using the inverse Legendre transform $\mathcal{L}^{-1}[\bar{Y}(s)](u)$.

4. EXAMPLE OF A SIMPLE COMMUNICATION NETWORK

Let us consider a simple network with a transfer characteristic:

$$H(s) = \mathcal{L}[h](s) = \left\{ \begin{array}{lll} 1 & \text{if} & s \ge 1/2 \\ \infty & \text{if} & s < 1/2 \end{array} \right. . \quad (8)$$

This transfer characteristic indicates that the network introduces a packet delay of 1 unit time, and that the maximum throughput is 2 packets per unit time. The transfer characteristic is shown in Figure 4. An example of a busy traffic is $\bar{x}(u) = 1/4$. The output in time domain and its Legendre transform are shown in Figure 5. In this case, the output is the sum of the input and the transfer characteristic. In case when the network is not congested and the source sends 4/3 packets per unit time, the output is shown in Figure 6. Thus, although both the congested and non-congested cases exhibit nonlinear phenomena in the sense of conventional system theory, we can represents the network system with a unique transfer characteristic in max-plus algebra.

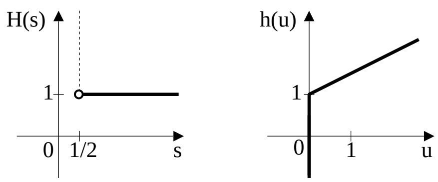

Figure 4: Example of a network transfer characteristic with 1 unit time delay and the maximum throughput of 2 packets per unit time.

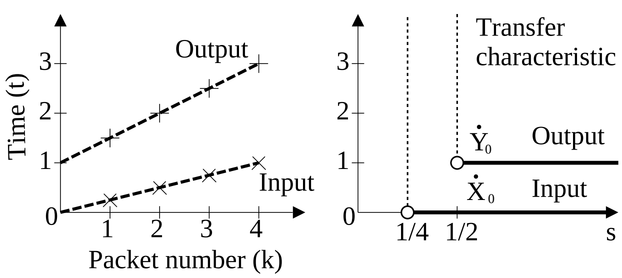

Figure 5: Input and output in the case of congested traffic. The network delay is 1 packet per unit time and the maximum throughput is 2 packets per unit time. The source sends 4 packets per unit time.

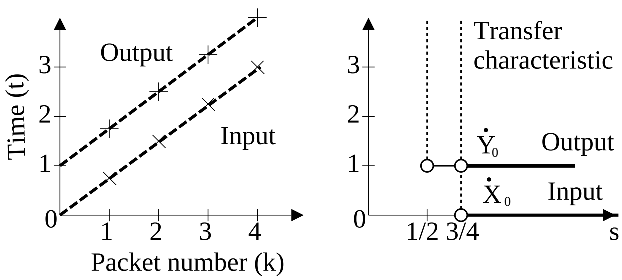

Figure 6: Input and output in the case of non-congested traffic. The network delay is 1 packet per unit time and the maximum throughput is 2 packets per unit time. The source sends 4/3 packets per unit time.

Next, we show an example of a system with variable bit rate input traffic. If the state of the network changes from a non-congested to a congested state, the output is represented by the first row of graphs in Figure 7. Hence, various states and features of the communication network can be captured by employing the Legendre transform. However, in some cases output cannot be calculated simply as the sum of the input and the transfer characteristic. The second row of graphs in Figure 7 shows the case when the state of the network changes from a congested to a non-congested state. In the non-congested state, backlog accumulated in network during period of congestion starts to drain. Transfer characteristic is not able to capture this phenomenon.

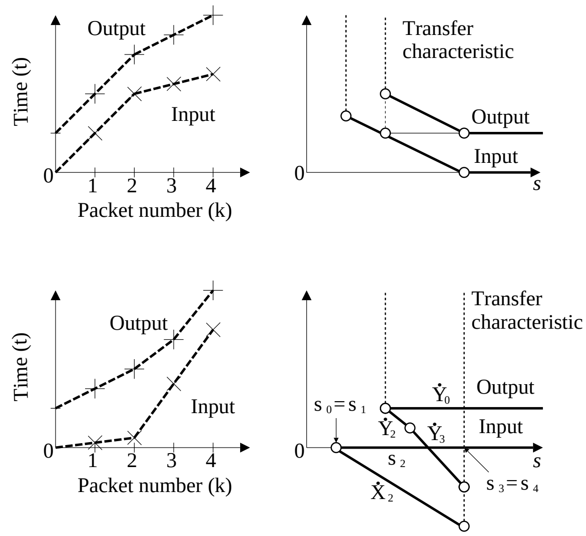

Figure 7: Network analysis for the case of variable traffic.

5. AN IDENTIFICATION METHOD

In order to identify transfer characteristic of a communication network, we use the system's input $x(k)$ and output $y(k)$. We assume that transfer characteristic is described by Equation (7). The procedure for finding the transfer characteristic of the network is:

1. Calculate the piecewise linear interpolations $\bar{x}(u)$ and $\bar{y}(u)$.
2. Obtain $\bar{X}(s)$ and $\bar{Y}(s)$ by applying the Legendre transform to $\bar{x}(u)$ and $\bar{y}(u)$.
3. Obtain the transfer characteristic $H(s)$ based on the difference of $\bar{Y}(s)$ and $\bar{X}(s)$. Obtain the transfer characteristic at s_{k-1} as $\dot{Y}_k(s_{k-1}) - \dot{X}_k(s_{k-1})$. Because $\mathcal{L}[y(k) - x(k)] = \dot{Y}_k(s) - \dot{X}_k(s)$, obtain the transfer characteristic $H(s)$ by plotting $(s_{k-1}, y(k) - x(k))$.

This procedure corresponds to the deconvolution in the Legendre domain.

In order to confirm the validity of the identification method, we consider a single input/single output model. We employ the ns-2 network simulator and we try to identify the transfer characteristic of the network. The model is shown in Figure 8. It consists of six nodes with the FIFO queuing scheme, without buffer overflows, and the transmission rates of 10, 1.5, 10, 1.5, and 10 Mbps. TCP packet size is 1,000 bytes. Our goal is to identify the transfer characteristic of the network using the proposed method.

We first consider an ON/OFF traffic trace. The result of the identification of the system's transfer characteristic is shown in Figure 9. The comparison between the simulated and calculated output is shown in Figure 10 (top). It confirms the successful identification.

We then consider the same network with traffic that follows the Pareto distribution. We calculate $\tilde{y}(u)$ using the same transfer characteristic. The result is shown in Figure 10 (bottom). The obtained transfer characteristic effectively predicts the output.

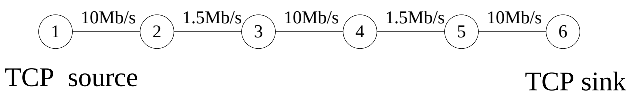

Figure 8: An example of a network with six nodes. Links between nodes have infinite buffers (queues). TCP packet size is 1,000 bytes.

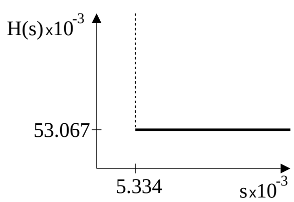

Figure 9: Identified transfer characteristic for TCP case. Minimum packet delay $d = 53.067$ msec and the maximum throughput $w = 187.48$ packets/sec (1.5 Mbps).

6. CONCLUSIONS

We proposed an application of the Legendre transform for analysis and identification of transfer characteristic of communication networks. Using the Legendre transform we described the features of a communication network with a very simple transfer characteristic. We applied the proposed method to a simple network model and we confirmed its effectiveness using the ns-2 network simulator. In the present form, the method is applicable only to simple single input/ single output communication systems.

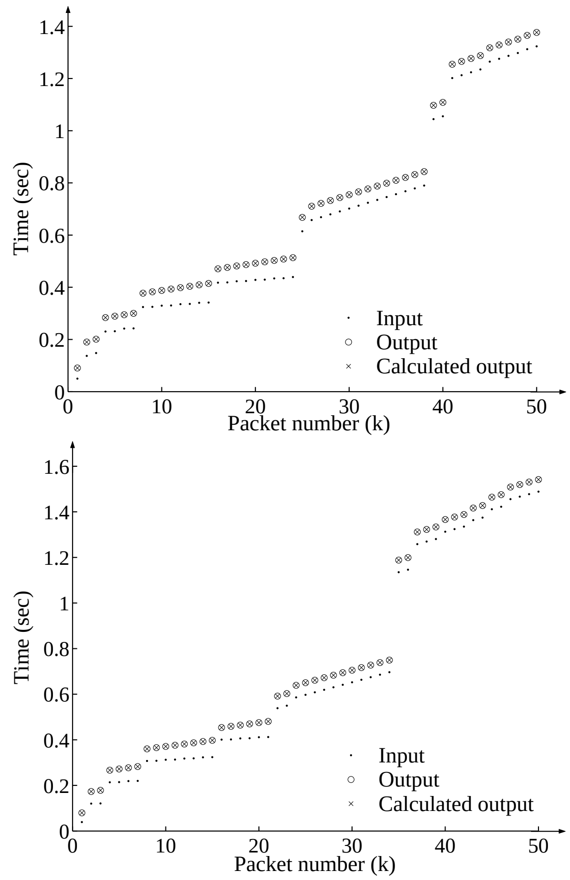

Figure 10: Analysis of a network with an ON/OFF traffic (top) and Pareto (bottom). ns-2 parameters: burst period 50 msec, idle period 50 msec, and bit rate 400 kbps. For the Pareto case: shape = 1.5.

7. REFERENCES

[1] G. Olsder, F. Baccelli, G. Cohen, and J. Quadrant, *Synchronization and Linearity*. Chichester: Wiley, 1992.

[2] J. Y. Le Boudec and P. Thiran, *Network Calculus*. New York: Springer Verlag, 2002.

[3] F. Baccelli and D. Hong, "TCP is max-plus linear, and what it tells us on its throughput" *in Proc. SIGCOMM'00*, Stockholm, Sweden, August 2000, pp. 219–230.

[4] V. P. Maslov and S. N. Samborskii, *Idempotent Analysis, Advances in Soviet Mathematics*, vol. 13, American Mathematical Society, Providence, RI, 1992.

[5] F. Baccelli and D. Hong, "AIMD, fairness and fractal scaling of TCP traffic," *in Proc. INFOCOM 2002*, New York, July 2002.

[6] J. Y. Le Boudec and P. Thiran, "Min-plus system theory applied to communication networks," *in Proc. MTNS'02*, South Bend, IN, 2002.

[7] L. Corrias, "Fast Legendre-Fenchel transform and applications to Hamilton-Jacobi equations and conservation laws," *SIAM J. Numer. Anal.*, vol. 33, no. 4, pp. 1534–1558, August 1996.

[8] Y. Lucet, "A fast computational algorithm for the Legendre-Fenchel transform," *Computational Optimization and Applications*, vol. 6, no. 1, pp. 27–57, 1996.

[9] Y. Lucet, "Faster than the fast Legendre transform, the linear-time Legendre transform," *Numer. Algorithms*, vol. 16, no. 2, pp. 171–185, 1997.

[10] L. Dorst and R. van den Boomgaard, "Morphological signal processing and the slope transform," *Signal Processing*, vol. 38, no. 1, pp. 79–98, July 1994.

[11] P. Maragos, "Slope transforms: theory and application to nonlinear signal processing," *IEEE Trans. Signal Processing*, vol. 43, no. 4, pp. 864–877, April 1995.

[12] R. T. Rockafellar, *Convex Analysis*. Princeton, NJ: Princeton University Press, 1972.

[13] A. D. Ioffe and V. M. Tihomirov, *Theory of Extremal Problems*. Amsterdam, The Netherlands: North-Holland, 1979.