Model Specification

We assume that for each observation i, we observe y_i (possibly censored) and an indicator Δ_i where:

$$\Delta_i = \begin{cases} 1, & \text{if } y_i \text{ is observed (uncensored);} \\ 0, & \text{if } y_i \text{ is censored.} \end{cases}$$

We work on the log-scale by defining a latent variable

$$\tilde{y}_i = \log(y_i) \quad \text{(or an adjusted value for censored observations).}$$

The regression model is

$$\tilde{y}_i = X_i\beta + r_{c(i)} + \varepsilon_i, \quad \varepsilon_i \sim N(0, \sigma^2),$$

where:

- X_i is the row vector of covariates for observation i.
- β is the vector of fixed effects.
- $r_{c(i)}$ is the random effect associated with the cluster $c(i)$ that observation i belongs to.
- σ^2 is the error variance.

The random effects for each cluster $j = 1, \ldots, I$ have a spatial structure through a weight matrix W. Let $D_j = \sum_{k=1}^{I} W_{jk}$ be the degree (row sum) for cluster j. A spatial prior is imposed by further introducing:

- τ^2: spatial variance.
- ρ: spatial correlation parameter satisfying $\rho \in [\rho_{\text{lower}}, \rho_{\text{upper}}]$.

Gibbs Sampler Steps

Step 1: Imputation of Latent Variables

For each observation i, if $\Delta_i = 0$, $\tilde{y}_i$ is missing (censored) and is imputed from a truncated normal distribution:

$$\tilde{y}_i \sim TN\left(a = \log(y_i),\, b = \infty;\, \mu_i,\, \sigma^2\right),$$

with

$$\mu_i = X_i\beta + r_{c(i)}.$$

For uncensored observations ($\Delta_i = 1$), we set

$$\tilde{y}_i = \log(y_i).$$

Step 2: Update of β (Fixed Effects)

Define an adjusted outcome

$$y_i^* = \tilde{y}_i - r_{c(i)}.$$

Assuming a conjugate normal prior for β, that is,

$$\beta \sim N(0, V_\beta) \quad \text{with } V_\beta = \text{diag}(\beta$$

the posterior for β is

$$\beta \mid \tilde{y}, r, \sigma^2 \sim N\left(\mu_\beta,\, V_\beta^{\text{post}}\right),$$

where

$$V_\beta^{\text{post}} = \left(\frac{X^T X}{\sigma^2} + V_\beta^{-1}\right)^{-1}, \quad \mu_\beta = V_\beta^{\text{post}}\left(\frac{X^T y^*}{\sigma^2}\right).$$

Step 3: Update of Random Effects r for Each Cluster

For each cluster j, let C_j be the set of observation indices belonging to cluster j, and let $n_j = |C_j|$. The full conditional for the random effect r_j is:

$$r_j \mid \cdot \sim N\left(m_j,\, v_j\right),$$

with

$$v_j = \left(\frac{n_j}{\sigma^2} + \frac{D_j}{\tau^2}\right)^{-1}, \quad m_j = v_j\left(\frac{1}{\sigma^2}\sum_{i \in C_j}(\tilde{y}_i - X_i\beta) + \frac{\rho}{\tau^2}\sum_{k=1}^{I} W_{jk} r_k\right).$$

Step 4: Update of σ^2 (Error Variance)

The residual for each observation is given by

$$e_i = \tilde{y}_i - X_i\beta - r_{c(i)}.$$

An inverse-gamma prior is assumed for σ^2. The posterior distribution is

$$\sigma^2 \sim \text{Inv-Gamma}\left(a_\sigma + \frac{n}{2},\, b_\sigma + \frac{1}{2}\sum_{i=1}^{n} e_i^2\right).$$

Step 5: Update of τ^2 (Spatial Variance)

Define a quadratic form involving the random effects:

$$Q = \sum_{j=1}^{I} D_j r_j^2 - \rho\, \mathbf{r}^T W \mathbf{r}.$$

Then, with an inverse-gamma prior for τ^2, the full conditional is

$$\tau^2 \sim \text{Inv-Gamma}\left(a_\tau + \frac{I}{2},\, b_\tau + \frac{Q}{2}\right).$$

Step 6: Update of ρ (Spatial Correlation Parameter)

A Metropolis–Hastings step is used for ρ. A proposal ρ_{prop} is drawn from:

$$\rho_{\text{prop}} \sim N(\rho,\, \sigma^2_{\text{proposal}}),$$

with $\sigma_{\text{proposal}} = 0.1$. If ρ_{prop} is outside the allowable range $[\rho_{\text{lower}}, \rho_{\text{upper}}]$, then the proposal is rejected. Otherwise, compute the quadratic forms:

$$Q(\rho) = \sum_{j=1}^{I} D_j r_j^2 - \rho\, \mathbf{r}^T W \mathbf{r},$$

and

$$Q(\rho_{\text{prop}}) = \sum_{j=1}^{I} D_j r_j^2 - \rho_{\text{prop}}\, \mathbf{r}^T W \mathbf{r}.$$

The log acceptance ratio is then given by:

$$\log \alpha = -\frac{1}{2\tau^2} \left[Q(\rho_{\text{prop}}) - Q(\rho) \right].$$

Accept ρ_{prop} with probability $\min(1, \exp(\log \alpha))$, otherwise retain the current ρ.

Summary

At each iteration, the sampler:

1. Imputes latent values $\tilde{y}_i$ for censored observations.
2. Updates the fixed effects β via a conjugate normal posterior.
3. Updates the cluster-level random effects r_j based on the observed data and the spatial structure.
4. Updates σ^2 and τ^2 using inverse-gamma full conditionals.
5. Updates ρ using a Metropolis–Hastings step.

Posterior samples for all these parameters are stored for later inference.