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A SURVEY ON SEMI-TENSOR PRODUCT OF MATRICES**

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Abstract Semi-tensor product of matrices is a generalization of conventional matrix product for the case when the two factor matrices do not meet the dimension matching condition. It was firstly proposed about ten years ago. Since then it has been developed and applied to several different fields. In this paper we will first give a brief introduction. Then give a survey on its applications to dynamic systems, to logic, to differential geometry, to abstract algebra, respectively.

Key words Abstract algebra, differential geometry, dynamic systems, logic, semi-tensor product.

1 Introduction

Consider two matrices $A \in M_{m\times n}$ and $B \in M_{p\times q}$. For statement ease, when $n = p$, A and B are said to satisfy matching dimension condition, as n is a factor of p or p is a factor of n, they are said to satisfy factor dimension condition, otherwise, they have general dimensions. From Linear Algebra, it is well known that as A and B have matching dimension, the conventional matrix product AB is well defined. Otherwise, it is not defined. Of course, we have some other matrix products, such as Kronecker product, which can be used for two matrices with arbitrary dimensions[1]; Hadamard product for two matrices with the same sides[2]. But they are different products, which have nothing to do with the conventional product.

Now we are facing two basic questions: 1) Is it possible to extend the conventional matrix product to more general cases, say, factor dimension case or even general dimension case? 2) Is it necessary to extend it? The second question is the same as: Is the extended product useful? The purpose of this paper is to answer these two questions.

The answer to the first question is positive. We defined the semi-tensor product (STP) of matrices for both factor dimension case and general dimension case. In matching dimension case, them coincide with conventional one. For the second question, we found many applications of the semi-tensor product. But so far, all of them are of factor dimension case. So, we should say that we didn't find a meaningful application for general dimension case.

The paper is organized as follows. Section 2 gives a brief introduction to semi-tensor product of matrices. Section 3 gives some simple motivating examples.

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⋆Dedicated to Academician Han-Fu Chen on the occasion of his 70th birthday.

2 Semi-Tensor Product

This section is a brief review on semi-tensor product of matrices. It plays a fundamental rule in the following discussion. We restrict it to the definitions and some basic properties, which are useful in the sequel. In addition, only left semi-tensor product for factor dimension case is discussed in the paper. We refer to [3,4] for right semi-tensor product, general dimension case and much more details.

Definition 2.1 1) Let X be a row vector of dimension np, and Y be a column vector with dimension p. Then we split X into p equal-size blocks as $X^1, X^2, \cdots, X^p$, which are $1\times n$ rows. Define the left STP, denoted by $\ltimes$, as

$$\begin{cases} X \ltimes Y = \sum_{i=1}^{p} X^i y_i \in \mathbb{R}^n, \\ Y^{\mathrm{T}} \ltimes X^{\mathrm{T}} = \sum_{i=1}^{p} y_i (X^i)^{\mathrm{T}} \in \mathbb{R}^n. \end{cases} \tag{1}$$

2) Let $A \in M_{m\times n}$ and $B \in M_{p\times q}$. If either n is a factor of p, say $nt = p$ and denote it as $A \prec_t B$, or p is a factor of n, say $n = pt$ and denote it as $A \succ_t B$, then we define the left STP of A and B, denoted by $C = A \ltimes B$, as the following: C consists of $m \times q$ blocks as $C = (C^{ij})$ and each block is

$$C^{ij} = A^i \ltimes B_j, \quad i = 1, 2, \cdots, m, \quad j = 1, 2, \cdots, q,$$

where A^i is i-th row of A and B_j is the j-th column of B.

We use some simple numerical examples to describe it.

Example 2.2 1) Let $X = \begin{bmatrix} 1 & 2 & 3 & -1 \end{bmatrix}$ and $Y = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Then

$$X \ltimes Y = \begin{bmatrix} 1 & 2 \end{bmatrix} \cdot 1 + \begin{bmatrix} 3 & -1 \end{bmatrix} \cdot 2 = \begin{bmatrix} 7 & 0 \end{bmatrix}.$$

2) Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 3 & 1 & 2 \\ 3 & 2 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}.$$

Then

$$A \ltimes B = \begin{bmatrix} \begin{pmatrix} 1 & 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ -1 \end{pmatrix} \\ \begin{pmatrix} 2 & 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} & \begin{pmatrix} 2 & 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} -2 \\ -1 \end{pmatrix} \\ \begin{pmatrix} 3 & 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} & \begin{pmatrix} 3 & 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} -2 \\ -1 \end{pmatrix} \end{bmatrix} = \begin{bmatrix} 3 & 4 & -3 & -5 \\ 4 & 7 & -5 & -8 \\ 5 & 2 & -7 & -4 \end{bmatrix}.$$

Remark Note that when $n = p$ the left STP coincides with the conventional matrix product. Therefore, the left STP is only a generalization of the conventional product. For convenience, we may omit the product symbol $\ltimes$.

Some fundamental properties of the left STP are collected in the following:

Proposition 2.3 *The left STP satisfies* (*as long as the related products are well defined*)

1) (*Distributive rule*)

$$
\begin{aligned}
&A \ltimes (\alpha B + \beta C) = \alpha A \ltimes B + \beta A \ltimes C,\\
&(\alpha B + \beta C) \ltimes A = \alpha B \ltimes A + \beta C \ltimes A, \quad \alpha, \beta \in \mathbb{R};
\end{aligned} \tag{2}
$$

2) (*Associative rule*)

$$
\begin{aligned}
&A \ltimes (B \ltimes C) = (A \ltimes B) \ltimes C,\\
&(B \ltimes C) \ltimes A = B \ltimes (C \ltimes A).
\end{aligned} \tag{3}
$$

Proposition 2.4 *Let $A \in M_{p\times q}$ and $B \in M_{m\times n}$. If $q = km$, then*

$$A \ltimes B = A(B \otimes I_k); \tag{4}$$

If $kq = m$, then

$$A \ltimes B = (A \otimes I_k)B. \tag{5}$$

Proposition 2.5 1) *Assume A and B are of proper dimensions such that $A \ltimes B$ is well defined. Then*

$$(A \ltimes B)^{\mathrm{T}} = B^{\mathrm{T}} \ltimes A^{\mathrm{T}}. \tag{6}$$

2) *In addition, assume both A and B are invertible, then*

$$(A \ltimes B)^{-1} = B^{-1} \ltimes A^{-1}. \tag{7}$$

Proposition 2.6 *Assume $A \in M_{m\times n}$ is given.*
1) *Let $Z \in \mathbb{R}^t$ be a row vector. Then*

$$A \ltimes Z = Z \ltimes (I_t \otimes A); \tag{8}$$

2) *Let $Z \in \mathbb{R}^t$ be a column vector. Then*

$$Z \ltimes A = (I_t \otimes A) \ltimes Z. \tag{9}$$

Note that when $\xi \in \mathbb{R}^n$ is a column or a row, then $\underbrace{\xi \ltimes \cdots \ltimes \xi}_{k}$ is well defined. We denote it briefly as

$$\xi^k := \underbrace{\xi \ltimes \cdots \ltimes \xi}_{k}.$$

In general, let $A \in M_{m\times n}$ and assume either m is a factor of n or n is a factor of m. Then

$$A^k := \underbrace{A \ltimes \cdots \ltimes A}_{k}$$

is well defined.

Next, we define the swap matrix, which is also called the permutation matrix and is defined implicitly in [5]. Many properties can be found in [3,4]. The swap matrix, $W_{[m,n]}$ is an $mn \times mn$ matrix constructed in the following way: label by $(11, 12, \cdots, 1n, \cdots, m1, m2, \cdots, mn)$ its columns and by $(11, 21, \cdots, m1, \cdots, 1n, 2n, \cdots, mn)$ its rows. Then its element in the position $((I, J), (i, j))$ is assigned as

$$w_{(IJ),(ij)} = \delta_{i,j}^{I,J} = \begin{cases} 1, & I = i \text{ and } J = j, \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

When $m = n$ we simply denote by $W_{[n]}$ for $W_{[n,n]}$.

Example 2.7 Let $m = 2$ and $n = 3$, the swap matrix $W_{[2,3]}$ is constructed as

$$W_{[2,3]} = \begin{array}{c} \begin{array}{cccccc} (11) & (12) & (13) & (21) & (22) & (23) \end{array} \\ \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array} \begin{array}{c} \\ (11) \\ (21) \\ (12) \\ (22) \\ (13) \\ (23) \end{array}.$$

Let $A \in M_{m\times n}$, i.e., A is an $m \times n$ matrix. Denote by $V_r(A)$ the row stacking form of A, that is,

$$V_r(A) = (a_{11} \cdots a_{1n} \ \cdots \ a_{m1} \cdots a_{mn})^{\mathrm{T}},$$

and by $V_c(A)$ the column stacking form of A, that is,

$$V_c(A) = (a_{11} \cdots a_{m1} \ \cdots \ a_{1n} \cdots a_{mn})^{\mathrm{T}}.$$

The following "swap" property shows the meaning of the name.

Proposition 2.8 1) *Let $X \in \mathbb{R}^m$ and $Y \in \mathbb{R}^n$ be two columns. Then*

$$W_{[m,n]} \ltimes X \ltimes Y = Y \ltimes X, \quad W_{[n,m]} \ltimes Y \ltimes X = X \ltimes Y. \tag{11}$$

2) *Let $A \in M_{m\times n}$. Then*

$$W_{[m,n]} V_r(A) = V_c(A), \quad W_{[n,m]} V_c(A) = V_r(A). \tag{12}$$

3) *Let $X_i \in \mathbb{R}^{n_i}$, $i = 1, 2, \cdots, m$. Then*

$$\begin{aligned} &\left(I_{n_1+\cdots+n_{k-1}} \otimes W_{[n_k,n_{k+1}]} \otimes I_{n_{k+2}+\cdots+n_m}\right) X_1 \ltimes \cdots \ltimes X_k \ltimes X_{k+1} \ltimes \cdots \ltimes X_m \\ &= X_1 \ltimes \cdots \ltimes X_{k+1} \ltimes X_k \ltimes \cdots \ltimes X_m. \end{aligned} \tag{13}$$

Proposition 2.9 *Let $A \in M_{m\times n}$ and $B \in M_{p\times q}$. Then*

$$W_{[m,n]}^{\mathrm{T}} = W_{[m,n]}^{-1} = W_{[n,m]}. \tag{14}$$

Proposition 2.10

$$W_{[pq,r]} = \left(W_{[p,r]} \otimes I_q\right)\left(I_p \otimes W_{[q,r]}\right). \tag{15}$$

Taking transpose on both sides of (15) yields

$$W_{[r,pq]} = \left(I_p \otimes W_{[r,q]}\right)\left(W_{[r,p]} \otimes I_q\right). \tag{16}$$

The swap matrix can be constructed in the following method: Denote the i-th canonical basic element in $\mathbb{R}^n$ by δ_i^n. That is, δ_i^n is the i-th column of I_n. Then we have

Proposition 2.11

$$W_{[m,n]} = \begin{pmatrix} \delta_1^n \ltimes \delta_1^m & \cdots & \delta_n^n \ltimes \delta_1^m & \cdots & \delta_1^n \ltimes \delta_m^m & \cdots & \delta_n^n \ltimes \delta_m^m \end{pmatrix}. \tag{17}$$

In [5], (17) is used as the definition.

Using swap matrix, we can prove that

Proposition 2.12 *Let $A \in M_{m\times n}$ and $B \in M_{s\times t}$. Then*

$$A \otimes B = W_{[s,m]} \ltimes B \ltimes W_{[m,t]} \ltimes A = (I_m \otimes B) \ltimes A. \tag{18}$$

Particularly, if $X \in \mathbb{R}^n$, $Y^{\mathrm{T}} \in \mathbb{R}^m$, then

$$XY = Y \ltimes W_{[n,m]} \ltimes X. \tag{19}$$

Since $\ltimes$ is a generalization of the conventional matrix product, hereafter, we omit the notation $\ltimes$.

Denote $X = (x_1, x_2, \cdots, x_n)^{\mathrm{T}}$, then X^k is a (redundant) basis of the k-th degree homogeneous polynomials. That is, if $P_k(x)$ is a k-th order homogeneous polynomial, then there exists a numerical matrix $F \in 1 \times n^k$, such that $P_k(x) = FX^k$. Note that since X^k is redundant, F is not unique.

Next, we define a differential of a matrix of functions.

Definition 2.13 Let $M(x)$ be a $p \times q$ matrix with entries $m_{i,j}(x)$ as functions of $x \in \mathbb{R}^n$. Then the differential of $M(x)$ is defined as a $p \times qn$ matrix with $m_{i,j}(x)$ be replaced by $dm_{i,j}(x)$.

Now if $f(x)$ is an analytic function, then we can use Taylor series expansion to expand it as

$$f(x) = F_0 + F_1 X + F_2 X^2 + \cdots.$$

So if we want to find a formula for the differential of $f(x)$, the key is to find DX^k. We construct an $n^{k+1} \times n^{k+1}$ matrix Φ_k as

$$\Phi_k = \sum_{s=0}^{k} I_{n^s} \otimes W_{[n^{k-s},n]}. \tag{20}$$

Then we have the following differential form of X^k, which is fundamental in later approach.

Proposition 2.14

$$D(X^{k+1}) = \Phi_k \ltimes X^k. \tag{21}$$

3 Motivating Examples

This section gives some motivating examples to show the motivations for semi-tensor products.

Example 3.1 (Incompleteness of conventional matrix product)

1) Consider $X, Y, Z, W \in \mathbb{R}^n$ as column vectors. Then

$$(XY^{\mathrm{T}})(ZW^{\mathrm{T}}) \in M_n, \tag{22}$$

where M_n is the set of $n \times n$ matrix. Now by associativity of matrix product and considering $Y^T Z$ is a scalar, we have

$$(XY^{\mathrm{T}})(ZW^{\mathrm{T}}) = X(Y^T Z)W^{\mathrm{T}} = Y^T ZXW^{\mathrm{T}} = Y^{\mathrm{T}}(ZX)W^{\mathrm{T}}. \tag{23}$$

But now what is ZX? It is not defined.

2) Consider $X, Y \in \mathbb{R}^n$, $W \in M_m$. Then $(X^T Y)W$ is well defined. Using associativity, we have

$$(X^T Y)W = X^T YW = X^{\mathrm{T}}(YW). \tag{24}$$

Again (24) is nonsense.

But when we generalize the conventional matrix product to semi-tensor product, both (23) and (24) are meaningful and the resulting matrices are the same as the original ones.

Next example shows semi-tensor product may much simplify the computation.

Example 3.2 Assume the inputs $u(i) \in \mathbb{R}^m$ and the state $x(i) \in \mathbb{R}^n$ have the following linear relation:

$$x(i+1) = Ax(i) + Bu(i), \quad i = 1, 2, \cdots . \tag{25}$$

We want to estimate A, B from input-state data. Using column stacking form, we have

$$x(i+1) = (x(i)^{\mathrm{T}}, u(i)^{\mathrm{T}}) \ltimes \begin{bmatrix} V_c(A) \\ V_c(B) \end{bmatrix}.$$

Define

$$W = \begin{bmatrix} x(2) \\ \vdots \\ x(N+1) \end{bmatrix}, \quad H = \begin{bmatrix} x(1)^{\mathrm{T}} & u(1)^{\mathrm{T}} \\ \vdots & \\ x(N)^{\mathrm{T}} & u(N)^{\mathrm{T}} \end{bmatrix}, \quad Y = \begin{bmatrix} V_c(A) \\ V_c(B) \end{bmatrix},$$

Then we have

$$W = HY.$$

Assume H has full column rank, we have a least square estimation of parameters A, B as

$$\widehat{Y} = (H^T H)^{-1} H^{\mathrm{T}} \ltimes W. \tag{26}$$

Here the use of semi-tensor product simplified the computation a lot.

The following example shows an advantage of semi-tensor product over conventional product plus Kronecker product:

Example 3.3 Let V be an n dimensional vector space, its dual space is denoted by V^*. Suppose $\{e_1, e_2, \cdots, e_n\}$ be a given basis of V, with its dual basis on V^* as $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$, where dual means $\langle \alpha_i, e_j \rangle = {}_{i,j}$. Let $\sigma \in T_s^r(V)$, i.e., σ is a tensor on V with covariant order r and contra-variant order s[6].

Denote by

$$\sigma(e_{i_1}, e_{i_2}, \cdots, e_{i_r}, \alpha_{j_1}, \alpha_{j_2}, \cdots, \alpha_{j_s}) = \sigma_{j_1 \cdots j_s}^{i_1 \cdots i_r}, \quad 1 \le i_p, j_q \le n.$$

Then we construct a matrix, called the structure matrix, as

$$M_\sigma = \begin{bmatrix} \sigma_{1\cdots 11}^{1\cdots 11} & \sigma_{1\cdots 11}^{1\cdots 12} & \cdots & \sigma_{1\cdots 11}^{1\cdots 1n} & \cdots & \sigma_{1\cdots 11}^{n\cdots nn} \\ \sigma_{1\cdots 12}^{1\cdots 11} & \sigma_{1\cdots 12}^{1\cdots 12} & \cdots & \sigma_{1\cdots 12}^{1\cdots 1n} & \cdots & \sigma_{1\cdots 12}^{n\cdots nn} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sigma_{n\cdots nn}^{1\cdots 11} & \sigma_{n\cdots nn}^{1\cdots 12} & \cdots & \sigma_{n\cdots nn}^{1\cdots 1n} & \cdots & \sigma_{n\cdots nn}^{n\cdots nn} \end{bmatrix}. \tag{27}$$

Express vector and co-vector as column and row vectors respectively as $X = \sum_{i=1}^{n} a_i e_i := (a_1, a_2, \cdots, a_n)^{\mathrm{T}}$, $\omega = \sum_{i=1}^{n} b_i \alpha_i := (b_1, b_2, \cdots, b_n)$. Using Kronecker product with conventional product, we have

$$\sigma(X_1, X_2, \cdots, X_r; \omega_1, \omega_2, \cdots, \omega_s) = (\omega_1 \otimes \cdots \otimes \omega_s) M_\sigma (X_1 \otimes \cdots \otimes X_r). \tag{28}$$

Using semi-tensor product, we have

$$\sigma(X_1, X_2, \cdots, X_r; \omega_1, \omega_2, \cdots, \omega_s) = \omega_s \ltimes \cdots \ltimes \omega_1 M_\sigma X_1 \ltimes \cdots \ltimes X_r. \tag{29}$$

The advantage of (29) over (28) is that since semi-tensor product has the property of associativity, we can manipulate it easily. For instance, let $X \in V$, $i_X : T_s^r(V) \to T_s^{r-1}(V)$ is defined as:

$$i_X(\sigma)(X_1, X_2, \cdots, X_{r-1}; \omega_1, \omega_2, \cdots, \omega_s) = \sigma(X, X_1, X_2, \cdots, X_{r-1}; \omega_1, \omega_2, \cdots, \omega_s)$$

Now the structure matrix of $i_X(\sigma)$ can be obtained from (29) immediately as

$$M_{i_X(\sigma)} = M_\sigma X. \tag{30}$$

To see another application, let $\sigma \in T_x^r(V)$ and $\chi \in T_q^p(V)$, we want to find the structure matrix of $\sigma \otimes \chi$. Using (8), (9), (19), and (11), we have

$$\begin{aligned}
&\sigma \otimes \chi(X_1, X_2, \cdots, X_r, X_{r+1}, \cdots, X_{r+p}; \omega_1, \omega_2, \cdots, \omega_s, \omega_{s+1}, \cdots, \omega_{s+q}) \\
&= \omega_s \cdots \omega_1 M_\sigma X_1 \cdots X_r \omega_{s+q} \cdots \omega_{s+1} M_\chi X_{r+1} \cdots X_{r+p} \\
&= \omega_s \cdots \omega_1 M_\sigma \omega_{s+q} \cdots \omega_{s+1} W_{[n^r, n^q]} X_1 \cdots X_r M_\chi X_{r+1} \cdots X_{r+p} \\
&= \omega_s \cdots \omega_1 \omega_{s+q} \cdots \omega_{s+1} (I_{n^q} \otimes M_\sigma) W_{[n^r, n^q]} (I_{n^r} \otimes M_\chi) X_1 \cdots X_r X_{r+1} \cdots X_{r+p} \\
&= \omega_{s+q} \cdots \omega_{s+1} \omega_s \cdots \omega_1 W_{[n^q, n^s]} (I_{n^q} \otimes M_\sigma) W_{[n^r, n^q]} (I_{n^r} \otimes M_\chi) X_1 \cdots X_r X_{r+1} \cdots X_{r+p}.
\end{aligned}$$

We conclude that

$$M_{\sigma \otimes \chi} = W_{[n^q, n^s]} (I_{n^q} \otimes M_\sigma) W_{[n^r, n^q]} (I_{n^r} \otimes M_\chi). \tag{31}$$

In fact, in statistics the so-called "cubic product" of matrices has been developed for 3-linear multiplication. Semi-tensor product can perform any finite multi-linear multiplication, and even in 3-linear case, it is much more general and convenient than "cubic product"[7].

4 Application to Dynamic (Control) Systems

4.1 Stability Region

Consider a smooth nonlinear system of the form

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \tag{32}$$

where $f(x)$ is an analytic vector field.

Suppose x_e is an equilibrium point of (32). The stable and unstable sub-manifolds of x_e are defined respectively as

$$\begin{aligned}
W^s(x_e) &= \Big\{ p \in \mathbb{R}^n \Big| \lim_{t \to \infty} x(t, p) \to x_e \Big\}, \\
W^u(x_e) &= \Big\{ p \in \mathbb{R}^n \Big| \lim_{t \to -\infty} x(t, p) \to x_e \Big\}.
\end{aligned} \tag{33}$$

Suppose x_s is a stable equilibrium point of (32). The region of attraction of x_s is defined as

$$A(x_s) = \left\{ p \in \mathbb{R}^n \,\middle|\, \lim_{t\to\infty} x(t,p) \to x_s \right\}. \tag{34}$$

The boundary of the region of attraction is denoted by $\partial A(x_s)$.

An equilibrium point x_e is hyperbolic if the Jacobian matrix of f at x_e, denoted by $J_f(x_e)$, has no eigenvalues with zero real part. A hyperbolic equilibrium point is said to be of type-k if $J_f(x_e)$ has k positive real part eigenvalues.

[8] and [9] proved that for a stable equilibrium point x_s the stability boundary is composed of the stability sub-manifolds of equilibrium points on the boundary of the region of attraction under the assumptions that

i) the equilibrium points on the stability boundary $\partial A(x_s)$ are hyperbolic;

ii) the stable and unstable sub-manifolds of the equilibrium points on the stability boundary $\partial A(x_s)$ satisfy the transversality condition;

iii) every trajectory on the stability boundary $\partial A(x_s)$ approaches one of the equilibrium points as $t \to \infty$.

It is well known that the stability boundary is of dimension $n-1$[9]. Therefore, the stability boundary is composed of the closure of stability sub-manifolds of type-1 equilibrium points on the boundary. Based on this fundamental fact, it is of significant meaning to calculate or estimate the stable sub-manifold of type-1 equilibrium points.

Our first result is to give a complete description of the stable sub-manifold of type-1 equilibrium points.

Theorem 4.1 *Assume $x_u = 0$ is a type-1 equilibrium point of system (32).*

$$W^s(e_u) = \{x \mid h(x) = 0\}. \tag{35}$$

Then $h(x)$ is uniquely determined by the following necessary and sufficient conditions (36)–(38).

$$h(0) = 0, \tag{36}$$

$$h(x) = \eta^{\mathrm{T}} x + 0(\|x\|^2), \tag{37}$$

$$L_f h(x) = \mu h(x), \tag{38}$$

where $L_f h(x)$ is the Lie derivative of $h(x)$ with respect to f; η is an eigenvector of $J_f^{\mathrm{T}}(0)$ with respect to its only positive eigenvalue μ.

Using semi-tensor product, we can find the quadratic approximation of $h(x)$ as

Theorem 4.2 *The stable sub-manifold of x_u, expressed as $h(x) = 0$, can be expressed as*

$$h(x) = H_1 x + \frac{1}{2} x^{\mathrm{T}} \Psi x + O(\|x\|^3), \tag{39}$$

where

$$\begin{cases} H_1 = \eta^{\mathrm{T}}, \\ \Psi = V_c^{-1} \left\{ \left[\left(\dfrac{\mu}{2} I_n - J^{\mathrm{T}} \right) \otimes I_n + I_n \otimes \left(\dfrac{\mu}{2} I_n - J^{\mathrm{T}} \right) \right]^{-1} V_c \left(\displaystyle\sum_{i=1}^{n} \eta_i \mathrm{Hess}(f_i(0)) \right) \right\}, \end{cases}$$

where μ and η are respect to $J = F_1$, $\mathrm{Hess}(f_i)$ is the Hessian matrix of the i-th component f_i of f.

Express $h(x)$ as

$$h(x) = H_1 x + H_2 x^2 + \cdots.$$

Since the coefficients are not unique, we convert them into symmetric form by

$$H_k = G_k T_B(n,k), \quad G_k = H_k T_N(n,k). \tag{40}$$

Then we have

Theorem 4.3 *Assume the matrices*

$$C_k := \mu I_d - T_B(n,k)\Phi_{k-1}(I_{n^{k-1}} \otimes F_1)T_N(n,k), \quad k \geq 3 \tag{41}$$

are non-singular, then

$$G_k = \left[\sum_{i=1}^{k-1} G_i T_B(n,i)\Phi_{i-1}(I_{n^{i-1}} \otimes F_{k-i+1})\right] T_N(n,k)C_k^{-1}. \tag{42}$$

We refer to [10–15] for details.

4.2 Singular Feedback Linearization

Consider a nonlinear system

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i. \tag{43}$$

Singular feedback linearization means find a (single input) feedback

$$\begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} = \begin{bmatrix} \alpha_1(x) \\ \vdots \\ \alpha_m(x) \end{bmatrix} v,$$

and a coordinate change $z = z(x)$ such that the closed-loop system is a linear control system.

Consider

$$\dot{x} = Ax + F_2x^2 + F_3x^3 + \cdots. \tag{44}$$

Assume

$$\mathrm{ad}_{Ax}\eta_k = F_k x^k.$$

Then we have

$$\eta_k = (\varGamma_k^n \odot F_k)\, x^k, \quad x \in \mathbb{R}^n. \tag{45}$$

Here $\odot$ is the Hadamard product of matrices[16]. $\varGamma_k^n$ can be constructed mechanically as

$$(\varGamma_k^n)_{ij} = \frac{1}{\left(\sum\limits_{s=1}^{n} \alpha_s^j \lambda_s\right) - \lambda_i}, \quad i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, n^k, \tag{46}$$

where α_1^j, α_2^j, $\cdots$, α_n^j are respectively the powers of x_1, x_2, $\cdots$, x_n of the j-th component of x^k.

Theorem 4.4 *Assume A is non-resonant. Then system* (44) *can be transformed into a linear form*

$$\dot{z} = Az \tag{47}$$

by the following coordinate transformation:

$$z = x - \sum_{i=2}^{\infty} E_i x^i, \tag{48}$$

where E_i are determined recursively by

$$\begin{aligned} &E_2 = \Gamma_2 \odot F_2, \\ &E_s = \Gamma_s \odot \left(F_s - \sum_{i=2}^{s-1} E_i \Phi_{i-1}(I_{n^{i-1}} \otimes F_{s+1-i}) \right), \ s \geq 3. \end{aligned} \tag{49}$$

Theorem 4.5 *System* (43) *is single-input linearizable, iff there exist an NR-type transformation and a constant vector b of non-zero component such that*

$$b \in \operatorname{Span} \left\{ \left(I - \sum_{i=2}^{\infty} E_i \Phi_{i-1} x^{i-1} \right) g_j \,\middle|\, j = 1, 2, \cdots, m \right\}. \tag{50}$$

We refer to [17,18] for more details.

4.3 Symmetry of Control Systems

Consider an analytic control system

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} f_i(x) u_i, \qquad x \in \mathbb{R}^n, \tag{51}$$

where $f_i(x)$, $i = 0, 1, \cdots, m$ are analytic vector fields. Let G be a Lie group acting on $\mathbb{R}^n$ (or an open subset $M \subset \mathbb{R}^n$).

Definition 4.6 System (51) is said to be state space(ss)-symmetric with respect to G (or has an ss-symmetry group G) if for each $\alpha \in G$

$$\theta(\alpha)_* f_i(x) = f_i(\theta(\alpha) x), \qquad i = 0, 1, \cdots, m,$$

where $\theta(\alpha)_*$ is the induced mapping of $\theta(\alpha)$, which is a diffeomorphism on $\mathbb{R}^n$. If $G < GL(n, \mathbb{R})$, it is called a linear symmetry.

Using semi-tensor product, many interesting symmetric results have been obtained. Then following is one of them.

Theorem 4.7 *System* (51) *with $n \geq 3$ has an ss-symmetry group $G = SO(n, \mathbb{R})$, iff*

$$f_j(x) = \sum_{i=0}^{\infty} a_i^j ||x||^{2i} x, \quad a_i^j \in \mathbb{R}, \quad j = 0, 1, \cdots, m. \tag{52}$$

Much more can be found in [19].

5 Application to Abstract Algebra

Semi-tensor product is a powerful tool in investigating algebraic structures.

Let $e_1, e_2, \cdots, e_n$ be a basis of a finite dimensional algebra, $\mathcal{L}$, where the product $* : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$. The structure matrix $M_{\mathcal{L}}$ is an $n \times n$ matrix with entries

$$m_{i,j} = e_i * e_j, \quad i, j = 1, 2, \cdots, n.$$

Definition 5.1 *An algebra, $\mathcal{L}$, is symmetric if*

$$X * Y = Y * X, \quad \forall X, Y \in \mathcal{L}; \tag{53}$$

$\mathcal{L}$ is skew-symmetric if

$$X * Y = -Y * X, \quad \forall X, Y \in \mathcal{L}; \tag{54}$$

$\mathcal{L}$ is associative if

$$(X * Y) * Z = X * (Y * Z), \quad \forall X, Y, Z \in \mathcal{L}. \tag{55}$$

Proposition 5.2 i) *An algebra, $\mathcal{L}$, is symmetric, iff*

$$M_{\mathcal{L}}(W_{[n]} - I_{n^2}) = 0; \tag{56}$$

ii) *$\mathcal{L}$ is skew-symmetric, iff*

$$M_{\mathcal{L}}(W_{[n]} + I_{n^2}) = 0. \tag{57}$$

iii) *$\mathcal{L}$ is associative, iff*

$$M_{\mathcal{L}}(M_{\mathcal{L}} \otimes I_n - I_n \otimes M_{\mathcal{L}}) = 0. \tag{58}$$

Next, we consider Lie algebra[20].

Proposition 5.3 *An algebra $\mathcal{L}$ is a Lie algebra, iff the structure matrix satisfies* i) (57)*;* ii) *the following* (59)*:*

$$M^2(I_{n^2} + W_{[n,n^2]} + W_{[n^2,n]}) = 0. \tag{59}$$

Example 5.4 Cross product defined in $\mathbb{R}^3$ is as follows: Let $X = x_1\boldsymbol{i} + x_2\boldsymbol{j} + x_3\boldsymbol{k}$, and $Y = y_1\boldsymbol{i} + y_2\boldsymbol{j} + y_3\boldsymbol{k}$. Then

$$X \times Y = \det \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}.$$

Then its structure matrix can be easily obtained as

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{60}$$

It is ready to verify that

$$M(I_9 + W_{[3]}) = 0,$$

and

$$M^2\left(I_{27} + W_{[3.9]} + W_{[9,3]}\right) = 0.$$

Therefore, $\mathbb{R}^3$ with cross product is a Lie algebra.

Since an algebra is uniquely determined by its structure matrix, we may search Lie algebras via structure matrices. Consider three dimensional case. Assume the algebra is skew-symmetric, its structure matrix should be

$$M_{\mathcal{L}_3} = \begin{bmatrix} 0 & a & d & -a & 0 & g & -d & -g & 0 \\ 0 & b & e & -b & 0 & h & -e & -h & 0 \\ 0 & c & f & -c & 0 & i & -f & -i & 0 \end{bmatrix}. \tag{61}$$

With the help of computer, we can calculate

$$M_{\mathcal{L}_3}^2(I_{27}+W_{[3,9]}+W_{[9,3]}),$$

which is a 3×27 matrix. Fortunately, there are very few different non-zero entries. They are

$$\begin{aligned}
m_{1,6}&=m_{1,16}=m_{1,22}=-m_{1,8}=-m_{1,12}=-m_{1,20}=bg+gf-ah-di;\\
m_{2,6}&=m_{2,16}=m_{2,22}=-m_{2,8}=-m_{2,12}=-m_{2,20}=ae-bd+hf-ei;\\
m_{3,6}&=m_{3,16}=m_{3,22}=-m_{3,8}=-m_{3,12}=-m_{3,20}=af+bi-cd-ch.
\end{aligned}$$

We conclude that

Theorem 5.5 *A three dimensional algebra is a Lie algebra, iff its structure matrix is as* (61) *with entries satisfying the following equations:*

$$\begin{cases}bg+gf-ah-di=0,\\ ae-bd+hf-ei=0,\\ af+bi-cd-ch=0.\end{cases}\tag{62}$$

Some interesting new Lie algebras have been constructed in [21]. Certain other properties, such as invertibility etc., have also been discussed there.

6 Application to Differential Geometry

We consider the computation of connection.

Definition 6.1 Let $f,g\in V(M)$ be two (C^∞) vector fields on M. An $\mathbb{R}$-bilinear mapping $\bigtriangledown$: $V(M)\times V(M)\to V(M)$ is called a connection, if

1)

$$\bigtriangledown_{rf}sg=rs\bigtriangledown_f g,\quad r,s\in\mathbb{R};\tag{63}$$

2)

$$\bigtriangledown_{hf}g=h\bigtriangledown_f g,\quad \bigtriangledown_f(hg)=L_f(h)g+h\bigtriangledown_f g,\quad h\in C^\infty(M).\tag{64}$$

By $\mathbb{R}$-linearity, as long as a connection is defined over a basis, it is well defined. Using local coordinates x, we have

$$\bigtriangledown_{\frac{\partial}{\partial x_i}}\left(\frac{\partial}{\partial x_j}\right)=\sum_{k=1}^n\gamma_{ij}^k\frac{\partial}{\partial x_k},$$

where γ_{ij}^k are called Christoffel symbol.

We call

$$\Gamma=\begin{bmatrix}\gamma_{11}^1&\cdots&\gamma_{1n}^1&\cdots&\gamma_{n1}^1&\cdots&\gamma_{nn}^1\\ \vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\ \gamma_{11}^n&\cdots&\gamma_{1n}^n&\cdots&\gamma_{n1}^n&\cdots&\gamma_{nn}^n\end{bmatrix}$$

Christoffel matrix. We give a matrix expression of connection.

Theorem 6.2 *Under new coordinates y, we have*

$$\widetilde{\Gamma}=D^2xDx+\Gamma\ltimes Dx(I\otimes Dx).\tag{65}$$

Let M be a Riemannian manifold with structure matrix $G = (g_{ij})_{n\times n}$. There exists a unique Riemannian connection on M[22]. The Christoffel symbols of this connection can be calculated from G by

$$\gamma_{ij}^{k} = \frac{1}{2}\sum_{s=1}^{n} g^{ks}\left(\frac{\partial g_{si}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_s} + \frac{\partial g_{js}}{\partial x_i}\right), \tag{66}$$

where g^{ij} is the (i,j) entry of G^{-1}.

It is known that [6] with this connection we have

$$[f,g] = \bigtriangledown_f g - \bigtriangledown_g f. \tag{67}$$

Christoffel matrix is said to be symmetric, if

$$\gamma_{ij}^{k} = \gamma_{ji}^{k}, \quad \forall\, i,j,k. \tag{68}$$

Then we have

Theorem 6.3 *If manifold N has symmetric Christoffel connection, then* (67) *holds.*

Since for Riemannian manifold, Christoffel matrix is symmentric, Theorem 6.3 is more general.

The structure matrices of curvature tensor and Riemann curvature tensor have also be constructed in [4].

7 Application to Mathematical Logic

In this section we consider the matrix expression of logic. Under matrix expression a general description of logical operators is proposed. Using semi-tensor product of matrices the logical inference can be simplified a lot. We refer to [4,23,24] for details.

First, we give some necessary notations and conclusions for multi-valued or k-valued logic, which remain true for $k = 2$ case, i.e., for the classical 2-valued logic.

Definition 7.1 A pure logical domain, denoted by D_l, is defined as

$$D_l = \{T = 1, F = 0\}; \tag{69}$$

A k-valued logical domain ($k \ge 2$), denoted by D_k, is defined as

$$D_k = \left\{T = 1,\ \frac{k-2}{k-1},\ \cdots,\ \frac{1}{k-1},\ F = 0\right\}; \tag{70}$$

A fuzzy logical domain, denoted by D_f, is defined as

$$D_f = \{r \mid 0 \le r \le 1\}. \tag{71}$$

Definition 7.2 An s-ary k-valued logical operator is a mapping $\sigma : \underbrace{D_k \times D_k \times \cdots \times D_k}_{s} \to D_k$.

To use matrix expression we identify the elements in D_k with a vector as

$$e_i = \frac{k-i}{k-1} \Longleftrightarrow {}_{i}^{k}, \quad i = 1, 2, \cdots, k-1, k,$$

where ${}_{i}^{k}$ is the i-th column of identity matrix I_k.

Let σ be an s-ary operator and denote

$$m_{i_1,i_2,\cdots,i_s} = \sigma(e_{i_1}, e_{i_2}, \cdots, e_{i_s}), \quad 1 \le i_1, i_2, \cdots, i_s \le k.$$

Now we can construct the structure matrix of σ as

$$M_\sigma = \begin{bmatrix} m_{1\cdots 11} & \cdots & m_{1\cdots 1k} & \cdots & m_{k\cdots k1} & \cdots & m_{k\cdots kk} \end{bmatrix}. \tag{72}$$

Using semi-tensor product, we have

Proposition 7.3 *If a $k \times k^s$ matrix M_σ is the structure matrix of an s-ary logical operator σ, then*

$$\sigma(P_1, P_2, \cdots, P_s) = M_\sigma \ltimes P_1 \ltimes \cdots \ltimes P_s. \tag{73}$$

Now we define a matrix, called the power-reducing matrix, as

$$M_r^k = \begin{bmatrix} \delta_1^k & 0_k & \cdots & 0_k \\ 0_k & \delta_2^k & \cdots & 0_k \\ \vdots & \vdots & \ddots & \vdots \\ 0_k & 0_k & \cdots & \delta_k^k \end{bmatrix}, \tag{74}$$

where 0_k is the zero vector in $\mathbb{R}^k$. Its name is from the following property.

Lemma 7.4 *Let $P \in D_k$. Then for any $p \times k^2 q$ matrix Ψ, we have*

$$\Psi P^2 = \Psi M_r^k P. \tag{75}$$

In a logic expression a logic variable is constant if its value is assigned in advance, it is called a free variable if its value can be arbitrary. Using this concept and above lemma, we have

Theorem 7.5 *Any logic expression $L(P_1, \cdots, P_s)$ with free logic variables $P_1, \cdots, P_s \in D_k$ can be expressed in a canonical form as*

$$L(P_1, P_2, \cdots, P_s) = M_L P_1 P_2 \cdots P_s, \tag{76}$$

where M_L is a $k \times k^s$ logic matrix.

Next, we give some examples in the classical 2-valued logic.

Example 7.6 Consider one fundamental unary operator: Negation, $\neg P$, and four fundamental binary operators[25]: Disjunction, $P \vee Q$; Conjunction, $P \wedge Q$; Implication, $P \to Q$; Equivalence, $P \leftrightarrow Q$. Their structure matrices are as follows:

$$\begin{aligned} &M_\neg := M_n = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \\ &M_\vee := M_d = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad M_\wedge := M_c = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}; \\ &M_\to := M_i = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}; \quad M_\leftrightarrow := M_e = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}. \end{aligned} \tag{77}$$

In fact, there are 2^{2^r} (k^{k^r}) r-ary 2-valued (correspondingly, k-valued) logical operators.

For any binary logical operator σ, we have

$$P \sigma Q = M_\sigma P Q.$$

Now we use the following example to show the application of Theorem 7.5.

Example 7.7 Person A said that person B is a liar, person B said person C is a liar, and person C said that both persons A and B are liars. Who is a liar ?

Denote A: person A is honest; B: person B is honest; and C: person C is honest. Then the logical expression of the statement is

$$(A \leftrightarrow \neg B) \wedge (B \leftrightarrow \neg C) \wedge (C \leftrightarrow \neg A \wedge \neg B).$$

Its matrix form, $L(A,B,C)$, is

$$M_c^2(M_eAM_nB)(M_eBM_nC)(M_eCM_cM_nAM_nB). \tag{78}$$

Its canonical form can be computed as

$$L(A,B,C) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \end{bmatrix} ABC.$$

L is true only if

$$A = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Conclusion: Only B is honest.

8 Safety Control of Power Systems

Direct applications of Theorems 4.1 and 4.2 are detailed in [14,15,26,27]. This section further reviews the application of Theorems 4.1 and 4.2 in power system dynamic security region (DSR)[28−33]. The concept of dynamic security region was first proposed by Felix F. Wu[28], and then series engineering work modifications were done to make it practical[29]. Recently, the theoretical foundation of the dynamic security region is revealed[30−32].

8.1 Power System Model

We review the classical model for transient stability analysis. Consider a power system consisting of n generators. Let the loads be modeled as constant impedances. Then the dynamics of the k-th generator can be written with the usual notation as

$$\begin{aligned} &\dot{\delta}_k = \omega_0\omega_k, \\ &2H_k\dot{\omega}_k = P_{mk} - P_{ek} - D_k\omega_k, \quad k = 1, 2, \cdots, n, \end{aligned} \tag{79}$$

where $\omega_0 = 2\pi f_B$, δ_k, and ω_k are the rotor angle and speed of machine k, D_k and H_k are the damping ratio and inertia constant of machine k, P_{mk} and P_{ek} are the mechanical power and the electrical power at machine $\#k$;

$$P_{ek} = \left\{ E_k^2 G_{kk} + E_k \sum_{j\neq k}^{n} E_j (G_{kj}\cos\delta_{kj} + B\sin\delta_{kj}) \right\},$$

where $\delta_{kj} = \delta_k - \delta_j$, E_k is the constant voltage behind direct axis transient reactance of machine $\#k$, and $Y = (G_{ij} + jB_{ij})_{n\times n}$ is the reduced admittance matrix.

Using the number n machine as the reference, (79) can be transformed into the form as follows:

$$\begin{aligned}&\dot{\delta}_{kn} = \omega_0\omega_{kn}, \quad k = 1, 2, \cdots n-1,\\&2H_k\dot{\omega}_k = P_{mk} - P_{ek} - D_i\omega_k, \quad k = 1, 2, \cdots, n.\end{aligned} \tag{80}$$

If, furthermore, as usual, uniform damping is assumed, i.e., $d_0 = \frac{D_k}{2H_k}$, $(k = 1, 2, \cdots, n)$, then using the n-th machine as the reference, (79) can be transformed into the form as follows:

$$\begin{aligned}&\dot{\delta}_{kn} = \omega_0\omega_{kn},\\&\dot{\omega}_{kn} = -d_0\omega_{kn} + \frac{P_{mk} - P_{ek}}{2H_k} - \frac{P_{mn} - P_{en}}{2H_n}, \quad k = 1, 2, \cdots, n-1.\end{aligned} \tag{81}$$

Let $\delta = (\delta_{1n}, \delta_{2n}, \cdots \delta_{n-1,n})^{\mathrm{T}}$, $m = 2n-2$, $x = (\delta^{\mathrm{T}}, \omega^{\mathrm{T}})^{\mathrm{T}}$, where $\omega = (\omega_{1,n}, \cdots, \omega_{n-1,n})^{\mathrm{T}}$ (or $m = 2n-1$ and $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^{\mathrm{T}}$ in the non-uniform damping case), and $u = (P_{m1}, P_{m2}, \cdots, P_{mn})^{\mathrm{T}}$ be the control variables, then the power system with the network reduction model has the following form

$$\dot{x} = f(x, u), \tag{82}$$

where f is twice differentiable, $x \in \mathbb{R}^m$.

8.2 Dynamic Security Region

Transient stability is the ability of the power system to maintain synchronism after a fault such as short circuit. Mathematically, the power system suffered from a fault has three stages: the pre-fault, fault-on, and post-fault stage. At the pre-fault stage, the system is operated at a stable equilibrium point $x_0(u)$ of the pre-fault system

$$\dot{x} = F_1(x, u), \quad t < 0. \tag{83}$$

At time $t = 0$, the system undergoes a fault that results in a structural change in the system. Suppose the fault is cleared at time $t = t_F$. Then during the fault-on stage, the system is governed by a fault-on dynamics described by

$$\dot{x} = F_2(x, u), \quad x(t) = \phi(t, x_0, u), \quad 0 \le t < t_F. \tag{84}$$

Once the fault is cleared, the system is henceforth governed by a post-fault dynamics described by the following differential equation (82). The initial condition of the post-fault system is the state of the fault-on system at the time of fault clearing, $\phi(t_F, x_0, u)$. Notice that since the clearing time is given and x_0 is a function of u, the system state at the time of clearing is really only a function of u, we therefore write $\phi(u) = \phi(t_F, x_0, u)$. The post-fault dynamics is described by

$$\dot{x} = f(x, u), \quad x(t_F) = \phi(u), \quad t \ge t_F. \tag{85}$$

Assuming the post-fault system has a (asymptotically) stable equilibrium point $x_s(u)$, then the transient stability analysis is to determine whether the initial point of the post-fault trajectory, $\phi(u)$, is located inside the stability region of the equilibrium point $x_s(u)$, $V(x_s(u))$. Furthermore, due to both the fault and its clearing time are fixed, the setting of the control variables u completely determines the transient stability of the system, therefore, mathematically, the dynamic security region (DSR), in the terms of control variables u in which the system is transiently stable (with respect to a given fault) can be described as follows:

$$\Omega_d = \{u \mid \phi(u) \in V(x_s(u))\}. \tag{86}$$

In the power system transient stability analysis, the concept of Controlling Unstable Equilibrium Point (CUEP) has been well recognized. The CUEP of a certain fault is the unstable equilibrium point whose stable manifold (which is a part of the boundary of the stability region) is crossed by the continuous faulted trajectory of the fault[33]. With the concept of the CUEP, the local boundary of dynamic security region that is of interest to the study of transient stability can therefore be written locally as

$$\{u \,|\, h(\phi(u), u) = 0\}, \tag{87}$$

where $h(x, u)$ is the implicitly function with which the local stable manifold of the CUEP x_e could be denoted as $\{x|h(x, u) = 0\}$, and furthermore, the function h is the solution of following partial differential equation:

$$f^{\mathrm{T}} \cdot \frac{\partial h}{\partial x} = \mu \cdot h(x, u), \quad h(x_e, u) = 0, \quad \mathrm{rank}\left(\frac{\partial h}{\partial x}\right) = 1, \tag{88}$$

where μ is the unstable eigenvalue of the Jacobian matrix $J(u) = D_x f(x, u)|_{x=x_e}$ at x_e.

8.3 Linear Approximation of Dynamic Security Region

Next, we briefly review one linear approximation for the DSR (for other approximation, please refer to [30]). The linear approximation of DSR is based on the linear approximation of stability region and sensitivities. The linear approximation of the stable manifold $h(x, u)$ in (88) is (see Theorem 4.1)

$$h_L(x, u) = [x - x_e(u)]^{\mathrm{T}} \eta(u), \tag{89}$$

where $\eta(u) = (\eta_1, \eta_2, \cdots, \eta_m)^{\mathrm{T}}$ is the left unstable eigen-vector of Jacobian matrix $J(u)$, i.e.,

$$J(u)^{\mathrm{T}} \eta(u) = \mu(u)\eta(u), \quad \eta(u)^{\mathrm{T}} \eta(u) = 1. \tag{90}$$

With the above linear approximation of stability region (89), one approximation for the boundary of DSR is

$$h_L(\phi(u), u) = [\phi(u) - x_e(u)]^{\mathrm{T}} \eta(u), \tag{91}$$

Furthermore, with the trajectory sensitivities and sensitivities of CUEP respected to control variable u, one linear approximations of DSR, which is called L_0-linear approximation can be obtained as follows

$$h_{L_0} = \{u \,|\, L_0 + L_1(u - u_0) = 0\}. \tag{92}$$

9 Conclusion

The tensor product of matrices was firstly proposed by D. Cheng about 8 years ago. Since then many colleagues and students have been worked with him on this new tool and its various applications. Some of them are Prof. Q. Lu, Prof. S. Mei, Prof. H. Qin, Prof. Y. Hong, Prof. W. Xie, Dr. Z. Xi, Dr. J. Ma, Dr. A. Xue, Dr. H. Qi, and many others.

It has been used to some problems on dynamic systems and dynamic control systems, such as stability and stabilization, linearization, symmetry etc.; to pure math problems such as computation of connections curvature tensors etc. on differential geometry; structure analysis of algebra etc. in abstract algebra. It has also been found some applications in physics[34], to power systems etc.

Now the authors are confident that semi-tensor product will survive and success in the further.

From very beginning, the research on this subject has been supported warmheartedly by Prof. H. F. Chen. So we dedicate this survey to him.

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