Variable-Shape Linear Algebra: Mathematical Foundations and High-Performance Implementation

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Abstract: Variable-Shape Linear Algebra (VSLA) treats *dimension* as intrinsic data rather than a rigid constraint. This paper makes five concrete contributions: (1) formalization of VSLA through equivalence classes of finite-dimensional vectors modulo trailing-zero padding; (2) construction of two semiring instantiations—convolution and Kronecker products—with complete algebraic characterization; (3) introduction of the stacking operator Σ that builds higher-rank tensors from collections of variable-shape tensors, forming a strict monoidal category and enabling tensor pyramid constructions for streaming data; (4) complexity analysis within the VSLA shape-semiring framework showing FFT convolution preserves $\mathcal{O}(mnd_{\max} \log d_{\max})$ efficiency despite heterogeneous shapes, versus $\mathcal{O}(mnd_{\max}^2)$ for naive approaches; (5) an open-source C99 library with Python bindings demonstrating significant performance improvements over existing approaches. Unlike existing ragged tensor frameworks (TensorFlow Ragged, PyTorch NestedTensors), VSLA provides mathematically rigorous semiring structures with provable algebraic identities, enabling principled dimension-aware computation for adaptive AI architectures, multi-resolution signal processing, and scientific computing applications.

Keywords: Variable-shape tensors, stacking operator, tensor pyramids, semiring algebra, automatic differentiation, high-performance computing, adaptive neural networks

1 Context and Motivation

1.1 The Dimension Problem

Traditional linear algebra fixes dimensions m, n *a priori*. Contemporary challenges—adaptive neural networks, multi-resolution signal analysis, dynamic meshes—demand structures whose shapes evolve in real time.

Running Example: Consider training a convolutional neural network where filter widths adapt dynamically based on input complexity. A standard 3×3 convolution kernel $K_1 = [1, -1, 2]$ might expand to $K_2 = [1, -1, 2, 0, 1]$ for high-resolution features. Traditional frameworks require manual padding: $K_1' = [1, -1, 2, 0, 0]$ before operations, losing semantic information and incurring unnecessary computation on artificial zeros.

Existing approaches fall short:

- **TensorFlow Ragged Tensors:** Handle variable-length sequences but lack rigorous algebraic structure and semiring properties.
- **PyTorch NestedTensors:** Provide dynamic shapes but without mathematical guarantees or efficient sparse representations.
- **Manual zero-padding:** Obscures mathematical structure, wastes computation, and lacks provable algebraic identities.

1.2 The VSLA Solution

VSLA incorporates the shape directly into every algebraic object through mathematically rigorous equivalence classes. Operations such as addition or convolution implicitly coerce operands to a common dimension while preserving sparsity and algebraic properties. In our example, $K_1 \oplus K_2 = [2, -2, 4, 0, 1]$ automatically, with provable semiring laws and efficient sparse computation.

2 Mathematical Foundations

VSLA is built on the concept of *dimension-aware vectors*, which treat dimension as intrinsic data rather than a fixed constraint.

Core Definition

A **dimension-aware vector** is an equivalence class $[(d, v)]$ where $d \in \mathbb{N}$ is the logical dimension and $v \in \mathbb{R}^d$ is the data vector. Two vectors (d_1, v_1) and (d_2, v_2) are equivalent iff their zero-padded extensions to $\max(d_1, d_2)$ dimensions are equal.

This equivalence relation enables automatic shape compatibility while preserving mathematical rigor.

3 Stacking Operator and Tensor Pyramids

VSLA extends beyond basic semiring operations with the *stacking operator* $\Sigma_k : (\mathbb{T}_r)^k \to \mathbb{T}_{r+1}$ that builds higher-rank tensors from collections of variable-shape tensors. Given k rank-r tensors with potentially different shapes, Σ_k computes their ambient shape (element-wise maximum dimensions), applies zero-padding equivalence, and concatenates along a fresh leading axis.

The stacking operator satisfies:

- **Zero-absorption:** Adding VSLA zeros preserves equivalence classes
- **Associativity:** Nested stacking operations compose predictably
- **Distributivity:** Stacking distributes over semiring operations

Combined with the *window-stacking operator* Ω_w, this enables construction of tensor pyramids for streaming data aggregation and multi-resolution analysis.

4 Implementation and Performance

Our C99 implementation provides two computational models:

Model A (Convolution Semiring): Uses FFT-based convolution for commutative operations. Achieves $\mathcal{O}(mnd_{\max} \log d_{\max})$ complexity for matrix-vector operations.

Model B (Kronecker Product Semiring): Uses tiled Kronecker products for non-commutative operations. Provides cache-optimal memory access patterns.

5 Experimental Results

We evaluated VSLA across 15 synthetic datasets with varying sparsity (10-90% zeros) on Intel i9-13900HX with RTX 4060 GPU, comparing against zero-padding, TensorFlow Ragged, and PyTorch NestedTensors.

Runtime Performance (ms):

- Vector operations: 8.7ms (VSLA) vs 45.2ms (zero-padding)
- FFT convolution: 24.6ms (VSLA) vs 128.7ms (zero-padding)
- Tensor stacking: 18.5ms (VSLA) vs 95.3ms (zero-padding)

Memory efficiency: 62-68% reduction vs zero-padding, 20-30% vs ragged tensors.

Real-world applications: 15% speedup in adaptive CNN training, 40% faster wavelet processing, 30% memory savings in transformer models.

6 Conclusion

VSLA provides the first mathematically rigorous framework for variable-shape linear algebra, enabling principled computation on dynamic tensor structures. The open-source implementation demonstrates both theoretical soundness and practical performance benefits.

Our approach enables new architectures for adaptive AI systems, multi-resolution signal processing, and scientific computing applications where traditional fixed-dimension approaches fall short.

AI Tools Disclosure

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