Learning Logs

Will Thomas

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The main goal of this document is to advance understanding of Log Math and how to efficiently solve Log problems, all while remembering a small number of first principles.
Let us begin by reviewing some of the things we know about Log, and more generally, exponents.

Review Section

1. $\forall x, x^0 = 1$ anything to the 0 power is 1. (This symbol $\forall$ means for all). So for all x values, x to the 0 power is 1!

2. $\forall x,\ a,\ b,\ x^a * x^b = x^{a+b}$ Let us quickly validate this with some easy examples that we can recall when we need to prove this for ourselves.
   Lets choose $x = 2, a = 3, b = 5$:
   $x^a * x^b = 2^3 * 2^5 = 256 = 2^8 = 2^{3+5} = x^{a+b}$
   Full circle, it really works! How about some problems to make sure this works even more.
   Solve the following the full way through like the above example:

   (a) Solve $3^5 * 3^3$
   (b) Solve $4^1 * 4^2$
   (c) Solve $5^2 * 5^3$

3. $\log_a b = c \iff a^c = b$ (This symbol $\iff$ means that the two are the equivalent. So if you know one, the other must be true also)

4. $\forall x, x^{-a} = \frac{1}{x^a}$ Negative exponents switch to the denominator

Extending our Knowledge

In this section, we will focus on proving things using only our review equations. Thus showing how we can derive more knowledge and equations, while remembering less.

1. The first thing we will prove is how to divide with common bases for exponents:
   $\forall x, a, b, \frac{x^a}{x^b} = x^{a-b}$
   To prove this, let us start by splitting apart the fraction into two fractions that are multiplied by one another:
   $\frac{x^a}{x^b} = \frac{x^a}{1} \cdot \frac{1}{x^b}$
   Then we can apply equation 4 from our review section $\forall x, x^{-a} = \frac{1}{x^a}$
   So $\frac{1}{x^b}$ turns into x^{-b} leaving us with $x^a \cdot x^{-b}$
   Here, it should be obvious that the equation to apply is equation 2
   $x^a * x^{-b} = x^{a+-b} = x^{a-b}$ $\square$ ($\square$ mean Done/Proven or Q.E.D)
   Thus we have proven that $\forall x, a, b, \frac{x^a}{x^b} = x^{a-b}$

   Now for some examples, go through the same process as above to prove the following:

   (a) $\frac{2^6}{2^7} = 2^{-1}$
   (b) $\frac{3^4}{3^3} = 3^1$
   (c) $\frac{10^{-3}}{10^{-6}} = 10^3$

2. The next thing we will prove is related to Logs, more specifically:
   $\forall \log_x a * b = \log_x a + \log_x b$
   First we should apply equation 3 to the right side of the equation to yield
   Note: When using this symbol $\exists$ below it means "There Exists" so $\exists c_1$ means there exists a value c_1 s.t. (such that) the following holds
   $\exists c_1$ s.t. $\log_x a = c_1 \iff x^{c_1} = a$ and $\exists c_2$ s.t. $\log_x b = c_2 \iff x^{c_2} = b$,
   multiplying the values for a and b together we see that $a * b = x^{c_1} * x^{c_2} = x^{c_1+c_2}$
   Taking the log base x of both sides we are left with $\log_x a * b = c_1 + c_2$
   substituting back in the values for c_1, c_2 we see that $\log_x a * b = \log_x a + \log_x b$ $\square$
   Thus we have proven this new property of logs, all by knowing how log operates and review equation 3.

3. Next we will prove the logarithmic identity.
   $\forall x, \log_x 1 = 0$ this is quite simple once we apply review equation 3
   $\forall x, \log_x 1 = 0 \iff x^0 = 1$
   As we can see, the right side perfectly matches review equation 1, as anything to the 0 power is 1.
   Thus we have proven it $\square$

4. We will now prove why the scientific notation approximation method for logarithms works well.
   The approximation states $\log_{10}(n \times 10^m) \approx m + 0.n$
   First we should apply equation 2 from this section in regards to log of products.
   This allows a reduction of $\log_{10}(n \times 10^m) = \log_{10} n + \log_{10} 10^m$
   Simplifying we see $\log_{10} n + m \approx m + 0.n$
   $\implies \log_{10} n \approx 0.n$. Stated more precisely $\log_{10} n \approx \frac{n}{10}$
   If we can prove that $\log_{10} n \approx \frac{n}{10}$ holds, then we will have proven the approximation works. We should work under the assumption that if n is less than 1, we would shift our scientific notation so that n is between 1 and 10. (Also shown as $n \in (1, 10)$).

   Take the values at each end of the problem $\log_{10} 1 = 0$ and $\log_{10} 10 = 1$