Homework 12 - MATH 791

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Problem 1:
Let S be any ring and R denote the ring of 2×2 matrices over S. Prove that $I \subseteq R$ is a two-sided ideal if and only if there exists a two-sided ideal $J \subseteq S$ such that $I = M_2(J)$.
Solution:
$I \subseteq R$ is a two-sided ideal $\implies \exists$ two-sided ideal $J \subseteq S,\ s.t.\ I = M_2(J)$
Let us define

$$J := \{x \mid \begin{pmatrix} x & * \\ * & * \end{pmatrix} \in I\}$$

Using this, the final conclusion that $I = M_2(J)$, so we only need to prove that J defined this way is a two-sided ideal.

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix},\ B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix},\ C = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}, A, C \in R,\ B \in I$$

$$ABC \in I = \begin{pmatrix} a_1b_1c_1 + a_2b_3c_1 + a_1b_2c_3 + a_2b_4c_3 & * \\ * & * \end{pmatrix}$$

$$\implies a_1b_1c_1 + a_2b_3c_1 + a_1b_2c_3 + a_2b_4c_3 \in J$$

Which means that J must be a two-sided ideal as for any arbitrary elements in R, the left and right sided conjugation will be in I which makes it in J.
Now for the converse: $\exists J \subseteq S, I = M_2(J) \implies I \subseteq R$ is a two-sided ideal
Since J is a two-sided ideal, that means that any elements $s_i, t_i \in S$, $s_i j_i t_i \in J$
Leveraging this, we know that if we are given any other elements $A, C \in R$ where $R = M_2(S)$, that for $B \in I \subseteq R$

$$ABC = \begin{pmatrix} a_1b_1c_1 + a_2b_3c_1 + a_1b_2c_3 + a_2b_4c_3 & a_1b_1c_2 + a_2b_3c_2 + a_1b_2c_4 + a_2b_4c_4 \\ a_3b_1c_1 + a_4b_3c_1 + a_3b_2c_3 + a_4b_4c_3 & a_3b_1c_2 + a_4b_3c_2 + a_3b_2c_4 + a_4b_4c_4 \end{pmatrix}$$

$$= \begin{pmatrix} a_1b_1c_1 & a_1b_1c_2 \\ a_3b_1c_1 & a_3b_1c_2 \end{pmatrix} + \begin{pmatrix} a_2b_3c_1 & a_2b_3c_2 \\ a_4b_3c_1 & a_4b_3c_2 \end{pmatrix} + \begin{pmatrix} a_1b_2c_3 & a_1b_2c_4 \\ a_3b_2c_2 & a_3b_2c_4 \end{pmatrix} + \begin{pmatrix} a_2b_4c_3 & a_2b_4c_4 \\ a_4b_4c_3 & a_4b_4c_4 \end{pmatrix}$$

Since each element of these mini-matrices is obvious in J, they are all elements of a two-sided ideal, and thus $I = M_2(J)$ is a two-sided ideal

Problem 2:
Let R be a ring and $X \subseteq R$ be a subset. Define $\langle X \rangle$, the *two-sided ideal generated by* X to be the intersection of all two-sided ideals of R containing X. First, show that $\langle X \rangle$ is a two-sided ideal of R containing X and then show $\langle X \rangle$ is the set of all finite expressions of the form $r_1x_1s_1 + \cdots + r_nx_ns_n$, with $r_i, s_j \in R$ and $x_i \in X$.

Solution:
We have two cases, either X is a two-sided ideal itself, or it needs to be extended to $X' := X \cup \{x'_1, \ldots, x'_m\}$ to be a two-sided ideal.
If X is a two-sided ideal itself, then the intersection of all two-sided ideals of R containing X will be some $\langle X \rangle = \bigcap(X \cup \cdots)$ and X will be the minimal element containing X, thus $\langle X \rangle$ will be the two-sided ideal generated by X and the intersection.
If X has to be extended to X', then that means $\exists x_k \in X$ such that $rx_ks \notin X$ for all $r, s \in R$, and that the additional elements $x'_1, \ldots, x'_m$ all $x_k \in X'$.
Any other two-sided ideal of R containing X will also have to contain $x'_1, \ldots, x'_m$ as well since they are what allow $x_k \in X'$

$$\implies \text{Any other two-sided ideal } H_i \text{ will have to contain } X'$$

$$\therefore \langle X \rangle := \bigcap(X' \cup H_i \cdots)$$

So any two-sided ideal generated by X will be the intersection of all two-sided ideals of R containing X

Problem 3:
Let R and S be rings. Let $R \times S$ denote $\{(r, s) \mid r \in R \text{ and } s \in S\}$.

(i) Show that $R \times S$ is a ring under coordinate-wise addition and multiplication.

(ii) Show that $K \subseteq R \times S$ is a two-sided ideal if and only if $K = I \times J$, for I a two-sided ideal in R and J a two-sided ideal in S.

Solution:

(i) This seems very obvious, define $+*$ as $(r_1, s_1) + *(r_2, s_2) = (r_1 + r_2, s_1 + s_2)$ which will always be in $R \times S$. Very similarly for $\times *$.
All other features should be directly inheirited except possibly the Abelian group feature. This becomes obvious though since it is $+*$ which is coordinate-wise, so the ring's R and S apply their own Abelian features.

$$\therefore R \times S \text{ is a ring}$$

(ii) Let us presume that I is not a two-sided ideal, this means that $\exists r, r' \in R$ such that $ri_1r' \notin I$, then the point $(i_1, j) \in K$ will not allow $(r, s) \times *(i_1, j) \times *(r', s') = (ri_1r', sjs') \notin K$. A very similar argument can be applied to J

$$\therefore K \text{ is a two-sided ideal } \iff K = I \times J \text{ where } I \text{ and } J \text{ are two-sided ideals}$$

Problem 4:
Let R and S be a commutative rings, so that $T := R \times S$ is also a commutative ring. Set $e_1 = (1, 0)$ and $e_2 = (0, 1)$

(i) Show that $Te_1 := \{te_1 \mid t \in T\}$ is both an ideal of T and a ring, in its own right. Similarly, for Te_2.

(ii) $T = Te_1 + Te_2$ and $Te_1 \cap Te_2 = 0$

(iii) Show that T is isomorphic to $Te_1 \times Te_2$

Solution:

(i) Te_1 will be an ideal of T if for any elements $(a1, b1), (a2, b2) \in T$,
$(a1, b1)(t1, 0)(a2, b2) \in Te_1$. This reduces to $(a1t1a2, 0) \in Te_1$ since Te_1 is define over all $t \in T$. Similar proof for Te_2
To see that it is a ring, we just recognize that we can create an isomorphism between $Te_1 \cong R$ and R is a commutative ring, thus Te_1 will be a commutative ring. Similar for Te_2 except for the comm. ring S. (Pick a trivial homomorphism $\phi((r, 0)) = r$ or $\phi((0, s)) = s$)

(ii) Any element $t \in T, t = (t1, t2) = t1 \in Te_1 + t2 \in Te_2$.
Let us presume an element $(t1, t2) \neq (0, 0) \in Te_1 \cap Te_2 \implies (t1, t2) \in Te_1 \implies t2 = 0 \wedge (t1, t2) \in Te_2 \implies t1 = 0$ thus no element other than $0 \in Te_1 \cap Te_2$

(iii) As we showed in part (i), $Te_1 \cong R$ and $Te_2 \cong S$ which can be combined to show $T \cong Te_1 \times Te_2$ just by direct application of the definition of T

Problem 5:
Let R be a commutative ring. An element $e \in R$ is called an *idempotent* if $e^2 = e$. We say that e is a *non-trivial idempotent* if $e \neq 0, 1$.

(i) Supposed that $e \in R$ is a non-trivial idempotent. Show that $1 - e$ is also a non-trivial idempotent and $e \cdot (1 - e) = 0$.

(ii) Show that Re is both an ideal and a ring. Similarly for $R(1 - e)$.

(iii) Show that $Re \cap R(1 - e) = 0$

(iv) Show that R is isomorphic to $Re \times R(1 - e)$

Solution:

(i) If e is non-trivial idempotent that means $e^2 = e$, if we take
$(1 - e)^2 = 1 - 2e + e^2 = (1 - e)$ so it is also a non-trivial idempotent.
$e \cdot (1 - e) = e - e^2 = e - e = 0$

(ii) Re will be an ideal if for any $r, s \in R$, $e' \in Re$, $re's \in Re$ since it is a commutative ring $rse' \in Re$ and since $e' = r'e$ we get $rsr'e \in Re$ and since $rsr' \in R$ we know this must hold. This ring property will hold as well since we have shown it is an ideal so the abelian subgroup holds, and multiplication will work by the argument over a comm. ring.
Reasoning about $R(1-e)$, $r, s \in R$, $e' \in R(1-e)$, $re's \in R(1-e)$ since it is comm. ring $rse' \in R(1-e)$ and $e' = r'(1-e)$ we get $rsr'(1-e) \in R(1-e)$ and $rsr' \in R$ trivially. Same argument for why it is a ring.

(iii) If $x \in Re \cap R(1-e)$ then that means that $\forall r \in R, rx \in Re$ and $rx \in R(1-e)$, however if we pick $r = 1$ then $x \in Re$ and $x \in R(1-e)$.
This means that $\exists r_1 \in R, x = r_1 e$ and $\exists r_2 \in R, x = r_2(1-e)$ Setting these two equal $r_1 e = r_2 - r_2 e \implies (r_1 + r_2)e = r_2 \implies r_1 r_2^{-1} e + e = 0 \implies (r_1 r_2^{-1} - 1)e = 0 \implies r_1 r_2^{-1} = e^{-1} + 1 = (1-e) = x r_2^{-1} \implies x = r_1 = r_2 \implies x = 0$

(iv) We know that any element $r \in R$ will be in either, so we can form this isomorphism by a cardinality argument (this obviously needs a lot of work).