Homework 22 - MATH 791

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Problem 1:
Let $F \subset K$ be fields and $U := \{u_1, ..., u_r\}$ a subset of K. Define $F(U)$ to be the intersection of all subfields of K containing F and U. We also denote this intersection as $F(u_1, ..., u_r)$.

i) Show that $F(U)$ is a field

ii) Show that

$$F(U) = \{a(u_1, ..., u_r)b(u_1, ..., u_r)^{-1} \mid a(x_1, ..., x_r), b(x_1, ..., x_r) \in F[x_1, ..., x_r], \text{ with } b(u_1, ..., u_r \neq 0)\}$$

Solution:
Proof of (i):
We have to prove that $F(U)$ is a field. Let

$$F(U) = \bigcap E_i, \text{ where } F \subseteq E_i \subseteq K, U \subseteq E_i$$

First, $F(U)$ inherits associativity, distributivity,
and commutativity in multiplication and addition from K
$f, g \in F(U) \Rightarrow f, g \in E_0, E_1, ...$
$\Rightarrow f + g \in E_0, E_1, ...$
$\Rightarrow f + g \in F(U)$

$f \in F(U) \Rightarrow f \in E_0, E_1, ...$
$\Rightarrow -f \in E_0, E_1, ...$
$\Rightarrow -f \in F(U)$
$\Rightarrow$ $F(U)$ is an abelian group under addition.

$$f, g \in F(U) \Rightarrow f, g \in E_0, E_1, ...$$
$$\Rightarrow fg \in E_0, E_1, ...$$
$$\Rightarrow fg \in F(U)$$

$$f \in F(U) \Rightarrow f \in E_0, E_1, ...$$
$$\text{If } f \neq 0,\ f^{-1} \in E_0, E_1, ...$$
$$\Rightarrow f^{-1} \in \bigcap E_i = F(U)$$

So $F(U)$ is a field.
Proof of (ii):
Now we have to prove that

$$F(U) = \{a(u_1, ..., u_r)b(u_1, ..., u_r)^{-1} \mid a(x_1, ..., x_r), b(x_1, ..., x_r) \in F[x_1, ..., x_r],\ \text{with } b(u_1, ..., u_r) \neq 0\}$$

Let $A = \{a(u_1, ..., u_r)b(u_1, ..., u_r)^{-1} \mid a(x_1, ..., x_r), b(x_1, ..., x_r) \in F[x_1, ..., x_r],\ \text{with } b(u_1, ..., u_r) \neq 0\}$
First we can prove that A is a field which contains F and U

A is closed under multiplication, multiplicative inverses :
$ab^{-1} \in A$
$a = a(u_1, ..., u_r)$
$b = b(u_1, ..., u_r)$
$\Rightarrow a(x_1, ..., x_r) \in F[x_1, ..., x_r], b(x_1, ..., x_r) \in F[x_1, ..., x_r]$

$cd^{-1} \in A$
$\Rightarrow c(x_1, ..., x_r) \in F[x_1, ..., x_r], d(x_1, ..., x_r) \in F[x_1, ..., x_r]$

$\Rightarrow (ca) \in F[x_1, ..., x_r]$
$\Rightarrow (bd) \in F[x_1, ..., x_r]$
$\Rightarrow (ca)(bd)^{-1} = (ab^{-1})(cd^{-1}) \in A$
So A is closed under multiplication

Also,

$ab^{-1} \in A$

$a = a(u_1, ..., u_r)$

$b = b(u_1, ..., u_r)$

$\Rightarrow a(x_1, ..., x_r) \in F[x_1, ..., x_r], b(x_1, ..., x_r) \in F[x_1, ..., x_r]$

$\Rightarrow ba^{-1} \in A$

So every nonzreo element in A has a multiplicative inverse

Closure under addition:

$ab^{-1} \in A, cd^{-1} \in A$

$a = a(u_1, ..., u_r)$

$b = b(u_1, ..., u_r)$

$c = c(u_1, ..., u_r)$

$d = d(u_1, ..., u_r)$

$\Rightarrow bd(u_1, ..., u_r)$ Has a corresponding $bd(x_1, ..., x_r) \in F[x_1, ..., x_r]$

$\Rightarrow (bd)^{-1} \in A$ with $b, d \neq 0$ since $ab^{-1}, cd^{-1} \in A$

$(ab + cd)(u_1, ..., u_r)$ also has a corresponding polynomial in $F[x_1, ..., x_r]$

Because it can be written as a polynomial in U

$\Rightarrow (ab + cd)(bd)^{-1} \in A$

$(ad + cb)(bd)^{-1} = add^{-1}b^{-1} + cbd^{-1}b^{-1}$

$= ab^{-1} + cd^{-1} \in A$

additive inverses:

$ab^{-1} \in A$

$a = a(u_1, ..., u_k) \Rightarrow a(x_1, ..., x_r) \in F[x_1, ..., x_r]$

$- a(x_1, ..., x_r) \in F[x_1, ..., x_r]$

$\Rightarrow -a(u_1, ...u_r)b^{-1} \in A$

$ab^{-1} + -ab^{-1} = (a - a)b^{-1} = 0 * b^{-1} = 0 \in A$

A inherits the rest of the field properties from K, since it is a subset of K.

Now we need to show that A contains F and U

$f \in F$

$\Rightarrow f \in F[x_1, ..., x_r]$ $f(x_1, ..., x_r)$ is degree 0

$\Rightarrow f(u_1, ..., u_r) = f$

$1 \in F, 1 \in F[x_1, ..., x_r]$

$f * 1^{-1} \in A$

$1^{-1} = 1$ in K

$\Rightarrow f \in A$

$\Rightarrow F \subseteq A$

$u_i \in U$

$\Rightarrow g(x_1, ..., x_r) = x_i \in F[x_1, ..., x_r]$

$\Rightarrow g(u_1, ..., u_r) = u_i$

$1 \in F, 1 \in F[x_1, ..., x_r]$

$g(u_1, ..., u_r) * 1^{-1} \in A$

$1^{-1} = 1$ in K

$\Rightarrow u_i \in A$

$\Rightarrow U \subseteq A$

Now we prove that A is a subfield of any field E containing
U and F

Let $E \supseteq F, E \supseteq U$, E a field
$ab^{-1} \in A$
$a = \sum_k f_{k_0}(u_1^{k_1} u_2^{k_2} ... u_r^{k_r})$
where $f_{k_i} \in F$

$b = \sum_y f_{y_0}(u_1^{y_1} u_2^{y_2} ... u_r^{y_r})$
where $f_{y_i} \in F$

E has closure under multiplication and addition
and E contains F and U
$\Rightarrow a \in E, b \in E$
E has multiplicative inverses :
$b^{-1} \in E \Rightarrow ab^{-1} \in E$
$\Rightarrow A \subseteq E$

So A is a field containing U and F, $A \subseteq E$ for every field E such that $F \subseteq E \subseteq K$ and $U \subseteq E$.

So
$\bigcap_i E_i = A$
$= \{a(u_1, ..., u_r)b(u_1, ..., u_r)^{-1} \mid a(x_1, ..., x_r), b(x_1, ..., x_r) \in F[x_1, ..., x_r], \text{ with } b(u_1, ..., u_r) \neq 0\}$

Problem 2:
Maintaining the notation from the previous problem

i) Suppose $r = 2$. Show that

$$F(u_1, u_2) = F(u_1)(u_2)$$

ii) Let $X_1 \cup ... \cup X_s$ with $s \le t$(r?) be a partition of U. Prove

$$F(U) = F(X_1)(X_2)...(X_S)$$

Solution:
Proof of (i):
I wasn't able to figure out a more straightforward way of doing this problem

$$\begin{aligned}
&\text{Let } a \in F(u_1)(u_2) \\
&\Rightarrow a = f(u_2)g(u_2)^{-1}, f(x), g(x) \in F(u_1)[x] \\
&\Rightarrow a = \frac{\sum_i \left(\frac{f_i'(u_1)}{g_i'(u_1)}\right) u_2^i}{\sum_j \left(\frac{y_j'(u_1)}{w_j'(u_1)}\right) u_2^j}, \text{ where } f_i'(x), g_i'(x), y_j'(x), w_j'(x) \in F[x] \\
&= \frac{\sum_i \left(\frac{f_i''(u_1,u_2)}{g_i'(u_1)}\right)}{\sum_j \left(\frac{y_j''(u_1,u_2)}{w_j'(u_1)}\right)} \\
&\text{A common denominator can be found by taking the product of all } g_i'(u_1): \\
&= \frac{\frac{\sum_i f_i'''(u_1,u_2)}{g''(u_1)}}{\frac{\sum_j y_j'''(u_1,u_2)}{w''(u_1)}} \\
&= \frac{\sum_i f_i'''(u_1,u_2)}{g''(u_1)} \frac{w''(u_1)}{\sum_j y_j'''(u_1,u_2)} \\
&= \frac{f_i''''(u_1,u_2)}{y''''(u_1,u_2)} \in F(u_1,u_2) \\
&\Rightarrow F(u_1)(u_2) \subseteq F(u_1,u_2)
\end{aligned}$$

$$
\begin{aligned}
&\text{Let } a \in F(u_1, u_2)\\
&a = \frac{f(u_1,u_2)}{g(u_1,u_2)}, f(x_1,x_2), g(x_1,x_2) \in F[x_1,x_2]\\
&= \frac{\sum_{i,j} f_{ij} u_1^i u_2^j}{\sum_{l,k} f_{lk} u_1^l u_2^k}\\
&= \frac{\sum_j \left(\sum_i f_{ij} u_1^i\right) u_2^j}{\sum_k \left(\sum_l f_{lk} u_1^l\right) u_2^k}\\
&= \frac{\sum_j f_j(u_1) u_2^j}{\sum_k g_l(u_1) u_2^k}\\
&= \frac{\sum_j (f_j(u_1)/1) u_2^j}{\sum_k (g_l(u_1)/1) u_2^k}\\
&1 \in F(u_1)\\
&= \frac{f'(u_2)}{g'(u_2)} \text{ where } f'(x), g'(x) \in F(u_1)[x]\\
&\frac{f'(u_2)}{g'(u_2)} \in F(u_1)(u_2)\\
&\Rightarrow F(u_1)(u_2) \supseteq F(u_1, u_2)
\end{aligned}
$$

$\therefore F(u_1)(u_2) = F(u_1, u_2)$.

Proof of (ii):

First we can do a similar proof to the previous part, and then apply induction to show that $F(u_1, ..., u_k) = F(u_1)(u_2)...(u_k)$.

$$a \in F(u_1, ..., u_k)$$

$$a = \frac{f(u_1, ..., u_k)}{g(u_1, ..., u_k')}, \text{ where } f(x_1, ..., x_k), g(x_1, ..., x_k) \in F[x_1, ..., x_k]$$

$$a = \frac{\sum_{i_k} ... \sum_{i_1} f_{i_1 ... i_k} u_1^{i_1} ... u_k^{i_k}}{\sum_{j_k} ... \sum_{j_1} f_{j_1 ... j_k} u_1^{j_1} ... u_k'^{j_k}}$$

$$a = \frac{\sum_{i_k} \left(\sum i_{k-1} ... \sum_{i_1} f_{i_1 ... i_k} u_1^{i_1} ... u_{k-1}^{i_{k-1}} \right) u_k^{i_k}}{\sum_{j_k} \left(\sum_{j_{k-1}} ... \sum_{j_1} f_{j_1 ... j_k} u_1^{j_1} ... u_{k-1}^{j_{k-1}} \right) u_k^{j_k}}$$

$$a = \frac{\sum_{i_k} f_{i_k}(u_1, ..., u_{k-1}) u_k^{i_k}}{\sum_{j_k} g_{j_k}(u_1, ..., u_{k-1}) u_k^{j_k}}$$

$$= \frac{\sum_{i_k} f_{i_k}(u_1, ..., u_{k-1})/1 u_k^{i_k}}{\sum_{j_k} g_{j_k}(u_1, ..., u_{k-1})/1 u_k^{j_k}}$$

$$1 \in F[u_1, ..., u_{k-1}]$$

$$\Rightarrow a \in F(u_1, ..., u_{k-1})(u_k)$$

$$F(u_1, ..., u_k) \subseteq F(u_1, ..., u_{k-1})(u_k)$$

Let $a \in F(u_1, ..., u_{k-1})(u_k)$

$\Rightarrow a = f(u_k)g(u_k)^{-1}, f(x), g(x) \in F(u_1, ..., u_{k-1})[x]$

$\Rightarrow a = \dfrac{\sum_i \left(\frac{f_i'(u_1,...,u_{k-1})}{g_i'(u_1,...,u_{k-1})}\right) u_2^i}{\sum_j \left(\frac{y_j'(u_1,...,u_{k-1})}{w_j'(u_1,...,u_{k-1})}\right) u_2^j}$, where $f_i'(x), g_i'(x), y_j'(x), w_j'(x) \in F[x_1, ..., x_{k-1}]$

$= \dfrac{\sum_i \left(\frac{f_i''(u_1,...,u_k)}{g_i'(u_1,...,u_{k-1})}\right)}{\sum_j \left(\frac{y_j''(u_1,...,u_{k-1},u_k)}{w_j'(u_1,...,u_{k-1})}\right)}$

A common denominator can be found by taking the product of all $g_i'(u_1, ..., u_{k-1})$:

$= \dfrac{\frac{\sum_i f_i'''(u_1,...,u_k)}{g''(u_1,...,u_{k-1})}}{\frac{\sum_j y_j'''(u_1,...,u_k)}{w''(u_1,...,u_{k-1})}}$

$= \dfrac{\sum_i f_i'''(u_1, ..., u_k)}{g''(u_1, ..., u_{k-1})} \dfrac{w''(u_1, ..., u_{k-1})}{\sum_j y_j'''(u_1, ..., u_k)}$

$= \dfrac{f_i''''(u_1, ..., u_k)}{y''''(u_1, ..., u_k)} \in F(u_1, ..., u_k)$

$\Rightarrow F(u_1, ..., u_{k-1})(u_k) \subseteq F(u_1, ..., u_k)$

So $F(u_1, ..., u_{k-1})(u_k) = F(u_1, ..., u_k)$. Now we can apply induction. With the base case proved in part (i), we can see

$$F(u_1, ..., u_k) = F(u_1)(u_2)...(u_k) \text{ For all } k$$

Now we can prove that $F(U) = F(X_1)(X_2)...(X_S)$.

We can use induction here:
Base case:
$F(u_1, ...u_y)$, where $X_1 = \{u_1, ..., u_y\}$
$F(u_1, ...u_y) = F(X_1)$, so the base case holds.

Inductive step:
WLOG Let $X_s = \{u_t, ..., u_r\}$ be the rightmost partition
$F(U) = F(u_1, ..., u_r)$
$= F(u_1)...(u_r)$

Let the field $F' = F(u_1)...(u_{t-1})$
$= F'(u_t)...(u_r)$
$= F'(X_s)$
From an induction hypothesis, we assume that
$F' = F(X_1)...(X_{s-1})$

$\therefore F(U) = F(X_1)(X_2)...(X_S)$.

Problem 3:
Show that if $u_1, ..., u_r$ are algebraically independent over F, then $F(u_1, ..., u_r)$ is isomorphic to the quotient field of $F[x_1, ..., x_r]$.
Solution:

If $u_1, ..., u_r$ are algebraically independent over F
We can define $\psi : F(u_1, ..., u_r) \to F[x_1, ..., x_r]$
$\psi(f(u_1, ..., u_r)g^{-1}(u_1, ..., u_r)) = (f(x_1, ..., x_r), g(x_1, ..., x_r))$
Show that ψ preserves operations

If $u_1, ..., u_r$ are not algebraically independent over F,
Then there exists a polynomial $p(x_1, ..., x_r)$ such that
$p(u_1, ..., u_r) = 0$
let $deg(p) = d$
For all a, if $def(a) \geq deg(p)$
$a = pq + r,$ where $deg(r) < deg(p)$
$\Rightarrow a(u_1, ..., u_r) = pq + r = (0)q + r = r(u_1, ..., u_r)$
So in $F(u_1, ..., u_r)$, there do not exist elements with exponent higher than d
While in $F[x_1, ..., x_r]$ polynomials can have any degree.