Homework 21 - MATH 791

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Problem 1:
Prove that $1, \sqrt[3]{2}, \sqrt[3]{4} \in \mathbb{Q}(\sqrt[3]{2})$ are linearly independent over $\mathbb{Q}$. Thus, $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$.
Solution:
Let us start with $1, \sqrt[3]{2}$ and assume for contradiction that they are not linearly independent.
That would mean that $\exists \lambda_1, \lambda_2 \in \mathbb{Q}$ such that $\lambda_1 * 1 + \lambda_2 * \sqrt[3]{2} = 0$. We can reduce this to $\lambda_1 + \lambda_2 * \sqrt[3]{2} = 0$, and since both $\lambda_1, \lambda_2 \in \mathbb{Q}$, we can factor out λ_2 from both to get.

$$\lambda_2 * (\lambda_1' + \sqrt[3]{2}) = 0$$

$$\implies \sqrt[3]{2} = -\lambda_1' \implies 2 = -(\lambda_1')^3$$

This would mean that there exists a $\lambda_1' \in \mathbb{Q}, \lambda_1' = \sqrt[3]{-2}$ which is a contradiction.

$$\therefore 1, \sqrt[3]{2} \text{ are linearly independent}$$

A very similar argument can be made when adding $\sqrt[3]{4}$ to the mix

$$\therefore 1, \sqrt[3]{2}, \sqrt[3]{4} \text{ are linearly independent over } \mathbb{Q}$$

Now to conclude that $[\mathbb{Q}[\sqrt[3]{2}] : \mathbb{Q}] = 3$, we need only realize that any element in $\mathbb{Q}[\sqrt[3]{2}]$ can be made of $1, \sqrt[3]{2}, \sqrt[3]{4}$. Thus the degree of $\mathbb{Q}[\sqrt[3]{2}]$ over $\mathbb{Q}$ is 3.

Problem 2:
Find the multiplicative inverse of $1 + 2\sqrt[3]{2}$ in $\mathbb{Q}(\sqrt[3]{2})$.
Solution:
The multiplicative inverse will be $(x + y\sqrt[3]{2} + z\sqrt[3]{4})$ such that

$$(1 + 2\sqrt[3]{2})(x + y\sqrt[3]{2} + z\sqrt[3]{4}) = 1$$

$$\implies (x + 4z) + (y + 2x)\sqrt[3]{2} + (z + 2y)\sqrt[3]{4} = 1$$

We could solve this via

$$\begin{pmatrix} 1 & 0 & 4 & 1 \\ 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & \frac{1}{17} \\ 0 & 1 & 0 & -\frac{2}{17} \\ 0 & 0 & 1 & \frac{4}{17} \end{pmatrix}$$

Thus the inverse element is $(\frac{1}{17} - \frac{2}{17}\sqrt[3]{2} + \frac{4}{17}\sqrt[3]{4})$

Problem 3:
Can you write down the multiplicative inverse of $1 + \sqrt[3]{2} + \sqrt[3]{4}$ in $\mathbb{Q}(\sqrt[3]{2})$ without doing any calculations?
Solution:
No I cannot, how am I supposed to without calculations.
Using calculations, we see $(x + 2y + 2z) + (x + y + 2z)\sqrt[3]{2} + (x + y + z)\sqrt[3]{4} = 1$.

$$\begin{pmatrix} 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Thus, the inverse is $(-1 + 1\sqrt[3]{2})$

Problem 4:
Let $F := \mathbb{Q}(\sqrt{2})$. Define $K := F(\sqrt{3})$ to be the set $\{a + b\sqrt{3} \mid a, b \in F\}$. Show that $[K : F] = 2$. Can you guess $[K : \mathbb{Q}]$? If so, give a proof validating your guess.
Solution:
I would guess that $[K : \mathbb{Q}] = 4$. This is because $\dim_{\mathbb{Q}} K = \dim_{\mathbb{Q}} F(\sqrt{3})$ And
$F := \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$, so
$F(\sqrt{3}) := \{a + b\sqrt{3} \mid a, b \in F\} = \{a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6} \mid a, b, c, d \in \mathbb{Q}\}$ To find a basis for this in $\mathbb{Q}$, we need the elements $1, \sqrt{2}, \sqrt{3}, \sqrt{6}$

$$\therefore [K : \mathbb{Q}] = 4$$

Problem 5:
Let $p(x) = x^2 + x + 1 \in \mathbb{Z}_2[x]$.

(i) Show that $p(x)$ is irreducible over $\mathbb{Z}_2$.

(ii) Show the the commutative ring $\mathbb{Z}_2[x]/\langle p(x)\rangle$ has just four elements.

(iii) Prove that the ring $\mathbb{Z}_2[x]/\langle p(x)\rangle$ is a field.

Solution:
Admitted