Homework 24 - MATH 791

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Problem 1:
Write out addition and multiplication tables for the field $\mathbb{Z}_2(\alpha)$ in problem 5 of the previous assignment.
Solution:
This problem is a bit confusing, we are supposed to not need to any explicit terms for α.

Problem 2:
Now let $p(x)$ and α be as in problems 1 and 2 from Homework 23. Determine whether or not $\mathbb{Z}_3(\alpha)$ the splitting field for $p(x)$ over $\mathbb{Z}_3$.
Solution:

Admitted

Problem 3:
Let $p, q \in \mathbb{Z}$ be distinct prime numbers. Show that $[\mathbb{Q}(\sqrt{p}, \sqrt{q}) : \mathbb{Q}] = 4$.
Solution:
We know that the minimal polynomial $f(x)$ such that $f(\sqrt{p}) = 0$ is $f(x) = x^2 - p$. Thus $[\mathbb{Q}(\sqrt{p}) : \mathbb{Q}] = 2$. Additionally, we know that $g(x) \in \mathbb{Q}(\sqrt{p})[x]$ such that $g(\sqrt{q}) = 0$ is $g(x) = x^2 - q$. This is due to the primes being distinct, and this being a UFD. Thus $[\mathbb{Q}(\sqrt{p})(\sqrt{q}) : \mathbb{Q}(\sqrt{p})] = 2$

$$\therefore [\mathbb{Q}(\sqrt{p}, \sqrt{q}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{p})(\sqrt{q}) : \mathbb{Q}(\sqrt{p})] \cdot [\mathbb{Q}(\sqrt{p}) : \mathbb{Q}]$$

$$\therefore [\mathbb{Q}(\sqrt{p}, \sqrt{q}) : \mathbb{Q}] = 4$$

Problem 4:
Let $n \geq 2$ and set $\epsilon := e^{\frac{2\pi i}{n}}$.

(i) Show that $\mathbb{Q}(\epsilon)$ is the splitting field for $x^n - 1$ over $\mathbb{Q}$.

(ii) If $n = p$ is prime, find $[\mathbb{Q}(\epsilon) : \mathbb{Q}]$. Hint: first make an educated guess for the minimal polynomial of ϵ over $\mathbb{Q}$, then show that the function $\phi : \mathbb{Q}[x] \to \mathbb{Q}[x]$ given by $\phi(f(x)) = f(x + 1)$ is an automorphism, and then apply/consider Eisenstein's criterion.

Solution:

Admitted