

Homework 18 - MATH 791

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The problems in this homework set deal with a special kind of PID. Let R be a principal ideal domain with the property that, given any two prime elements, π_1 and π_2 , $\langle \pi_1 \rangle = \langle \pi_2 \rangle$, i.e., up to a unit multiple, there is just one prime element, say $\pi \in R$. Such a ring is called a *discrete valuation ring*, denoted DVR, and $\pi \in R$ is called a *uniformizing parameter*.

Problem 1:

Fix a prime $p \in \mathbb{Z}$. Let R denote the set of rational numbers whose denominators is not divisible by p . First show that R is a subring of \mathbb{Q} , and then show that R is a DVR with uniformizing parameter p .

Solution:

Admitted

Problem 2:

Let R be a DVR with uniformizing parameter $\pi \in R$. Show that $\bigcap_{n \geq 1} \langle \pi^n \rangle = 0$.

Solution:

Admitted

Problem 3:

Let R be a DVR with uniformizing parameter $\pi \in R$. Show that every element in R can be written uniquely as $u\pi^n$ for some $n \geq 0$ and $u \in R$ a unit. Conclude that if K denotes the quotient field of R , then every element in K can be written uniquely in the form $u\pi^n$ for some $n \in \mathbb{Z}$ and $u \in R$, a unit.

Solution:

Admitted

Problem 4:

Let R be a DVR with uniformizing parameter $\pi \in R$, and quotient field K . Define $v : K \rightarrow \mathbb{Z} \cup \{\infty\}$ by $v(0) = \infty$ and for $\alpha \neq 0$, $v(\alpha) = n$, where $\alpha \in K$ and $\alpha = u\pi^n$, as in 3. Show that for all $\alpha, \beta \in K$:

$$(i) \quad v(\alpha + \beta) \geq \min\{v(\alpha), v(\beta)\}$$

$$(ii) \quad v(\alpha\beta) = v(\alpha) + v(\beta)$$

Observe that $R = \{a \in K \mid v(a) \geq 0\}$

Solution:

Admitted

Problem 5:

Let K be a field. Suppose $v : K \rightarrow \mathbb{Z} \cup \{\infty\}$ is a function such that for all $\alpha, \beta \in K$:

- (i) $v(\alpha) = \infty \iff \alpha = 0$
- (ii) $v(\alpha + \beta) \geq \min\{v(\alpha), v(\beta)\}$
- (iii) $v(\alpha\beta) = v(\alpha) + v(\beta)$

Such a function is called a *discrete valuation* on K . We assume that v takes values other than 0 and ∞ . Set $R := \{\alpha \in K \mid v(\alpha) \geq 0\}$. Prove that R is DVR by the following steps below:

- (i) Show that $u \in R$ is a unit $\iff v(u) = 0$. Hint: First show $v(1) = 0$.
- (ii) Show there exist element $r \in R$, with $v(r) > 0$.
- (iii) Prove that if $r \in R$, and $v(r) > 0$, then as an element of K , $v(\frac{1}{r}) = -v(r)$.
- (iv) Suppose $c := \min\{v(r) \mid r \in R \text{ and } v(r) > 0\}$. Show that the image of v is $c\mathbb{Z}$.
- (v) Show that if $\pi \in R$ and $v(\pi) = c$, then R is a DVR with uniformizing parameter π .

Solution: