3) R, S = rings

$R \times S = \{(r, s) \mid r \in R, s \in S\}$

i) $R \times S$ is ring under coordinate-wise addition & multiplication.

ans:

given $r_1, r_2 \in R$

$s_1, s_2 \in S$

$$(r_1, s_1) + (r_2, s_2) = (\underbrace{r_1 + r_2}_{\substack{\in R \\ \text{since } R \text{ is} \\ \text{closed under} \\ \text{addition.}}}, \underbrace{s_1 + s_2}_{\substack{\in S \\ \text{since } S \text{ is} \\ \text{closed under} \\ \text{addition}}}) \in R \times S$$

Closure for addition

Additive Associativity

$\forall r_1, r_2, r_3 \in R, \; r_1 + r_2 + r_3$

$\forall s_1, s_2, s_3 \in S, \; s_1 + s_2 + s_3$

$$(r_1, s_1) + ((r_2, s_2) + (r_3, s_3))$$

$= (r_1, s_1) + (r_2 + r_3, s_2 + s_3)$

$= (r_1 + r_2 + r_3, s_1 + s_2 + s_3)$

$= (r_1 + r_2, s_1 + s_2) + (r_3 + s_3)$

$= ((r_1, s_1) + (r_2, s_2)) + (r_3 + s_3)$

Additive Identity

$e_R = R$'s identity

$e_S = S$'s identity

$\forall r \in R, s \in S$

$(r, s) + (e_R, e_S) = (r + e_R, s + e_S) = (r, s)$

$(e_R, e_S) + (r, s) = (e_R + r, e_S + s) = (r, s)$

Additive Inverse

$\forall r, r' \in R, s, s' \in S$

$(r, s) + (r^{-1}, s^{-1}) = (r + r^{-1}, s + s^{-1}) = (e_R, e_S)$

$(r^{-1}, s^{-1}) + (r, s) = (r^{-1} + r, s^{-1} + s) = (e_R, e_S)$

Commutativity

$\forall r_1, r_2 \in R, s_1, s_2 \in R$

$r_1 + r_2 \,\&\, s_1 + s_2$

$$(r_1,s_1)+(r_2,s_2)=(\underbrace{r_1+r_2}_{\text{R's comm.}},\underbrace{s_1+s_2}_{\text{S's com}})=(r_2+r_1,s_2+s_1)$$

$$=(r_2,s_2)+(r_1,s_1)$$

Multiplicative Identity

Same method as additive . . .

ii) ans

$K \subseteq R \times S$ is a two-sided ideal $\Leftrightarrow K = I \times J$ where I is a two-sided ideal in R & J in S

Left: $\forall (j_1, k_1), (j_2, k_2) \in K$

$\Rightarrow$ $(j_1, k_1) + (j_2, k_2) = (\underbrace{j_1 + j_2}_{\text{Known its two-sided ideal} \to \in R}, \underbrace{k_1 + k_2}_{\text{Known its two-sided ideal} \to \in S})$

addition closure

Ring·Ideal closure

$\forall (r, s) \in R \times S$
$(j, k) \in K$

$(r, s)(j, k) = (\underbrace{r \cdot j}_{\text{closed under} \cdot \text{(r/c)}}, \underbrace{s \cdot k}_{\text{closed under} \cdot \text{ bc two sided}})$

two-sided I is J

Right: same as left

$\Leftarrow$ Ix is two-sided ideal