

Homework 21 - MATH 791

Will Thomas

Problem 1:

Prove that $1, \sqrt[3]{2}, \sqrt[3]{4} \in \mathbb{Q}(\sqrt[3]{2})$ are linearly independent over \mathbb{Q} . Thus, $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$.

Solution:

Let us start with $1, \sqrt[3]{2}$ and assume for contradiction that they are not linearly independent.

That would mean that $\exists \lambda_1, \lambda_2 \in \mathbb{Q}$ such that $\lambda_1 * 1 + \lambda_2 * \sqrt[3]{2} = 0$. We can reduce this to $\lambda_1 + \lambda_2 * \sqrt[3]{2} = 0$, and since both $\lambda_1, \lambda_2 \in \mathbb{Q}$, we can factor out λ_2 from both to get.

$$\begin{aligned}\lambda_2 * (\lambda'_1 + \sqrt[3]{2}) &= 0 \\ \implies \sqrt[3]{2} &= -\lambda'_1 \implies 2 = -(\lambda'_1)^3\end{aligned}$$

This would mean that there exists a $\lambda'_1 \in \mathbb{Q}, \lambda'_1 = \sqrt[3]{-2}$ which is a contradiction.

$\therefore 1, \sqrt[3]{2}$ are linearly independent

A very similar argument can be made when adding $\sqrt[3]{4}$ to the mix

$\therefore 1, \sqrt[3]{2}, \sqrt[3]{4}$ are linearly independent over \mathbb{Q}

Now to conclude that $[\mathbb{Q}[\sqrt[3]{2}] : \mathbb{Q}] = 3$, we need only realize that any element in $\mathbb{Q}[\sqrt[3]{2}]$ can be made of $1, \sqrt[3]{2}, \sqrt[3]{4}$. Thus the degree of $\mathbb{Q}[\sqrt[3]{2}]$ over \mathbb{Q} is 3.

Problem 2:

Find the multiplicative inverse of $1 + 2\sqrt[3]{2}$ in $\mathbb{Q}(\sqrt[3]{2})$.

Solution:

Admitted

Problem 3:

Can you write down the multiplicative inverse of $1 + \sqrt[3]{2} + \sqrt[3]{4}$ in $\mathbb{Q}(\sqrt[3]{2})$ without doing any calculations?

Solution:

Admitted

Problem 4:

Let $F := \mathbb{Q}(\sqrt{2})$. Define $K := F(\sqrt{3})$ to be the set $\{a + b\sqrt{3} \mid a, b \in F\}$. Show that $[K : F] = 2$. Can you guess $[K : \mathbb{Q}]$? If so, give a proof validating your guess.

Solution:

Admitted

Problem 5:

Let $p(x) = x^2 + x + 1 \in \mathbb{Z}_2[x]$.

- (i) Show that $p(x)$ is irreducible over \mathbb{Z}_2 .
- (ii) Show the the commutative ring $\mathbb{Z}_2[x]/\langle p(x) \rangle$ has just four elements.
- (iii) Prove that the ring $\mathbb{Z}_2[x]/\langle p(x) \rangle$ is a field.

Solution:

Admitted