Homework 13 - MATH 791

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Problem 1:
Let R be a ring and $I \subseteq R$ a two-sided ideal. Show that there is a one-to-ono correspondence between the right (left, two-sided) ideals of R containing I and right (left, two-sided) ideals of R/I. Conclude that every right (left, two-sided) ideal of R/I is of the form J/I for some right (left, two-sided) ideal of R containing I.
Solution:
First, let's show that there is a one-to-one correspondence between the right ideals of R containing I and the right ideals of R/I:
Let J be a right ideal of R containing I. Then, we claim that the set $\overline{J} = x + I : x \in J$ is a well-defined right ideal of R/I.
To show that $\overline{J}$ is a right ideal, let $\overline{x}, \overline{y} \in \overline{J}$, where $x, y \in J$. Then, $\overline{x} = \overline{x' + i}$ and $\overline{y} = \overline{y' + j}$ for some $x', y' \in R$ and $i, j \in I$. Since J is a right ideal of R, we have $x'y \in J$ and $ij \in I \subseteq J$. Therefore, $[\overline{xy} = \overline{x'y + i(y'x + yj)} \in \overline{J}.]$
Moreover, for any $r \in R$, we have $\overline{rx} = \overline{rx + ri} \in \overline{J}$ since $rx \in J$. Thus, $\overline{J}$ is a well-defined right ideal of R/I.
Conversely, suppose K is a right ideal of R/I. Then, we claim that the set $J = \{x \in R : x + I \in K\}$ is a well-defined right ideal of R containing I.
To show that J is a right ideal, let $x, y \in J$ and $r \in R$. Then, we have $(x + I)(y + I) = (xy + I) \in K$ by the definition of K. Therefore, $xy \in J$. Moreover, we have $(rx + I) = (r + I)(x + I) \in K$, so $rx \in J$. Finally, since I is a two-sided ideal of R, we have $I \subseteq J$.
It is clear that these constructions give us one-to-one correspondences between the right ideals of R containing I and the right ideals of R/I. Similar arguments hold for left ideals and two-sided ideals.
Now, let L be a right ideal of R/I. Then, there exists a right ideal J of R containing I such that $L = \overline{J} = x + I : x \in J$. Conversely, for any right ideal J of R containing I, we have J/I is a right ideal of R/I, and we have $J/I = \overline{J}$. Therefore, every right ideal of R/I is of the form J/I for some right ideal J of R containing I. Similar arguments hold for left ideals and two-sided ideals.

Problem 2:
Suppose $J \subseteq I$ are two-sided ideals in the ring R. Prove that $(R/J)/(I/J) \cong R/I$ as rings.
Solution:
Let $\pi : R \to R/J$ and $\rho : R \to R/I$ be the natural surjective ring homomorphisms given by $\pi(x) = x + J$ and $\rho(x) = x + I$, for all $x \in R$. Then, by the first isomorphism theorem, we have $\ker(\pi) = J$ and $\ker(\rho) = I$. Moreover, we have $\rho(J) \subseteq I$, so ρ induces a well-defined ring homomorphism $\overline{\rho} : R/J \to R/I$ given by $\overline{\rho}(x + J) = x + I$, for all $x \in R$.

Now, we claim that $\overline{\rho}$ is an isomorphism with inverse given by $\pi|_I : I \to J$.
First, let's show that $\overline{\rho}$ is well-defined. Suppose $x + J = y + J$ for some $x, y \in R$. Then, $x - y \in J \subseteq I$, so $\rho(x - y) = x + I - (y + I) = x + I - y - I \in I$. Therefore, $x + I = y + I$ and $\overline{\rho}$ is well-defined.
Next, let's show that $\overline{\rho}$ is a homomorphism. Let $x + J, y + J \in R/J$. Then, we have

$$\begin{aligned} \overline{\rho}((x + J)(y + J)) &= \overline{\rho}(xy + J) \\ &= xy + I \\ &= (x + I)(y + I) \\ &= \overline{\rho}(x + J)\overline{\rho}(y + J), \end{aligned}$$

so $\overline{\rho}$ is a homomorphism.
Finally, we need to show that $\pi|_I : I \to J$ is the inverse of $\overline{\rho}$. Let $x \in I$. Then, we have $\overline{\rho}(\pi(x) + J) = \rho(x) + I = x + I$, so $\overline{\rho} \circ \pi|_I = \mathrm{id}_I$. Moreover, for any $y + J \in R/J$, we have $\pi(y) \in J \subseteq I$, so

$$\begin{aligned} \overline{\rho}(y + J) &= y + I \\ &= (\pi|_I \circ \pi)(y) + J \\ &= \pi|_I(y + J), \end{aligned}$$

so $\pi|I \circ \overline{\rho} = \mathrm{id}\, R/J$. Therefore, $\overline{\rho}$ is an isomorphism with inverse given by $\pi|_I$.
Thus, we have shown that $(R/J)/(I/J) \cong R/I$ as rings.

Problem 3:
Supposed I, J are two-sided ideals in the ring R. Show that $I \cap J$ and $I + J := \{i + j \mid i \in I \text{ and } j \in J\}$ are two-sided ideals, and that there is an injective ring homomorphism $\phi : R/(I \cap J) \to R/I \times R/J$. Suppose R is commutative. Can you think of a sufficient condition on I and J that guarantees that ϕ is surjective? (Hint: If you know it, consider a ring version of the Chinese Remainder Theorem.)
Solution:
First, we will show that $I \cap J$ is a two-sided ideal.
Given $x \in (I \cap J) \implies x \in I \wedge x \in J$, if we take any
$r, s \in R,\ rxs \in I \wedge rxs \in J \implies rxs \in I \cap J$

$$\therefore I \cap J \text{ is a two-sided ideal}$$

Now to show that $I + J$ is a two-sided ideal:
Given $x \in (I + J) \implies x = (i_x + j_x)$ where $i_x \in I$ and $j_x \in J$ Take any
$r, s \in R,\ rxs = r(i_x + j_x)s = (ri_x + rj_x)s = ri_xs + rj_xs$ and $ri_xs \in I \wedge rj_xs \in J$ which implies that $ri_xs + rj_xs \in I + J$

$$\therefore I + J \text{ is a two-sided ideal}$$

Finally, we will show that $\phi : R/(I \cap J) \to R/I \times R/J$ is an injective ring homomorphism.
Define $\phi(x + I \cap J) = (x + I, x + J)$.
Injective:

$$\forall x_1 x_2 \in R,\ \phi(x_1 + I \cap J) = \phi(x_2 + I \cap J) \implies x_1 + I \cap J = x_2 + I \cap J$$

$$\phi(x_1 + I \cap J) = \phi(x_2 + I \cap J) \implies (x_1 + I, x_1 + J) = (x_2 + I, x_2 + J)$$
$$\implies x_1 + I = x_2 + I \wedge x_1 + J = x_2 + J$$
$$\implies (x_1 - x_2) \in I \wedge (x_1 - x_2) \in J \implies (x_1 - x_2) \in I \cap J$$
$$\implies x_1 + I \cap J = x_2 + I \cap J$$

Homomorphism:
Addition:

$$\forall x_1 x_2 \in R,\ \phi(x_1 + I \cap J + x_2 + I \cap J) = \phi(x_1 + I \cap J) + \phi(x_2 + I \cap J)$$

$$\phi(x_1 + I \cap J + x_2 + I \cap J) = \phi(x_1 + x_2 + I \cap J) = (x_1 + x_2 + I, x_1 + x_2 + J)$$
$$= (x_1 + I, x_1 + J) + (x_2 + I, x_2 + J) = \phi(x_1 + I \cap J) + \phi(x_2 + I \cap J)$$

Multiplication:

$$\forall x_1 x_2 \in R,\ \phi(x_1 + I \cap J \cdot x_2 + I \cap J) = \phi(x_1 + I \cap J) \cdot \phi(x_2 + I \cap J)$$

$$\phi(x_1 + I \cap J \cdot x_2 + I \cap J) = \phi(x_1 \cdot x_2 + I \cap J) = (x_1 \cdot x_2 + I, x_1 \cdot x_2 + J)$$
$$= (x_1 + I, x_1 + J) \cdot (x_2 + I, x_2 + J) = \phi(x_1 + I \cap J) \cdot \phi(x_2 + I \cap J)$$

In order for ϕ to be surjective, we need to map to all of $R/I \times R/J$, which mean that $\forall x + I \in R/I, y + J \in R/J$ we need a corresponding $z + (I \cap J) \in R/(I \cap J)$ such that $\phi(z + (I \cap J)) = (x + I, y + J)$
This will happen if $z \in x + I$ and $z \in y + J$, which is to essentially say that $z \sim x \sim y \implies$ that ϕ is surjective. This is guaranteed to occur if any element $z \in R$ is also automatically in $I + J$, so $R = I + J$ implies surjectivity.

Problem 4:
Let R be a ring and I, J, K be two-sided ideals. Define $IJ := \langle X \rangle$ where $X := \{ij \mid i \in I \text{ and } j \in J\}$.

(i) Show that IJ is a two-sided ideal.

(ii) Show that $I \cdot (J + K) = IJ + IK$

(iii) Show that if, in addition, R is commutative, $I + J = R \implies I \cap J = IJ$

Solution:

(i) To show that IJ is a two-sided ideal, we need to show that for any $r, s \in R$, $r(ij)s \in IJ$ Since I, J are two sided ideals, we know that $ri \in I$ and $js \in J$ which means that $(ri)(js) \in IJ$

(ii)

$$\begin{aligned} I \cdot (J + K) &:= \langle \{ i \cdot (j + k) \mid i \in I \text{ and } (j + k) \in (J + K) \} \rangle \\ &= \langle \{ i \cdot j + i \cdot k \mid i \in I, j \in J, \text{ and } k \in K \} \rangle = IJ + IK \end{aligned}$$

(iii) If R is commutative, and we know that $I + J = R$, then that implies $I \cap J = IJ$
$I + J = R \implies \forall r \in R,\ r = i + j$ for some $i \in I, j \in J$
First, we will show that $I \cap J \subseteq IJ$:

Given $x \in I \cap J$ we know that $x \in I \wedge x \in J$, $IJ := \{ij \mid i \in I \text{ and } j \in J\}$, if we just trivially pick $x \in I$ and $e \in J$, then $xe = x \in IJ$ for any $x \in I \cap J$

$$\therefore I \cap J \subseteq IJ$$

Next, we will show that $IJ \subseteq I \cap J$:

Given $x \in IJ \subseteq R$ we know that $x = ij$ for some $i \in I$ and $j \in J$, yet we also know that $1 \in R, 1 = (i_1 + j_1)$ for some $i_1 \in I$ and $j_1 \in J$.
Multiplying, we get $x = ij \cdot 1 = iji_1 + ijj_1$, yet $iji_1 \in I$ and $ijj_1 \in J$ so $x \in I \cap J$

$$\therefore IJ \subseteq I \cap J$$

Thus $I + J = R \implies I \cap J = IJ$