Homework 21 - MATH 791

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Problem 1:
Prove that $1, \sqrt[3]{2}, \sqrt[3]{4} \in \mathbb{Q}(\sqrt[3]{2})$ are linearly independent over $\mathbb{Q}$. Thus, $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$.
Solution:
Admitted

Problem 2:
Find the multiplicative inverse of $1 + 2\sqrt[3]{2}$ in $\mathbb{Q}(\sqrt[3]{2})$.
Solution:
Admitted

Problem 3:
Can you write down the multiplicative inverse of $1 + \sqrt[3]{2} + \sqrt[3]{4}$ in $\mathbb{Q}(\sqrt[3]{2})$ without doing any calculations?
Solution:
Admitted

Problem 4:
Let $F := \mathbb{Q}(\sqrt{2})$. Define $K := F(\sqrt{3})$ to be the set $\{a + b\sqrt{3} \mid a, b \in F\}$. Show that $[K : F] = 2$. Can you guess $[K : \mathbb{Q}]$? If so, give a proof validating your guess.
Solution:
Admitted

Problem 5:
Let $p(x) = x^2 + x + 1 \in \mathbb{Z}_2[x]$.

(i) Show that $p(x)$ is irreducible over $\mathbb{Z}_2$.

(ii) Show the the commutative ring $\mathbb{Z}_2[x]/\langle p(x)\rangle$ has just four elements.

(iii) Prove that the ring $\mathbb{Z}_2[x]/\langle p(x)\rangle$ is a field.

Solution:
Admitted