Homework 20 - MATH 791

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Throughout this assignment R denotes a commutative ring.

Problem 1:
An ideal $P \neq R$ is said to be a *prime ideal* if for $a, b \in R$, $ab \in P \Rightarrow a \in P$ or $b \in P$. Prove that P is a prime ideal iff R/P is an integral domain.
Solution:
For the first direction, R/P is an ID $\Rightarrow$ P is a prime ideal:

$$\text{Suppose } r_1r_2 + P = 0_{R/P} = P \Rightarrow r_1r_2 \in P$$
$$r_1r_2 + P = (r_1 + P)(r_2 + P) = 0_{R/P} = P$$
$$\Rightarrow (r_1 + P) = 0_{R/P} = Por(r_2 + P) = 0_{R/P} = P \text{ Since } R/P \text{ is an ID}$$

$$\text{If } r_1 + P = P :$$
$$\text{then } r_1 \in P$$
$$\text{And if } r_2 + P = P :$$
$$\text{then } r_2 \in P$$

So $r_1r_2 \in P \Rightarrow r_1 \in P$ or $r_2 \in P$, so P is a prime ideal. Now the other direction:

$$\text{Suppose } P \text{ is a prime ideal}$$
$$\text{Suppose } r_1r_2 \in P$$
$$\Rightarrow r_1r_2 + P = P = 0_{R/P} = (r_1 + P)(r_2 + P)$$
$$\text{Either } r_1 \in P \text{ or } r_2 \in P \text{ since } P \text{ is a prime ideal}$$
$$\Rightarrow (r_1 + P) = 0_{R/P} \text{ or } (r_2 + P) = 0_{R/P}$$

So if P is a prime ideal, then $r_1r_2 + P = 0_{R/P} \Rightarrow (r_1 + P) = 0_{R/P}$ or $(r_2 + P) = 0_{R/P}$, so R/P is an integral domain.

Problem 2:
An ideal $M \neq R$ is a *maximal ideal* if whenever $J \subseteq R$ is an ideal satisfying $M \subseteq J \subseteq R$, then $J = M$ or $J = R$. In other words, M is maximal among the proper ideals of R. It follows from Zorn's Lemma, that if $I \subset R$ is an ideal, then there exists a maximal ideal $M \subseteq R$ with $I \subseteq M$. In particular, every commutative ring has at least one maximal

ideal. Prove that M is a maximal ideal iff R/M is a field. Conclude that every maximal ideal is a prime ideal, and give an example of a prime ideal that is not a maximal ideal.
Solution:
First we can prove that M being maximal implies that R/M is a field. We have to show that every element in R/M other than the additive identity has a multiplicative inverse.

let $r + M \in R/M$
If $r \in M \Rightarrow r + M = M$ is the additive identity and doesn't have a multiplicative inverse
If $r \notin M \Rightarrow (\langle r \rangle + M) \supset M$
$\Rightarrow (\langle r \rangle + M) = R$ since M is maximal
$\Rightarrow 1 \in R, 1 \in (\langle r \rangle + M)$
$\Rightarrow 1 = rr' + m'$
$\Rightarrow 1 + (-m') = rr'$
$\Rightarrow rr' \in M + 1 \Rightarrow rr' = m_2 + 1$
$\Rightarrow rr' + M = 1 + M$
$\Rightarrow (r + M)(r' + M) = (1 + M)$
So $(r + M)$ has the multiplicative inverse $(r' + M)$, so R/M is a field

Now for the other direction, we show that R/M being a field implies that M is maximal.

R/M is a field, so every nonzero element has a multiplicative inverse
Suppose $J \supset M$is an ideal that is a strict superset of M
$\exists j \in J, j \notin M$
(j + M) has an inverse :
$(j + M)(r' + M) = (1 + M)$
$\Rightarrow jr' + M = 1 + M$
$\Rightarrow jr' + m' = 1$ for some m'
$m' = jr''$ since $M \subset J$
$= jr' + jr'' = 1 = j(r' + r'') = 1$
$\Rightarrow j(r' + r'') * r = r$ for all $r \in R$

So $J = R$, which means that M is a maximal ideal.

We can notice that all fields are integral domains. Suppose F is a field.

$$
\begin{aligned}
&a, b \in F \\
&ab = 0 \\
&\text{If } a = 0 \text{ we are done. If } a \neq 0\text{, it has an inverse} \\
&ab = 0 \\
&a^{-1}ab = a^{-1}0 \\
&b = 0
\end{aligned}
$$

So either a or b are the additive inverse. So if M is a maximal ideal:

$$
\begin{aligned}
&M \text{ maximal} \Rightarrow R/M \text{ is a field} \Rightarrow R/M \text{ is an ID} \Rightarrow M \text{ is a prime ideal from problem 1} \\
&M \text{ maximal} \Rightarrow M \text{ is a prime ideal}
\end{aligned}
$$

Finally, consider the ideal $\langle 0 \rangle$ in the ring $\mathbb{Z}$. $\mathbb{Z}$ is an ID, so if $ab \in < 0 > \Rightarrow ab = 0 \Rightarrow a \in < 0 >$ or $b \in < 0 >$. But $< 0 > \subseteq < r >$ for all $r \in R$, so it is not maximal.

Problem 3:
Let R be a commutative ring. Ideals $I, J \subseteq R$ are said to be comaximal if $I + J = R$. Prove that I and J are comaximal iff there is no maximal ideal M containing both I and J.
Solution:
First we can show that if $I + J = R$, then there is no maximal ideal that is a superset of both I and J.

$$
\begin{aligned}
&I + J = R \\
&\Rightarrow i + j = 1 \text{ For some } i \in I, j \in J \\
\\
&\text{Suppose } M \text{ is an ideal, and that } I \subseteq M, J \subseteq M \\
&i + j \in M \Rightarrow 1 \in M \Rightarrow r \in M \\
&\Rightarrow M = R
\end{aligned}
$$

Since M must be the whole ring, there is no maximal ideal containing both I and J. Now the other direction, where we assume there is no maximal ideal containing I and J:

If $I \subset M, J \subset M$, and M is an ideal, then $M = R$
Let $M = I + J$
$I + J$ Is an ideal, which we can show through closure under addition, multiplication, etc.
It inherits associativity and distributivity from R
$I \subseteq I + J, J \subseteq I + J \Rightarrow I + J = R$

So I, J must be comaximal.

Problem 4:
Suppose I, J are comaximal ideals in the commutative ring R. Show that $I \cap J = IJ$.
Solution:
First we can show that if $a \in IJ \Rightarrow a \in I \cap J$:

$$\begin{aligned}
&a \in IJ \\
&a = r_1 i_1 j_1 r_1' + ... + r_k i_k j_k r_k' \\
&= i_1(r_1 j_1 r_1') + ... + i_k(r_k j_k r_k') \in I \\
&= j_1(r_1 i_1 r_1') + ... + j_k(r_k i_k r_k') \in J \\
&\Rightarrow r_1 i_1 j_1 r_1' + ... + r_k i_k j_k r_k' \in I \cap J
\end{aligned}$$

Now the other direction:

$$\begin{aligned}
&a \in I \cap J \\
&\text{We use the fact that } I + J = R \Rightarrow 1 = i' + j' \\
&a = a * 1 = a(i' + j') = ai' + aj' = i'a + aj' \\
&= i' j_a + i_a j' \text{ since } a \text{ can be considered an element of both } I \text{ and } J \\
&= i' j_a + i_a j' \in IJ
\end{aligned}$$

So $I \cap J = IJ$

Problem 5:
For I and J as in 4, prove that the natural map $\phi : R \Rightarrow (R/I) \times (R/J)$ given by $\phi(r) = (r + I, r + J)$ is a surjective ring homomorphism whose kernel equals $I \cap J$. Conclude that $R/IJ \cong (R/I) \times (R/J)$. When $R = \mathbb{Z}$, this isomorphism is one version of the *Chinese Remainder Theorem.*
Solution:

First we should show that ϕ is a surjective homomorphism:

$$\phi(a_1) + \phi(a_2) = (a_1 + I, a_1 + J) + (a_2 + I, a_2 + J)$$
$$= (a_1 + a_2 + I, a_1 + a_2 + J)$$
$$= \phi(a_1 + a_2)$$

$$\phi(a_1) * \phi(a_2) = (a_1 + I, a_1 + J) * (a_2 + I, a_2 + J)$$
$$= (a_1a_2 + I, a_1a_2 + J) = \phi(a_1a_2)$$

Now we have to show that the homomorphism is surjective:

Since I, J are comaximal:
$I + J = R \Rightarrow 1 = i' + j'$ for some $i' \in I, j' \in J$
So for any $r \in R :$
$r = ri' + rj'$
$\Rightarrow \forall r \in R$
$r + I = ri' + rj' + I = rj' + I$ because $ri' \in I$
Similarly,
$r + J = ri' + rj' + J = ri' + J$ because $ri' \in I$

For any element in $(R/I) \times (R/J), (a_1 + I, a_2 + J)$
Let $r \in R, r = a_2i' + a_1j',$ with i', j' as above

$\phi(r) = (a_2i' + a_1j' + I, a_2i' + a_1j' + J)$
$a_2i' \in I, a_1j' \in J$
$\Rightarrow \phi(r) = (a_1j' + I, a_2i' + J) = (a_1 + I, a_2 + J)$ from what we proved earlier

Suppose $k \in R$ s.t. $\phi(k) = (I, J)$
((I, J) is the additive identity of the ring $(R/I) \times (R/J)$)
$\phi(k) = (k + I, k + J) = (I, J)$
$\Rightarrow k \in I, k \in J \Rightarrow k \in I \cap J$
So $ker\phi = I \cap J$

From the first isomorphism theorem, we see that

$$R/(I \cap J) \cong (R/I) \times (R/J)$$

But $I \cap J = IJ$ since I, J are comaximal

$$\Rightarrow R/(IJ) \cong (R/I) \times (R/J)$$