Homework 19 - MATH 791

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Throught this assignment R denotes a commutative ring.

Problem 1:
Let $I \subseteq R$ be an ideal, and $R[x]$ denote the polynomial ring in x over R. Let $I[x]$ denote the set of polynomials in R with coefficients in I and let $\langle I \rangle$ denote the ideal of $R[x]$ generated by the set I. Show that $I[x] = \langle I \rangle$.
Solution:
Let us start by defining the sets:

$$I[x] := \{i_n x^n + \cdots + i_0 \mid i_j \in I\}$$
$$\langle I \rangle := \{i_n x^n + \cdots + i_0 \mid i_j \in I\}$$

These sets are definitionally equal as is straightforward to see, with the ideal generated by I being equivalent due to the closured of the ideal.

Problem 2:
Maintaining the notation from 1, show that the rings $R[x]/I[x]$ and $(R/I)[x]$ are isomorphic.
Solution:
We can show this isomorphism by showing there is an onto ring homomorphism $\phi : R[x] \to (R/I)[x]$ and that $\ker(\phi) = I[x]$
Let us define $\phi(f(x)) = [f(x)]$ ($[f(x)]$ is the equivalence class).
First, we want to show the ring homomorphism property holds:

$$\forall f_1(x), f_2(x) \in R[x], \phi(f_1(x) + f_2(x)) = [f_1(x) + f_2(x)]$$

Due to it being an equivalence class, we can unfold this

$$[f_1(x) + f_2(x)] = [f_1(x)] + [f_2(x)] = \phi(f_1(x)) + \phi(f_2(x))$$

Addition holds, as for multiplication:

$$\forall f_1(x), f_2(x) \in R[x], \phi(f_1(x)f_2(x)) = [f_1(x)f_2(x)]$$

Due to it being an equivalence class, we can unfold this

$$[f_1(x)f_2(x)] = [f_1(x)][f_2(x)] = \phi(f_1(x))\phi(f_2(x))$$

Onto property:

$$\forall [f(x)] \in (R/I)[x],\ \exists f(x) \in R,\ \text{s.t. } \phi(f(x)) = [f(x)]$$

Showing $\ker(\phi) = I[x]$:

$$\ker(\phi) = \{f(x) \in R \mid \phi(f(x)) = [0]\}$$

$$= \{[f(x)] = [0] \mid f(x) \in R\}$$

This is $= I[x]$ as $[f(x)] = [0] \iff f(x) \in I[x]$
Using the first isomorphism theorem for rings, we can conclude then that

$$\therefore R[x]/I[x] \cong (R/I)[x]$$

Problem 3:
Let $R[[x]]$ denote the formal power series ring over R, i.e., the set of expressions of the form $\sum_{i=0}^{\infty} a_i x^i$, with $a_i \in R$. Note this is purely an algebraic expression and does not involve any notion of convergence. We add and multiply element of $R[[x]]$ in the expected way: If $f = \sum_{i=0}^{\infty} a_i x^i$ and $g = \sum_{i=0}^{\infty} b_i x^i$, then: $f + g = \sum_{i=0}^{\infty}(a_i + b_i)x^i$ and $fg = \sum_{k=0}^{\infty} c_k x^k$, where $c_k = \sum_{i+j=k} a_i b_j$. For $I \subseteq R$, let $I[[x]]$ denote the elements in $R[[x]]$, all of whose coefficients belong to I.

(i) Verify that $R[[x]]$ is a ring and $I[[x]]$ is an ideal of $R[[x]]$.

(ii) Show that if I is finitely generated, then $\langle I \rangle = I[[x]]$ as ideals of $R[[x]]$.

(iii) Can you give an example where $I[[x]] \neq \langle I \rangle$

Solution:
Admitted

Here is Eisentsteain's Criterion, which is an important test for irreducibility of polynomials over a UFD.
Eisenstein's Criterion: Let R be a UFD with quotient field K. Suppose $f(x) = a_n x^n + \cdots + a_0 \in R[x]$ is a primitive polynomial. Let $p \in R$ be a prime element and suppose: (i) $p \mid a_i$, for all $0 \leq i < n$. (ii) $p \nmid a_n$, and (iii) $p^2 \nmid a_0$. Then $f(x)$ is irreducible over K (equivalently, over R). For example, $x^6 + 10x^2 + 5x + 15$ is irreducible over $\mathbb{Q}$, by using Eisenstein's criterion and $p = 5$.

Problem 4:
Let $p \in \mathbb{Z}$ be prime and $f_p(x) = x^{p-1} + x^{p-2} + \cdots + x + x \in \mathbb{Z}[x]$. Use Eisenstein's criterion, together with the following fact to show that $f_p(x)$ is irreducible over $\mathbb{Q}[x]$:
$f_p(x)$ is irreducible over $\mathbb{Q}$ if and only if $f_p(x+1)$ is irreducible over $\mathbb{Q}$.
Solution:
Admitted

Problem 5:
Use Eisenstein's criterion and the fact that $\mathbb{Q}[x]$ is a UFD to show that $x^2 + y^2 - 9$ is irreducible in $\mathbb{Q}[x, y]$.
Solution:
Admitted.