Homework 20 - MATH 791

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Throughout this assignment R denotes a commutative ring.

Problem 1:
An ideal $P \neq R$ is said to be a *prime ideal* if for $a, b \in R$, $ab \in P \Rightarrow a \in P$ or $b \in P$. Prove that P is a prime ideal iff R/P is an integral domain.
Solution:
Solution

Problem 2:
An ideal $M \neq R$ is a *maximal ideal* if whenever $J \subseteq R$ is an ideal satisfying $M \subseteq J \subseteq R$, then $J = M$ or $J = R$. In other words, M is maximal among the proper ideals of R. It follows from Zorn's Lemma, that if $I \subset R$ is an ideal, then there exists a maximal ideal $M \subseteq R$ with $I \subseteq M$. In particular, every commutative ring has at least one maximal ideal. Prove that M is a maximal ideal iff R/M is a field. Conclude that every maximal ideal is a prime ideal, and give an example of a prime ideal that is not a maximal ideal.
Solution:
Solution

Problem 3:
Let R be a commutative ring. Ideals $I, J \subseteq R$ are said to be comaximal if $I + J = R$. Prove that I and J are comaximal iff there is no maximal ideal M containing both I and J.
Solution:
Solution

Problem 4:
Suppose I, J are comaximal ideals in the commutative ring R. Show that $I \cap J = IJ$.
Solution:
Solution

Problem 5:
For I and J as in 4, prove that the natural map $\phi : R \Rightarrow (R/I) \times (R/J)$ given by $\phi(r) = (r + I, r + J)$ is a surjective ring homomorphism whose kernel equals $I \cap J$. Conclude that $R/IJ \cong (R/I) \times (R/J)$. When $R = \mathbb{Z}$, this isomorphism is one version of the *Chinese Remainder Theorem.*
Solution:
Solution