Philosophy 160
Predicate logic with functions and identity.

Translate 1-5 into predicate logic using function symbols and identity where appropriate. Read relatives as blood relatives. e.g., mother = biological mother.

1. Jenny's mother is mean.

   $Mm(j)$

2. Jenny's mother is nicer than Adnan's mother.

   $Nm(j)m(a)$

3. Jenny's father is Adnan's maternal grandfather.

   $f(j) = f(m(a))$

4. Everyone who loves Jenny's mother loves Adnan's father.

   $\forall x(Lxm(j) \rightarrow Lxf(a))$

5. Jenny's brother is Adnan's mother.

This one is a little tricky. Since 'brother is not a function, we can not write b(j). But if we write Bj, then this can not be an argument for the function m. Here is the correct answer, where we interpret the above as saying that Jenny has only one brother.

   $\exists x(Bxj \ \& \ \forall y(Byj \rightarrow y=x) \ \& \ x=m(a))$

6. All siblings have the same mother.

   $\forall x \forall y(Sxy \rightarrow m(x) = m(y))$

Translate 7-9 treating height and combined height as the functions c and h.

7. The combined height of Adrian and Michael is less than the height of Jane.

   $Lc(a,m)\ h(j)$

8. The height of Michael's father is identical to the combined height of Michael's mother and Jenny.

   $h(f(i)) = c(m(i), j)$ (Mike = i.)

9. The combined height of Jenny's sister and Jenny is greater than the height of Marvin's mother.

   $\exists x(Sxj \ \& \ \forall y(y=Syj \rightarrow x=y) \ \& \ Gc(x,j)h(m(a))$ (Marvin = a)

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10. $Fa \vdash \forall x(x{=}a \to Fx)$

1.	Fa	A
2.	$b = a$	H
3.	Fb	1,2 =E
4.	$b{=}a \to Fb$	2-3, →I
5.	$\forall x(x{=}a \to Fx)$	4, ∀I

11. $\forall x \forall y(\sim Pxy \to \sim x{=}y) \vdash \exists x \exists y Pxy$

1.	$\forall x \forall y(\sim Pxy \to \sim x{=}y)$	A
2.	$\forall y(\sim Pay \to \sim a{=}y)$	1, ∀E
3.	$\sim Paa \to \sim a{=}a$	2, ∀E
4.	$\sim \exists x \exists y Pxy$	H ~I
5.	$\forall x \sim \exists y Pxy$	4, QE
6.	$\sim \exists y Pay$	5, ∀E
7.	$\forall y \sim Pay$	6, QE
8.	$\sim Paa$	7, ∀E
9.	$\sim a{=}a$	3,8 →E
10.	$a{=}a$	=I
11.	$P \& \sim P$	9,10 CON
12.	$\sim\sim \exists x \exists y Pxy$	4-11 ~I
13.	$\exists x \exists y Pxy$	12, DN

Note: This is an unnecessarily long proof. You can also do it directly by simply introducting a=a on line.

12. $\vdash \forall x \forall y\ x{=}y \to (\exists v Fv \to \forall z Fz)$

1.	$\forall x \forall y\ x{=}y$	H →I
2.	$\exists v Fv$	H →I
3.	$\sim \forall z Fz$	H ~I
4.	$\exists z \sim Fz$	3, QE
5.	$\sim Fa$	H ∃E
6.	Fb	H ∃E
7.	$\forall y\ a{=}y$	1, ∀E
8.	$a{=}b$	7, ∀E
9.	Fa	6,8 =E
10.	$P \& \sim P$	5,9 CON
11.	$P \& \sim P$	2, 6-10 ∃E
12.	$P \& \sim P$	4, 5-11 ∃E
13.	$\sim\sim \forall z Fz$	3-12 ~I
14.	$\forall z Fz$	13, DN
15.	$\exists v Fv \to \forall z Fz$	2-14, →I
16.	$\forall x \forall y\ x{=}y \to (\exists v Fv \to \forall z Fz)$	1-15, →I

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13. ∃x∀y(y=x ↔ Gy) ├ ∀xGx → ∀x∀y x=y

1.	∃x∀y(y=x ↔ Gy)	A
2.	│ ∀xGx	H
3.	│ │ ∀y(y=a ↔ Gy)	H, ∃E
4.	│ │ b=a ↔ Gb	3, ∀E
5.	│ │ Gb → b=a	4, ↔E
6.	│ │ Gb	2, ∀E
7.	│ │ b=a	5,6 →E
8.	│ │ c=a ↔ Gc	3, ∀E
9.	│ │ Gc	2, ∀E
10.	│ │ Gc → c=a	8, ↔E
11.	│ │ c=a	9,10 →E
12.	│ │ c=b	7,11 =E
13.	│ c=b	1, 3-12 ∃E
14.	│ ∀y c=y	13, ∀I
15.	│ ∀x∀yx=y	14, ∀I
16.	∀xGx → ∀x∀y x=y	2-15 →I

This is a bit tricky. We can't do ∀I or punch out of the ∃E hypothesis at step 7 because 'a' is in an undischarged hypothesis. However, we can eliminate a by doing ∀E's again on lines 2 and 3 to a different name. Then we employ =E to produce an indentity without 'a' in it.

14. ∃x x = f(j), ∀y(y=f(j) → y=a) ├ a = f(j)

1.	∃x x = f(j)	A
2.	∀y(y=f(j) → y=a)	A
3.	│ ~a=f(j)	H ~I
4.	│ │ b= f(j)	H ∃E
5.	│ │ b = f(j) → b=a	2, ∀E
6.	│ │ b = a	4,5 →E
7.	│ │ a = f(j)	4,6 =E
8.	│ │ P & ~P	3,7 CON
9.	│ P & ~P	1, 4-8 ∃E
10.	~~a=f(j)	3-9, ~I
11.	a=(fj)	10, DN

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15. Translate the following into English using ‘m’ to mean midpoint. Then prove it.

$\forall\forall x\forall y\forall z(z= m(x,y) \leftrightarrow z = m(y,x)) \vdash (a=m(c,d) \;\&\; b= m(d,c)) \rightarrow a=b$

Something is identical to a midpoint between an ordered pair of objects if and only if it is identical to a midpoint of the same two objects in reverse order. So if a is identical to a midpoint between the ordered pair (c,d) and b is identical to a midpoint between the ordered pair (d,c) then a and b are identical.

This proof can be confusing until you realize that it doesn’t actually require the first assumption. The reason it doesn’t require it because the definition of a function itself guarantees the uniqueness of the midpoint.

1.	$\forall x\forall y\forall z(z= m(x,y) \leftrightarrow z = m(y,x))$	A
2.	$(a=m(c,d) \;\&\; b= m(d,c))$	H
3.	$a=m(c,d)$	2, &E
4.	$b= m(d,c)$	3, &E
5.	$\forall y\forall z(z= m(c,y) \leftrightarrow z = m(y,c))$	1, $\forall$E
6.	$\forall z(z= m(c,d) \leftrightarrow z = m(d,c))$	5, $\forall$E
7.	$b= m(c,d) \leftrightarrow b = m(d,c)$	6, $\forall$E
8.	$b = m(d,c) \rightarrow b= m(c,d)$	7, $\leftrightarrow$E
9.	$b= m(c,d)$	4,8 $\rightarrow$E
10.	$a=b$	3,9 =E
11.	$(a=m(c,d) \;\&\; b= m(d,c)) \rightarrow a=b$	2-10 $\rightarrow$I