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Derivations in modal logic

Prove the following in Leibnizian modal logic. In addition to all the standard inference rules of propositional logic, you have the following for Leibnizian modal logic.

Duality (Dual)	$\Box A \dashv\vdash \sim\Diamond\sim A$ and $\Diamond A \dashv\vdash \sim\Box\sim A$
K rule (K)	$\Box(A \rightarrow B) \vdash (\Box A \rightarrow \Box B)$
T rule (T)	$\Box A \vdash A$
S4 Rule (S4)	$\Box A \vdash \Box\Box A$
Brouwer Rule (B)	$A \vdash \Box\Diamond A$
N Rule (N)	You may introduce any previously derived theorem of propositional logic as a necessary truth. In other words, if $\vdash A$ has been previously demonstrated, then you may state $\Box A$.

Keep in mind that the key to doing some proofs is to use K in combination with N. To do these proofs, you may introduce anything that is transparently a theorem in propositional logic by the theorem introduction rule (TI). Additionally, you may introduce any such theorem as a necessity in accord with N. For example, since $P \rightarrow (P \vee Q)$ is a theorem you may introduce $\Box(P \rightarrow (P \vee Q))$ by N.

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1. $\Box P \vdash \Diamond P$

1. $\Box P$	A
2. P	1, T
3. $\Box\Diamond P$	2, B
4. $\Diamond P$	3, T

2. $\Box P \vdash \Box(Q \rightarrow P)$

1. $\Box P$	A
2. $\Box(P \rightarrow (Q \rightarrow P))$	N, theorem easily derived by $\rightarrow$I
3. $\Box P \rightarrow \Box(Q \rightarrow P)$	2, K
4. $\Box(Q \rightarrow P)$	1,3 $\rightarrow$E

3. $\Diamond\Box S \vdash S$

1. $\Diamond\Box S$	A
2. $\sim S$	H
3. $\Box\Diamond{\sim}S$	2, B
4. $\Box{\sim}\Box{\sim}{\sim}S$	3, Dual
5. $\Box{\sim}\Box S$	4, DN
6. ${\sim}\Diamond{\sim}{\sim}\Box S$	5,Dual
7. ${\sim}\Diamond\Box S$	6, DN
8. P & ${\sim}P$	1,7 Con
9. ${\sim}{\sim}S$	2-8 ~I
10. S	9, DN

To make this sort of proof shorter, you can combine DN and Dual in one step. Just be sure to cite both rules on the same line.

4. $\Diamond\Box R$ & $\Diamond\Box(R \rightarrow M) \vdash \Box\Diamond M$ (Hint: use result from 3)

1. $\Diamond\Box R$	A
2. $\Diamond\Box(R \rightarrow M)$	A
3. $\Diamond\Box R \rightarrow R$	TI
4. $\Diamond\Box(R \rightarrow M) \rightarrow (R \rightarrow M)$	TI
5. R	1,3 $\rightarrow$E
6. $R \rightarrow M$	2,4 $\rightarrow$E
7. M	5,6 $\rightarrow$E
8. $\Box\Diamond M$	7, B

Notice that steps 3 and 4 are not using the N rule, but just the theorem introduction rule from propositional logic.

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5. $\Box(P \rightarrow Q) \vdash \sim\Diamond\sim P \rightarrow \sim\Diamond\sim Q$

1.	$\Box(P \rightarrow Q)$	A
2.	$\Box P \rightarrow \Box Q$	K
3.	$\mid\ \sim\Diamond\sim P$	H $\rightarrow$I
4.	$\mid\ \Box P$	3, Dual
5.	$\mid\ \Box Q$	2,4 $\rightarrow$E
6.	$\mid\ \sim\Diamond\sim Q$	5, Dual
7.	$\sim\Diamond\sim P \rightarrow \sim\Diamond\sim Q$	3-6, $\rightarrow$I

6. $\Box(\sim Q \rightarrow \sim P), \Box P \vdash \Diamond Q$

1.	$\Box(\sim Q \rightarrow \sim P)$	A
2.	$\Box P$	A
3.	$\Box\sim Q \rightarrow \Box\sim P$	1, K
4.	$\mid\ \Box\sim Q$	H ~I
5.	$\mid\ \Box\sim P$	3,4 $\rightarrow$E
6.	$\mid\ \sim P$	T
7.	$\mid\ P$	2, T
8.	$\mid\ P \,\&\, \sim P$	6,7 &I
9.	$\sim\Box\sim Q$	4-8, ~I
10.	$\Diamond Q$	9. Dual

This is equivalent to the $\sim\Diamond Q$ and saves a step.

7. $\Box(\sim P \rightarrow \sim Q) \vdash \Box Q \rightarrow \Box P$

1.	$\Box(\sim P \rightarrow \sim Q)$	A
2.	$\mid\ \Box Q$	H
3.	$\mid\ \Box((\sim P \rightarrow \sim Q) \rightarrow (Q \rightarrow P))$	N (Trans equivalence)
4.	$\mid\ \Box(\sim P \rightarrow \sim Q) \rightarrow \Box(Q \rightarrow P))$	3, K
5.	$\mid\ \Box(Q \rightarrow P)$	1,4 $\rightarrow$E
6.	$\mid\ \Box Q \rightarrow \Box P$	5, K
7.	$\mid\ \Box P$	2,6 $\rightarrow$E
8.	$\Box Q \rightarrow \Box P$	2-7, $\rightarrow$E

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8. $\Box(\sim P \to \sim Q), \Diamond\sim P \vdash \Diamond\sim Q$ (Hint: use result from 7)

1.	$\Box(\sim P \to \sim Q)$	A
2.	$\Diamond\sim P$	A
3.	$\Box Q$	H ~I
4. .	$\Box(\sim P \to \sim Q) \to \Box Q \to \Box P$	TI, proven in 7 above
5.	$\Box Q \to \Box P$	1,4 →E
6.	$\Box P$	3,5 →E
7.	$\sim\Diamond\sim P$	6, Dual
8.	K &~K	2,6 Con
9.	$\sim\Box Q$	3-7, ~I
10.	$\Diamond\sim Q$	8, Dual, DN

9. $\Diamond(P \,\&\, Q) \vdash \Diamond P$

1.	$\Diamond(P \,\&\, Q)$	A
2.	$\Box\sim P$	H
3.	$\Box(\sim P \to \sim(P \,\&\, Q))$	N (easily derived theorem)
4.	$\Box\sim P \to \Box\sim(P \,\&\, Q)$	3, K
5.	$\Box\sim(P \,\&\, Q)$	2,4 →E
6.	$\sim\Diamond(P \,\&\, Q)$	5, DN, Dual
7.	K & ~K	1,6 Con
8.	$\sim\Box\sim P$	2-7, ~I
9.	$\Diamond P$	8, Dual

10. $\Box(P \vee Q) \vdash \Box\sim P \to \Box Q$

1.	$\Box(P \vee Q)$	A
2.	$\Box((P \vee Q) \to (\sim P \to Q))$	N (material implication equivalence)
3.	$\Box(P \vee Q) \to \Box(\sim P \to Q)$	2, K
4.	$\Box(\sim P \to Q)$	1,3 →E
5.	$\Box\sim P \to \Box Q$	4, K

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11. $\Diamond P \vdash \Diamond(K \rightarrow P)$

1.	$\Diamond P$	A
2.	$\Box{\sim}(K \rightarrow P)$	H ~I
3.	$\Box({\sim}(K \rightarrow P) \rightarrow {\sim}P)$	N (easily derived theorem)
4.	$\Box{\sim}(K \rightarrow P) \rightarrow \Box{\sim}P$	3, K
5.	$\Box{\sim}P$	2,4 →E
6.	${\sim}\Diamond P$	5, Dual, DN
7.	$\Diamond P \;\&\; {\sim}\Diamond P$	1,6 &I
8.	${\sim}\Box{\sim}(K \rightarrow P)$	2-7, ~I
9.	$\Diamond(K \rightarrow P)$	8, Dual

12. $\Box P, \Box Q \vdash \Box(P \;\&\; Q)$

1.	$\Box P$	A
2.	$\Box Q$	A
3.	$\Box(P \rightarrow (Q \rightarrow (P \;\&\; Q)))$	N
4.	$\Box P \rightarrow \Box(Q \rightarrow (P \;\&\; Q))$	3, K
5.	$\Box(Q \rightarrow (P \;\&\; Q))$	1,4 →E
6.	$\Box Q \rightarrow \Box(P \;\&\; Q)$	5, K
7.	$\Box(P \;\&\; Q)$	2,6 &I

13. $\Box(\Box P \vee \Box Q), {\sim}P \vdash \Box Q$

1.	$\Box(\Box P \vee \Box Q)$	A
2.	${\sim}P$	A
3.	$\Box((\Box P \vee \Box Q) \rightarrow ({\sim}\Box P \rightarrow \Box Q))$	N (disjunctive syllogism)
4.	$\Box(\Box P \vee \Box Q) \rightarrow \Box({\sim}\Box P \rightarrow \Box Q)$	3, K
5.	$\Box({\sim}\Box P \rightarrow \Box Q)$	1,4 →E
6.	$\Box{\sim}\Box P \rightarrow \Box\Box Q$	5, K
7.	$\Box\Diamond{\sim}P$	2, B
8.	$\Box{\sim}\Box P$	7, Dual, DN
9.	$\Box\Box Q$	6,8 →E
10.	$\Box Q$	9, T

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14. $\Box(P \rightarrow \sim P) \vdash \Box\sim P$

1.	$\Box(P \rightarrow \sim P)$	A
2.	$\Box((P \rightarrow \sim P) \rightarrow \sim P)$	N
3.	$\Box(P \rightarrow \sim P) \rightarrow \Box\sim P$	2, K
4.	$\Box\sim P$	1,3 $\rightarrow$E

15. $\Box P \vee \Box Q, \Box(P \rightarrow R), \Box(Q \rightarrow S) \vdash \Box(R \vee S)$

1.	$\Box P \vee \Box Q$	A
2.	$\Box(P \rightarrow R)$	A
3.	$\Box(Q \rightarrow S)$	A
4.	$\Box P$	H
5.	$\Box(P \rightarrow ((P \rightarrow R) \rightarrow (R \vee S)))$	N
6.	$\Box P \rightarrow \Box((P \rightarrow R) \rightarrow (R \vee S))$	5, K
7.	$\Box((P \rightarrow R) \rightarrow (R \vee S))$	4,6 $\rightarrow$E
8.	$\Box(P \rightarrow R) \rightarrow \Box(R \vee S)$	7, K
9.	$\Box(R \vee S)$	2,8 $\rightarrow$E
10.	$\Box P \rightarrow \Box(R \vee S)$	4-9, $\rightarrow$E
11.	$\Box Q$	H
12.	$\Box(Q \rightarrow ((Q \rightarrow S) \rightarrow (R \vee S)))$	N
13.	$\Box Q \rightarrow \Box((Q \rightarrow S) \rightarrow (R \vee S))$	12, K
14.	$\Box((Q \rightarrow S) \rightarrow (R \vee S))$	11,13 $\rightarrow$E
15.	$\Box(Q \rightarrow S) \rightarrow \Box(R \vee S)$	14, K
16.	$\Box(R \vee S)$	3,15 $\rightarrow$E
17.	$\Box Q \rightarrow \Box(R \vee S)$	11-16, $\rightarrow$E
18.	$\Box(R \vee S)$	1,10,17 $\vee$E