SOLUTIONS - EXERCISES ON PUMPING LEMMA

Using the pumping Lemma, prove that the following languages are not regular:

1. L = $\{a^n b^{2n}\}$

   Assume L is regular. From the pumping lemma there exists a p such that every w ∈ L such that $|w| \geq p$ can be represented as x y z with $|y| \neq 0$ and $|xy| \leq p$. Let us choose $a^p b^{2p}$. Its length is $3p \geq p$. Since the length of xy cannot exceed p, y must be of the form a^k for some $k > 0$. From the pumping lemma $a^{p-k} b^{2p}$ must also be in L but it is not of the right form. Hence the language is not regular.

2. L = $\{0^n 1^m 2^n$ I n, m □ 0 $\}$

   Assume L is regular. From the pumping lemma there exists a p such that every w ∈ L such that $|w| \geq p$ can be represented as x y z with $|y| \neq 0$ and $|xy| \leq p$. Let us choose $0^p 1 2^p$. Its length is $2p+1 \geq p$. Since the length of xy cannot exceed p, y must be of the form 0^p for some $p > 0$. From the pumping lemma $0^{n-p} 1 2^n$ must also be in L but it is not of the right form. Hence the language is not regular.
3. L = $\{a^{2k} w$ I $w \in \{a, b\}^*, |w| = k\}$

   Assume L is regular. From the pumping lemma there exists a p such that every w ∈ L such that $|w| \geq p$ can be represented as x y z with $|y| \neq 0$ and $|xy| \leq p$. Let us choose $a^{2p} b^p$. Its length is $3p \geq p$. Since the length of xy cannot exceed p, y must be of the form a^k for some $k > 0$. From the pumping lemma $a^{2p-k} b^p$ must also be in L but it is not of the right form since the number of a's cannot be twice the number of b's (Note that you must subtract not add , otherwise some a's could be shifted into w). Hence the language is not regular.
4. L = $\{a^n b^l \mid n \leq l\}$

   Assume L is regular. From the pumping lemma there exists a p such that every w ∈ L such that $|w| \geq p$ can be represented as x y z with $|y| \neq 0$ and $|xy| \leq p$. Let us choose $a^p b^p$. Its length is $2p \geq p$. Since the length of xy cannot exceed p, y must be of the form a^k for some $k > 0$. From the pumping lemma $a^{p+k} b^p$ must also be in L but it is not of the right form since the number of a's exceeds the number of b's (Note that you must add not subtract, otherwise the string would be OK). Hence the language is not regular.
5. L = $(v a^{2k} \mid v \in \{a, b\}^*, |v| = k\}$

   Assume L is regular. From the pumping lemma there exists a n such that every w ∈ L such that $|w| \geq n$ can be represented as x y z with $|y| \neq 0$ and $|xy| \leq n$. Let us choose $b^n a^{2n}$. Its length is $3n \geq n$. Since the length of xy cannot exceed n, y must be of the form b^k for some $k > 0$. From the pumping lemma $b^{n+k} a^n$ must also be in L but it is not of the right form since the number of a's exceeds the number of b's and we cannot move any b's on the a side (Note that you must add not subtract, otherwise the string would be OK by shifting a's to the b side). Hence the language is not regular.
6. L = $\{ww \mid w \in \{a, b\}^*\}$

   Assume L is regular. From the pumping lemma there exists an n such that every w ∈ L such that $|w| \geq n$ can be represented as x y z with $|y| \neq 0$ and $|xy| \leq n$. Let us choose $a^n b^n a^n b^n$. Its length is $4n \geq n$. Since the length of xy cannot exceed n, y must be of the form a^k for some $k > 0$. From the pumping lemma $a^{n+k} b^n a^n b^n$ must also be in L but it is not of the right form since the middle of the string would be in the middle of the b which prevents a match with the beginning of the string. Hence the language is not regular.

7. $L = \{a^{n!} \mid n$ □ $0\}$

Proof by contradiction:
Let us assume L is regular. From the pumping lemma, there exists a number p such that any string w of length greater than p has a “repeatable” substring generating more strings in the language L. Let us consider $a^{p!}$ (unless $p < 3$ in which case we chose $a^{3!}$). From the pumping lemma the string w has a “repeatable” substring. We will assume that this substring is of length $k \geq 1$. From the pumping lemma $a^{p!-k}$ must also be in L. For this to be true there must be j such that $j! = m! - k$ But this is not possible since when $p > 2$ and $k \leq m$ we have
$m! - k > (m - 1)!$
Hence L is not regular.

8. $L = \{a^k \mid k$ is a prime number$\}$

Let us assume L is regular. Clearly L is infinite (there are infinitely many prime numbers). From the pumping lemma, there exists a number n such that any string w of length greater than n has a “repeatable” substring generating more strings in the language L. Let us consider the first prime number p $\geq$ n. For example, if n was 50 we could use $p = 53$. From the pumping lemma the string of length p has a “repeatable” substring. We will assume that this substring is of length $k \geq 1$. Hence:

$a^p \in L$ and
$a^{p+k} \in L$ as well as
$a^{p+2k} \in L$, etc.

It should be relatively clear that $p + k$, $p + 2k$, etc., cannot all be prime but let us add k p times, then we must have:

$a^{p+pk} \in L$, of course $a^{p+pk} = a^{p(k+1)}$

so this would imply that $(k + 1)p$ is prime, which it is not since it is divisible by both p and $k + 1$.

Hence L is not regular.