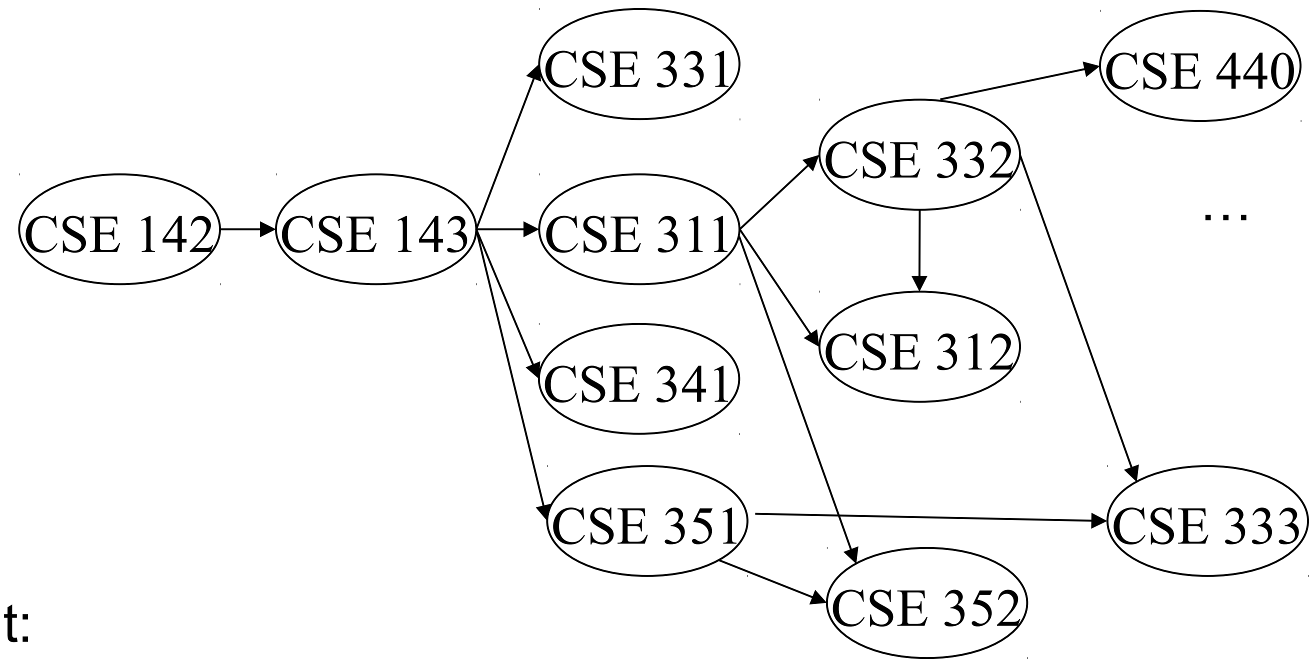
Topological Sort

Problem: Given a DAG `G=(V,E)`, output all vertices in an order such that no vertex appears before another vertex that has an edge to it

Example input:

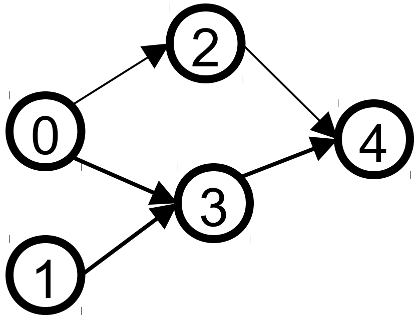


Example output:

142, 143, 311, 331, 332, 312, 341, 351, 333, 440, 352

Questions and comments

- Why do we perform topological sorts only on DAGs?
  - Because a cycle means there is no correct answer
- Is there always a unique answer?
  - No, there can be 1 or more answers; depends on the graph
  - Graph with 5 topological orders:



- What DAGs have exactly 1 answer?
  - Lists
- Terminology: A DAG represents a partial order and a topological sort produces a total order that is consistent with it

Uses

- Figuring out how to finish your degree
- Computing the order in which to recompute cells in a spreadsheet
- Determining the order to compile files using a Makefile
- In general, using a dependency graph to find an order of execution
- …

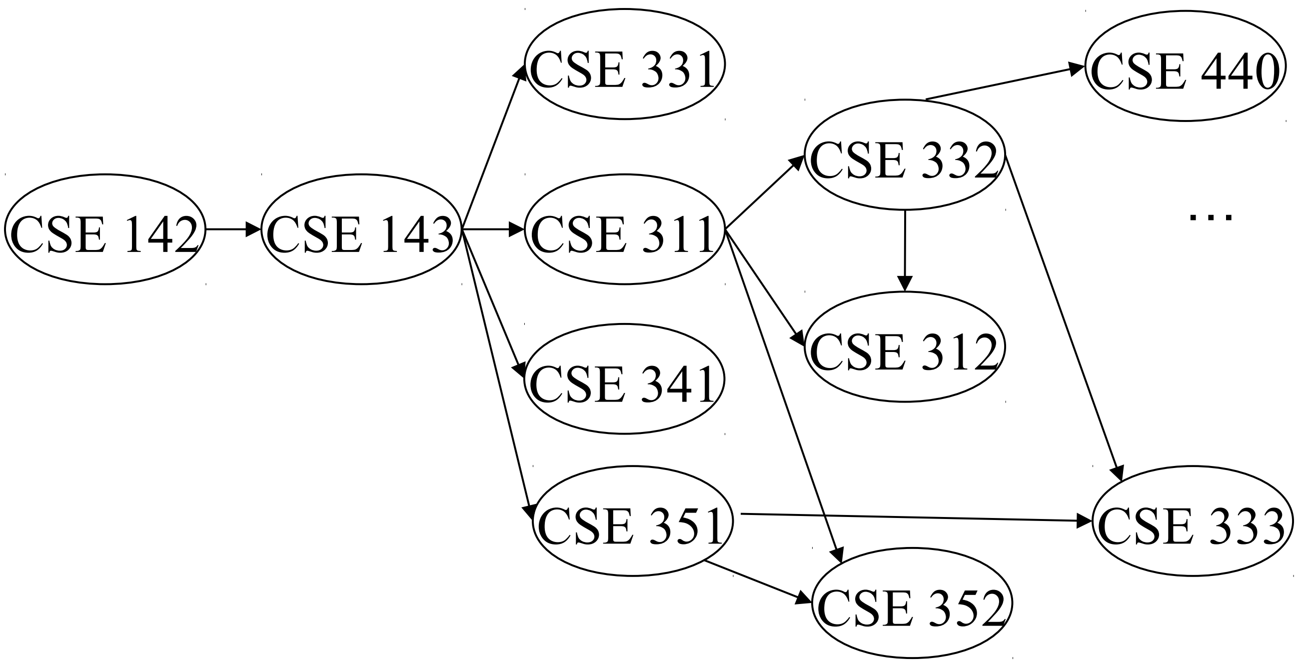
A First Algorithm for Topological Sort

1. Label (“mark”) each vertex with its in-degree
   – Think “write in a field in the vertex”
   – Could also do this via a data structure (e.g., array) on the side

2. While there are vertices not yet output:
   a) Choose a vertex **v** with labeled with in-degree of 0
   b) Output **v** and *conceptually* remove it from the graph
   c) For each vertex **u** adjacent to **v** (i.e. **u** such that (**v**,**u**) in **`E`**), decrement the in-degree of **u**

Example

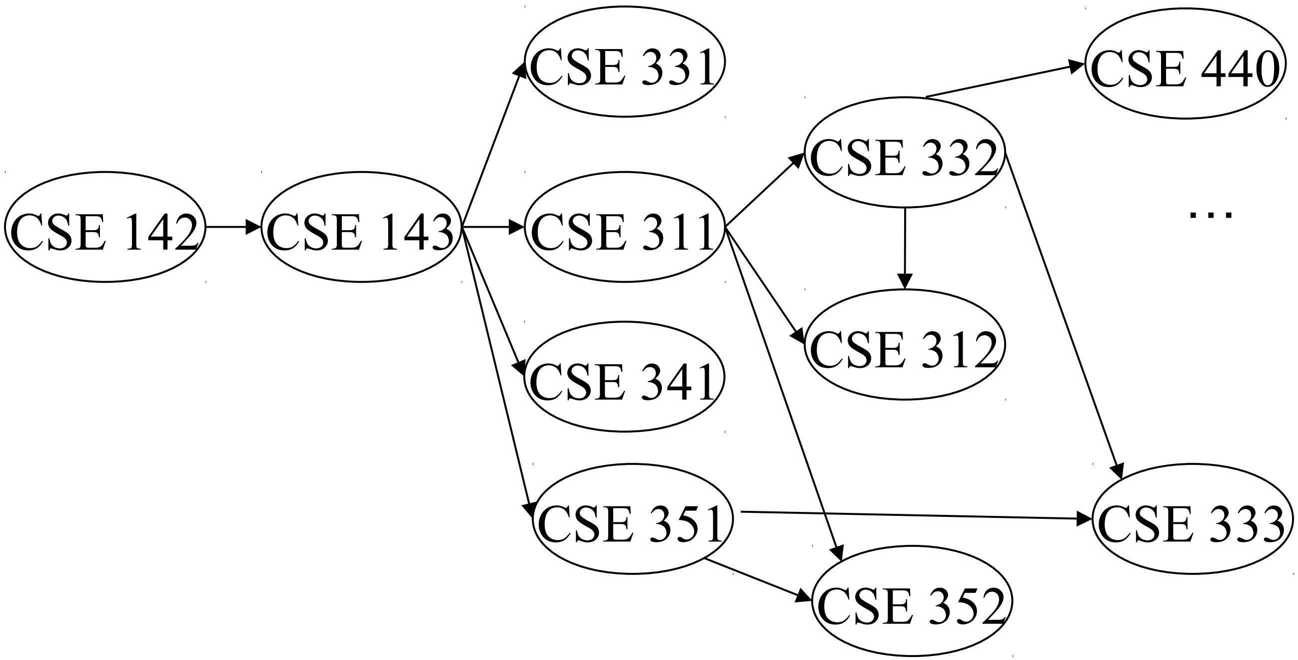
Output:



Node:	142	143	311	312	331	332	333	341	351	352	440
Removed?											
In-degree:	0	1	1	2	1	1	2	1	1	2	1

Example

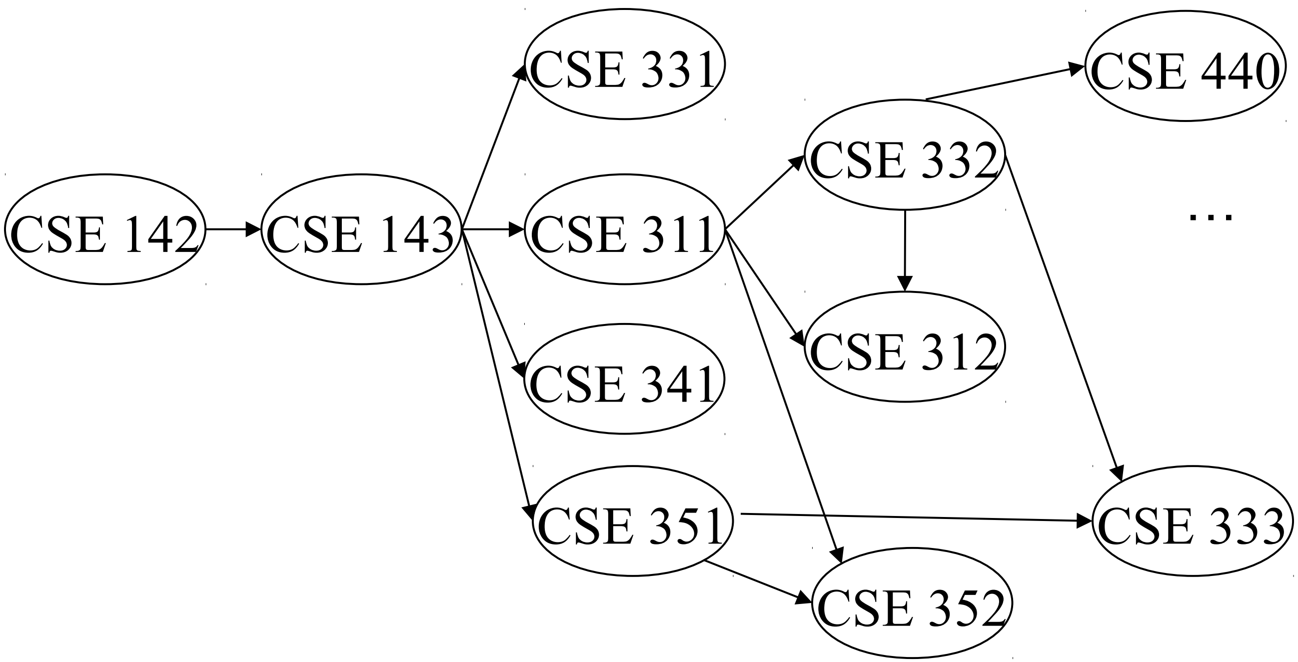
Output: 142



Node:	142	143	311	312	331	332	333	341	351	352	440
Removed?	x										
In-degree:	0	1 0	1	2	1	1	2	1	1	2	1

Example

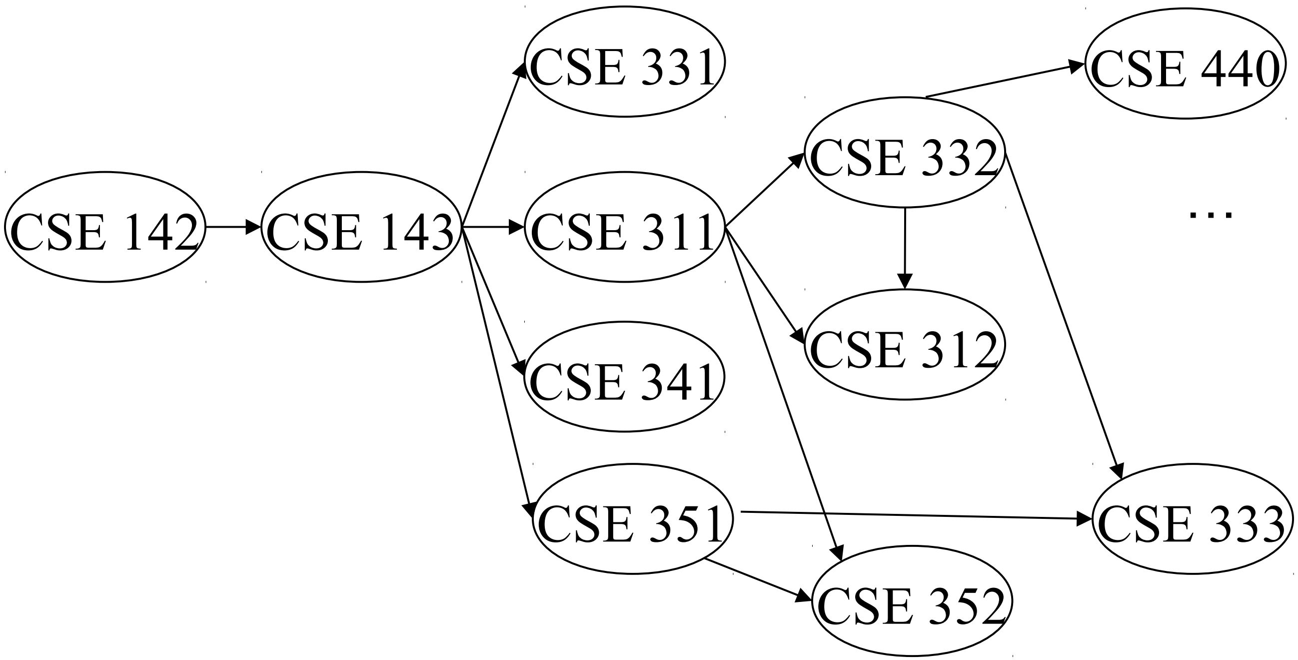
Output: 142
143



Node:	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x									
In-degree:	0	1	1	2	1	1	2	1	1	2	1
		0	0		0			0	0		

Example

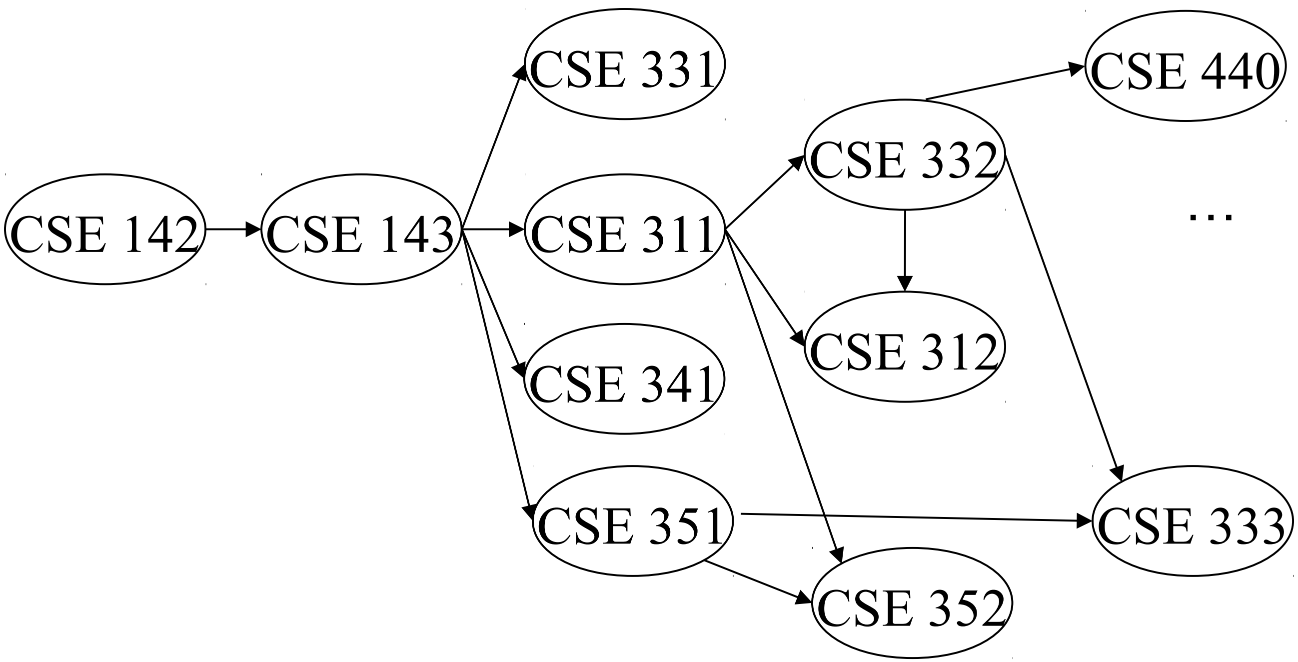
Output: 142
143
311



Node:	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x	x								
In-degree:	0	1	1	2	1	1	2	1	1	2	1
		0	0	1	0	0		0	0	1	

Example

Output: 142
143
311
331



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Example

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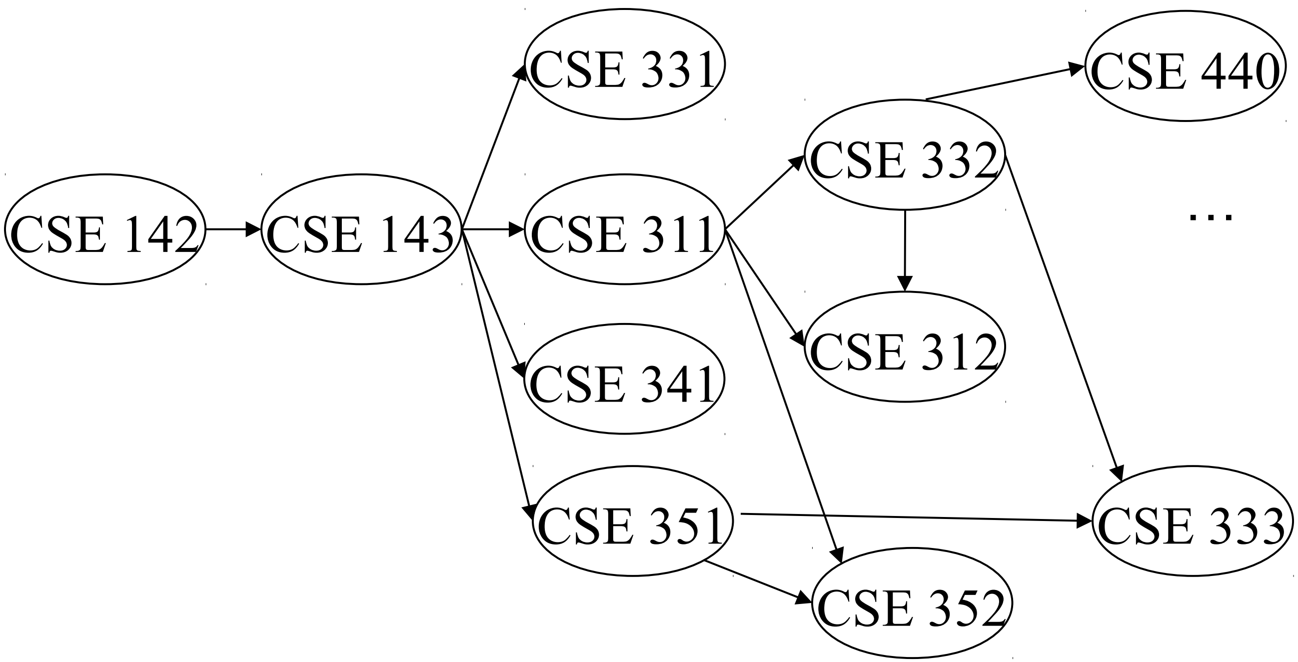

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In-degree:	0	1	1	2	1	1	2	1	1	2	1
		0	0	1	0	0	1	0	0	1	0
				0							

Example

Output: 142
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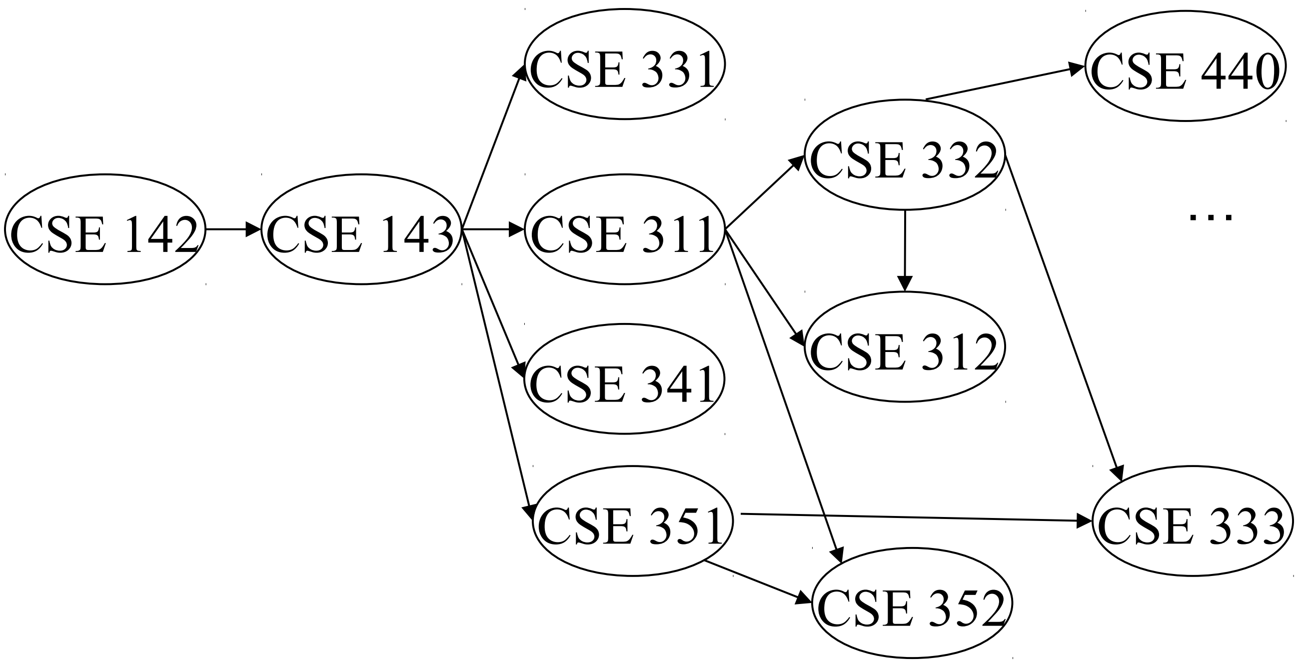

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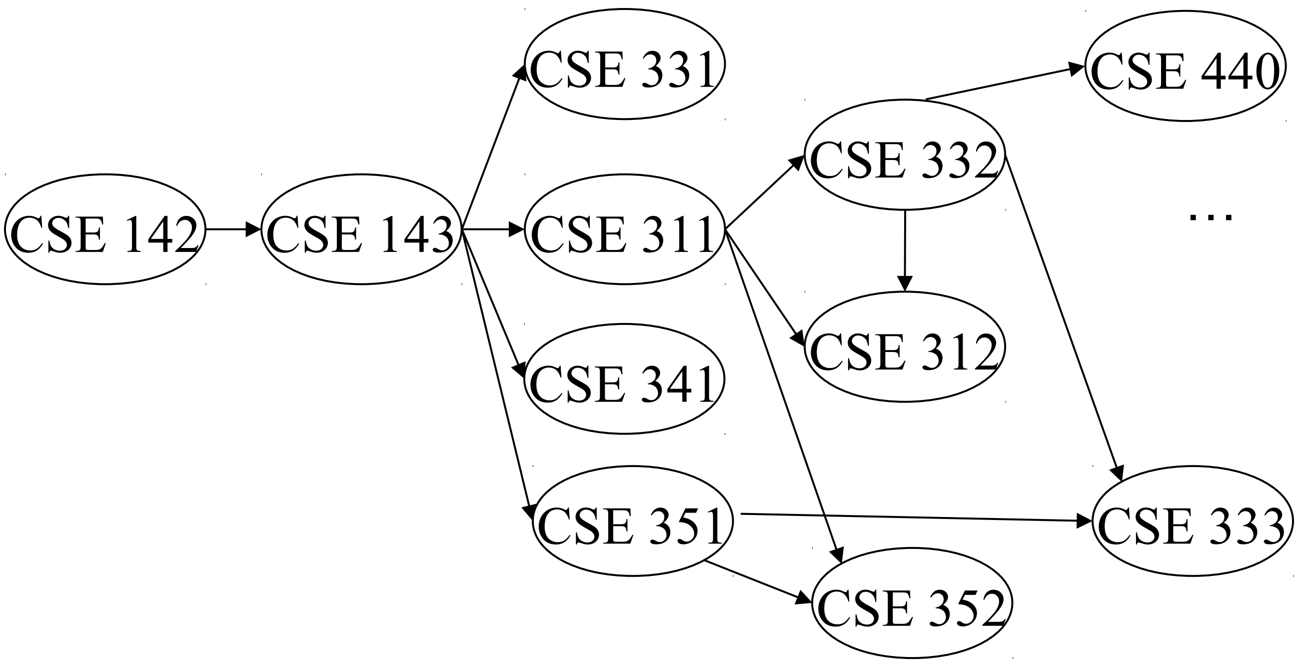
In-degree:	0	1	1	2	1	1	2	1	1	2	1
		0	0	1	0	0	1	0	0	1	0
				0							

Example



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Removed?	x	x	x	x	x	x		x			
In-degree:	0	1	1	2	1	1	2	1	1	2	1
		0	0	1	0	0	1	0	0	1	0
				0							

Example

Output: 142
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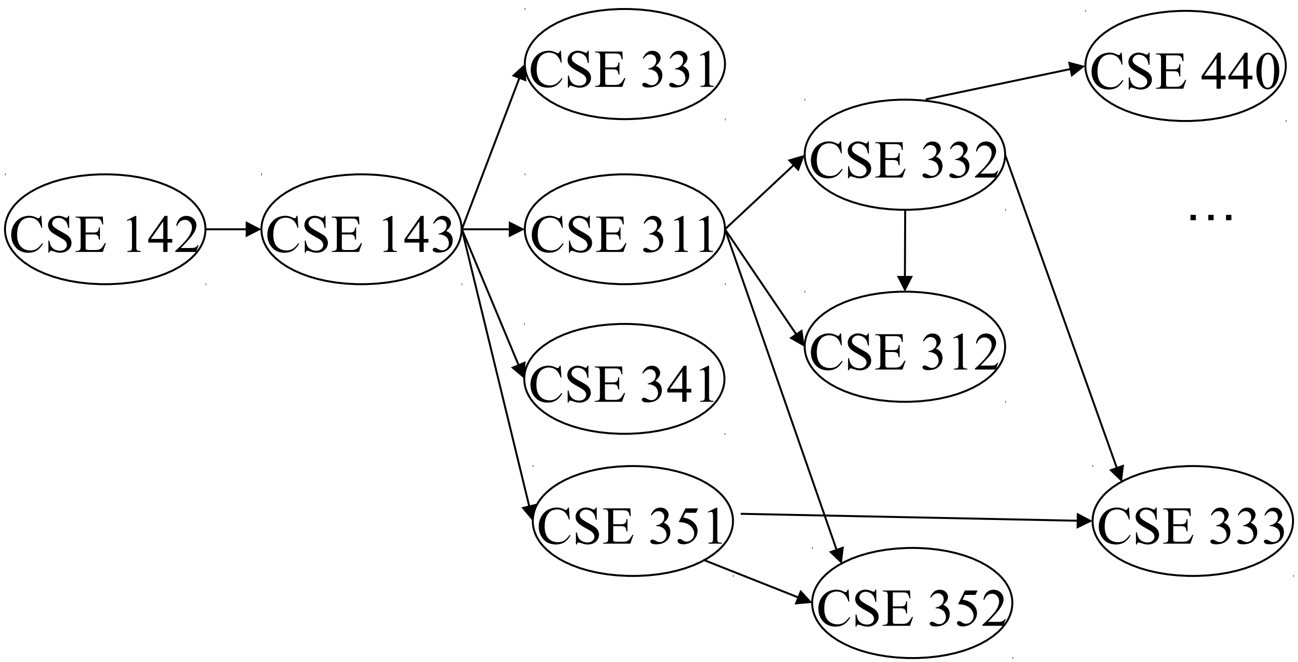

Node:	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x	x	x	x	x		x	x		
In-degree:	0	1	1	2	1	1	2	1	1	2	1
		0	0	1	0	0	1	0	0	1	0
				0			0			0	

Example

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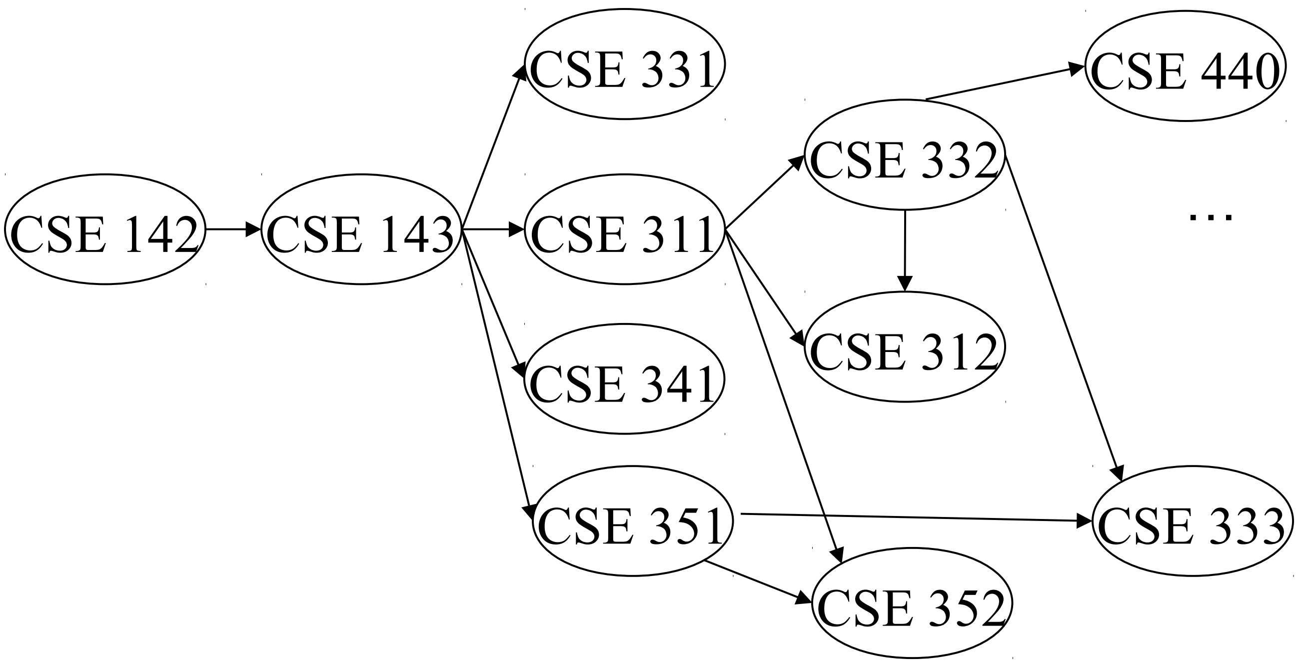


Node:	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x	x	x	x	x	x	x	x	x	x
In-degree:	0	1	1	2	1	1	2	1	1	2	1
		0	0	1	0	0	1	0	0	1	0
				0			0			0	

Running time?

```
labelEachVertexWithItsInDegree();
    for(ctr=0; ctr < numVertices; ctr++){
  v = findNewVertexOfDegreeZero();
  put v next in output
      for each w adjacent to v
    w.indegree--;
}
```

Running time?

```
labelEachVertexWithItsInDegree();
    for(ctr=0; ctr < numVertices; ctr++){
  v = findNewVertexOfDegreeZero();
  put v next in output
      for each w adjacent to v
    w.indegree--;
}
```

- What is the worst-case running time?
  - Initialization $O(|V|+|E|)$ (assuming adjacency list)
  - Sum of all find-new-vertex $O(|V|^2)$ (because each $O(|V|)$)
  - Sum of all decrements $O(|E|)$ (assuming adjacency list)
  - So total is $O(|V|^2)$ – not good for a sparse graph!

Doing better

The trick is to avoid searching for a zero-degree node every time!

- Keep the "pending" zero-degree nodes in a list, stack, queue, bag, table, or something
- Order we process them affects output but not correctness or efficiency provided add/remove are both $O(1)$

Using a queue:

1. Label each vertex with its in-degree, enqueue 0-degree nodes
2. While queue is not empty
   - **v** = dequeue()
   - Output **v** and remove it from the graph
   - For each vertex **u** adjacent to **v** (i.e. **u** such that (**v**,**u**) in **`E`**), decrement the in-degree of **u**, if new degree is 0, enqueue it

Running time?

```
labelAllAndEnqueueZeros();
    for(ctr=0; ctr < numVertices; ctr+
+){
  v = dequeue();
  put v next in output
      for each w adjacent to v {
    w.indegree--;
    if(w.indegree==0)
      enqueue(v);
  }
}
```

Running time?

```
labelAllAndEnqueueZeros();
    for(ctr=0; ctr < numVertices; ctr+
+){
  v = dequeue();
  put v next in output
      for each w adjacent to v {
    w.indegree--;
    if(w.indegree==0)
      enqueue(v);
  }
}
```

- What is the worst-case running time?
  - Initialization: $O(|V|+|E|)$ (assuming adjacenty list)
  - Sum of all enqueues and dequeues: $O(|V|)$
  - Sum of all decrements: $O(|E|)$ (assuming adjacency list)
  - So total is $O(|E| + |V|)$ – much better for sparse graph!

Graph Traversals

Next problem: For an arbitrary graph and a starting node **v**, find all nodes *reachable* from **v** (i.e., there exists a path)

- Possibly “do something” for each node
- Examples: print to output, set a field, return from iterator, etc.

Related problems:

- Is an undirected graph connected?
- Is a directed graph weakly / strongly connected?
  - For strongly, need a cycle back to starting node

Basic idea:

- Keep following nodes
- But “mark” nodes after visiting them, so the traversal terminates and processes each reachable node exactly once

Abstract Idea

```
traverseGraph(Node start) {
        Set pending = emptySet();
        pending.add(start)
    mark start as visited
    while(pending is not empty) {
      next = pending.remove()
      for each node u adjacent to next
          if(u is not marked) {
            mark u
            pending.add(u)
          }
    }
}
```

Running Time and Options

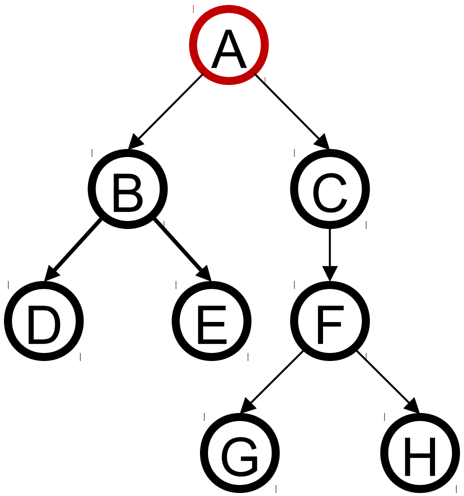
- Assuming add and remove are $O(1)$, entire traversal is $O(|E|)$
  - Use an adjacency list representation

- The order we traverse depends entirely on add and remove
  - Popular choice: a stack "depth-first graph search" "DFS"
  - Popular choice: a queue "breadth-first graph search" "BFS"

- DFS and BFS are "big ideas" in computer science
  - Depth: recursively explore one part before going back to the other parts not yet explored
  - Breadth: explore areas closer to the start node first

Example: trees

- A tree is a graph and DFS and BFS are particularly easy to "see"




```
DFS(Node start) {
  mark and process start
  for each node u adjacent to start
    if u is not marked
      DFS(u)
}
```

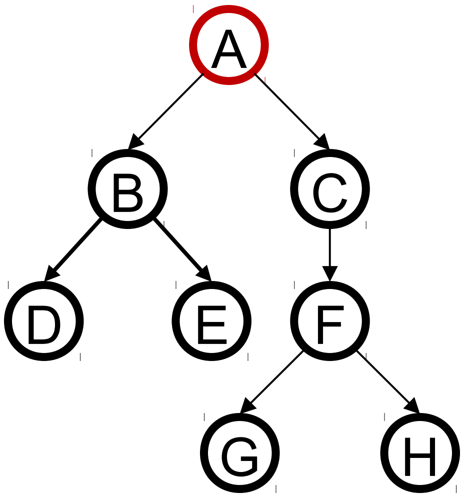
A, B, D, E, C, F, G, H

Exactly what we called a "pre-order traversal" for trees

– The marking is because we support arbitrary graphs and we want to process each node exactly once

Example: trees

- A tree is a graph and DFS and BFS are particularly easy to "see"

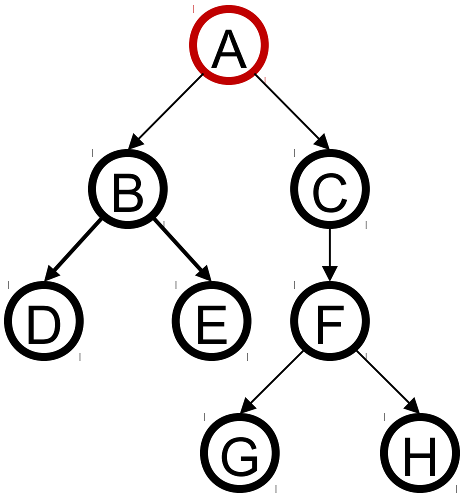



```
DFS2(Node start) {
  initialize stack s to hold start
  mark start as visited
  while(s is not empty) {
    next = s.pop() // and "process"
    for each node u adjacent to next
     if(u is not marked)
       mark u and push onto s
  }
}
```

A, C, F, H, G, B, E, D
A different but perfectly fine traversal

Example: trees

- A tree is a graph and DFS and BFS are particularly easy to "see"




```
BFS(Node start) {
   initialize queue q to hold start
   mark start as visited
   while(q is not empty) {
      next = q.dequeue() // and "process"
      for each node u adjacent to next
       if(u is not marked)
          mark u and enqueue onto q
   }
}
```

A, B, C, D, E, F, G, H
A "level-order" traversal

Comparison

- Breadth-first always finds shortest paths, i.e., "optimal solutions"
  - Better for "what is the shortest path from `x` to `y`"

- But depth-first can use less space in finding a path
  - If *longest path* in the graph is `p` and highest out-degree is `d` then DFS stack never has more than `d*p` elements
  - But a queue for BFS may hold $O(|V|)$ nodes

- A third approach:
  - *Iterative deepening (IDFS)*:
    - Try DFS but disallow recursion more than `K` levels deep
    - If that fails, increment `K` and start the entire search over

a) Like BFS, finds shortest paths. Like DFS, less space.

Saving the Path

- Our graph traversals can answer the reachability question:
  - "Is there a path from node x to node y?"
- But what if we want to actually output the path?
  - Like getting driving directions rather than just knowing it's possible to get there!
- Easy:
  - Instead of just "marking" a node, store the previous node along the path (when processing **`u`** causes us to add **`v`** to the search, set **`v.path`** field to be **`u`**)
  - When you reach the goal, follow **`path`** fields back to where you started (and then reverse the answer)
  - If just wanted path *length*, could put the integer distance at each node instead

Example using BFS

What is a path from Seattle to Tyler

- Remember marked nodes are not re-enqueued
- Note shortest paths may not be unique