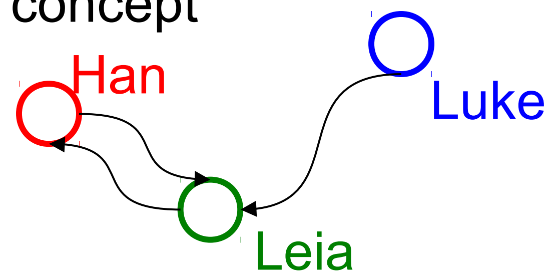
Graphs

- A graph is a formalism for representing relationships among items
  - Very general definition because very general concept




```
V = {Han,Leia,Luke}
E = {(Luke,Leia),
     (Han,Leia),
     (Leia,Han)}
```

- A graph is a pair

  $G = (V,E)$

  - A set of vertices, also known as nodes

    $V = \{v_1, v_2, \ldots, v_n\}$

  - A set of edges

    $E = \{e_1, e_2, \ldots, e_m\}$

    - Each edge e_i is a pair of vertices

      (v_j, v_k)

    - An edge "connects" the vertices

- Graphs can be directed or undirected

An ADT?

- Can think of graphs as an ADT with operations like
  `isEdge((vj,vk))` ($isEdge((v_j,v_k))$)

- But it is unclear what the "standard operations" are

- Instead we tend to develop algorithms over graphs and then use data structures that are efficient for those algorithms

- Many important problems can be solved by:
  1. Formulating them in terms of graphs
  2. Applying a standard graph algorithm

- To make the formulation easy and standard, we have a lot of *standard terminology* about graphs

Some Graphs

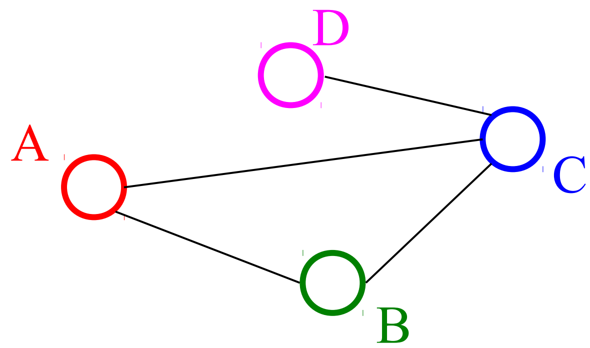
For each, what are the vertices and what are the edges?

- Web pages with links
- Facebook friends
- "Input data" for the Kevin Bacon game
- Methods in a program that call each other
- Road maps (e.g., Google maps)
- Airline routes
- Family trees
- Course pre-requisites
- …

Wow: Using the same algorithms for problems across so many domains sounds like "core computer science and engineering"

Undirected Graphs

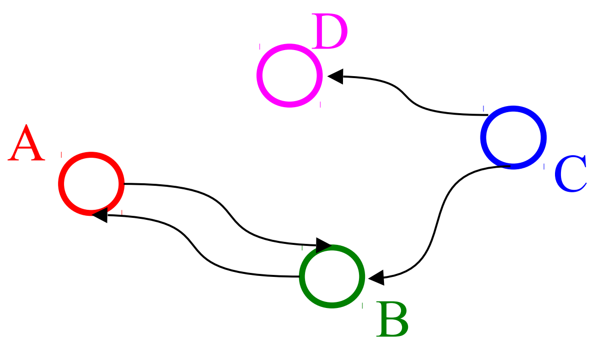
- In undirected graphs, edges have no specific direction
  - Edges are always “two-way”



- Thus, `(u,v)` × `E` implies `(v,u)` × `E`
  - Only one of these edges needs to be in the set
  - The other is implicit, so normalize how you check for it
- Degree of a vertex: number of edges containing that vertex
  - Put another way: the number of adjacent vertices

Directed Graphs

- In directed graphs (sometimes called digraphs), edges have a direction



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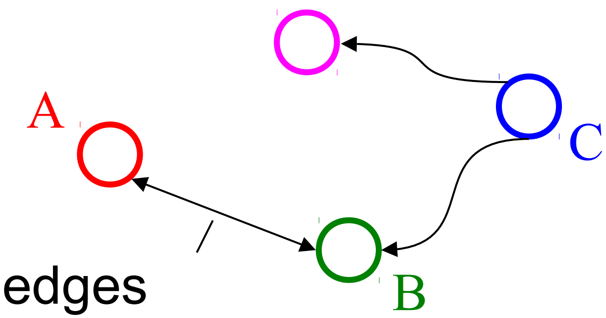


2 edges
here

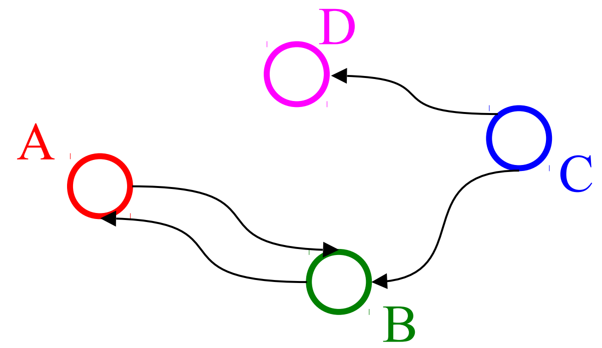
- Thus, `(u,v)` × `E` does *not* imply `(v,u)` × `E`.
  - Let `(u,v)` × `E` mean u → v
  - Call `u` the source and `v` the destination
- In-Degree of a vertex: number of in-bound edges, i.e., edges where the vertex is the destination
- Out-Degree of a vertex: number of out-bound edges i.e., edges where the vertex is the source

Self-Edges, Connectedness

- A self-edge a.k.a. a loop is an edge of the form **`(u,u)`**
  - Depending on the use/algorithm, a graph may have:
    - No self edges
    - Some self edges
    - All self edges (often therefore implicit, but we will be explicit)
- A node can have a degree / in-degree / out-degree of zero
- A graph does not have to be connected
  - Even if every node has non-zero degree

More Notation

For a graph **`G = (V,E)`**

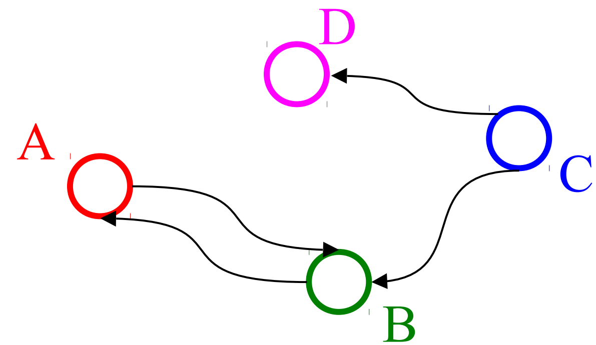


- **`|V|`** is the number of vertices
- **`|E|`** is the number of edges
  - Minimum?
  - Maximum for undirected?
  - Maximum for directed?

```
V = {A, B, C, D}
E = {(C, B),
     (A, B),
     (B, A)
     (C, D)}
```

- If **`(u,v)`** $\times$ **`E`**
  - Then **`v`** is a neighbor of **`u`**, i.e., **`v`** is adjacent to **`u`**
  - Order matters for directed edges
    - **`u`** is not adjacent to **`v`** unless **`(v,u)`** $\times$ **`E`**

More notation



For a graph **G=(V,E)**:

- **|V|** is the number of vertices
- **|E|** is the number of edges
  - Minimum? 0
  - Maximum for undirected? $|V||V+1|/2 \times O(|V|^2)$
  - Maximum for directed? $|V|^2 \times O(|V|^2)$

(assuming self-edges allowed, else subtract **|V|**)

- If $(u,v) \times E$
  - Then **v** is a neighbor of **u**, i.e., **v** is adjacent to **u**
  - Order matters for directed edges
    - **u** is not adjacent to **v** unless $(v,u) \times E$

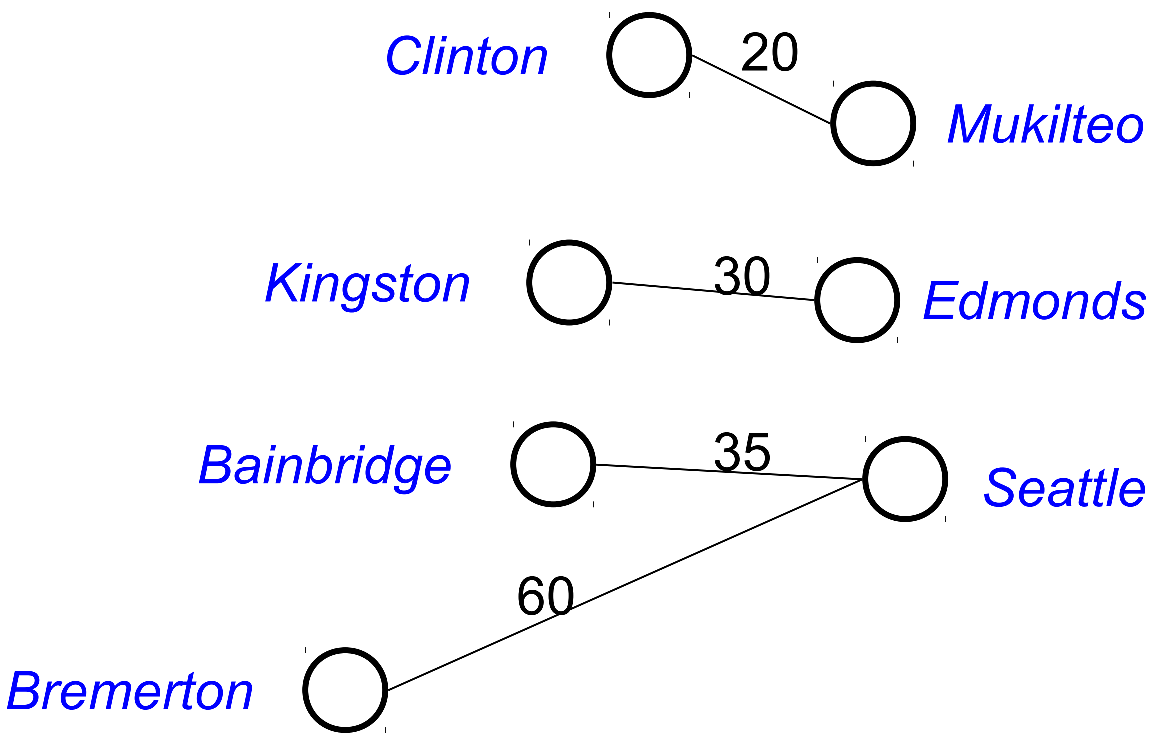
Examples again

Which would use directed edges? Which would have self-edges? Which would be connected? Which could have 0-degree nodes?

- Web pages with links
- Facebook friends
- “Input data” for the Kevin Bacon game
- Methods in a program that call each other
- Road maps (e.g., Google maps)
- Airline routes
- Family trees
- Course pre-requisites
- …

Weighted Graphs

- In a weighed graph, each edge has a weight a.k.a. cost
  - Typically numeric (most examples use ints)
  - *Orthogonal* to whether graph is directed
  - Some graphs allow *negative weights*; many do not



Examples

What, if anything, might weights represent for each of these?
Do negative weights make sense?

- Web pages with links
- Facebook friends
- “Input data” for the Kevin Bacon game
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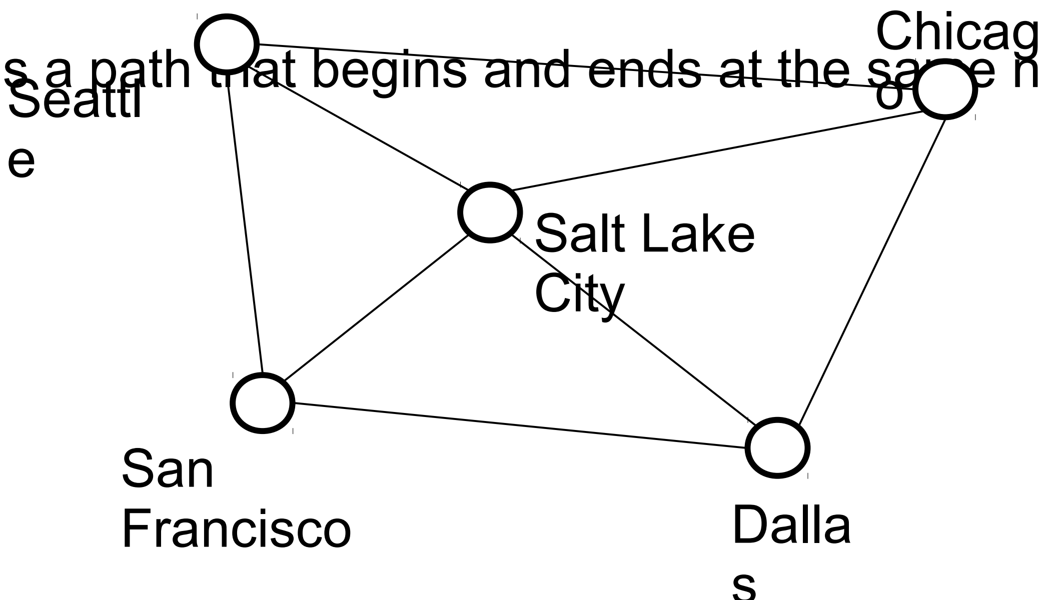
Paths and Cycles

- A path is a list of vertices $\mathbf{[v_0,v_1,\ldots,v_n]}$ such that $\mathbf{(v_i,v_{i+1})} \times \mathbf{E}$ for all $\mathbf{0}$ □ $\mathbf{i < n.}$ Say *"a path from* $\mathbf{v_0}$ *to* $\mathbf{v_n}$*"*

- A cycle is a path that begins and ends at the same node ($\mathbf{v_0 == v_n}$)



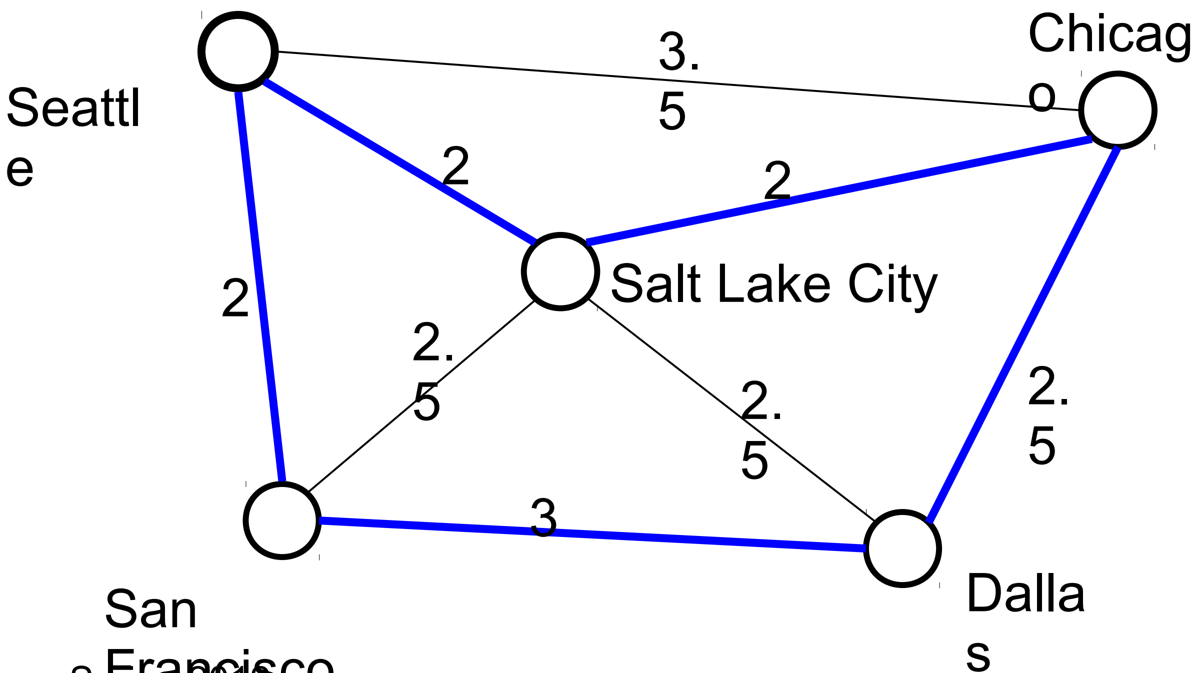
Example: [Seattle, Salt Lake City, Chicago, Dallas, San Francisco, Seattle]

Path Length and Cost

- Path length: Number of *edges* in a path
- Path cost: Sum of *weights* of edges in a path

Example where

P= [Seattle, Salt Lake City, Chicago, Dallas, San Francisco, Seattle]



length(P) = 5
cost(P) = 11.5

Simple Paths and Cycles

- A simple path repeats no vertices, except the first might be the last

  [Seattle, Salt Lake City, San Francisco, Dallas]

  [Seattle, Salt Lake City, San Francisco, Dallas, Seattle]

- Recall, a cycle is a path that ends where it begins

  [Seattle, Salt Lake City, San Francisco, Dallas, Seattle]

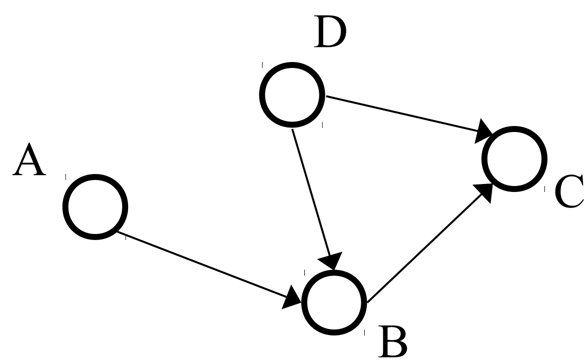
  [Seattle, Salt Lake City, Seattle, Dallas, Seattle]

- A simple cycle is a cycle and a simple path

  [Seattle, Salt Lake City, San Francisco, Dallas, Seattle]

Paths and Cycles in Directed Graphs

Example:

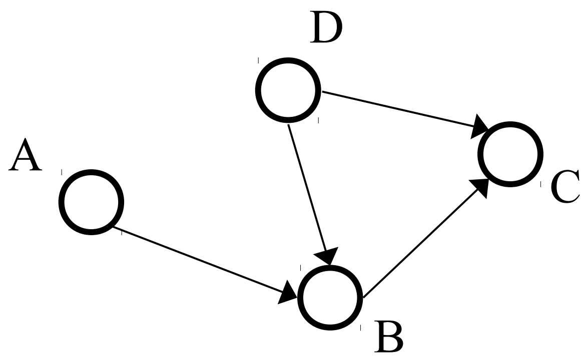


Is there a path from A to D?

Does the graph contain any cycles?

Paths and Cycles in Directed Graphs

Example:

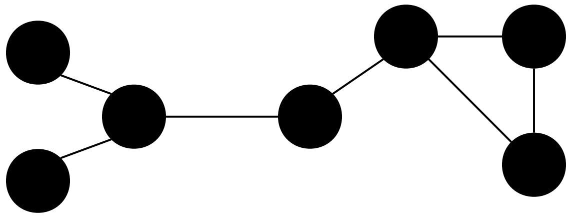


Is there a path from A to D? No

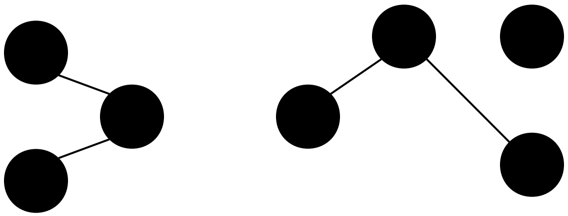
Does the graph contain any cycles? No

Undirected-Graph Connectivity

- An undirected graph is connected if for all
  pairs of vertices **u,v**, there exists a *path* from **u** to **v**

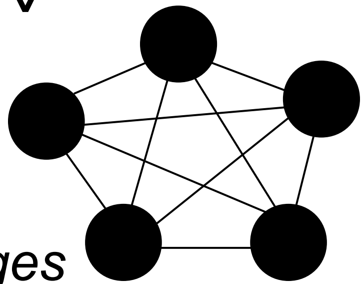


Connected graph



Disconnected graph

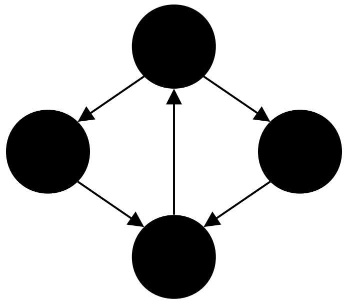
- An undirected graph is complete, a.k.a. fully connected if for *all* pairs of vertices **u,v**, there exists an *edge* from **u** to **v**



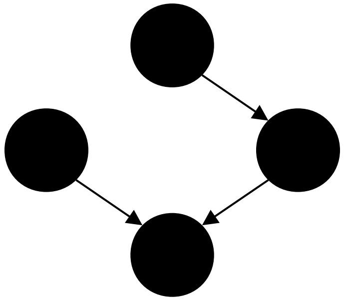
plus self edges

Directed-Graph Connectivity

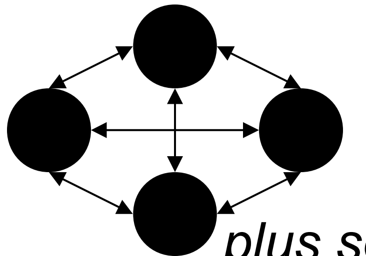
- A directed graph is strongly connected if there is a path from every vertex to every other vertex



- A directed graph is weakly connected if there is a path from every vertex to every other vertex *ignoring direction of edges*



- A complete a.k.a. fully connected directed graph has an edge from every vertex to every other vertex



plus self edges

Examples

For undirected graphs: connected?

For directed graphs: strongly connected? weakly connected?

- Web pages with links
- Facebook friends
- “Input data” for the Kevin Bacon game
- Methods in a program that call each other
- Road maps (e.g., Google maps)
- Airline routes
- Family trees
- Course pre-requisites
- …

Trees as Graphs

When talking about graphs,

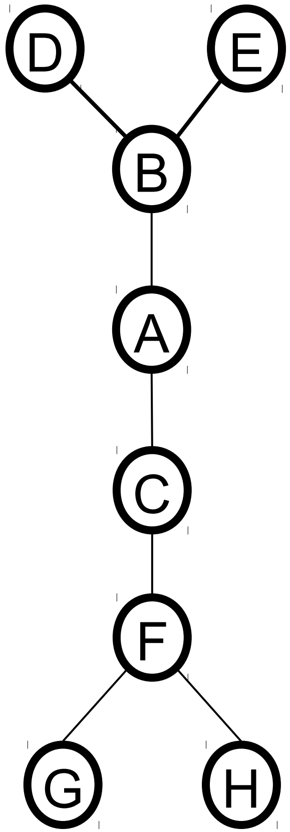
we say a tree is a graph that is:

- undirected
- acyclic
- connected

So all trees are graphs, but not all graphs are trees

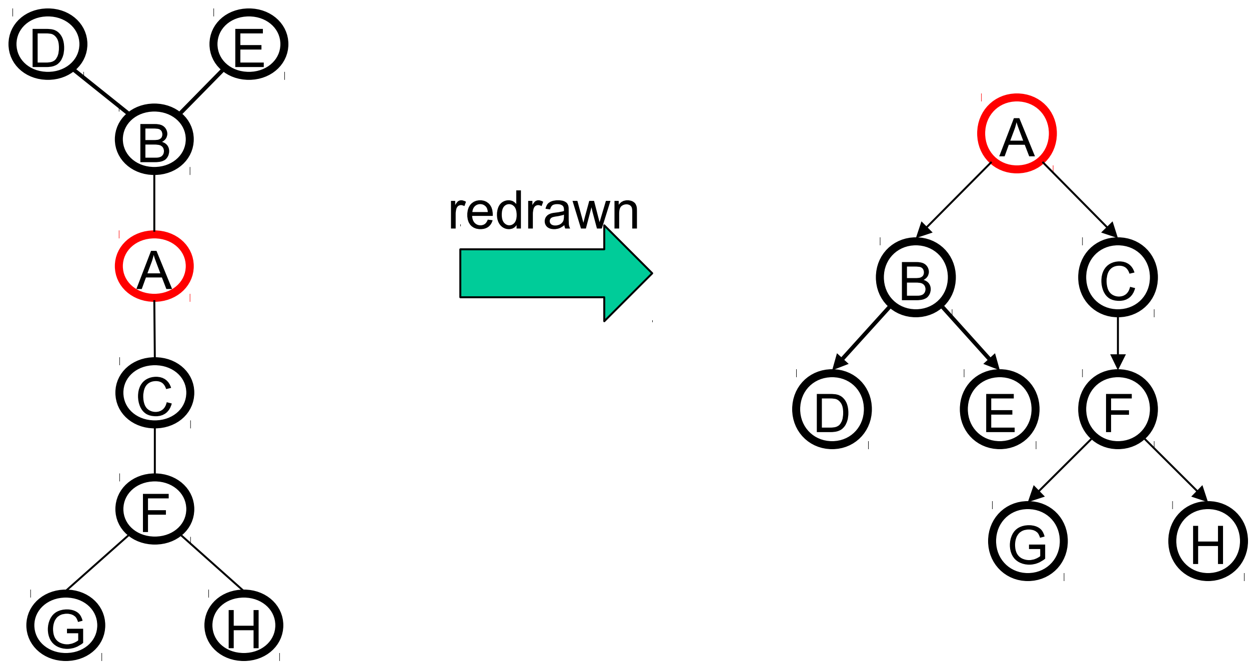
How does this relate to the trees we know and love?...

Example:



Rooted Trees

- We are more accustomed to rooted trees where:
  - We identify a unique root
  - We think of edges are directed: parent to children
- Given a tree, picking a root gives a unique rooted tree
  - The tree is just drawn differently and with undirected edges



Rooted Trees

- We are more accustomed to rooted trees where:
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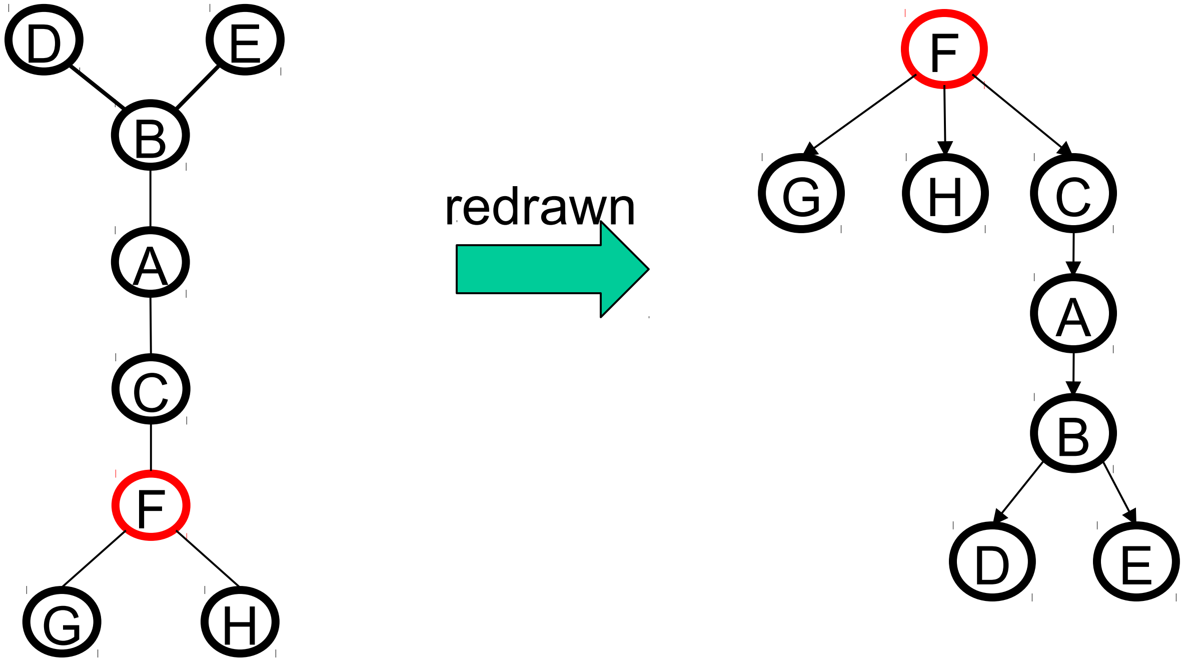
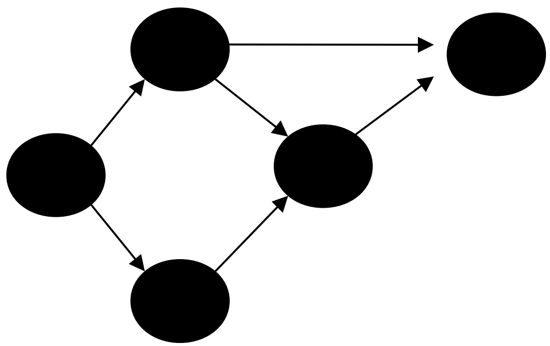
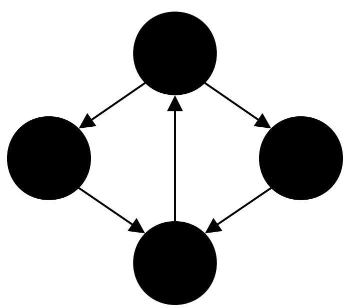


Directed Acyclic Graphs (DAGs)

- A DAG is a directed graph with no (directed) cycles
  - Every rooted directed tree is a DAG
  - But not every DAG is a rooted directed tree



1. Every DAG is a directed graph
2. But not every directed graph is a DAG



Examples

Which of our directed-graph examples do you expect to be a DAG?

- Web pages with links
- “Input data” for the Kevin Bacon game
- Methods in a program that call each other
- Airline routes
- Family trees
- Course pre-requisites
- …

Density / Sparsity

- Recall: In an undirected graph, $0 \leq |E| < |V|^2$
- Recall: In a directed graph: $0 \leq |E| \leq |V|^2$
- So for any graph, $O(|E|+|V|^2)$ is $O(|V|^2)$

- Another fact: If an undirected graph is *connected*, then $|V|-1 \leq |E|$

- Because $|E|$ is often much smaller than its maximum size, we do not always approximate $|E|$ as $O(|V|^2)$
  - This is a correct bound, it just is often not tight
  - If it is tight, i.e., $|E|$ is $\square\ (|V|^2)$ we say the graph is dense
    - More sloppily, dense means “lots of edges”
  - If $|E|$ is $O(|V|)$ we say the graph is sparse
    - More sloppily, sparse means “most possible edges missing”

What is the Data Structure?

- So graphs are really useful for lots of data and questions
  - For example, “what’s the lowest-cost path from x to y”

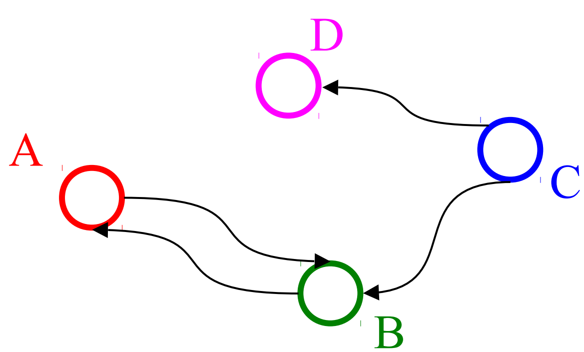
- But we need a data structure that represents graphs

- The “best one” can depend on:
  - Properties of the graph (e.g., dense versus sparse)
  - The common queries (e.g., “is **`(u,v)`** an edge?” versus “what are the neighbors of node **`u`**?”)

- So we’ll discuss the two standard graph representations
  1. Adjacency Matrix and Adjacency List
  - Different trade-offs, particularly time versus space

Adjacency Matrix

- Assign each node a number from `0` to `|V|-1`
- A `|V|` x `|V|` matrix (i.e., 2-D array) of Booleans (or 1 vs. 0)
  - If `M` is the matrix, then `M[u][v]` being `true` means there is an edge from `u` to `v`



	A	B	C	D
A	F	T	F	F
B	T	F	F	F
C	F	T	F	T
D	F	F	F	F

Adjacency Matrix Properties

	A	B	C	D
A	F	T	F	F
B	T	F	F	F
C	F	T	F	T
D	F	F	F	F

- Running time to:
  - Get a vertex's out-edges:
  - Get a vertex's in-edges:
  - Decide if some edge exists:
  - Insert an edge:
  - Delete an edge:

- Space requirements:

- Best for sparse or dense graphs?

Adjacency Matrix Properties

	A	B	C	D
A	F	T	F	F
B	T	F	F	F
C	F	T	F	T
D	F	F	F	F

- Running time to:
  - Get a vertex's out-edges: $O(|V|)$
  - Get a vertex's in-edges: $O(|V|)$
  - Decide if some edge exists: $O(1)$
  - Insert an edge: $O(1)$
  - Delete an edge: $O(1)$

- Space requirements:
  1. $|V|^2$ bits

- Best for sparse or dense graphs?
  1. Best for dense graphs

Adjacency Matrix Properties

- How will the adjacency matrix vary for an *undirected graph*?

- How can we adapt the representation for *weighted graphs*?

	A	B	C	D
A	F	T	F	F
B	T	F	F	F
C	F	T	F	T
D	F	F	F	F

Adjacency Matrix Properties

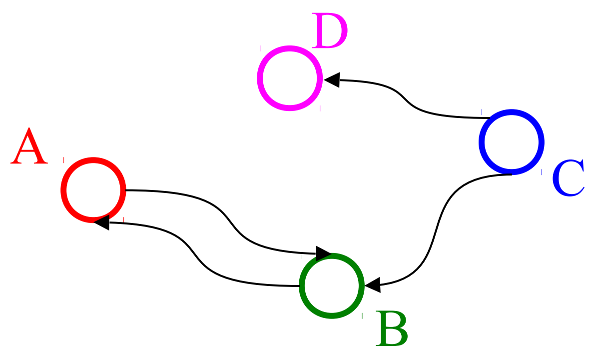
- How will the adjacency matrix vary for an *undirected graph*?
  1. Undirected will be symmetric about diagonal axis

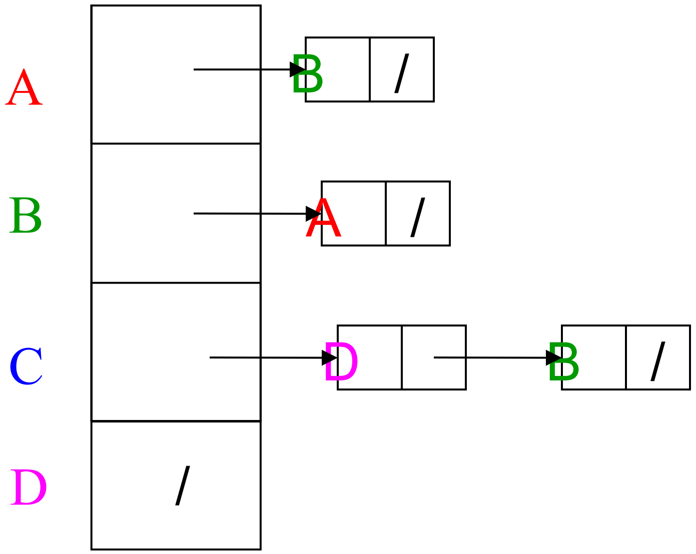
- How can we adapt the representation for *weighted graphs*?
  1. Instead of a Boolean, store a number in each cell
  2. Need some value to represent ‘not an edge’
     - In *some* situations, 0 or -1 works

	A	B	C	D
A	F	T	F	F
B	T	F	F	F
C	F	T	F	T
D	F	F	F	F

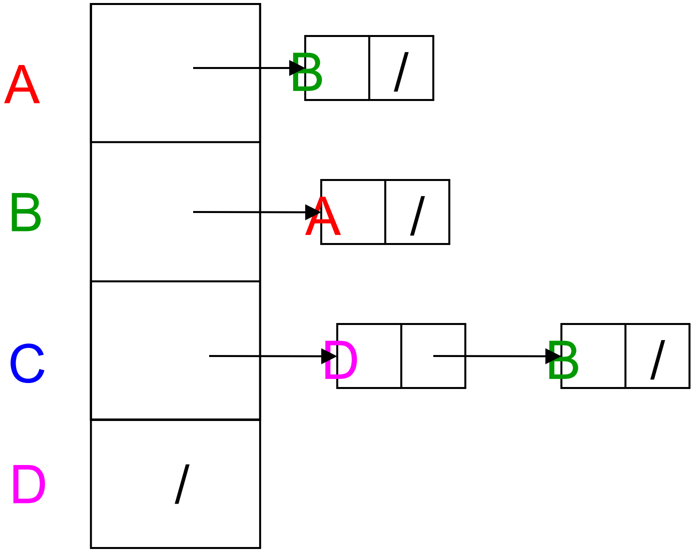
Adjacency List

- Assign each node a number from **`0`** to **`|V|-1`**
- An array of length **`|V|`** in which each entry stores a list of all adjacent vertices (e.g., linked list)



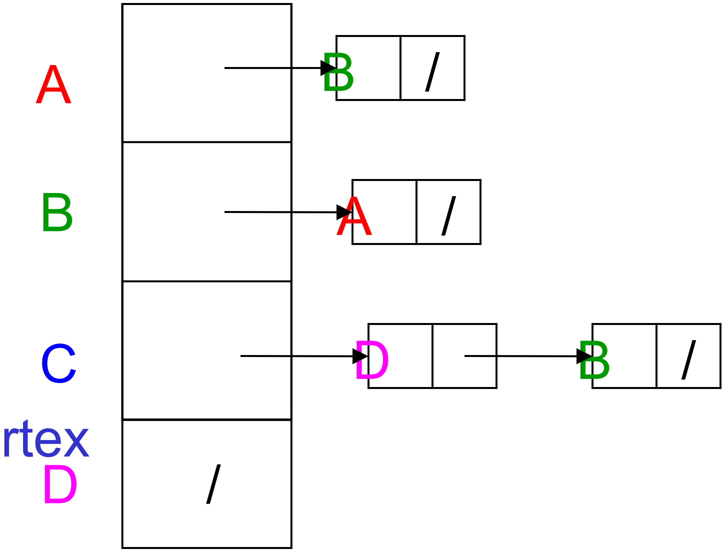


Adjacency List Properties



- Running time to:
  - Get all of a vertex's out-edges:
  - Get all of a vertex's in-edges:
  - Decide if some edge exists:
  - Insert an edge:
  - Delete an edge:
- Space requirements:
- Best for dense or sparse graphs?

Adjacency List Properties



- Running time to:
  - Get all of a vertex's out-edges:

    $O(d)$ where d is out-degree of vertex
  - Get all of a vertex's in-edges:

    O(|E|) (but could keep a second adjacency list for this!)
  - Decide if some edge exists:

    $O(d)$ where d is out-degree of source
  - Insert an edge: $O(1)$ (unless you need to check if it's there)
  - Delete an edge: $O(d)$ where d is out-degree of source

- Space requirements:
  1. $O(|V|+|E|)$

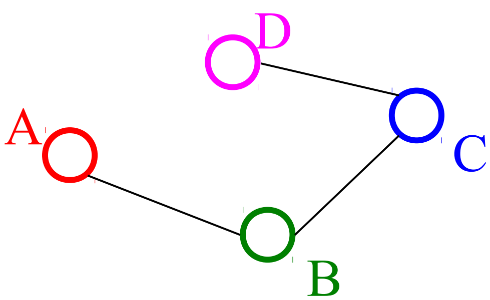
- Best for dense or sparse graphs?
  1. Best for sparse graphs, so usually just stick with linked lists

Undirected Graphs

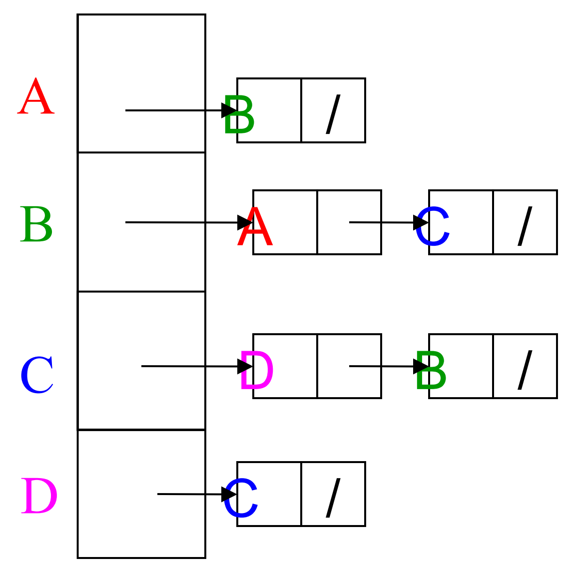
Adjacency matrices & adjacency lists both do fine for undirected graphs

- Matrix: Can save roughly 2x space
  - But may slow down operations in languages with "proper" 2D arrays (not Java, which has only arrays of arrays)
  - How would you "get all neighbors"?
- Lists: Each edge in two lists to support efficient "get all neighbors"

Example:



	A	B	C	D
A	F	T	F	F
B	T	F	T	F
C	F	T	F	T
D	F	F	T	F



Next…

Okay, we can represent graphs

Now let's implement some useful and non-trivial algorithms

- Topological sort: Given a DAG, order all the vertices so that every vertex comes before all of its neighbors
- Shortest paths: Find the shortest or lowest-cost path from x to y
  – Related: Determine if there even is such a path