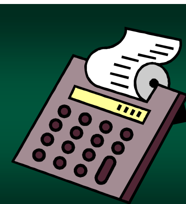
Counting

Part 9

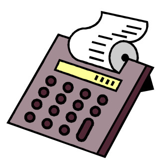


Counting

And a 1 and a 2…

Counting

- A major study of discrete mathematics is how, well, numbers actually work
- This is *counting theory* and is fundamental to math



Basic Principles

- Counting problems come in many forms:
  - "how many different basketball teams can be selected from a bench of 20 players?"
  - "how many unique 7 characters passwords are there?"
- So, counting is used to compute the probabilities of events

Sum Rule

- Many tasks may be part of the same overall scenario, but are not dependent on each other
- The *sum rule* is used to combine two mutually exclusive tasks

Sum Rule

1. Task T_1 can be done in n_1 ways and task T_2 can be done in n_2 ways
2. These two tasks are not dependent on each other
3. Then there are $n_1 + n_2$ ways to do both

Sum Rule Example

- The University will award a free tablet to either a computer science student or art student
- There are 187 computer science students and 102 art students.
- How many different choices are there?

Sum Rule Examples

Since the number of computer science and art students are independent from each other, we will use the Sum Rule.

So, there are 187 + 102 = 289 choices

Product Rule

- Many tasks are dependent on each other
- The second task can have a different possibilities *based on the first task*
- The *product rule* combines two related tasks

Product Rule

1. Task T_1 can be done in n_1 ways and a task T_2 can be done in n_2 ways
2. For each of the n_1 ways to do T_1 – there are n_2 different ways to do T_2
3. Then there are $n_1 \times n_2$ ways to do both

Product Rule Example

- A computer system is implementing a very, very simple security code. It will use PIN passwords consisting of 4 lowercase letters.
- How many unique PINs can be created?

Product Rule Example

Since, a PIN of "ab" is not the same as "ba", the two tasks are dependent on one another. As a result, we use the product rule.

So there are 26 × 26 × 26 × 26 = 456,976 total unique pins

Sum and Product in Sets

Back to the Beginning
(of the semester)

Sum and Product in Sets

- Both the sum and product rules can be defined using set theory
- When you think about it, all the different values (e.g. PIN digit) is a set of values

Sum and Product in Sets

- And, previously, we have covered operations on a set
- However, we are more interested in the number of elements

Sum Rule in Sets

- The sets A_1, A_2, …, A_k be disjoint – which means they do not intersect
- To combine the sets, we will union and retrieve the number of items

$$| A_1 \cup A_2 \cup \dots \cup A_k |$$

Sum Rule in Sets

- Since they are disjoint, there are no element in common
- For example, The union of the sets A_1 and A_2 will product | A1 | + | A2 | elements

$$|A_1 \cup A_2 \dots A_k| = |A_1| + |A_2| \dots |A_k|$$

Product Rule in Sets

- Assume that A_1, A_2, …, A_k are finite sets
- Each set is dependent on the previous set (e.g. a password)

$$| A_1 \times A_2 \times \dots A_k |$$

Product Rule in Sets

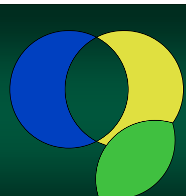
- Given two sets, the number items is the cross-product of two sets
- So, the number of items would be $| A_1 \times A_2 |$

$$| A_1 \times A_2 \times \ldots A_k |$$

Product Rule in Sets

- Since we are not worried about duplicate elements (for now), we can multiply the number of elements be each to get the same result

$$| A_1 \times A_2 \times \ldots A_k | = |A_1| \times |A_2| \times \ldots |A_k|$$

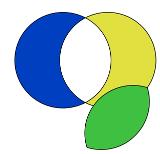


Inclusion-Exclusion in Counting

Removing values

Inclusion-Exclusion

- Sometimes the number of different ways a task can be performed is not straight forward
- As a result, the problem needs to be dissected to get a proper result



Example

How many 8 bit strings start with 1 or end with 11?

Example Analysis

- To analyze this problem, we need to recognize that this problem consists of several sub-problems
- Task 1: Construct a 8-bit strings of length 8 that starts with a 1
- Task 2: Construct a string of length 8 that ends with 11.

Example: Task 1

```
1 way  to pick the bit 7 (only 1)
2 ways to pick the bit 6 (0 or 1)
2 ways to pick the bit 5 (0 or 1)
2 ways to pick the bit 4 (0 or 1)
2 ways to pick the bit 3 (0 or 1)
2 ways to pick the bit 2 (0 or 1)
2 ways to pick the bit 1 (0 or 1)
2 ways to pick the bit 0 (0 or 1)
```


Example: Task 1

Using the product rule, Task 1 can be done

1 * 2 * 2 * 2 * 2 * 2 * 2 * 2
= 2^7 = 128 ways

Example: Task 2

```
2 ways to pick the bit 7 (0 or 1)
2 ways to pick the bit 6 (0 or 1)
2 ways to pick the bit 5 (0 or 1)
2 ways to pick the bit 4 (0 or 1)
2 ways to pick the bit 3 (0 or 1)
2 ways to pick the bit 2 (0 or 1)
1 way  to pick the bit 1 (only 1)
1 way  to pick the bit 0 (only 1)
```


Example: Task 2

Using the product rule, Task 2 can be done

2 * 2 * 2 * 2 * 2 * 2 * 1 * 1
= 2^6 = 64 ways

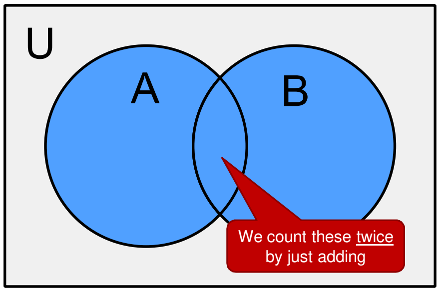
Example: Combining the Results

- So, there are of 128 ways to do Task 1
- …and 64 ways to do Task 2
- Does that mean that there are 128 + 64 ways to do both?
- **No…**

Why Not?

- Many of the bit-strings created by Task 1 will end in the 11 that is used in Task 2.
- *They are not mutually exclusive*
- The sum rule cannot be simply used!

Venn Diagram View



Example: Exclusion

- To find the result, we need to find the cases where Task 1 and Task 2 overlap
- Basically, we need to find this number and subtract it (exclude it) from the total
- *How many bit-strings start with 1 and end with 11?*

Example: Exclusion

```
1 way  to pick the bit 7 (only 1)
2 ways to pick the bit 6 (0 or 1)
2 ways to pick the bit 5 (0 or 1)
2 ways to pick the bit 4 (0 or 1)
2 ways to pick the bit 3 (0 or 1)
2 ways to pick the bit 2 (0 or 1)
1 way  to pick the bit 1 (only 1)
1 way  to pick the bit 0 (only 1)
```


Example: Exclusion

Using the product rule, the overlap is:

$1 * 2 * 2 * 2 * 2 * 1 * 1$

$= 2^5 = 32$ **overlaps**

Example: Exclusion

- So, the total number of values can be computed as the total of Task 1 + number of Task 2, but excluding the overlap.
- We add the first two, and remove overlap

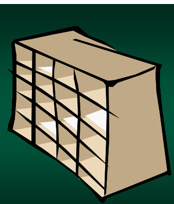
128 + 64 – 32 = 160 ways

Inclusion-Exclusion

- In set theory, this corresponds to sets A_1 and A_2 that are not disjoint.
- Just like before, we need to remove the overlap

$$|A \cup B| = |A| + |B| - |A \cap B|$$

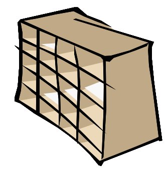
Pigeon-Hole Principle



Only So Many Boxes

Pigeon Hole Principle

- The *pigeonhole principle* states that if (k + 1) objects are placed into k boxes, then there is *at least* one box with two or more of the objects
- i.e. pigeons fitting into their nesting boxes



Example: Classes

If you have 6 classes from Monday to Friday…

Then there must be at least one day where you have at least two classes

Example: Hockey

If there are 11 players in a hockey team that wins 12-0…

Then there must be at least one player in the team who scored at least twice

Example: Birthday

Given 8 people…

At least one pair share a birthday, on the same day of the week this year.

Example: Birth Month

Given 100 people, how many will have the same birth month?

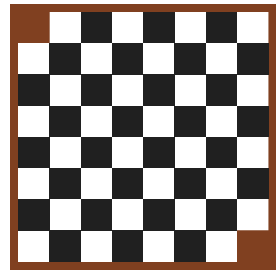
At least 9 will be born in the same month. At max, all will be born in the same month.

Example: Chessboard

Consider a chess board with two of the diagonally opposite corners removed.

Is it possible to cover the board with pieces of domino whose size is exactly two board squares

Example: Chessboard

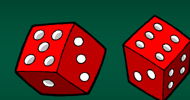


Example: Chessboard Analysis

No, it's not possible.

There are 62 squares. To cover them all you'd need 31 dominos. Each would cover 1 white 1 one black square. But, there are only 30 black squares and 32 white.

The pigeonhole principle: at least one of the black squares is covered twice

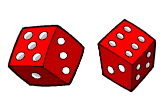


Probability

Don't Play The Lottery

Probability

- Often, we can calculate the probability that some event happens
- This is the ratio between the event occurring and all the possible outcomes



Probability

- A is the set of all outcomes that qualify the event as happening
- S is the set of all possible outcomes

```
P(A) = |A| / |S|
```

Probability

- We already have the tools to solve some pretty tough questions about the sizes of sets

```
P(A) = |A| / |S|
```


Example

How many integers in the range 1 to 300 are divisible by 3, 5 or 7?

Example

```
To compute the total, we need to
find | A ∪ B ∪ C | where:

A = numbers divisible by 3
B = numbers divisible by 5
C = numbers divisible by 7
```


Example: Find Counts

```
A = 300 div 3 → 100
B = 300 div 5 → 60
C = 300 div 7 → 42
```


Example

- It is more complex than it looks…
- Because 3, 5 and 7 are all prime…
  - only numbers divisible by both 3 and 5 are those numbers divisible by 15
  - Similarly for 3×7, 5×7 and $3\times5\times7$
- So, there is overlap between A, B, and C

Example: Find Overlaps

```
A ∩ B = 300 div 15 → 20
A ∩ C = 300 div 21 → 14
B ∩ C = 300 div 35 → 8
A ∩ B ∩ C = 300 div 105 → 2
```

Example: Inclusion-Exclusion

```
| A ∪ B ∪ C | =

   |A| + |B| + |C|
   - |A ∩ B|
   - |A ∩ C|
   - |B ∩ C|
   - |A ∩ B ∩ C|
```


Example: Inclusion-Exclusion

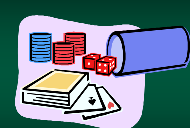
```
= 100 + 60 + 42
  - 20
  - 14
  - 8
  - 2

= 158
```


Probability

- So, out of a total of 300 possibilities, 158 numbers will be factors of either 3, 5 or 7
- If we were to randomly select a number…

```
158 out of 300 chance
```

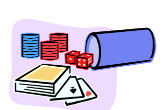



Permutations and Combinations

Don't Play The Lottery

Permutations and Combinations

- How many ways is there of choosing r items from a collection of n items?
- That depends on what you mean by "choose"!



Permutations and Combinations

- Does the order in which the selection occurs matter?
- Does the collection of n items have any repeats?

Permutations and Combinations

- The answers to these questions makes a great deal of difference
- The study of these problems are covered by either a *permutation* or *combination*

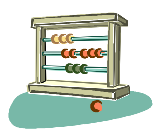
When Items are Retained

- Often the order of selection matters and items *go back into the collection*
- Total number of possible values is the range of values to the power of the number selected

$$T(n, r) = n^r$$

Permutations

- A *permutation* is a case where multiple items are selected from a group
- … and, the selected item is removed from the group
- Most importantly: *the order of the selection matters*



Permutations

- The order of the selection matters and
- Items do not go back into the collection
- So, each new items is 1 less

$$P(n, r) = n(n-1)(n-2)...(n-r+1)$$

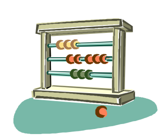
Permutations

- The order of the selection matters and
- Items do not go back into the collection
- So, each new items is 1 less

$$P(n, r) = \frac{n!}{(n-r)!}$$

Combinations

- Often order doesn't matter
- In this case, a permutation will over-count cases that don't matter
- e.g. ab, ba are the same!



Combinations

- This means, the permutation logic will create duplicates
- To remove them, we divide the permutation logic by the permutation of the items selected

```
C(n,r)  =  P(n,r)
             r!
```


Combinations

- So the mathematics are a tad more complex
- Fortunately, in practice, it is not that difficult

```
C(n,r)  =      n!
           r!(n-r)!
```


Example

If you have a basketball team consisting of 5 players, how many different teams can you create given a bench (collection) of 12 players?

Example Permutation

```
If order mattered…

P(12,5)  = 12! / (12 – 5)!
         = 12! / 7!

         = 12 * 11 * 10 * 9 * 8
         = 95,040
```


Example Combination

```
If order doesn't matter…

C(12,5)  = P(12,5) / 5!
         = (12 * 11 * 10 * 9 * 8)/ 5!
         = 95,040 / 5!
         = 95,040 / 120
         = 792
```


Example Combination:
Using Merged Formula

```
C(12,5)  = 12! / (5! (12 – 5)!)
         = 12! / (5! 7!)

         = (12 * 11 * 10 * 9 * 8)/ 5!
         = 95,040 / 120
         = 792
```

Example: California Lotto

- 54 total balls placed in a giant container
- 6 are taken – *(well, actually, they randomly fall into a hole)*
- C(54,6) = 25,827,165

Example: California Mega Millions

- 75 total balls where 5 are taken
- An additional "power ball" with 15 values is selected from another collection
- C(75,5) * 15 = 258,890,850