Chapter 14 Addendum
Appendix E

Bode Diagrams

Text: *Electric Circuits* by J. Nilsson and S. Riedel

Prentice Hall

EEE 117 Network Analysis

Instructor: Russ Tatro

Preview

A Bode diagram is a graphical technique that gives a feel for the frequency response of the circuit.

Diagrams are named after Hendrick W. Bode.

Dr. Bode was an engineer for Bell Telephone Laboratories. He proposed this approach now known by his name.

We can approximate the complete frequency response by selecting a few appropriate frequencies. Then we draw a straight-line approximation to the complete response.

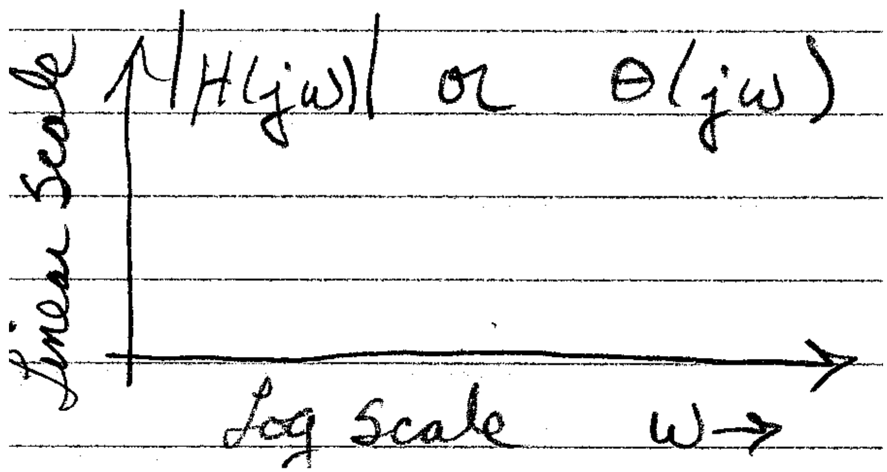
There are many software programs that can plot the frequency response very accurately. But as always, the hand analysis confirms the validity of the computer program.

A Bode diagrams consists of two separate plots.

One plot shows how the amplitude of H(jω) varies with frequency.

The other plot shows how the phase θ(jω) varies with frequency.

The plots are made on semilog paper so that a wide range of frequency values may be plotted.



Decibel

The magnitude is given in decibels. $dB \equiv 10\log_{10}\left(\frac{P_1}{P_2}\right)$

Properties of logarithms.

$$\log\left(P_1 P_2\right) = \log P_1 + \log P_2$$

$$\log\left(\frac{P_1}{P_2}\right) = \log P_1 - \log P_2$$

$$\log P^N = N\log P$$

$$\log_{10}\left(1\right) = \log_{10}\left(10^0\right) = 0$$

Decibel

$$dB \equiv 10\log_{10}\left(\frac{P_1}{P_2}\right)$$

Definition of the decibel is in terms of power!

dB for a voltage

$$10\log_{10}\left(\frac{P_1}{P_2}\right) = 10\log_{10}\left(\frac{\frac{V_1^2}{R_1}}{\frac{V_2^2}{R_2}}\right) \quad = 10\log_{10}\left(\frac{V_1^2}{V_2^2}\frac{R_2}{R_1}\right)$$

$$= 10\log_{10}\left(\frac{V_1}{V_2}\right)^2 + 10\log_{10}\left(\frac{R_2}{R_1}\right)$$

$$= 20\log_{10}\left(\frac{V_1}{V_2}\right) + 10\log_{10}\left(\frac{R_2}{R_1}\right)$$

If $R_1 = R_2$ then

$$dB = 20\log_{10}\left(\frac{V_1}{V_2}\right) + 10\log_{10}(1) \quad = 20\log_{10}\left(\frac{V_1}{V_2}\right)$$

Decibel

$$dB \equiv 10\log_{10}\left(\frac{P_1}{P_2}\right)$$

dB for a current

$$10\log_{10}\left(\frac{P_1}{P_2}\right) = 10\log_{10}\left(\frac{I_1^2 R_1}{I_2^2 R_2}\right)$$

$$= 10\log_{10}\left(\frac{I_1}{I_2}\right)^2 + 10\log_{10}\left(\frac{R_1}{R_2}\right)$$

If $R_1 = R_2$ then

$$dB = 20\log_{10}\left(\frac{I_1}{I_2}\right) + 10\log_{10}(1) \quad = 20\log_{10}\left(\frac{I_1}{I_2}\right)$$

Magnitude in terms of Linear versus dB

Linear A Voltage or Current	A_{dB}
10^{-6}	-120
10^{-5}	-100
10^{-4}	-80
10^{-3}	-60
10^{-2}	-40
10^{-1}	-20
$\frac{1}{2}$	-6
$\frac{1}{\sqrt{2}} = 0.707$	-3
$10^{0} = 1$	0
$1.41 = \sqrt{2}$	3
2	6
10^{1}	20
10^{2}	40
10^{3}	60
10^{4}	80
10^{5}	100
10^{6}	120

Let the transfer function be a real, first-order system with non-repeated roots.

$$H(s) = \frac{K(s+z_1)}{s(s+p_1)} \qquad H(j\omega) = \frac{K(j\omega+z_1)}{j\omega(j\omega+p_1)}$$

To make the plot easier to create, we put the polynomial in a standard form.

$$H(j\omega) = K\frac{z_1}{p_1}\frac{(1+\frac{j\omega}{z_1})}{j\omega(1+\frac{j\omega}{p_1})}$$

Create a new constant K_0.

$$K_0 = K\frac{z_1}{p_1}$$

Now express the transfer function in polar form.

$$H(j\omega) = K_0 \frac{\left|1+\frac{j\omega}{z_1}\right| \angle \tan^{-1}\left(\frac{\frac{\omega}{z_1}}{1}\right)}{\left|j\omega\right| \angle 90^\circ \left[\left|1+\frac{j\omega}{p_1}\right| \angle \tan^{-1}\left(\frac{\frac{\omega}{p_1}}{1}\right)\right]}$$

Thus

$$\left|H(j\omega)\right| = \left|K_0 \frac{(1+\frac{j\omega}{z_1})}{j\omega(1+\frac{j\omega}{p_1})}\right| \qquad = K_0 \frac{\left|1+\frac{j\omega}{z_1}\right|}{\omega\left|1+\frac{j\omega}{p_1}\right|}$$

$$\theta(j\omega) = \angle \tan^{-1}\left(\frac{\omega}{z_1}\right) - 90^\circ - \angle \tan^{-1}\left(\frac{\omega}{p_1}\right)$$

Now can form the straight line approximation plots.

$$A\,dB = 20\log_{10} K_0 + 20\log_{10}\left|1+\frac{j\omega}{z_1}\right| - 20\log_{10}\omega - 20\log_{10}\left|1+\frac{j\omega}{p_1}\right|$$

First term is a constant

$$20\log_{10} K_0 = \text{a constant}$$

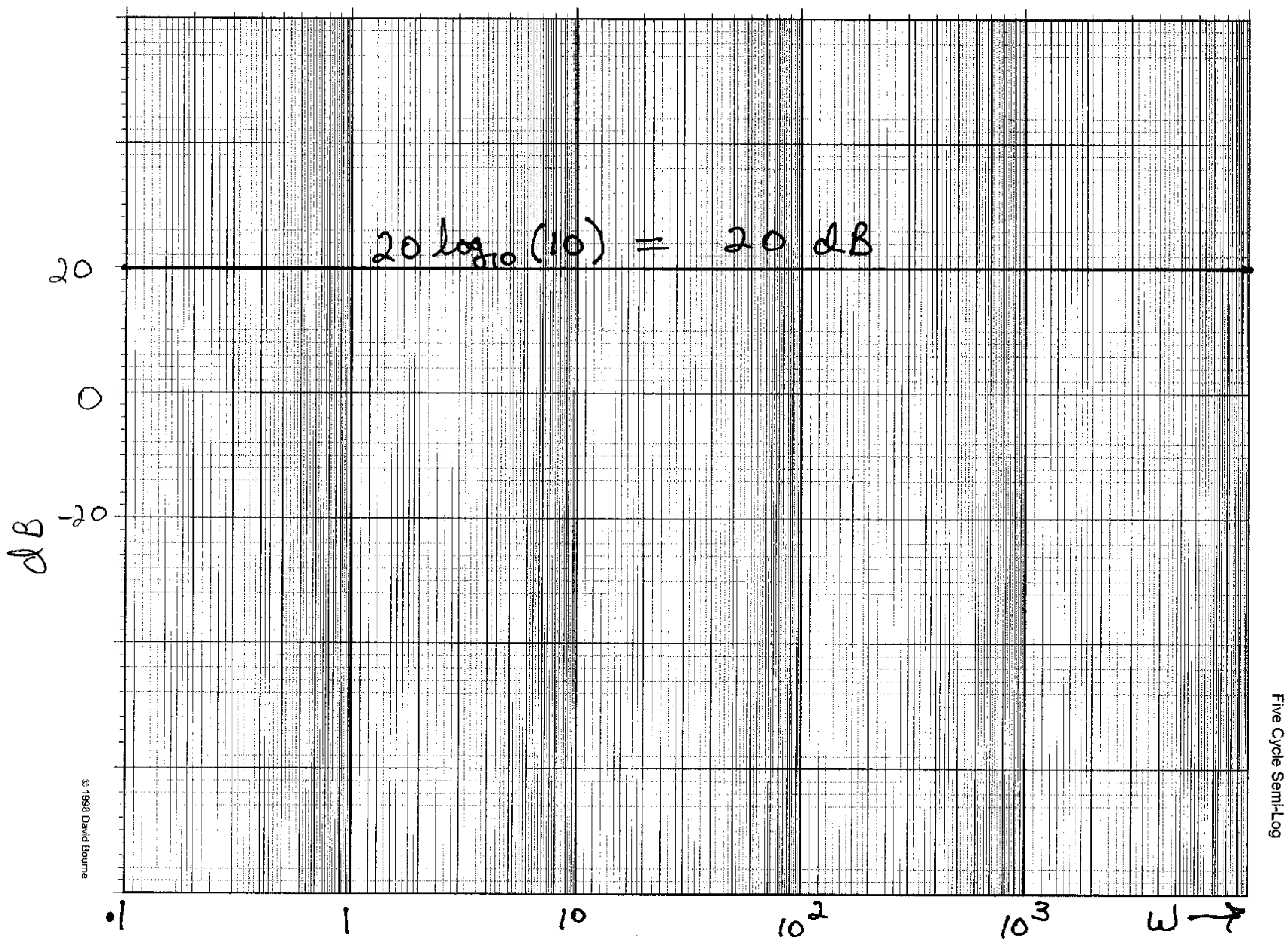
As we go through the process of looking at each piece of a bode plot, we will use the following transfer function.

$$H(s) = \frac{100(s+10)}{s(s+100)} \qquad H(j\omega) = \frac{100(j\omega+10)}{j\omega(j\omega+100)}$$

First step is to put the polynomial in standard form.

$$H(j\omega) = \frac{100(10)(1+\frac{j\omega}{10})}{j\omega(100)(1+\frac{j\omega}{100})} \qquad = \frac{10(1+\frac{j\omega}{10})}{j\omega(1+\frac{j\omega}{100})}$$

$$H_{dB} = 20\log_{10}(10) + 20\log_{10}\left|1+\frac{j\omega}{10}\right| - 20\log_{10}\omega - 20\log_{10}\left|1+\frac{j\omega}{100}\right|$$


$20 \log_{10}(10) = 20$ dB
20
0
-20
dB
·1
1
10
10^2
10^3
ω →
Five Cycle Semi-Log
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Section E.1 Real, First-Order Poles and Zeros

$$A\,dB = 20\log_{10} K_0 + 20\log_{10}\left|1+\tfrac{j\omega}{z_1}\right| - 20\log_{10}\omega - 20\log_{10}\left|1+\tfrac{j\omega}{p_1}\right|$$

The second term is a zero of the transfer function.

$$20\log_{10}\left|1+\tfrac{j\omega}{z_1}\right| \quad = 20\log_{10}\left(\sqrt{1^2+\left[\tfrac{\omega}{z_1}\right]^2}\right)$$

This term is zero at ω = 0.

$$20\log_{10}\left(\sqrt{1^2+\left[\tfrac{0}{z_1}\right]^2}\right) \quad = 20\log_{10}\left(\sqrt{1^2}\right) \quad = 20\log_{10}(1) \quad = 20(0) \;= 0\,dB$$

As $\omega \rightarrow$, the magnitude increases by 20 db/decade.

$$20\log_{10}\left|1+\frac{j\omega}{z_1}\right| \approx 20\log_{10}\left|\frac{\omega}{z_1}\right|$$

When $\omega = \omega_c = z_1$, the magnitude is

$$20\log_{10}\left(\sqrt{1^2+\left[\frac{\omega_c = z_1}{z_1}\right]^2}\right) = 20\log_{10}\left(\sqrt{1^2+1^2}\right) = 20\log_{10}\left(\sqrt{2}\right)$$

We will use an approximation and let the magnitude equal 1.
This is the essence of the straight line approximation.

$$20\log_{10}\left(\sqrt{1^2+\left[\frac{\omega_c = z_1}{z_1}\right]^2}\right) \approx 20\log_{10}(1) = 0\,dB$$

at $\omega = \omega_c$ $\quad 20\log_{10}\left(\frac{\omega_c}{z_1}\right) = 20\log_{10}\left(\frac{10}{10}\right) = 0\text{ dB}$

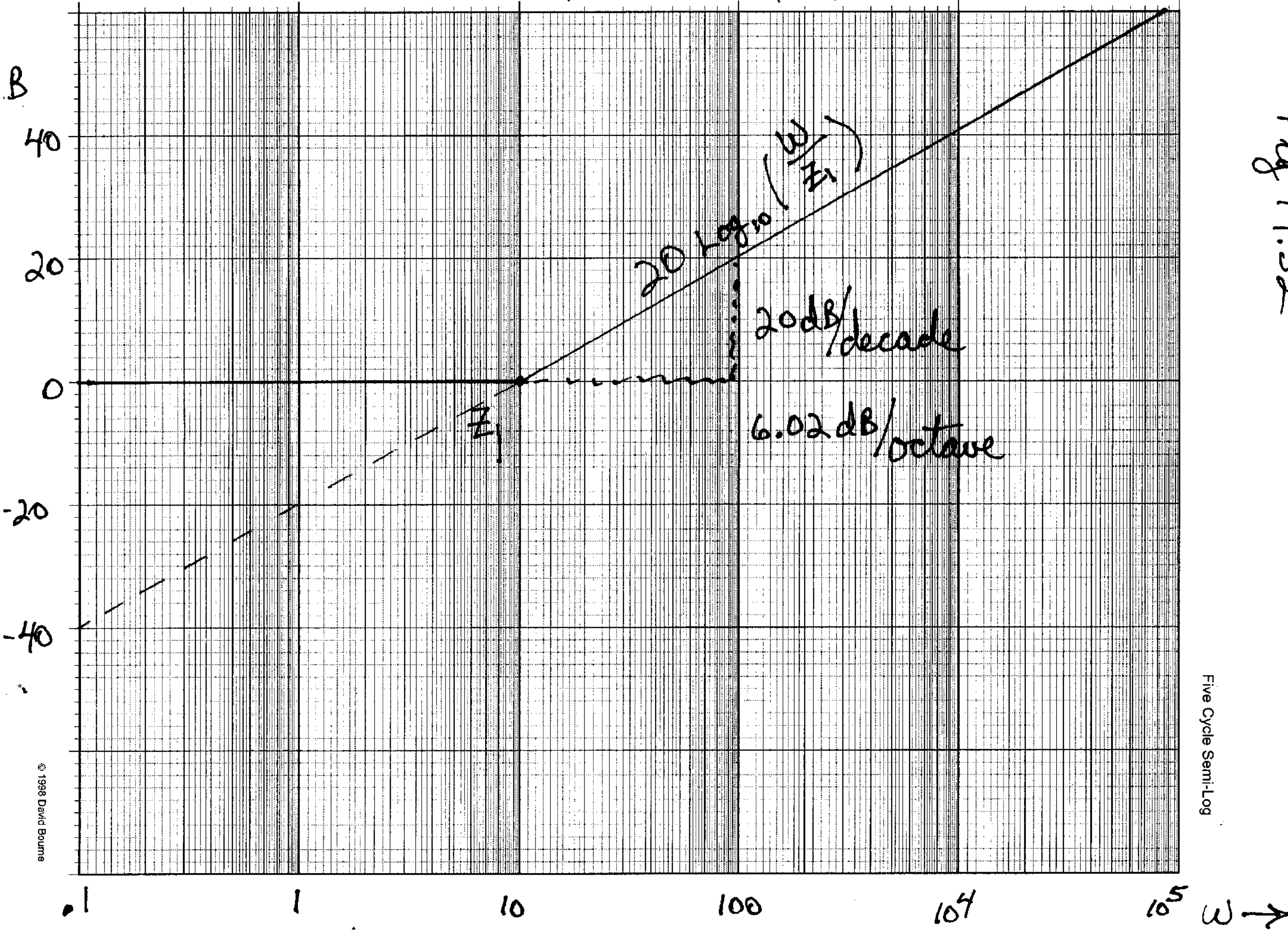


Fig 14.32

Now for the poles of the transfer function.

One pole is at the origin $\omega = 0$.

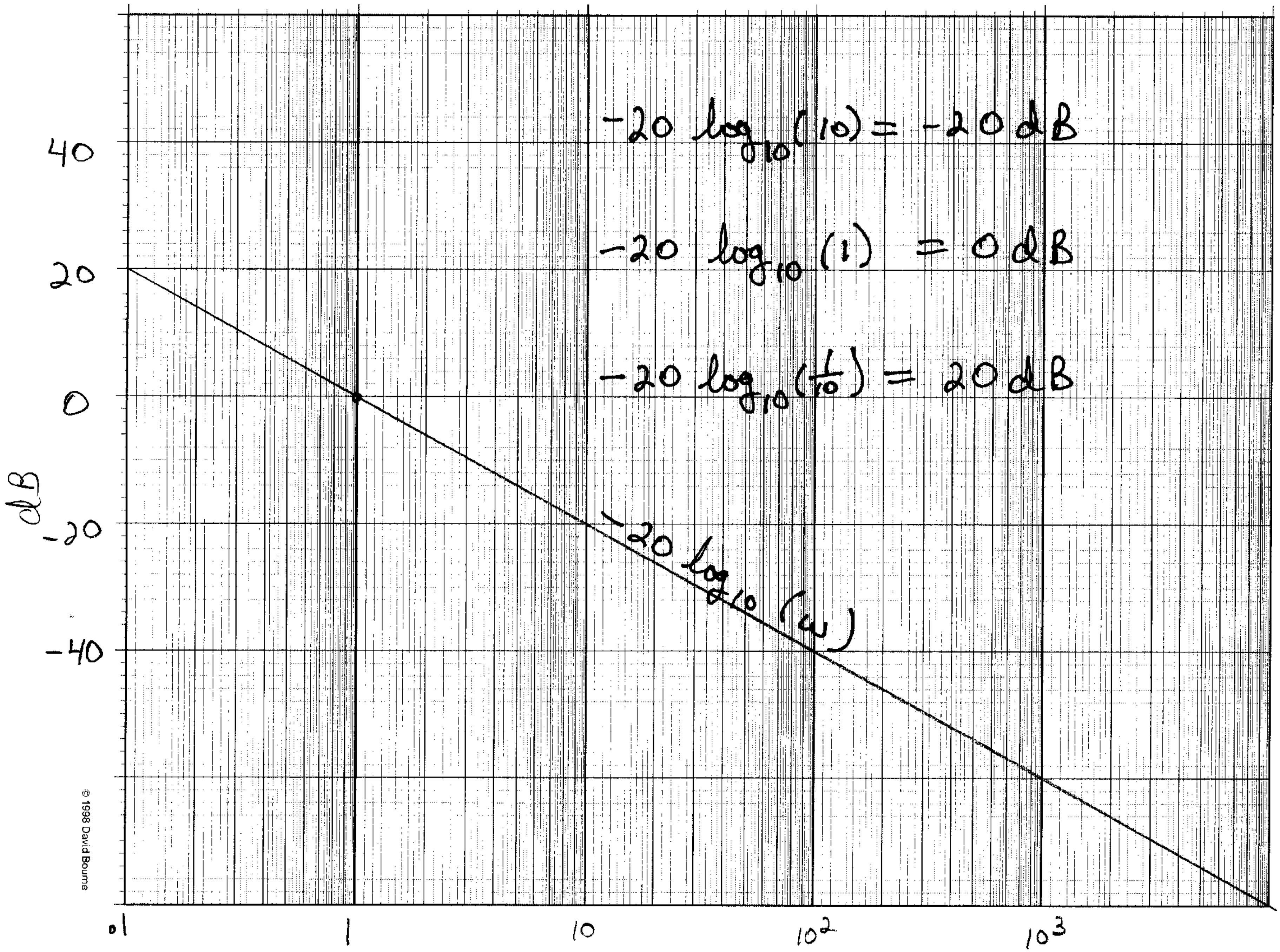
The term is negative since it is in the denominator.
Recall the properties of logarithms.

$$-20\log_{10}\left(\sqrt{\omega^2}\right) = -20\log_{10}(\omega)$$

This pole is undefined in terms of dB at $\omega = 0$.

At $\omega = 1$, the function equals 0 dB.

As ω increases above $\omega = 1$, the function decreases at -20 dB/decade.



$-20 \log_{10}(10) = -20\,dB$
$-20 \log_{10}(1) = 0\,dB$
$-20 \log_{10}(\frac{1}{10}) = 20\,dB$
$-20 \log_{10}(\omega)$
dB
40
20
0
-20
-40
.1
1
10
10^2
10^3
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Now for the other pole.

$$-20\log_{10}\left|1+\frac{j\omega}{p_1}\right| = -20\log_{10}\left(\sqrt{1^2+\left(\frac{\omega}{p_1}\right)^2}\right)$$

This pole at $\omega = 0$ is

$$-20\log_{10}\left(\sqrt{1^2+\left(\frac{0}{p_1}\right)^2}\right) = -20\log_{10}(1) = 0\ dB$$

At $\omega = p_1$, the function equals

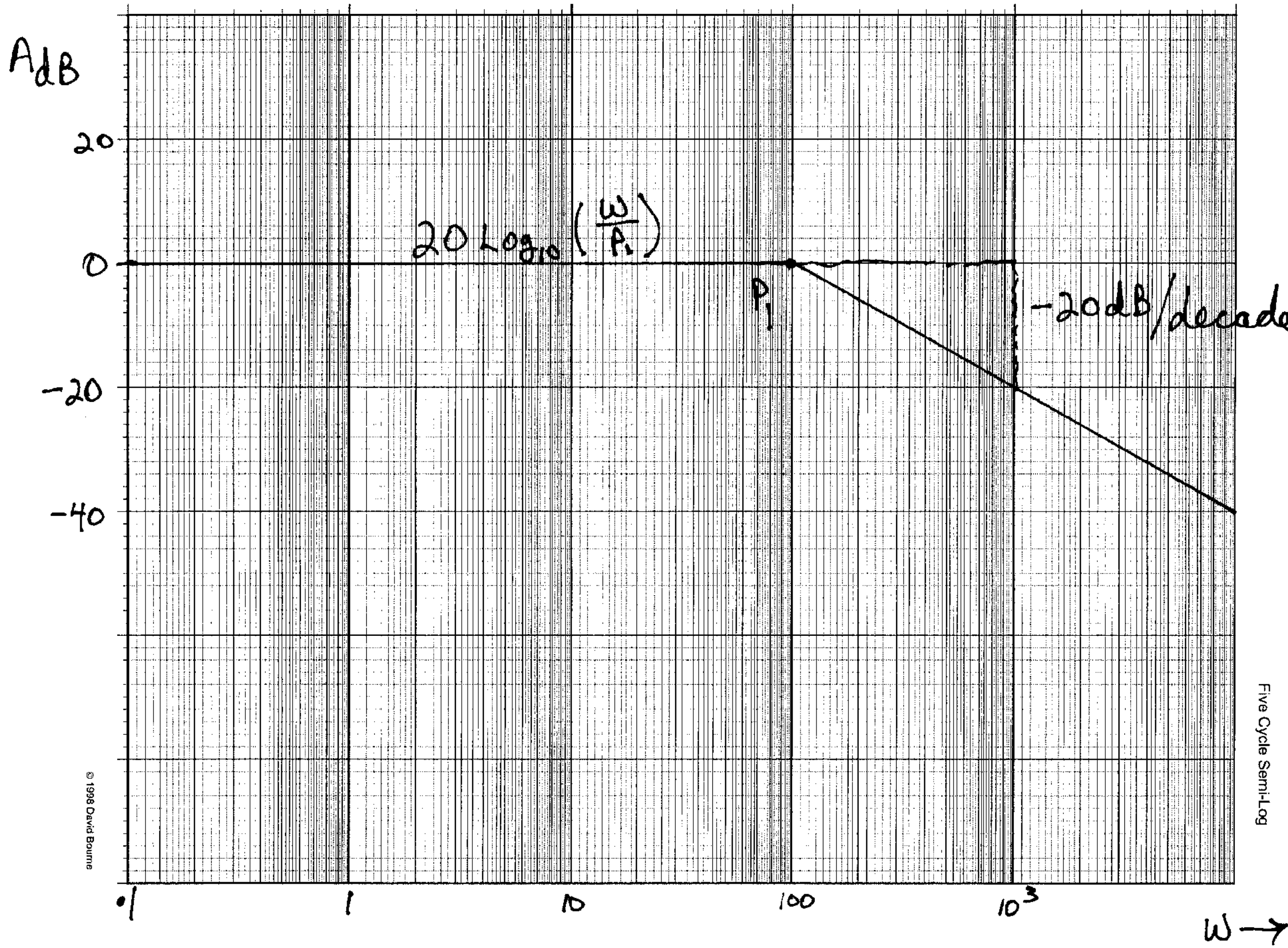
$$-20\log_{10}\left(\sqrt{1^2+\left(\frac{\omega=p_1}{p_1}\right)^2}\right) = -20\log_{10}\left(\sqrt{1^2+1^2}\right) = -3\ dB$$

As ω increases above $\omega = p_1$, the function decreases at -20 dB/decade.

Thus for the straight-line approximation, we let the term equal

$$-20\log_{10}\left|1+\frac{j\omega}{p_1}\right| \equiv \begin{cases} 0 \quad for \quad \omega \le p_1 \\ -20\log_{10}(\frac{\omega}{p_1}) \quad for \quad \omega \ge p_1 \end{cases}$$

With the approximation, we simplify the behavior the corner frequency in order to plot the function more easily.



A_{dB}
20
0
-20
-40
$20 \log_{10}\left(\frac{\omega}{P_1}\right)$
P_1
-20dB/decade
.1
1
10
100
10^3
$\omega \rightarrow$
Five Cycle Semi-Log
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Fig 14.33

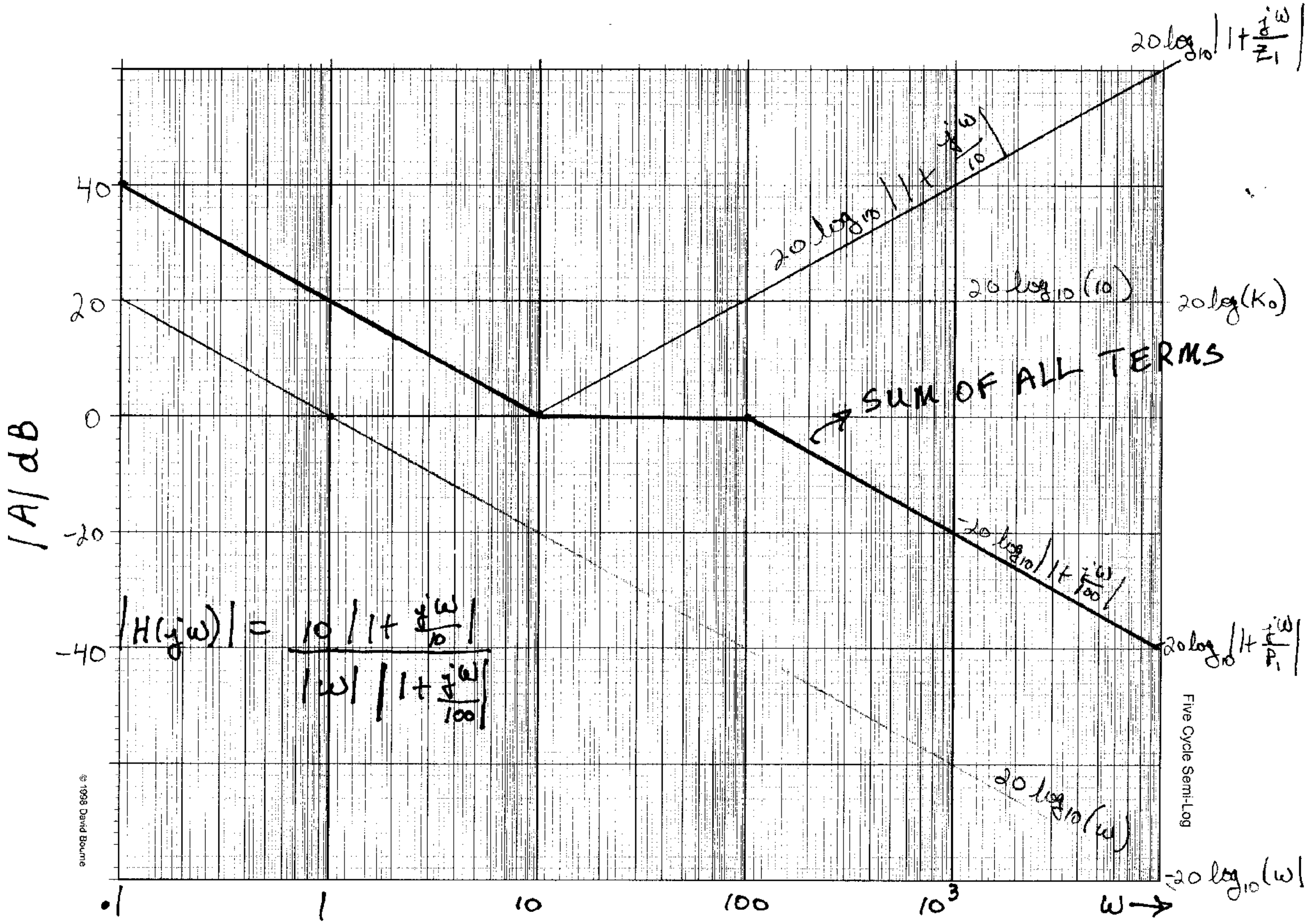
Now we need to gather up these individual terms and plot the magnitude function all at once.

For this example we have

$$H_{dB} = 20\log_{10}(10) + 20\log_{10}\left|1+\tfrac{j\omega}{10}\right| - 20\log_{10}\omega - 20\log_{10}\left|1+\tfrac{j\omega}{100}\right|$$

$$= 20 + \begin{cases} 0\ dB\ \ for\ \ \omega \le 10 \\ 20\log_{10}(\tfrac{\omega}{10})\ \ for\ \ \omega \ge 10 \end{cases} - 20\log_{10}\omega - \begin{cases} 0\ dB\ \ for\ \ \omega \le 100 \\ 20\log_{10}(\tfrac{\omega}{10})\ \ for\ \ \omega \ge 100 \end{cases}$$

Thus we have four terms which sum to the one line drawn at any particular frequency.



40
20
0
-20
-40
|A| dB
.1
1
10
100
10^3
$\omega \rightarrow$
$20 \log_{10} \left| 1 + \frac{j\omega}{Z_1} \right|$
$20 \log_{10} \left| 1 + \frac{j\omega}{10} \right|$
$20 \log_{10}(10)$
$20 \log(K_0)$
SUM OF ALL TERMS
$-20 \log_{10} \left| 1 + \frac{j\omega}{100} \right|$
$20 \log_{10} \left| 1 + \frac{j\omega}{P_1} \right|$
$20 \log_{10}(\omega)$
$-20 \log_{10} |\omega|$
$|H(j\omega)| = \frac{10 \left| 1 + \frac{j\omega}{10} \right|}{|\omega| \left| 1 + \frac{j\omega}{100} \right|}$
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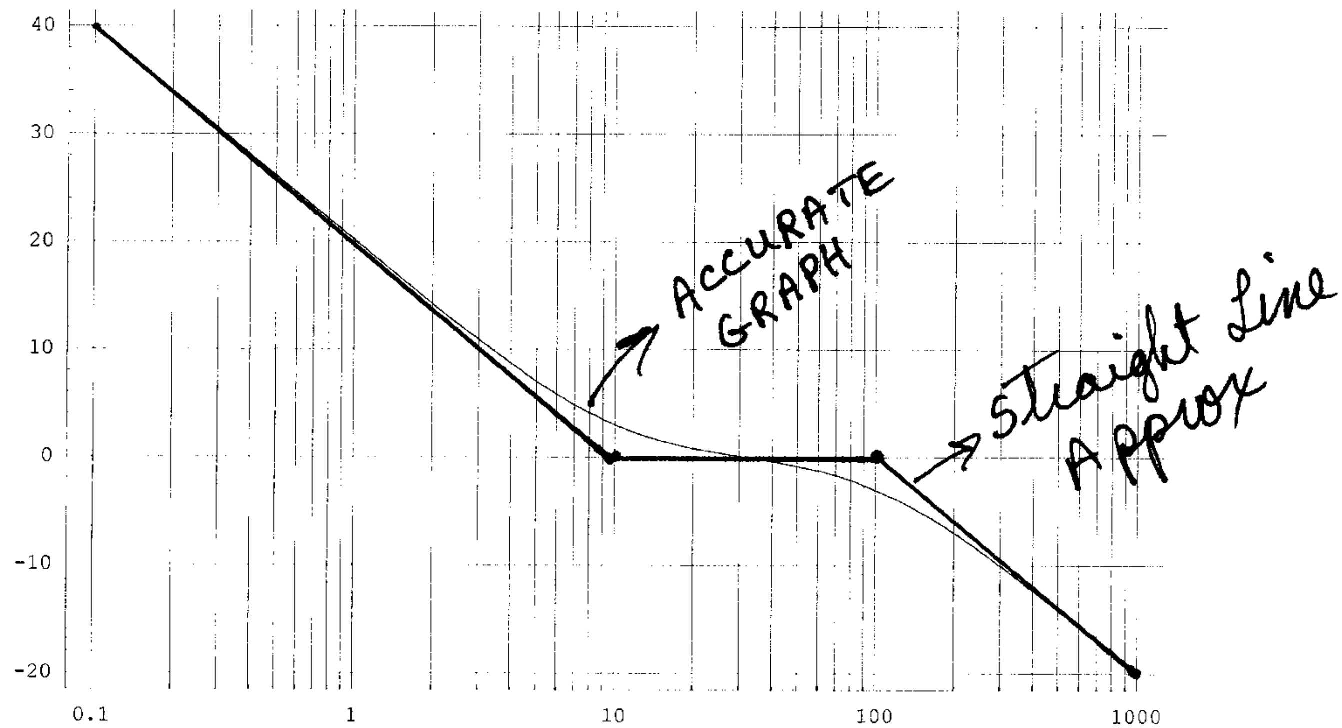


$$|H(j\omega)| = \frac{10\,\left|1+\frac{j\omega}{10}\right|}{|\omega|\,\left|1+\frac{j\omega}{100}\right|}$$

The text makes brief mention on how to “improve” the approximation.

These more accurate amplitude plots may not be worth the hand effort. If you want an exact plot, use a software program to plot the best graph possible usually in a fraction of the time of twiddling with a hand drawing.

However, the approximate plots give the engineer an excellent sense of the frequency behavior of the circuit.

Any automated program is thus validated by the hand effort!

We are only half done. Now we need the straight-line phase angle plots.

$$H(j\omega) = K_0 \frac{\left|1+\frac{j\omega}{z_1}\right| \angle \tan^{-1}\left(\frac{\frac{\omega}{z_1}}{1}\right)}{|j\omega|\angle 90^\circ \left[\left|1+\frac{j\omega}{p_1}\right| \angle \tan^{-1}\left(\frac{\frac{\omega}{p_1}}{1}\right)\right]}$$

$$\theta(j\omega) = \angle \tan^{-1}\left(\frac{\omega}{z_1}\right) - 90^\circ - \angle \tan^{-1}\left(\frac{\omega}{p_1}\right)$$

The phase associated with the constant term K_0 is zero.

Section E.1 Real, First-Order Poles and Zeros

$$H(j\omega) = K_0 \frac{\left|1+\frac{j\omega}{z_1}\right| \angle \tan^{-1}\left(\frac{\frac{\omega}{z_1}}{1}\right)}{\left|j\omega\right| \angle 90^\circ \left[\left|1+\frac{j\omega}{p_1}\right| \angle \tan^{-1}\left(\frac{\frac{\omega}{p_1}}{1}\right)\right]}$$

The phase for the zero of the transfer function is

$$\theta(j\omega) = \angle \tan^{-1}\left(\tfrac{\omega}{z_1}\right)$$

We always look at low, corner and high frequencies to evaluate the phase.

$$\angle \tan^{-1}\left(\tfrac{\omega}{z_1}\right) = \begin{cases} 0^\circ & for\ \ \omega = 0 \\ 45^\circ & for\ \ \omega = z_1 \\ 90^\circ & for\ \ \omega \to \infty \end{cases} = \begin{cases} 0^\circ & for\ \ \omega \le \frac{1}{10}\omega_c \\ 45^\circ & for\ \ \omega = z_1 \\ 90^\circ & for\ \ \omega \ge 10\omega_c \end{cases}$$

The one tenth and ten times the corner frequency is a convention that eases the phase plot.

Section E.1 Real, First-Order Poles and Zeros

$$H(j\omega) = K_0 \frac{\left|1+\frac{j\omega}{z_1}\right| \angle \tan^{-1}\left(\frac{\frac{\omega}{z_1}}{1}\right)}{\left|j\omega\right| \angle 90^\circ \left[\left|1+\frac{j\omega}{p_1}\right| \angle \tan^{-1}\left(\frac{\frac{\omega}{p_1}}{1}\right)\right]}$$

The phase for the pole of the transfer function at the origin is a constant.

$$\theta(j\omega) = -\angle \tan^{-1}\left(\tfrac{\omega}{0}\right) \quad = -\angle \tan^{-1}\left(\infty\right) \quad = -90^\circ$$

In a capacitive circuit the current leads the voltage. This transfer circuit appears to have a term which acts as a capacitor.

Section E.1 Real, First-Order Poles and Zeros

$$H(j\omega) = K_0 \frac{\left|1+\frac{j\omega}{z_1}\right| \angle \tan^{-1}\left(\frac{\frac{\omega}{z_1}}{1}\right)}{\left|j\omega\right| \angle 90^\circ \left[\left|1+\frac{j\omega}{p_1}\right| \angle \tan^{-1}\left(\frac{\frac{\omega}{p_1}}{1}\right)\right]}$$

The phase for the other pole of the transfer function is

$$\theta(j\omega) = -\angle \tan^{-1}\left(\tfrac{\omega}{p_1}\right)$$

Look at low, corner and high frequencies to evaluate the phase.

$$-\angle \tan^{-1}\left(\tfrac{\omega}{p_1}\right) = \begin{cases} 0^\circ & \text{for } \omega = 0 \\ -45^\circ & \text{for } \omega = z_1 \\ -90^\circ & \text{for } \omega \to \infty \end{cases} = \begin{cases} 0^\circ & \text{for } \omega \le \tfrac{1}{10}\omega_c \\ -45^\circ & \text{for } \omega = p_1 \\ -90^\circ & \text{for } \omega \ge 10\omega_c \end{cases}$$

Section E.1 Real, First-Order Poles and Zeros

$$H(j\omega) = K_0 \frac{\left|1+\frac{j\omega}{z_1}\right| \angle \tan^{-1}\left(\frac{\frac{\omega}{z_1}}{1}\right)}{\left|j\omega\right| \angle 90^\circ \left[\left|1+\frac{j\omega}{p_1}\right| \angle \tan^{-1}\left(\frac{\frac{\omega}{p_1}}{1}\right)\right]}$$

The phase is then the sum of these four terms:

the constant K_0 has zero phase.

the zero of H(s) goes from zero to +90°.

the pole at the origin is a constant -90°.

the other pole of H(s) goes from zero to -90°.

$$\theta(j\omega) = \angle \tan^{-1}\left(\frac{\omega}{z_1}\right) - 90^\circ - \angle \tan^{-1}\left(\frac{\omega}{p_1}\right)$$

It is helpful to build a table of inflection points where the phase curve changes.

For all frequencies below 1/10 ω_{z1}, the phase = zero.

In other words, z_1 is the lowest frequency in the transfer function.

We also need to remember 10 ω_{z1} which is where the phase becomes +90°.

Likewise for the pole of H(s) at p_1.

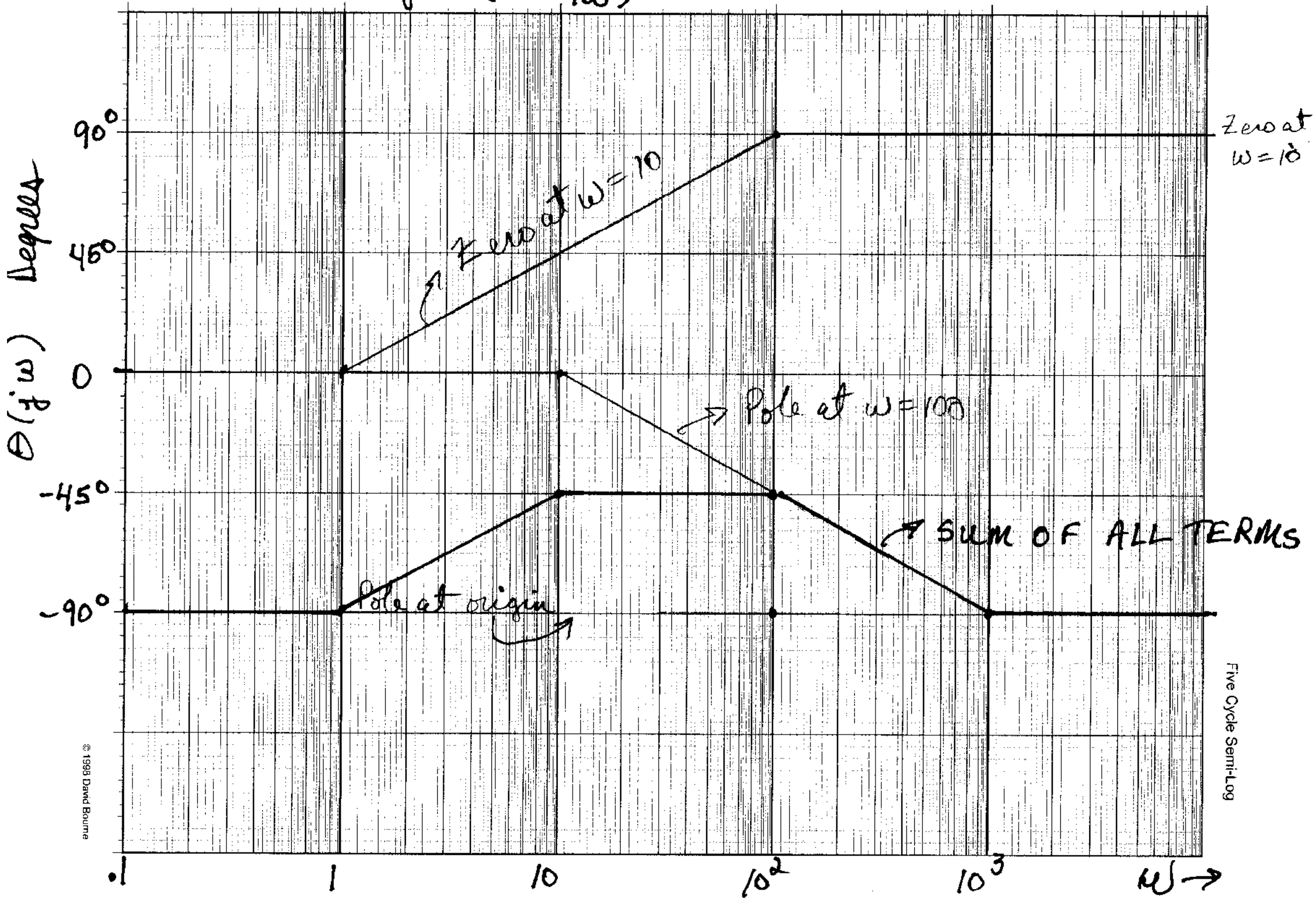
$$H(j\omega) = \frac{10(1+\frac{j\omega}{10})}{j\omega(1+\frac{j\omega}{100})}$$

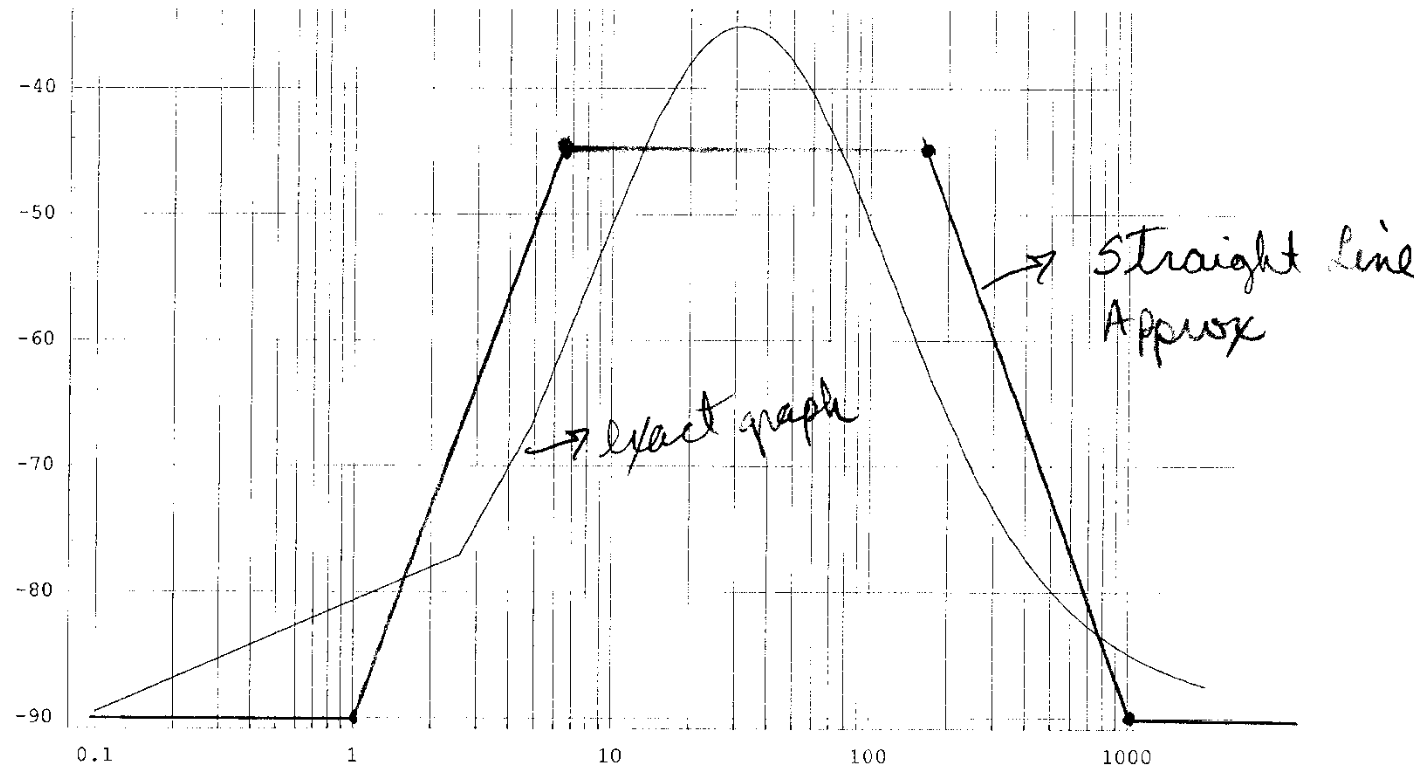
Now back to the specific example we have been graphing.

$$\theta(j\omega) = \angle \tan^{-1}\left(\tfrac{\omega}{z_1}\right) - 90^\circ - \angle \tan^{-1}\left(\tfrac{\omega}{p_1}\right) \quad = \angle \tan^{-1}\left(\tfrac{\omega}{10}\right) - 90^\circ - \angle \tan^{-1}\left(\tfrac{\omega}{100}\right)$$

ω	θ
0 to 1	-90°
1 to 10	Rising at + 45°/decade
10	-45°
10 to 100	One term rising and one term falling equally So = a constant (-45°)
100	-45° But now the zero has ended its phase effect.
100 to 1000	Pole still falling by -45°/decade
1000	-90° Pole has finished its effect.
>1000	No force exists to change the phase further.

$$H(j\omega) = \frac{10\left(1 + \frac{j\omega}{10}\right)}{(j\omega)\left(1 + \frac{j\omega}{100}\right)}$$



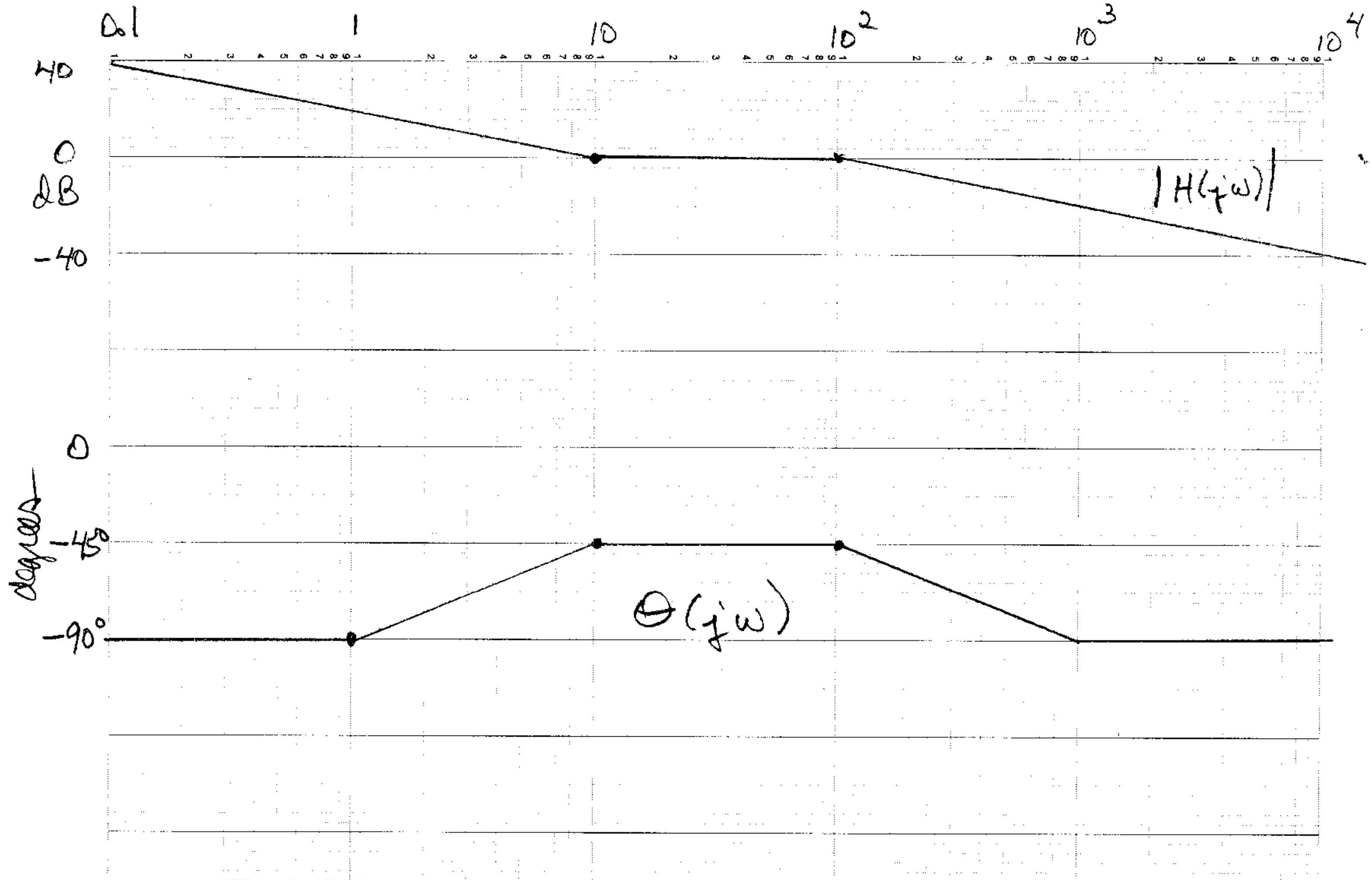


-40
-50
-60
-70
-80
-90
0.1
1
10
100
1000
exact graph
Straight line Approx

We have presented the magnitude and phase as separate plots.

While it is helpful to see the two plots individually, the most helpful graph is when we plot both on the some graph.

Thus we can see the circuit behavior most fully on a combined magnitude and phase plot.



0.1
1
10
10^2
10^3
10^4
40
0
dB
-40
|H(jw)|
0
degrees
-45°
-90°
Θ(jw)

Example

Given H(s) as follows, find the magnitude and phase plots. Prepare a combined magnitude and phase graph.

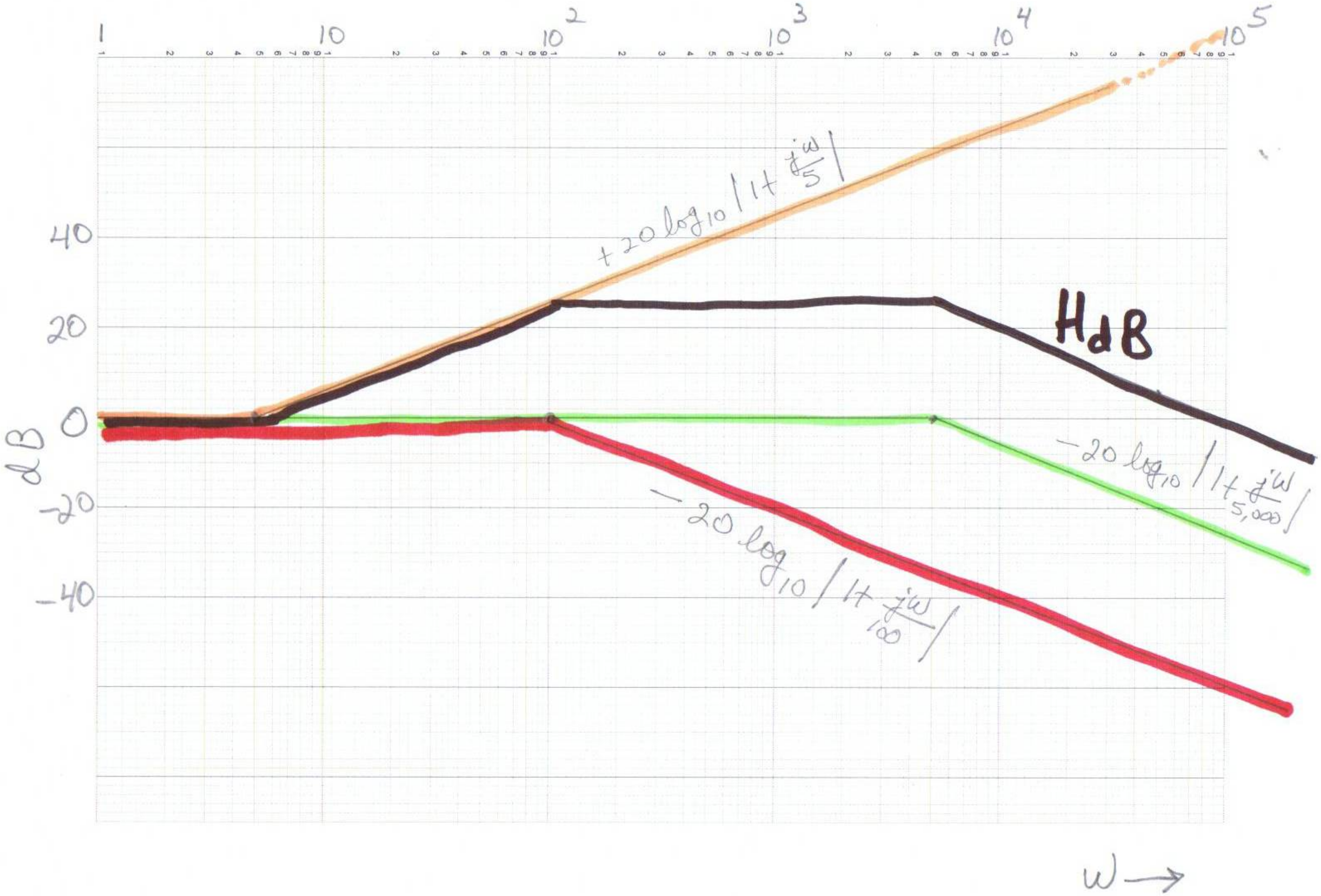
$$H(s) = \frac{10^5(s+5)}{(s+100)(s+5,000)}$$

$$H(j\omega) = \frac{10^5(5)(1+\frac{j\omega}{5})}{100(1+\frac{j\omega}{100})(5,000)(1+\frac{j\omega}{5,000})} = \frac{5\times10^5}{5\times10^5}\frac{(1+\frac{j\omega}{5})}{(1+\frac{j\omega}{100})(1+\frac{j\omega}{5,000})} = \frac{(1+\frac{j\omega}{5})}{(1+\frac{j\omega}{100})(1+\frac{j\omega}{5,000})}$$

$$= \frac{\left|1+\frac{j\omega}{5}\right|\angle\tan^{-1}\left(\frac{\omega}{5}\right)}{\left|1+\frac{j\omega}{100}\right|\angle\tan^{-1}\left(\frac{\omega}{100}\right)\left[\left|1+\frac{j\omega}{5,000}\right|\angle\tan^{-1}\left(\frac{\omega}{5,000}\right)\right]}$$

$$H_{dB} = 20\log\left|1+\frac{j\omega}{5}\right| - 20\log\left|1+\frac{j\omega}{100}\right| - 20\log\left|1+\frac{j\omega}{5,000}\right|$$

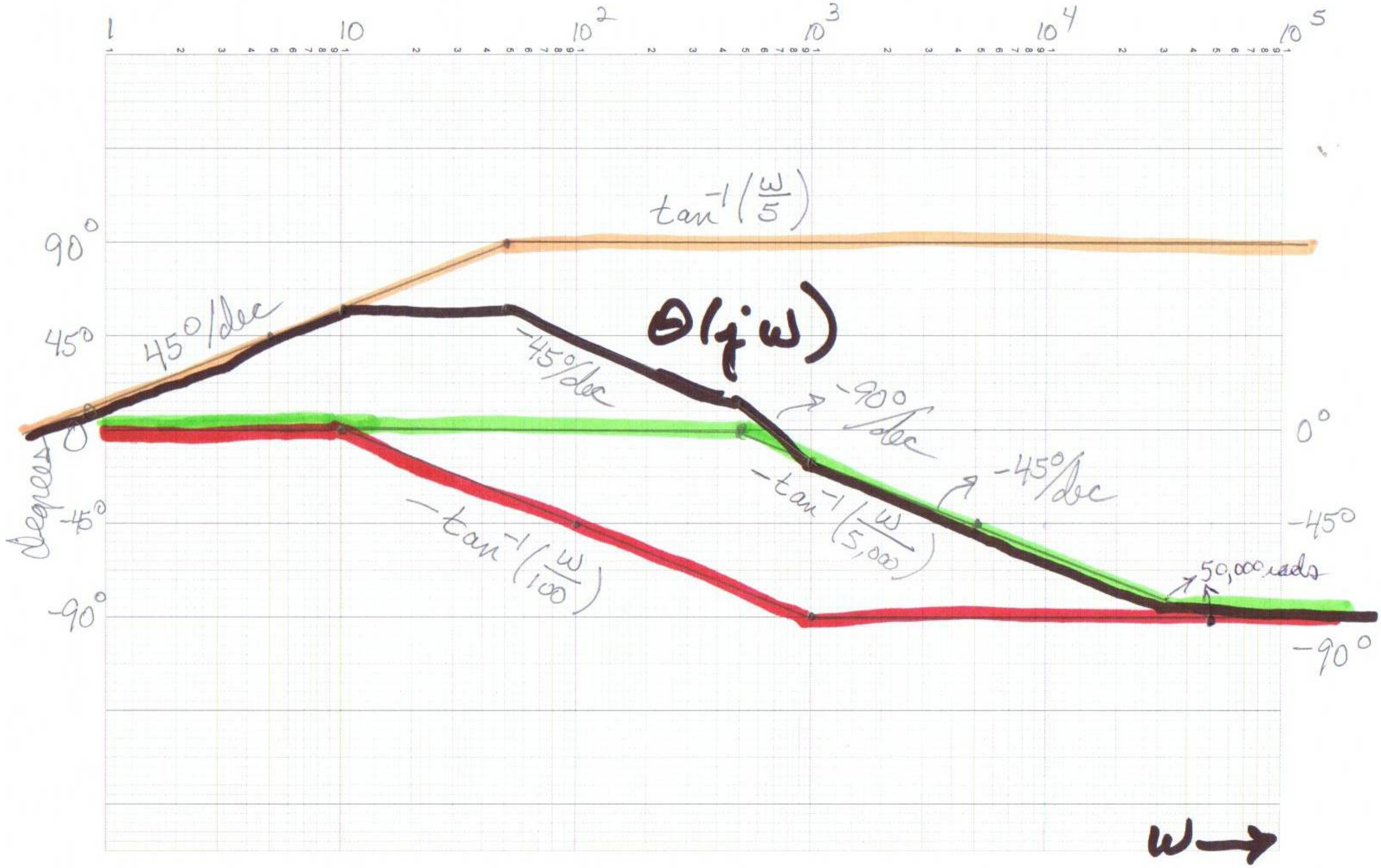
$$\theta(j\omega) = \angle\tan^{-1}\left(\frac{\omega}{5}\right) - \angle\tan^{-1}\left(\frac{\omega}{100}\right) - \angle\tan^{-1}\left(\frac{\omega}{5,000}\right)$$


1
10
10^2
10^3
10^4
10^5
40
20
0
-20
-40
dB
$+20\log_{10}\left|1+\frac{j\omega}{5}\right|$
HdB
$-20\log_{10}\left|1+\frac{j\omega}{5{,}000}\right|$
$-20\log_{10}\left|1+\frac{j\omega}{100}\right|$
$\omega \rightarrow$

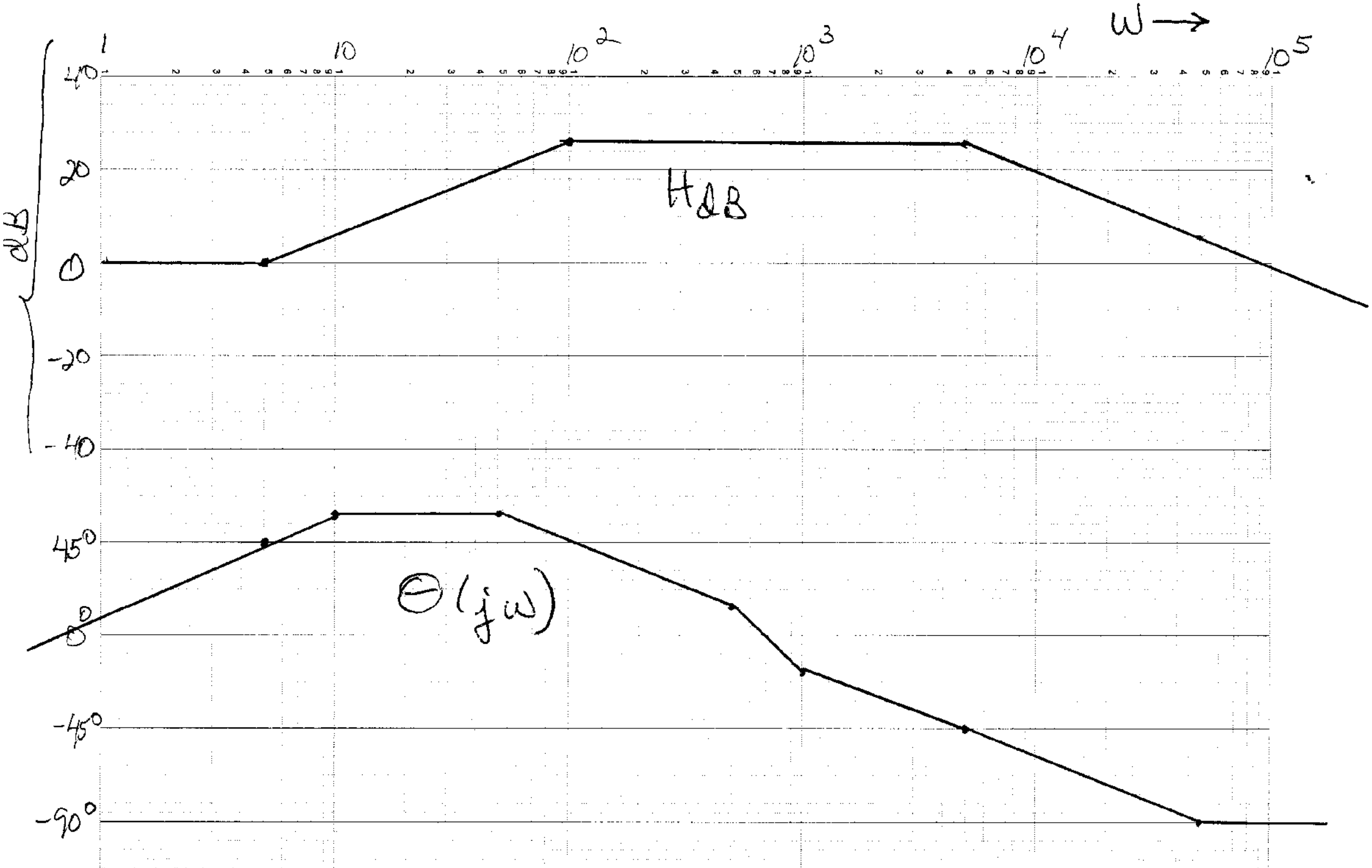
Example

The phase is given by $\theta(j\omega) = \angle \tan^{-1}\left(\frac{\omega}{5}\right) - \angle \tan^{-1}\left(\frac{\omega}{100}\right) - \angle \tan^{-1}\left(\frac{\omega}{5{,}000}\right)$

0°	± 45°	± 90°	θ(jω)
1/5 rads	5 rads	50 rads	$\tan^{-1}(\omega/5)$
10	100	1,000	$-\tan^{-1}(\omega/100)$
500	5,000	50,000	$-\tan^{-1}(\omega/5{,}000)$



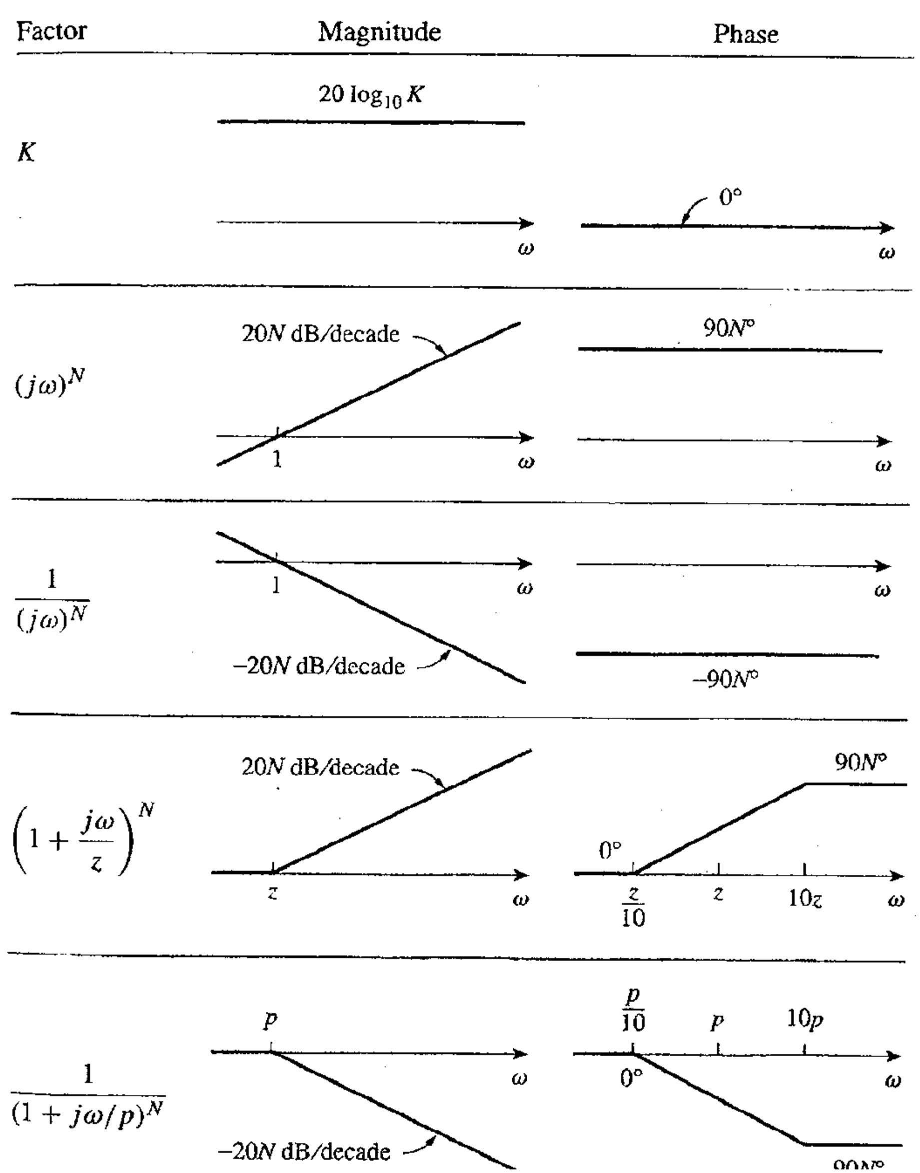
1
10
10^2
10^3
10^4
10^5
$\tan^{-1}\left(\frac{\omega}{5}\right)$
90°
45°
45°/dec
$\theta(j\omega)$
-45°/dec
-90°/dec
0°
degrees
-45°
$-\tan^{-1}\left(\frac{\omega}{5,000}\right)$
-45°/dec
$-\tan^{-1}\left(\frac{\omega}{100}\right)$
-45°
50,000 rads
-90°
-90°
ω



ω →
1
10
10^2
10^3
10^4
10^5
dB
40
20
0
-20
-40
H_{dB}
45°
0°
-45°
-90°
Θ(jω)

Summary – Alexander and Sadiku, Fundamental of Electric Circuits, 2nd Edition, McGraw Hill

TABLE 14.3 Summary of Bode straight-line magnitude and phase plots.



Summary

The text covers non-repeated complex roots.

For complex roots and multiple repeated roots, I would use Matlab/Mathematica/Maple or some other mathematical software to accurately plot the transfer function.

Note that this straight-line approximation works reasonably well when the roots are fairly widely separately.

But for a first order look at the frequency behavior of the transfer function it is surprisingly helpful.

Chapter 14 Addendum
Appendix E

Bode Diagrams

Text: *Electric Circuits* by J. Nilsson and S. Riedel

Prentice Hall

EEE 117 Network Analysis

Instructor: Russ Tatro