Propositional logic basic inference rules

Negation Elimination (~E)

$\sim\sim\mathcal{A} \vdash \mathcal{A}$

Example

1. ~~P
2. P 1 ~E

Conditional Elimination (→E)

$\mathcal{A} \rightarrow \mathcal{B},\ \mathcal{A} \vdash \mathcal{B}$

Example

1. P → Q
2. P
3. Q 1,2 →E

Conjunction Introduction (&I)

$\mathcal{A},\ \mathcal{B} \vdash \mathcal{A}\ \&\ \mathcal{B}$
$\mathcal{A},\ \mathcal{B} \vdash \mathcal{B}\ \&\ \mathcal{A}$

Example

1. P
2. Q
3. P & Q 1,2 &I

Conjunction Elimination (&E)

$\mathcal{A}\ \&\ \mathcal{B} \vdash \mathcal{A}$
$\mathcal{A}\ \&\ \mathcal{B} \vdash \mathcal{B}$

Example

1. P & Q
2. Q 1 &E

Disjunction Introduction (vI)

$\mathcal{A} \vdash \mathcal{A} \vee \mathcal{B}$
$\mathcal{A} \vdash \mathcal{B} \vee \mathcal{A}$

Example

1. P
2. P v Q 1,2 vI

Disjunction Elimination (vE)

$\mathcal{A} \vee \mathcal{B},\ \mathcal{A} \rightarrow \mathcal{C},\ \mathcal{B} \rightarrow \mathcal{C} \vdash \mathcal{C}$

Example

1. P v Q
2. P → R
3. Q → R
4. R 1,2,3 vE

Biconditional Elimination (↔E)

$\mathcal{A} \leftrightarrow \mathcal{B} \vdash \mathcal{A} \rightarrow \mathcal{B}$
$\mathcal{A} \leftrightarrow \mathcal{B} \vdash \mathcal{B} \rightarrow \mathcal{A}$

Example

1. P ↔ Q
2. P → Q 1 ↔E

Biconditional Introduction (↔I)

$\mathcal{A} \rightarrow \mathcal{B},\ \mathcal{B} \rightarrow \mathcal{A} \vdash \mathcal{A} \leftrightarrow \mathcal{B}$
$\mathcal{A} \rightarrow \mathcal{B},\ \mathcal{B} \rightarrow \mathcal{A} \vdash \mathcal{B} \leftrightarrow \mathcal{A}$

Example

1. P → Q
2. Q → P
3. P ↔ Q 1,2 ↔I

Important propositional logic equivalences

De Morgan's Rules (DM)

$\sim(\mathcal{A}\ \&\ \mathcal{B}) \dashv\vdash \sim\mathcal{A} \vee \sim\mathcal{B}$

$\sim(\mathcal{A} \vee \mathcal{B}) \dashv\vdash \sim\mathcal{A}\ \&\ \sim\mathcal{B}$

Transposition (TRANS)

$\mathcal{A} \rightarrow \mathcal{B} \dashv\vdash \sim\mathcal{B} \rightarrow \sim\mathcal{A}$

Material Implication (MI)

$\mathcal{A} \rightarrow \mathcal{B} \dashv\vdash \sim\mathcal{A} \vee \mathcal{B}$

Negation Conditional (~→)

$\sim(\mathcal{A} \rightarrow \mathcal{B}) \dashv\vdash \mathcal{A}\ \&\ \sim\mathcal{B}$

Commutation (COM)

$\mathcal{A} \vee \mathcal{B} \dashv\vdash \mathcal{B} \vee \mathcal{A}$
$\mathcal{A}\ \&\ \mathcal{B} \dashv\vdash \mathcal{B}\ \&\ \mathcal{A}$

Double Negation (DN)

$\mathcal{A} \dashv\vdash \sim\sim\mathcal{A}$

Association (ASS)

$\mathcal{A}\ \&\ (\mathcal{B}\ \&\ \mathcal{C}) \dashv\vdash (\mathcal{A}\ \&\ \mathcal{B})\ \&\ \mathcal{C}$

$\mathcal{A} \vee (\mathcal{B} \vee \mathcal{C}) \dashv\vdash (\mathcal{A} \vee \mathcal{B}) \vee \mathcal{C}$

Distribution (DIST)

$\mathcal{A}\ \&\ (\mathcal{B} \vee \mathcal{C}) \dashv\vdash (\mathcal{A}\ \&\ \mathcal{B}) \vee (\mathcal{A}\ \&\ \mathcal{C})$

$\mathcal{A} \vee (\mathcal{B}\ \&\ \mathcal{C}) \dashv\vdash (\mathcal{A} \vee \mathcal{B})\ \&\ (\mathcal{A} \vee \mathcal{C})$

Schröder–Bernstein theorem

If *A* and *B* are sets and $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

Important propositional logic derived rules

Hypothetical Syllogism (HS)

$\mathcal{A} \rightarrow \mathcal{B},\ \mathcal{B} \rightarrow \mathcal{C} \vdash \mathcal{A} \rightarrow \mathcal{C}$

Example

1. P → Q
2. Q → R
3. P → R 1,2 HS

Modus Tolens (MT)

$\mathcal{A} \rightarrow \mathcal{B},\ \sim\mathcal{B} \vdash \sim\mathcal{A}$

Example

1. P → Q
2. ~Q
3. ~P 1,2 MT

Disjunctive Syllogism (DS)

$\mathcal{A} \vee \mathcal{B},\ \sim\mathcal{A} \vdash \mathcal{B}$
$\mathcal{A} \vee \mathcal{B},\ \sim\mathcal{B} \vdash \mathcal{A}$

Example

1. P v Q
2. ~P
3. Q 1,2 DS

Contradiction (CON)

$\mathcal{A},\ \sim\mathcal{A} \vdash \mathcal{B}$

Example

1. P
2. ~P
2. Q 1,2 CON

Repeat (RE)

$\mathcal{A} \vdash \mathcal{A}$

Example

1. P
2. P 1 RE

Constructive Dilemma (CD)

$\mathcal{A} \vee \mathcal{B},\ \mathcal{A} \rightarrow \mathcal{C},\ \mathcal{B} \rightarrow \mathcal{D} \vdash \mathcal{C} \vee \mathcal{D}$

Example

1. P v Q
2. P → R
3. Q → S
4. R v S 1,2,3 CD

Theorem Introduction (TI) Any substitution instance of a previously derived theorem may be introduced at any time.

Leibnizian modal logic inference rules

Duality (Dual): $\Box\mathcal{A} \dashv\vdash \sim\Diamond\sim\mathcal{A}$ and $\Diamond\mathcal{A} \dashv\vdash \sim\Box\sim\mathcal{A}$

K Rule (K) $\Box(\mathcal{A} \rightarrow \mathrm{B}) \vdash (\Box\mathcal{A} \rightarrow \Box\mathrm{B})$

T Rule (T) $\Box\mathcal{A} \vdash \mathcal{A}$

S4 Rule (S4) $\Box\mathcal{A} \vdash \Box\Box\mathcal{A}$

Brouwer Rule (B) $\mathcal{A} \vdash \Box\Diamond\mathcal{A}$

Necessitation (N) Any substitution instance of a previously derived theorem may be introduced as a necessary truth at any time.

Meinongian free logic inference rules (summary)

Free Existential Introduction (F∃I)

Introduce existential quantifier on one or more occurrences of *a* only if ∃x *a*=x exists on available line of proof.

Free Existential Elimination (F∃E)

Hypothesize to arbitrary: F*a* & ∃x *a*=x, then follow procedure for classical ∃E.

Free Universal Introduction (F∀I)

Hypothesize to arbitrary ∃x *a*=x. Discharge hypothesis on any line containing *a* by introducing universal quantifier on all occurrences of *a*.

Free Universal Elimination (F∀E)

Instantiate to *a* only when ∃x *a*=x

Quantifier Equivalence (QE) is valid in free logic.

Set theoretic definitions

Membership: $\forall x(x \in A \leftrightarrow (x=a \vee x=b \ldots \vee x=n))$	{}∈
Identity: $A = B \leftrightarrow \forall x(x\in A \leftrightarrow x\in B)$	{} =
Subset: $A \subseteq B \leftrightarrow \forall x(x\in A \rightarrow x\in B)$	Def $\subseteq$
Proper subset: $A \subset B \leftrightarrow (A \subseteq B \ \& \ A \neq B)$	Def $\subset$
Power set: $\forall x(x\in \mathcal{P}(A) \leftrightarrow x \subseteq A)$	Def $\mathcal{P}(A)$
Union: $\forall x\ (x\in(A \cup B) \leftrightarrow (x\in A \ \vee x\in B))$	Def $\cup$
Intersection: $\forall x\ (x\in(A \cap B) \leftrightarrow (x\in A \ \& \ x\in B))$	Def $\cap$
Cartesian Prod: $A\mathrm{x}B = \{(a,b) \mid a\in A \ \& \ b\in B\}$	Def AxB
<u>Functions f: A→B, g: B→C</u>	
Injective: $\forall x\forall y(f(x) = f(y) \leftrightarrow x=y)$	Def Inj
Surjective: $\forall y(y\in B \leftrightarrow \exists x(x\in A \ \& \ f(x) = y))$	Def Sur
Comp.: $(f\circ g) =\{z \mid z\in C \ \& \ \exists x(x\in A \ \& \ f(g(x)) = z)\}$	Def $f\circ g$