Chapter 14

Frequency Selective Circuits

Text: *Electric Circuits* by J. Nilsson and S. Riedel
Prentice Hall

EEE 117 Network Analysis
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Preview

Up to this point, our circuit analysis was at a fixed frequency.

We now begin the analysis of circuits when the input frequency is varied.

Frequency selective circuits are called *filters*.

The transfer function can be used to find the steady state response of a circuit to sinusoidal inputs.

We will now let the input be a variable frequency but with constant amplitude and phase unless we specifically wish to change the amplitude or phase.

Thus for linear, lumped parameter, time-invariant systems, if the input is a sinusoid of frequency ω, then the response will also be a sinusoid at the same frequency ω.

Note that there may a decaying exponential involved (transient response), but the resulting steady state sinusoid still has the frequency ω.

We will look at the transfer functions and define them as the ratio of some input to some output.

In the case of input/output voltages this can written as

$$H(s) = \frac{V_{output}(s)}{V_{input}(s)}$$

Or in terms of current as

$$H(s) = \frac{I_{output}(s)}{I_{input}(s)}$$

There are many other possible forms for the ratio of an arbitrary input to an arbitrary output.

Definitions:

Passband – the band of frequencies that appear in the output.

Stopband – Frequencies that do not appear in the output.

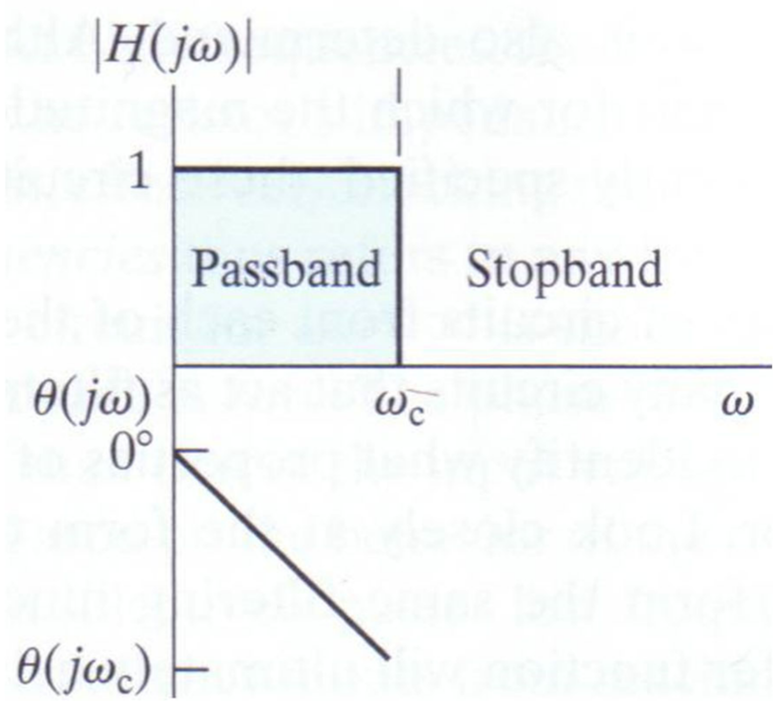
Frequency selective circuits are usually characterized by the location of their passband.

The frequency response plot shows how a circuit's transfer function (both amplitude and phase) changes as the source frequency changes.

In this chapter, we will examine passive filters only. The *gain* will thus be equal to or less than 1.

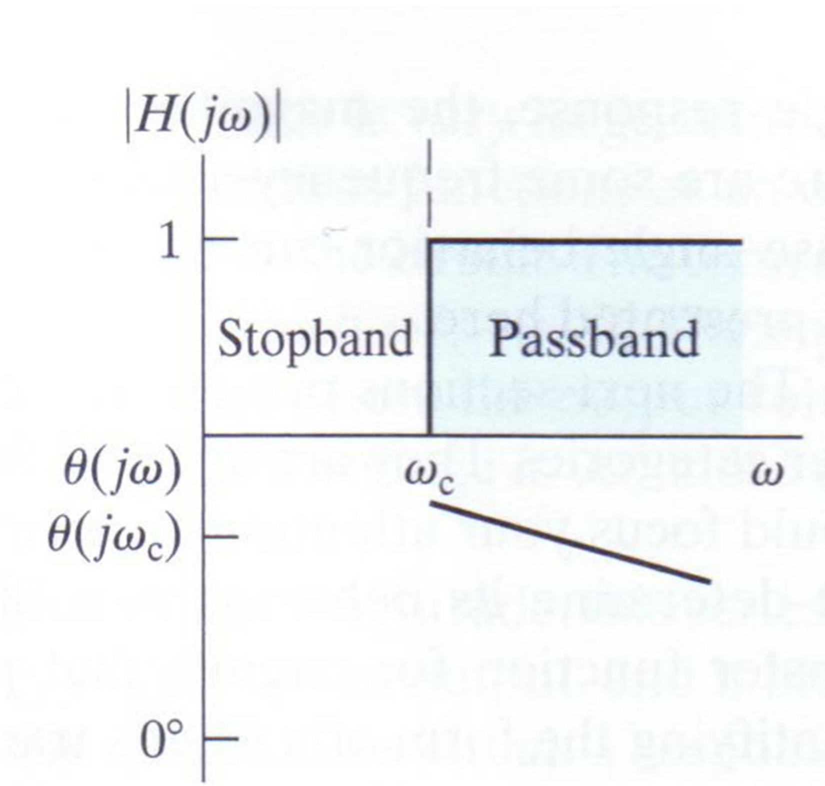
The frequency response plot shows the magnitude $|H(j\omega)|$ and the phase $\theta(j\omega)$.

Ideal Low Pass Filter frequency plot.

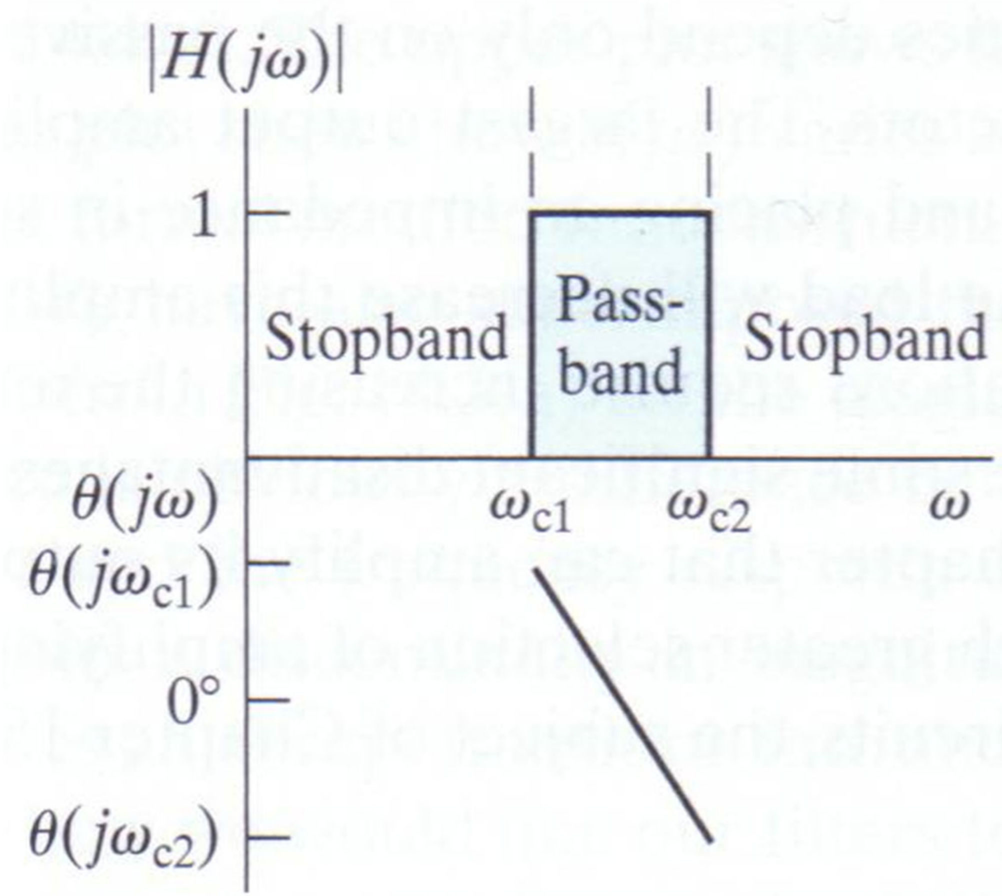


Section 14.1 High Pass Filter

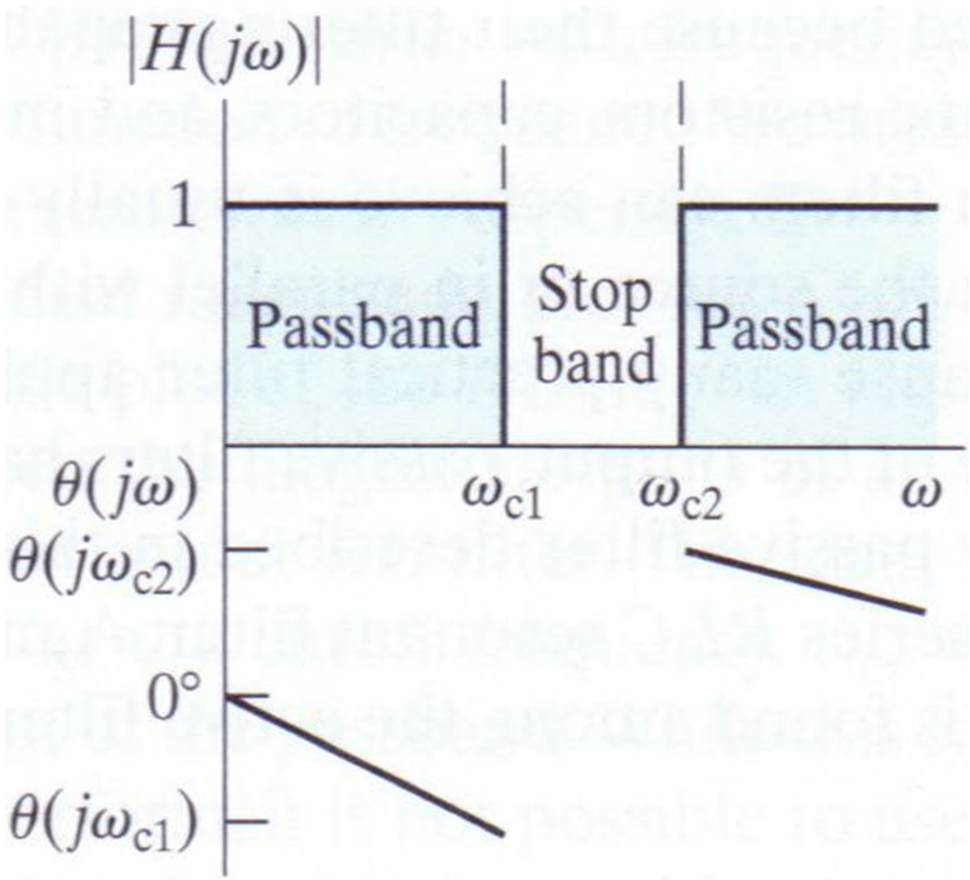
Ideal High Pass Filter frequency plot.



Ideal Band Pass Filter frequency plot.



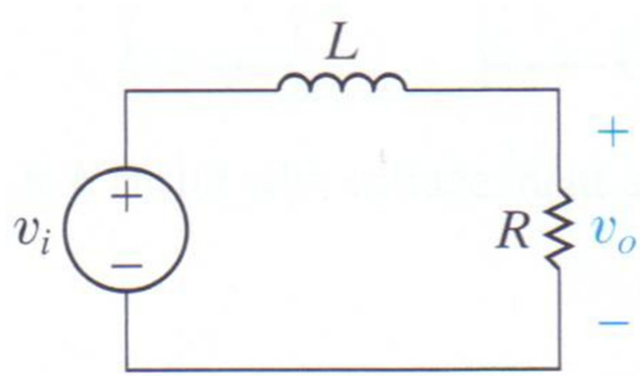
Ideal Band Reject Filter frequency plot.



Let's start with a look at a low-pass filter.

A low-pass filter selects the *low* frequencies $0 \leq \omega \leq \omega_c$ for the pass-band.

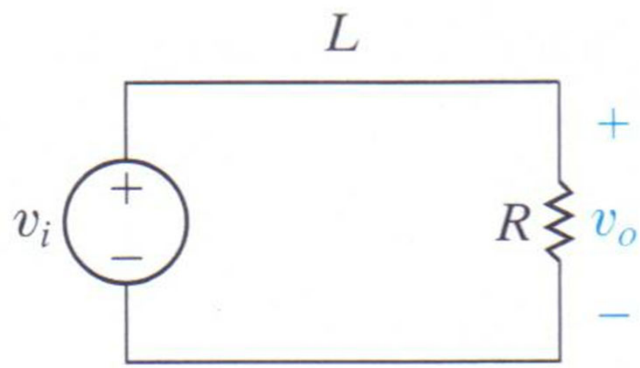
One implementation of a low-pass filter is with a RL circuit.



Recall that the impedance of the inductor is given by

$$Z_L = j\omega L$$

For ω = zero, and $Z_L = j(0)$, we have the following equivalent circuit.

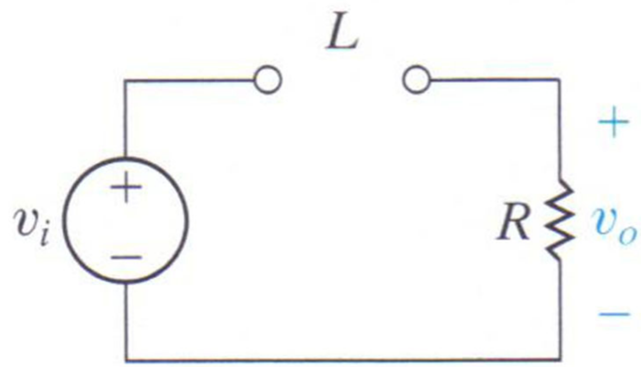


As the frequency rises, the behavior of the inductor will change. However, the behavior of the resistor is the same for all frequencies where linear, lumped parameter applies.

So a working definition of low frequency in this circuit is when

$$|Z_L| \ll |R| \qquad or \qquad \omega L \ll R$$

For $\omega = \infty$, and $Z_L = j(\infty)$, we have the following equivalent circuit.

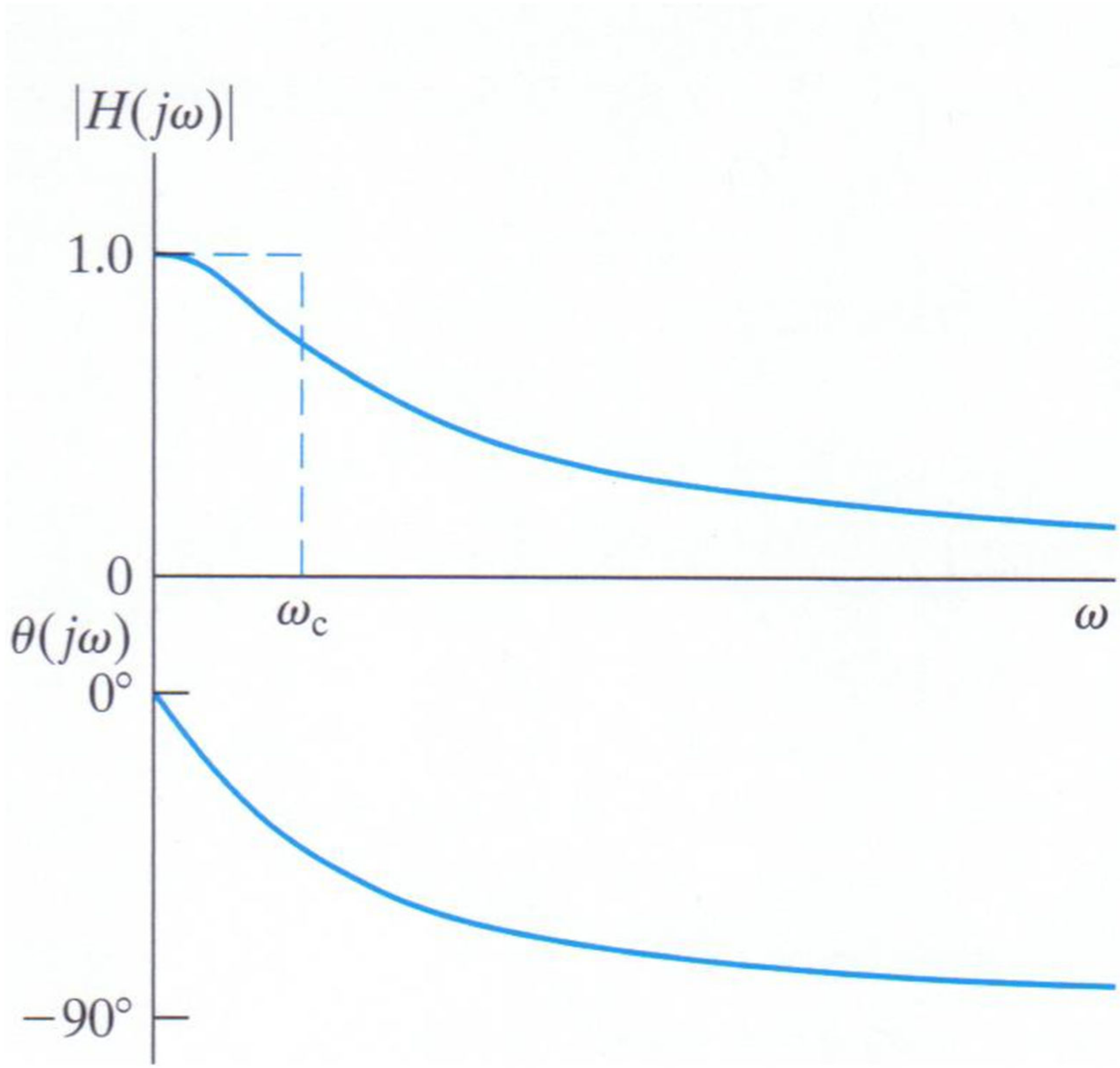


Now the inductor acts as an open circuit. Thus no current flows through the resistor. V_R = zero.

We know from earlier studies that for an inductor the *current lags the voltage*. Thus we see a phase shift as the frequencies rise.

This phase shift in this circuit approaches -90° at "high" frequency.

The frequency response plot is then



Section 14.2 Cutoff Frequency

As shown in the previous figure, the transition from passband to stopband is not abrupt but occupies a certain range of frequencies.

Thus we need to define a cutoff frequency that identifies where the circuit behavior changes.

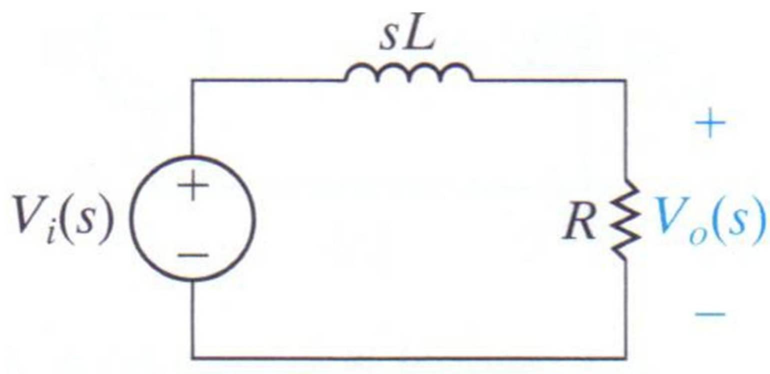
We will call this frequency the *half-power point*.

In terms of voltage or current magnitude, this is

$$H(j\omega_c) = \frac{1}{\sqrt{2}} H_{\max}$$

In other words, the pass band is where $H(j\omega) > 70.7\%$ of the maximum input value.

In the s-domain, the series RL circuit is



$$I(s)R = V_o(s)$$

$$I(s) = \frac{V_o(s)}{R}$$

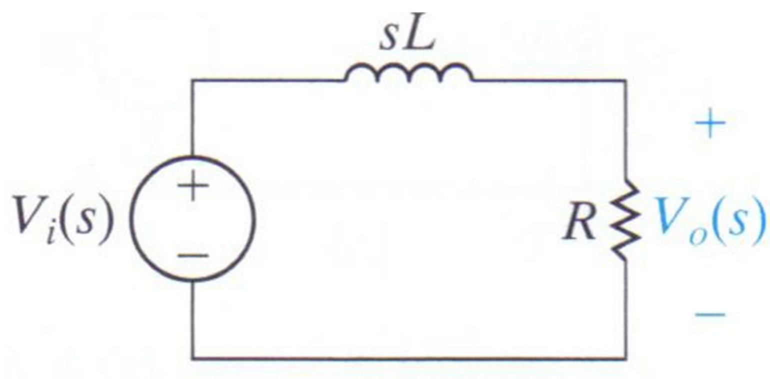
By KVL, we have

$$-V_i(s) + I(s)sL + I(s)R = 0$$

$$V_i(s) = I(s)sL + I(s)R \quad = I(s)\left(sL + R\right) \quad = \frac{V_o(s)}{R}\left(sL + R\right)$$

$$H(s) = \frac{V_o(s)}{V_i(s)} \quad = \frac{R}{sL + R} \quad = \frac{\frac{R}{L}}{s + \frac{R}{L}}$$

In the s-domain, the series RL circuit is



We can also find H(s) by the use of a voltage divider.

$$V_o(s) = \frac{R}{sL + R} V_i(s)$$

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{R}{sL + R} = \frac{\frac{R}{L}}{s + \frac{R}{L}}$$

To see the frequency effects for steady-state, substitute s = jω in the transfer function.

$$H(s) = \frac{V_o(s)}{V_i(s)} \qquad = \frac{\frac{R}{L}}{s + \frac{R}{L}}$$

$$H(j\omega) = \frac{V_o(j\omega)}{V_i(j\omega)} \quad = \frac{\frac{R}{L}}{j\omega + \frac{R}{L}}$$

Now we must find the magnitude and phase of H(jω).

Recall that $\left|a + jb\right| = \sqrt{a^2 + b^2} \qquad and \quad \theta = \tan^{-1}(\frac{b}{a})$

$$\frac{1}{\left|a + jb\right| \angle \theta} = \frac{1}{\sqrt{a^2 + b^2} \angle \tan^{-1}(\frac{b}{a})} \qquad = \frac{1}{\sqrt{a^2 + b^2}} \angle -\tan^{-1}(\frac{b}{a})$$

$$H(j\omega) = \frac{\frac{R}{L}}{j\omega + \frac{R}{L}}$$

Thus

$$H(j\omega) = \left|H(j\omega)\right| \theta(j\omega) = \frac{\frac{R}{L}}{\sqrt{\omega^2 + \left(\frac{R}{L}\right)^2} \angle \tan^{-1}\left(\frac{\omega}{\frac{R}{L}}\right)}$$

Where θ(jω) is

$$\theta(j\omega) = -\tan^{-1}\left(\frac{\omega}{\frac{R}{L}}\right) \quad = -\tan^{-1}\left(\frac{\omega L}{R}\right)$$

The phase limits are found at ω = 0 and ω = ∞

$$\theta(j\omega = 0) = -\tan^{-1}\left(\frac{(\omega = 0)L}{R}\right) = -\tan^{-1}(0) \quad = 0^\circ$$

$$\theta(j\omega = \infty) = -\tan^{-1}\left(\frac{(\omega = \infty)L}{R}\right) = -\tan^{-1}(\infty) \quad = -90^\circ$$

Section 14.2 Series RL Circuit – Quantitative Analysis

Thus at ω = 0

$$H(j\omega = 0) = \frac{\frac{R}{L}}{\sqrt{0^2 + \left(\frac{R}{L}\right)^2}\angle \tan^{-1}\left(\dfrac{0}{\frac{R}{L}}\right)} = \frac{\frac{R}{L}}{\sqrt{\left(\frac{R}{L}\right)^2}\angle 0^\circ} = 1\angle 0^\circ = H_{\max}$$

at ω = ∞

$$H(j\omega = \infty) = \frac{\frac{R}{L}}{\sqrt{\infty^2 + \left(\frac{R}{L}\right)^2}\angle \tan^{-1}\left(\dfrac{\infty}{\frac{R}{L}}\right)} = \frac{\frac{R}{L}}{\infty\angle 90^\circ} = 0\angle -90^\circ$$

Now we can find the cutoff frequency ω_c.

$$\left|H(j\omega_c\right| \equiv \frac{1}{\sqrt{2}} H_{Max} \quad = \frac{1}{\sqrt{2}}(1) \quad = \frac{\frac{R}{L}}{\sqrt{\omega_c^2 + \left(\frac{R}{L}\right)^2}}$$

$$\frac{1}{\sqrt{2}} = \frac{\frac{R}{L}}{\sqrt{\omega_c^2 + \left(\frac{R}{L}\right)^2}} \quad \Rightarrow \frac{1}{2} = \frac{\left(\frac{R}{L}\right)^2}{\omega_c^2 + \left(\frac{R}{L}\right)^2} \quad \Rightarrow \omega_c^2 + \left(\frac{R}{L}\right)^2 = 2\left(\frac{R}{L}\right)^2$$

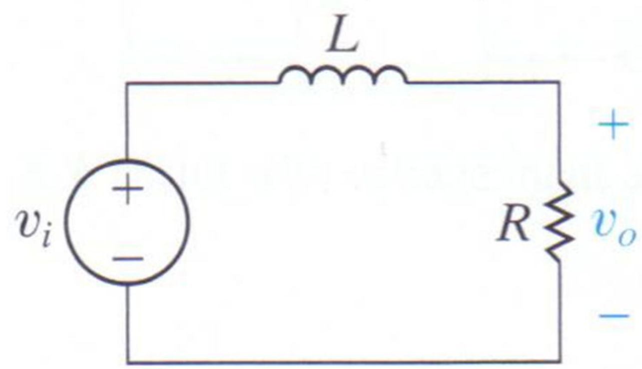
$$\omega_c^2 = 2\left(\frac{R}{L}\right)^2 - \left(\frac{R}{L}\right)^2 \quad = \left(\frac{R}{L}\right)^2$$

$So\ we\ \ have \quad \omega_c = \frac{R}{L}$ For the series RL circuit.

Example 14.1 on page 574

Design a low pass filter with a corner frequency of 10 Hz.

Use the RL series circuit.



The corner frequency was specified in Hertz. In radians, the frequency is

$$\omega_c = 2\pi f_c \quad = 2\pi(10) \quad = 20\pi \ \tfrac{rads}{\text{sec}}$$

$$\omega_c = \frac{R}{L}$$

So we have one equation and two unknowns, R and L.

Often in engineering, there are component values that relate well in the day to day service. Usually, we will start with one or a few assumed values and simulate the resulting circuit to determine suitability to the application.

Example 14.1

So let L = 100 mH. $\omega_c = \frac{R}{L} \quad \Rightarrow \omega_c L = R$

$$R = 20\pi \tfrac{rad}{\sec}\left(100\times10^{-3}H\right) \quad \approx 6.283\ \Omega$$

So with the component values determined, we can find the frequency response at 1 Hz, 10 Hz and 60 Hz.

$$\left|H(j\omega)\right| = \frac{\left|V_o(j\omega)\right|}{\left|V_i(j\omega)\right|} \quad \Rightarrow \left|V_o(\omega)\right| = \left|H(\omega)\right|\cdot\left|V_i(\omega)\right|$$

$$\left|V_o(\omega)\right| = \frac{\frac{R}{L}}{\sqrt{\omega^2+\left(\frac{R}{L}\right)^2}}\left|V_i(\omega)\right| \quad = \frac{\frac{R}{L}}{\sqrt{\omega^2+\omega_c^2}}\left|V_i(\omega)\right|$$

$$= \frac{20\pi}{\sqrt{\omega^2+400\pi^2}}\left|V_i(\omega)\right|$$

Example 14.1

f = 1Hz $\qquad \omega = 2\pi$ rad/sec

f = 10 Hz $\qquad \omega = 20\pi$ rad/sec

f = 60 Hz $\qquad \omega = 120\pi$ rad/sec

$$\left|V_o(\omega)\right| = \frac{20\pi}{\sqrt{\omega^2 + 400\pi^2}}\left|V_i(\omega)\right|$$

$$\left|V_o(\omega = 2\pi)\right| = \frac{20\pi}{\sqrt{(2\pi)^2 + 400\pi^2}}\left|V_i(\omega)\right| \quad = 0.99504\left|V_i(\omega)\right|$$

$$\left|V_o(\omega = 20\pi)\right| = \frac{20\pi}{\sqrt{(20\pi)^2 + 400\pi^2}}\left|V_i(\omega)\right| \quad = \frac{1}{\sqrt{2}}\left|V_i(\omega)\right| \quad = 0.70711\left|V_i(\omega)\right|$$

$$\left|V_o(\omega = 120\pi)\right| = \frac{20\pi}{\sqrt{(120\pi)^2 + 400\pi^2}}\left|V_i(\omega)\right| \quad = 0.16440\left|V_i(j\omega)\right|$$

Example 14.1 Bode Diagram

Create the straight-line Bode Diagram for this low-pass filter.

R = 6.283 Ω L = 100 mH

$$H(s) = \frac{V_o(s)}{V_i(s)} \qquad = \frac{\frac{R}{L}}{s + \frac{R}{L}}$$

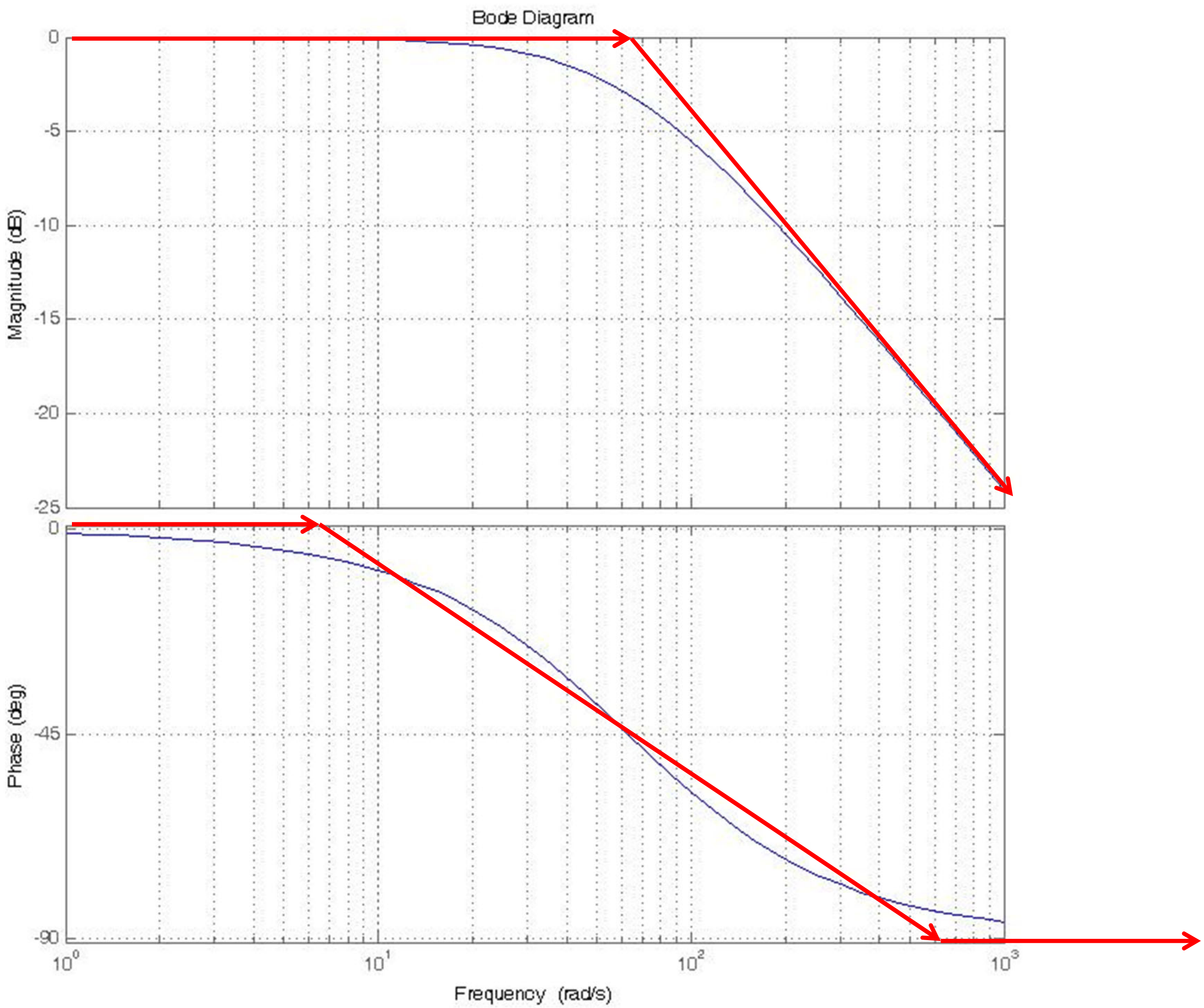
$$\frac{R}{L} = \frac{6.283\Omega}{100mH} = 62.83\,\text{sec}$$

$$H(j\omega) = \frac{62.83}{j\omega + 62.83} = \frac{1}{1 + \frac{j\omega}{62.83}}$$

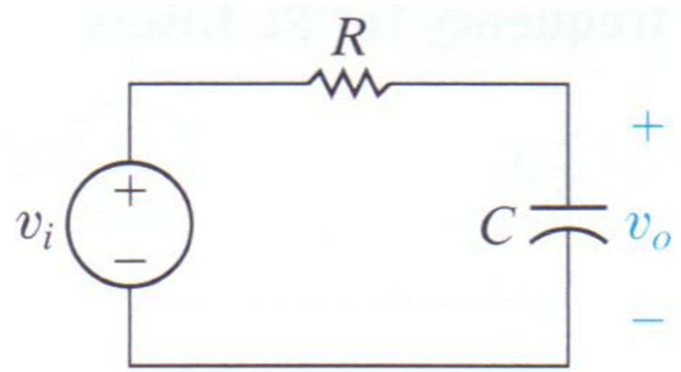
$$\left|H(j\omega)\right| = -20\log_{10}\left|1 + \frac{j\omega}{62.83}\right|$$

$$\theta(j\omega) = \begin{cases} 0^\circ \;\; for\ \omega \le 6.283\,\frac{rad}{\text{sec}} \\ -45^\circ \;\; for\ \omega \le 62.83\,\frac{rad}{\text{sec}} \\ -90^\circ \;\; for\ \omega \ge 628.3\,\frac{rad}{\text{sec}} \end{cases}$$

Example 14.1 Bode Diagram



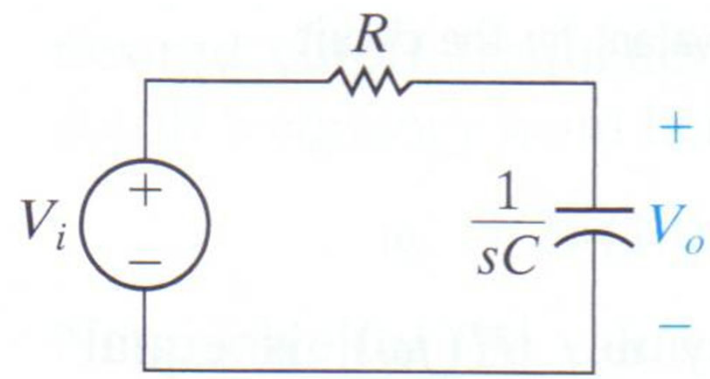
The series RC circuit also acts as a low-pass filter when in the following configuration and assuming the output is defined as the voltage across the capacitor.



When $\omega = 0$, the capacitor acts as an open circuit. $V_i = V_o$.

When $\omega = \infty$, the capacitor acts as an short circuit. $V_o = 0$.

The s-domain circuit is



We can find the output voltage V_o quickly by using the voltage divider.

$$V_o(s) = \frac{\frac{1}{sC}}{R + \frac{1}{sC}} V_i(s) = \frac{\frac{1}{SRC}}{1 + \frac{1}{sRC}} V_i(s) = \frac{\frac{1}{RC}}{s + \frac{1}{RC}} V_i(s)$$

From the previous derivation and the form of H(s), we know

$$H(s) = \frac{\omega_c}{s + \omega_c} \qquad \omega_c = \frac{1}{RC}$$

We know that for a series RL circuit, the time constant in seconds is

$$\tau = \frac{L}{R}$$

We now know that for a series RC circuit, the time constant is

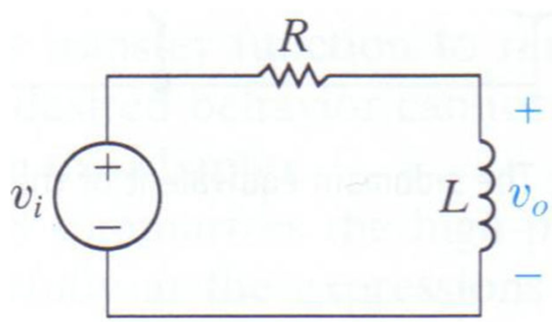
$$\tau = RC$$

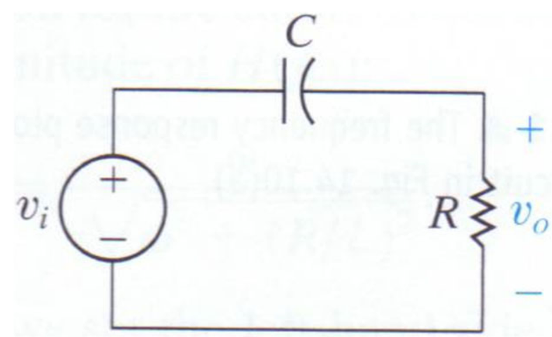
The time constant is related to the corner frequency in the frequency domain by

$$\tau = \frac{T}{2\pi} = \frac{1}{2\pi f_c} = \frac{1}{\omega_c} \qquad Thus \quad \omega_c = \frac{1}{\tau}$$

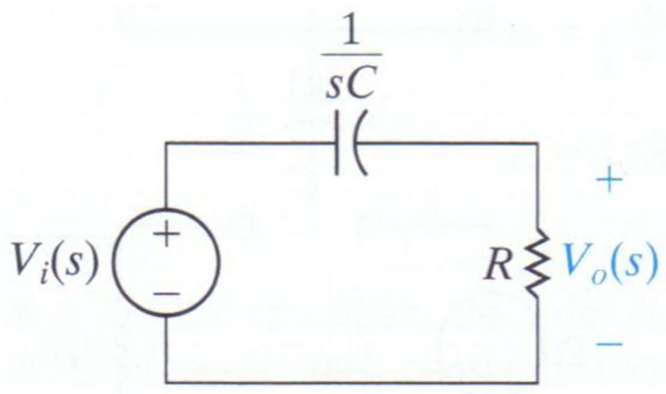
In the beginning of the chapter, we said the nature of the filter can depend on how the output is defined.

Both of the following circuits are high-pass filters. They look very much like the previous series RL and RC circuits.





For the RC high-pass filter, first construct the s-domain circuit.



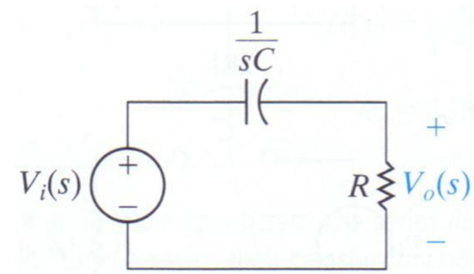
$$V_o(s) = \frac{R}{\frac{1}{sC} + R} V_i(s)$$

As before, we can find the transfer function by the voltage divider.

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{R}{\frac{1}{sC} + R} = \frac{sR}{\frac{1}{C} + sR} = \frac{s}{s + \frac{1}{RC}}$$

We can evaluate specific frequencies by using the phasor form:

$$H(j\omega) = \frac{j\omega}{j\omega + \frac{1}{RC}}$$



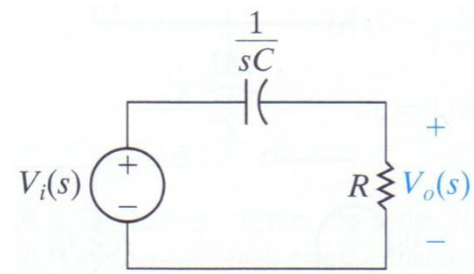
We need both the magnitude and phase responses.

For the magnitude

$$|H(j\omega)| = \frac{|j\omega|}{\left|j\omega + \frac{1}{RC}\right|} = \frac{\sqrt{\omega^2}}{\sqrt{\omega^2 + (\frac{1}{RC})^2}} = \frac{\omega}{\sqrt{\omega^2 + (\frac{1}{RC})^2}}$$

For the phase response, we need to note that there are two phase components.

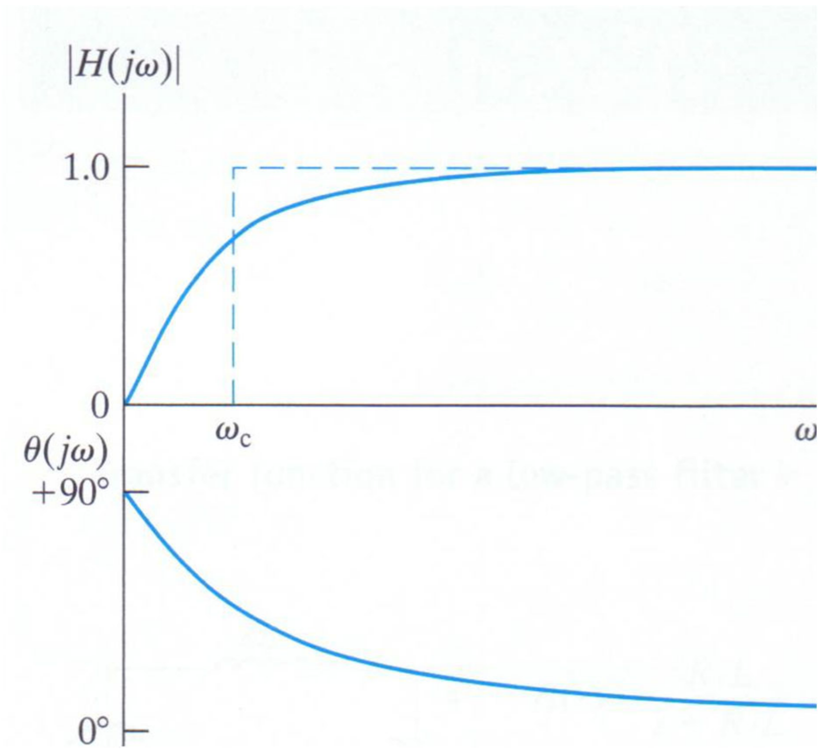
$$\theta(j\omega) = \frac{\angle \tan^{-1}\left(\frac{\omega}{0}\right)}{\angle \tan^{-1}\left(\frac{\omega}{\frac{1}{RC}}\right)} = \frac{\angle \tan^{-1}(\infty)}{\angle \tan^{-1}(\omega RC)} = 90^\circ - \tan^{-1}(\omega RC)$$



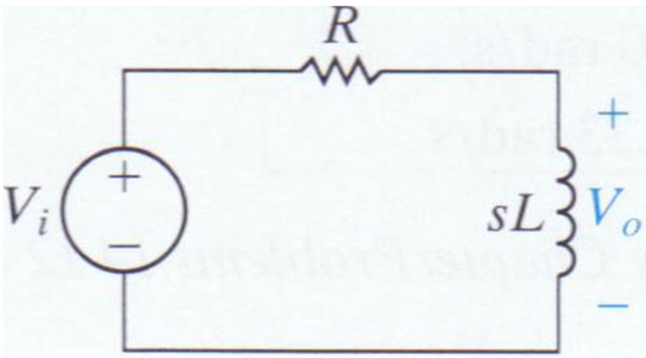
The cutoff frequency ω_c easily seen from the transfer function

$$H(j\omega) = \frac{j\omega}{j\omega + \frac{1}{RC}} = \frac{j\omega}{j\omega + \omega_c} \qquad \omega_c = \frac{1}{RC}$$

The frequency response plots is



For the RL high-pass filter, first construct the s-domain circuit.

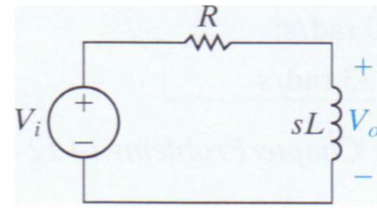


$$V_o(s) = \frac{sL}{R+sL} V_i(s)$$

Once again, we can find the transfer function by the voltage divider.

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{sL}{R+sL} = \frac{s}{s+\frac{R}{L}} = \frac{s}{s+\frac{1}{\frac{L}{R}}}$$

$$H(j\omega) = \frac{j\omega}{j\omega+\frac{R}{L}}$$

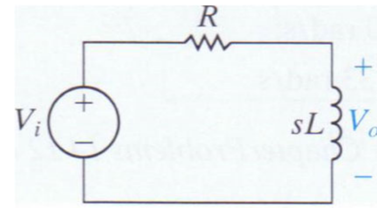


Find the magnitude and phase response.

$$|H(j\omega)| = \frac{|j\omega|}{\left|j\omega + \frac{R}{L}\right|} = \frac{\sqrt{\omega^2}}{\sqrt{\omega^2 + (\frac{R}{L})^2}} = \frac{\omega}{\sqrt{\omega^2 + (\frac{R}{L})^2}}$$

For the phase response

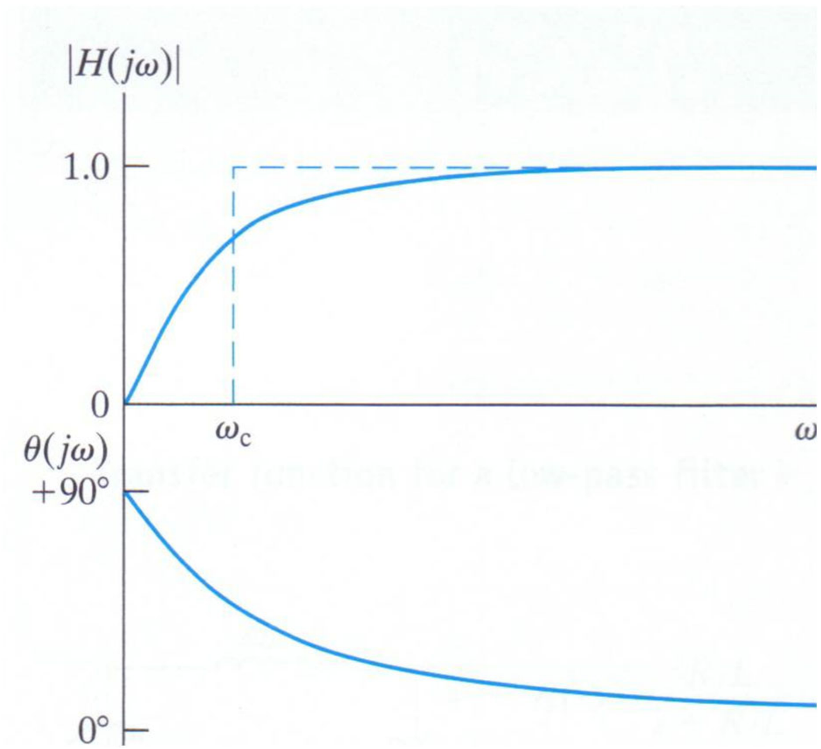
$$\theta(j\omega) = \frac{\angle \tan^{-1}\left(\frac{\omega}{0}\right)}{\angle \tan^{-1}\left(\dfrac{\omega}{\frac{R}{L}}\right)} = \frac{\angle \tan^{-1}(\infty)}{\angle \tan^{-1}\left(\frac{\omega L}{R}\right)} = 90^\circ - \tan^{-1}\left(\frac{\omega L}{R}\right)$$



The cutoff frequency ω_c is

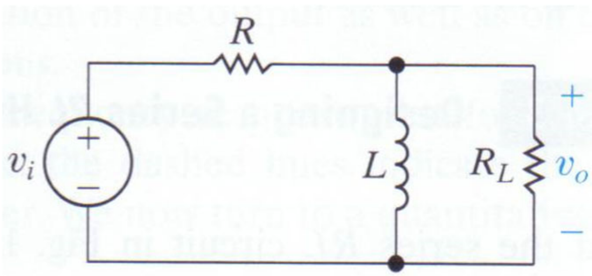
$$H(j\omega)=\frac{j\omega}{j\omega+\frac{1}{\frac{L}{R}}}=\frac{j\omega}{j\omega+\omega_c} \qquad \omega_c=\frac{1}{\tau}=\frac{1}{\frac{L}{R}}=\frac{R}{L}$$

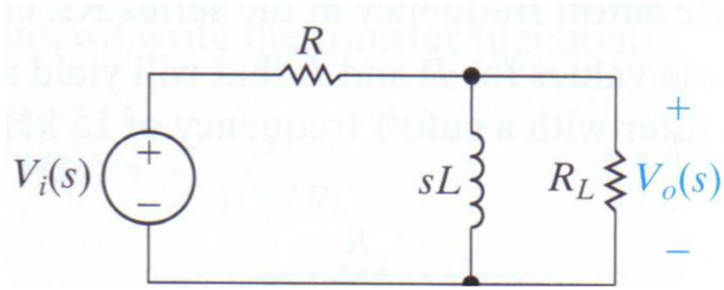
The frequency response plot is



Example 14.4 on page 580

Examine the effect of placing a load resistor in parallel with the inductor in the RL high-pass filter.





First, determine the transfer function for the loaded circuit.

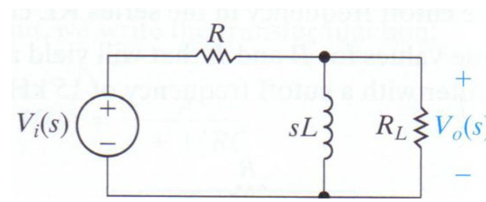
$$V_o = \frac{\frac{R_L sL}{R_L + sL}}{R + \frac{R_L sL}{R_L + sL}} V_i$$

By the use of the parallel R_L and L combination in a voltage divider.

$$= \frac{s\frac{R_L L}{R_L + sL}}{R + s\frac{R_L L}{R_L + sL}} V_i = \frac{R_L + sL}{R_L + sL}\frac{s\frac{R_L L}{R_L + sL}}{R + s\frac{R_L L}{R_L + sL}} V_i = \frac{sR_L L}{sR_L L + R(R_L + sL)} V_i$$

$$= \frac{sR_L L}{sL(R + R_L) + RR_L} V_i = \frac{s\frac{R_L L}{L(R + R_L)}}{s + \frac{RR_L}{L(R + R_L)}} V_i = \frac{s\frac{R_L}{R + R_L}}{s + \frac{R_L}{R + R_L}\frac{R}{L}} V_i$$

Example 14.4 on page 580



$$V_0 = \frac{s\,\frac{R_L}{R+R_L}}{s+\frac{R_L}{R+R_L}\,\frac{R}{L}}V_i$$

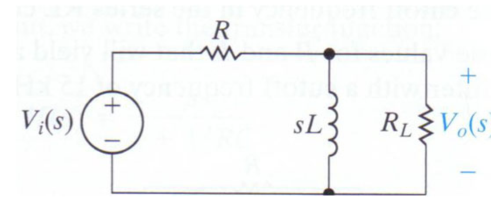
Thus the transfer function is

$$H(s) = \frac{V_0(s)}{V_i(s)} = \frac{s\,\frac{R_L}{R+R_L}}{s+\frac{R_L}{R+R_L}\,\frac{R}{L}} \quad = \frac{Ks}{s+K\,\frac{R}{L}}$$

$$where \quad K = \frac{R_L}{R+R_L}$$

$$and \quad \omega_c = K\,\frac{R}{L}$$

Example 14.4_10ed



Let's use the values for the unloaded circuit of example 14.3.
R = 500 Ω and L = 5.3052 mH, and $\omega_c = 2\pi(15{,}000)$.

And let the loading resistor $R_L = 500\ \Omega$

$$K = \frac{R_L}{R+R_L} = \frac{500}{500+500} = \frac{500}{1{,}000} = \frac{1}{2}$$

So for these resistor values, the gain K is half the unloaded gain.

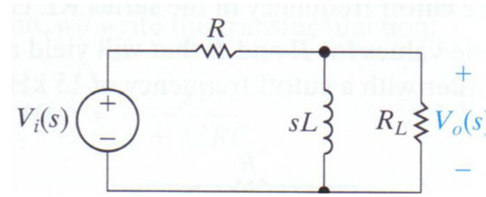
Now let's see the effect of the loading resistor on the corner frequency.

$$\omega_c = K\frac{R}{L} = \frac{R_L}{R+R_L}\frac{R}{L} = \frac{500}{500+500}\frac{500}{5.3052\times10^{-3}} = \frac{1}{2}(2\pi)(15{,}000)\,\frac{rad}{\sec}$$

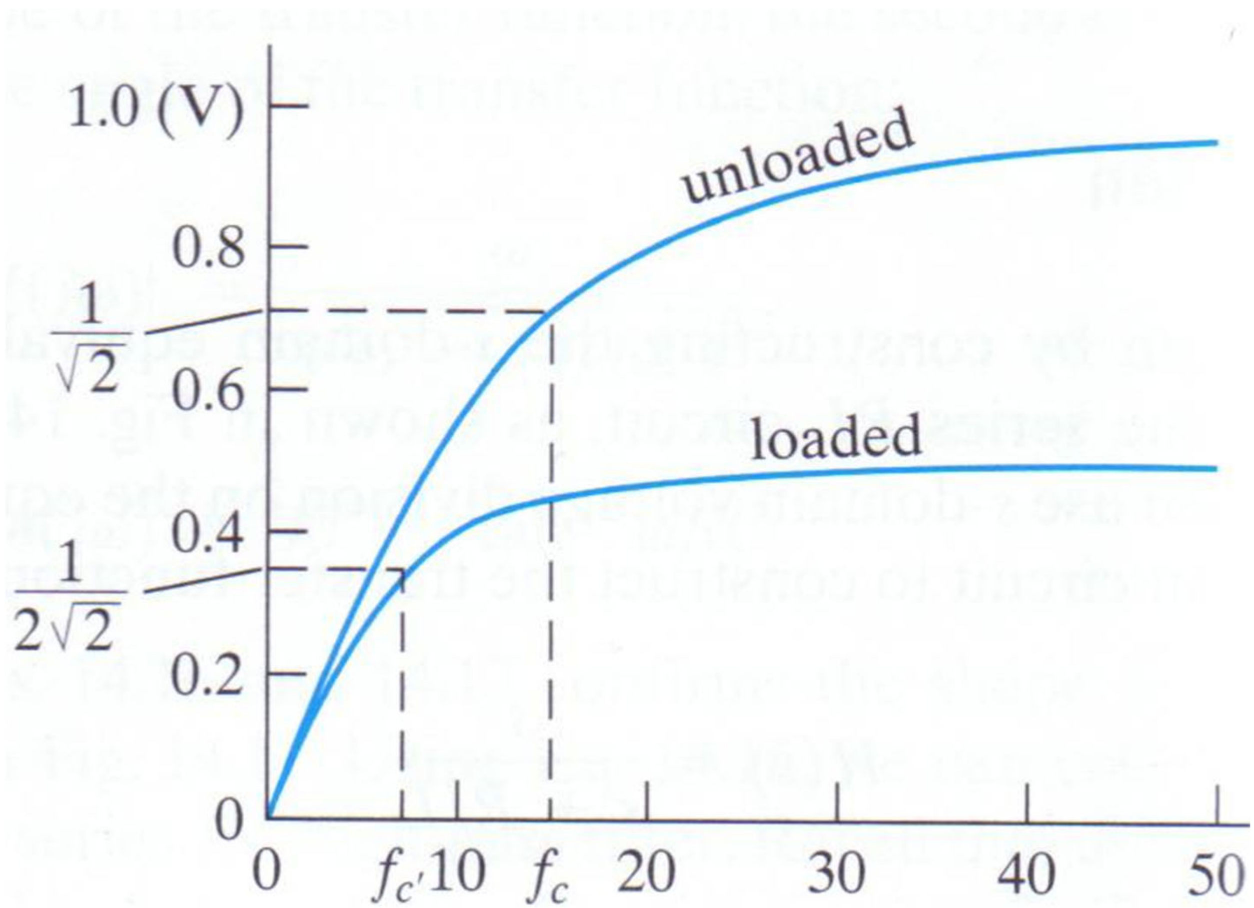
$$f_c = \frac{1}{2}\frac{\omega_c}{2\pi} = \frac{1}{2}\frac{2\pi(15{,}000)}{2\pi} = \frac{1}{2}15{,}000\ Hz \quad = 7{,}500\ Hz$$

So the loaded cutoff frequency went down by ½.

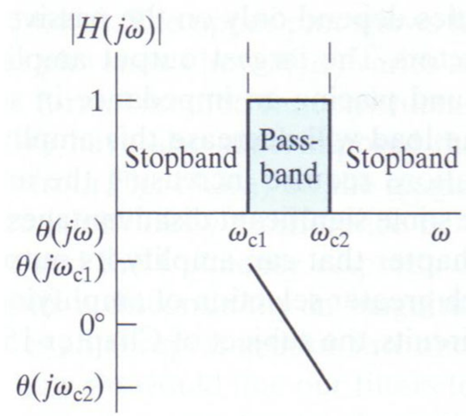
Example 14.4



The loaded and unloaded magnitude plots are



The ideal band-pass filter allows a limited region of frequencies to pass through the filter.



ω_0 is defined as the center frequency of the pass-band region.
It also called the resonant frequency.

ω_0 is the geometric center of the pass-band.

$$\omega_0 = \sqrt{\omega_{c1}\ \omega_{c2}}$$

The maximum of the filter's transfer function H_{max} occurs at ω_0.

$$H_{\max} = \left| H(j\omega_0) \right|$$

The bandwidth B of the filter is expressed as the width of the passband (separation of the two cutoff frequencies).

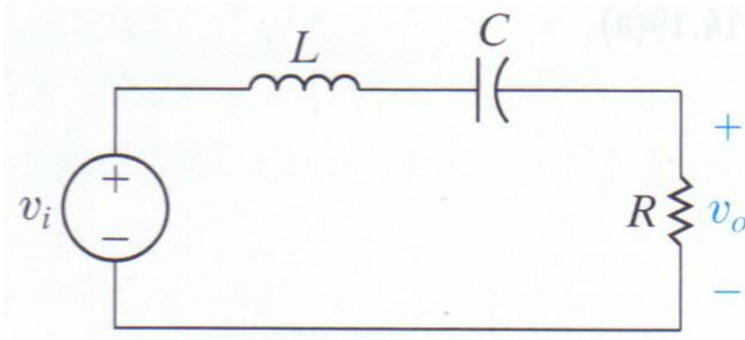
$$B = \omega_{c2} - \omega_{c1} \ in \tfrac{rad}{\sec} \qquad \Rightarrow B = f_{c2} - f_{c1} \ in \ Hz$$

The quality factor Q is a measure of how strongly the filter resonates at the center frequency.

$$Q = \frac{\omega_0}{\beta \, in \tfrac{rad}{\sec}} \quad = \frac{f_0}{\beta \, in \ Hz}$$

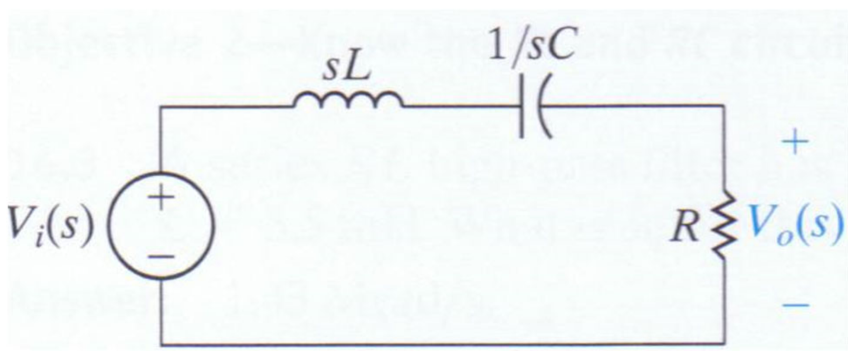
You can think of Q as an indicator of the power of the input signal versus width of the passband.

The series RLC circuit can act as a bandpass filter.



The output is defined as the voltage across the resistor R.

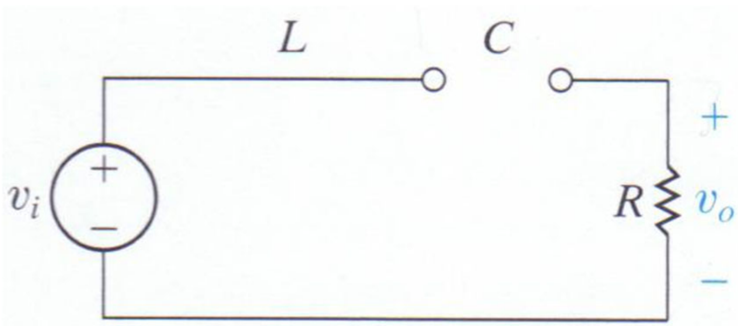
We immediately draw the s-domain circuit.



$$H(s) = \frac{V_o(s)}{V_i(s)}$$

We can see some qualitative circuit behavior.

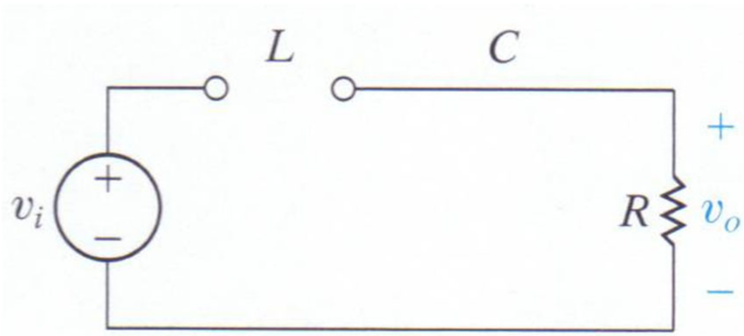
At DC ($\omega = 0$), the capacitor acts as an open circuit and the inductor acts like a short circuit.



$$v_0 = 0$$

At DC ($\omega = 0$), no current flows through the resistor R.

At "high" frequency ($\omega \to \infty$), the inductor acts as an open circuit and the capacitor acts like a short circuit.



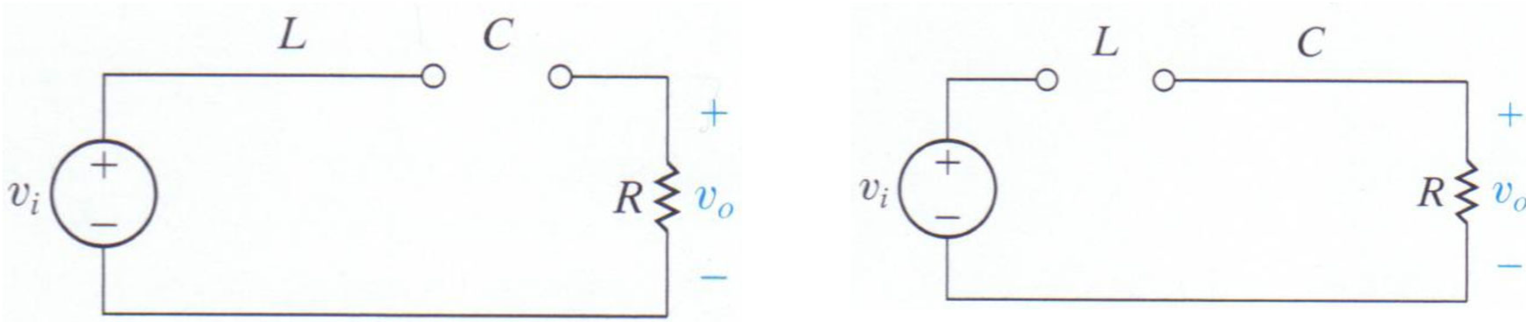
$$v_0 = 0$$

Above a certain frequency we again see that no current flows through the resistor R.

Thus qualitatively we see that there are two stop-bands. One stopband in the low frequency area, and one stopband in the high frequency region.

Intuitively, we know that at some frequency ω_0, the impedances Z_L and Z_C will be of the same magnitude and opposite sign. Thus the output voltage will be only real at ω_0 and is the maximum output voltage.

The phase can also be deduced qualitatively from the time domain circuits.

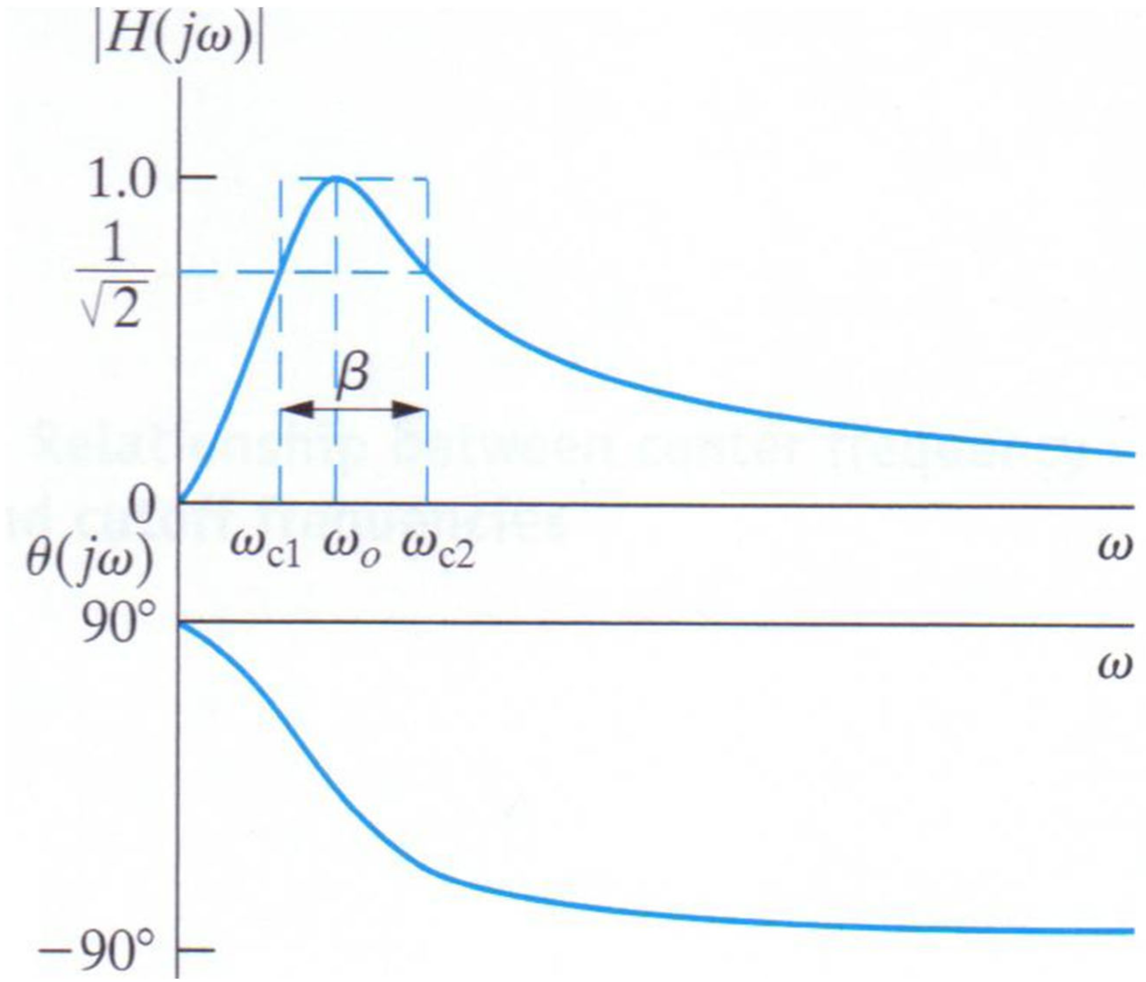


At low frequency, the capacitor behavior has the current leading the voltage by 90°.

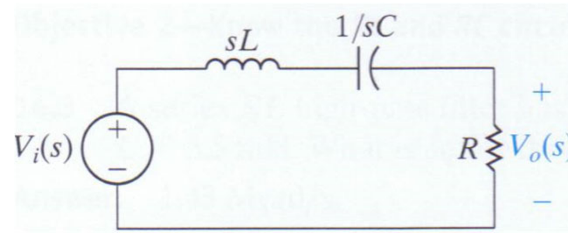
At high frequency, the inductor behavior has the current lagging the voltage by 90°.

At ω_0, the phase is zero due to the reactance = 0 at that frequency.

Thus the magnitude and phase frequency response plot is



Section 14.4 The Series RLC Circuit



The transfer function can be found by venerable voltage divider.

$$V_o(s) = \frac{R}{sL + \frac{1}{sC} + R} V_i(s)$$

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{R}{sL + \frac{1}{sC} + R} = \frac{Rs}{s^2 L + \frac{1}{C} + Rs} = \frac{\frac{R}{L} s}{s^2 + \frac{1}{LC} + s\frac{R}{L}} = \frac{\frac{R}{L} s}{s^2 + s\frac{R}{L} + \frac{1}{LC}}$$

The transfer function H(s) is a second order system as expected.

Depending on the placement of the poles, we will expect exponential and sinusoidal circuit behavior.

It is the specific component values that place the poles. The transfer function then shows the circuit response to a driving input frequency.

The transfer function can be written in terms of jω.

$$H(s) = \frac{\frac{R}{L}s}{s^2 + s\frac{R}{L} + \frac{1}{LC}}$$

$$H(s = j\omega) = \frac{\frac{R}{L}j\omega}{(j\omega)^2 + j\omega\frac{R}{L} + \frac{1}{LC}} = \frac{\frac{R}{L}j\omega}{-\omega^2 + j\omega\frac{R}{L} + \frac{1}{LC}} = \frac{\frac{R}{L}j\omega}{(\frac{1}{LC} - \omega^2) + j\omega\frac{R}{L}}$$

The magnitude and phase of H(jω) are

$$|H(j\omega)| = \frac{\frac{R}{L}\omega}{\sqrt{(\frac{1}{LC} - \omega^2)^2 + (\omega\frac{R}{L})^2}}$$

$$\theta(j\omega) = \tan^{-1}\left(\frac{\frac{R}{L}\omega}{0}\right) - \tan^{-1}\left(\frac{\frac{R}{L}\omega}{\frac{1}{LC} - \omega^2}\right) = 90^\circ - \tan^{-1}\left(\frac{\frac{R}{L}\omega}{\frac{1}{LC} - \omega^2}\right)$$

We can now calculate the five parameters that characterize this RLC band-pass filter.

First, we can find the center frequency ω_0 by understanding this frequency is where $Z_L + Z_C$ = zero.

$$j\omega_0 L + \frac{1}{j\omega_0 C} = 0 \qquad \Rightarrow (j\omega_0 C)\, j\omega_0 L + j\omega_0 C \frac{1}{j\omega_0 C} = 0$$

$$-\omega_0^2 LC + 1 = 0 \qquad \Rightarrow \omega_0^2 LC = 1 \qquad \Rightarrow \omega_0^2 = \frac{1}{LC}$$

$$\omega_0 = \sqrt{\frac{1}{LC}}$$

The text now shows the derivation of the cutoff frequencies. Please see the text for that derivation.

$$\omega_{C1} = \frac{-R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$\omega_{C2} = \frac{+R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

The bandwidth β is

$$\beta = \omega_{C2} - \omega_{C1} \quad = \frac{R}{2L} - \frac{-R}{2L} \quad = \frac{R}{L}$$

The quality factor Q is

$$Q = \frac{\omega_0}{\beta} \quad = \frac{\sqrt{\frac{1}{LC}}}{\frac{R}{L}} \quad = \sqrt{\frac{L^2}{LCR^2}} \quad = \sqrt{\frac{L}{CR^2}}$$

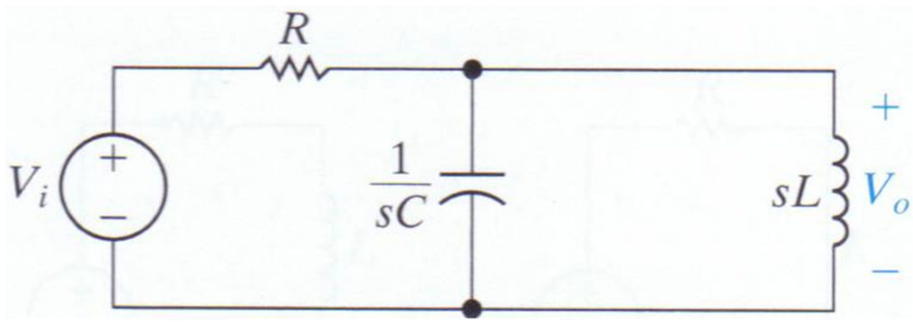
Thus we have found all five parameters ω_0, ω_{c1}, ω_{C2}, β and Q.

The general form of a band-pass filter is

$$H(s) = \frac{K\beta s}{s^2 + \beta s + \omega_0^2}$$

Where K is used to handle a nonzero finite voltage source impedance. See the text.

There is a parallel form of the RLC band-pass filter.

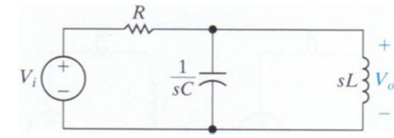


The transfer function H(s) is identical to the series form.

$$H(s) = \frac{K\beta s}{s^2 + \beta s + \omega_0^2}$$

Where the five parameters are only slightly different.

Section 14.4 The Parallel RLC Bandpass Filter



Parallel RLC transfer function $H(s) = \dfrac{K\beta s}{s^2 + \beta s + \omega_0^2}$

$$\omega_0 = \sqrt{\frac{1}{LC}}$$

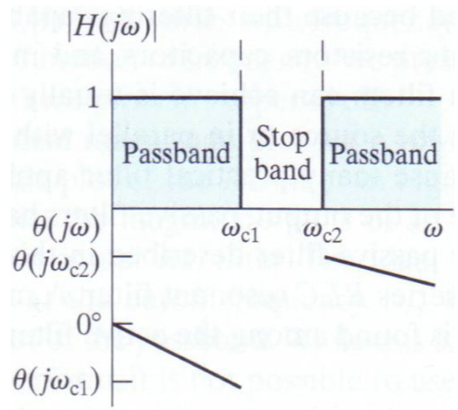
$$\omega_{C1} = \frac{-1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$

$$\omega_{C2} = \frac{+1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$

$$\beta = \omega_{C2} - \omega_{C1} = \frac{1}{RC}$$

$$Q = \frac{\omega_0}{\beta} = \sqrt{\frac{R^2C}{L}}$$

The bandreject filter passes source voltages outside the band between two cutoff frequencies.

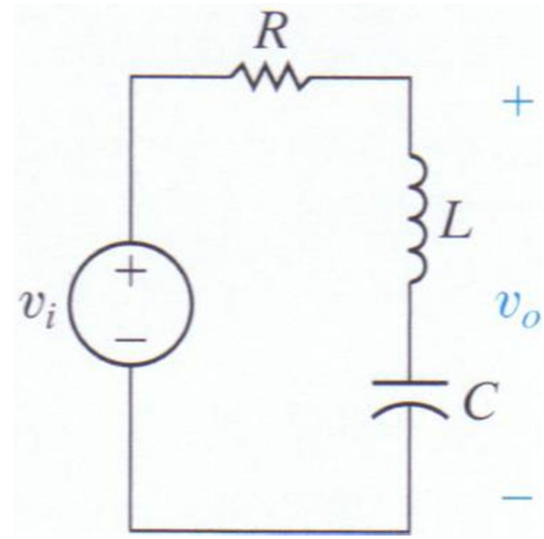


The transfer function H(s) is also a second-order polynomial somewhat similar to the bandpass filter.

$$H(s) = \frac{s^2 + \frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

This filter also uses the same five parameters ω_0, ω_{c1}, ω_{C2}, β and Q.

The series RLC circuit can also be a bandreject filter when the output voltage is defined as that across the inductor and capacitor pair.



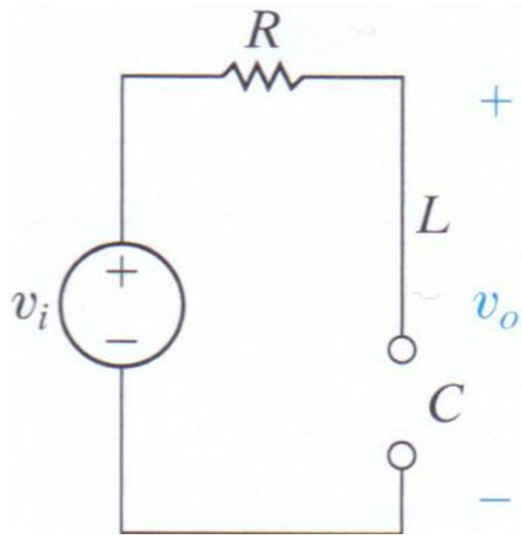
Since we take the output as the voltage across these two elements, we see a complementary frequency behavior.

In the low frequencies, there is a voltage across the capacitor.

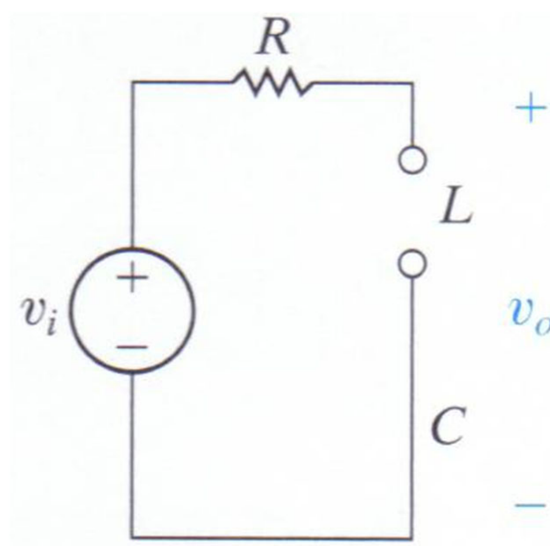
In the high frequencies, there is a voltage across the inductor.

At ω_0, the impedances Z_L and Z_C cancel and there is zero voltage!

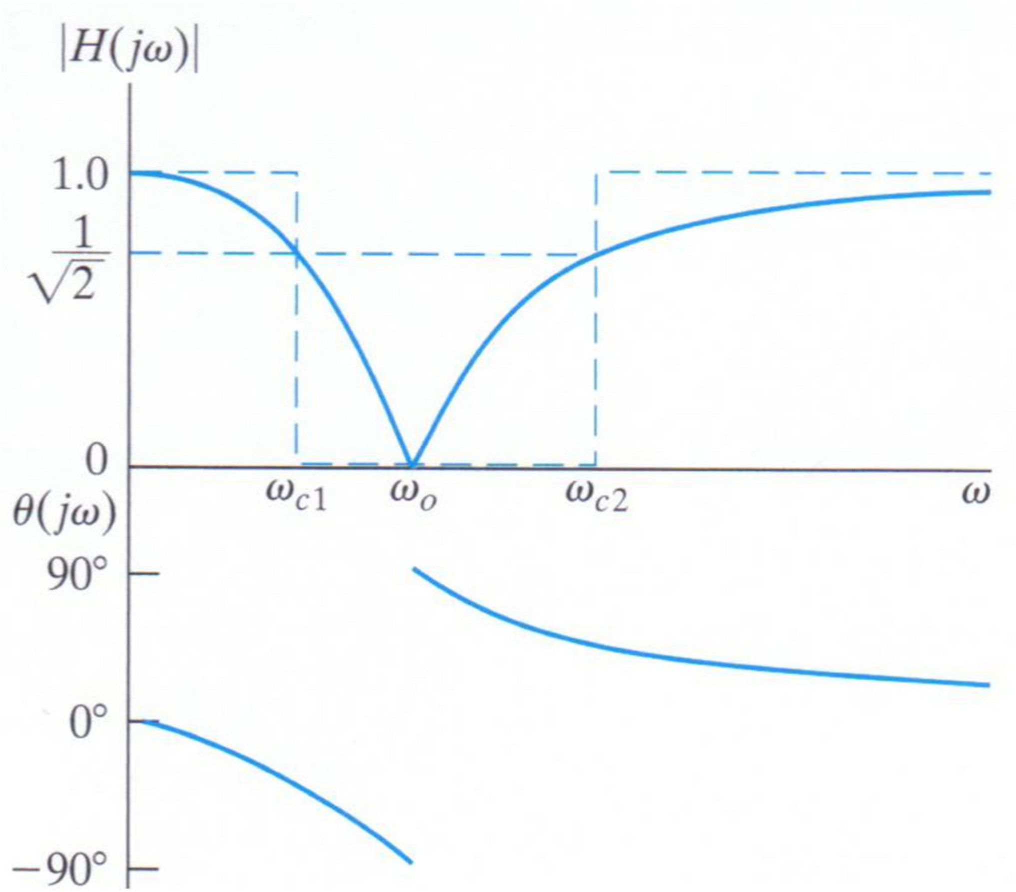
For low frequencies the capacitor is an open circuit.

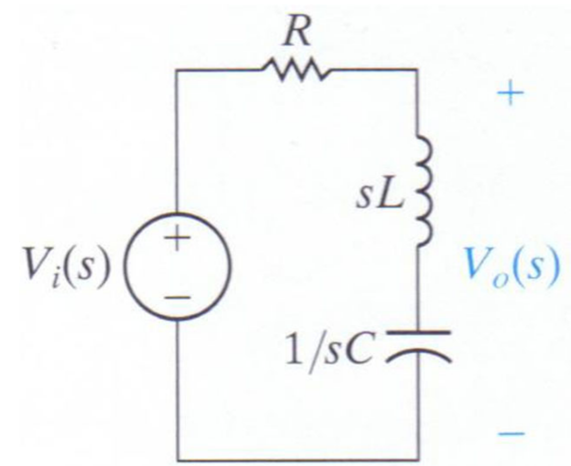


For high frequencies the inductor acts like an open circuit.



The frequency response plot is





As usual, we start the analysis with the s-domain circuit. From the voltage divider we have

$$V_o(s) = \frac{sL + \frac{1}{sC}}{R + sL + \frac{1}{sC}} V_i(s)$$

The transfer function is quickly simplified to

$$H(s) = \frac{sL + \frac{1}{sC}}{R + sL + \frac{1}{sC}} = \frac{s^2 L + \frac{1}{C}}{s^2 L + sR + \frac{1}{C}} = \frac{s^2 + \frac{1}{LC}}{s^2 + s\frac{R}{L} + \frac{1}{LC}}$$

If we only need the steady-state response, then we can let s = j ω and write the transfer function in the phasor domain:

$$H(j\omega) = \frac{(j\omega)^2 + \frac{1}{LC}}{(j\omega)^2 + j\omega\frac{R}{L} + \frac{1}{LC}} = \frac{-\omega^2 + \frac{1}{LC}}{-\omega^2 + j\omega\frac{R}{L} + \frac{1}{LC}} = \frac{\frac{1}{LC} - \omega^2}{\frac{1}{LC} - \omega^2 + j\omega\frac{R}{L}}$$

Section 14.5 Series RLC Bandreject Filters

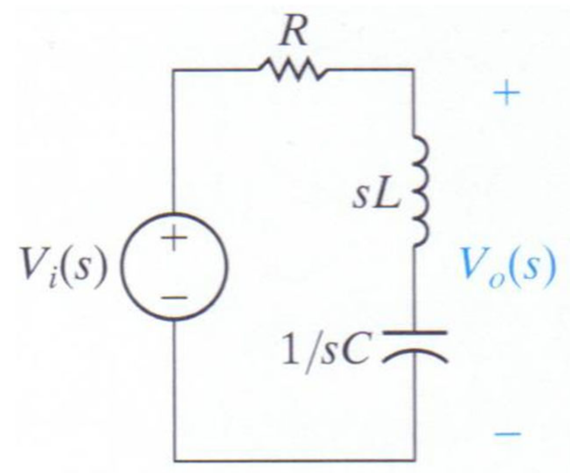
$$H(j\omega) = \frac{-\omega^2 + \frac{1}{LC}}{-\omega^2 + j\omega\frac{R}{L} + \frac{1}{LC}}$$

The magnitude and phase are

$$|H(j\omega)| = \frac{\left|\frac{1}{LC} - \omega^2\right|}{\sqrt{(\frac{1}{LC} - \omega^2)^2 + (\omega\frac{R}{L})^2}}$$

$$\theta(j\omega) = \tan^{-1}\left(\frac{0}{\frac{1}{LC} - \omega^2}\right) - \tan^{-1}\left(\frac{\omega\frac{R}{L}}{\frac{1}{LC} - \omega^2}\right) \quad = -\tan^{-1}\left(\frac{\omega\frac{R}{L}}{\frac{1}{LC} - \omega^2}\right)$$

Section 14.5 Series RLC Bandreject Filter Parameters



The center frequency ω_0 is

$$\omega_0 = \sqrt{\frac{1}{LC}}$$

The cutoff frequencies are

$$\omega_{C1} = \frac{-R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$\omega_{C2} = \frac{+R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

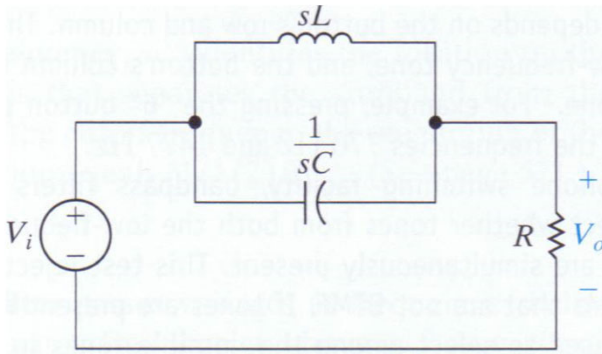
The bandwidth β is

$$\beta = \omega_{C2} - \omega_{C1} \quad = \frac{R}{L}$$

The quality factor Q is

$$Q = \frac{\omega_0}{\beta} \quad = \sqrt{\frac{L}{CR^2}}$$

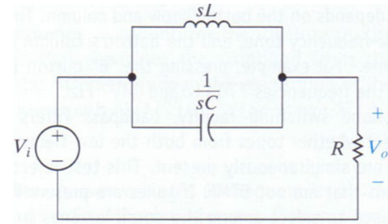
There is a parallel form of the RLC bandreject filter.



The transfer function H(s) for this parallel RLC bandreject circuit is (derivation on next page)

$$H(s) = \frac{s^2 + \frac{1}{LC}}{s^2 + s\frac{1}{RC} + \frac{1}{LC}} \quad = \frac{s^2 + \omega_0^2}{s^2 + \beta s + \omega_0^2}$$

Section 14.5 Parallel RLC Bandreject Filters



The derivation of the s-domain circuit starts with the parallel impedance.

$$Z_{\|} = \frac{sL(\frac{1}{sC})}{sL + \frac{1}{sC}} = \frac{sL}{s^2LC + 1}$$

From the voltage divider, the output voltage is

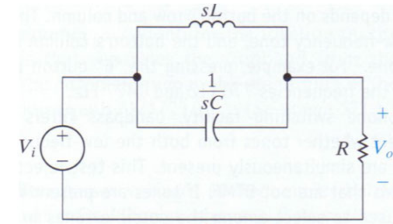
$$V_0(s) = V_i(s)\frac{R}{Z_{\|} + R}$$

The s domain transfer function is

$$H(s) = \frac{V_0(s)}{V_i(s)} = \frac{R}{\frac{sL}{s^2LC+1} + R} = \frac{R(s^2LC+1)}{sL + R(s^2LC+1)} = \frac{RLC(s^2 + \frac{1}{LC})}{RLC\left[s\frac{1}{RC} + s^2 + \frac{1}{LC}\right]}$$

$$= \frac{s^2 + \frac{1}{LC}}{s^2 + s\frac{1}{RC} + \frac{1}{LC}} = \frac{s^2 + \omega_0^2}{s^2 + s\beta + \omega_0^2}$$

Section 14.5 Parallel RLC Bandreject Filters



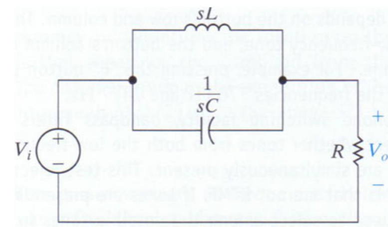
$$H(s) = \frac{s^2 + \frac{1}{LC}}{s^2 + s\frac{1}{RC} + \frac{1}{LC}}$$

If we only need the steady-state response, then we can let s = j ω and write the transfer function in the phasor domain:

$$H(j\omega) = \frac{(j\omega)^2 + \frac{1}{LC}}{(j\omega)^2 + j\omega\frac{R}{L} + \frac{1}{LC}} = \frac{-\omega^2 + \frac{1}{LC}}{-\omega^2 + j\omega\frac{R}{L} + \frac{1}{LC}} = \frac{\frac{1}{LC} - \omega^2}{\frac{1}{LC} - \omega^2 + j\omega\frac{R}{L}}$$

I leave it to you to write the magnitude and phase response of the phasor domain version.

Section 14.4 The Parallel RLC Bandreject Parameters



Parallel RLC Bandreject filter $H(S) = \dfrac{s^2 + \omega_0^2}{s^2 + \beta s + \omega_0^2}$

$$\omega_0 = \sqrt{\frac{1}{LC}}$$

$$\omega_{C1} = \frac{-1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}} \qquad \omega_{C2} = \frac{+1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$

$$\beta = \omega_{C2} - \omega_{C1} = \frac{1}{RC}$$

$$Q = \frac{\omega_0}{\beta} = \sqrt{\frac{R^2 C}{L}}$$

Chapter 14

Frequency Selective Circuits

Text: *Electric Circuits* by J. Nilsson and S. Riedel
Prentice Hall

EEE 117 Network Analysis
Instructor: Russ Tatro