Chapter 13

The Laplace Transform in Circuit Analysis

Text: *Electric Circuits* by J. Nilsson and S. Riedel
Prentice Hall

EEE 117 Network Analysis
Instructor: Russ Tatro

Preview

$$\mathscr{L}\left\{f(t)\right\} = \int_0^\infty f(t)e^{-st}dt$$

The Laplace transform takes a set of linear constant coefficient integrodifferential equations into a set of linear polynomial equations.

The polynomials are much easier to manipulate!

The Laplace transform also inherently includes the initial conditions as part of the transform process.

Time Domain	Freq Domain
Volts	Volts sec
Amps	Amps sec
Ω	Ω
$\left(\frac{V}{A}\right)$	$\left(\frac{V\cdot s}{A\cdot s}\right) = \left(\frac{V}{A}\right)$

Section 13.1 Circuit Elements in the s Domain

The procedure for developing an s-domain equivalent for each circuit element is simple and straightforward.

Just find Laplace transform of each element.

For a resistor, we have

$$v = Ri$$

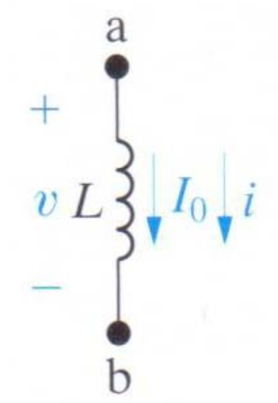
The value of R is a scalar (constant), so the Laplace transform is

$$V = RI$$

$$Where \;\; V = \mathscr{L}\{v\} \qquad And \;\; I = \mathscr{L}\{i\}$$

Section 13.1 An Inductor in the s Domain

The voltage across an inductor is given by the differential equation



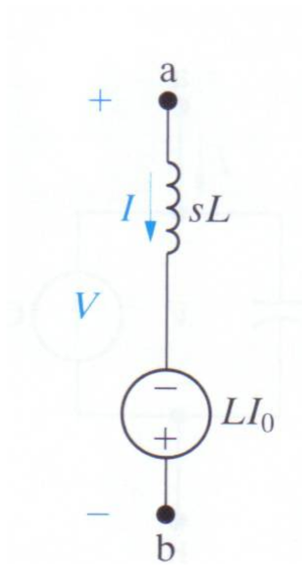
$$v = L\frac{di}{dt}$$

The inductor may be carrying an initial current of I_0 amperes at t = 0.

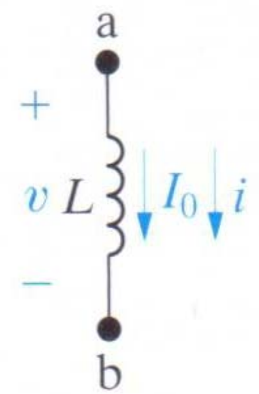
Thus the Laplace transform is

$$V = L\left[sI - i(0^-)\right] \quad = sLI - Li(0^-)$$

$$= sLI - LI_0$$



This is the series form for the inductor.

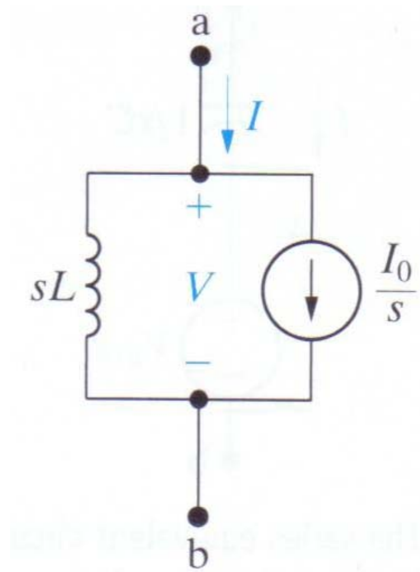


The same inductor element can be modeled by a parallel connection by solving the previous equation for I.

$$V = sLI - LI_0 \quad \Rightarrow I = \frac{V + LI_0}{sL} \quad = \frac{V}{sL} + \frac{I_0}{s}$$

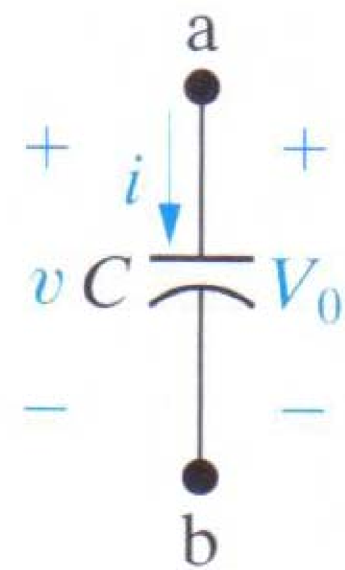
Thus the inductor can be shown as

$$I = \frac{V}{sL} + \frac{I_0}{s}$$



Note that the passive sign convention determines the direction of current flow.

Section 13.1 A Capacitor in the s Domain



The current displaced around a capacitor is given by the differential equation

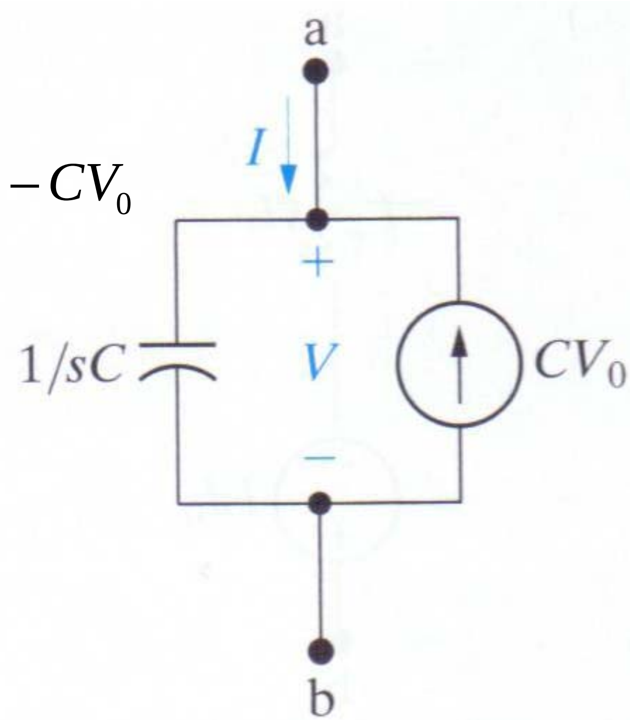
$$i = C\frac{dv}{dt}$$

The capacitor may have an initial voltage of V_0 across it at t = 0.

Thus the Laplace transform is

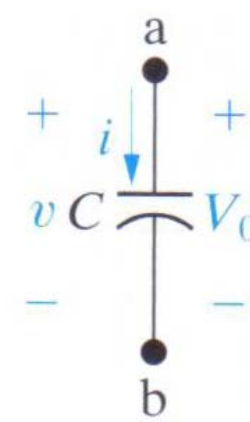
$$I = \frac{V}{\frac{1}{sC}} - CV_0$$

$$I = C\left[sV - v(0^-)\right] \quad = \frac{V}{\frac{1}{sC}} - CV_0$$



Note that Z_C = 1/(sC).

This is the parallel form for the capacitor.

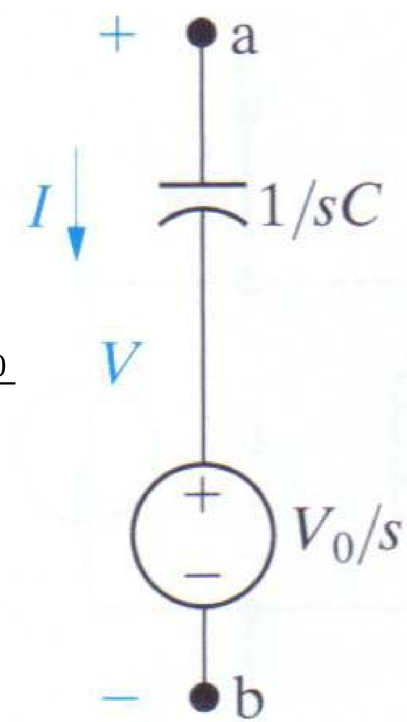


The same capacitor element can be modeled by a series connection by solving the previous equation for V.

$$I = sCV - CV_0 \Rightarrow V = \frac{I + CV_0}{sC} = I\frac{1}{sC} + \frac{V_0}{s}$$

Thus the capacitor can be shown as

$$V = I\frac{1}{sC} + \frac{V_0}{s}$$



Note that the passive sign convention again determines the direction of current flow.

When the capacitor does not have an initial voltage across it at $t = 0^-$, the s-domain representation is simply

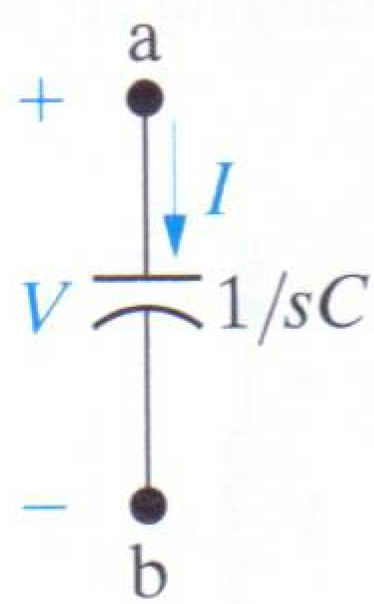


Table 13.1 on page 510 Resistor

Time Domain

Freq Domain

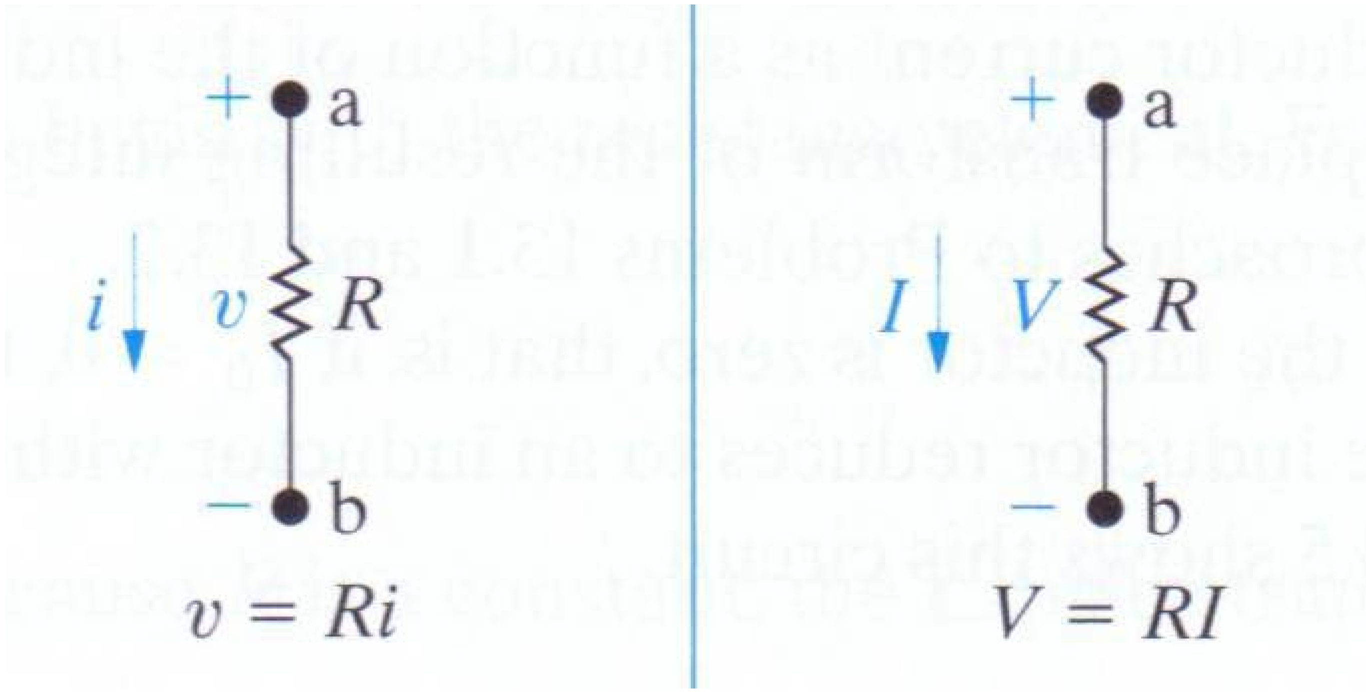


Table 13.1 Inductor

Time Domain

Freq Domain

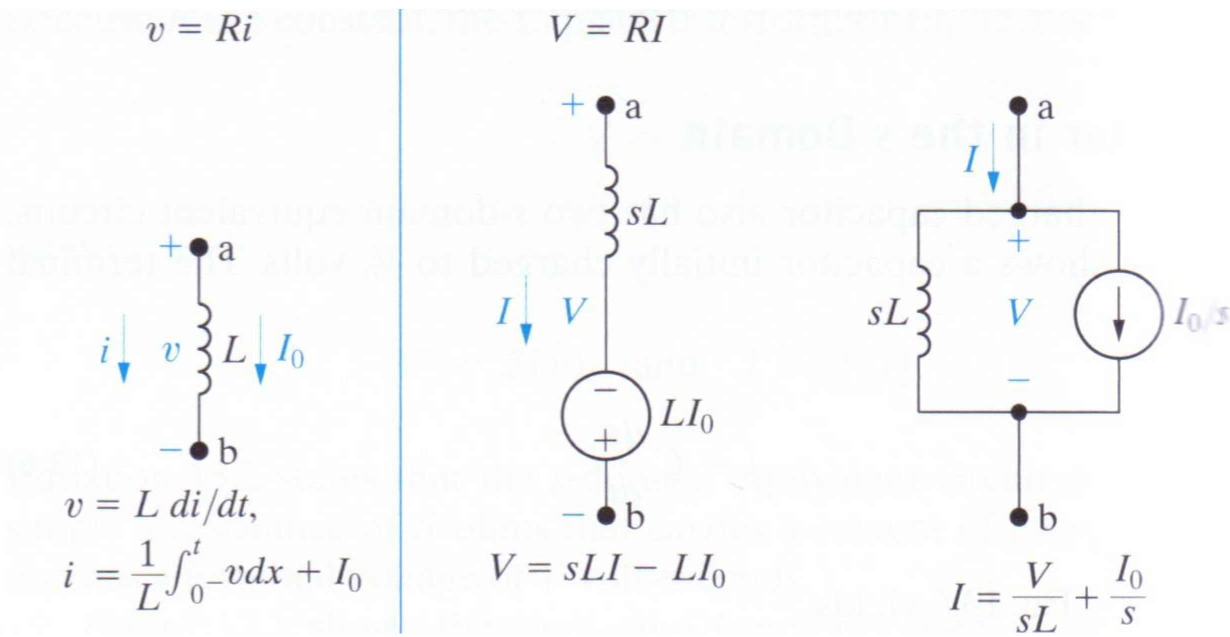
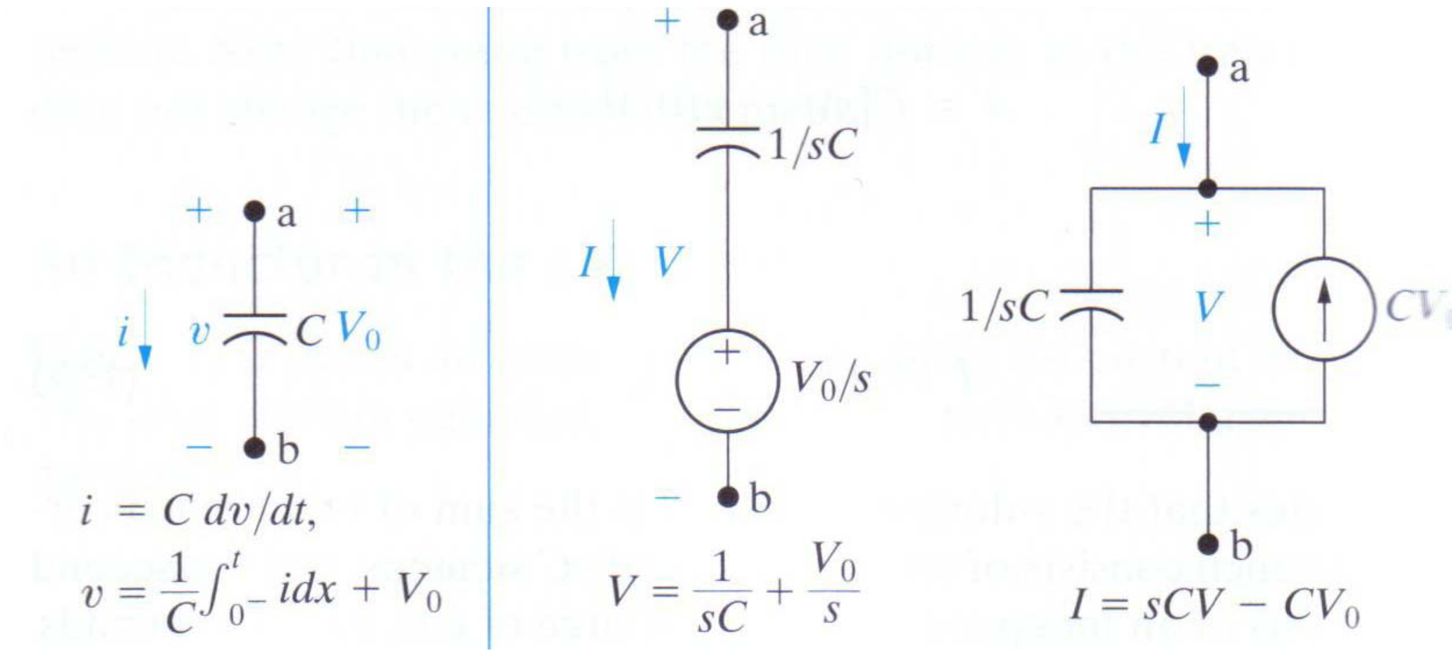


Table 13.1 Capacitor

Time Domain

Freq Domain



When no initial conditions are present, we can write the following equation in the s domain for any of the three circuit elements.

$$V = ZI$$

For the resistor Z = R ohms.

For the inductor Z = sL ohms.

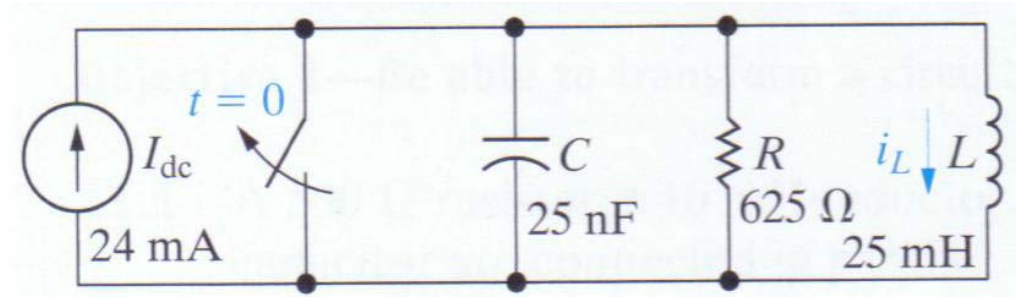
For the capacitor Z = 1/(sC) ohms.

There is now a good news and bad news moment.

Good News! All the circuit analysis techniques used in DC circuits can be directly applied to the s domain circuit.

Bad News! All the circuit analysis techniques used in DC circuits can be directly applied to the s domain circuit. Ya gotta know it!

Example - Step Response of a Parallel RLC Circuit



$$v_C(t<0)=0$$

$$i_L(t<0)=0$$

$$v_R(t<0)=0$$

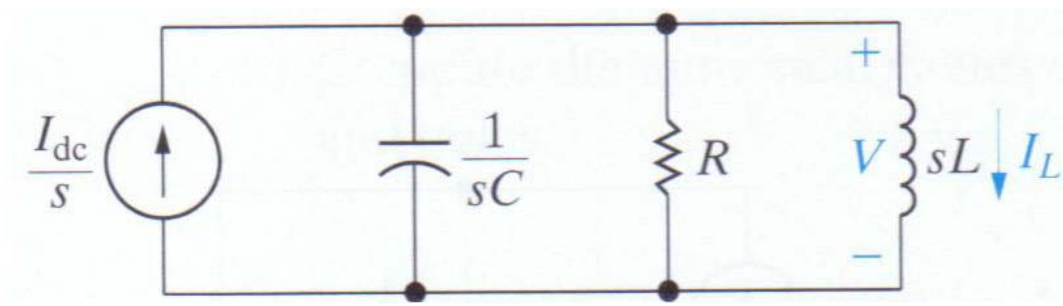
Find the current i_L through the inductor for t > 0.

First rewrite the circuit in the frequency domain.
What are the initial conditions?

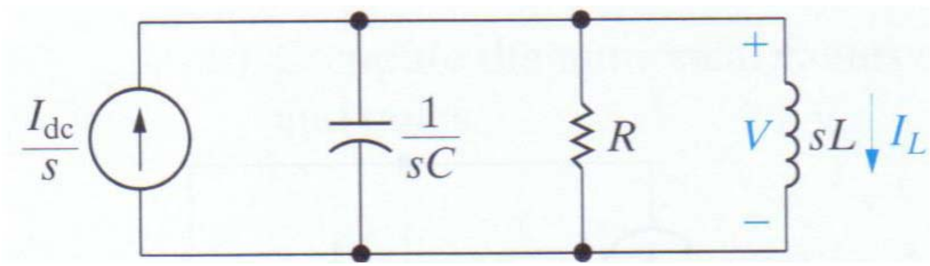
$$I_{DC}u(t) \Leftrightarrow \frac{I_{DC}}{s}$$

$$Z_C = \frac{1}{sC}$$

$$Z_L = sL$$



Example - Step Response of a Parallel RLC Circuit



Determine I_L in the frequency domain.

$$I_L = \frac{V}{sL}$$

Thus we need to find V (lots of parallel elements) and can write the KCL equation.

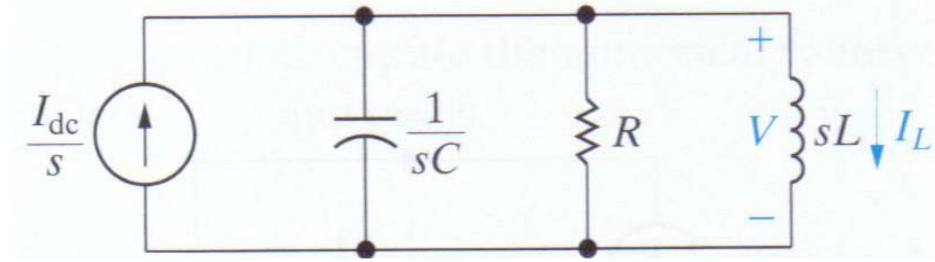
$$-\frac{I_{DC}}{s} + \frac{V}{Z_C} + \frac{V}{R} + \frac{V}{Z_L} = 0$$

$$V\left(\frac{1}{\frac{1}{sC}} + \frac{1}{R} + \frac{1}{sL}\right) = \frac{I_{DC}}{s}$$

$$V = \frac{I_{DC}}{s\left(sC + \frac{1}{R} + \frac{1}{sL}\right)} \quad = \frac{\frac{I_{DC}}{C}}{s^2 + s\frac{1}{RC} + \frac{1}{LC}}$$

Example - Step Response of a Parallel RLC Circuit

$$V = \frac{\frac{I_{DC}}{C}}{s^2 + s\frac{1}{RC} + \frac{1}{LC}}$$



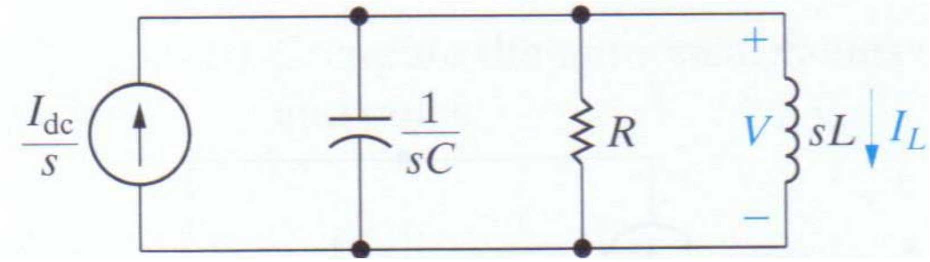
Now that we know the voltage V, we can write the current I_L as

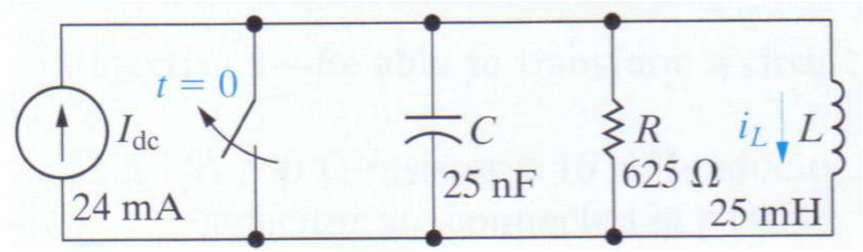
$$I_L = \frac{V}{sL} = \frac{1}{sL}\frac{\frac{I_{DC}}{C}}{s^2 + s\frac{1}{RC} + \frac{1}{LC}} = \frac{\frac{I_{DC}}{LC}}{s\left(s^2 + s\frac{1}{RC} + \frac{1}{LC}\right)}$$

Notice that I was not in a hurry to substitute in the numeric values. The above result is generic. Need to change the variables – no problem!

Example - Step Response of a Parallel RLC Circuit

$$I_L = \frac{\frac{I_{DC}}{LC}}{s\left(s^2 + s\frac{1}{RC} + \frac{1}{LC}\right)}$$





$$\frac{I_{DC}}{LC} = \frac{24\times10^{-3}}{(25\times10^{-3})(25\times10^{-9})} = 384\times10^5$$

$$\frac{1}{RC} = \frac{1}{(625)(25\times10^{-3})} = 64,000$$

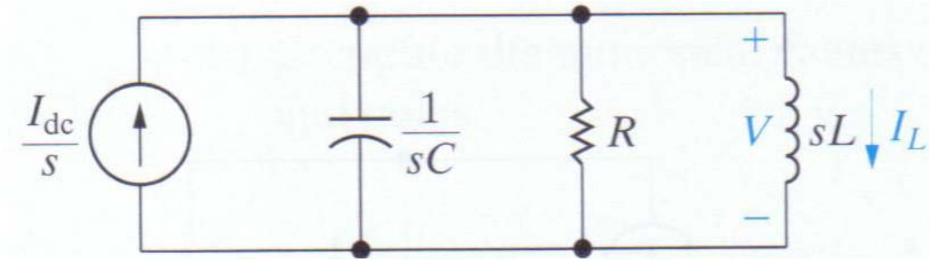
$$\frac{1}{LC} = \frac{1}{(25\times10^{-3})(25\times10^{-9})} = 16\times10^8$$

The numeric value of I_L is

$$I_L = \frac{384\times10^5}{s\left(s^2 + 64,000s + 16\times10^8\right)}$$

Example - Step Response of a Parallel RLC Circuit

$$I_L = \frac{384\times10^5}{s\left(s^2+64,000s+16\times10^8\right)}$$



Now is a good time to test the results.

We know as t $\rightarrow \infty$, the time domain current i_L = 24 mA.

Test this with the Final Value Theorem.

$$\underbrace{\lim}_{s=0} sI_L = s\frac{384\times10^5}{s\left(s^2+64,000s+16\times10^8\right)} = \frac{384\times10^5}{s^2+64,000s+16\times10^8}$$

$$= \frac{384\times10^5}{16\times10^8} = 24\times10^{-3}$$

Answer checks.

Example - Step Response of a Parallel RLC Circuit

$$I_L = \frac{384\times10^5}{s\left(s^2+64,000s+16\times10^8\right)}$$

Now find the partial fraction expansion. I found the roots using Matlab “roots”.

$$\frac{384\times10^5}{s\left(s^2+64,000s+16\times10^8\right)} = \frac{A}{s} + \frac{B}{s+32,000-j24,000} + \frac{B^*}{s+32,000+j24,000}$$

$$A = \frac{384\times10^5}{s^2+64,000s+16\times10^8}\bigg|_{s=0} = \frac{384\times10^5}{16\times10^8} = 24\times10^{-3}$$

$$B = \frac{384\times10^5}{s\left(s+32,000+j24,000\right)}\bigg|_{s=-32,000+j24,000}$$

$$= \frac{384\times10^5}{\left(-32,000+j24,000\right)\left(-32,000+j24,000+32,000+j24,000\right)}$$

$$= \frac{384\times10^5}{\left(-32,000+j24,000\right)\left(j48,000\right)} = \frac{384\times10^5}{-1.152\times10^9 - j1.536\times10^9}$$

$$= \frac{384\times10^5}{1.92\times10^9\angle-126.87^\circ} = 20\times10^{-3}\angle126.87^\circ$$

Example - Step Response of a Parallel RLC Circuit

The PFE is

$$I_L = \frac{24\times10^{-3}}{s} + \frac{20\times10^{-3}\angle 126.87^\circ}{s+32,000 - j24,000} + \frac{20\times10^{-3}\angle -126.87^\circ}{s+32,000 + j24,000}$$

So now write $i_L(t)$ from the inverse Laplace transform.

$$i_L(t) = \left[24 + 2(20e^{-32,000t}\cos(24,000t + 126.87^\circ)\right]u(t)\,mA$$

$$= \left[24 + 40e^{-32,000t}\cos(24,000t + 126.87^\circ\right]u(t)\,mA$$

Now, wasn’t that easier than solving all those nasty integrodifferential equations?

Section 13.4 The Transfer Function

The transfer function is defined as the s-domain ratio of the (Laplace transformed) output to the input.

$$H(s) = \frac{Y(s)}{X(s)}$$

Y(s) is called the response of the system.

X(s) is the input to the system.

Thus a system can have many responses – it just depends on what is *defined* as the output.

For linear lumped parameter circuits (which is most of your undergraduate study), H(s) is always a rational function of s.

$$H(s) = \frac{Y(s)}{X(s)}$$

Complex poles and zeros *always* appear in conjugate pairs.

The poles (from the denominator) of H(s) must lie in the left hand plane (LHP) for a bounded response.

The zeros (from the numerator) of H(s) may lie anywhere.

We can write the circuit output Y(s) as the product of the transfer function H(s) and the driving function X(s).

$$Y(s) = H(s)X(s)$$

For the inputs of common use in circuits, the input X(s) is also a rational function.

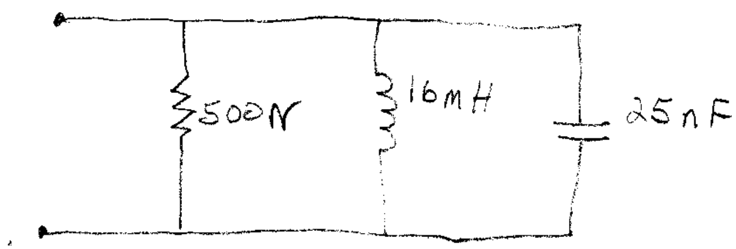
Thus by partial fraction expansion, we can form the poles of H(s) and X(s) into a summation of terms.

By definition, the poles of X(s) give rise to the steady state solution.

The poles of H(s) give rise to the transient solution.

Assessment Problem 13.1 on page 512

A 500 Ω resistor, a 16 mH inductor, and a 25 nF capacitor are connected in parallel.



Express the admittance of this parallel combination of elements as a rational function of s.

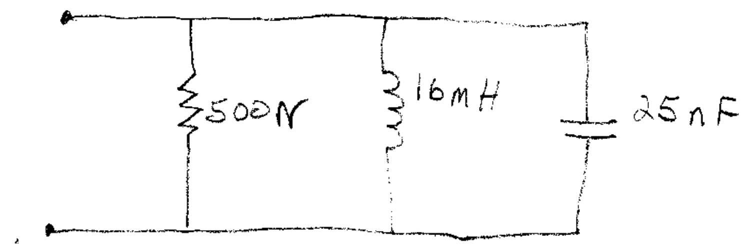
Admittance is given by

$$Y = \frac{1}{Z} \qquad = \frac{1}{R} + \frac{1}{sL} + \frac{1}{\frac{1}{sC}} \qquad = \frac{1}{R} + \frac{1}{sL} + sC$$

$$= \frac{\frac{s}{R} + \frac{1}{L} + s^2 C}{s} \qquad = \frac{C\left(\frac{s}{RC} + \frac{1}{LC} + s^2\right)}{s} \qquad = \frac{C\left(s^2 + s\frac{1}{RC} + \frac{1}{LC}\right)}{s}$$

Assessment Problem 13.1

$$Y = \frac{C\left(s^2 + s\frac{1}{RC} + \frac{1}{LC}\right)}{s}$$



Now find the numeric result.

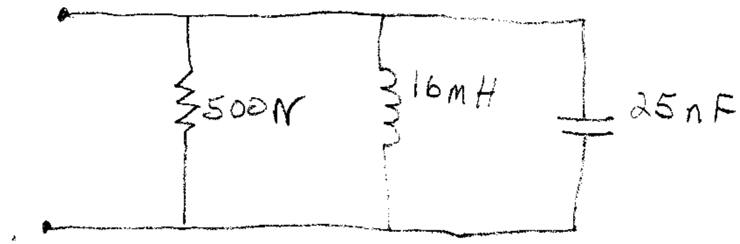
$$\frac{1}{RC} = \frac{1}{(500\Omega)(25\times10^{-9}F)} \qquad = 80,000\,\tfrac{1}{\text{sec}}$$

$$\frac{1}{LC} = \frac{1}{(16\times10^{-3}H)(25\times10^{-9}F)} \qquad = 25\times10^{8}\,\tfrac{1}{\text{sec}^2}$$

Thus the numerical admittance is

$$Y = \frac{(25\times10^{-9})\left(s^2 + 80,000s + 25\times10^{8}\right)}{s}$$

Assessment Problem 13.1



Now find the zeros and poles of the admittance Y.

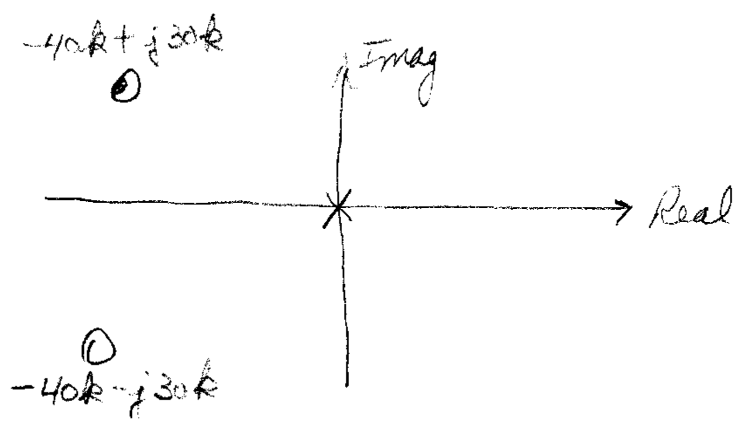
$$Y = \frac{(25\times 10^{-9})\left(s^2 + 80,000s + 25\times 10^8\right)}{s}$$

$$= \frac{(25\times 10^{-9})(s + 40,000 + j30,000)(s + 40,000 - j30,000)}{s}$$

There is a single pole at s = 0.

There is a complex zero.

$$s = -40,000 \pm j30,000$$



Once we have the transfer function, we do not need to perform a separate analysis to determine the circuit's steady-state response.

We will use the transfer function to relate the steady-state response to the excitation source. Let

$$x(t) = A\cos(\omega t + \phi)$$

Expand this sinusoid by using the following trig identity.

$$\cos(\omega t + \phi) = \cos\omega t\cos\phi - \sin\omega t\sin\phi$$

Thus

$$x(t) = A\cos\omega t\cos\phi - A\sin\omega t\sin\phi \quad = \left(A\cos\phi\right)\cos\omega t - \left(A\sin\phi\right)\sin\omega t$$

Recall the following Laplace transform pairs.

$$\cos(\omega t) \Leftrightarrow \frac{s}{s^2+\omega^2} \qquad \sin(\omega t) \Leftrightarrow \frac{\omega}{s^2+\omega^2}$$

So we can write the original sinusoid as

$$X(s) = \frac{(A\cos\phi)s}{s^2+\omega^2} - \frac{(A\sin\phi)\omega}{s^2+\omega^2} = \frac{A(s\cos\phi - \omega\sin\phi)}{s^2+\omega^2}$$

$$= \frac{A(s\cos\phi - \omega\sin\phi)}{(s-j\omega)(s+j\omega)}$$

So we see that, not unexpectedly, the input transforms to the s-domain complex conjugate.

So now we need to form the response Y(s).

$$Y(s) = H(s)X(s) \quad = H(s)\frac{A\left(s\cos\phi - \omega\sin\phi\right)}{\left(s - j\omega\right)\left(s + j\omega\right)}$$

The text now does some hand waving and states that H(s) contributes only to the transient response.

Recall that for a real signal (realizable in the lab), the poles of H(s) must lie in the LHP. Thus h(t) in the time domain must exponentially decay to zero.

Since we are interested in the steady-state solution at the moment, this allows us to simplify the derivation.

$$Y(s) = \frac{K}{s - j\omega} + \frac{K^*}{s + j\omega} + \sum \text{terms generated by the poles of } H(s)$$

By the partial fraction expansion, we have K equal to

$$K = H(s)\frac{A\left(s\cos\phi - \omega\sin\phi\right)}{\left(s + j\omega\right)}\Bigg|_{s=j\omega} \quad = H(j\omega)\frac{A\left(j\omega\cos\phi - \omega\sin\phi\right)}{\left(j\omega + j\omega\right)}$$

$$= H(j\omega)\frac{A\left(j\omega\cos\phi - \omega\sin\phi\right)}{2j\omega} \quad = H(j\omega)\frac{A\left(\cos\phi + j\sin\phi\right)}{2}$$

By Euler's Identity $e^{j\phi} = \cos\phi + j\sin\phi$

$$= H(j\omega)\frac{Ae^{j\phi}}{2}$$

In general, H(jω) is a complex quantity.

$$H(j\omega) = |H(j\omega)| e^{j\theta(\omega)}$$

Since the phase angle θ varies with frequency ω, we write the phase of H(jω) as θ(ω).

So now we can write the coefficient K as

$$K = H(j\omega)\frac{Ae^{j\phi}}{2} \quad = \frac{1}{2}|H(j\omega)| e^{j\theta(\omega)} Ae^{j\phi} \quad = \frac{A}{2}|H(j\omega)| e^{j(\theta(\omega)+\phi)}$$

For a linear, time-invariant, lumped parameter system, the frequency of the response must equal the frequency of the input. So now all the pieces are in place to write the steady-state response.

See Table 12.3 on page 493 for the transform pair for this complex conjugate PFE.

$$2|K|e^{-\alpha t}\cos(\beta t+\theta)u(t) \Leftrightarrow \frac{K}{s+\alpha-j\beta}+\frac{K^*}{s+\alpha+j\beta}$$

Thus

$$Y(s)=\frac{\frac{A}{2}|H(j\omega)|e^{j(\theta(\omega)+\phi)}}{s+\alpha-j\beta}+\frac{\frac{A}{2}|H(j\omega)|e^{-j(\theta(\omega)+\phi)}}{s+\alpha+j\beta}$$

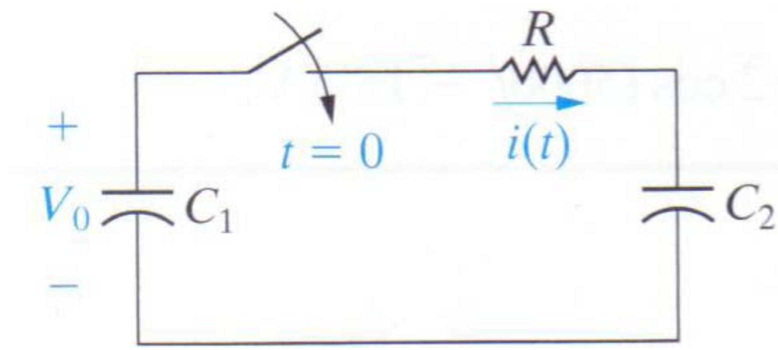
$$y_{ss}(t)=2\left(\frac{A}{2}|H(j\omega)|\right)e^{-0t}\cos(\omega t+\theta(\omega)+\phi) = A|H(j\omega)|\cos(\omega t+\theta(\omega)+\phi)$$

Section 13.8 The Impulse Function in Circuit Analysis

Impulse functions occur in circuit analysis either because of a switching operation or because a circuit is excited by an impulsive source.

If we create the frequency domain equivalent circuit correctly, these impulse functions are handled as a routine matter in the course of solving the frequency domain circuit equations.

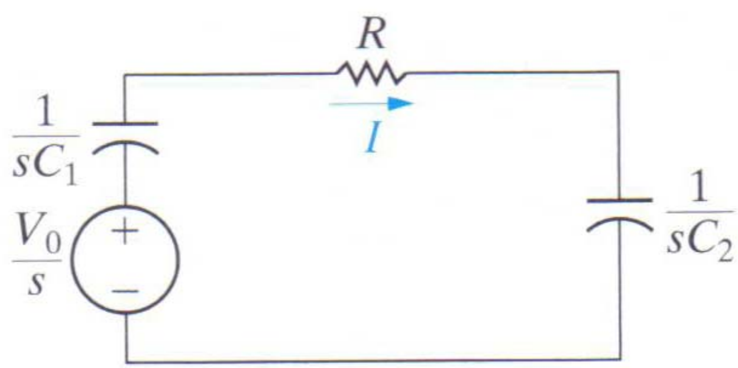
Example



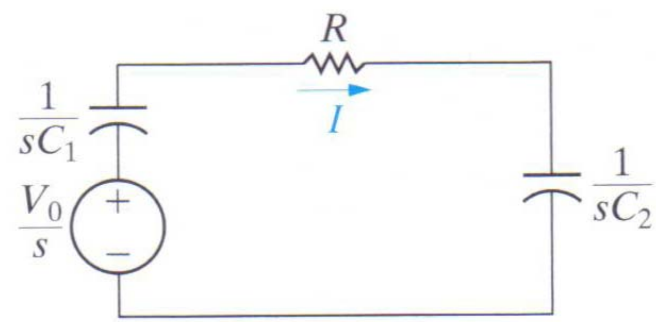
In the circuit above, the capacitor C_1 is charged to an initial voltage V_0 prior to closing the switch at t = 0.

The capacitor C_2 is discharged so that $V_{C2} = 0$ prior to closing the switch at t = 0.

The s-domain equivalent circuit for $t > 0^-$ is



Example



By KVL.

$$-\frac{V_0}{s} + V_{C1} + IR + V_{C2} = 0 \qquad V_{C1} = I\frac{1}{sC_1} \qquad V_{C2} = I\frac{1}{sC_2}$$

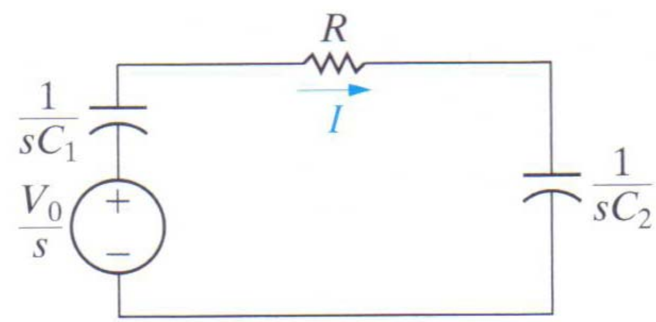
$$\frac{V_0}{s} = I\left(\frac{1}{sC_1} + R + \frac{1}{sC_2}\right) = I\left(R + \frac{1}{sC_1} + \frac{1}{sC_2}\right) = I\left[R + \frac{1}{s}\left(\frac{1}{C_1} + \frac{1}{C_2}\right)\right]$$

Capacitors add in parallel. So for this series connection we have

$$\frac{1}{C_{Equivalent}} = \frac{1}{C_1} + \frac{1}{C_2}$$

$$\frac{1}{sC_1} + \frac{1}{sC_2} = \frac{1}{s}\left(\frac{1}{C_1} + \frac{1}{C_2}\right) = \frac{1}{s}\left(\frac{1}{C_{Equiv}}\right)$$

Example



Thus

$$I\left(R+\frac{1}{sC_{Equiv}}\right)=\frac{V_0}{s}$$

$$I\left(\frac{RsC_{Equiv}+1}{sC_{Equiv}}\right)=\frac{V_0}{s} \quad \Rightarrow I=\frac{V_0}{s}\left(\frac{sC_{Equiv}}{RsC_{Equiv}+1}\right)$$

$$I=\frac{V_0}{R}\left(\frac{1}{s+\frac{1}{RC_{Equiv}}}\right)$$

Thus the inverse Laplace transform is

$$i(t)=\frac{V_0}{R}e^{-\frac{t}{RC_{Equiv}}}u(t)$$

Section 13.8 The Impulse Function in Circuit Analysis

Read the rest of section 13.8 on your own.

Students report difficulty in dealing with the frequency domain circuits. In my opinion, this is due to a lack of confidence or knowledge that was supposed to be gained in Engr 17.

At this point work lots of problems. If you are still weak, come see me. “Reading” the problems just does get the skills ingrained.

Using Matlab with symbolic equations

The algebra in the s domain can be tedious.

Node and Mesh analysis results in symbolic equations in terms of a variable and “s”.

If the Matlab “Symbolic Toolbox” is available, there is a “solve” function that might speed up your work.

Using Matlab with symbolic equations

For example, Given

$$V_1(5)+V_2(-2)=\frac{150}{s} \qquad and \quad V_1(-25s)+V_2\left[s^2+25s+50\right]=75s$$

In Matlab, enter

```
syms V1 V2 s
[Sol_V1,Sol_V2]=solve([V1*5+V2*-2==150/s, V1*(-25*s)+V2*(s^2+25*s+50)==75*s],[V1,V2])
```

Matlab returns the symbolic solution as

```
Sol_V1 = (30*(2*s + 5))/(s*(s + 5))
Sol_V2 = 75/(s + 5)
```

Using Linear Algebra with symbolic equations

The algebra in the s domain can also be solved using Linear Algebra and the use of Cramer's Rule.

This course will not explain how to use Cramer's rule! But for those who have the needed background can review the appendix of the textbook.

Find the determinant of the Node or Mesh set of equations.

$$\Delta = 5(s^2 + 25s + 50) - 50s \Rightarrow 5(s^2 + 15s + 50) = 5(\mathrm{s}+5)(\mathrm{s}+10)$$

Then follow Cramer's Rule to find either V1 or V2. For example, V2 is

$$|A_2| = 5(75s) + 25s\left(\frac{75}{s}\right) = 375\mathrm{s} + 3,750 = 375(\mathrm{s}+10)$$

$$V_2 = \frac{|A_2|}{\Delta} = \frac{375(\mathrm{s}+10)}{5(s+5)(s+10)} = \frac{75}{s+5}$$

Chapter 13

The Laplace Transform in Circuit Analysis

Text: *Electric Circuits* by J. Nilsson and S. Riedel
Prentice Hall

EEE 117 Network Analysis
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