Philosophy 60
Predicate calculus worksheet

Derive the following using all known inferences rules and equivalences from propositional logic as well as the ∀E, ∀I, ∃I and ∃E.

1. $\exists y Gya, \forall z(Gza \rightarrow Gaz) \vdash \exists x Gax$

1.	$\exists y Gya$	A
2.	$\forall z(Gza \rightarrow Gaz)$	A
3.	$\quad$ Gba	H for ∃E
4.	$\quad Gba \rightarrow Gab$	2, ∀E
5.	$\quad$ Gab	3,4 →E
6.	$\quad \exists x Gax$	5, ∃I
7.	$\exists x Gax$	1, 3-6 ∃E

Note, that step 7 is permissible here because b does not occur in any assumption, in line 1 (which in this case is an assumption), or in line 7 itself. The name ‘a’ does occur in line 7 but that is ok, because ‘a’ is not the name we hypothesized.

2. $\forall x \forall y(Fxy \rightarrow Fyx)$, $\exists xFxa \vdash \exists xFax$

1.	$\forall x \forall y(Fxy \rightarrow Fyx)$	A
2.	$\exists xFxa$	A
3.	Fba	H
4.	$\forall y(Fby \rightarrow Fyb)$	1, $\forall$E
5.	$Fba \rightarrow Fab$	4, $\forall$E
6.	Fab	3,5 $\rightarrow$E
7.	$\exists xFax$	6, $\exists$I
8.	$\exists xFax$	2, 3-7, $\exists$E

3. $\forall xFx \lor \forall xGx \vdash \exists x(Fx \lor Gx)$

1.	$\forall xFx \lor \forall xGx$	A
2.	$\forall xFx$	H
3.	Fa	2, $\forall$E
4.	$Fa \lor Ga$	3, vI
5.	$\exists x(Fx \lor Gx)$	4, $\exists$I
6.	$\forall xFx \rightarrow \exists x(Fx \lor Gx)$	2-5 $\rightarrow$I
7.	$\forall xGx$	H
8.	Ga	7, $\forall$E
9.	$Fa \lor Ga$	8, vI
10.	$\exists x(Fx \lor Gx)$	9, $\exists$I
11.	$\forall xGx \rightarrow \exists x(Fx \lor Gx)$	7-10 $\rightarrow$I
12.	$\exists x(Fx \lor Gx)$	1,6,11 vE

4. $\exists x(Fx \lor Gx) \vdash \exists xFx \lor \exists xGx$

1.	$\exists x(Fx \lor Gx)$	A
2.	$Fa \lor Ga$	H
3.	Fa	H
4.	$\exists xFx$	3, $\exists$I
5.	$\exists xFx \lor \exists xGx$	4, $\lor$I
6.	$Fa \rightarrow (\exists xFx \lor \exists xGx)$	3-5 $\rightarrow$I
7.	Ga	H
8.	$\exists xGx$	7, $\exists$I
9.	$\exists xFx \lor \exists xGx$	8, $\lor$I
10.	$Ga \rightarrow (\exists xFx \lor \exists xGx)$	7-9 $\rightarrow$I
11.	$\exists xFx \lor \exists xGx$	2,6,10 $\lor$E
12.	$\exists xFx \lor \exists xGx$	1, 2-11 $\exists$E

5. $\exists x Fx \vdash \sim\forall x \sim Fx$

1.	$\exists x Fx$	A
2.	$\mid\ Fa$	H $\exists$E
3.	$\mid\ \mid\ \forall x \sim Fx$	H ~I
4.	$\mid\ \mid\ \sim Fa$	3,$\forall$E
5.	$\mid\ \mid\ Fa\ \&\ \sim Fa$	2,4 &I
6.	$\mid\ \sim\forall x \sim Fx$	3-5, ~I
7.	$\sim\forall x \sim Fx$	1, 2-6 $\exists$E

Alternatively,

1.	$\exists x Fx$	A
2.	$\mid\ \forall x \sim Fx$	H ~I
3.	$\mid\ \sim Fa$	2, $\forall$E
4.	$\mid\ \mid\ Fa$	H, $\exists$E
5.	$\mid\ \mid\ P\ \&\ \sim P$	3,4 CON
6.	$\mid\ P\ \&\ \sim P$	1, 4-5 $\exists$E
7.	$\sim\forall x \sim Fx$	2-6, ~I

Note carefully the importance in this case of using P&~P by CON since you can not end the proof with Fa &~Fa, since a is in the hypothesis and must not remain in the last line. You can avoid mistakes just by always using this method whenever you require a contradiction to discharge any hypothesis at all.

6. ~∃x~Fx ├ ∀xFx

1.	~∃x~Fx	A
2.	~∀xFx	H for ~I
3.	~Fa	H for ~I
4.	∃x~Fx	3, ∃I
5.	P & ~P	1,4 CON
6.	~~Fa	3-5, ~I
7.	Fa	6, DN
8.	∀xFx	7, ∀I
9.	P & ~P	2,8 CON
10.	~~∀xFx	2-9, ~I
11.	∀xFx	10, DN

Note that step 7 is permissible only because a does not occur in an <u>undischarged</u> hypothesis or an assumption.

7. $\forall x(Fx \rightarrow Gx), \exists x{\sim}Gx \vdash {\sim}\forall xFx$

1.	$\forall x(Fx \rightarrow Gx)$	A
2.	$\exists x{\sim}Gx$	A
3.	$\quad {\sim}Ga$	H
4.	$\quad Fa \rightarrow Ga$	1, $\forall$E
5.	$\quad {\sim}Fa$	3,4 MT
6.	$\qquad \forall xFx$	H ~I
7.	$\qquad Fa$	6, $\forall$E
8.	$\qquad P \,\&\, {\sim}P$	5,7 CON
9.	$\quad {\sim}\forall xFx$	6-8, ~I
10.	${\sim}\forall xFx$	2, 3-9 $\exists$E

Note that on line 6 you can not simply do a ∀I because this would result in ∀x~Fx, not ~∀xFx. You also can not do this because a is in an undischarged hypothesis.

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8. $\forall x(Fx \rightarrow \forall y(Gy \rightarrow Hxy)), \exists x(Fx \,\&\, \exists y{\sim}Hxy) \vdash \exists x{\sim}Gx$

1.	$\forall x(Fx \rightarrow \forall y(Gy \rightarrow Hxy))$	A
2.	$\exists x(Fx \,\&\, \exists y{\sim}Hxy)$	A
3.	$Fa \,\&\, \exists y{\sim}Hay$	H, $\exists$E
4.	Fa	3, &E
5.	$\exists y{\sim}Hay$	3, &E
6.	${\sim}Hab$	H, $\exists$E
7.	$Fa \rightarrow \forall y(Gy \rightarrow Hay)$	1, $\forall$E
8.	$\forall y(Gy \rightarrow Hay)$	1,7 $\rightarrow$E
9.	$Gb \rightarrow Hab$	8, $\forall$E
10.	${\sim}Gb$	6,9 MT
11.	$\exists x{\sim}Gx$	10, $\exists$I
12.	$\exists x{\sim}Gx$	5, 6-10 $\exists$E
13.	$\exists x{\sim}Gx$	2, 3-12 $\exists$E

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9. $\forall x(\exists yFxy \rightarrow \exists zFzx) \vdash \exists xFax \rightarrow \sim\forall z\sim Fza$

1.	$\forall x(\exists yFxy \rightarrow \exists zFzx)$	A
2.	$\exists xFax$	H $\rightarrow$I
3.	$\exists yFay \rightarrow \exists xFza$	1, $\forall$E
4.	Fab	H, $\exists$E
5.	$\exists yFay$	4, $\exists$I
6.	$\exists yFay$	2, 4-5 $\exists$E
7.	$\exists xFza$	3,6 $\rightarrow$E
8.	Fca	H, $\exists$E
9.	$\forall z\sim Fza$	H, ~I
10.	~Fca	9, $\forall$E
11.	P & ~P	8,10 CON
12.	$\sim\forall z\sim Fza$	9-11, ~I
13.	$\sim\forall z\sim Fza$	7, 8-12 $\exists$E
14.	$\exists xFax \rightarrow \sim\forall z\sim Fza$	2-12, $\rightarrow$I

10. $\exists x(Fx \vee \forall yGxy) \vdash \exists yFy \vee \exists y(Gyy \,\&\, Gyb)$

1.	$\exists x(Fx \vee \forall yGxy)$	A
2.	$Fa \vee \forall yGay$	H, $\exists$E
3.	Fa	H $\rightarrow$I
4.	$\exists yFy$	3, $\exists$I
5.	$\exists yFy \vee \exists y(Gyy \,\&\, Gyb)$	4, $\forall$I
6.	$Fa \rightarrow (\exists yFy \vee \exists y(Gyy \,\&\, Gyb))$	3-5 $\rightarrow$I
7.	$\forall yGay$	H, $\rightarrow$I
8.	Gab	7, $\forall$E
9.	Gaa	7, $\forall$E
10.	$Gaa \,\&\, Gab$	8,9 &I
11.	$\exists yGyy \,\&\, Gyb$	10, $\exists$I
12.	$\exists yFy \vee \exists y(Gyy \,\&\, Gyb)$	11, vI
13.	$\forall yGay \rightarrow \exists yFy \vee \exists y(Gyy \,\&\, Gyb)$	7-12, $\rightarrow$I
14.	$\exists yFy \vee \exists y(Gyy \,\&\, Gyb)$	2,6,13 vE
15.	$\exists yFy \vee \exists y(Gyy \,\&\, Gyb)$	1, 2-14 $\exists$E