Recursion (8 points)

1. [2 points] Using the master method in Section 4.5, you can show that the solution to the recurrence $T(n) = 4T(n/3) + n$ is $T(n) = \Theta\left(n^{\log_3 4}\right)$. Show that a substitution proof with the assumption $T(n) \leq cn^{\log_3 4}$ fails. Then show how to subtract off a lower-order term to make a substitution proof work.

> **Solution:** We first try the substitution proof $T(n) \leq cn^{\log_3 4}$
>
> $$T(n) = 4T(n/3) + n \leq 4c(n/3)^{\log_3 4} + n = cn^{\log_3 4} + n$$
>
> This clearly will not be $\leq cn^{\log_3 4}$ as required. Now, suppose instead that we make our inductive hypothesis $T(n) \leq cn^{\log_3 4}$− 3n. $T(n) = 4T(n/3) + n \leq 4\left(c(n/3)^{\log_3 4} - n\right) + n = cn^{\log_3 4} - 4n + n = cn^{\log_3 4} - 3n$ as desired.

2. [2 points] How would you modify Strassen's algorithm to multiply $n \times n$ matrices in which n is not an exact power of 2? Show that the resulting algorithm runs in time $\Theta\left(n^{\lg 7}\right)$. (Read the details about the Strassen's algorithm in Section 4.2. Hint: Pad out the matrix to an exact power of 2.)

> **Solution:** We can zero-pad out the input matrices to be powers of two and then run the given algorithm. Padding out the the next largest power of two (call it m) will at most double the value of n because each power of two is off from each another by a factor of two. So, this will have runtime
>
> $$m^{\lg 7} \leq (2n)^{\lg 7} = 7n^{\lg 7} \in O\left(n^{\lg 7}\right)$$
>
> and
>
> $$m^{\lg 7} \geq n^{\lg 7} \in \Omega\left(n^{\lg 7}\right)$$
>
> Putting these together, we get the runtime is $\Theta\left(n^{\lg 7}\right)$

3. Give asymptotic upper and lower bounds for $T(n)$ in each of the following recurrences. Assume that $T(n)$ is constant for $n \leq 2$. Make your bounds as tight as possible, and justify your answers.
   (a) [1 point] $T(n) = 2T(n/2) + n^4$

   > **Solution:** By Master Theorem, $T(n) \in \Theta\left(n^4\right)$

   (b) [1 point] $T(n) = T(7n/10) + n$

   > **Solution:** By Master Theorem, $T(n) \in \Theta(n)$

   (c) [1 point] $T(n) = 16T(n/4) + n^2$

Solution: By Master Theorem, $T(n) \in \Theta\left(n^2 \lg(n)\right)$

(d) [1 point] $T(n) = T(\sqrt{n}) + \Theta(\lg \lg n)$

Solution: Change of variables: let $m = \lg n$. Recurrence becomes $S(m) = S(m/2) + \Theta(\lg m)$. Case 2 of master's theorem applies, so $T(n) = \Theta\left((\lg \lg n)^2\right)$