Philosophy 160

Derive the following in free logic.

1. ├ ∀y∃x x=y

1.	| ∃x x=a	H (F∀I)
2.	∀y∃x x=y	1-1 F∀I

Note: Consider the fact that one of the main motivations behind free logic is to create a system in which nothing exists necessarily. This is not achieved in classical predicate logic since both ∃x x=a and ∃x x=x are theorems. In free logic, these are not theorems, but the formula above is a theorem. Do you think this means that this system of free logic has failed to achieve its aims? Why or why not?

Answer: The fact that this is a theorem is consistent with the aims of free logic because the universal quantifier applies only to existing objects. But from this theorem one can not deduce that some object actually exists, since the rule for universal elimination can be executed only on objects that exist.

2. ∀xFx, ∃x~Fx ├ P (Don't use QE)

1.	∀xFx	A
2.	∃x~Fx	A
3.	| ~Fa & ∃x x=a	H (F∃E)
4.	| ~Fa	3, &E
5.	| ∃x x=a	3, &E
6.	| Fa	1,5 F∀E
7.	| P	4,6 Con
8.	P	2, 3-7 F∃E

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Derive the following in free logic.

3. ∃xFx ├ ∃x x=x

1.	∃xFx	A
2.	Fa & ∃x x=a	H (F∃E)
4.	∃x x=a	2, &E
5.	a=a	=I
6.	∃x x=x	F∃I 4,5
7.	∃x x=x	1, 2-6 F∃E

4. ∃xFx ├ ∀x x=x

1.	∃xFx	A
2.	∃x x=a	H (F∀I)
3.	a=a	=I
4.	∀x x=x	2-3 (F∀I)

5. ├ ∀x(Fx → (Fx v Gx))

1.	∃x x=a	H (F∀I)
2.	Fa	H (→I)
3.	Fa v Ga	2, vI
4.	Fa → (Fa v Ga)	2-3, →I
5.	∀x(Fx → (Fx v Gx))	1,4 F∀I

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Derive the following in free logic.

6. $\forall x(Fx \rightarrow Gx), \exists xFx \vdash \exists xGx$

1.	$\forall x(Fx \rightarrow Gx)$	A
2.	$\exists xFx$	A
3.	$Fa \,\&\, \exists x\, x=a$	H ($F\exists E$)
4.	$\exists x\, x=a$	3, &E
5.	Fa	3, &E
6.	$Fa \rightarrow Ga$	1,4 $F\forall E$
7.	Ga	5,6 $\rightarrow E$
8.	$\exists xGx$	4,7 $F\exists I$
9.	$\exists xGx$	2, 3-8 $F\exists E$

7. $\vdash \sim\exists x\Diamond\sim x=x$

1.	$\exists x\Diamond\sim x=x$	H (~I)
2.	$\Diamond\sim a=a \;\&\; \exists x\, x=a$	H ($F\exists E$)
3.	$\Diamond\sim a=a$	2, &E
4.	$\sim\Box a=a$	3, Dual, DN
5.	$\Box a=a$	N
6.	$P \,\&\, \sim P$	4,5 Con
7.	$P \,\&\, \sim P$	1, 2-6 $F\exists E$
8.	$\sim\exists x\Diamond\sim x=x$	1-7, ~I

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Derive the following in free logic.

8. $\Box \forall x Fx, \Box \exists x\, x=a \vdash \Box Fa$

1.	$\Box \forall x Fx$	A
2.	$\Box \exists x\, x=a$	A
3.	$\Box(\forall x Fx \rightarrow (\exists x\, x=a \rightarrow Fa))$	N (rule of F∀E)
4.	$\Box \forall x Fx \rightarrow \Box(\exists x\, x=a \rightarrow Fa)$	3, K
5.	$\Box(\exists x\, x=a \rightarrow Fa)$	1,3 →E
6.	$\Box \exists x\, x=a \rightarrow \Box Fa$	5, K
7.	$\Box Fa$	2,6 →E

9. $\Box \forall x(Fx \rightarrow Gx), \Diamond \exists x Fx \vdash \Diamond \exists x Gx$ (Hint: Use result from 6 above)

1.	$\Box \forall x(Fx \rightarrow Gx)$	A
2.	$\Diamond \exists x Fx$	A
3.	$\Box((\forall x(Fx \rightarrow Gx) \rightarrow (\exists x Fx \rightarrow \exists x Gx))$	N (theorem derived in problem 6)
4.	$\Box \forall x(Fx \rightarrow Gx) \rightarrow \Box(\exists x Fx \rightarrow \exists x Gx)$	3, K
5.	$\Box(\exists x Fx \rightarrow \exists x Gx)$	1,4 →E
6.	│ $\Box \sim \exists x Gx$	H~I
7.	│ $\Box(\sim \exists x Gx \rightarrow \sim \exists x Fx)$	5, Trans
8.	│ $\Box \sim \exists x Gx \rightarrow \Box \sim \exists x Fx$	7, K
9.	│ $\Box \sim \exists x Fx$	6,8 →E
10.	│ $\sim \Diamond \exists x Fx$	9, Dual, DN
11.	│ $P \,\&\, \sim P$	Con 2,10
12.	$\sim \Box \sim \exists x Gx$	6-11, ~I
13.	$\Diamond \exists x Gx$	12, Dual

This turned out just to be a propositonal modal logic proof. No quantifier rules are involved.

Note, however, that line 3 was only possible because the theorem was derived in free logic.

In free logic, every theorem introduced involving quantifiers must be a theorem in free logic. Hence, e.g., you can not introduce

$\Box(\forall x Fx \rightarrow Fa)$

since this is not a theorem in free logic.

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Derive the following in free logic.

10. $\exists y \Box(Fy \,\&\, \exists x\ y=x) \vdash \Box\exists yFy$

1.	$\exists y\Box(Fy \,\&\, \exists x\ y=x)$	A
2.	$\Box(Fa \,\&\, \exists x\ a=x) \,\&\, \exists y\ y=a$	H F∃E
3.	$\Box((Fa \,\&\, \exists x\ a=x) \rightarrow \exists yFy)$	N (rule of F∃I in free logic)
4.	$\Box(Fa \,\&\, \exists x\ a=x) \rightarrow \Box\exists yFy$	3, K
5.	$\Box(Fa \,\&\, \exists y\ a=y)$	2, &E
6.	$\Box\exists yFy$	4,5 →E
7.	$\Box\exists yFy$	1, 2-6 F∃E

11. $\exists x\Box Fx, \forall x\Box(Fx \rightarrow Gx) \vdash \exists x\Box Gx$

1.	$\exists x\Box Fx$	A
2.	$\forall x\Box(Fx \rightarrow Gx)$	A
3.	$\Box Fa \,\&\, \exists x\ x=a$	H (F∃E)
4.	$\exists x\ x=a$	3, &E
5.	$\Box(Fa \rightarrow Ga)$	2,4 F∀E
6.	$\Box Fa$	3, &E
7.	$\Box Fa \rightarrow \Box Ga$	5, K
8.	$\Box Ga$	6,7 →E
9.	$\exists x\Box Gx$	4,8 F∃I
10.	$\exists x\Box Gx$	1, 3-9 F∃E

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Derive the following in free logic.

12. $\exists x(Fx \lor Gx) \vdash \exists xFx \lor \exists xGx$

1.	$\exists x(Fx \lor Gx)$	A
2.	$(Fa \lor Ga) \,\&\, \exists x\, x=a$	H (F∃E)
3.	$Fa \lor Ga$	2, &E
4.	$\exists x\, x=a$	2, &E
5.	Fa	H →I
6.	$\exists xFx$	4,6 F∃I
7.	$\exists xFx \lor \exists xGx$	6, vI
8.	$Fa \rightarrow (\exists xFx \lor \exists xGx)$	5-7 →I
9.	Ga	H →I
10.	$\exists xGx$	4,9 F∃I
11.	$\exists xFx \lor \exists xGx$	10, vI
12.	$Ga \rightarrow (\exists xFx \lor \exists xGx)$	5-7 →I
13.	$\exists xFx \lor \exists xGx$	3,8,12 vE
14.	$\exists xFx \lor \exists xGx$	1, 2-13 (F∃E)

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Derive the following in free logic.

13. $\exists x \exists y Gxy \vdash \forall x \forall y\ x=y \rightarrow \forall z Gzz$

1.	$\exists x \exists y Gxy$	A
2.	$\forall x \forall y\ x=y$	H $\rightarrow$I
3.	$\exists x\ x=a$	H F$\forall$I
4.	$\exists y Gby\ \&\ \exists x\ x=b$	H F$\exists$E
5.	$\exists y Gby$	4, &E
6.	$Gbc\ \&\ \exists y\ y=c$	H F$\exists$E
7.	$\exists x\ x=b$	4, &E
8.	$\forall y\ b=y$	2,7 F$\forall$E
9.	$\exists y\ y=c$	6, &E
10.	$b=c$	8,9 F$\forall$E
11.	Gbc	6, &E
12.	Gcc	10, 11 =E
13.	$\forall y\ a=y$	2,3 F$\forall$E
14.	$a=c$	9,13 F$\forall$E
15.	Gaa	12,14 =E
16.	Gaa	5, 6-15 F$\exists$E
17.	Gaa	1, 4-16 F$\exists$E
18.	$\forall z Gzz$	3-17 $\forall$I
19.	$\forall x \forall y\ x=y \rightarrow \forall z Gzz$	2-18 $\rightarrow$I

We could have gone to Gbb here, but for the $\forall$FE step on line 14 we would then require $\exists y\ y=b$, so you would need to take a step to switch the variable from x to y on line 7.

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Derive the following in free logic.