Chapter 12

Introduction to the Laplace Transform

Text: *Electric Circuits* by J. Nilsson and S. Riedel

Prentice Hall

EEE 117 Network Analysis

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Preview

The Laplace transform is a tool for analyzing *linear, time-invariant, lumped parameter systems*.

$$\mathscr{L}\left\{f(t)\right\} = \int_0^\infty f(t)e^{-st}\,dt$$

We will use this tool as a transformation from the time-domain, in which inputs and outputs are functions of time, to the frequency-domain, where the same inputs and outputs are functions of complex angular frequency.

Given a simple mathematical or functional description of an input or output to a system, the Laplace transform provides an alternative functional description that often simplifies the process of analyzing the behavior of the system, or in synthesizing a new system based on a set of specifications.

Preview

Motivation to use the Laplace transform.

1. Discovery of the transient conditions in multi-node or multi-mesh systems. (remember: Phasor method is steady state)

2. Reduce the math complexity of sets of linear differential equations in multi-node or multi-mesh systems.

3. Discovery of the transient conditions in the presence of more complicated signal sources.

4. Use *Transfer Functions* in a system where the frequency of the input varies.

Section 12.1 Definition of the Laplace Transform

The Laplace transform of a function is given by the expression

$$\mathscr{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$$

Where the symbol $\mathscr{L}$\{f(t)\} means "the Laplace transform of f(t)".

The Laplace transform is also denoted as F(s).

$$F(s) = \mathscr{L}\{f(t)\}$$

In circuit analysis, we use the Laplace transform to take the integrodifferential equations of the time-domain *into* a set of algebraic equations in the frequency domain.

$$\textit{Time Domain} \rightarrow \text{t in seconds}$$

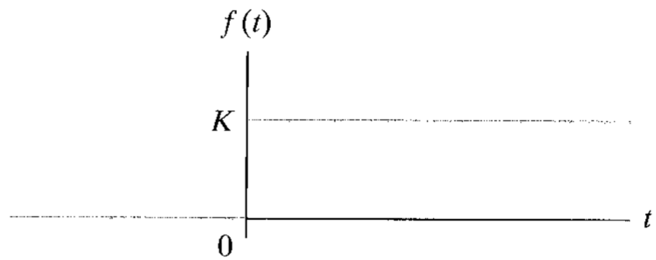
$$\textit{Frequency Domain} \rightarrow s \text{ in} \frac{1}{\text{seconds}} \quad \textit{i.e.Hertz}$$

In this course we only use the unilateral or one-sided Laplace transform.

Thus we integrate from zero to infinity. We will be careful to use sources where the Laplace integral converges.

The result of circuit behavior prior to t = zero is accounted for by *initial conditions*.

Arbitrary signals may have discontinuities. We will look at step and impulse functions so that we can handle these abrupt changes in signals.



Let the abrupt change at t = 0 be given as

$f(t) = K\ u(t)$ Where K u(t) = 0 for t < 0

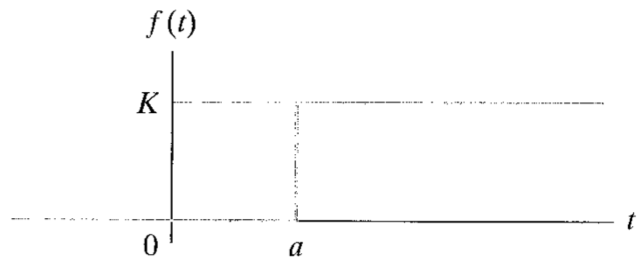
And K u(t) = K for t > 0

If K = 1, then we have the unit step function.

The step function is not defined at t = 0.
Read the text for the mathematical background.

Abrupt changes can occur at other times as well.

$$f(t) = K\,u(t-a)$$



K u(t - a) = 0 for t < a

K u(t - a) = K for t > a

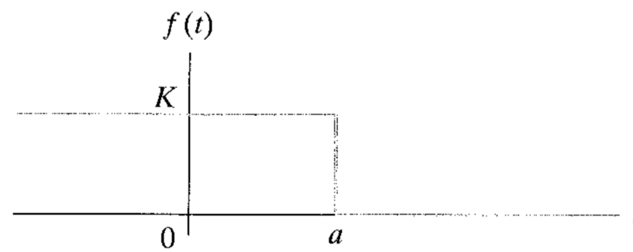
We say the signal is advanced or retarded with respect to the origin depending of the value of a.

Signal happens earlier in time (advanced) for a < 0.

Signal happens later in time (retarded) for a > 0.

The step function can also exist for t < 0 and turn-off abruptly.

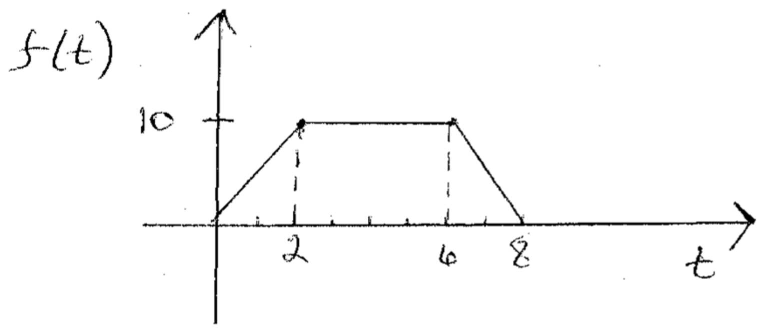
$$f(t) = K\,u\left[-(t-a)\right] = K\,u(a-t)$$



K u(a - t) = K for t < a

K u(a - t) = 0 for t > a

Example



Use step equations (and a little creativity) to write the function f(t).

Time $0 < t \leq 2$ slope of the line is 5t so first part is $5t\left[u(t)-u(t-2)\right]$

Time $2 \leq t \leq 6$ function is a constant f(t) = 10 so we have $10\left[u(t-2)-u(t-6)\right]$

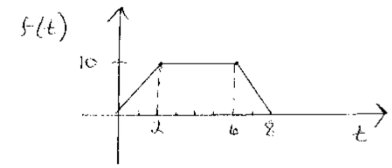
Time $6 \leq t \leq 8$ slope of the line is -5t

At t = 6, f(t) = 10
-5t = -t(6) = -30 but we need 10!
So use f(t) = 40 - 5t.

$$\left(40-5t\right)\left[u(t-6)-u(t-8)\right]$$

f(t) = 0 for t > 8

Example



Using the slope-intercept form to determine the equation of a line.

$$y = mx + b$$

m is the slope of the line.

b is the y-coordinate at which the line crosses the Y-axis (at x = 0)

For example

$$m = \frac{10-0}{6-8} = \frac{10}{-2} = -5$$

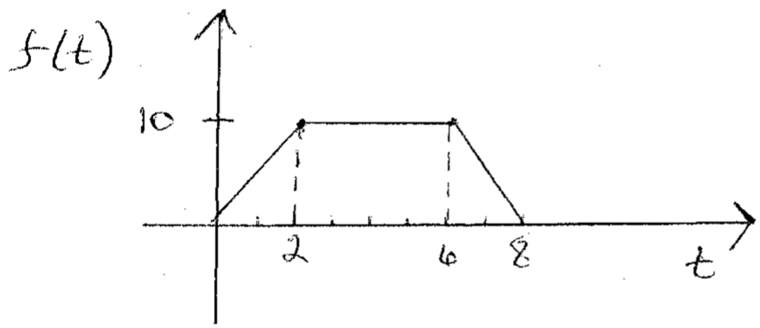
At x = 8, y = 0 thus

$$y = mx + b \Rightarrow b = y - mx = 0 - (-5) \times 8 = 40$$

Thus we have

$f(t) = -5t + 40 = 40 - 5t$ as we found before.

Example



Thus the function f(t) can be represented by

$$f(t) = 5t\left[u(t) - u(t-2)\right] + 10\left[u(t-2) - u(t-6)\right] + \left(40 - 5t\right)\left[u(t-6) - u(t-8)\right]$$

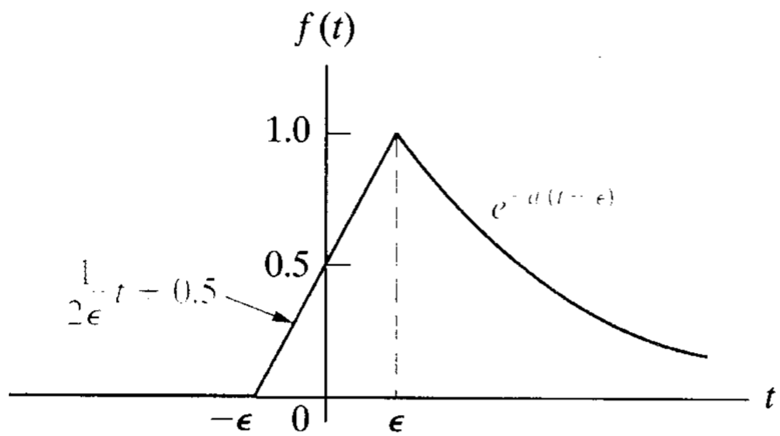
We can simplify.

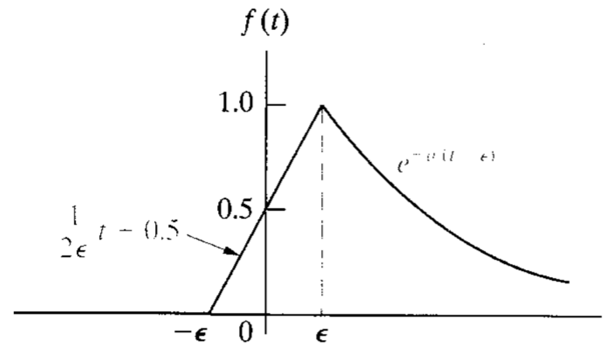
$$f(t) = 5t\,u(t) + \left(10 - 5t\right)u(t-2) + \left(30 - 5t\right)u(t-6) - \left(40 - 5t\right)u(t-8)$$

The Impulse function is defined as a signal of infinite amplitude and zero duration.

While a "pure" impulse signal does not exist in nature, the mathematical use of the concept is very valuable.

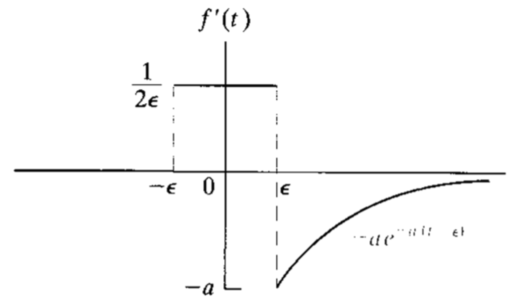
Let us start by assuming some signal has a linear discontinuity at the origin.





As ε approaches zero in width, the value of the derivative f'(t) approaches infinity.

Or in other words, the area under the curve equals a constant. Which it must for a linear time-invariant system.



$$f'(t=0) \rightarrow \delta(t) \quad as\ \varepsilon \rightarrow 0$$

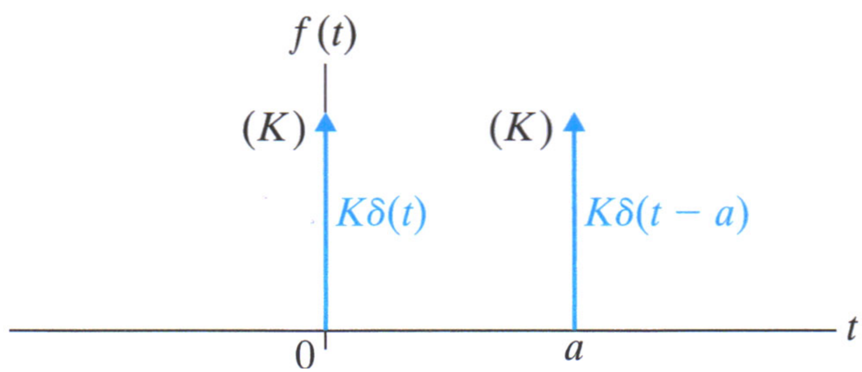
Thus for our impulse function

1. The amplitude approaches infinity.
2. The duration (ε) of the function approaches zero.
3. The area under the curve is constant as the duration ε changes.

The Impulse function is mathematically defined as

$$\int_{-\infty}^{\infty} K\delta(t)\,dt = \begin{cases} K & for \;\; t = 0 \\ 0 & for \;\; t \neq 0 \end{cases}$$

The impulse can be advanced or retarded in time. Thus it can have a value at some other time a (but only at that other time).



Since the impulse only exists at a single point in time, it may be used to "sift" the values of another function.

The sifting property is defined as

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)\,dt = \quad = f(a)$$

We can use the sifting property to find the Laplace transform of δ(t).

$$\mathscr{L}\{\delta(t)\} \equiv \int_{0^-}^{\infty} \delta(t=0)\,e^{-st}\,dt = \underbrace{\left(e^{-st}\Big|^{t=0}\right)}_{fct\ only\ exists\ for\ t=0} \int_{0^-}^{\infty} \delta(t)\,dt$$

$$= e^{-0}(1) \qquad = 1$$

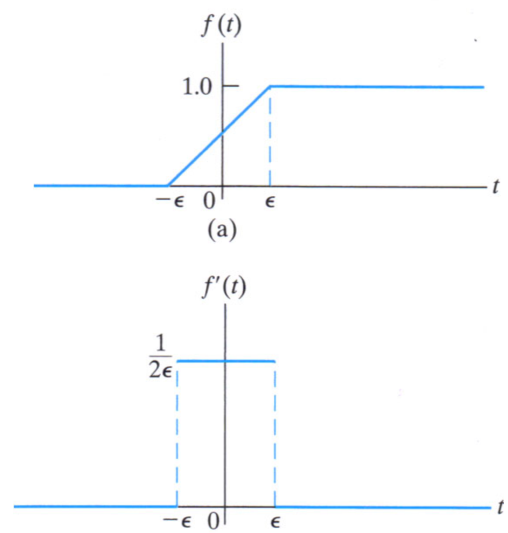
The derivative of the impulse function can also be found. See the text for the derivation.

$$\mathscr{L}\left\{\delta'(t)\right\} = s$$

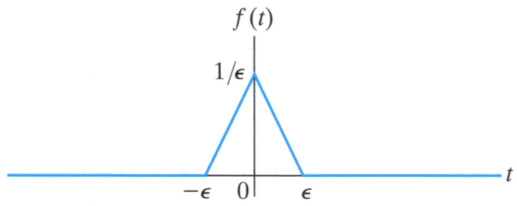
$$\mathscr{L}\left\{\delta^n(t)\right\} = s^n$$

Finally note that the impulse function itself can be found from the discontinuity of the step function u(t).

$$\delta(t) = \frac{du}{dt}$$



Example



(a)

Find the area under the figure above.

Area of triangle = 1/2 (base)(height)

$$Area = \tfrac{1}{2}(2\varepsilon)\left(\tfrac{1}{\varepsilon}\right) \qquad = 1$$

What is the function’s duration when ε = 0?

Duration = zero!

What is the magnitude of f(0) when ε → 0?

Since we required that the area under the curve be constant, the magnitude → ∞ when ε → 0.

Example

Find f(t) by taking the inverse Laplace transform of the following function.

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\left(\frac{4+j\omega}{9+j\omega}\right)\pi\delta(\omega)}_{F(\omega)} e^{j\omega t}\,d\omega$$

Here δ(ω) means δ(ω = 0). So we evaluate F(ω) at ω = 0.

$$\frac{4+j\omega}{9+j\omega}e^{j\omega t} = \frac{4+j0}{9+j0}e^{0} = \frac{4}{9}$$

So the integral becomes

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4}{9}\pi\,\delta(\omega)\,d\omega = \frac{1}{2\pi}\frac{4}{9}\pi(1) = \frac{2}{9}$$

We used the sifting property.
ω = 0 is the only value to be found due to the impulse function at the origin.

A functional transform is simply the Laplace transform of a specified function of t.

Since we will use only the unilateral Laplace transform, we will define all our functions = zero for t < 0^-.

TABLE 12.1 An Abbreviated List of Laplace Transform Pairs

Type	$f(t)\ (t > 0-)$	$F(s)$
(impulse)	$\delta(t)$	1
(step)	$u(t)$	$\frac{1}{s}$
(ramp)	t	$\frac{1}{s^2}$
(exponential)	e^{-at}	$\frac{1}{s+a}$
(sine)	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
(cosine)	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
(damped ramp)	te^{-at}	$\frac{1}{(s+a)^2}$
(damped sine)	$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
(damped cosine)	$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

Example

$$\mathscr{L}\{f(t)\} = \int_{0^-}^{\infty} f(t)e^{-st}dt$$

Show that the unit step function Laplace transform is 1/s.

$$\mathscr{L}\{u(t)\} = \int_{0^+}^{\infty} u(t)e^{-st}dt \quad = \int_{0^+}^{\infty} 1\, e^{-st}dt$$

$Let\ u = -st \qquad then\ du = -s\,dt$

$\int e^u du = e^u + \text{constant of integration}$

$$= \int_{0^+}^{-\infty} e^u \frac{du}{-s}$$

Change the limits of integration due to the change in variable.

$$= \left.\frac{e^u}{-s}\right|_{0^+}^{-\infty} \quad = \frac{e^{-\infty}}{-s} - \frac{e^{-0}}{-s} \qquad = \frac{0}{-s} - \frac{1}{-s}$$

$$= \frac{1}{s} \quad Q.E.D.$$

Q.E.D. = quod erat demonstrandum = "which was to be demonstrated".

Euler's Identity

See appendix B for the discussion on complex notation and Euler's Identity.

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Proof:

$$e^{j\theta} - e^{-j\theta} = \left(\cos\theta + j\sin\theta\right) - \left(\cos\theta - j\sin\theta\right) \quad = 2j\sin\theta$$

$$\mathit{Thus} \quad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

See the next example for the use of the identity.

Example

$$\mathscr{L}\{f(t)\} = \int_{0^-}^{\infty} f(t)\, e^{-st}\, dt$$

Find the Laplace transform of sin ωt.

$$\mathscr{L}\{\sin\omega t\} = \int_{0^-}^{\infty} (\sin\omega t) e^{-st}\, dt \quad = \int_{0^-}^{\infty} \left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right) e^{-st}\, dt$$

$$= \int_{0^-}^{\infty} \left(\frac{e^{-(s-j\omega)t} - e^{-(s+j\omega)t}}{2j}\right) dt \;\; = \frac{1}{2j}\int_{0^-}^{\infty} \left(e^{-(s-j\omega)t} - e^{-(s+j\omega)t}\right) dt$$

We can now solve the last equation as two separate integrals.

$$= \frac{1}{2j}\left\{\int_{0^-}^{\infty} e^{-(s-j\omega)t}\, dt - \int_{0^-}^{\infty} e^{-(s+j\omega)t}\, dt\right\} \;\; = \frac{1}{2j}\left\{\int_{0^-}^{-\infty} e^{u} \frac{du}{-(s-j\omega)} - \int_{0^-}^{-\infty} e^{u} \frac{du}{-(s+j\omega)}\right\}$$

Where u = - (s $\pm$ jω) t $\Rightarrow$ du = -(s $\pm$ jω) dt

$$= \frac{1}{2j}\left(\frac{-1}{s-j\omega}\underbrace{\left(e^{-\infty} - e^{0}\right)}_{-1} + \frac{1}{s+j\omega}\underbrace{\left(e^{-\infty} - e^{0}\right)}_{-1}\right) \quad = \frac{1}{2j}\left(\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right)$$

Example

$$\mathscr{L}\{f(t)\} = \int_{0^-}^{\infty} f(t)e^{-st}dt$$

Find the Laplace transform of sin ωt.

$$\mathscr{L}\{\sin \omega t\} = \int_{0^-}^{\infty} (\sin \omega t)e^{-st}dt \quad = \frac{1}{2j}\left(\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right)$$

Now clean up the fractions.

$$= \frac{1}{2j}\left(\frac{1}{s-j\omega}\frac{s+j\omega}{s+j\omega} - \frac{1}{s+j\omega}\frac{s-j\omega}{s-j\omega}\right)$$

$$= \frac{1}{2j}\frac{(s+j\omega)-(s-j\omega)}{s^2+\omega^2} \quad = \frac{1}{2j}\frac{2j\omega}{s^2+\omega^2} \quad = \frac{\omega}{s^2+\omega^2}$$

$$\mathscr{L}\{\sin \omega t\} = \frac{\omega}{s^2+\omega^2}$$

Operational transforms indicate how mathematical operations performed on either f(t) or F(s) are converted to the other domain.

Multiplication by a Constant

If $\mathscr{L}\{f(t)\} = F(s)$

Then

$$\mathscr{L}\{K\,f(t)\} = K\,F(s)$$

This can be quickly proven by simple integration.

Addition is the same in both domains.

Addition (or Subtraction)

If $\mathscr{L}\left\{f_1(t)\right\} = F_1(s)$

And $\mathscr{L}\left\{f_2(t)\right\} = F_2(s)$

Then

$$\mathscr{L}\left\{f_1(t) \pm f_2(t)\right\} = F_1(s) \pm F_1(s)$$

Differentiation in the time domain corresponds to multiplying F(s) by s then subtracting the initial value of f(t).

$$\mathscr{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0^-)$$

See the text and my handwritten notes for the proof.

Integration in the time domain corresponds to dividing F(s) by s.

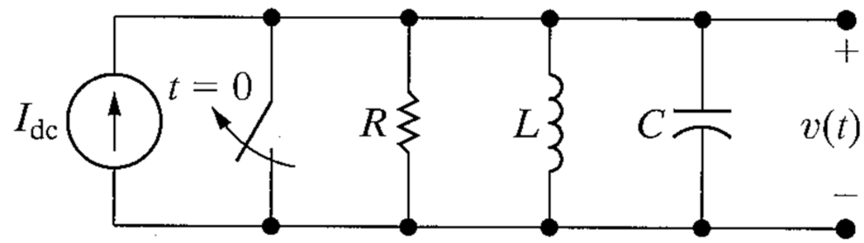
$$\mathscr{L}\left\{\int_{0^-}^{t} f(x)dx\right\} = \frac{F(s)}{s}$$

There are several other operations of which you should be familiar.
See Table 12.2 on page 480.

Table 12.2 Operational Transforms

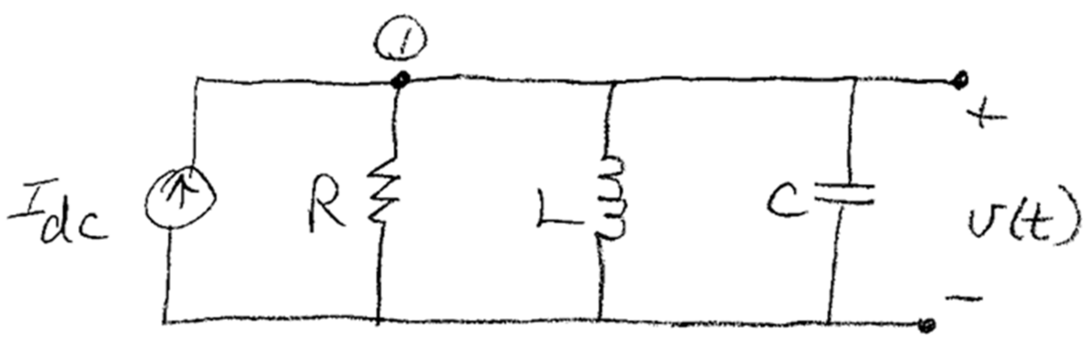
Operation	$f(t)$	$F(s)$
Multiplication by a constant	$Kf(t)$	$KF(s)$
Addition/subtraction	$f_1(t) + f_2(t) - f_3(t) + \cdots$	$F_1(s) + F_2(s) - F_3(s) + \cdots$
First derivative (time)	$\dfrac{df(t)}{dt}$	$sF(s) - f(0^-)$
Second derivative (time)	$\dfrac{d^2f(t)}{dt^2}$	$s^2F(s) - sf(0^-) - \dfrac{df(0^-)}{dt}$
nth derivative (time)	$\dfrac{d^nf(t)}{dt^n}$	$s^nF(s) - s^{n-1}f(0^-) - s^{n-2}\dfrac{df(0^-)}{dt} - s^{n-3}\dfrac{df^2(0^-)}{dt^2} - \cdots - \dfrac{d^{n-1}f(0^-)}{dt^{n-1}}$
Time integral	$\displaystyle\int_0^t f(x)\,dx$	$\dfrac{F(s)}{s}$
Translation in time	$f(t-a)u(t-a), a > 0$	$e^{-as}F(s)$
Translation in frequency	$e^{-at}f(t)$	$F(s+a)$
Scale changing	$f(at), a > 0$	$\dfrac{1}{a}F\left(\dfrac{s}{a}\right)$
First derivative (s)	$tf(t)$	$-\dfrac{dF(s)}{ds}$
nth derivative (s)	$t^nf(t)$	$(-1)^n\dfrac{d^nF(s)}{ds^n}$
s integral	$\dfrac{f(t)}{t}$	$\displaystyle\int_s^\infty F(u)\,du$

The following shows an example of using the Laplace Transform.



What is the time domain behavior of this circuit?
At $t = 0^-$? At $t = 0^+$? At $t \to \infty$?

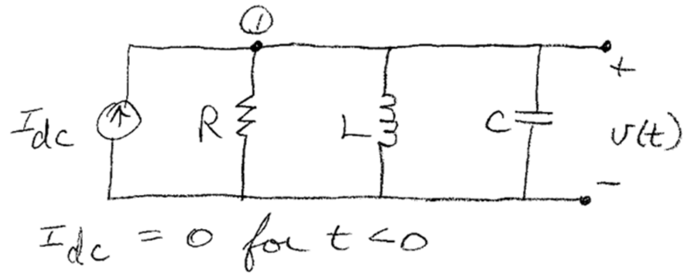
Here is my circuit for $t > 0$.



$I_{dc} = 0$ for $t < 0$

Section 12.6 Applying The Laplace Transform

Recall $\mathrm{i}_L = \frac{1}{L}\int_0^t v(x)dx$ and $i_C = C\frac{dv(t)}{dt}$



At node 1, the node equation is (by convention in = - and out = +)

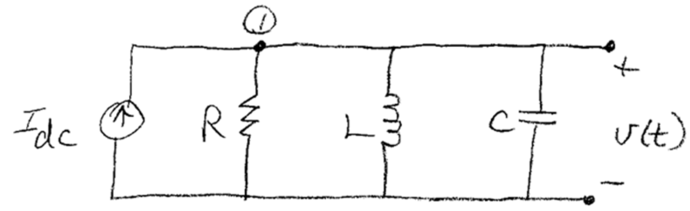
$$-I_{DC}u(t) + \frac{v(t)}{R} + \frac{1}{L}\int_0^t v(x)dx + C\frac{dv(t)}{dt} = 0$$

$$\frac{v(t)}{R} + \frac{1}{L}\int_0^t v(x)dx + C\frac{dv(t)}{dt} = I_{DC}u(t)$$

This same node equation in the frequency domain is

$$\frac{V(s)}{R} + \frac{1}{L}\frac{V(s)}{s} + C\left[sV(s) - v(0^-)\right] = I_{DC}\left(\frac{1}{s}\right)$$

$$\frac{V(s)}{R}+\frac{1}{L}\frac{V(s)}{s}+C\left[sV(s)-\underbrace{v(0^-)}_{=0}\right]=I_{DC}\left(\frac{1}{s}\right)$$



$I_{dc} = 0$ for $t < 0$

$v(t = 0^-)$ = zero as we said in the time domain preview.

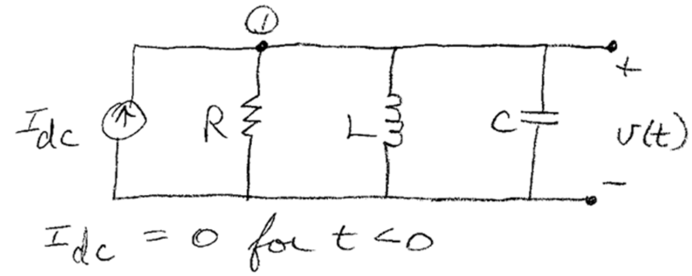
We also used the following transforms.

$$\int_0^t v(x)dx \Leftrightarrow \frac{F(s)}{s}$$

$$\frac{df(t)}{dt} \Leftrightarrow sF(s)-f(0^-)$$

$$K\,f(t) \Leftrightarrow K\,F(s)$$

$$u(t) \Leftrightarrow \frac{1}{s}$$



Thus we can write

$$V(s)\left(\frac{1}{R}+\frac{1}{sL}+Cs\right)=\frac{I_{DC}}{s}$$

$$V(s)=\frac{I_{DC}}{s\left(\frac{1}{R}+\frac{1}{sL}+Cs\right)} = \frac{\frac{I_{DC}}{C}}{s^2+s\frac{1}{RC}+\frac{1}{LC}}$$

The last equation was introduced in Chapter 8 as the natural response of a parallel RLC circuit. See page 287.

Take a moment to read section 8.1 and 8.2 and note the over/under/critically damped form of the response.
We will do a full analysis of this circuit later in this course.

In general, the Laplace transform resulting from a circuit analysis will yield a rational function in the form:

$$F(s) = \frac{N(s)}{D(s)} \quad = \frac{a_n s^n + a_{n-1} s^{n-1} + \bullet\bullet\bullet + a_1 s + a_0}{b_m s^m + b_{m-1} s^{m-1} + \bullet\bullet\bullet + b_1 s + b_0}$$

If m > n, this is a proper rational function.

If m < n, the improper rational function must be divided to create a proper rational function.

The task now is to simplify the proper rational function into a form which can be identified in the Laplace transform pair Table 12.1

A proper rational function can be expanded into a sum of partial fractions by a process detailed by Oliver Heaviside – the "cover up method".

Another way to expand the partial fractions is the equate coefficients method (which, due to past student experience, I don't recommend).

For example, the following function can be expanded as

$$F(s) = \frac{s+6}{s(s+3)(s+1)^2} \equiv \frac{a}{s} + \frac{b}{s+3} + \frac{c}{(s+1)^2} + \frac{d}{s+1}$$

In this case, the denominator has four roots and thus the four fractions on the right hand side.

For example, using Table 12.1 the "transform pairs" are

$$F(s) = \frac{a}{s} + \frac{b}{s+3} + \frac{c}{(s+1)^2} + \frac{d}{s+1}$$

$$a\,u(t) \Leftrightarrow \frac{a}{s}$$

$$b\,e^{-3t}u(t) \Leftrightarrow \frac{b}{s+3}$$

$$c\,t\,e^{-t}u(t) \Leftrightarrow \frac{c}{(s+1)^2}$$

$$d\,e^{-t}u(t) \Leftrightarrow \frac{d}{s+1}$$

$$f(t) = (a + be^{-3t} + cte^{-t} + de^{-t})u(t)$$

$$= \left[a + be^{-3t} + (ct+d)e^{-t}\right]u(t)$$

There are four possible partial fraction expansions (PFE).

The roots of D(s) are real and distinct.

The roots of D(s) are real and repeated.

The roots of D(s) are complex.

The roots of D(s) are complex and repeated.

Software such as Matlab might be helpful in finding the partial fraction expansions but Matlab Laplace solutions will not be covered in EEE 117.

Assessment Problem 12.8 on page 494

Find f(t) for $F(s)=\dfrac{5s^2+29s+32}{(s+2)(s+4)}=\dfrac{5s^2+29s+32}{s^2+6s+8}$

In this case the order of the denominator m is equal to the order of the numerator n.

m = n

First divide the numerator by the denominator to make this a proper rational function.

$$F(s)=\frac{5s^2+29s+32}{s^2+6s+8} \quad =5+\frac{-s-8}{(s+2)(s+4)} \quad =5+\frac{a}{s+2}+\frac{b}{s+4}$$

Now we can find the coefficients *a* and *b*.

Assessment Problem 12.8

$$\frac{-s-8}{(s+2)(s+4)} = \frac{a}{s+2} + \frac{b}{s+4}$$

To find the coefficient *a* multiply both sides by s + 2 and then let s = -2

$$\left.\frac{-s-8}{s+4}\right|_{s=-2} = a + \frac{b}{s+4}\underbrace{(s+2)}_{=0\ for\, s=-2}$$

$$a = \frac{-(-2)-8}{-2+4} \quad = \frac{-6}{2} \quad = -3$$

To find the coefficient *b* multiply both sides by s + 4 and then let s = -4

$$\left.\frac{-s-8}{s+2}\right|_{s=-4} = \frac{a}{s+2}\underbrace{(s+4)}_{=0\ for\, s=-4} + b$$

$$b = \frac{-(-4)-8}{-4+2} \quad = \frac{-4}{-2} \quad = 2$$

Assessment Problem 12.8

Thus $$F(s) = \frac{5s^2 + 29s + 32}{(s+2)(s+4)} \quad = 5 - \frac{3}{s+2} + \frac{2}{s+4}$$

Check this solution. Let s = zero.

$$\frac{5(0)^2 + 29(0) + 32}{(0+2)(0+4)} = 5 - \frac{3}{0+2} + \frac{2}{0+4} \quad \Rightarrow \frac{32}{8} = 5 - \frac{3}{2} + \frac{2}{4}$$

$$\Rightarrow 4 = \frac{20 - 6 + 2}{4} \quad \Rightarrow 4 = \frac{16}{4} = 4$$

The solution checks!

Thus we can use Table 12.1 to write f(t).

$$f(t) = 5\delta(t) - 3e^{-2t}u(t) + 2e^{-4t}u(t) \quad \underline{\underline{= 5\delta(t) - (3e^{-2t} - 2e^{-4t})u(t)}}$$

Example – Distinct Complex Roots

Note: complex roots ALWAYS appear as complex conjugate pairs in real circuits with physically realizable sources.

Given: $$F(s) = \frac{10(s^2+119)}{(s+5)(s^2+10s+169)} = \frac{10(s^2+119)}{(s+5)(s+5-j12)(s+5+j12)}$$

$$= \frac{a}{(s+5)} + \frac{b}{(s+5-j12)} + \frac{b^*}{(s+5+j12)}$$

$$a = \left.\frac{10(s^2+119)}{(s^2+10s+169)}\right|_{s=-5} = \frac{10\left[(-5)^2+119\right]}{\left[(-5)^2+10(-5)+169\right]} = \frac{10[144]}{[144]} = 10$$

$$b = \left.\frac{10(s^2+119)}{(s+5)(s+5+j12)}\right|_{s=-5+j12} = \frac{10\left[(-5+j12)^2+119\right]}{[(-5+j12)+5][(-5+j12)+5+j12]}$$

$$= \frac{10[(25-j120-144)+119]}{(j12)(j24)} = \frac{10[-j120]}{-288} = j4.1667 = 4.1667\langle 90^\circ$$

Example – Distinct Complex Roots

$$F(s) = \frac{10(s^2+119)}{(s+5)(s^2+10s+169)} \qquad = \frac{10}{(s+5)} + \frac{4.17\langle 90^\circ}{(s+5-j12)} + \frac{4.17\langle -90^\circ}{(s+5+j12)}$$

Use Tables 12.1 and 12.3 $\qquad \dfrac{K\langle\theta}{s+\alpha-j\beta} + \dfrac{K\langle-\theta}{s+\alpha+j\beta} \Leftrightarrow 2\left|K\right|e^{-\alpha t}\cos(\beta t+\theta)$

$$f(t) = \left[10e^{-5t} + 2(4.17)e^{-5t}\cos(12t+90^\circ)\right]u(t)$$

$$= \underline{\underline{\left[10e^{-5t} - 2(4.17)e^{-5t}\sin(12t)\right]u(t)}}$$

Example – Distinct Complex Roots

We can also verify the transform pair in Table 12.3

$$F(s)=\frac{10(s^2+119)}{(s+5)(s^2+10s+169)} \qquad =\frac{10}{(s+5)}+\frac{4.17\langle 90^\circ}{(s+5-j12)}+\frac{4.17\langle -90^\circ}{(s+5+j12)}$$

Write the inverse Laplace transform in terms of exponentials.

$$f(t)=\left[10e^{-5t}+(4.17e^{j90^\circ})e^{(-5+j12)t}+(4.17e^{j(-90^\circ)})e^{(-5-j12)t}\right]u(t)$$

$$=\left\{10e^{-5t}+4.17e^{-5t}\left[e^{j90^\circ}e^{j12t}+e^{j(-90^\circ)}e^{-j12t}\right]\right\}u(t)$$

$$=\left\{10e^{-5t}+4.17e^{-5t}\left[2\frac{e^{j(12t+90^\circ)t}+e^{-j(12t+90^\circ)}}{2}\right]\right\}u(t)$$

$$=\underline{\underline{\left[10e^{-5t}-2(4.17)e^{-5t}\cos(12t+90^\circ)\right]u(t)}}$$

The rational function of the Laplace transform can be expressed as the ratio of two factored polynomials.

$$F(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_n)}{(s+p_1)(s+p_2)\cdots(s+p_m)}$$

The roots of the denominator polynomials $-p_1, -p_2, \ldots, -p_m$ are called the *poles of F(s)*.

The roots of the numerator polynomials $-z_1, -z_2, \ldots, -z_n$ are called the *zeros of F(s)*.

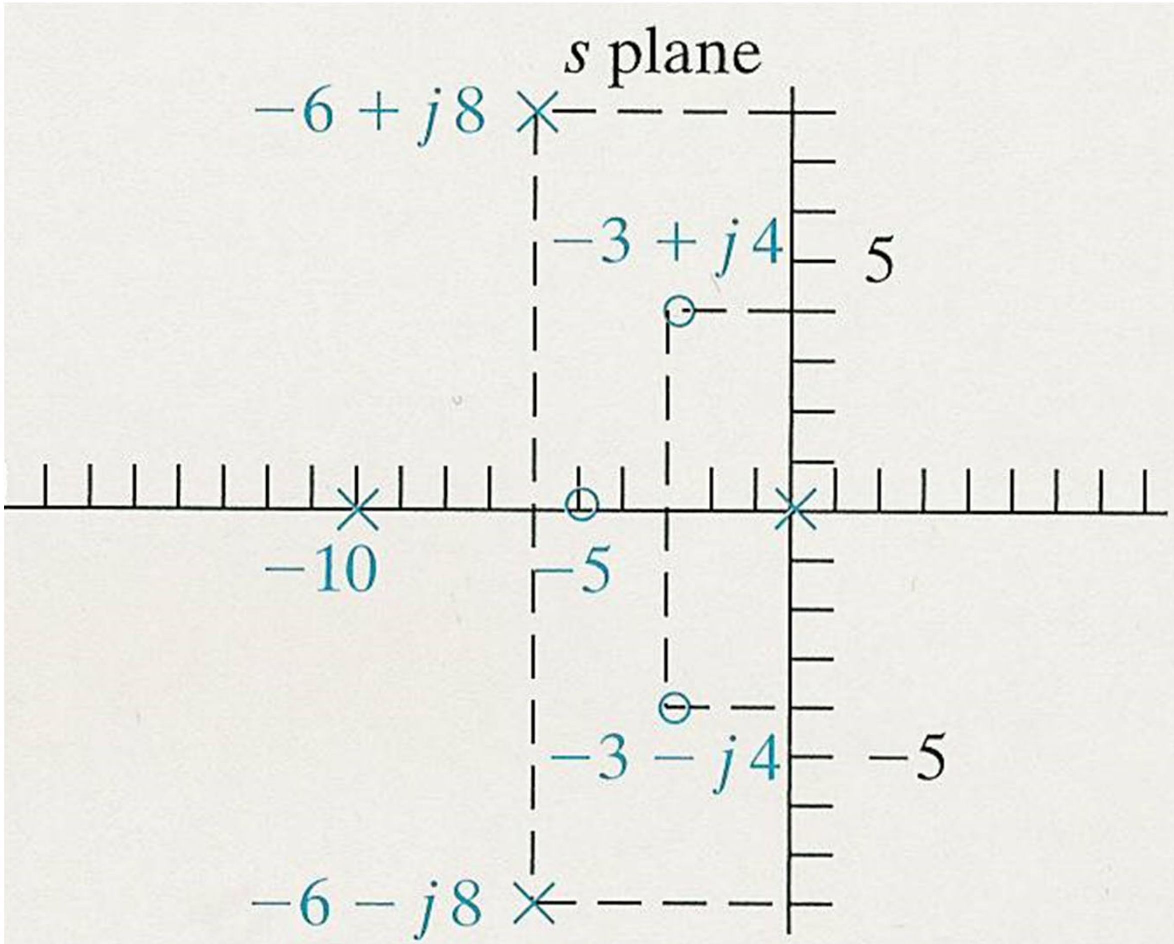
The zeros of F(s) are the values of s at which F(s) becomes zero.

Plot the poles and zeros of the following rational function.

$$F(s)=\frac{10(s+5)(s+3-j4)(s+3+j4)}{s(s+10)(s+6-j8)(s+6+j8)}$$

The zeros of F(s) are
-5, -3 + j4, -3 - j4.

The poles of F(s) are
0, -10, -6 + j8, -6 – j8.



Section 12.9 Initial-value and Final-value Theorems

The initial-value and final-value theorems are useful because they enable us to determine from F(s) the behavior of f(t) at t = 0 and $t \to \infty$.

The initial-value theorem $$\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s)$$

The final-value theorem $$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

The theorems assume (require) that:

f(t) contains no impulses

All the poles f(t) lie in the left-hand plane.

A non-repeated pole (first order) of f(t) may exist at the origin.

Example: Initial-value and Final-value Theorems

Given: $$f(t) = \left[-12e^{-6t} + 20e^{-3t}\cos(4t - 53.13^\circ)\right]u(t)$$

$$F(s) = \frac{100(s+3)}{(s+6)(s^2+6s+25)}$$

The initial-value theorem states

$$\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} s\frac{100(s+3)}{(s+6)(s^2+6s+25)} \approx \lim_{s\to\infty}\frac{s^2}{s^3} = 0$$

You can divide through by s to avoid the ∞^2/∞^3 ambiguity or accept that $\infty^2/\infty^3 = 0$.

$$f(t=0^+) = \left[-12e^0 + 20e^0\cos(-53.13^\circ)\right](1)$$

$$= -12 + 20(0.6) \quad = -12 + 12 \quad = 0$$

Answer checks!

Example: Initial-value and Final-value Theorems

Given: $$f(t) = \left[-12e^{-6t} + 20e^{-3t}\cos(4t - 53.13^\circ)\right]u(t)$$

$$F(s) = \frac{100(s+3)}{(s+6)(s^2+6s+25)}$$

The final-value theorem states

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) \quad = \lim_{s\to 0} s\frac{100(s+3)}{(s+6)(s^2+6s+25)} \quad = \frac{0}{6(25)} \qquad = 0$$

Check this with the known f(t).

$$f(t\to\infty) = \left[-12e^{-\infty} + 20e^{-\infty}\cos(\infty - 53.13^\circ)\right](1)$$

$$= 0 + 0(0.6) \qquad = 0$$

Answer checks!

L'Hôpital's Rule

Suppose we have an indeterminate form to resolve such as

$$\lim_{x\to a}\frac{f(\mathrm{x})}{g(\mathrm{x})}=\frac{0}{0} \qquad \text{or} \qquad \lim_{x\to a}\frac{f(\mathrm{x})}{g(\mathrm{x})}=\frac{\pm\infty}{\pm\infty}$$

Sometimes we have knowledge of the function and can properly interpret the indeterminate form.

Otherwise we need to use L'Hôpital's Rule by differentiating both the numerator and the denominator and then take the limit.

For example: $\lim_{x\to\infty}\frac{e^x}{x^2}=\frac{\infty}{\infty}$

Differentiate both numerator and denominator as many times as needed.

$$\lim_{x\to\infty}\frac{e^x}{x^2}=\lim_{x\to\infty}\frac{e^x}{2x}=\frac{\infty}{\infty} \text{ still indeterminate} \qquad \text{so again } \lim_{x\to\infty}\frac{e^x}{2x}=\lim_{x\to\infty}\frac{e^x}{2}=\infty$$

Chapter 12

Introduction to the Laplace Transform

Text: *Electric Circuits* by J. Nilsson and S. Riedel

Prentice Hall

EEE 117 Network Analysis

Instructor: Russ Tatro