Chapter 16

Fourier Series

Text: *Electric Circuits* by J. Nilsson and S. Riedel

Prentice Hall

EEE 117 Network Analysis

Instructor: Russ Tatro

Preview

Nonsinusoidal but periodic excitations are common.

This chapter begins the study of periodic functions and introduces the Fourier Series

To evaluate the Fourier coefficients, the following integrals will be helpful.

$$\int \cos at \; dt = \frac{1}{a} \sin at$$

$$\int t \cos at \; dt = \frac{1}{a^2} \cos at + \frac{1}{a} t \sin at$$

$$\int \sin at \; dt = \frac{-1}{a} \cos at$$

$$\int t \sin at \; dt = \frac{1}{a^2} \sin at - \frac{1}{a} t \cos at$$

Values of cosine, sine and exponential functions for integral multiples of π.

$$\cos 2n\pi = 1 \qquad \cos n\pi = (-1)^n \qquad \cos\frac{n\pi}{2} = \begin{cases} (-1)^{\frac{n}{2}} & \text{for n even} \\ 0 & \text{for n odd} \end{cases}$$

$$\sin 2n\pi = 0 \qquad \sin n\pi = 0 \qquad \sin\frac{n\pi}{2} = \begin{cases} (-1)^{\frac{n-1}{2}} & \text{for n odd} \\ 0 & \text{for n even} \end{cases}$$

$$e^{j2n\pi} = 1 \qquad e^{jn\pi} = (-1)^n \qquad e^{\frac{jn\pi}{2}} = \begin{cases} (-1)^{\frac{n}{2}} & \text{for n even} \\ j(-1)^{\frac{n-1}{2}} & \text{for n odd} \end{cases}$$

A periodic function is one that satisfies the relationship

$$f(t) = f(t \pm nT)$$

Where n is an integer 1, 2, 3, ….
And T is the period.

The smallest value of T that satisfies the periodicity condition is the *fundamental period* of f(t).

Thus the function repeats with period T.

$$f(t_0) = f(t_0 - T) \quad = f(t_0 + T) = f(t_0 - 2T) \quad = f(t_0 + 2T)$$ and so on...

Fourier discovered that a periodic function can be represented by an infinite sum of sine or cosine functions that are harmonically related (as long as certain conditions are met).

$$f(t) = a_v + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

The Fourier coefficients are a_v, a_n and b_n.

The fundamental frequency is given by ω_0.

The harmonic frequencies of f(t) are the integer multiples of ω_0.

Section 16.1 Dirichlet's Conditions (Dirichlet – German Mathematician 1805 to 1859)

The required conditions for a convergent Fourier series are

Dirichlet's Conditions:

1. f(t) is single-valued (at any one point).
2. f(t) has a finite number of discontinuities in one period.
3. f(t) has a finite number of maxima and minima in one period.
4. The integral $\int_{t_0}^{t_0+T} |f(t)|\,dt$ exists.

Any periodic function generated by a physically realizable source satisfies Dirichlet's conditions.

The “process” - We will determine f(t).

Then we will calculate the coefficients a_v, a_n and b_n.

It this course, linear lumped-parameter, time-invariant applies.
Thus so does superposition.

The procedure is straightforward and involves no new techniques of circuit analysis.

The analysis results in the Fourier series representation of the steady-state response of the circuit.

The Fourier coefficients are defined by the following integrals.

$$a_v = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\,dt$$

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos(\mathrm{n}\,\omega_0 t)\,dt$$

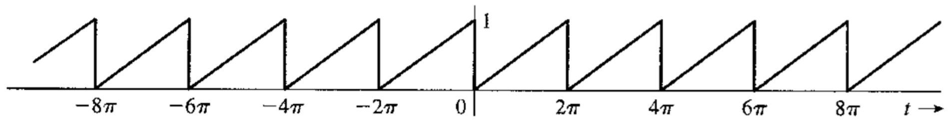
$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin(\mathrm{n}\,\omega_0 t)\,dt$$

Recall that $\omega_0 = \dfrac{2\pi}{T} \quad = 2\pi f$

The authors awkwardly use the subscript "k" for the calculation of the integrals and "n" in the resulting Fourier Series. You can assume k = n for Chapter 16.

Example

Find the Fourier Series for the following signal.



The period of this signal is T = 2π.

$$\omega_0 = \frac{2\pi}{T} \quad = \frac{2\pi}{2\pi} \quad = 1 \tfrac{rad}{\sec}$$

By inspection, the waveform function is $f(t) = \dfrac{t}{2\pi} \quad for\ 0 \le t \le 2\pi$

Find the Fourier coefficient for the average value

$$a_v = \frac{1}{T}\int_0^T f(t)\,dt \quad = \frac{1}{2\pi}\int_0^{2\pi} \frac{t}{2\pi}\,dt \quad = \frac{1}{4\pi^2}\left(\tfrac{1}{2}t^2\right)\Big|_0^{2\pi}$$

$$= \frac{1}{8\pi^2}\left[(2\pi)^2 - 0\right] \quad = \frac{4\pi^2}{8\pi^2} \quad = \frac{1}{2}$$

Example

$$a_k = \frac{2}{T}\int_0^T f(t)\cos k\omega_0 t\,dt \qquad = \frac{2}{2\pi}\int_0^{2\pi} \frac{t}{2\pi}\cos kt\,dt \qquad = \frac{2}{4\pi^2}\int_0^{2\pi} t\cos kt\,dt$$

Recall the integral solution:

$$\int x\cos ax\,dx = \tfrac{1}{a^2}(\cos ax + ax\sin ax)$$

Thus

$$a_k = \frac{1}{2\pi^2}\left[\tfrac{1}{k^2}\left(\cos kt + kt\sin kt\right)\Big|_0^{2\pi}\right]$$

$$= \frac{1}{2\pi^2}\left[\tfrac{1}{k^2}\left(\cos k2\pi + k2\pi\sin k2\pi\right) - \tfrac{1}{k^2}\left(\cos k0 + 0\sin k0\right)\right]$$

$$= \frac{1}{2\pi^2}\left[\tfrac{1}{k^2}(1+0) - \tfrac{1}{k^2}(1+0)\right] \qquad = \frac{1}{2\pi^2}\left[\tfrac{1}{k^2} - \tfrac{1}{k^2}\right] \qquad = \frac{1}{2\pi^2}[0] \qquad = 0$$

Example

$$b_k = \frac{2}{T}\int_0^T f(t)\sin k\omega_0 t\, dt \qquad = \frac{2}{2\pi}\int_0^{2\pi} \frac{t}{2\pi}\sin kt\, dt \qquad = \frac{1}{2\pi^2}\int_0^{2\pi} t\sin kt\, dt$$

A helpful integral solution is

$$\int x\sin ax\, dx = \tfrac{1}{a^2}(\sin ax - ax\cos ax)$$

Thus

$$b_k = \frac{1}{2\pi^2}\left[\tfrac{1}{k^2}\left(\sin kt - kt\cos kt\right)\Big|_0^{2\pi}\right]$$

$$= \frac{1}{2\pi^2}\left[\tfrac{1}{k^2}\left(\sin k2\pi - k2\pi\cos k2\pi\right) - \tfrac{1}{k^2}\left(\sin k0 - 0\cos k0\right)\right]$$

$$= \frac{1}{2\pi^2}\left[\tfrac{1}{k^2}(0 - k2\pi) - \tfrac{1}{k^2}(0+0)\right] \quad = \frac{1}{2\pi^2}\left[\frac{-k2\pi}{k^2}\right] \quad = \frac{-1}{\pi k}$$

Example

The Fourier series for this signal is then

$$f(t) = a_v + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \qquad a_v = \frac{1}{2} \qquad a_n = 0 \qquad b_n = \frac{-1}{n\pi}$$

$$= \tfrac{1}{2} - \tfrac{1}{\pi} \sum_{n=1}^{\infty} \tfrac{1}{n} \sin nt$$

$$= \tfrac{1}{2} - \tfrac{1}{\pi}\left(\sin t + \tfrac{1}{2}\sin 2t + \tfrac{1}{3}\sin 3t + \tfrac{1}{4}\sin 4t + \ldots\right)$$

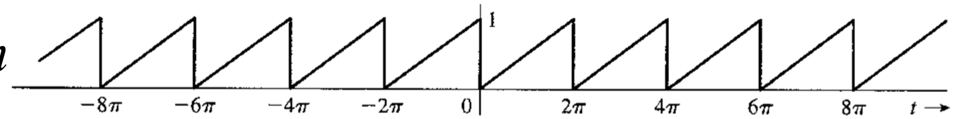
$$= \tfrac{1}{2} + \tfrac{1}{\pi}\left[\cos(t + \tfrac{\pi}{2}) + \tfrac{1}{2}\cos(2t + \tfrac{\pi}{2}) + \tfrac{1}{3}\cos(3t + \tfrac{\pi}{2}) + \ldots\right]$$

The sine term conversion to the cosine form can be proved by using the following trigonometric identity

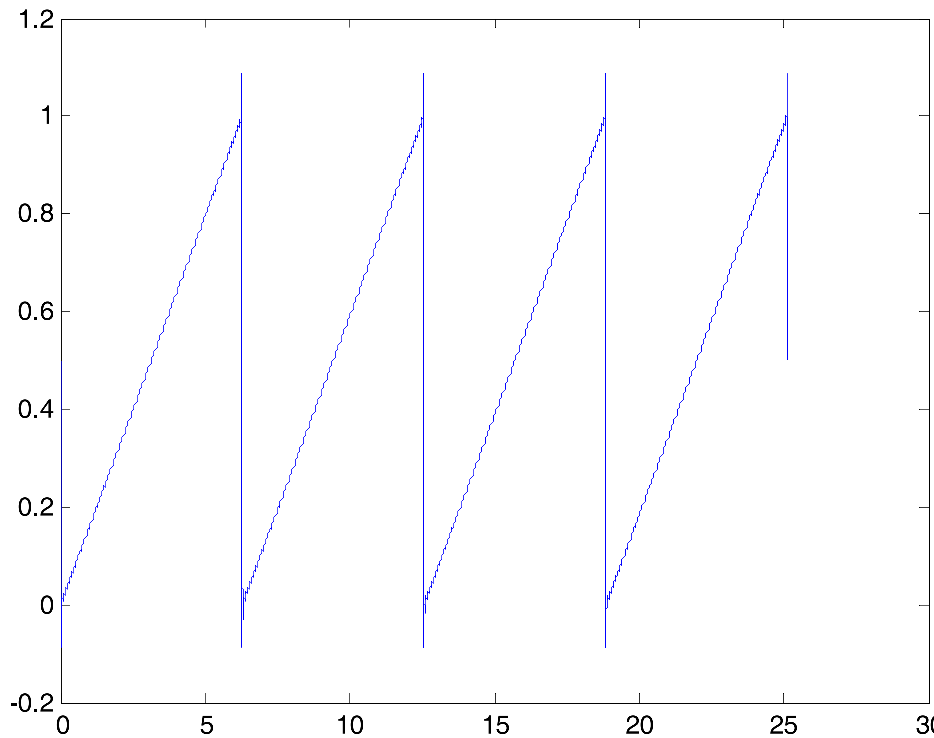
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\cos(x + \tfrac{\pi}{2}) = \cos x \underbrace{\cos \tfrac{\pi}{2}}_{=0} - \sin x \underbrace{\sin \tfrac{\pi}{2}}_{=1} \quad = (\cos x)(0) - (\sin x)(1) \quad = -\sin x$$

Original Waveform

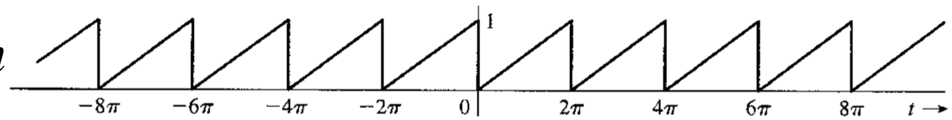


Fourier Series $f(t) = \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{1,000} \frac{1}{n}\sin nt$

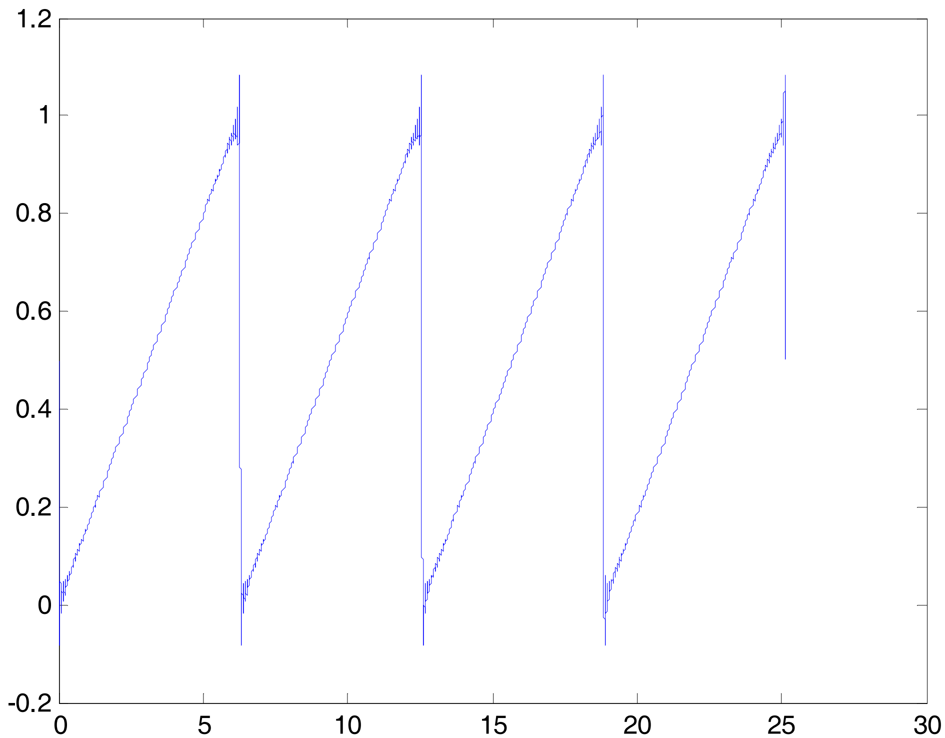


Matlab plot of the first 1,001 terms of the Fourier Series for $0 \le t \le 8\pi$

Original Waveform

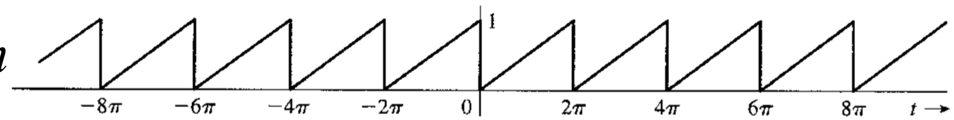


Fourier Series $f(t) = \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{100}\frac{1}{n}\sin nt$

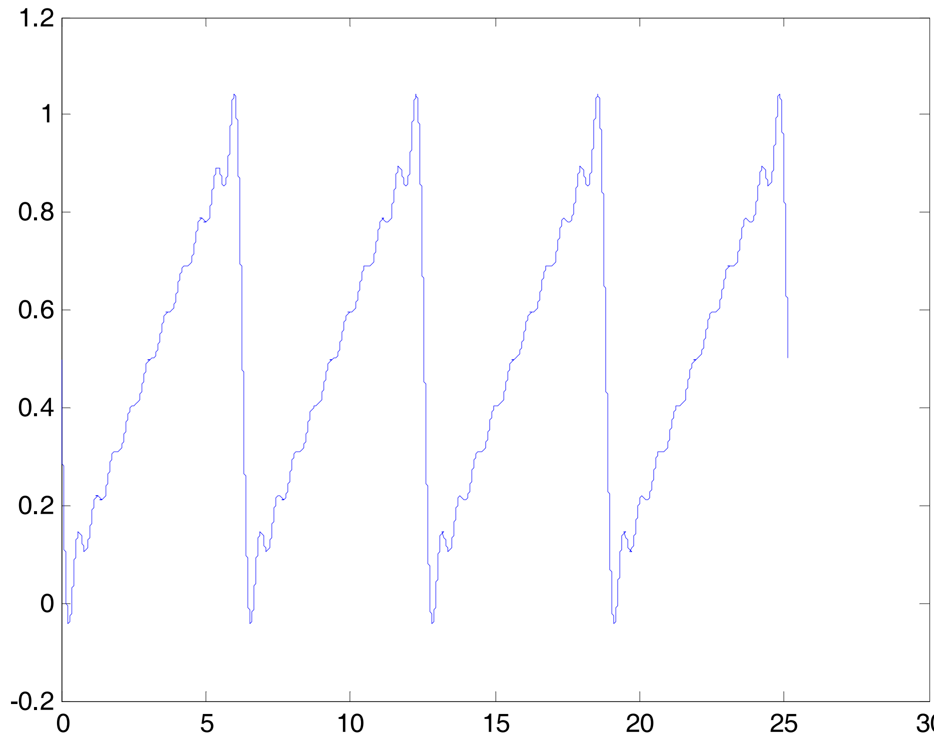


Matlab plot of the first 101 terms of the Fourier Series for $0 \leq t \leq 8\pi$

Original Waveform



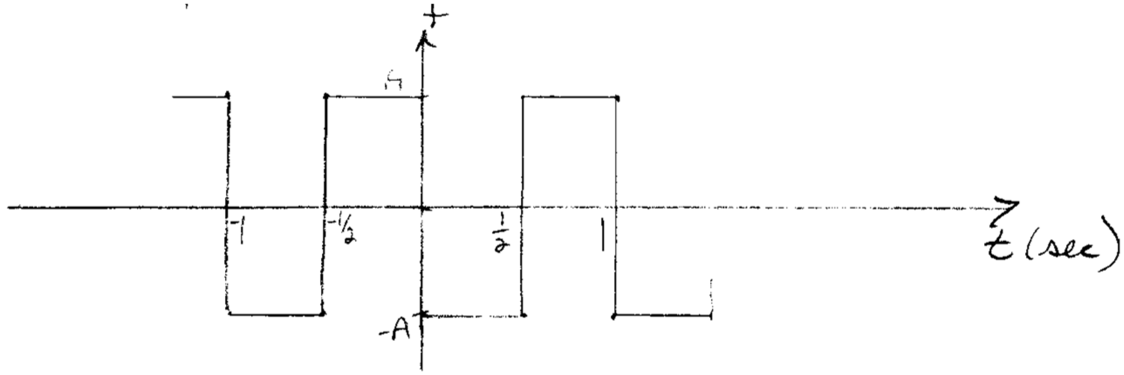
Fourier Series $f(t) = \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{10}\frac{1}{n}\sin nt$



Matlab plot of the first 11 terms of the Fourier Series for $0 \leq t \leq 8\pi$

Another Example – Fourier Series of Symmetric Square Wave

Find the Fourier Series for the following signal. (This is NOT the signal shown on the back cover of the textbook!)



The period of this signal is T = 1.

$$\omega_0 = \frac{2\pi}{T} \quad = \frac{2\pi}{1} \quad = 2\pi \tfrac{rad}{\sec}$$

$$a_v = \frac{1}{T}\int_0^T f(t)\,dt \quad = \frac{1}{1}\left\{\int_0^{\frac{1}{2}} (-A)\,dt + \int_{\frac{1}{2}}^1 A\,dt\right\} \quad = A\left[-t\Big|_0^{\frac{1}{2}} + t\Big|_{\frac{1}{2}}^1\right]$$

$$= A\left[-\left(\tfrac{1}{2}-0\right)+\left(1-\tfrac{1}{2}\right)\right] \quad = A(1-1) \quad = 0$$

Example – Fourier Series of Symmetric Square Wave

$$a_k = \frac{2}{T}\int_0^T f(t)\cos k\omega_0 t\, dt \qquad = \frac{2}{1}\int_0^{\frac{1}{2}} (-A)\cos k2\pi t\, dt + \frac{2}{1}\int_{\frac{1}{2}}^{1} A\cos k2\pi t\, dt$$

$$= \frac{-2A}{k2\pi}\sin k2\pi t\Big|_0^{\frac{1}{2}} + \frac{2A}{k2\pi}\sin k2\pi t\Big|_{\frac{1}{2}}^{1}$$

$$= \frac{-2A}{k2\pi}\left(\underbrace{\sin k\pi}_{=0\ for\ all\ k} - \underbrace{\sin 0}_{=0}\right) + \frac{2A}{k2\pi}\left(\underbrace{\sin k2\pi}_{=0\ for\ all\ k} - \underbrace{\sin k\pi}_{=0\ for\ all\ k}\right) \qquad = 0$$

$$a_k = 0$$

Example – Fourier Series of Symmetric Square Wave

$$b_k = \frac{2}{T}\int_0^T f(t)\sin k\omega_0 t\,dt \quad = \frac{2}{1}\int_0^{\frac{1}{2}}(-A)\sin k2\pi t\,dt + \frac{2}{1}\int_{\frac{1}{2}}^{1} A\sin k2\pi t\,dt$$

$$= \frac{2(-A)}{k2\pi}(-)\cos k2\pi t\Big|_0^{\frac{1}{2}} + \frac{2A}{k2\pi}(-)\cos k2\pi t\Big|_{\frac{1}{2}}^{1}$$

$$= \frac{A}{k\pi}\left(\cos k\pi - \underbrace{\cos 0}_{=1}\right) - \frac{A}{k\pi}\left(\underbrace{\cos k2\pi}_{=1\ for\ all\ k} - \cos k\pi\right)$$

$$= \frac{A}{k\pi}\left(\cos k\pi - 1 - 1 + \cos k\pi\right) \quad = \frac{A}{k\pi}\left(2\cos k\pi - 2\right) \quad = \frac{2A}{k\pi}\left(\cos k\pi - 1\right)$$

$$\text{where } \cos k\pi = \begin{cases} 1 & for\ k\ even \\ -1 & for\ k\ odd \end{cases}$$

$$Thus\ \ b_k = \frac{2A}{k\pi}\left(\cos k\pi - 1\right) \ = \begin{cases} 0 & for\ k\ even \\ \dfrac{-4A}{k\pi} & for\ k\ odd \end{cases}$$

Example – Fourier Series of Symmetric Square Wave

Thus the Fourier series for this signal is

$$f(t) = a_v + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \qquad a_v = 0 \qquad a_n = 0 \qquad b_n = \begin{cases} 0 \;\; for\ n\ even \\ \dfrac{-4A}{n\pi} \;\; for\ n\ odd \end{cases}$$

$$= 0 + 0 + \sum_{n=1,3,5,\ldots}^{\infty} \frac{-4A}{n\pi} \sin n2\pi t$$

$$= \frac{-4A}{\pi} \sin 2\pi t - \frac{4A}{3\pi} \sin 6\pi t - \frac{4A}{5\pi} \sin 10\pi t - \ldots.$$

This can be put into the cosine form by use of the following elementary trigonometric relations:

$$\sin(\omega t + \theta + 90^\circ) = \cos(\omega t + \theta + 0^\circ) \qquad \Rightarrow \sin(\omega t + \theta) = \cos(\omega t + \theta - 90^\circ)$$

Thus

$$f(t) = -\tfrac{4A}{\pi}\left[\cos(1 \times 2\pi t - \tfrac{\pi}{2}) + \tfrac{1}{3}\cos(3 \times 2\pi t - \tfrac{\pi}{2}) + \tfrac{1}{5}\cos(5 \times 2\pi t - \tfrac{\pi}{2}) + \ldots\right]$$

$$= -\tfrac{4A}{\pi}\left[\cos(2\pi t - \tfrac{\pi}{2}) + \tfrac{1}{3}\cos(6\pi t - \tfrac{\pi}{2}) + \tfrac{1}{5}\cos(10\pi t - \tfrac{\pi}{2}) + \ldots\right]$$

Example – Fourier Series of Symmetric Square Wave

$$f(t) = \frac{-4A}{\pi}\sin 2\pi t - \frac{4A}{3\pi}\sin 6\pi t - \frac{4A}{5\pi}\sin 10\pi t - ....$$

Or we can use the equivalent relation:

$$\cos(\omega t + \theta + 90^\circ) = -\sin(\omega t + \theta + 0^\circ) \quad \Rightarrow \sin(\omega t + \theta + 0^\circ) = -\cos(\omega t + \theta + 90^\circ)$$

Thus

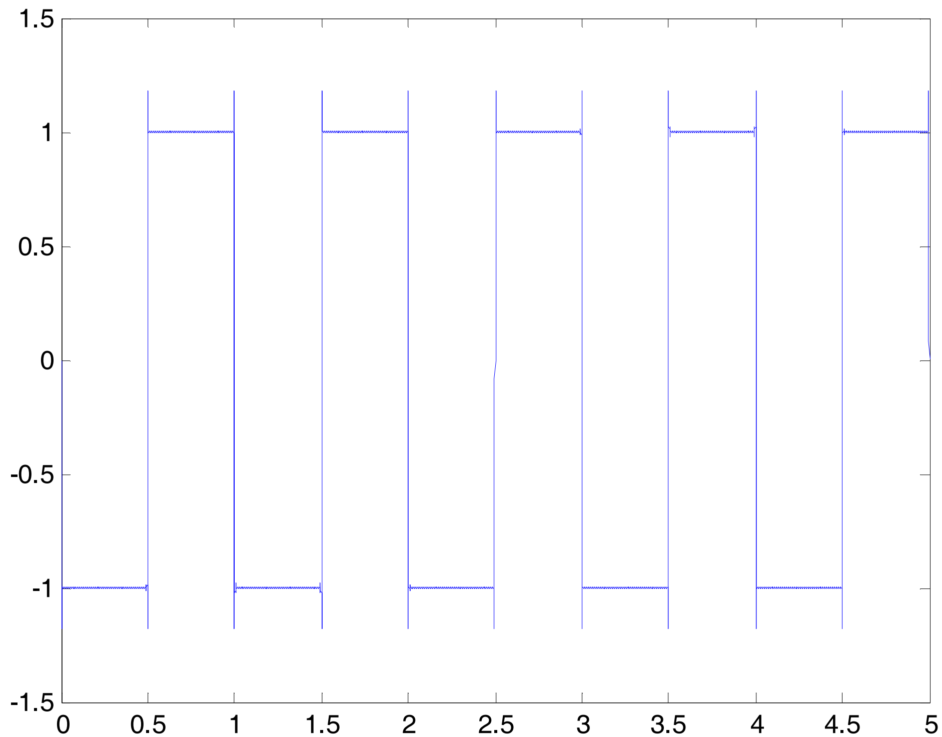
$$f(t) = (-)\tfrac{(-)4A}{\pi}\left[\cos(1\times 2\pi t + \tfrac{\pi}{2}) + \tfrac{1}{3}\cos(3\times 2\pi t + \tfrac{\pi}{2}) + \tfrac{1}{5}\cos(5\times 2\pi t + \tfrac{\pi}{2}) + ...\right]$$

$$= \tfrac{4A}{\pi}\left[\cos(1\times 2\pi t + \tfrac{\pi}{2}) + \tfrac{1}{3}\cos(3\times 2\pi t + \tfrac{\pi}{2}) + \tfrac{1}{5}\cos(5\times 2\pi t + \tfrac{\pi}{2}) + ...\right]$$

$$= \tfrac{4A}{\pi}\left[\cos(2\pi t + \tfrac{\pi}{2}) + \tfrac{1}{3}\cos(6\pi t + \tfrac{\pi}{2}) + \tfrac{1}{5}\cos(10\pi t + \tfrac{\pi}{2}) + ...\right]$$

Fourier Series with 1,000 terms

$$f(t) = \sum_{n=1,3,5,\ldots}^{2,000} \frac{-4A}{n\pi} \sin(n2\pi t)$$

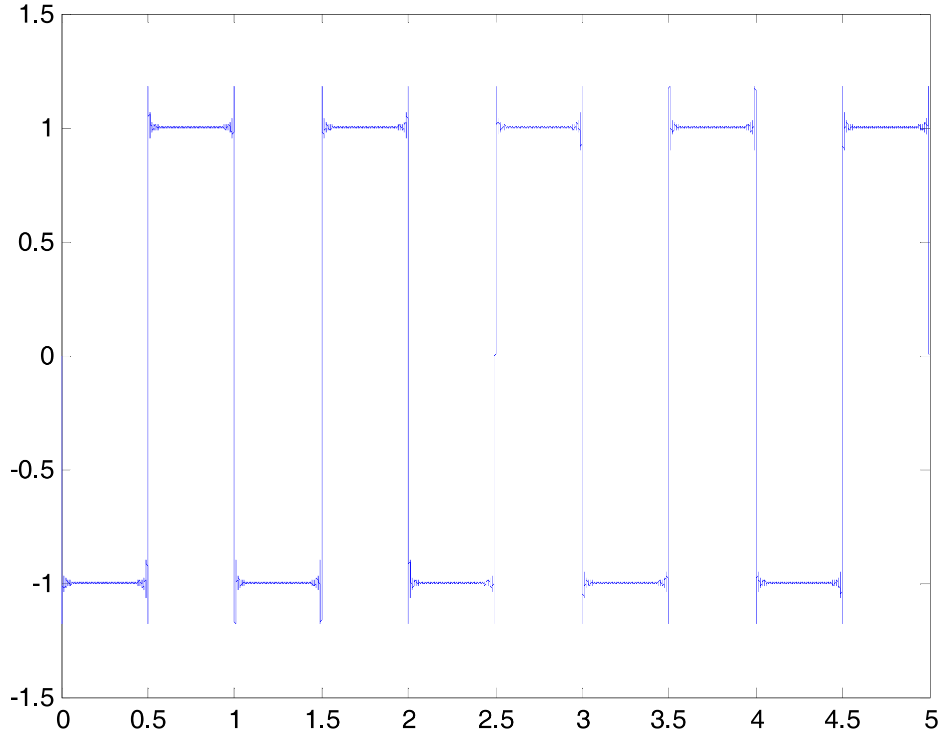


Matlab plot of the first 1,000 terms of the Fourier Series for $0 \leq t \leq 5$ sec

The overshoot at the transition discontinuity is called the *Gibbs Phenomenon.*

Fourier Series with 100 terms

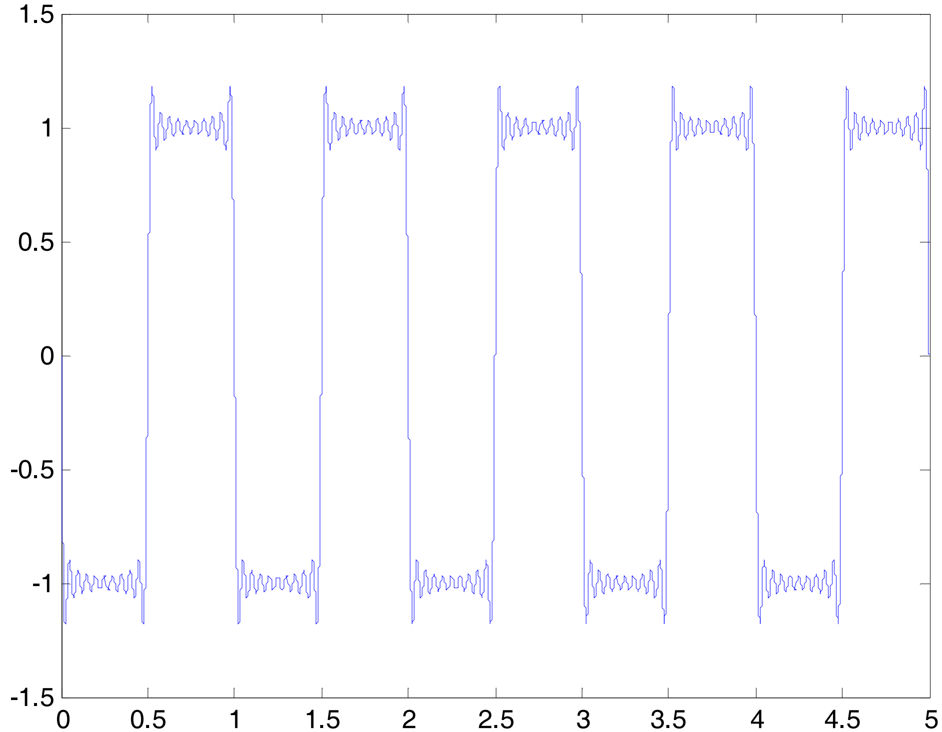
$$f(t)=\sum_{n=1,3,5,\ldots}^{200}\frac{-4A}{n\pi}\sin(n2\pi t)$$



Matlab plot of the first 100 terms of the Fourier Series for $0 \leq t \leq 5$ sec

Fourier Series with 10 terms

$$f(t) = \sum_{n=1,3,5,...}^{20} \frac{-4A}{n\pi} \sin(n2\pi t)$$



Matlab plot of the first 10 terms of the Fourier Series for $0 \leq t \leq 5$ sec

Section 16.3 The Effect of Symmetry on the Fourier Coefficients

Four types of symmetry may be used to simplify the task of evaluating the Fourier coefficients.

Even function $f(t) = f(-t)$

Odd function $f(t) = -f(-t)$

Half-wave symmetry $f(t) = -f(t - T/2)$

Quarter-wave symmetry which has symmetry at the midpoint of both the positive and negative half cycles.

Use symmetry at your own peril at this point. On tests or quizzes, the use of incorrect symmetry is no excuse. You will have time to solve the coefficients without the use of symmetry of any kind.

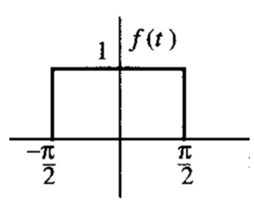
Any even periodic function f(t) consists of cosine terms only.

A function is an **even** function if f(t) = f(-t).

Examples of even functions.

$$f(x) = x^2 \qquad \Rightarrow f(-x) = (-x)^2 = x^2$$

$$f(x) = \cos(45^\circ) = \frac{1}{\sqrt{2}} \qquad \Rightarrow f(-x) = \cos(-45^\circ) = \frac{1}{\sqrt{2}}$$



When a function is **even** we can use the follow abbreviated equations:

$$a_v = \frac{2}{T}\int_0^{\frac{T}{2}} f(t)\,dt$$

$$a_n = \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\cos n\omega_0 t\,dt$$

$$b_n = 0$$

Cosine is even (and shows above) and sine is odd (and thus is not included above).

Section 16.3 Odd Symmetry

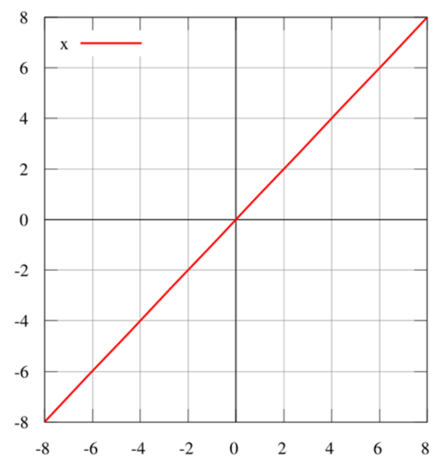
Any odd periodic function f(t) consists of sine terms only.

A function is an odd function if f(t) = -f(-t).

Examples of odd functions.

$$f(x) = x^3 \qquad \Rightarrow f(-x) = (-x)^3 = -x^3$$

$$f(x) = \sin(45^\circ) = \frac{1}{\sqrt{2}} \qquad \Rightarrow f(-x) = \sin(-45^\circ) = -\frac{1}{\sqrt{2}}$$



When a function is **odd** we can use the follow abbreviated equations:

$$a_v = a_n = 0$$

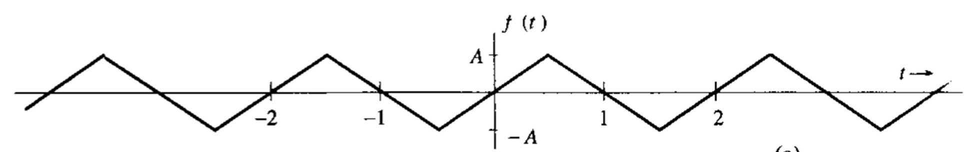
$$b_n = \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\sin n\omega_0 t\, dt$$

Sine is odd (and shows above) and cosine is even (and thus is not included above).

Also note that odd functions do not have a “dc” value.

If a periodic signal f(t), shifted by half the period, remains unchanged except for a sign (-) then the signal is said to have half-wave symmetry.

$$f(t) = -f(t - \frac{T}{2})$$



(a)

In a signal with half-wave symmetry, all the even numbered harmonics vanish.

Section 16.3 Quarter-wave symmetry

Quarter-wave symmetry is really reaching down into the bag just to avoid some calculus.

Quarter-wave symmetry will not be covered and will not be tested
Read the section on your own.

Section 16.4 Compact Form of the Fourier Series

We can combine the a_n and b_n coefficients into a single cosine (or sine) term. The resulting series is called the compact Fourier Series.

$$f(t) = a_v + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t - \theta_n)$$

$$A_n \cos(n\omega_o t - \theta_n) = a_n \cos n\omega_o t + b_n \sin n\omega_o t \quad = a_n \cos n\omega_o t + b_n \cos(n\omega_o t - 90^\circ)$$

$$A_n \angle -\theta_n = a_n \angle 0^\circ + b_n \angle -90^\circ \quad = a_n - j\, b_n$$

$$\text{where } A_n = \sqrt{a_n^2 + b_n^2} \quad and \quad -\theta_n = -\tan^{-1}\left(\frac{b_n}{a_n}\right)$$

The dc term ($\omega = 0$) is given by a_v.

Section 16.4 Compact Form of the Fourier Series

We can combine the a_n and b_n coefficients into a single cosine (or sine) term. The resulting series is called the compact Fourier Series.

$$f(t) = a_v + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t - \theta_n)$$

$$A_n \cos(n\omega_o t - \theta_n) = a_n \cos n\omega_o t + b_n \sin n\omega_o t \quad = a_n \cos n\omega_o t + b_n \cos(n\omega_o t - 90^\circ)$$

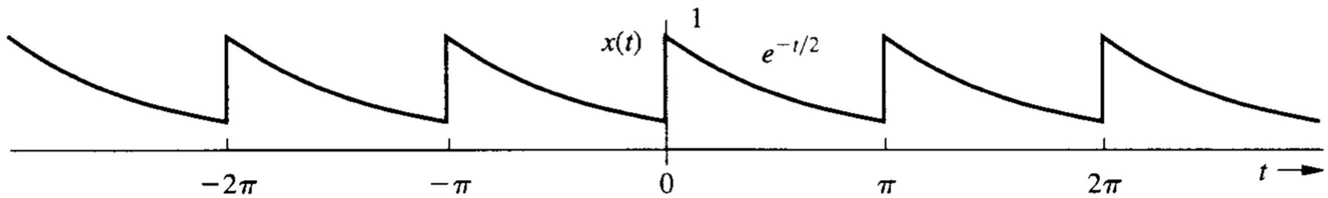
$$\text{where } A_n = \sqrt{a_n^2 + b_n^2} \quad and \quad \theta_n = \tan^{-1}\left(\frac{b_n}{a_n}\right)$$

The dc term ($\omega_0 = 0$) is given by a_v.

Note that in this textbook, the phase term can be misleading. The authors incorporated a minus sign in the Fourier Series definition for the phase.

Example – Compact Fourier Series

Find the compact trigonometric Fourier Series for the periodic signal x(t).



In this case $T_0 = \pi$.

$$Fundamental\ \ frequency\ f_0 = \frac{1}{T_0} \quad = \frac{1}{\pi} Hz$$

$$\omega_0 = \frac{2\pi}{T_0} \quad = \frac{2\pi}{\pi \sec} \quad = 2\frac{rad}{\sec}$$

Therefore

$$f(t) = a_v + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \quad = a_v + \sum_{n=1}^{\infty} a_n \cos n2t + b_n \sin n2t$$

Example – Compact Fourier Series

The coefficients are

$$a_v = \frac{1}{T_0}\int_{T_0} f(t)\,dt \;= \frac{1}{\pi}\int_0^{\pi} e^{-\frac{t}{2}}\,dt \;= \frac{1}{\pi}\left(-\frac{1}{\frac{1}{2}}\right)e^{-\frac{t}{2}}\Big|_0^{\pi} \qquad = \frac{1}{\pi}(-2)\left(e^{-\frac{\pi}{2}} - e^{0}\right)$$

$$= \left(-\frac{2}{\pi}\right)(0.2079 - 1) \qquad = 0.5043$$

Now for a_n.

$$a_n = \frac{2}{T_0}\int_{t_1}^{t_1+T_0} f(t)\cos n\omega_0 t\,dt \quad = \frac{2}{\pi}\int_0^{\pi} e^{-\frac{t}{2}}\cos n2t\,dt$$

Recall $\displaystyle\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\cos bx + b\sin bx\right)$

Where a = -1/2 and b = 2n for the integral pair above.

Example – Compact Fourier Series

Therefore

$$a_n = \frac{2}{\pi}\left[\frac{e^{-\frac{t}{2}}}{\left(-\tfrac{1}{2}\right)^2 + (2n)^2}\left(-\tfrac{1}{2}\cos 2nt + 2n\sin 2nt\right)\right]\Bigg|_0^{\pi}$$

$$= \frac{2}{\pi}\left\{\left[\frac{e^{-\frac{\pi}{2}}}{\frac{1}{4}+4n^2}\left(-\tfrac{1}{2}\cos 2n\pi + 2n\sin 2n\pi\right)\right] - \left[\frac{e^{0}}{\frac{1}{4}+4n^2}\left(-\tfrac{1}{2}\cos 0 + 2n\sin 0\right)\right]\right\}$$

$$= \frac{2}{\pi}\frac{4}{4(\frac{1}{4}+4n^2)}\left\{e^{-\frac{\pi}{2}}\left[-\tfrac{1}{2}(1)+2n(0)\right] - \left[-\tfrac{1}{2}(1)+2n(0)\right]\right\}$$

$$= \frac{8}{\pi(1+16n^2)}\left\{e^{-\frac{\pi}{2}}\left[-\tfrac{1}{2}\right] - \left[-\tfrac{1}{2}\right]\right\} \quad = \frac{8}{\pi(1+16n^2)}\left(-\frac{e^{-\frac{\pi}{2}}}{2} + \tfrac{1}{2}\right)$$

$$a_n = \frac{8}{\pi(1+16n^2)}\left(-\frac{0.2079}{2} + \tfrac{1}{2}\right) = \frac{2}{(1+16n^2)}\frac{4}{\pi}\left(0.3961\right) \approx \frac{2}{1+16n^2}\left(0.5043\right)$$

Example – Compact Fourier Series

Now for b_n $$b_n = \frac{2}{T_0}\int_{t_1}^{t_1+T_0} f(t)\sin n\omega_0 t\, dt \quad = \frac{2}{\pi}\int_0^{\pi} e^{-\frac{t}{2}} \cos n2t\, dt$$

Recall $\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}\left(a\sin bx - b\cos bx\right)$ Where a = -1/2 and b = 2n.

Therefore

$$b_n = \frac{2}{\pi}\left[\frac{e^{-\frac{t}{2}}}{\left(-\frac{1}{2}\right)^2+(2n)^2}\left(-\tfrac{1}{2}\sin 2nt - 2n\cos 2nt\right)\right]\Bigg|_0^{\pi}$$

$$= \frac{8}{\pi(1+16n^2)}\left\{\left[e^{-\frac{\pi}{2}}\left(-\tfrac{1}{2}\sin 2n\pi - 2n\cos 2n\pi\right)\right] - \left[\left(-\tfrac{1}{2}\sin 0 - 2n\cos 0\right)\right]\right\}$$

$$= \frac{8}{\pi(1+16n^2)}\left\{\left[e^{-\frac{\pi}{2}}(0-2n)\right] - \left[(0-2n)\right]\right\} \quad = \frac{8n}{\pi(1+16n^2)}\underbrace{\left(e^{-\frac{\pi}{2}}(-2)+2\right)}_{=1.5842}$$

$$b_n = \frac{8n}{(1+16n^2)}\left(\frac{1.5842}{\pi}\right) \approx \frac{8n}{(1+16n^2)}(0.5403)$$

Example – Compact Fourier Series

The calculations show that the “standard” Fourier Series coefficient are

$$a_v = 0.5043 \qquad a_n = 0.5043\frac{2}{1+16n^2} \qquad b_n = 0.5043\frac{8n}{1+16n^2}$$

Thus the coefficient for A_n are

$$A_n = \sqrt{a_n^2 + b_n^2} \quad = 0.5043\sqrt{\left(\frac{2}{1+16n^2}\right)^2 + \left(\frac{8n}{1+16n^2}\right)^2} \quad = 0.5043\sqrt{\frac{4+64n^2}{(1+16n^2)^2}}$$

$$= 0.5043\sqrt{\frac{4(1+16n^2)}{(1+16n^2)^2}} \quad = 0.5043\sqrt{\frac{4}{1+16n^2}} \quad = 0.5043\frac{2}{\sqrt{1+16n^2}}$$

$$\theta_n = \tan^{-1}\left(\frac{b_n}{a_n}\right) \quad = \tan^{-1}\left(\frac{\frac{8n}{1+16n^2}}{\frac{2}{1+16n^2}}\right) \quad = \tan^{-1}\left(\frac{8n}{2}\right) \quad = \tan^{-1}(4n)$$

Example – Compact Fourier Series

$$f(t) = a_v + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t - \theta_n) \quad \text{where } \omega_0 = 2 \text{ rad/sec}$$

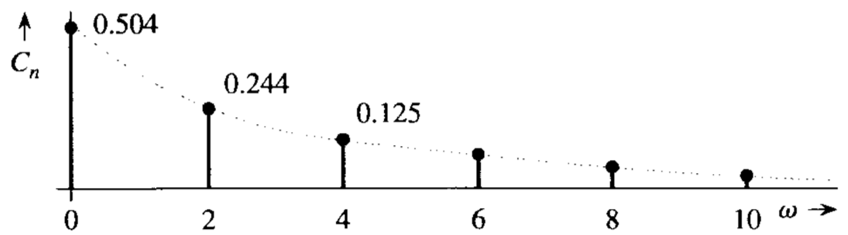
$$f(t) = 0.504 + 0.504 \sum_{n=1}^{\infty} \frac{2}{\sqrt{1+16n^2}} \cos(2nt - \tan^{-1} 4n)$$

We can calculate A_n and θ_n for a few harmonic terms.

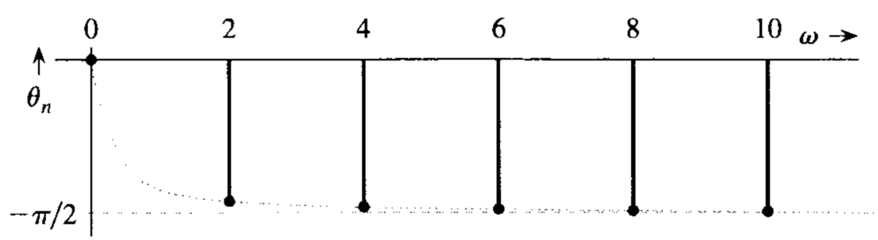
n	A_n	θ_n
0	0.504	0
1	0.244	75.96
2	0.125	82.87
3	0.084	85.24
4	0.063	86.42
5	0.0504	87.14
6	0.042	87.61
7	0.036	87.95

Example – Compact Fourier Series

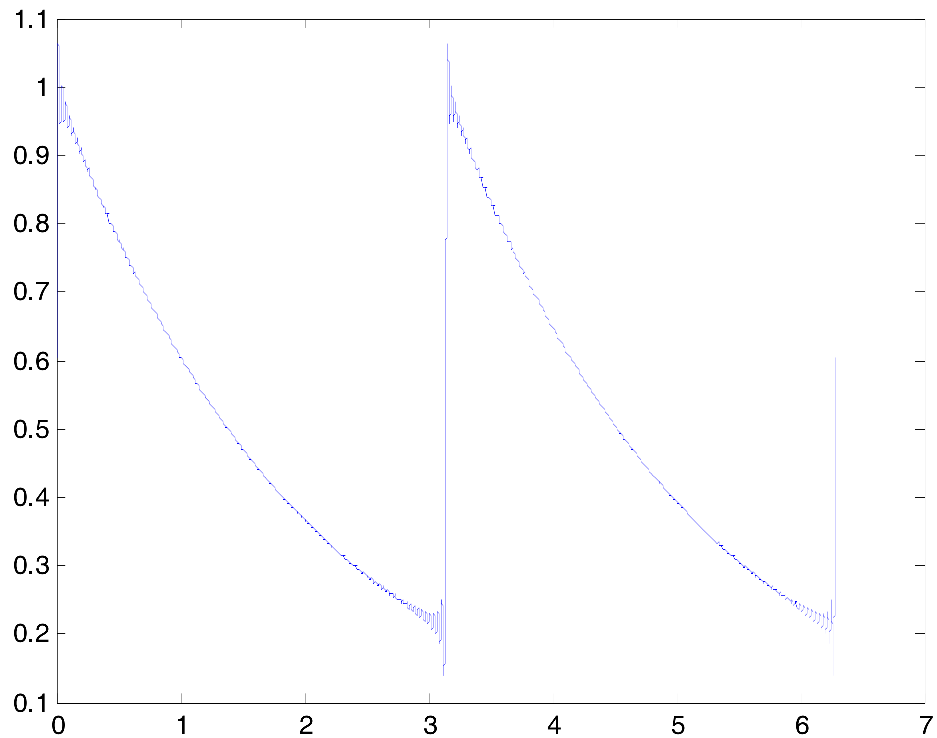
$$f(t) = 0.504$$
$$+0.244\cos(2t - 75.96^\circ)$$
$$+0.125\cos(4t - 82.87^\circ)$$
$$+0.084\cos(6t - 85.24^\circ)$$
$$+0.084\cos(8t - 86.42^\circ) + \ldots$$



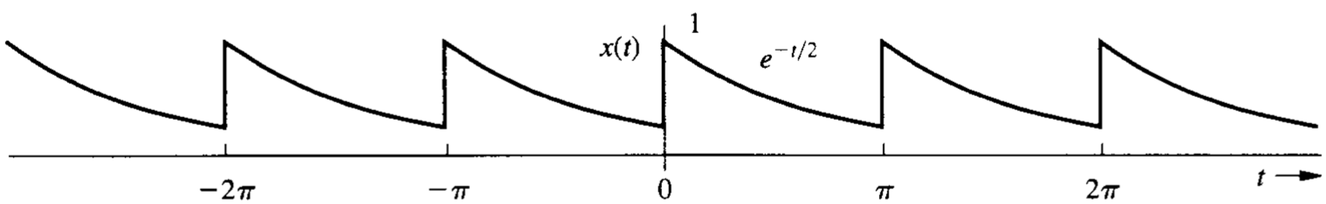
(b)



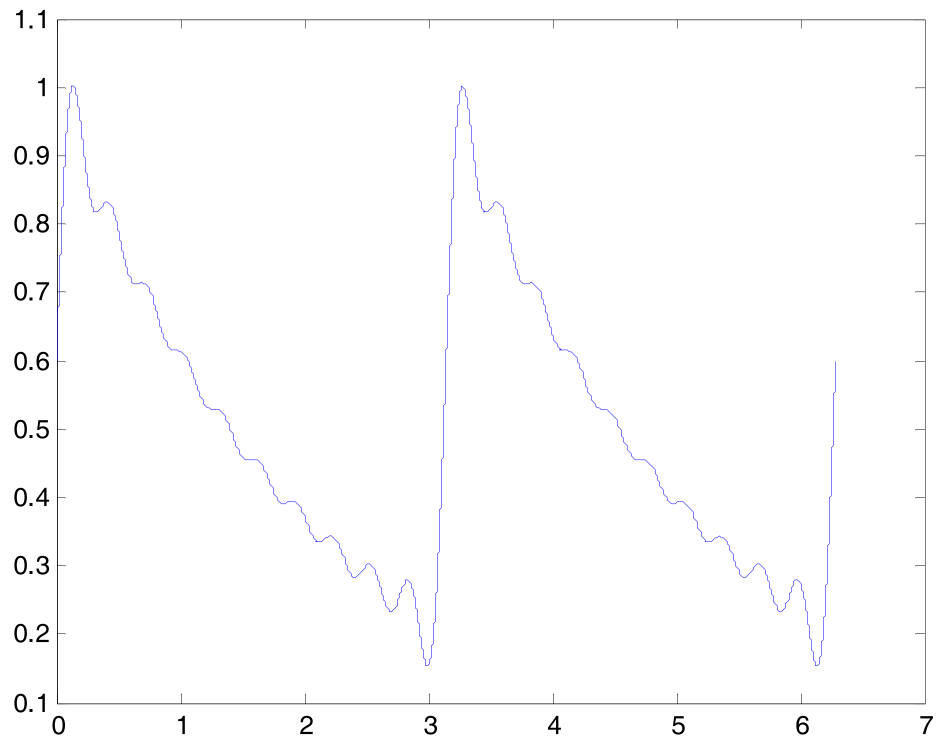
$$f(t) = 0.504 + 0.504\sum_{n=1}^{100} \frac{2}{\sqrt{1+16n^2}} \cos(2nt - \tan^{-1} 4n)$$



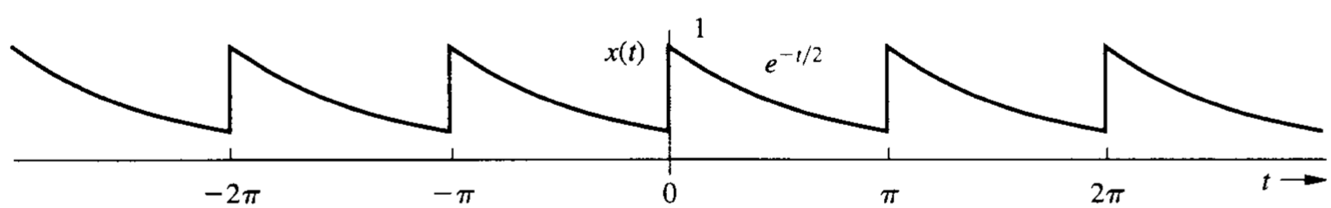
Matlab plot of the first 101 terms of the Fourier Series for $0 \leq t \leq 2\pi$ sec



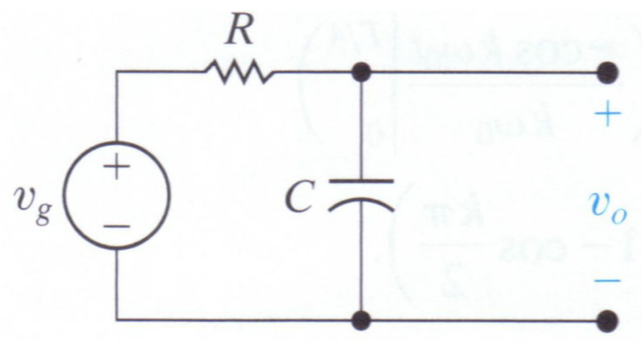
$$f(t) = 0.504 + 0.504\sum_{n=1}^{10} \frac{2}{\sqrt{1+16n^2}} \cos(2nt - \tan^{-1} 4n)$$

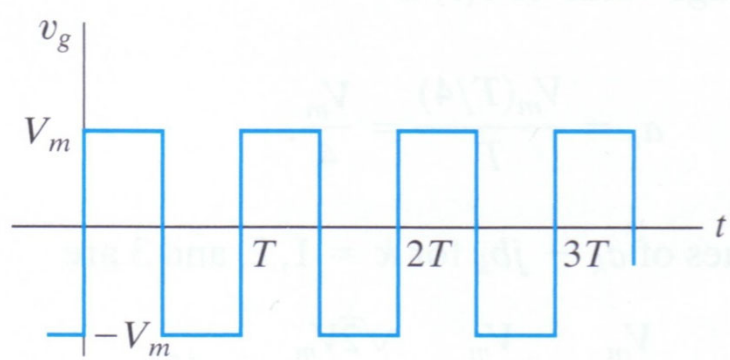


Matlab plot of the first 11 terms of the Fourier Series for $0 \leq t \leq 2\pi$ sec



A low pass filter is excited by a square signal.

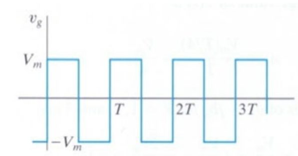




$$\omega_0 = \frac{2\pi}{T}$$

We want to find the steady-state response of the output signal.

First find the Fourier series of the input signal.



The square wave has odd, half-wave and quarter-wave symmetry.

I will use odd and half-wave symmetry in finding the Fourier coefficients.

$$a_v = 0 \; by \; odd \; symmetry$$

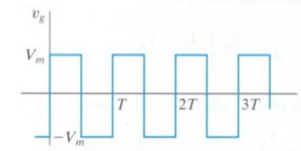
$$a_k = 0 \; by \; odd \; symmetry$$

$$b_k = 0 \; for \; k \, even \; by \; half-wave \; symmetry$$

$$b_k = \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\sin k\omega_0 t \; dt \; for \; k \, odd \; by \; half-wave \; symmetry$$

$$= \frac{4}{T}\int_0^{\frac{T}{2}} V_m \sin k\frac{2\pi}{T}t \; dt \quad = \frac{4V_m}{T\,\frac{k2\pi}{T}}(-)\cos k\omega_0 t\Big|_0^{\frac{T}{2}}$$

$$= \frac{-2V_m}{k\pi}\left[\underbrace{\cos k\frac{2\pi}{T}\frac{T}{2}}_{=-1} - \underbrace{\cos k\frac{2\pi}{T}0}_{=1}\right] \quad = \frac{-2V_m}{k\pi}(-2) \quad = \frac{4V_m}{k\pi}$$



The input square wave has the following Fourier series.

$$v_g = \frac{4V_m}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \tfrac{1}{n} \sin n\omega_0 t$$

$$= \frac{4V_m}{\pi} \sin \omega_0 t \; + \frac{4V_m}{3\pi} \sin 3\omega_0 t \; + \frac{4V_m}{5\pi} \sin 5\omega_0 t + \ldots.$$

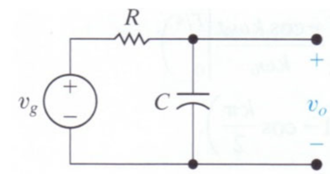
The system is assumed to be linear, lumped parameter, time invariant so superposition applies.

Now write the transfer function of the circuit and find the output voltage.

Section 16.5 An Application of Fourier Series

$$v_g = \frac{4V_m}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \tfrac{1}{n} \sin n\omega_0 t$$

The output is the voltage across the capacitor.



$$H(j\omega) = \frac{v_0}{v_g}$$

$$v_0 = \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + R} v_g \quad = \frac{1}{1 + j\omega RC} v_g$$

For the first term of the Fourier series (n = 1) we have

$$v_{o1} = \frac{v_{g1}}{1 + j\omega RC} \quad = \frac{\frac{4V_m}{\pi}\angle 0^\circ}{1 + j\omega RC} \quad = \frac{\frac{4V_m}{\pi}\angle 0^\circ}{\sqrt{1^2 + (\omega_0 RC)^2}\angle \tan^{-1}(\frac{\omega_0 RC}{1})}$$

$$= \left[\frac{4V_m}{\pi}\angle -\tan^{-1}(\omega_0 RC)\right] \frac{1}{\sqrt{1 + (\omega_0 RC)^2}}$$

These are phasors.

In this case the phasors are referenced to the sine function rather than the usual cosine function. Thus

$$v_{o1} = \frac{\frac{4V_m}{\pi} \sin\left[\omega_0 t - \angle \tan^{-1}(\omega_0 RC)\right]}{\sqrt{1+(\omega_0 RC)^2}}$$

Similarly the next term of the Fourier series (n = 3) is

$$v_{o3} = \frac{\frac{4V_m}{3\pi} \sin\left[3\omega_0 t - \angle \tan^{-1}(3\omega_0 RC)\right]}{\sqrt{1+(3\omega_0 RC)^2}}$$

For the general k^th case we have

$$v_{ok} = \frac{\dfrac{4V_m}{k\pi}\sin\left[k\omega_0 t - \angle\tan^{-1}(k\omega_0 RC)\right]}{\sqrt{1+(k\omega_0 RC)^2}}$$

So we can now write the Fourier series (steady-state) of the output signal v_o.

$$v_o = \frac{4V_m}{\pi}\sum_{n=1,3,5,\ldots}^{\infty}\frac{\sin\left[n\omega_0 t - \angle\tan^{-1}(n\omega_0 RC)\right]}{n\sqrt{1+(n\omega_0 RC)^2}}$$

We can now draw some circuit behavior conclusions from the Fourier series for the output signal.

$$v_o = \frac{4V_m}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{\sin\left[n\omega_0 t - \angle \tan^{-1}(n\omega_0 RC)\right]}{n\sqrt{1+(n\omega_0 RC)^2}}$$

As the frequency $\omega = n\omega_0 \rightarrow \infty$, the magnitude of the response $\rightarrow 0$.

$$v_o = \frac{4V_m}{\pi} \frac{\sin\left[n\omega_0 t - \angle \tan^{-1}(n\omega_0 RC)\right]}{n\infty} = 0$$

The Fourier series does describe a low pass filter as expected!

What about the effects of varying the capacitance C?

For large C

$$v_o = \frac{4V_m}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{\sin\left[n\omega_0 t - \angle \tan^{-1}(n\omega_0 RC)\right]}{n\sqrt{1+(n\omega_0 RC)^2}}$$

$$\approx \frac{4V_m}{\pi\omega_0 RC} \sum_{n=1,3,5,\ldots}^{\infty} \frac{\sin\left[n\omega_0 t - 90^\circ\right]}{n^2} \qquad = \frac{4V_m}{\pi\omega_0 RC} \sum_{n=1,3,5,\ldots}^{\infty} \frac{-\cos(n\omega_0 t)}{n^2}$$

The output harmonics are decreasing by $1/n^2$.

But the input harmonics are decreasing by $1/n$.

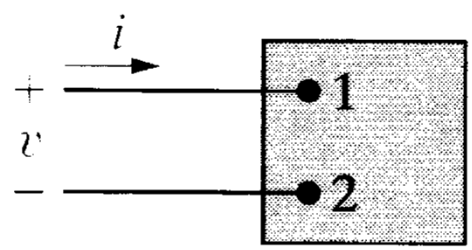
Again, we see the effects of a low pass filter.

Given the Fourier series representations of the voltage and current at a pair of terminals.

$$v = V_{dc} + \sum_{n=1}^{\infty} V_n \cos(n\omega_0 t - \theta_{vn})$$

$$i = I_{dc} + \sum_{n=1}^{\infty} I_n \cos(n\omega_0 t - \theta_{in})$$

Write v & i in accordance with the passive sign convention.
Instantaneous power p = vi.



(a) $p = vi$

The average power is given by

$$P_{avg} = \frac{1}{T}\int_{t_0}^{t_0+T} p\ dt \qquad = \frac{1}{T}\int_{t_0}^{t_0+T} vi\ dt$$

Only terms of the same frequency (of v and i) survive the integration. Thus

$$P_{avg} = \frac{1}{T} V_{dc} I_{dc}\, t \Big|_{t_0}^{t_0+T} \quad \textit{for the dc components}$$

$$+\sum_{n=1}^{\infty} \frac{1}{T} \int_{t_0}^{t_0+T} V_n I_n \cos(n\omega_0 t - \theta_{vn}) \cos(n\omega_0 t - \theta_{in})\, dt \quad \textit{for the harmonics.}$$

Section 16.6 Average-Power Calculations with Periodic Functions

$$P_{avg} = V_{dc} I_{dc} + \sum_{n=1}^{\infty} \frac{1}{T} \int_{t_0}^{t_0+T} V_n I_n \cos(n\omega_0 t - \theta_{vn}) \cos(n\omega_0 t - \theta_{in})\, dt$$

Use the following trig identity to simplify.

$$\cos\alpha \cos\beta = \tfrac{1}{2}\cos(\alpha - \beta) + \tfrac{1}{2}\cos(\alpha + \beta)$$

$$\alpha = n\omega_0 t - \theta_{in} \qquad\qquad \beta = n\omega_0 t - \theta_{vn}$$

$$\alpha - \beta = (n\omega_0 t - \theta_{in}) - (n\omega_0 t - \theta_{vn}) = \theta_{vn} - \theta_{in}$$

$$\alpha + \beta = (n\omega_0 t - \theta_{in}) + (n\omega_0 t - \theta_{vn}) = 2n\omega_0 t - \theta_{vn} - \theta_{in}$$

Then

$$P_{avg} = V_{dc} I_{dc} + \sum_{n=1}^{\infty} \frac{1}{T} \frac{V_n I_n}{2} \int_{t_0}^{t_0+T} [\cos(\theta_{vn} - \theta_{in}) + \underbrace{\cos(2n\omega_0 t - \theta_{vn} - \theta_{in})}_{\text{integral of cosine over one full period = zero}}]\, dt$$

$$= V_{dc} I_{dc} + \sum_{n=1}^{\infty} \frac{1}{T} \frac{V_n I_n}{2} [\cos(\theta_{vn} - \theta_{in})] T \quad = V_{dc} I_{dc} + \sum_{n=1}^{\infty} \frac{V_n I_n}{2} \cos(\theta_{vn} - \theta_{in})$$

The total average power is the superposition of the average powers associated with each harmonic.

Example (Assessment Problem 16.3 and 16.7 in the 8th edition)

Input the following voltage waveform into the RLC circuit.

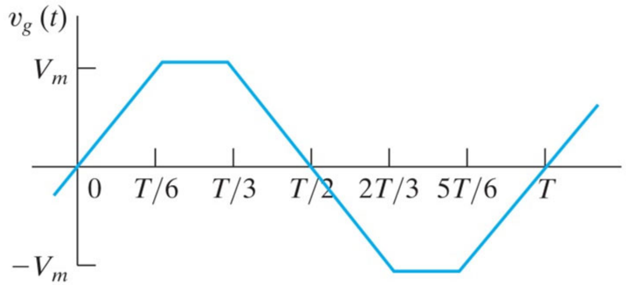


Figure: 16-10-01AO1-16.3

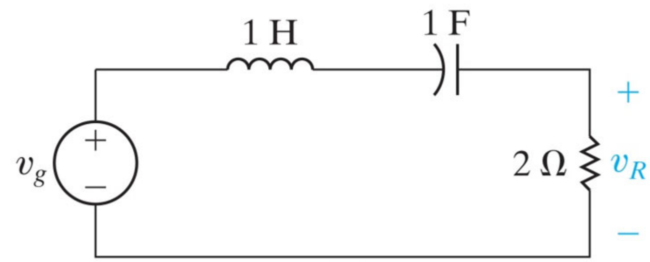


Figure: 16-14-031,2AO2-16.7

The circuit is a series RLC band-pass filter.

The Fourier series of the input signal was found in AP16.3 as

$$v_g = \frac{12V_m}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} \frac{\sin(\frac{n\pi}{3})}{n^2} \sin(n\omega_0 t)$$

Example

$$v_g = \frac{12V_m}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{\sin(\frac{n\pi}{3})}{n^2}\sin(n\omega_0 t)$$

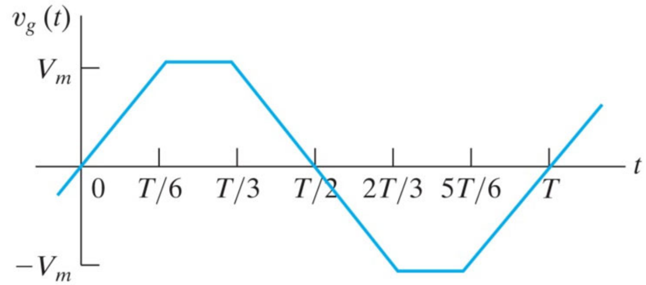


Figure: 16-10-01AO1-16.3

Let the period of the trapezoidal signal T = 2094.4 ms and 12 V_m = 296.09V.

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2094.4\times 10^{-3}\ \sec} = \frac{2\pi}{2.0944\,\sec} = 3\,\tfrac{rad}{\sec}$$

$$n\omega_0 = n3\,\tfrac{rad}{\sec}$$

Our goal is to estimate the average power P_{avg} delivered to the 2 Ω resistor.

Note that the input signal does not contain a dc component as can be seen by inspection of the waveform.

Example

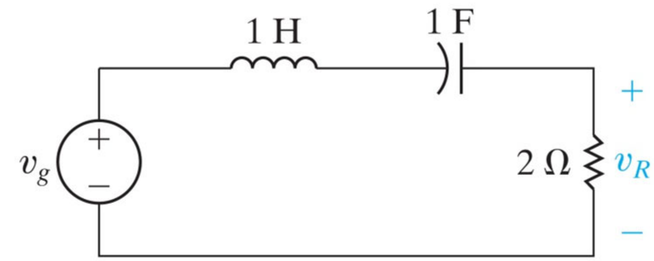


Figure: 16-14-031,2AO2-16.7

The transfer function for the circuit is

$$H(s) = \frac{V_{out}}{V_{in}} = \frac{V_{resistor}}{V_g}$$

The voltage across the resistor can be found by the voltage divider:

$$V_{resistor} = \frac{R}{sL + \frac{1}{sC} + R} V_g \quad = \frac{2}{s(1H) + \frac{1}{s(1F)} + 2} V_g \quad = \frac{s(2)}{s(s + \frac{1}{s} + 2)} V_g$$

Thus

$$H(s) = \frac{2s}{s^2 + 2s + 1}$$

For the steady-state solution, substitute jω = s to create H(jω)

$$H(j\omega = 3n \tfrac{rad}{\sec}) = \frac{2(j3n)}{(j3n)^2 + 2(j3n) + 1} \quad = \frac{j6n}{-9n^2 + j6n + 1} \quad = \frac{j6n}{(1 - 9n^2) + j6n}$$

Example

$$v_g = \frac{12V_m}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} \frac{\sin(\frac{n\pi}{3})}{n^2} \sin(n\omega_0 t)$$

Find the parameters for n = 1.

$$v_{g1} = \frac{12V_m}{\pi^2} \frac{\sin(\frac{\pi}{3})}{1^2} \sin(\omega_0 t)$$

$$= \frac{296.09}{\pi^2} \sin(\tfrac{\pi}{3}) \sin(\omega_0 t) \qquad = 25.98 \sin(\omega_0 t)\ Volts$$

$$= 25.98 \angle 0^\circ\ Volts$$

$$H(jn\omega_0) = H(j3) = \frac{j6(1)}{(1-9(1)^2) + j6(1)} \quad = \frac{j6}{-8+j6} \quad = \frac{6\angle 90^\circ}{\sqrt{(-8)^2 + 6^2} \angle \tan^{-1}(\frac{6}{-8})}$$

$$= \frac{6\angle 90^\circ}{10\angle 143.13^\circ} \qquad = 0.6\angle -53.13^\circ$$

Example 16.7

The output voltage (across the resistor) for n = 1 is then

$$V_{R1} = H(j3)v_{g1} \quad = \left(0.6\angle -53.13^\circ\right)\left(25.98\angle 0^\circ\right) \quad = 15.59\angle -53.13^\circ$$

$$= 15.59\cos(3t - 53.13^\circ)\ Volts$$

The average power in the resistor for n = 1 is then

$$P_{R,1} = \frac{\left[V_{R1,\text{rms}}\right]^2}{2\Omega} \quad = \frac{\left(\frac{15.59}{\sqrt{2}}\right)^2}{2\Omega} \quad = 60.746\ Watts$$

Follow the same analysis steps for n = 3, 5, …

$$P_3 = 0\ W \qquad\qquad P_5 = 4.76\ mW$$

Since the higher order terms are decreasing by $1/n^2$, they decrease very rapidly and will be ignored in this estimate.

So a good estimate of total power is $P_R \approx 60.75$ W.

Section 16.7 The rms Value of a Periodic Function

The rms value of a periodic function can be expressed in terms of the Fourier coefficients.

$$F_{rms} \equiv \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} f(t)^2\,dt} \qquad = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T}\left[a_v + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t - \theta_n)\right]^2 dt}$$

The only terms to survive the integration are those at the same frequency. All other cross-frequency terms equal zero.

$$F_{rms} = \sqrt{\frac{1}{T}\left(a_v^2 T + \sum_{n=1}^{\infty} A_n^2 \frac{T}{2}\right)} \qquad = \sqrt{a_v^2 + \sum_{n=1}^{\infty} \frac{A_n^2}{2}} \qquad = \sqrt{a_v^2 + \sum_{n=1}^{\infty}\left(\frac{A_n}{\sqrt{2}}\right)^2}$$

Example

Given the Fourier series for a voltage.

$$v = 10 + 30\cos(\omega_o t - \theta_1) + 20\cos(2\omega_o t - \theta_2) + 5\cos(3\omega_o t - \theta_3) + 2\cos(5\omega_o t - \theta_5)$$

Since only a finite number of Fourier terms are given, we can only estimate the rms voltage.

$$V_{rms} = \sqrt{a_v^2 + \sum_{n=1}^{\infty}\left(\frac{A_n}{\sqrt{2}}\right)^2} \quad = \sqrt{10^2 + \left(\frac{30}{\sqrt{2}}\right)^2 + \left(\frac{20}{\sqrt{2}}\right)^2 + \left(\frac{5}{\sqrt{2}}\right)^2 + \left(\frac{2}{\sqrt{2}}\right)^2}$$

$$= \sqrt{100 + 450 + 200 + 12.5 + 2} \quad = \sqrt{764.5}$$

$$= 27.65\ V_{rms}$$

Chapter 16

Fourier Series

Text: *Electric Circuits* by J. Nilsson and S. Riedel
Prentice Hall

EEE 117 Network Analysis
Instructor: Russ Tatro