CSC 130: AVL Trees

Binary Search Trees

- A binary search tree has these properties for each of its nodes:
  - All the values in the node's left subtree is less than the value of the node itself.
  - All the values in the node's right subtree is greater than the value of the node itself.

AVL Trees

- An AVL tree is a binary search tree (BST) with a balance condition.
  - Named after its inventors, Adelson-Velskii and Landis.
- For each node of the BST, the heights of its left and right subtrees can differ by at most 1.
  - Remember that the height of a tree is the length of the longest path from the root to a leaf.
  - The height of the root = the height of the tree.
  - The height of an empty tree is -1.

AVL Trees, *cont'd*

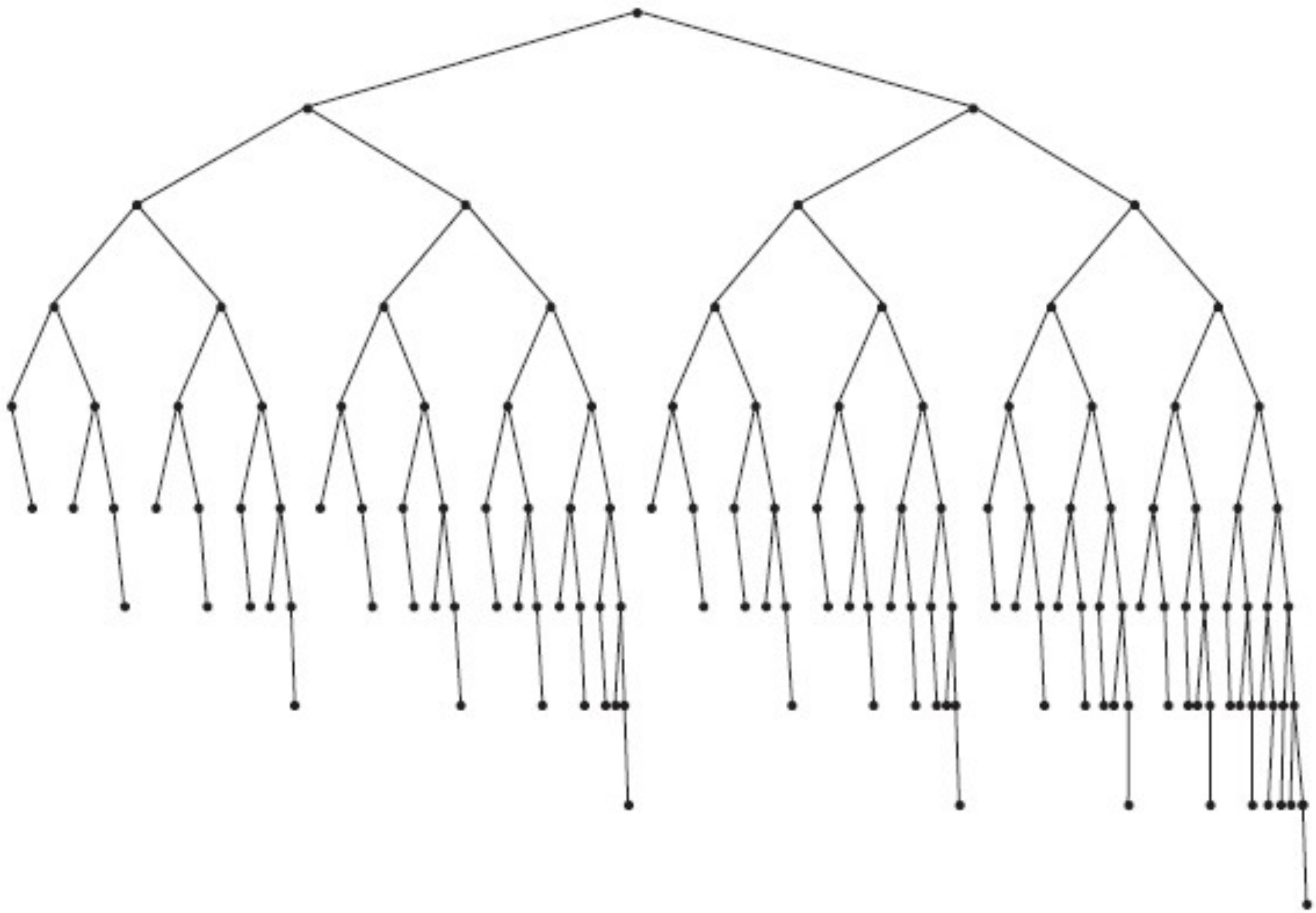

Figure 4.30 Smallest AVL tree of height 9

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Balancing AVL Trees

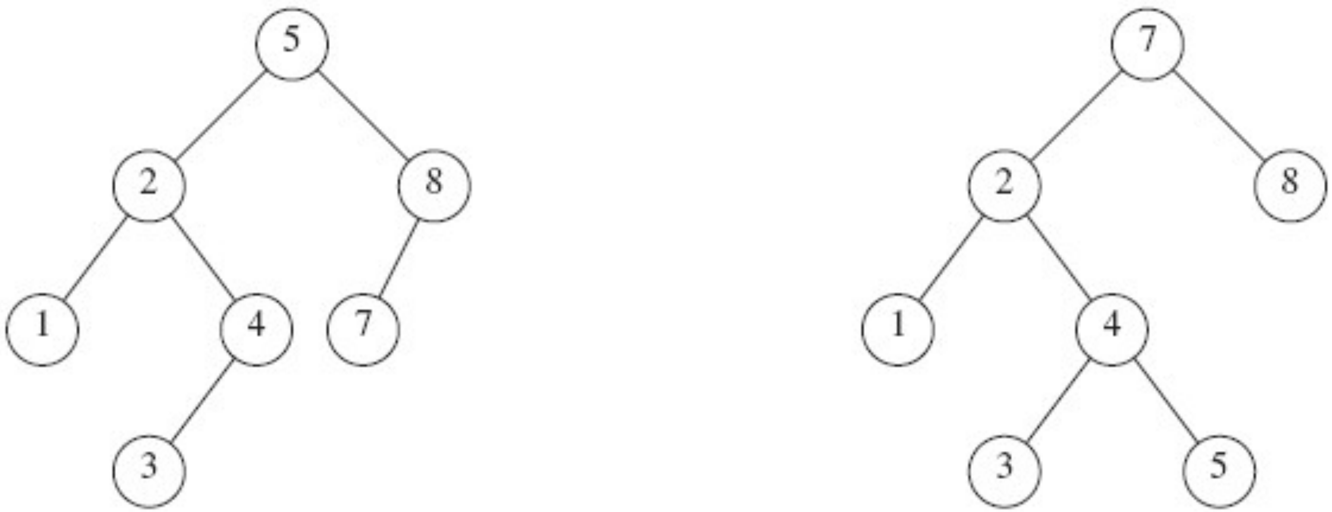


Figure 4.29 Two binary search trees. Only the left tree is AVL

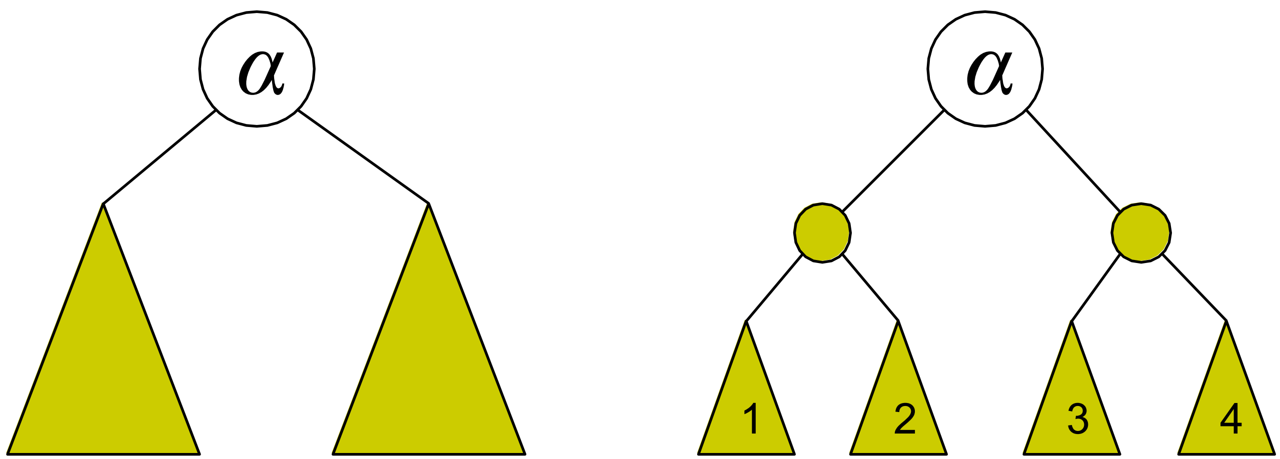
- We need to rebalance the tree whenever the balance condition is violated.
  - We need to check after every insertion and deletion.

Balancing AVL Trees, *cont'd*

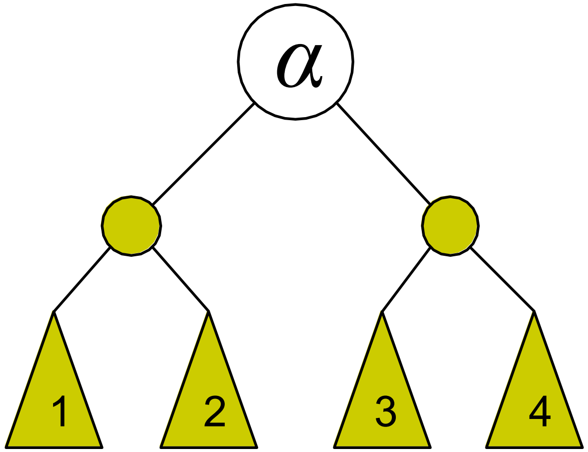
- Assume the tree was balanced before an insertion.
- If it became unbalanced due to the insertion, then the inserted node must have caused some nodes between itself and the root to be unbalanced.
- An unbalanced node must have the height of one of its subtrees exactly 2 greater than the height its other subtree.

Balancing AVL Trees, *cont'd*

- Let the deepest unbalanced node be α.
- Any node has at most two children.
- A new height imbalance means that the heights of α's two subtrees now differ by 2.



Balancing AVL Trees, *cont'd*



- Therefore, one of the following had to occur:
  - Case 1 (outside left-left): The insertion was into the left subtree of the left child of α.
  - Case 2 (inside left-right): The insertion was into the right subtree of the left child of α.
  - Case 3 (inside right-left): The insertion was into the left subtree of the right child of α.
  - Case 4 (outside right-right): The insertion was into the right subtree of the right child of α.

Cases 1 and 4 are mirrors of each other, and cases 2 and 3 are mirrors of each other.

Balancing AVL Trees: Case 1

- Case 1 (outside left-left):
  Rebalance with a single right rotation.

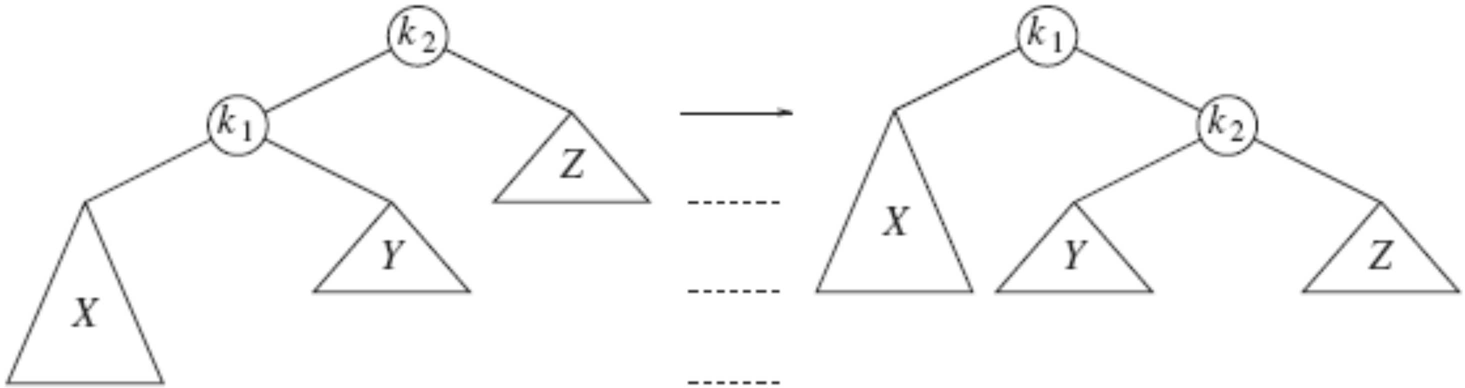
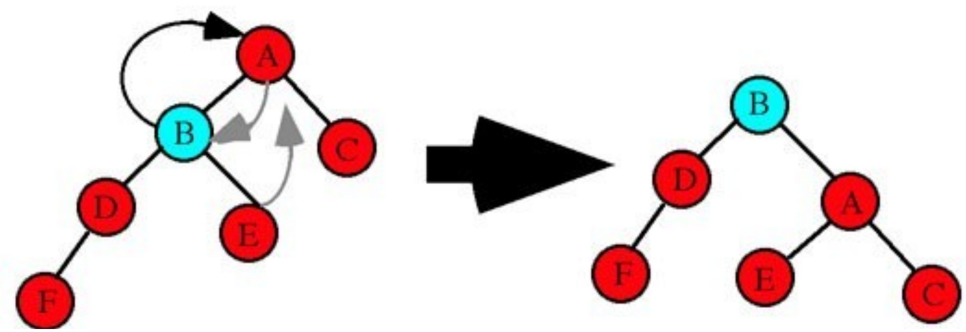


Figure 4.31 Single rotation to fix case 1

Balancing AVL Trees: Case 1, *cont'd*

- Case 1 (outside left-left):
  Rebalance with a single right rotation.



Node A is unbalanced.
Single right rotation: A's left child B becomes the new root of the subtree.
Node A becomes the right child and adopts B's right child as its new left child.

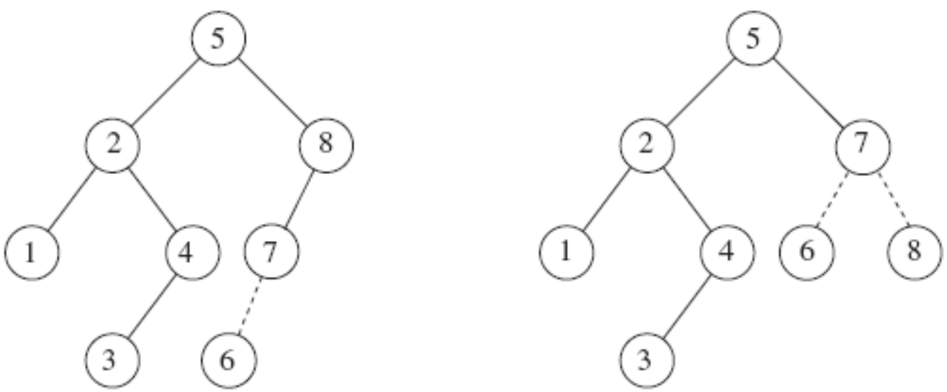


Figure 4.32 AVL property destroyed by insertion of 6, then fixed by a single rotation

Node 8 is unbalanced.
Single right rotation: 8's left child 7 becomes the new root of the subtree.
Node 8 is the right child.

http://www.cs.uah.edu/~rcoleman/CS221/Trees/AVLTree.html

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Balancing AVL Trees: Case 4

- Case 4 (outside right-right):
  Rebalance with a single left rotation.

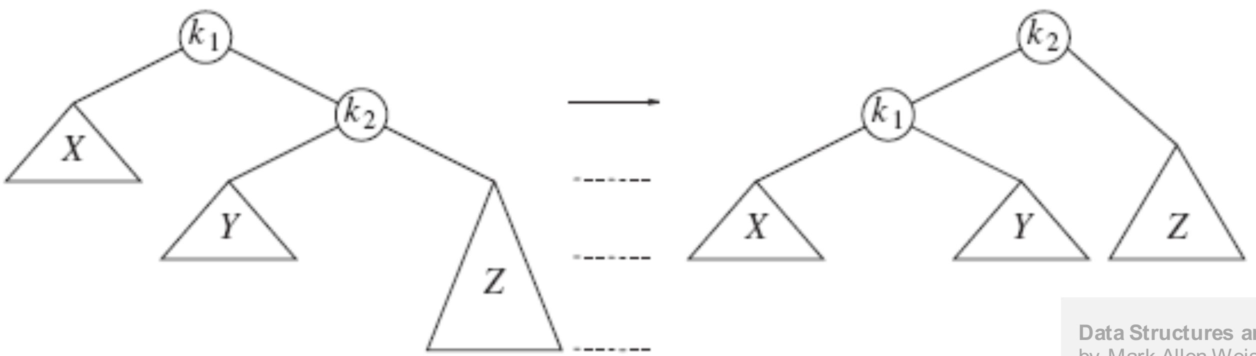

Figure 4.33 Single rotation fixes case 4

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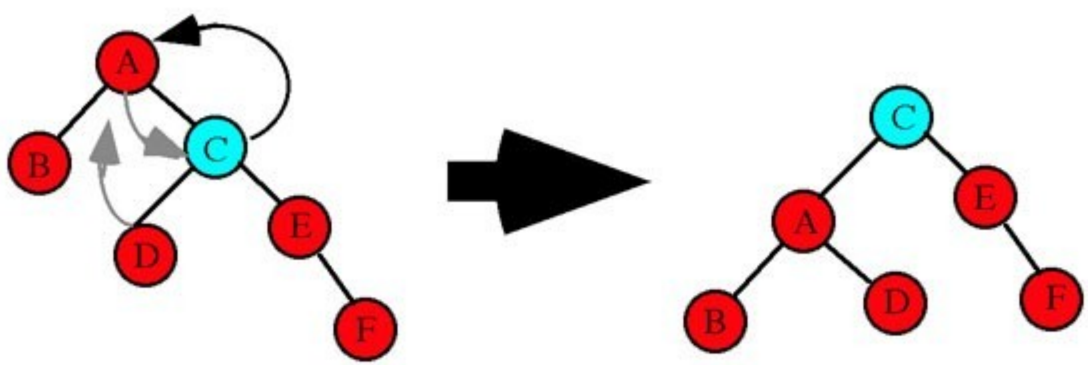
Node A is unbalanced.
Single left rotation: A's right child C becomes the new root of the subtree.
Node A becomes the left child and adopts C's left child as its new right child.



http://www.cs.uah.edu/~rcoleman/CS221/Trees/AVLTree.html

Balancing AVL Trees: Case 2

- Case 2 (inside left-right):
  Rebalance with a double left-right rotation.

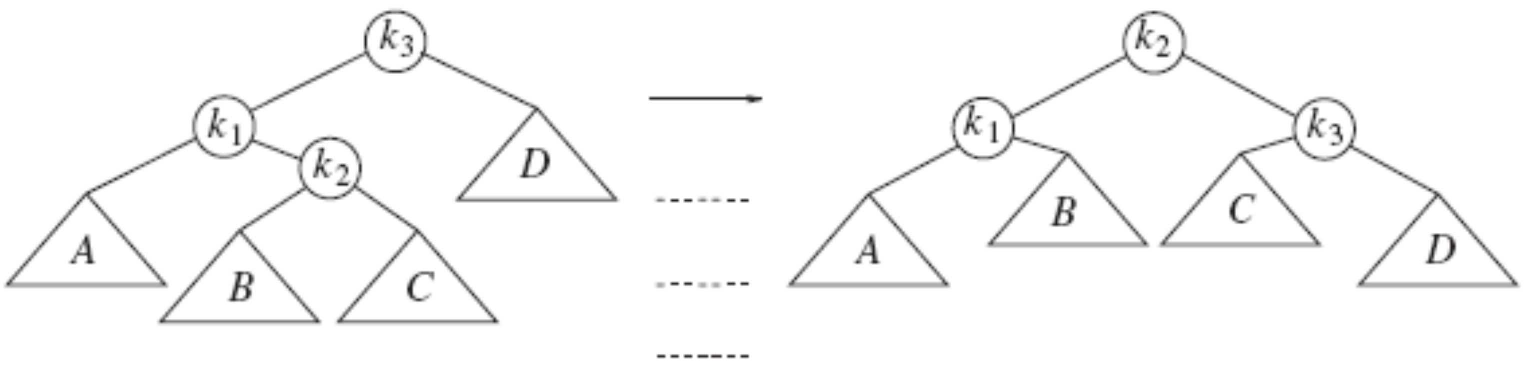


Figure 4.35 Left–right double rotation to fix case 2

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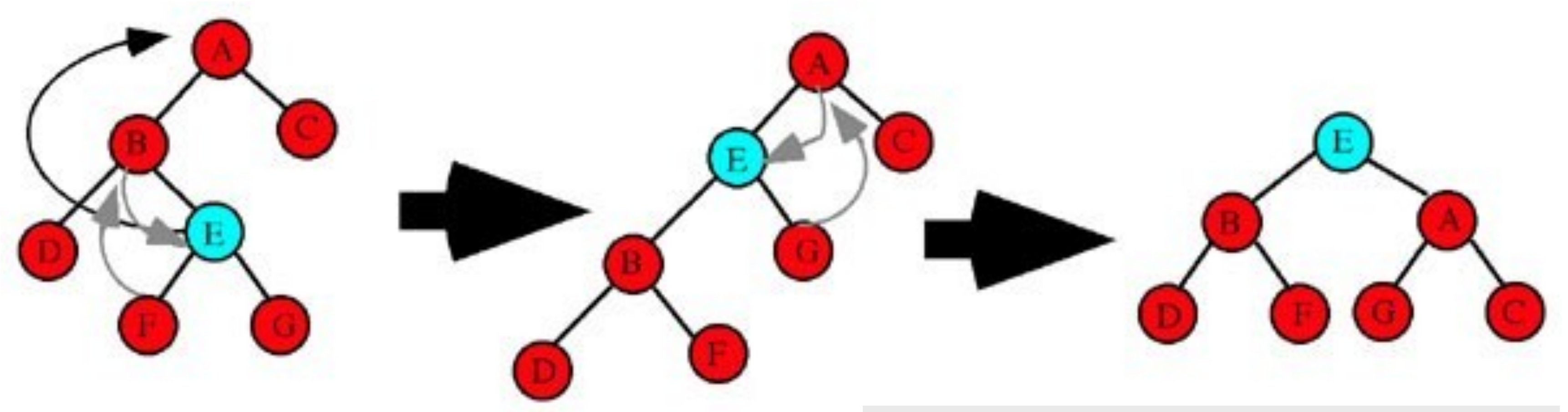
Balancing AVL Trees: Case 2, *cont'd*

- Case 2 (inside left-right):
  Rebalance with a double left-right rotation.



http://www.cs.uah.edu/~rcoleman/CS221/Trees/AVLTree.html

Node A is unbalanced.
Double left-right rotation: E becomes the new root of the subtree after two rotations. Step 1 is a single left rotation between B and E. E replaces B as the subtree root. B becomes E's left child and B adopts E's left child F as its new right child. Step 2 is a single right rotation between E and A. E replaces A is the subtree root. A becomes E's right child and A adopts E's right child G as its new left child.

Balancing AVL Trees: Case 3

- Case 3 (inside right-left):
  Rebalance with a double right-left rotation.

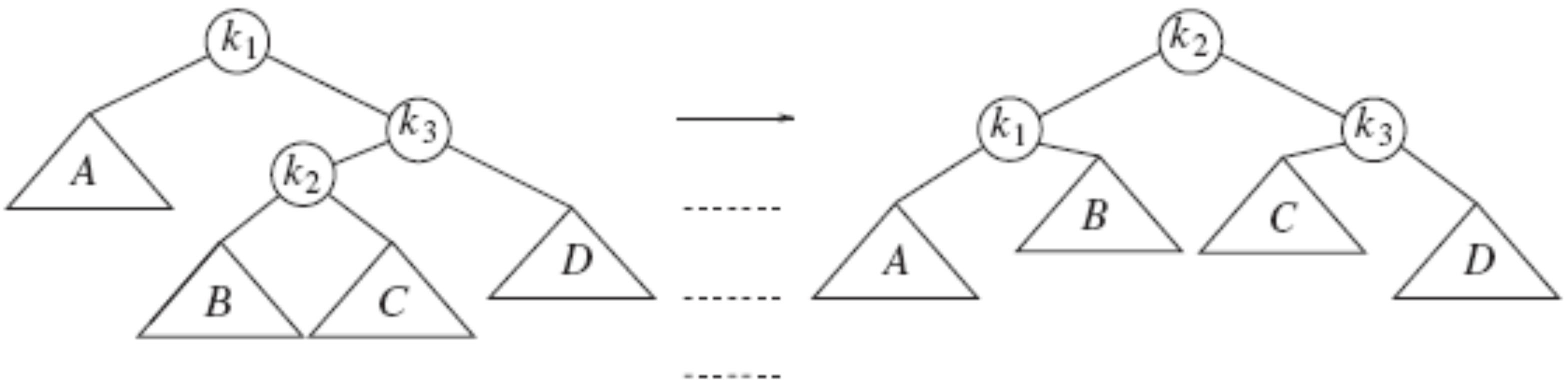
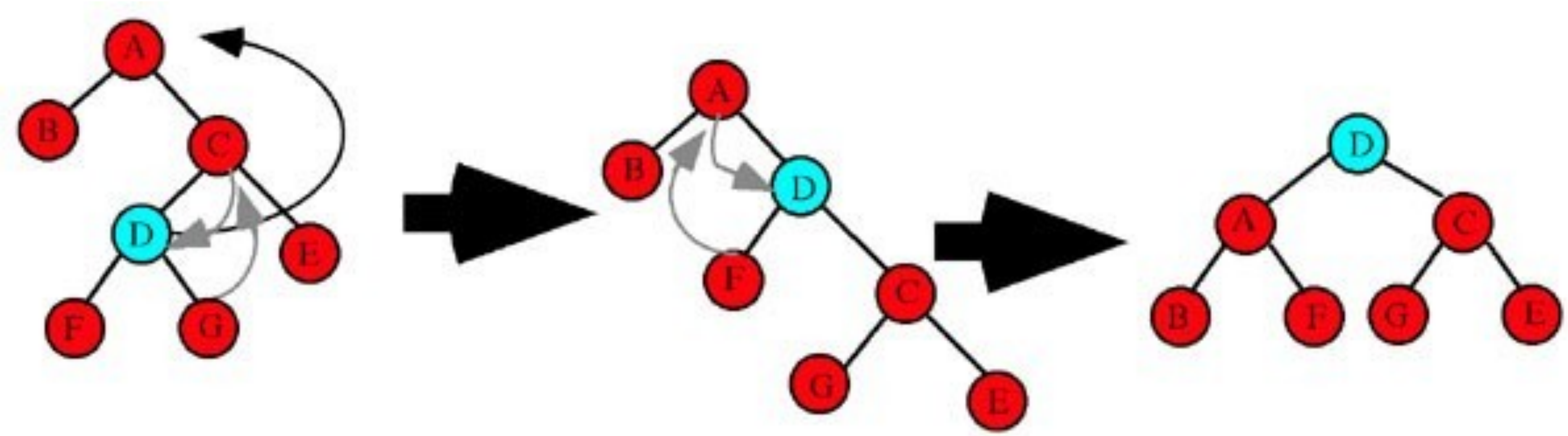


Figure 4.36 Right–left double rotation to fix case 3

Balancing AVL Trees: Case 3, *cont'd*

http://www.cs.uah.edu/~rcoleman/CS221/Trees/AVLTree.html

- Case 3 (inside right-left):
  Rebalance with a double right-left rotation.



Node A is unbalanced.
Double right-left rotation: D becomes the new root of the subtree after two rotations. Step 1 is a single right rotation between C and C. D replaces C as the subtree root. C becomes D's right child and C adopts D's right child G as its new left child. Step 2 is a single left rotation between D and A. D replaces A is the subtree root. A becomes D's left child and A adopts D's left child F as its new right child.

AVL Tree Implementation

- Since an AVL tree is just a BST with a balance condition, it makes sense to make the AVL tree class a subclass of the BST class.

```
public class AvlTree extends BinarySearchTree
```

- Both classes can share the same `BinaryNode` class.

The AVL Tree Node

- With so many height calculations, it makes sense to store each node's height in the node itself

```
public class BinaryNode
{
    private int data;             // data in this node
    private int height;           // height of this node
    private BinaryNode left;      // left child
    private BinaryNode right;     // right child

    ...
}
```

AVL Tree Implementation, *cont'd*

- Class `AVLTree` overrides the `insert()` and `remove()` methods of class `BinarySearchTree`.
  - Each method calls the superclass's method and wraps the result in a call to the `balance()` method.

```
protected BinaryNode insert(int data, BinaryNode node)
{
    return balance(super.insert(data, node));
}

protected BinaryNode remove(int data, BinaryNode node)
{
    return balance(super.remove(data, node));
}
```

AVL Tree Implementation, *cont'd*

- The private **`AVLTree`** method **`balance()`** checks whether the balance condition still holds, and rebalances the tree with rotations whenever necessary.

AVL Tree Implementation, *cont'd*

```
private BinaryNode balance(BinaryNode node)
{
    ...
    if (height(node.getLeft()) - height(node.getRight()) > 1) {
        if (height(node.getLeft().getLeft())
                >= height(node.getLeft().getRight())) {
            node = singleRightRotation(node);              Case 1
        }
        else {
            node = doubleLeftRightRotation(node);          Case 2
        }
    }
    else if (height(node.getRight()) - height(node.getLeft()) > 1) {
        if (height(node.getRight().getRight())
                >= height(node.getRight().getLeft())) {
            node = singleLeftRotation(node);               Case 4
        }
        else {
            node = doubleRightLeftRotation(node);          Case 3
        }
    }
    ...
    return node;
}
```

AVL Tree Implementation, *cont'd*

```
private BinaryNode singleRightRotation(BinaryNode k2)
{
    BinaryNode k1 = k2.getLeft();
    k2.setLeft(k1.getRight());
    k1.setRight(k2);
    k2.setHeight(Math.max(height(k2.getLeft()), height(k2.getRight())) + 1);
    k1.setHeight(Math.max(height(k1.getLeft()), k2.getHeight()) + 1);

    return k1;
}
```

Case 1

```
private BinaryNode singleLeftRotation(BinaryNode k1)
{
    BinaryNode k2 = k1.getRight();
    k1.setRight(k2.getLeft());
    k2.setLeft(k1);
    k1.setHeight(Math.max(height(k1.getLeft()), height(k1.getRight())) + 1);
    k2.setHeight(Math.max(height(k2.getRight()), k1.getHeight()) + 1);

    return k2;
}
```

Case 4

AVL Tree Implementation, *cont'd*

```
private BinaryNode doubleLeftRightRotation(BinaryNode k3)
{
    k3.setLeft(singleLeftRotation(k3.getLeft()));
    return singleRightRotation(k3);
}

private BinaryNode doubleRightLeftRotation(BinaryNode k1)
{
    k1.setRight(singleRightRotation(k1.getRight()));
    return singleLeftRotation(k1);
}
```

Case 2

Case 3

Break

Project 2

- This assignment will give you practice with binary search trees (BST) and AVL trees.
- You are provided a **`TreePrinter`** class that has a **`print()`** method that will print any arbitrary binary tree.
  - A template for how it prints a tree:

```
                                                                      xx
                                                                      /\
                                     -------------------------------     -------------------------------
                                    /                                                                    \
                                   xx                                                                    xx
                                   /\                                                                    /\
                    --------------   --------------                                       --------------   --------------
                  /                                 \                                    /                                 \
                 xx                                 xx                                  xx                                 xx
                 /\                                 /\                                  /\                                 /\
          ------   ------                    ------   ------                    ------   ------                     ------   ------
         /                \                  /                \                 /                \                  /               \
        xx                xx               xx                xx                xx                xx               xx                xx
        /\                /\               /\                /\                /\                /\               /\                /\
      --  --           --   --           --   --           --  --           --   --           --   --           --  --           --   --
     /       \        /       \         /       \         /       \         /       \        /       \         /       \         /       \
    xx      xx       xx       xx       xx       xx       xx       xx      xx       xx       xx       xx       xx       xx      xx       xx
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 xx   xx  xx   xx      xx    xx  xx   xx  xx   xx  xx  xx   xx  xx       xx   xx  xx  xx      xx    xx  xx   xx  xx  xx   xx  xx   xx   xx
xx                                            xx
```

- **`TreePrinter`** is able to print trees with height up to 5, i.e., 32 node values on the bottom row.
  - An example of an actual printed tree:

TreePrinter.java

```
                            24
                            /\
              --------------  --------------
             /                              \
            18                              73
            /\                              /\
      ------  ------                  ------  ------
     /              \                /              \
    12              19              38              87
    /                               /\               \
  --                              --  --              --
 /                               /      \               \
10                              37      41              90
                                         \
                                          \
                                          64
```

- The first part of the assignment makes sure that you can successfully insert nodes into, and delete nodes from, a binary search tree (BST) and an AVL tree.

- First generate a random BST that has height 5 and contains random values from 10 through 99.
  - You may have to generate dozens of trees until you get one that's exactly height 5.
  - Don't worry that the tree is unbalanced.
  - Print the tree.
- Now repeatedly delete the root of the tree.
  - Print the tree after each deletion to verify that you did the deletion correctly.
  - Stop when the tree becomes empty.

- Second, create an AVL tree node by node.
  - Generate 35 unique random integers 10-99 to insert into the tree.
  - Print the tree after each insertion to verify that you are keeping it balanced.
  - Each time you do a rebalancing, print a message indicating which rotation operation and which node.
    - Example:

```
Double left-right rotation: 76
```

- As you did with the BST, repeatedly delete the root of your AVL tree.
  - Print the tree after each deletion to verify that you are keeping it balanced.

- A handy AVL tree balance checker:

```
private int checkBalance(BinaryNode node) throws Exception
{
    if (node == null) return -1;

    if (node != null) {
        int leftHeight  = checkBalance(node.getLeft());
        int rightHeight = checkBalance(node.getRight());
        if (   (Math.abs(height(node.getLeft()) - height(node.getRight())) > 1
)
            || (height(node.getLeft())  != leftHeight)
            || (height(node.getRight()) != rightHeight)) {
            throw new Exception("Unbalanced trees.");
        }
    }

    return height(node);
}
```

Demo

- First, generate n random integers.
  - n is some large number, explained on the next slide.
- Time and print how long it takes to insert the random integers one at a time into an initially empty BST.
  - Do not print the tree after each insertion.
- Time and print how long it takes to insert the same random integers one at a time into an initially empty AVL tree.
  - Do not print the tree after each insertion.

- You can use any code from the lectures or from the textbook.
- You do not have to use parameterized generic types.
  - You can use raw (nongeneric) types, or `<Integer>`.

- You may choose a partner to work with you on this assignment.
  - Both of you will receive the same score.
- Create a zip file containing:
  - Your Java source files.
  - Any instructions on how to build and run your code.
  - Text files containing your outputs.
  - A 1- or 2-page that briefly describes your conclusions from doing this assignment.

Splay Trees

- Not a new type of tree, but a reimplementation of the BST insert, delete, and search methods.
  - The goal is to improve their performance.
- No single operation on a splay tree is guaranteed to have better performance.
  - But a series of m operations will take $O(m \log n)$ time for a tree of n nodes, whenever $m > n$.
- Not highly balanced like an AVL tree.
  - Lowering the cost of an entire series of operations is more important then keeping the tree balanced.

Splay Trees, *cont'd*

- Whenever a splay tree node is accessed, the tree performs splaying operations that moves the accessed node to the root of the tree.
- Splaying a node consists of a series of rotations.
  - Similar to AVL tree rotations.
- The goal is to move the accessed node to the root.
- A side benefit is to make the tree more balanced.

Splay Trees, *cont'd*

- The theory is that once a node has been accessed, it will soon be accessed again.
- Future accesses are fast if the node is the root.

Splay Trees, *cont'd*

- How is a worst-case BST created?
  - When all the nodes are entered in sorted order.
  - Suppose the bottom node is accessed in such a tree:

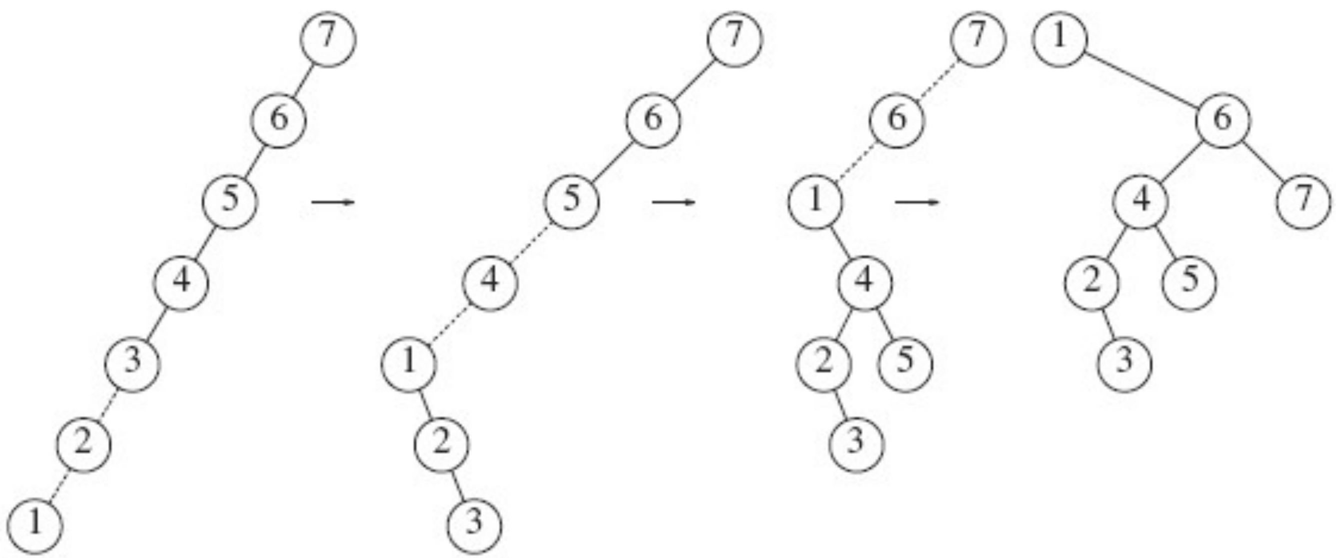

Figure 4.47 Result of splaying at node 1

Splay Trees, *cont'd*

- If a node hasn't been accessed in a while, then the next time it's accessed, you pay the performance penalty of splaying.
- But accesses of that node in the near future will be very fast.
- And so we amortize the cost of splaying over future operations.

Splay Trees, *cont'd*

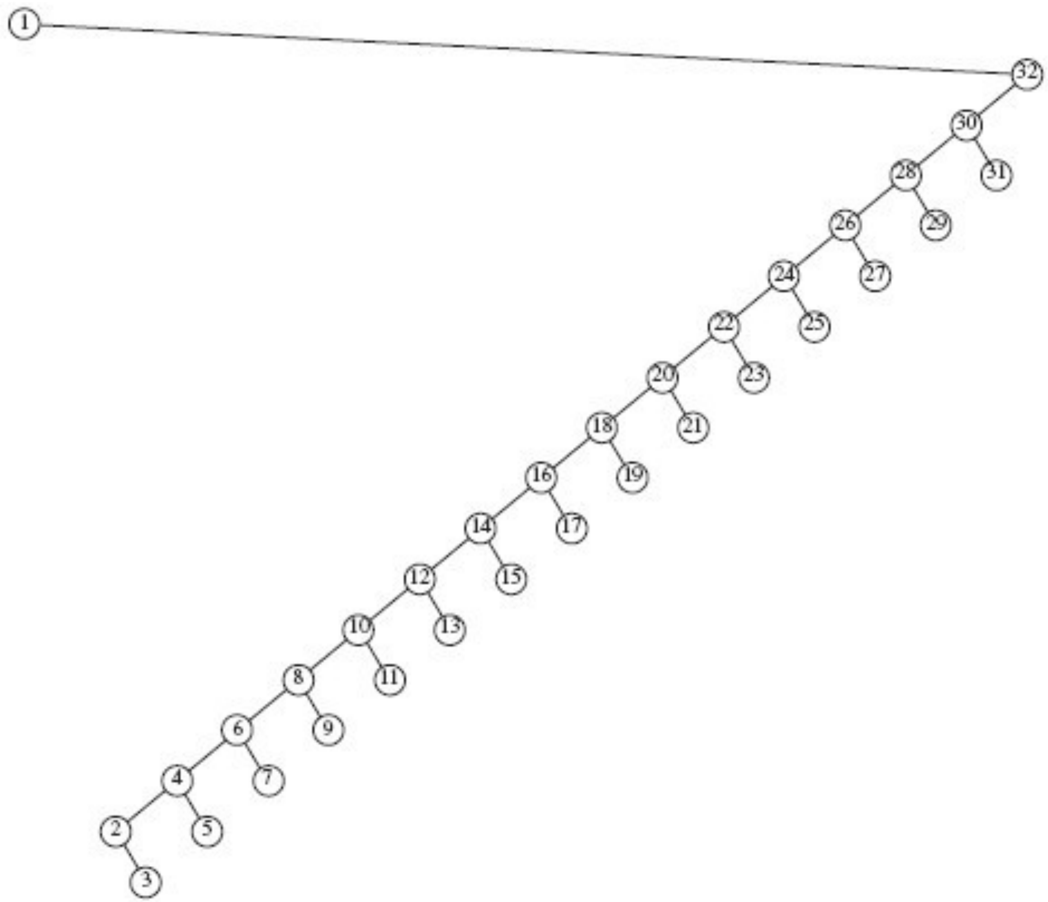

Figure 4.48 Result of splaying at node 1 a tree of all left children

Splay Trees, *cont'd*

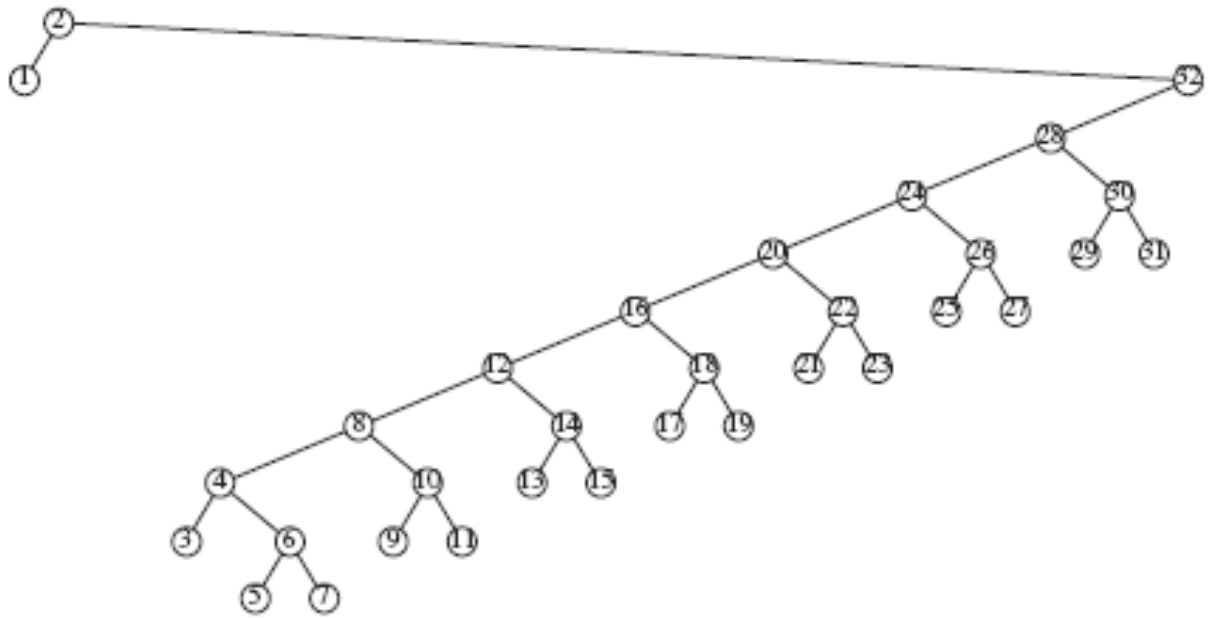

Figure 4.49 Result of splaying the previous tree at node 2

Splay Trees, *cont'd*

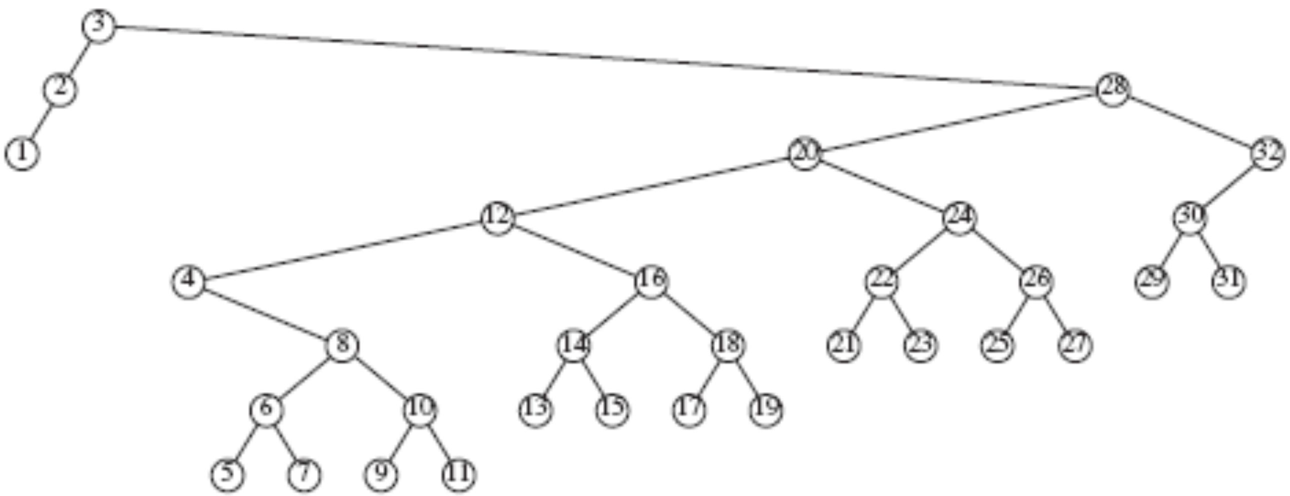

Figure 4.50 Result of splaying the previous tree at node 3

Splay Trees, *cont'd*

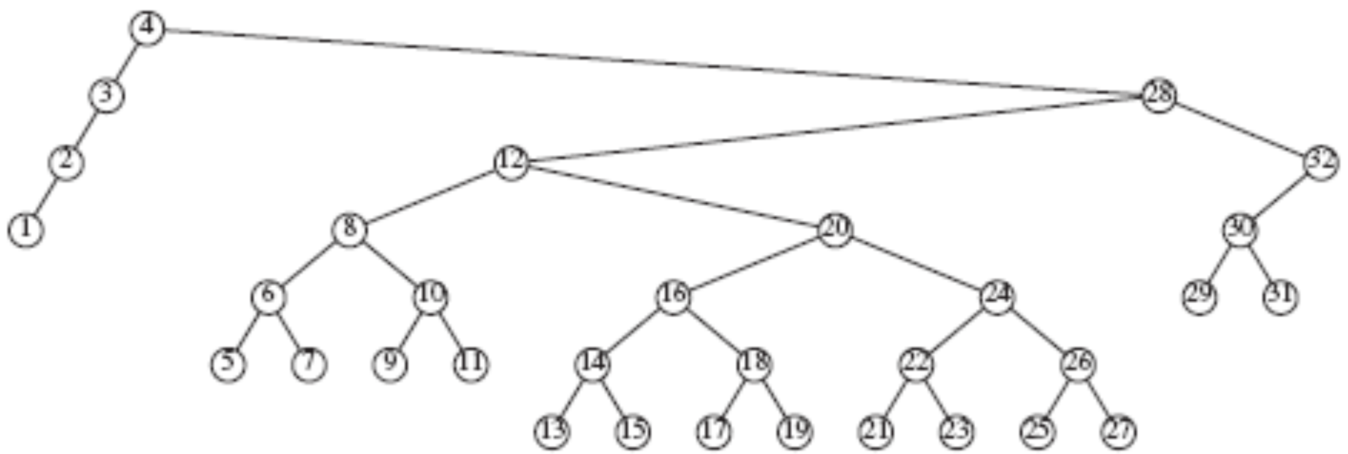

Figure 4.51 Result of splaying the previous tree at node 4

Splay Trees, *cont'd*

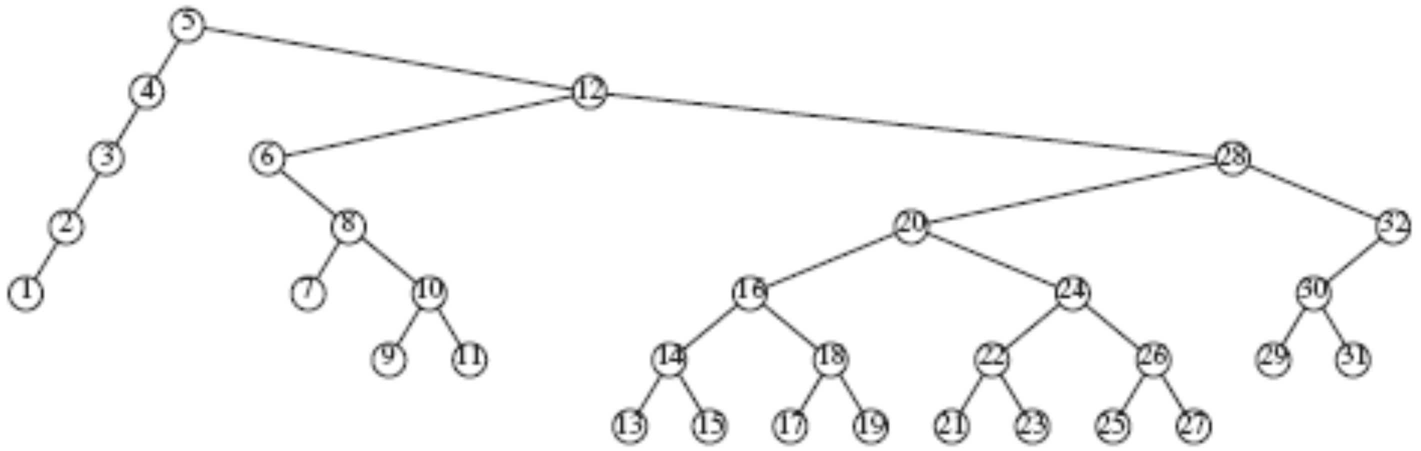

Figure 4.52 Result of splaying the previous tree at node 5

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Splay Trees, *cont'd*

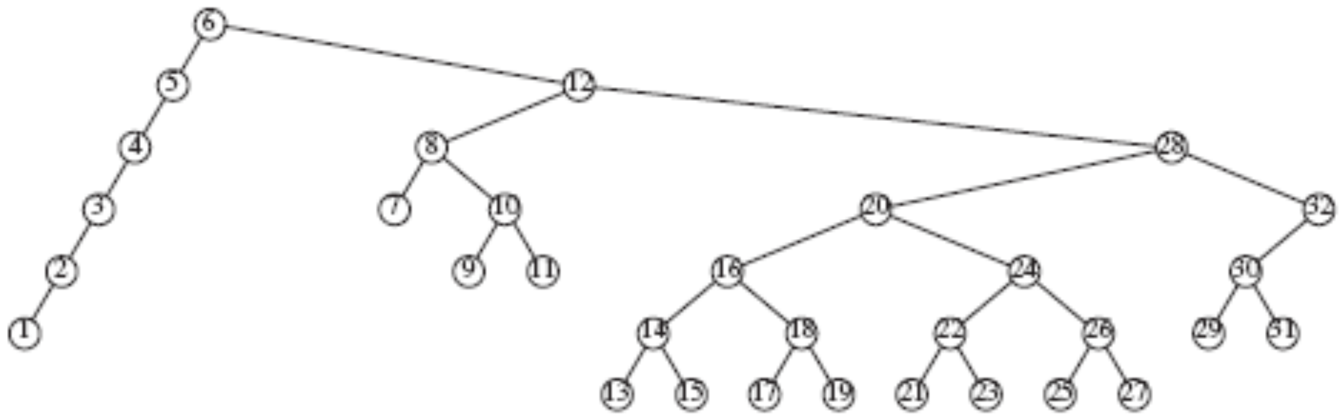

Figure 4.53 Result of splaying the previous tree at node 6

Splay Trees, *cont'd*

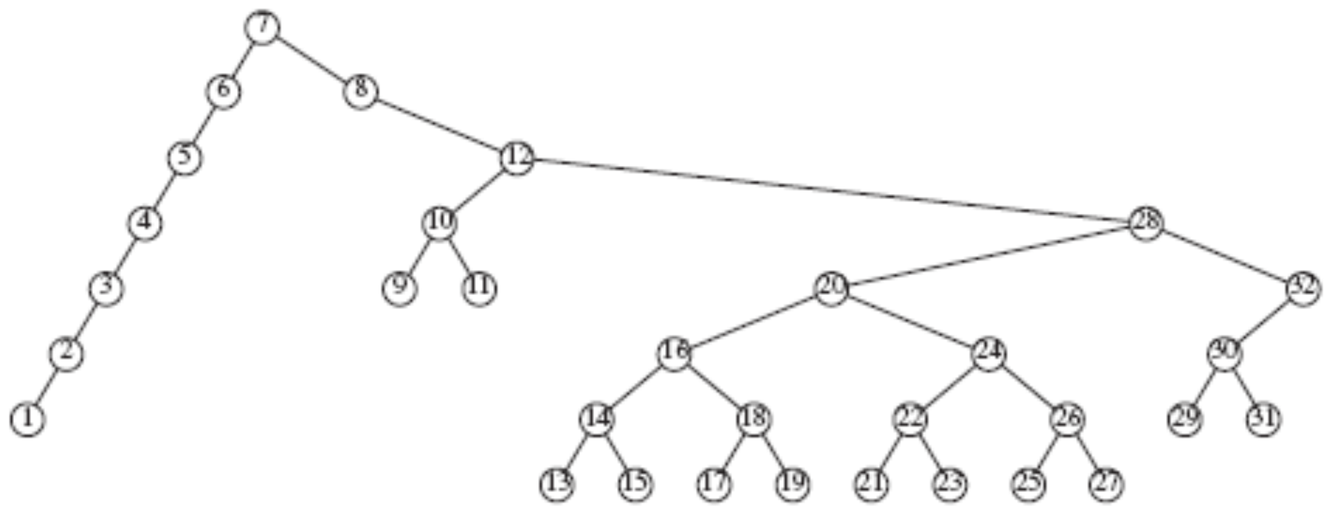

Figure 4.54 Result of splaying the previous tree at node 7

Splay Trees, *cont'd*

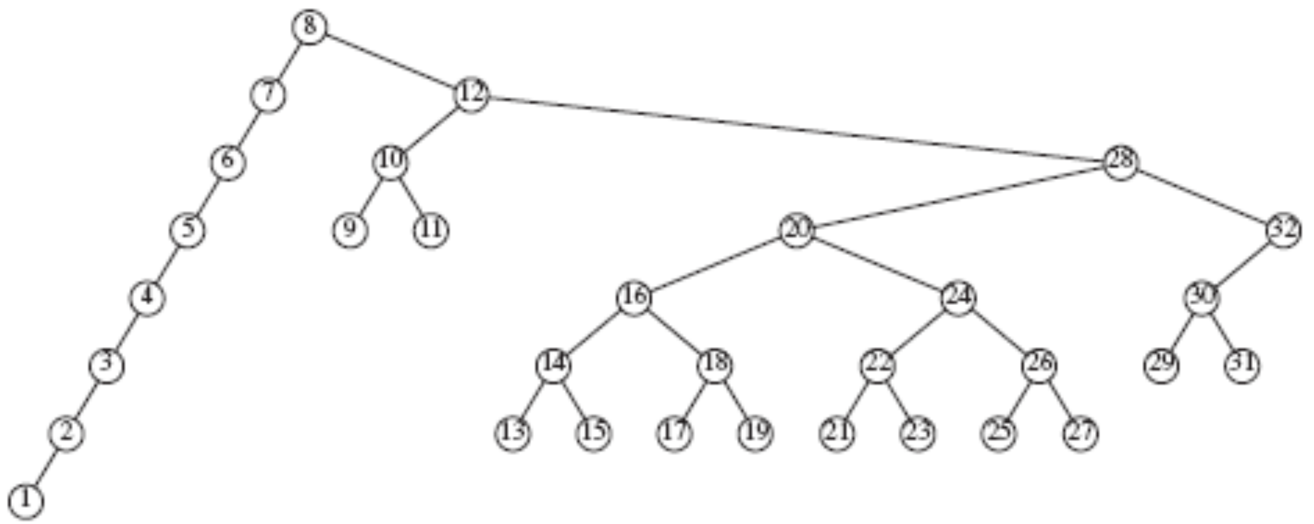

Figure 4.55 Result of splaying the previous tree at node 8

Splay Trees, *cont'd*

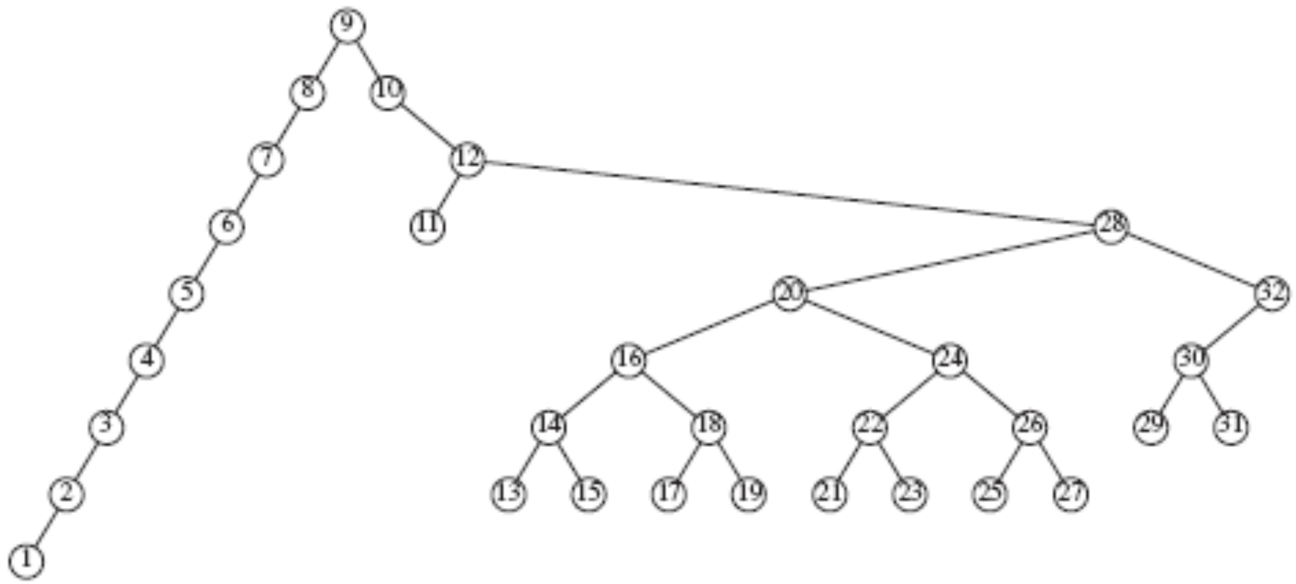

Figure 4.56 Result of splaying the previous tree at node 9

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