009x_23

March 18, 2017

Stochastic Processes: Data Analysis and Computer Simulation

Distribution function and random number
-The central limit theorem-

1 Binomial distribution $\rightarrow$ Gauss distribution

1.1 From the previous lesson

- The binomial distribution becomes equivalent to the Gaussian distribution in the limit $n, M \gg 1$, as shown in the 1st plot of this week.

$$P(n) = \frac{M!}{n!(M-n)!} p^n (1-p)^{M-n} \tag{C6}$$

$$\xrightarrow[n \to cont.]{n, M \gg 1} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(n-\mu_1)^2}{2\sigma^2}\right] \tag{C1}$$

$$\mu_1 = Mp, \quad \sigma^2 = Mp(1-p) \tag{C7, C8}$$

1.2 Numerical experiment 1

- While the proof for the equivalence has been given in the supplemental note, let us examine this by performing numerical experiments for various values of $M = 1, 2, 4, 10, 100$ and 1000.

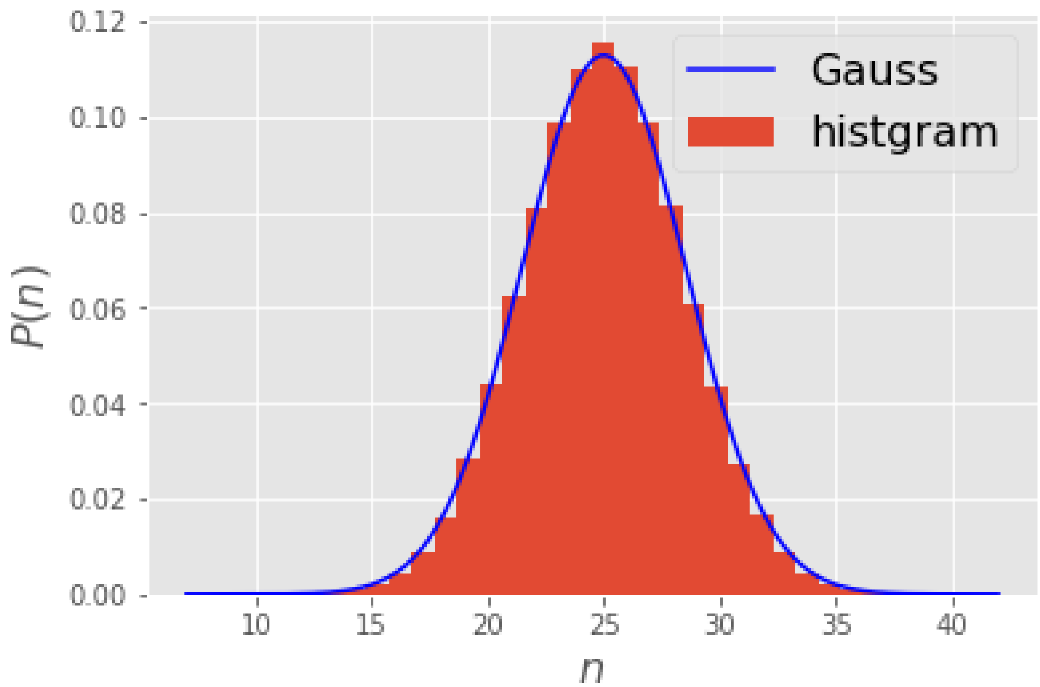
1.2.1 Include libraries

```
In [1]: % matplotlib inline
        import numpy as np # import numpy library as np
        import math # use mathematical functions defined by the C standard
        import matplotlib.pyplot as plt # import pyplot library as plt
        plt.style.use('ggplot') # use "ggplot" style for graphs
```

```
In [2]: p = 0.5     # set p, propability to obtain "head" from a coin toss
        M = 50       # set M, number of tosses in one experiment
        N = 100000 # number of experiments
        ave = M*p
```

```
std = np.sqrt(M*p*(1-p))
print('p =',p,'M =',M)
np.random.seed(0) # initialize the random number generator with seed=0
X = np.random.binomial(M,p,N) # generate the number of head come up N times
nmin=np.int(ave-std*5)
nmax=np.int(ave+std*5)
nbin=nmax-nmin+1
plt.hist(X,range=[nmin,nmax],bins=nbin,normed=True) # plot normalized histg
x = np.arange(nmin,nmax,0.01/std) # create array of x from nmin to nmax wit
y = np.exp(-(x-ave)**2/(2*std**2))/np.sqrt(2*np.pi*std**2) # calculate the
plt.plot(x,y,color='b') # plot y vs. x with blue line
plt.xlabel(r'$n$',fontsize=16) # set x-label
plt.ylabel(r'$P(n)$',fontsize=16) # set y-label
plt.legend([r'Gauss',r'histgram'], fontsize=16) # set legends
plt.show() # display plots
```

```
p = 0.5 M = 50
```



1.3 What we can learn from the experiment

- Stochastic variable "s" is a result of single binary choice,

$$s = 0 \text{ or } 1 \tag{1}$$

and Stochastic variable “n^M” is a sum of M independent binary choices s, with the index j representing the j-th choice.

$$n^M = \sum_{j=1}^{M} s_j \tag{2}$$

1.3.1 For $M = 1$

$$n^{M=1} = s_1 = s = 0 \text{ or } 1 \tag{D1}$$

- Distribution function → Binary choice, \$P^{M=1}(0)=1-p, P^{M=1}(1)=p, \$ with

$$\mu_1^{M=1} = p, \qquad \sigma^2_{M=1} = p(1-p) \tag{D2, D3}$$

1.3.2 For $M \gg 1$

$$n^M = \sum_{j=1}^{M} s_j = \sum_{j=1}^{M} n_j^{M=1} \tag{D4}$$

- Distribution function → Gaussian with

$$\mu_1^{M\gg1} = M\mu_1^{M=1}, \qquad \sigma^2_{M\gg1} = M\sigma^2_{M=1} \tag{D5, D6}$$

2 The central limiting theorem (CLT)

2.1 Generalization of Eqs. (D4-D6) for $M \gg 1$

2.1.1 CLT for sum of stochastic variables

- Stochastic variable “n^M” as a SUM of any M independent stochastic variables $n^{M=1}$ with $\mu_1^{M=1}$ and $\sigma^2_{M=1}$,

$$n^M = \sum_{j=1}^{M} n_j^{M=1} \tag{D7}$$

- Distribution function → Gauss with

$$\mu_1^{M\gg1} = M\mu_1^{M=1}, \qquad \sigma^2_{M\gg1} = M\sigma^2_{M=1} \tag{D8, D9}$$

2.1.2 CLT for average of stochastic variables

- Stochastic variable "n^M" as an AVERAGE of any M independent stochastic variables with $\mu^{M=1}$ and $\sigma^2_{M=1}$,

$$n^M = \frac{1}{M}\sum_{j=1}^{M} n_j^{M=1} \tag{D10}$$

- Distribution function $\rightarrow$ Gauss with

$$\mu_1^{M\gg 1} = \mu_1^{M=1}, \qquad \sigma^2_{M\gg 1} = \frac{\sigma^2_{M=1}}{M} \tag{D11, D12}$$

- Eqs. (D7-D12) is called "the central limiting theorem".

3 Uniform distribution $\rightarrow$ Gauss distribution

3.1 From CLT

3.1.1 For $M = 1$

- Stochastic variable "x" is uniformly distributed between 0 and 1,

$$x^{M=1} \in [0:1] \tag{D13}$$

- Distribution function: $P^{M=1}(x) = 1\ (\text{for } 0 \leq x < 1),\ P^{M=1}(x) = 0\ (\text{otherwise})$

$$\mu_1^{M=1} = \frac{1}{2}, \qquad \sigma^2_{M=1} = \frac{1}{12} \tag{D14, D15}$$

3.1.2 For $M \gg 1$

- Stochastic variable "x" is a sum of M independent uniform random numbers

$$x^M = \sum_{j=1}^{M} x_j^{M=1} \tag{D16}$$

- Distribution function $\rightarrow$ Gauss with

$$\mu_1^{M\gg 1} = M\mu_1^{M=1} = \frac{M}{2}, \qquad \sigma^2_{M\gg 1} = M\sigma^2_{M=1} = \frac{M}{12} \tag{D17, D18}$$

3.2 Numerical experiment 2

```
In [3]: M = 10     # set M, the number of random variables to add
        N = 100000 # number samples to draw, for each of the random variables
        ave = M/2
        std = np.sqrt(M/12)
        print('M =',M)
        np.random.seed(0)          # initialize the random number generator with seed
        X = np.zeros(N)
        for i in range(N):
            X[i] += np.sum(np.random.rand(M)) # draw a random numbers for each of t
        nmin=np.int(ave-std*5)
        nmax=np.int(ave+std*5)
        plt.hist(X,range=[nmin,nmax],bins=50,normed=True) # plot normalized histgra
        x = np.arange(nmin,nmax,0.01/std) # create array of x from nmin to nmax wit
        y = np.exp(-(x-ave)**2/(2*std**2))/np.sqrt(2*np.pi*std**2) # calculate the
        plt.plot(x,y,color='b') # plot y vs. x with blue line
        plt.xlabel(r'$n$',fontsize=16) # set x-label
        plt.ylabel(r'$P(n)$',fontsize=16) # set y-label
        plt.legend([r'Gauss',r'histgram'], fontsize=16) # set legends
        plt.show() # display plots
```

```
M = 10
```