Supplemental note for Week 3 Part 2

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1 Derivation of the mean square displacement

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\begin{aligned}
\langle |\Delta \mathbf{R}(t)|^2 \rangle &= 2\int_0^t dt_1 \int_{t_1}^t dt_2 \frac{3\tilde{D}}{\zeta m} \exp\left(-\frac{\zeta}{m}(t_2 - t_1)\right) && (1)\\
&= \frac{6\tilde{D}}{\zeta m}\int_0^t dt_1 \left[\exp\left(\frac{\zeta}{m}t_1\right)\int_{t_1}^t dt_2 \exp\left(-\frac{\zeta}{m}t_2\right)\right] && (2)\\
&= \frac{6\tilde{D}}{\zeta m}\int_0^t dt_1 \left[\exp\left(\frac{\zeta}{m}t_1\right)\left(-\frac{m}{\zeta}\left(\exp\left(-\frac{\zeta}{m}t\right) - \exp\left(-\frac{\zeta}{m}t_1\right)\right)\right)\right] && (3)\\
&= \frac{6\tilde{D}}{\zeta^2}\int_0^t dt_1 \left[\exp\left(\frac{\zeta}{m}t_1\right)\left(\exp\left(-\frac{\zeta}{m}t_1\right) - \exp\left(-\frac{\zeta}{m}t\right)\right)\right] && (4)\\
&= \frac{6\tilde{D}}{\zeta^2}\int_0^t dt_1 \left[1 - \exp\left(\frac{\zeta}{m}t_1\right)\exp\left(-\frac{\zeta}{m}t\right)\right] && (5)\\
&= \frac{6\tilde{D}}{\zeta^2}\left[t - \exp\left(-\frac{\zeta}{m}t\right)\int_0^t dt_1 \exp\left(\frac{\zeta}{m}t_1\right)\right] && (6)\\
&= \frac{6\tilde{D}}{\zeta^2}\left[t - \exp\left(-\frac{\zeta}{m}t\right)\frac{m}{\zeta}\left(\exp\left(\frac{\zeta}{m}t\right) - 1\right)\right] && (7)\\
&= \frac{6\tilde{D}}{\zeta^2}\left[t - \frac{m}{\zeta} + \frac{m}{\zeta}\exp\left(-\frac{\zeta}{m}t\right)\right] && (8)\\
&\simeq \frac{6\tilde{D}}{\zeta^2}t \qquad (t \to \infty) && (9)
\end{aligned}
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