- In this plot, we study basic theories of Brownian motion.

- Let us start with writing down the celebrated Langevin equation that describes the Brownian motion of a particle diffusing in a fluid.

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- Consider a spherical particle of radius a and mass m in a solvent fluid.

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- Assuming the size of the Brownian particle is much larger than the size of the fluid molecules, we can treat the fluid as a continuum medium with viscosity $¥eta$.

- Let $R(t)$ be the temporal position of the particle at time t and V its velocity.

- To write down Newton’s equation of motion for the Brownian particle, the mass times acceleration should equal the total force acting on the particle.

- First, if a body is moving relative to a fluid, it experiences a friction force, colored in blue,

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- which will be proportional to the velocity with the constant of proportionality called the friction constant $¥zeta$.

- For a spherical particle the friction constant ζ is given by $6\pi\eta a$, known as Stokes law.

- And second, in addition to the friction force, we know that there must be another type of force which gives rise to the irregular motion of the Brownian particle.

- We call this the random force $F(t)$ colored in red, which represents the effects of the many collisions taking place between the Brownian particle and the fluid molecules.

- Finally, by putting all of this together, we can write down the Langevin equation shown here as Eq.(21).

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- Now, let us characterize this random force in more detail.

- Assuming three-dimensional Cartesian coordinates, the random force has three components along the x, y, and z directions.

- Without loss of generality, we can assume that the average of the force along any direction is zero as shown in Eq,(22).

- If this was not the case, we could always separate the non-zero part as an extra drift force to be added separately in the Langevin equation.

- We still need to specify how the forces at different times, or along different directions are correlated.

- We note that we are interested in the dynamics of the Brownian particle at time scales much larger than the time-scales of the collisions with the fluid molecules.

- Therefore, we can assume that the successive random forces are uncorrelated on the time scale of the Brownian particle.

- This is expressed mathematically in Eq.(23) using the auto-correlation function for the random force.

- Here, $¥delta of ¥alpha¥beta$ is the Kroenecker’s delta, it is 1 if $¥alpha = ¥beta$ and zero otherwise.

- $¥delta(t)$ is the Dirac delta function, it is zero everywhere except at the origin t=0, where it diverges.

- The crucial point is that the integral of the delta function equals to one.

- Noise which obeys equations (22) and (23) is called white noise or the Gaussian noise.

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- Now, let us calculate the power spectrum of the random force $F(t)$.

- Using the Wiener-Kintchine theorem, we can write the power spectrum S_F (¥omega) in terms of the random force auto-correlation function ¥phi_F(t).

- Using the properties of the white noise, this auto-correlation Eq.(23) is proportional to a delta function, which kills the integral.

- Thus, we arrive at the simple result that the power spectrum is a constant proportional to D tilde, which determines the amplitude of the random force in Eq. (23).

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- Let us try and see what the power spectrum and correlation functions look like.

- On the left, we plot the power spectrum of the random force, on the right the auto-correlation function.

- S_F(¥omega) is a constant, as we have just proved, and ¥phi(t) is proportional to a delta function.

- Next, let us characterize the properties of the particle velocity V.

- Taking the Fourier transform of the Langevin equation Eq (1), we obtain a simple algebraic equation for V(¥omega).

- This can be easily solved, to give the following equation for V(¥omega) as a function of F(¥omega).

- Now, we are in a position to calculate the power spectrum of the velocity.

- By definition, S_V(¥omega) is given by the square norm of the Fourier transform of the velocity.

- Writing $V(¥omega)$ in terms of $F(¥omega)$, and using Eq.(24) for the power spectrum of the random forces, we obtain Eq.(25).

- We note that the form of this equation appears so often in Physics and Mathematics that it has its own name. It is the Lorentzian function.

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- Once we have computed the power spectrum, we can use the Wiener-Khintchine theorem to obtain the velocity auto-correlation function.

- In the last step, we have used a well known-result, that the Fourier transform of a Lorentzian is a two-sided decaying exponential.

- This means that it is an even function of time t, and therefore depends only on the absolute value of t.

- You can look this up in any table of integrals, use a computer algebra system, or do it yourself by hand using Cauchy’s integral formula.

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- Let us now visualize the previously derived results for the velocity of the Brownian particle.

- On the left, we have the Lorentzian function describing the Power spectrum,

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- and on the right we have the two-sided decaying exponential which gives us the auto-correlation of the velocities.

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- With the results we have obtained so far, we are in a position to derive a useful relation named the fluctuation-dissipation theorem.

- Furthermore, using the equipartition theorem of classical statistical mechanics, we also know that this average should be equal to 3 times kbT divided by m.

- Solving for D tilde, we finally obtain the fluctuation-dissipation theorem, which relates the amplitude of the fluctuating random forces with the magnitude of the dissipative friction forces.

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- Let us now turn our attention to looking at the temporal particle positions or displacements, in order to characterize its diffusive motion.

- By definition, the displacement of the particle after some time t can be expressed as the time integral of the velocity V from 0 to t.

- The mean square displacement of the particle is defined as in this equation.

- By rewriting the displacement as a time integral of the velocity,

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- we are left with a double integral of the auto-correlation of the velocity, which has been calculated in Eq.(26), at two distinct times, t_1 and t_2.

- The integral can be performed analytically, but care must be taken to properly handle the absolute value that appears in the exponential.

- We have drawn the integration domain on the right.

- The blue outline gives the original integration limits for t1 and t2, which is represented by a square domain of side length t.

- Since the function we are integrating depends only on the absolute value of $t_2 – t_1$,

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- we can divide this domain into an upper triangular part, the red region, where t_2 is larger than t_1, and a lower triangular part where t_1 is larger than t2.

- Note that the value of the integral over the upper triangular domain must be equal to the value of the integral over the lower domain,

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- and it should be half of the integral over the square domain.

- Thus, we can rearrange the integration limits to go only over the red domain, where t_2 is larger than t1.

- With this, we can get rid of the annoying absolute value from the equation, and the integrals over t1 and t2 can now be separated and easily performed.

- After doing all this, we see that the mean square displacement is increasing lineally with time t.

- At the end of the plot, we will define the self-diffusion constant of Brownian particles, as the long-time limit of the mean-square displacement divided by 6 t.

- We then obtain that the Diffusion constant is D tilde divided by ¥zeta^2 as shown in Eq.(30).

- Using the fluctuation-dissipation theorem Eq.(29), we arrive at the Einstein relation shown as Eq.(31).

- Finally, from the Einstein relation, together with Stokes law, we obtain the Stokes-Einstein relation shown as Eq.(32),

- which gives a good estimate for the self diffusion constant of spherical particles of radius a, diffusing in a fluid of viscosity ¥eta at temperature T.

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