009x_24

March 18, 2017

Stochastic Processes: Data Analysis and Computer Simulation

Distribution function and random number
-Random walk-

1 1D random walk

1.1 The model system

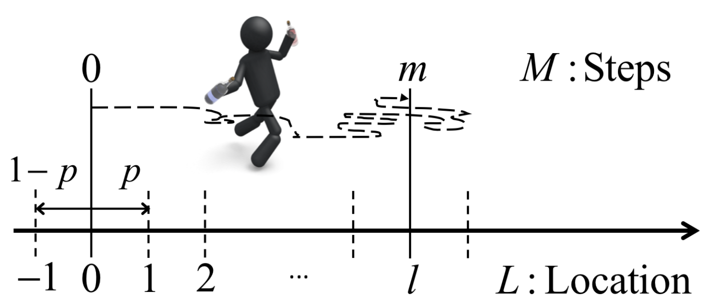
1.2 As binomial distribution

- The total number of steps to the right: n_+
- The total number of steps to the left: n_-
- The total number of steps: $m = n_+ + n_-$
- The current location: $l = n_+ - n_-$

$$\therefore \quad n_+ = \frac{m+l}{2}, \qquad n_- = \frac{m-l}{2} \tag{E1}$$

$$P(l,m) \to P_{\text{Binomial}}(n_+;m) = P_{\text{Binomial}}(n_-;m) \tag{E2}$$

$$= \frac{m!}{n_+!(m-n_+)!} p^{n_+}(1-p)^{m-n_+} \tag{E3}$$



1.3 As normal distribution (for $n_+, m \gg 1$)

$$P_{\text{Binomial}}(n_+; m) \simeq \frac{1}{\sqrt{2\pi\sigma'^2}} \exp\left[-\frac{(n_+ - \mu'_1)^2}{2\sigma'^2}\right] \tag{E4}$$

$$\text{with} \quad \mu'_1 = \langle n_+ \rangle = mp, \quad \sigma'^2 = \langle n_+^2 \rangle - \langle n_+ \rangle^2 = mp(1-p) \tag{E5, E6}$$

Recall that $n_+ = (m + l) / 2$

$$P_{\text{Binomial}}(n_+; m) \simeq \exp\left[-\frac{(l - m(2p-1))^2}{8mp(1-p)}\right] \tag{E7}$$

$$\therefore \quad P(l, m) = P_{\text{Binomial}}(n_+; m)\frac{dn_+}{dl} = P_{\text{Binomial}}(n_+; m)\frac{1}{2} \tag{E8}$$

$$\simeq \frac{1}{\sqrt{2\pi\sigma''^2}} \exp\left[-\frac{(l - \mu''_1)^2}{2\sigma''^2}\right] \tag{E9}$$

$$\text{with} \quad \mu''_1 = \langle l \rangle = m(2p-1), \quad \sigma''^2 = \langle l^2 \rangle - \langle l \rangle^2 = 4mp(1-p) \tag{E10, E11}$$

1.4 By computer simulation

```
In [1]: % matplotlib inline
        import numpy as np # import numpy library as np
        import math # use mathematical functions defined by the C standard
        import matplotlib.pyplot as plt # import pyplot library as plt
        plt.style.use('ggplot') # use "ggplot" style for graphs

In [2]: p = 0.5 # set p, propability to take a step to the right
        M = 1000 # M = number of total steps
        N = 100000 # N = number of independent random walkers
        ave = M*(2*p-1) # average of the location L after M steps Eq.(E10)
        std = np.sqrt(4*M*p*(1-p)) # standard deviation of L after M steps Eq.(E11)
        print('p =',p,'M =',M)
        L = np.zeros(N)
        np.random.seed(0) # initialize the random number generator with seed=0
        for i in range(N): # repeat independent random walks N times
            step=np.random.choice([-1,1],M) # generate random sampling from -1 or
            L[i]=np.sum(step) # calculate l after making M random steps and store
        nmin=np.int(ave-std*5)
        nmax=np.int(ave+std*5)
        nbin=np.int((nmax-nmin)/4)
        plt.hist(L,range=[nmin,nmax],bins=nbin,normed=True) # plot normalized histg
        x = np.arange(nmin,nmax,0.01/std) # create array of x from nmin to nmax wit
        y = np.exp(-(x-ave)**2/(2*std**2))/np.sqrt(2*np.pi*std**2) # calculate the
        plt.plot(x,y,lw=2,color='b') # plot y vs. x with blue line
        plt.xlabel(r'$l$',fontsize=16) # set x-label
        plt.ylabel(r'$P(l,m)$',fontsize=16) # set y-label
        plt.legend([r'Gauss',r'histgram'], fontsize=16) # set legends
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    plt.xlim(ave-250,ave+250) # set x-range
    plt.show() # display plots
```
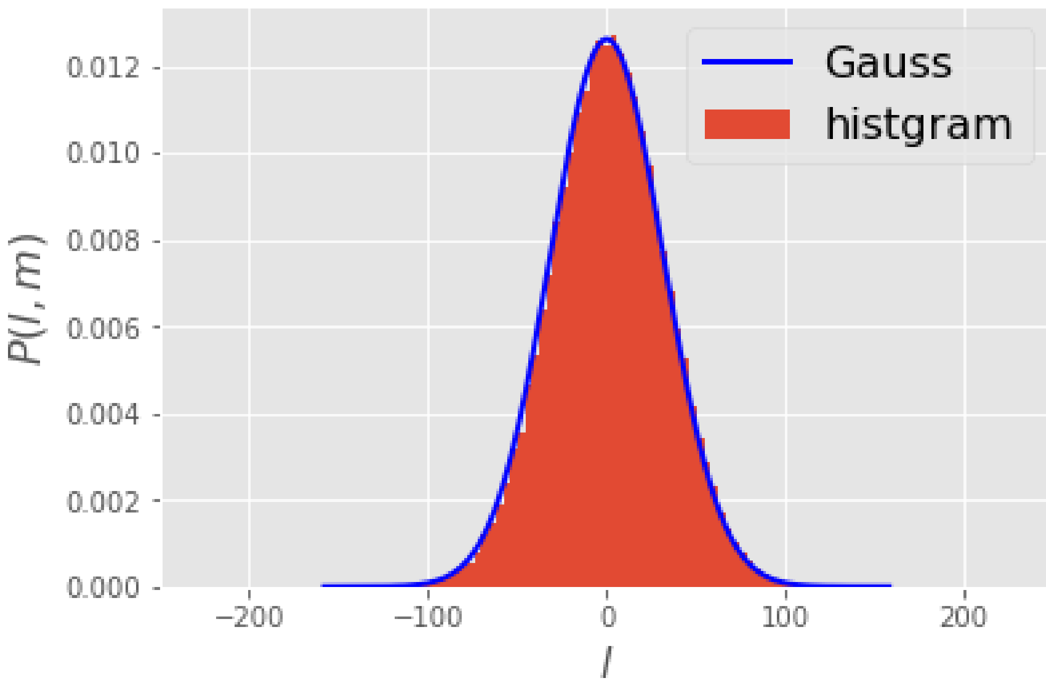

```
p = 0.5 M = 1000
```



- You may repeat the same simulation by choosing different values of total steps, for example $M = 100$, $1,000$, $10,000$, and $100,000$ to see how the distribution changes with the total number of steps.

1.5 Connection with the diffusion constant D

1.5.1 $P(x,t)$ from random walk

- Define a as the length of a single step and t_s as the time between subsequent steps.
- Define $x = al$ as the position of the random walker and $t = t_s m$ as the duration of time needed to take m steps.
- Here we consider a drift free case $p = 0.5$, i.e., $\mu_1 = \langle l \rangle = m(2p-1) = 0$.

$$P(x,t) = P(l,m)\frac{dl}{dx} = P(l,m)\frac{1}{a} \tag{E12}$$

$$= \frac{1}{a\sqrt{8\pi mp(1-p)}} \exp\left[-\frac{l^2}{8mp(1-p)}\right] \tag{E13}$$

$$= \frac{1}{\sqrt{8\pi a^2 p(1-p)t/t_s}} \exp\left[-\frac{x^2}{8a^2 p(1-p)t/t_s}\right] \tag{E14}$$

$$\text{with} \quad \mu_1 = \langle x \rangle = 0, \;\; \sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 = 4a^2 p(1-p)t/t_s \qquad \text{(E15, E16)}$$

1.5.2 $P(x,t)$ from the diffusion equation

- Consier the 1-D diffusion equation with diffusion constant D

$$\frac{\partial}{\partial t}P(x,t) = D\frac{\partial^2}{\partial x^2}P(x,t) \tag{E17}$$

$$\text{with} \quad P(x, t=0) = \delta(x) \tag{E18}$$

- The solution is given by

$$P(x,t) = \frac{1}{\sqrt{4\pi Dt}} \exp\left[-\frac{x^2}{4Dt}\right] \tag{E19}$$

- By comparing Eqs.(E14) and (E19) we can relate the diffusion constant D to the variance of the position of random walkers

$$D = \frac{2a^2 p(1-p)}{t_s} = \frac{\sigma^2}{2t} \tag{E20}$$

- In this case, σ^2 is also referred to as the mean-square displacement