**Department of Mathematics and Statistics**

UNIVERSITY OF CANTERBURY
*Te Whare Wānanga o Waitaha*
CHRISTCHURCH NEW ZEALAND

# CSE Exercises Wk 1

(1) The sensitivity and specificity of a medical diagnostic test for a disease are defined as

sensitivity = P(test is positive | patient has the disease),

specificity = P(test is negative | patient does not have the disease).

Suppose that a medical test has a sensitivity of 0·7 and a specificity of 0·95. If the prevalence of the disease in the general population is 1%, find

(a) the probability that a patient who tests positive actually has the disease,

(b) the probability that a patient who tests negative is free from the disease.

(2) The detection rate and false alarm rate of an intrusion sensor are defined as

detection rate = P(detection declared | intrusion),

false alarm rate = P(detection declared | no intrusion).

If the detection rate is 0·999 and the false alarm rate is 0·001, and the probability of an intrusion occurring is 0·01, find

(a) the probabity that there is an intrusion when a detection is declared,

(b) the probability that there is no intrusion when no detection is declared.

③ Let $A$ and $B$ be events such that $P(A) \neq 0$ and $P(B) \neq 0$. When $A$ and $B$ are disjoint, are they also independent? Explain clearly why or why not.

④ Let $a, b \in \mathbb{R}$, such that $a < b$. Let $X$ be a Uniform$(a,b)$ random variable whose PDF is given by

$$f(x) = c\, \mathbb{1}_{[a,b]}(x) = \begin{cases} c, & a \leq x \leq b, \\ 0, & \text{otherwise}, \end{cases}$$

where $c$ is a constant.

(a) Find the following in terms of $a$ and $b$:

(i) $c$,

(ii) $E(X)$,

(iii) $E(X^2)$,

(iv) $V(X)$.

(b) Plot the Uniform $(a, b)$ PDF and CDF for $a = -1$ and $b = 1$.

⑤ Starting from the definition of the variance of a random variable (Definition 20), show that

$$V(X) = E(X^2) - E(X)^2 .$$

⑥ Let $X$ be a discrete random variable with PMF given by

$$f(x) = \begin{cases} \frac{x}{10}, & x \in \{1, 2, 3, 4\}, \\ 0, & \text{otherwise}. \end{cases}$$

(a) Find (i) $P(X = 0)$,

(ii) $P(2.5 < X < 5)$,

(iii) $E(X)$,

(iv) $V(X)$.

(b) Write down the CDF of $X$.

(b) Plot the PMF and CDF of $X$.

(7) Let $X$ be a discrete random variable whose CDF is given by

$$F(x) = \begin{cases} 0, & x < 1.1, \\ 0.5, & 1.1 \leq x < 2.3, \\ 0.7, & 2.3 \leq x < 3.7, \\ 1, & x \geq 3.7. \end{cases}$$

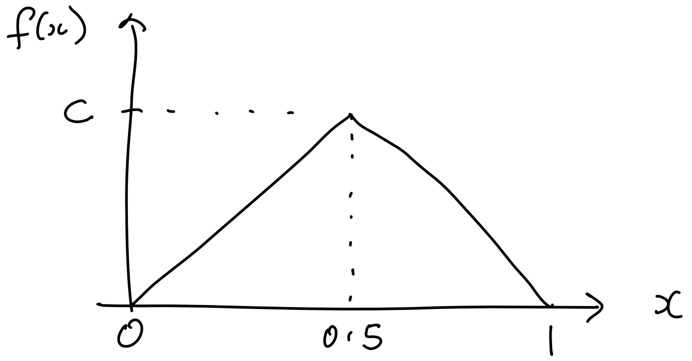
(a) Write down the PMF of $X$.

(b) Plot the PMF and CDF of $X$.

(c) Find (i) $E(X)$,

(ii) $V(X)$.

(8) Let $X$ be a random variable with PDF given in the figure below:



(a) What is the value of $c$ ?

(b) Write down the formulas for the PDF and CDF :

$$f(x) = ?$$

$$F(x) = ?$$

(c) Plot the CDF.

(d) Find (i) $E(X)$,

(ii) $V(X)$.

9 The covariance of two random variables $X$ and $Y$ is defined as

$$\mathrm{Cov}(X,Y) := E\big((X-E(X))(Y-E(Y))\big).$$

(a) Show, starting from the definition, that

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y).$$

(b) When $\mathrm{Cov}(X,Y) = 0$, $X$ and $Y$ are said to be "uncorrelated". Show that if $X$ and $Y$ are independent, then they are also uncorrelated, i.e. show that

$X$ and $Y$ independent $\Longrightarrow$ $X$ and $Y$ uncorrelated.

(Note, however, that the converse is generally not true.)

(10) Let $X_1, \ldots, X_n$ be random variables. Their joint CDF is defined as

$$F(x_1, \ldots, x_n) := P(X_1 \leq x_1, \ldots, X_n \leq x_n).$$

By repeated application of the definition of conditional probability, show that the joint CDF admits the following "telescopic" representation:

$$\begin{aligned} F(x_1, \ldots, x_n) &= F(x_n | x_1, \ldots, x_{n-1}) F(x_{n-1} | x_1, \ldots, x_{n-2}) \cdots \\ &\qquad \cdots F(x_2 | x_1) F(x_1) \\ &= F(x_1) \prod_{i=2}^{n} F(x_i | x_1, \ldots, x_{i-1}), \end{aligned}$$

where $F(x_i | x_1, \ldots, x_{i-1})$ denotes the conditional probability, $P(X_i \leq x_i | X_1 \leq x_1, \ldots, X_{i-1} \leq x_{i-1})$.

Note that the same telescopic representation exists for a joint PMF (defined in the obvious way), and even for a joint PDF (which will be defined later on).