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CSE Exercises - Week 4

(1) In this exercise, we show analytically and computationally that adding random variables is different from adding distributions and so the two operations should not be confused.

Consider two normal distributions, Normal(μ_1, σ_1^2) and Normal(μ_2, σ_2^2).

(a) Adding random variables.

Let X and Y be independent random variables such that $X \sim$ Normal(μ_1, σ_1^2) and $Y \sim$ Normal(μ_2, σ_2^2). Let $Z = X + Y$.

(i) Show that $E(Z) = \mu_1 + \mu_2$,
and $V(Z) = \sigma_1^2 + \sigma_2^2$.

(ii) Now let $\mu_1 = -2$, $\mu_2 = 2$ and $\sigma_1^2 = \sigma_2^2 = 1$. We will use simulation to show, as expected, that the distribution of Z is Normal$(0, 2)$:

- Let $n = 10{,}000$.
- Generate $X_1, \ldots, X_n \sim$ Normal$(-2, 1)$.
- Generate $Y_1, \ldots, Y_n \sim$ Normal$(2, 1)$.

- Compute $Z_i = X_i + Y_i$, $i = 1, \ldots, n$.
- Plot a density histogram for $Z_1, \ldots, Z_n$.
- On the same plot, superimpose the curve for the Normal$(0,2)$ density.

(b) Adding distributions.

Let $f(x; \mu_1, \sigma_1^2)$ denote the Normal(μ_1, σ_1^2) density and $f(x; \mu_2, \sigma_2^2)$ denote the Normal(μ_2, σ_2^2) density. Define a 2-component normal mixture density as

$$g(x) := \omega f(x; \mu_1, \sigma_1^2) + (1-\omega) f(x; \mu_2, \sigma_2^2)$$

where $\omega \in (0,1)$.

(i) Show that $g(x)$ integrates to 1 (as it should to be a proper density).

(ii) Let X be a random variable with density $g(x)$. Show that

$$E(X) = \omega\mu_1 + (1-\omega)\mu_2$$

and

$$V(X) = \omega\sigma_1^2 + (1-\omega)\sigma_2^2 + \omega(1-\omega)(\mu_1 - \mu_2)^2.$$

Notice the differences compared to part (a)(i).

(iii) We can generate a random variable, X, with density $g(x)$ by the following algorithm:

Generate $U \sim$ Uniform $(0,1)$.
If $U \leq \omega$
    generate $X \sim$ Normal (μ_1, σ_1^2),
else
    generate $X \sim$ Normal (μ_2, σ_2^2).
Return X.

Now let $\mu_1 = -2$, $\mu_2 = 2$ and $\sigma_1^2 = \sigma_2^2 = 1$ as in part (a)(ii), and let $\omega = 0.5$. We will use simulation to show that, unlike part (a)(ii), $g(x)$ is not a normal density.

- Let $n = 10{,}000$.
- Implement the algorithm given above and use it to generate $X_1, \ldots, X_n$ with density $g(x)$.
- Plot a density histogram for $X_1, \ldots, X_n$.
- On the same plot, superimpose the curve for $g(x)$.

(2) Page 142 Exercise 105.

(3) In this exercise, we investigate the approximation of the Binomial(n, θ) distribution by the Poisson(λ) distribution with $\lambda = n\theta$.

(a) Let $n = 20$ and $\theta = 0{\cdot}1$. Produce a plot of the Binomial$(20, 0{\cdot}1)$ PMF, similar to the one given in Figure 6·13. On the same plot, superimpose the Poisson(2) PMF (use a different symbol or colour so that the two PMFs are easily distinguishable).

(b) Repeat part (a) with $\theta = 0{\cdot}5$.

(c) Repeat part (a) with $\theta = 0{\cdot}9$.

(d) What can you conclude from the plots from (a), (b) and (c)?

(e) Describe how, for $\theta > 0{\cdot}5$, the approximation of the binomial distribution by the Poisson distribution can be made as accurate as for $\theta < 0{\cdot}5$.

(4) In this exercise, we investigate the approximation of the Binomial(n, θ) distribution by the Normal(μ, σ^2) distribution with $\mu = n\theta$ and $\sigma^2 = n\theta(1-\theta)$, when n is large and p is not too close to 0 or 1.

It may come as a surprise that a discrete distribution can be approximated by a continuous one but this exercise shows that this is possible.

(a) Let $n = 20$ and $\theta = 0.1$. Produce a plot of the Binomial$(20, 0.1)$ PMF, similar to the one given in Figure 6.13. Compute the Normal$(2, 1.8)$ densities for $x = 0, 1, \ldots, 20$. Superimpose these onto the same plot.

(b) Repeat part (a) with $\theta = 0.5$.

(c) Repeat part (a) with $\theta = 0.9$.

(d) What can you conclude from the plots from (a), (b) and (c)?

5 The Poisson(λ) distribution can also be approximated by the Normal(μ, σ^2) distribution with $\mu = \lambda$ and $\sigma^2 = \lambda$, when λ is large. To see this,

(a) Let $\lambda = 2$ and plot the Poisson(2) PMF for a range of x values, say $x = 0, 1, \ldots, 10$. Compute the Normal$(2, 2)$ densities for the same x values and superimpose them onto the plot.

(b) Repeat part (a) with $\lambda = 20$ and for $x = 0, 1, \ldots, 40$.

(c) Repeat part (a) with $\lambda = 100$ and for $x = 70, 71, \ldots, 130$.

(d) What can you conclude from the plots?

(6) Like the normal random varibles, Poisson random variables have the property that a sum of independent Poisson random variables is also a Poisson random variable. Thus, if $X_1, \ldots, X_n$ are independent random variables such that $X_i \sim \text{Poisson}(\lambda_i)$, then

$$X_1 + \cdots + X_n \sim \text{Poisson}(\lambda_1 + \cdots + \lambda_n).$$

We shall demonstrate this for a sum of two independent Poisson random variables.

- Let $n = 10{,}000$.
- Generate $X_1, \ldots, X_n \sim \text{Poisson}(3)$.
- Generate $Y_1, \ldots, Y_n \sim \text{Poisson}(6)$.
- Compute $Z_i = X_i + Y_i$, $i = 1, \ldots, n$.
- Compare the relative frequencies from the resulting $Z_1, \ldots, Z_n$ against the PMF of the Poisson(9) distribution.
- Estimate $E(Z)$ and $V(Z)$ using $Z_1, \ldots, Z_n$ and comment on your estimated values.

Solutions

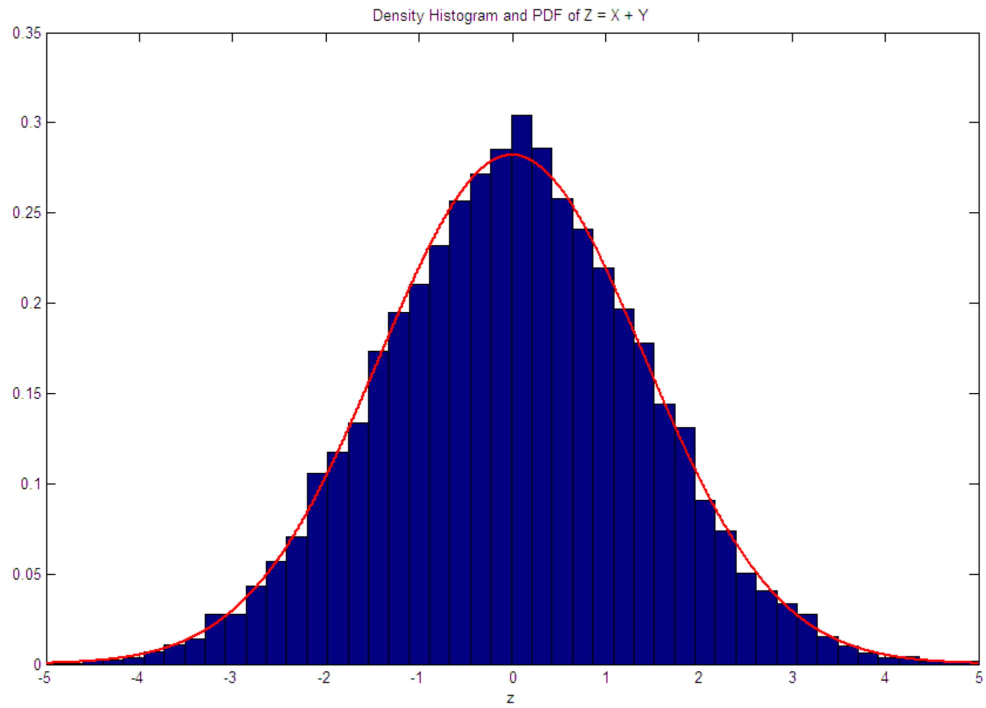
① (a) $X \sim \text{Normal}(\mu_1, \sigma_1^2)$, $Y \sim \text{Normal}(\mu_2, \sigma_2^2)$,

$Z = X + Y$, X and Y independent.

(i) $E(Z) = E(X+Y) = E(X) + E(Y) = \mu_1 + \mu_2$.

$V(Z) = V(X+Y) = V(X) + V(Y) = \sigma_1^2 + \sigma_2^2$
(since X and Y are independent).

(ii)



(b) $g(x) := \omega f(x;\mu_1,\sigma_1^2) + (1-\omega) f(x;\mu_2,\sigma_2^2)$, $\omega \in (0,1)$.

(i) $\int_{-\infty}^{\infty} g(x)\, dx$

$$= \omega \underbrace{\int_{-\infty}^{\infty} f(x;\mu_1,\sigma_1^2)\,dx}_{=1} + (1-\omega) \underbrace{\int_{-\infty}^{\infty} f(x;\mu_2,\sigma_2^2)\,dx}_{=1}$$

$$= \omega + (1-\omega) = 1.$$

(ii) Let $X \sim g(x)$.

$$E(X) = \int_{-\infty}^{\infty} x\, g(x)\, dx$$

$$= \omega \underbrace{\int_{-\infty}^{\infty} x f(x;\mu_1,\sigma_1^2)\,dx}_{=\mu_1} + (1-\omega) \underbrace{\int_{-\infty}^{\infty} x f(x;\mu_2,\sigma_2^2)\,dx}_{=\mu_2}$$

$$= \omega\mu_1 + (1-\omega)\mu_2.$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 g(x)\, dx$$

$$= \omega \underbrace{\int_{-\infty}^{\infty} x^2 f(x;\mu_1,\sigma_1^2)\,dx}_{\mu_1^2+\sigma_1^2} + (1-\omega) \underbrace{\int_{-\infty}^{\infty} x^2 f(x;\mu_2,\sigma_2^2)\,dx}_{\mu_2^2+\sigma_2^2}$$

$$= \omega(\mu_1^2+\sigma_1^2) + (1-\omega)(\mu_2^2+\sigma_2^2).$$

$$V(X) = E(X^2) - E(X)^2$$

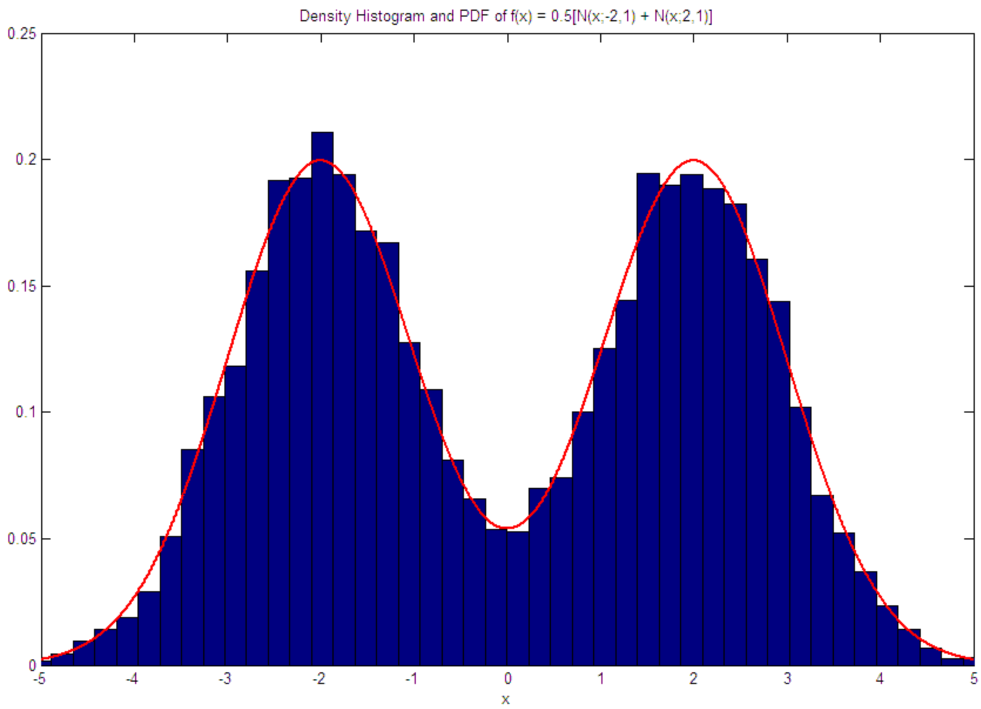
$$= \omega(\mu_1^2+\sigma_1^2) + (1-\omega)(\mu_2^2+\sigma_2^2) - [\omega\mu_1 + (1-\omega)\mu_2]^2$$

$$= \omega\sigma_1^2 + (1-\omega)\sigma_2^2 + \omega\mu_1^2 + (1-\omega)\mu_2^2 - \omega^2\mu_1^2 - (1-\omega)^2\mu_2^2 - 2\omega(1-\omega)\mu_1\mu_2$$

$$= \omega\sigma_1^2 + (1-\omega)\sigma_2^2 + \omega(1-\omega)\mu_1^2 + \omega(1-\omega)\mu_2^2 - 2\omega(1-\omega)\mu_1\mu_2$$

$$= \omega\sigma_1^2 + (1-\omega)\sigma_2^2 + \omega(1-\omega)(\mu_1-\mu_2)^2 \, .$$

(iii)



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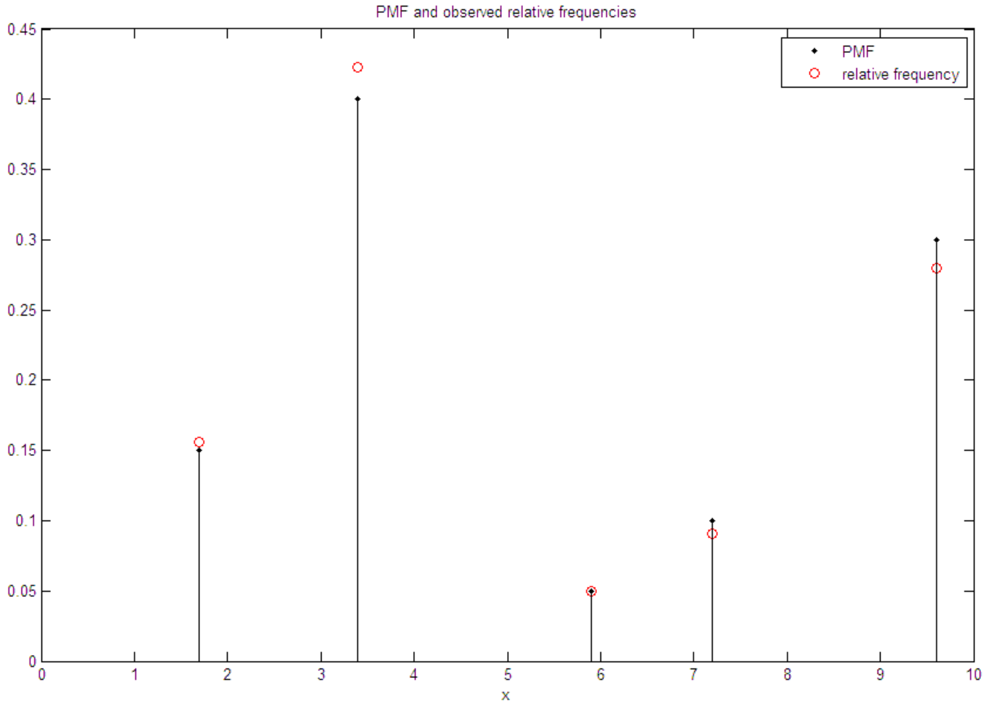
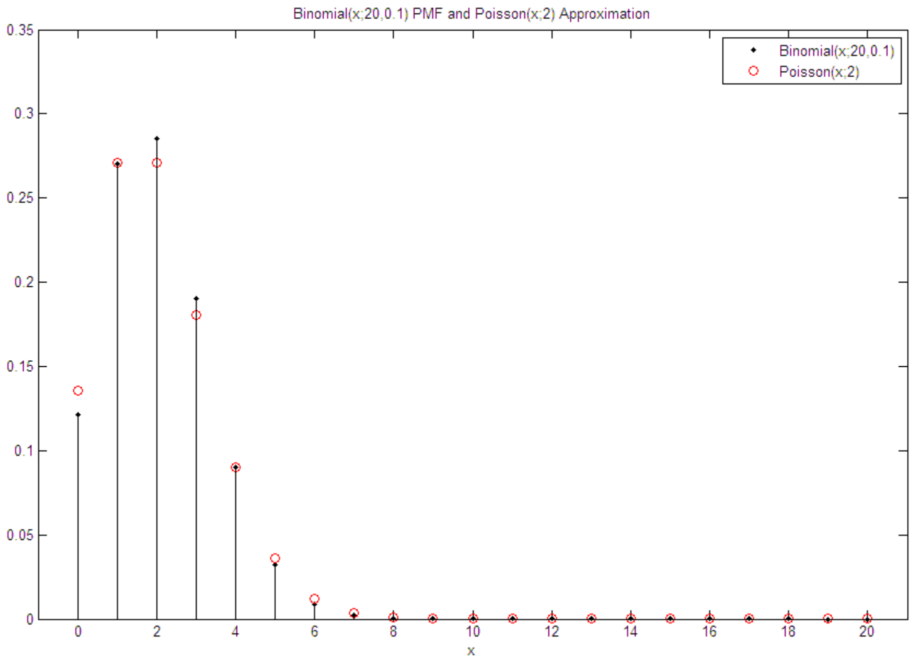
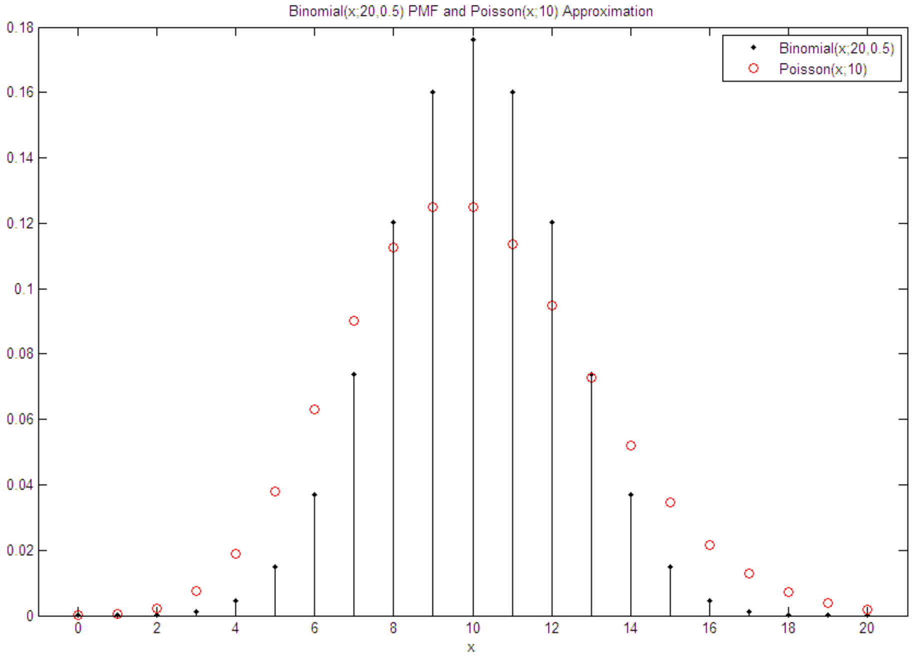


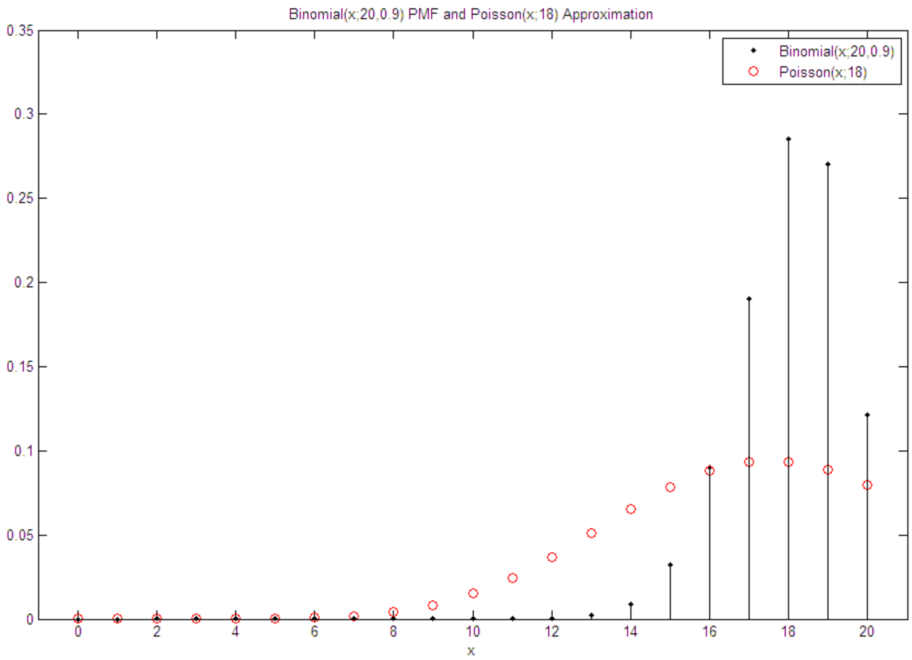
3 (a) Approximate Binomial (20, 0.1) by Poisson (2).



(b) Approximate Binomial(20, 0.5) by Poisson (10).



(c) Approximate Binomial (20, 0.9) by Poisson (18).



(d) Approximation of Binomial (n, θ) by Poisson $(n\theta)$ is good when θ is small but degrades when θ gets bigger.

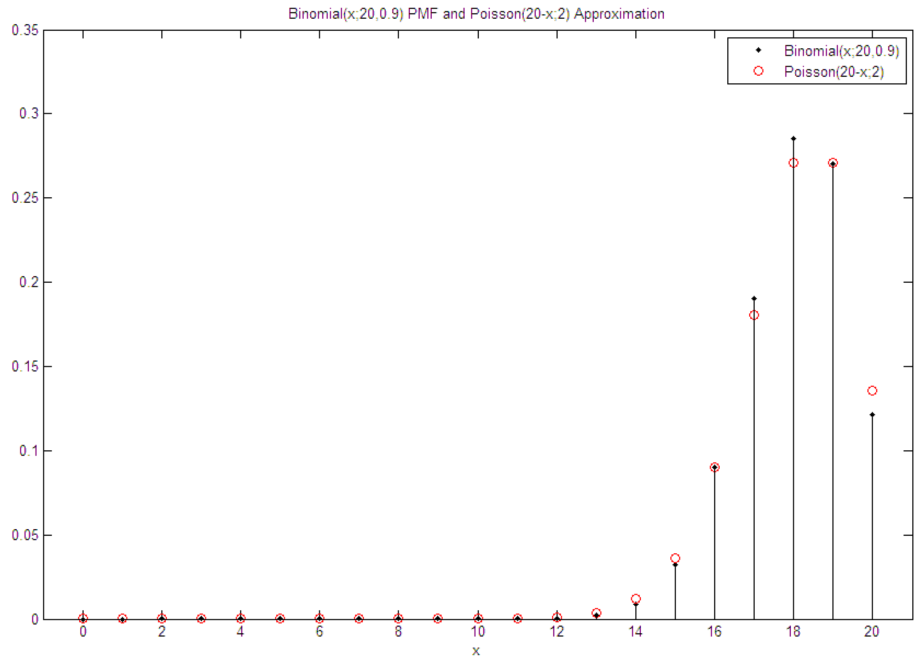
(e) Notice the symmetry of the binomial PMF in parts (a) and (c). Mathematically,

$$\text{Binomial}(x; n, \theta) = \frac{n!}{x!(n-x)!}\theta^x(1-\theta)^{n-x}$$

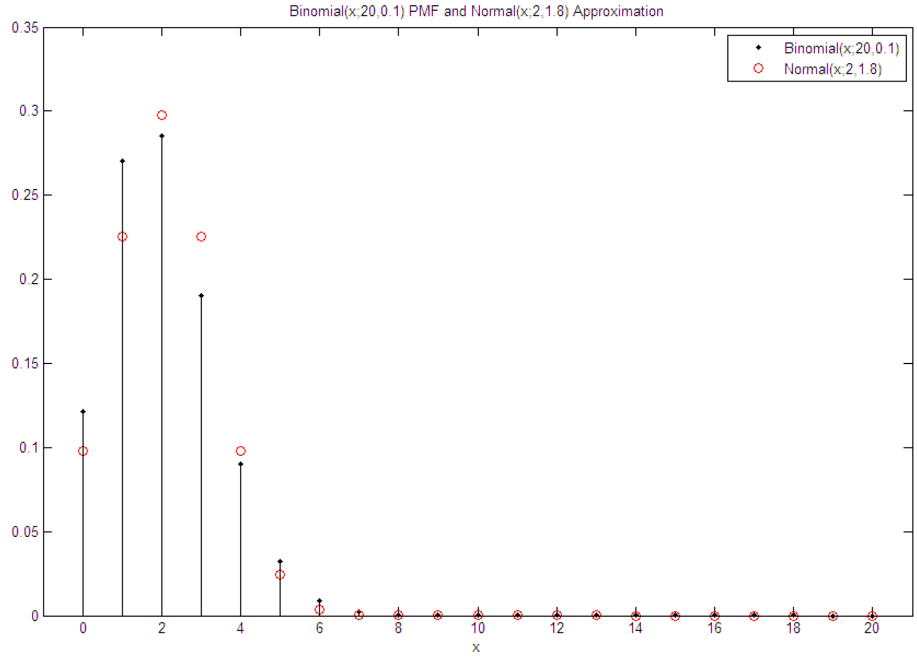
$$= \text{Binomial}(n-x; n, 1-\theta).$$

Hence, for $\theta > 0.5$, instead of approximating Binomial $(x; n, \theta)$ by Poisson $(x; n\theta)$, we can approximate it by Poisson $(n-x; n(1-\theta))$.

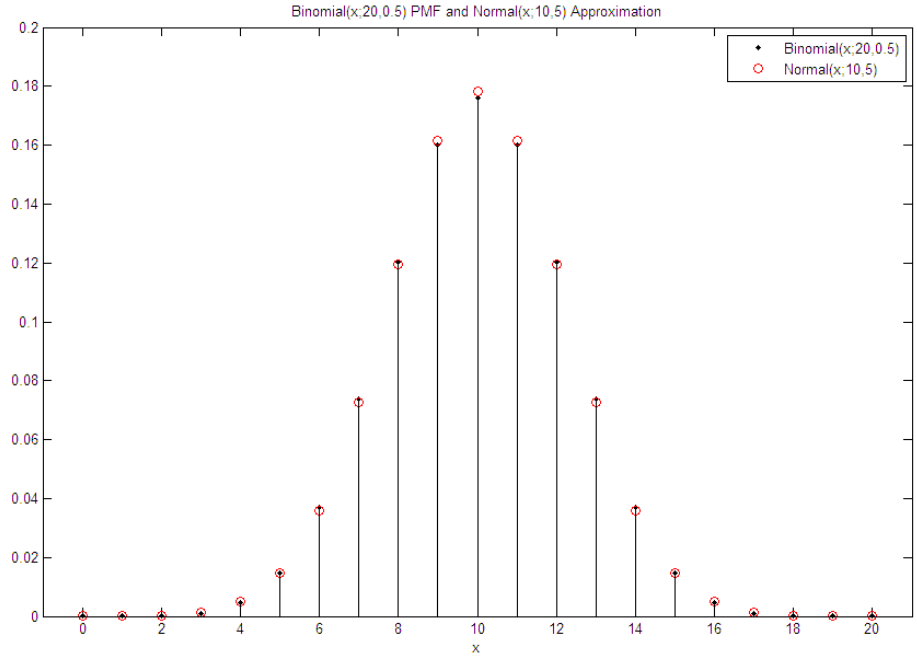
Thus, in part (c), we can approximate Binomial $(x; 20, 0.9)$ by Poisson $(20-x; 2)$.



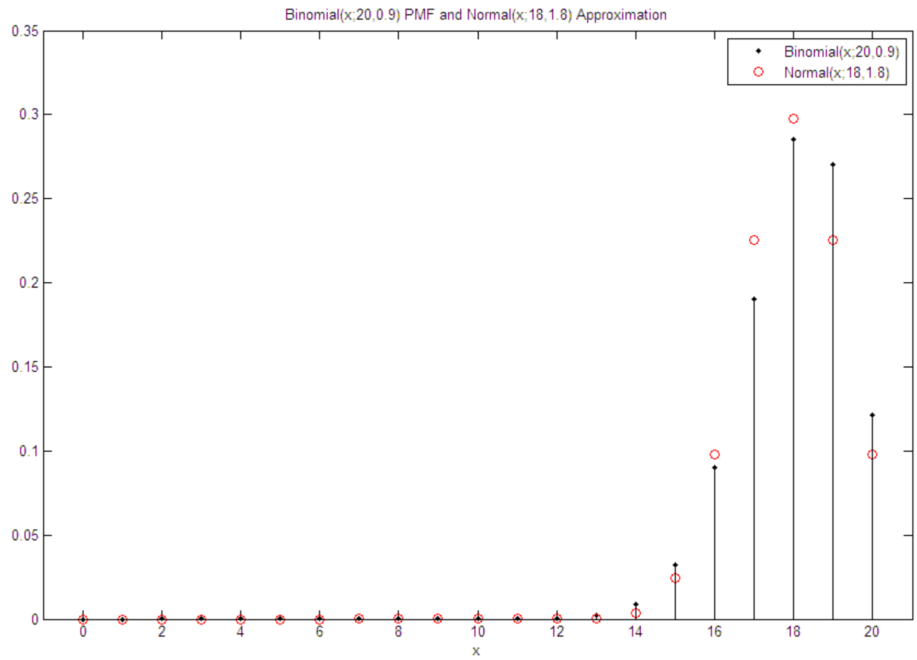
4 (a) Approximate Binomial $(20, 0.1)$ by Normal $(2, 1.8)$.



(b) Approximate Binomial $(20, 0.5)$ by Normal$(10, 5)$.

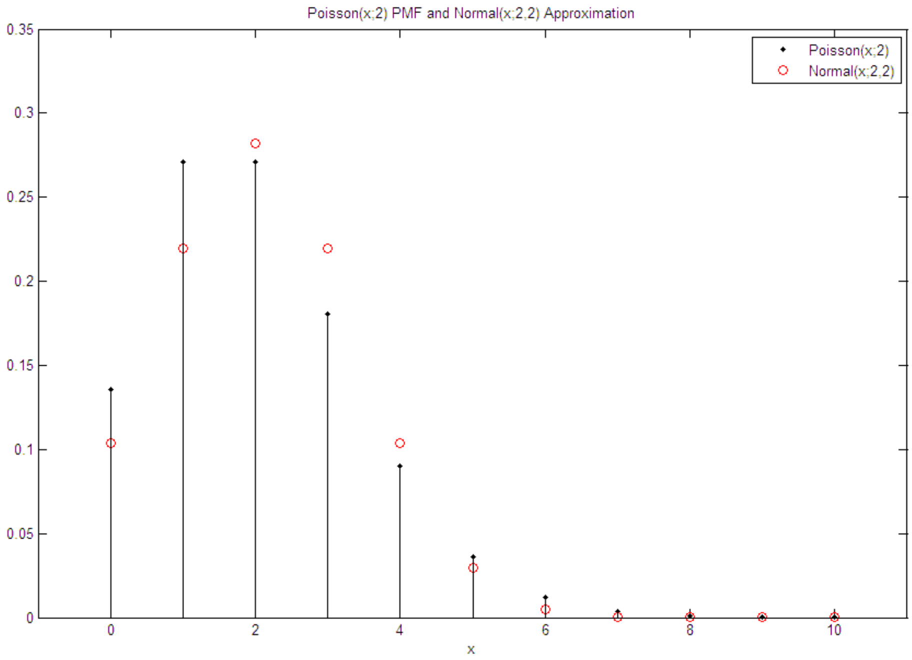


(c) Approximate Binomial$(20, 0.9)$ by Normal$(18, 1.8)$.

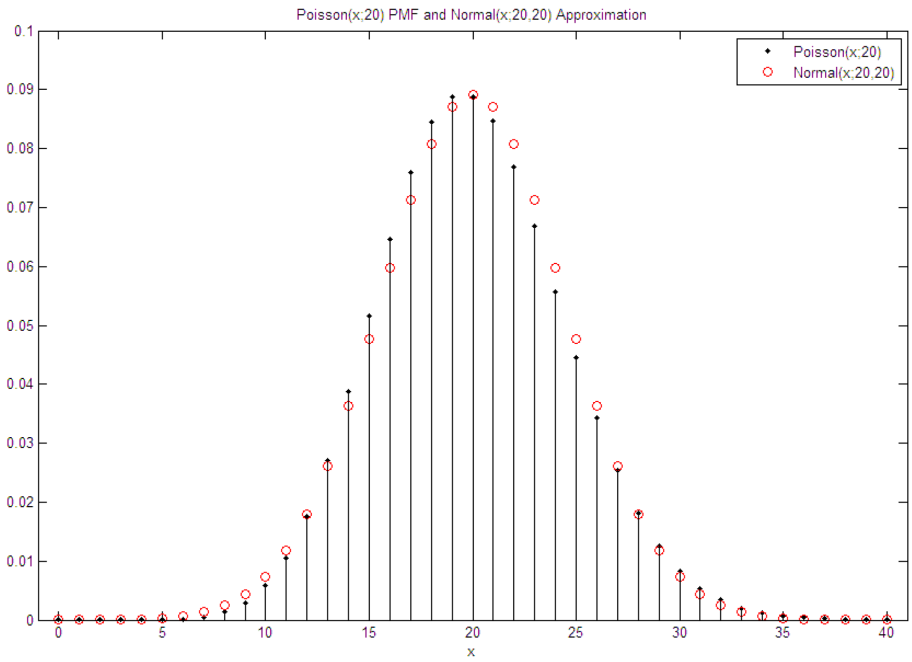


(d) The approximation of Binomial(n, θ) by Normal$(n\theta, n\theta(1-\theta))$ is good when n is large and θ is not too close to 0 or 1. For $n = 20$, we can see clearly that the approximation is best for $\theta = 0.5$ as compared to $\theta = 0.1$ or 0.9.

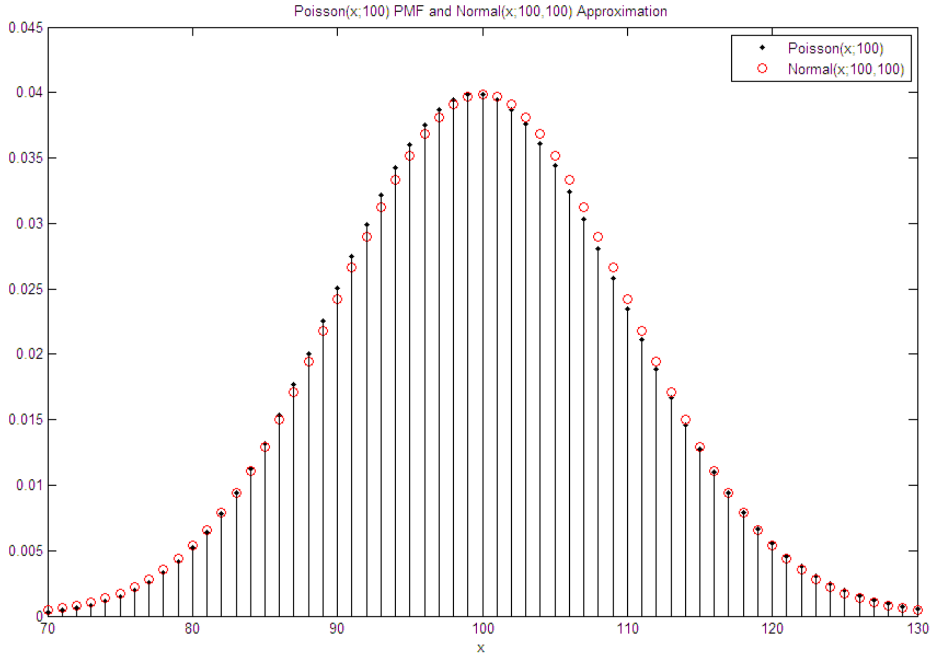
⑤ (a) Approximate Poisson (2) by Normal (2,2).



(b) Approximate Poisson (20) by Normal (20, 20).



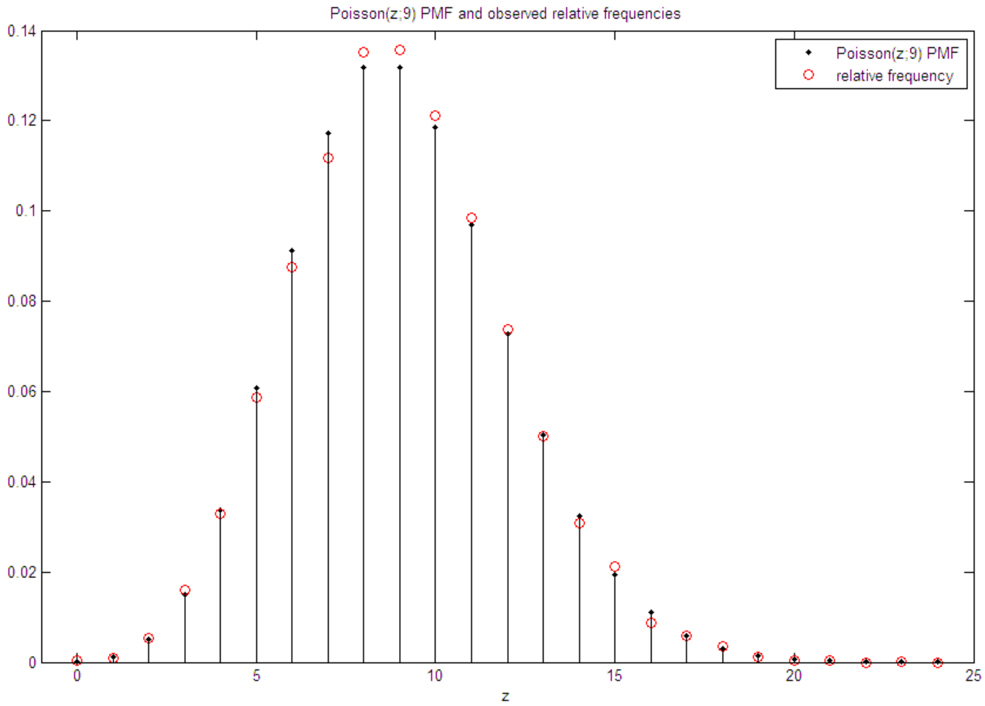
(c) Approximate Poisson (100) by Normal (100, 100).



(d) Approximation of Poisson (λ) by Normal (λ, λ) improves as λ gets larger.

(6) $X_1, \ldots, X_n \overset{IID}{\sim} \text{Poisson}(3)$, $Y_1, \ldots, Y_n \overset{IID}{\sim} \text{Poisson}(6)$.

$Z_i = X_i + Y_i$, $i = 1, \ldots, n$.



Sample mean and variance of my Z values are

$$\bar{Z}_n = 9.0215$$

$$S_n^2 = 8.9227$$

which are close to 9 as expected because $Z \sim \text{Poisson}(9)$, for which $E(Z) = V(Z) = 9$.