CSE Exercises – Week 5

① Page 143 Exercise 107.

Change "Referring to Example 2.2.3 ..." to "Referring to Simulation 97 (page 133) ...".

② Page 143 Exercise 108.

③ Page 143 Exercise 109.

④ Page 137 Labwork 100.

⑤ X has a Beta(α, β) distribution with parameters $\alpha > 0$ and $\beta > 0$ if its PDF is

$$f(x; \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1},$$

for $0 \leq x \leq 1$, and 0 elsewhere.

(a) Let (X_1, X_2) be a continuous random vector with joint PDF $f_X(x_1, x_2)$ and for which there exists an open subset $\mathcal{U} \subseteq \mathbb{R}^2$ such that $P((X_1, X_2) \in \mathcal{U}) = 1$. Let $(Y_1, Y_2) = g(X_1, X_2) = (g_1(X_1, X_2), g_2(X_1, X_2))$ be such that $g = (g_1, g_2)$ is a one-to-one function on $\mathcal{U}$ and

$$\det \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{pmatrix} \neq 0,$$

where $\det A$ denotes the determinant of matrix A. Let $h = (h_1, h_2)$ be the pointwise inverse of g so that $(X_1, X_2) = h(Y_1, Y_2) = (h_1(Y_1, Y_2), h_2(Y_1, Y_2))$. Then (Y_1, Y_2) is a continuous random vector and its joint PDF is given by

$$f_Y(y_1, y_2) = f_X(h(y_1, y_2)) \left| \det \begin{pmatrix} \frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} \\ \frac{\partial h_2}{\partial y_1} & \frac{\partial h_2}{\partial y_2} \end{pmatrix} \right|, \quad (y_1, y_2) \in g(U).$$

Now if X_1 and X_2 are independent random variables such that $X_1 \sim \text{Gamma}(a_1, 1)$ and $X_2 \sim \text{Gamma}(a_2, 1)$, use the above result, known as the transformation theorem, to show that $Y_1 = X_1/(X_1 + X_2)$ has a Beta (a_1, a_2) distribution. Hint: Consider the transformation, $(Y_1, Y_2) = g(X_1, X_2) = (X_1/(X_1+X_2), X_2)$.

(b) Use the result in part (a) to generate 10,000 samples from the Beta $(2,4)$ distribution, starting from samples with gamma distributions. Use the Matlab function "gamrnd" to generate gamma distributed samples.

(c) Plot the density histogram for your generated Beta samples and superimpose on it the curve for the Beta$(2,4)$ PDF.