Quotient Law

Claim: $\lim_{x\to a}[f(x) - g(x)] = M - N$

Proof. Provided that $\lim_{x\to c} f(x) = M, \quad \lim_{x\to c} g(x) = N, \quad$ and $N \neq 0$. Using the ε–δ definition of a limit, we will directly prove that:

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{M}{N}$$

The definition tell us, that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

Suppose $0 < |x - a| < \delta$. Let us analyze the distance between the function and its limit value:

$$|f(x)/g(x) - M/N| = |f(x)(N - M) + M(f(x) - g(x)|/|g(x)N|$$

Then using the triangle inequality, we can estimate each part as follows:

$$\left|\frac{f(x)(N-M) + M(f(x) - g(x))}{g(x)N}\right| \leq \frac{|f(x)||N-M| + |M||f(x) - g(x)|}{|g(x)N|}$$

Hence, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - g(x) - (M - N)| < \varepsilon$. Thus, we can conclude:

$$\lim_{x\to c}[f(x) - g(x)] = M - N.$$

To put a lower bound on $|g(x)|$, choose $\delta_1 = \frac{|N|^2}{2}$ such that if $|g(x)-N| < \delta_1$, then:

$$|g(x)||N| > \frac{|N|^2}{2} \quad \Rightarrow \quad \frac{1}{|g(x)||N|} < \frac{2}{|N|^2}$$

Then we can represent the total as:

$$\frac{|f(x)||N-M| + |M||f(x) - g(x)|}{|g(x)||N|} < (|f(x)||N-M| + |M||f(x) - g(x)|) \cdot \frac{2}{|N|^2} < \epsilon$$

Since the total expression is to be bounded by ϵ, choose $\delta_2 = \frac{\epsilon|N|}{4}$. We can bound each part individually as follows:

$$|f(x)||N-M| < \frac{\epsilon|N|}{4} \quad \text{and} \quad |M||f(x) - g(x)| < \frac{\epsilon|N|}{4}$$

Bounding $|f(x)|$, choose $\delta_3 = 1 + |L|$ such that:

$$|f(x)| < 1 + |L|$$

Since $|f(x)|$ is bounded. Choose $\delta_4 = e|N|/4(1 + |L|)$, it follows that:

$$|f(x)||N-M| < \frac{\epsilon|N|}{4} \quad \Rightarrow \quad |N-M| < \frac{\epsilon|N|}{4(1+|L|)}$$

Now choose $\delta = \min\left(\frac{2}{|N|}, \quad \frac{\epsilon|N|}{4|M|}, \quad 1+|L|, \quad \frac{\epsilon|N|}{4(1+|L|)}\right)$ such that:

$$\frac{|f(x)||N-M| + |M||f(x)-g(x)|}{|g(x)N|} < \left[(1+|L|)\cdot\frac{\epsilon|N|}{4(1+|L|)} \; + \; |M|\cdot\frac{\epsilon|N|}{4|M|}\right]\cdot\frac{2}{|N|}$$

Simplifying the right side:

$$= \left[\frac{\epsilon|N|}{4} + \frac{\epsilon|N|}{4}\right]\cdot\frac{2}{|N|} = \frac{\epsilon|N|}{2}\cdot\frac{2}{|N|} = \epsilon$$

Hence, for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x-c| < \delta$, then:

$$\left|\frac{f(x)}{g(x)} - \frac{L}{M}\right| < \epsilon$$

Thus, we conclude:

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{L}{M}$$

□