Constant Multiple Law

Claim: $\lim_{x \to a} (a \cdot f(x)) = a \cdot \lim_{x \to a} f(x)$

Proof. We will directly prove that $\lim_{x \to a} af(x) = a \cdot L$ using the ε–δ definition.

Let us recall the ε–δ definition: For every $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

Then in our case, given $\varepsilon > 0$, we want to find a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|af(x) - aL| < \varepsilon$.

Now suppose $0 < |x - a| < \delta$. Let us analyze $|af(x) - aL|$:

$$|af(x) - aL| = |a||f(x) - L| < \varepsilon$$

To guarantee this, we need:

$$|f(x) - L| < \frac{\varepsilon}{|a|}$$

So, let us define $\delta = \frac{\varepsilon}{|a|}$. Then, whenever $0 < |x - a| < \delta$, it follows that:

$$|f(x) - L| < \frac{\varepsilon}{|a|} \Rightarrow |a||f(x) - L| < \varepsilon \Rightarrow |af(x) - aL| < \varepsilon$$

Thus, for every $\varepsilon > 0$, we have found a δ such that if $0 < |x - a| < \delta$, then $|af(x) - aL| < \varepsilon$.

Hence, $\lim_{x \to a}(a \cdot f(x)) = a \cdot L$.

□