Sum Law

Claim: $\lim_{x\to a}[f(x) - g(x)] = M - N$

Proof. Using the ε–δ definition of a limit, we will directly prove that:

$$\lim_{x\to a}[f(x) - g(x)] = M - N$$

The definition tell us, that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

Suppose $0 < |x - a| < \delta$. Let us analyze the distance between the function and its limit value:

$$|(f(x) - g(x)) - (M - N)| = |(f(x) - M) - (g(x) - N)|$$

Using the triangle inequality, we estimate the difference as follows:

$$|(f(x) - M) - (g(x) - N)| \leq |f(x) - M| + |g(x) - N| < \varepsilon$$

Since the whole, and the sum of its two parts is below ε. We can estimate each part by saying they are below $\frac{\varepsilon}{2}$:

$$|f(x) - M| < \frac{\varepsilon}{2}$$

$$|g(x) - N| < \frac{\varepsilon}{2}$$

So let δ_1 and δ_2 both equal to $\frac{\varepsilon}{2}$.

Now choose $\delta = \min(\delta_1, \delta_2)$:

$$\begin{aligned}|(f(x) - g(x)) - (M - N)| &= |(f(x) - M) - (g(x) - N)| \\ &\leq |f(x) - M| + |g(x) - N| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Hence, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - g(x) - (M - N)| < \varepsilon$. Thus, we can conclude:

$$\lim_{x\to c}[f(x) - g(x)] = M - N.$$

Hence, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$. Thus, we can conclude:

$$\lim_{x\to c}[f(x) - g(x)] = M - N$$

□