Sum Law

Claim: $\lim_{x\to a}[f(x)+g(x)] = M+N$

Proof. Using the ε–δ definition of a limit, we will directly prove that:

$$\lim_{x\to a}[f(x)+g(x)] = M+N$$

The definition tell us, that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|f(x)-L| < \varepsilon$.

Suppose $0 < |x-a| < \delta$. Let us analyze the distance from the limit point:

$$|(f(x)+g(x))-(M+N)| \leq |f(x)-M| + |g(x)-N|$$

We're look for when:

$$|f(x)-M| < \delta 1$$

$$|g(x)-N| < \delta 2$$

Now choose $\delta = \min(\delta_1, \delta_2)$:

$$\begin{aligned}|(f(x)+g(x))-(M+N)| &= |(f(x)-M)+(g(x)-N)| \\ &\leq |f(x)-M| + |g(x)-N| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

Hence, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x-c| < \delta$, then $|f(x)-L| < \varepsilon$. Thus, we can conclude:

$$\lim_{x\to c}[f(x)+g(x)] = M+N$$

□