Sum Law

Claim: $\lim_{x\to a} f(x)^n = M^n$

Proof. Provided that $\lim_{x\to c} f(x) = M, \quad \lim_{x\to c} g(x) = N, \quad$ and $N \neq 0$. Using the ε–δ definition of a limit, we will prove by induction that for all n $\in \mathbb{N}$:

$$\lim_{x\to a} f(x)^n = M^n$$

The definition tell us, that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

Suppose $0 < |x - a| < \delta$. In a base case of $n = 1$:

$$|f(x)^1 - M^1| = |f(x) - M|$$

Now suppose $|x - a| < \delta$, it follows that:

$$|f(x) - M| < \epsilon$$

Hence, the base case of $n = 1$ holds.

Assume $\lim_{x\to a} f(x)^n = M^n$ holds for all $n \in \mathbb{N}$.

Now suppose $|x - a| < \delta$, it follows that:

$$|f(x)^n - M^n| < \epsilon$$

Using this assumption, we must show that $\lim_{x\to a} f(x)^{n+1} = M^{n+1}$ holds as well.

Let us first analyse the distance between the function and its limit point:

$$|f(x)^{n+1} - M^{n+1}| = |f(x) \cdot f(x)^n - M \cdot M^n|$$

$$= |f(x) \cdot f(x)^n - f(x) \cdot M^n + f(x) \cdot M^n - M \cdot M^n|$$

$$= |f(x)(f(x)^n - M^n) + M^n(f(x) - M)|$$

Now apply the triangle inequality:

$$\leq |f(x)| \cdot |f(x)^n - M^n| + |M^n| \cdot |f(x) - M|$$

Such that:

$$|f(x)| \cdot |f(x)^n - M^n| < \frac{\epsilon}{2}$$

$$|M^n| \cdot |f(x) - M| < \frac{\epsilon}{2}$$

Since $|f(x)-M| < \epsilon_1$ whenever $|x-a| < \delta_1$, and $|f(x)^n - M^n| < \epsilon_2$ whenever $|x-a| < \delta_2$, choose

$$\delta = \min(\delta_1, \delta_2)$$

Then, for all x such that $0 < |x-a| < \delta$, we have:

$$|f(x)^{n+1} - M^{n+1}| \leq |f(x)| \cdot |f(x)^n - M^n| + |M^n| \cdot |f(x) - M| < |f(x)| \cdot \epsilon_2 + |M^n| \cdot \epsilon_1$$

Now choose $\epsilon_1 = \frac{\epsilon}{2|M^n|}$ and $\epsilon_2 = \frac{\epsilon}{2|f(x)|}$ so the total is less than ϵ.

Hence, the formula holds for $n+1$.

$\square$