Limit Sum Law

Claim: $lim_{x \to a}[f(x) + g(x)] = L + M$

Proof. Using the ε–δ definition, we will prove that:

$$\lim_{x \to a}[f(x) + g(x)] = L + M$$

First, let us recall the ε–δ definition:

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \Rightarrow |[f(x) + g(x)] - [L + M]| < \varepsilon.$$

Then to prove our claim, we must find some $\delta > 0$ such that $0 < |x - a| < \delta$ using our givens. According to the ε–δ definition that $|f(x) - L| < \varepsilon$. So in our case, $|[f(x) + g(x)] - [L + M]| < \varepsilon$. Now consider the triangle inequality:

$$|a + b| \leq |a| + |b|$$

Re-arranging the components of our given found earlier, we get:

$$|f(x) - L + g(x) + M|$$

This re-arrangement allows us to easily choose values to replace the arbitrary variables in the theorem's definition, so let:

$$a = f(x) - L \quad \text{and} \quad b = g(x) - M$$

Then it follows according to the triangle inequality:

$$|[f(x) + g(x)] - [L + M]| \leq |f(x) - L| + |g(x) - M| < \varepsilon$$

With some interval in place, we can control each term. Consider the ε–δ definition and an arbitrary $\varepsilon > 0$. It follows that:

$$|f(x) - L| < \varepsilon \quad \text{and} \quad |g(x) - M| < \varepsilon$$

Since the sum of the absolute differences satisfies that:

$$|f(x) - L| + |g(x) - M| < \varepsilon$$

Then by the triangle inequality:

$$|f(x) - L| < \frac{\varepsilon}{2} \quad \text{and} \quad |g(x) - M| < \frac{\varepsilon}{2}$$

Ensuring that:

$$|[f(x) + g(x)] - [L + M]| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

So let $\delta = \frac{\varepsilon}{2}$ such that:

$$|[f(x) + g(x)] - [L + M]| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus, we proved that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|[f(x) + g(x)] - [L + M]| < \varepsilon$. Hence, we can conclude that $\lim_{x \to a}[f(x) + g(x)] = L + M$.

$\square$