Product Law

Claim: $lim_{x \to a}[f(x) \cdot g(x)] = M \cdot N$

Proof. Using the ε–δ definition of a limit, we will directly prove that:

$$\lim_{x \to a}[f(x) \cdot g(x)] = M \cdot N$$

The definition tell us, that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

Suppose $0 < |x - a| < \delta$. Let us analyze the distance between the function and its limit value:

$$|f(x)g(x) - MN| = |f(x)(g(x) - N) + N(f(x) - M)| \leq |f(x)||g(x) - N| + |N||f(x) - M|$$

Since $|f(x)g(x) - MN| \leq |f(x)||g(x) - N| + |N||f(x) - M|$, we can ensure the whole is less than ε by making each part less than $\frac{\varepsilon}{2}$, thus we require:

$$|f(x)||g(x) - N| < \frac{\varepsilon}{2}, \quad \text{and} \quad |N||f(x) - M| < \frac{\varepsilon}{2}.$$

Choose $\delta 1 = e/2|N|$ so it satisfies that:

$$|f(x) - M| < e/2|N|$$

Now we bound $|f(x)|$, choose $\delta_2 = \frac{\varepsilon + 2|N||M|}{2|N|}$, then:

$$|f(x)| < e/2|N| + |M|$$

And choose $\delta_3 = \frac{\varepsilon|N|}{\varepsilon + 2|N||M|}$ such that:

$$|g(x) - N| < \frac{\varepsilon|N|}{\varepsilon + 2|N||M|}$$

$$\text{Let } \delta = \min\left(\frac{\varepsilon}{2|N|}, \frac{\varepsilon + 2|N||M|}{2|N|}, \frac{\varepsilon|N|}{\varepsilon + 2|N||M|}\right)$$

Then, for all x such that $0 < |x - a| < \delta$, we have:

$$|f(x)g(x) - MN| \leq |f(x)||g(x) - N| + |N||f(x) - M|$$

Now, since $|f(x)| < \dfrac{\varepsilon + 2|N||M|}{2|N|}$ and $|g(x) - N| < \dfrac{\varepsilon|N|}{\varepsilon + 2|N||M|}$, we get:

$$|f(x)||g(x) - N| < \left(\frac{\varepsilon + 2|N||M|}{2|N|}\right)\left(\frac{\varepsilon|N|}{\varepsilon + 2|N||M|}\right) = \frac{\varepsilon}{2}$$

Also, since $|f(x) - M| < \dfrac{\varepsilon}{2|N|}$, we have:

$$|N||f(x) - M| < |N| \cdot \frac{\varepsilon}{2|N|} = \frac{\varepsilon}{2}$$

Thus,

$$|f(x)g(x) - MN| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Therefore, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$, then:

$$|f(x)g(x) - MN| < \varepsilon$$

Hence, by the ε–δ definition of limit:

$$\lim_{x \to a} [f(x) \cdot g(x)] = M \cdot N$$

$\square$