

### Constant Multiple Law

**Claim:**  $\lim_{x \rightarrow a} (a \cdot f(x)) = a \cdot L$

*Proof.* Using the  $\varepsilon$ - $\delta$  definition of a limit, we will directly prove that:

$$\lim_{x \rightarrow a} af(x) = a \cdot L$$

The definition tell us, that for every  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that if  $0 < |x - a| < \delta$ , then  $|f(x) - L| < \varepsilon$ .

Suppose  $0 < |x - a| < \delta$ . Let us analyze the distance from the limit point:

$$|af(x) - aL| = |a||f(x) - L|.$$

We're looking for when:

$$|af(x) - aL| < \varepsilon.$$

From scratch work, this will happen if:

$$|f(x) - L| < \frac{\varepsilon}{|a|}.$$

So we choose:

$$\delta = \frac{\varepsilon}{|a|}.$$

Then, for every  $\varepsilon > 0$ , there exists a  $\delta > 0$  (namely,  $\delta = \varepsilon/|a|$ ) such that if  $0 < |x - a| < \delta$ , then

$$|f(x) - L| < \frac{\varepsilon}{|a|} \quad \Rightarrow \quad |af(x) - aL| < \varepsilon.$$

Hence, we can conclude from the definition of a limit:

$$\lim_{x \rightarrow a} af(x) = a \cdot L.$$

□