Quotient Law

Claim: $\lim \left[\frac{f(x)}{g(x)}\right] = \frac{\lim f(x)}{\lim g(x)}$

Proof. Using the ε–δ definition, we will prove that:

$$\lim \left[\frac{f(x)}{g(x)}\right] = \frac{\lim f(x)}{\lim g(x)}$$

First, let us recall the ε–δ definition:

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

Since we already know $\lim f(x) = L$ and $\lim g(x) = M$, then we know $\lim f(x)/\lim g(x) = L/M$, it follows:

$$\lim \frac{f(x)}{g(x)} = \frac{L}{M}.$$

Then in our case, we must find some δ where if $0 < |x - c| < \delta$, then:

$$\left|\frac{f(x)}{g(x)} - \frac{L}{M}\right| < \varepsilon.$$

To find a delta which satisfies that, let us re-arrange our equation to be more manageable:

$$\left|\frac{f(x)M - g(x)L}{g(x)M}\right| < \varepsilon$$

$$\frac{|f(x)M - g(x)L|}{|g(x)M|} < \varepsilon$$

$$\frac{|f(x)M - f(x)L + f(x)L - g(x)L|}{|g(x)||M|}$$

$$\frac{|f(x)(M - L)| + |L(f(x) - g(x))|}{|g(x)M|} < \varepsilon$$

Now let us bound the denominator from above, ensuring its something positive above 0:

$$|g(x)|\,|M| > \frac{|M|^2}{2}$$

$$|g(x)| > \frac{|M|}{2}$$

Second, let us give an upper bound to $f(x)$ since it's not a constant. Since I previously stated we know $\lim f(x) = L$, it follows that $|f(x) - L| < \varepsilon$. Let that epsilon range be, say, 1, then:

$$|f(x) - L| < 1$$

$$|f(x)| < |L| + 1$$

Now let us control the size of the nominator using the triangle inequality:

$$|f(x)M - g(x)L| \leq |f(x)(M - L)| + |L(f(x) - g(x))| < \varepsilon$$

$$|f(x)|\,|M - L| + |L|\,|f(x) - g(x)| < \varepsilon$$

We make sure that each piece is a bit below $\frac{\varepsilon}{2}$, to ensure they vibe with the lower bound of the denominator, let each piece be less than $\frac{\varepsilon|M|^2}{4}$:

$$|f(x)|\,|M - L| < \frac{\varepsilon|M|^2}{4} \quad (1)$$

$$(|L| + 1)|M - L| < \frac{\varepsilon|M|^2}{4} \quad (2)$$

$$|M - L| < \frac{\varepsilon|M|^2}{4(|L| + 1)} \quad (3)$$

$$|L|\,|f(x) - g(x)| < \frac{\varepsilon|M|^2}{4} \quad (4)$$

$$|f(x) - g(x)| < \frac{\varepsilon|M|^2}{4|L|} \quad (5)$$

Let $\delta = \min\left(\frac{\varepsilon|M|^2}{4(|L|+1)}, \frac{\varepsilon|M|^2}{4|L|}\right)$ such that:

$$\begin{aligned}
|f(x)M - f(x)L + f(x)L - g(x)L| &\leq |f(x)(M - L)| + |L(f(x) - g(x))| < \varepsilon \\
&< \left((|L| + 1) \cdot \frac{\varepsilon|M|^2}{4(|L| + 1)} + |L| \cdot \frac{\varepsilon|M|^2}{4|L|}\right) \cdot \frac{2}{|M|^2} \\
&= \left(\frac{\varepsilon|M|^2}{4} + \frac{\varepsilon|M|^2}{4}\right) \cdot \frac{2}{|M|^2} \\
&= \frac{\varepsilon|M|^2}{2} \cdot \frac{2}{|M|^2} \\
&= \varepsilon
\end{aligned}$$

Since for every $\varepsilon > 0$, there exist some δ_1 and δ_2 such that if $0 < |x - a| < \delta = \min(\delta_1, \delta_2)$, then $\left|\frac{f(x)}{g(x)} - \frac{L}{M}\right| < \varepsilon$. Hence, we have proven that the limit of a quotient is the same as the quotient of the two limits:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}.$$

$\square$