**Difference Law**

# Claim: $\lim_{x \to a}[f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

*Proof.* Using the $\varepsilon$–$\delta$ definition, we will prove that:

$$\lim_{x \to a}[f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$

First, let us recall the $\varepsilon$–$\delta$ definition:

$$\forall \varepsilon > 0, \ \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

Then to prove our claim, we must find some $\delta > 0$ such that $0 < |x - a| < \delta$ using our givens. According to the $\varepsilon$–$\delta$ definition that $|f(x) - L| < \varepsilon$. In our case case:

$$\left| [f(x) - g(x)] - \left[ \lim_{x \to a} f(x) - \lim_{x \to a} g(x) \right] \right| < \varepsilon$$

Let us re-arrange this by distributing the difference of factors, such that:

$$\left| f(x) - \lim_{x \to a} f(x) - g(x) + \lim_{x \to a} g(x) \right| < \varepsilon$$

Then by re-arranging the absolute values:

$$\left| f(x) - \lim_{x \to a} f(x) \right| - \left| g(x) - \lim_{x \to a} g(x) \right| < \varepsilon$$

Given the $\varepsilon$–$\delta$ definition, consider an arbitrary limit. For every $\varepsilon > 0$, there exists some $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$. It follows that $\lim_{x \to a} f(x) = L$.

Then we can re-express our equality using the fact that:

$$\lim_{x \to a} f(x) = L \quad \text{and} \quad \lim_{x \to a} g(x) = M.$$

This allows us to easily bound our factors to control them. Then there exists some $\delta_1 > 0$ such that if $0 < |x - c| < \delta_1$, then $|f(x) - L| < \varepsilon$.

Similarly, there exists some $\delta_2 > 0$ such that if $0 < |x - c| < \delta_2$, then $|g(x) - M| < \varepsilon$. Let $\delta_1 = \frac{\varepsilon}{2}$ and $\delta_2 = \frac{\varepsilon}{2}$. Meaning, $\delta = \min(\delta_1, \delta_2)$, so let $\delta = \frac{\varepsilon}{2}$. So it follows that:

$$|f(x) - L| + |g(x) - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

It clearly follows:

$$\left| [f(x) - g(x)] - \left[ \lim_{x \to a} f(x) - \lim_{x \to a} g(x) \right] \right| < \varepsilon$$

Hence, we can conclude that for every $\varepsilon > 0$, there exists some $\delta > 0$ such that if $0 < |x - a| < \delta$, then:

$$\left| [f(x) - g(x)] - \left[ \lim_{x \to a} f(x) - \lim_{x \to a} g(x) \right] \right| < \varepsilon$$

$\square$