Limit Difference Law

Claim: $\lim_{x\to a}[f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

Proof. Using the ε–δ definition, we will prove that:

$$\lim_{x\to a}[f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$$

First, let us recall the ε–δ definition:

$$\forall\varepsilon > 0,\ \exists\delta > 0 \text{ such that } 0 < |x - a| < \delta \Rightarrow |[f(x) + g(x)] - [L + M]| < \varepsilon.$$

Then to prove our claim, we must find some $\delta > 0$ such that $0 < |x - a| < \delta$ using our givens. According to the ε–δ definition that $|f(x) - L| < \varepsilon$. In our case case:

$$\left|[f(x) - g(x)] - \left[\lim_{x\to a} f(x) - \lim_{x\to a} g(x)\right]\right| < \varepsilon$$

Let us re-arrange this by distributing the difference of factors, such that:

$$\left|f(x) - \lim_{x\to a} f(x) - g(x) + \lim_{x\to a} g(x)\right| < \varepsilon$$

Then by re-arranging the absolute values:

$$\left|f(x) - \lim_{x\to a} f(x)\right| - \left|g(x) - \lim_{x\to a} g(x)\right| < \varepsilon$$

Given the ε–δ definition, consider an arbitrary limit. For every $\varepsilon > 0$, there exists some $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$. It follows that $\lim_{x\to a} f(x) = L$.

Then we can re-express our equality using the fact that:

$$\lim_{x\to a} f(x) = L \quad \text{and} \quad \lim_{x\to a} g(x) = M.$$

This allows us to easily bound our factors to control them. Then there exists some $\delta_1 > 0$ such that if $0 < |x - c| < \delta_1$, then $|f(x) - L| < \varepsilon$.

Similarly, there exists some $\delta_2 > 0$ such that if $0 < |x - c| < \delta_2$, then $|g(x) - M| < \varepsilon$. Let $\delta_1 = \frac{\varepsilon}{2}$ and $\delta_2 = \frac{\varepsilon}{2}$. Meaning, $\delta = \min(\delta_1, \delta_2)$, so let $\delta = \frac{\varepsilon}{2}$. So it follows that:

$$|f(x) - L| + |g(x) - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

It clearly follows:

$$\left|[f(x) - g(x)] - \left[\lim_{x\to a} f(x) - \lim_{x\to a} g(x)\right]\right| < \varepsilon$$

Hence, we can conclude that for every $\varepsilon > 0$, there exists some $\delta > 0$ such that if $0 < |x - a| < \delta$, then:

$$\left|[f(x) - g(x)] - \left[\lim_{x\to a} f(x) - \lim_{x\to a} g(x)\right]\right| < \varepsilon$$

□