Constant Multiple Law

Claim: $\lim_{x\to a}(a\cdot f(x)) = a\cdot\lim_{x\to a} f(x)$

Proof. Using the ε–δ definition, we will prove that:

$$\lim_{x\to a}(a\cdot f(x)) = a\cdot\lim_{x\to a} f(x)$$

First, let us recall the ε–δ definition:

$$\forall\varepsilon > 0,\ \exists\delta > 0 \text{ such that } 0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

Since $\lim_{x\to a} f(x) = L$, this is the same as saying $\lim_{x\to a} af(x) = a\lim_{x\to a} f(x) = aL$ in our case. Then to prove our claim, we must satisfy these cases:

$$1.\forall\varepsilon 1 > 0,\ \exists\delta 1 > 0 \text{ such that } 0 < |x-a| < \delta 1 \Rightarrow |af(x) - aL| < \varepsilon 1.$$

$$2.\forall\varepsilon 2 > 0,\ \exists\delta 2 > 0 \text{ such that } 0 < |x-a| < \delta 2 \Rightarrow |a\,[f(x) - L]| < \varepsilon 2.$$

Suppose we re-arrange $|a\,[f(x) - L]|$, then:

$$|a\,[f(x) - L]| \Rightarrow |af(x) - aL|$$

Then we need only satisfy that:

$$1.\forall\varepsilon 1 > 0,\ \exists\delta 1 > 0 \text{ such that } 0 < |x-a| < \delta 1 \Rightarrow |af(x) - aL| < \varepsilon 1.$$

Re-distributing the absolute value we get:

$$|a|\,|f(x) - L| < \varepsilon$$

It follows that $|f(x) - L| < \frac{\varepsilon}{|a|}$. So let $\delta = \frac{\varepsilon}{|a|}$, such that:

$$|af(x) - aL| < \frac{\varepsilon}{|a|} \implies |f(x) - L| < \varepsilon$$

Thus, for every $\varepsilon > 0$, there exists some $\delta > 0$ such that if $0 < |x-a| < \delta$ then

$$|a(f(x) - L)| < \varepsilon \quad \text{and} \quad |af(x) - aL| < \varepsilon.$$

Hence, we can conclude that for every $\varepsilon > 0$, there exists some $\delta > 0$ such that if $0 < |x-a| < \delta$, then:

$$\lim_{x\to a} af(x) = a\lim_{x\to a} f(x) = aL.$$

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