**Constant Multiple Law**

## Claim: $\lim_{x\to a}(a \cdot f(x)) = a \cdot L$

*Proof.* Using the $\varepsilon$–$\delta$ definition of a limit, we will directly prove that:

$$\lim_{x\to a} af(x) = a \cdot L$$

The definition tell us, that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

Suppose $0 < |x - a| < \delta$. Let us analyze the distance from the limit point:

$$|af(x) - aL| = |a||f(x) - L|.$$

We're looking for when:

$$|a||f(x) - L| < \varepsilon.$$

From scratch work, this will happen if:

$$|f(x) - L| < \frac{\varepsilon}{|a|}.$$

So we choose:

$$\delta = \frac{\varepsilon}{|a|}.$$

Then, for every $\varepsilon > 0$, there exists a $\delta > 0$ (namely, $\delta = \varepsilon/|a|$) such that if $0 < |x - a| < \delta$, then

$$|f(x) - L| < \frac{\varepsilon}{|a|} \quad \Rightarrow \quad |af(x) - aL| < \varepsilon.$$

Hence, we can conclude from the definition of a limit:

$$\lim_{x\to a} af(x) = a \cdot L.$$

□