Product Law

Claim: $\lim_{x\to a}[f(x)\cdot g(x)] = M\cdot N$

Proof. Using the ε–δ definition, we will prove that:

$$\lim_{x\to a}[f(x)\cdot g(x)] = M\cdot N$$

First, let us recall the ε–δ definition:

$$\forall\varepsilon > 0,\ \exists\delta > 0 \text{ such that } 0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

Then, let's consider in our case that:

$$|f(x)g(x) - MN| < \varepsilon$$

Then let us re-arrange this into more manageable parts:

$$\begin{aligned}|f(x)g(x) - f(x)N + f(x)N - MN| &= |f(x)(g(x)-N) + N(f(x)-M)| \\ &\leq |f(x)|\cdot|g(x)-N| + |N|\cdot|f(x)-M| \\ &< \varepsilon\end{aligned}$$

Let us use the triangle inequality to bound our expression:

$$|f(x)g(x) - MN| \leq |f(x)|\cdot|g(x)-N| + |N|\cdot|f(x)-M|$$

Bounding each part to below $\frac{\varepsilon}{2}$, where $|N| \neq 0$:

$$|f(x)|\cdot|g(x)-N| < \frac{\varepsilon}{2} \Rightarrow |g(x)-N| < \frac{\varepsilon}{2|f(x)|}$$

$$|N|\cdot|f(x)-M| < \frac{\varepsilon}{2} \Rightarrow |f(x)-M| < \frac{\varepsilon}{2|N|}$$

Then let $\delta = \min\left(\frac{\varepsilon}{2|f(x)|}, \frac{\varepsilon}{2|N|}\right)$. Such that:

$$|f(x)|\cdot|g(x)-N| < \frac{\varepsilon}{2|f(x)|} < \frac{\varepsilon|f(x)|}{2|f(x)|} = \frac{\varepsilon}{2}$$

$$|N|\cdot|f(x)-M| < \frac{\varepsilon}{2|N|} < \frac{\varepsilon|N|}{2|N|} = \frac{\varepsilon}{2}$$

Hence, there exists a δ such that both factors $|x-c|$ are below. It follows that:

$$|f(x)g(x) - MN| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x-c| < \delta$, then $|f(x)g(x) - MN| < \varepsilon$.

Thus, we can conclude:

$$\lim_{x\to a}[f(x)\cdot g(x)] = M\cdot N.$$

□