Fourier transforms of N_s and Lee's formula

Yoann Pradat

August 25, 2019

Contents

I The proof

Notations

- $m, r \in \mathbb{N}^*$ with $m \geq r$
- $N_s := N_{2m,r,s}$, $s = 0, \ldots, r-1$ are Hermite B-splines
- $L_s := L_{2m,r,s}$, $s = 0, \ldots, r-1$ are Schoenberg fundamental splines

Given these notations, Lee stated and proved the following theorem

Theorem 1 (Lee76b, Theorem 1)**.** *The Fourier transform of the fundamental splines is*

$$\hat{L}_{2m,r,s}(u) = (-j)^s \frac{H_{r,s}(\alpha_{2m}(u))}{H_r(\alpha_{2m}(u))} \tag{1}$$

Proof. Let's first prove a property of the Hankel determinant that we will use later. For any sequence $(a_n)_n$ and any complex μ the following holds

$$H_r(a_n \mu^n) = \mu^{r(n-r+1)} H_r(a_n) \tag{2}$$

This is easily proved using the Leibniz formula for the determinant as follows

$$\begin{aligned}
H_r(a_n \mu^n) &= \sum_{\sigma \in \mathfrak{S}_r} \epsilon(\sigma) \prod_{j=1}^{r} \mu^{n-\sigma(j)-j+2} a_{n-\sigma(j)-j+2} \\
&= \mu^{rn - (\sum_{j=1}^{r} \sigma(j)+j) + 2r} \sum_{\sigma \in \mathfrak{S}_r} \epsilon(\sigma) \prod_{j=1}^{r} a_{n-\sigma(j)-j+2} \\
&= \mu^{rn - r(r+1) + 2r} \sum_{\sigma \in \mathfrak{S}_r} \epsilon(\sigma) \prod_{j=1}^{r} a_{n-\sigma(j)-j+2} \\
&= \mu^{r(n-r+1)} H_r(a_n)
\end{aligned}$$

Let $m, r \in \mathbb{N}^*$ such that $m \geq r$ and let $s \in [\![0, r-1]\!]$. The key observation for the proof of the Fourier transform of $L_{2m,r,s}$ is that the $(s+1)^{th}$ fundamental spline can be written as the integral of the $(s+1)^{th}$ exponential Euler-Hermite splines of order $n+1 = 2m$. By definition, the $(s+1)^{th}$ exponential Euler-Hermite splines of order n for the base λ ($\Pi_{n,r}(\lambda) \neq 0$) is given by

$$S_{n+1,r,s}(x;\lambda) = \frac{A_{n,r,s}(x;\lambda)}{A^{(s)}_{n,r,s}(x;\lambda)}, \quad 0 \leq x \leq 1 \tag{3}$$

$$S_{n+1,r,s}(x+1;\lambda) = \lambda S_{n+1,r,s}(x;\lambda), \quad \forall x \in \mathbb{R} \tag{4}$$

Consider the r functions

$$I_{2m,r,s}(x)=\begin{cases}\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}S_{2m,r,s}(x;e^{ju})du & \text{if } r \text{ even}\\ \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}S_{2m,r,s}(x;e^{ju})du & \text{if } r \text{ odd}\end{cases}\tag{5}$$

The functions $S_{2m,r,s}(\cdot;\lambda)$ being in $\mathscr{S}^{(s)}_{2m,r}$ so is $I_{2m,r,s}$. Given the properties of the derivatives of $S_{2m,r,s}$, the following holds

$$\forall k\in\mathbb{Z},\qquad \begin{cases}I^{(\rho)}_{2m,r,s}(k)=0 & \rho=0,\dots,r-1,\rho\neq s,\\ I^{(s)}_{2m,r,s}(k)=\delta_k\end{cases}$$

However there is only one element in $\mathscr{S}_{2m,r}$ that satisfies such conditions and that element is $L_{2m,r,s}$ by definition. As a consequence,

$$L_{2m,r,s}=I_{2m,r,s},\quad s=0,\dots,r-1\tag{6}$$

From (Sch72b, (7.14)), the Euler-Frobenius polynomial can be written as

$$\frac{A_n(x;e^{ju})}{n!}=(e^{ju}-1)e^{-ju}e^{jux}\sum_{k=-\infty}^{\infty}\frac{e^{2j\pi kx}}{(ju+2jk\pi)^{n+1}},\quad e^{ju}\neq 1, 0\leq x\leq 1\tag{7}$$

Using the multilinearity of the determinant $A_{n,r,s}$ and using the equation (2), the numerator of $S_{n+1,r,s}$ can be written as

$$\begin{aligned}A_{n,r,s}(x;e^{ju})&=\frac{(e^{ju}-1)}{e^{ju}}e^{jux}\sum_{k=-\infty}^{\infty}e^{2j\pi kx}\begin{vmatrix}\frac{A_n(0;e^{ju})}{n!} & \cdots & \frac{1}{(ju+2jk\pi)^{n+1}} & \cdots & \frac{A_{n-r+1}(0;e^{ju})}{(n-r+1)!}\\ \vdots & & \vdots & & \vdots\\ \frac{A_{n-r+1}(0;e^{ju})}{(n-r+1)!} & \cdots & \frac{1}{(ju+2jk\pi)^{n-r+2}} & \cdots & \frac{A_{n-2r+2}(0;e^{ju})}{(n-2r+2)!}\end{vmatrix}\\ &=\frac{(e^{ju}-1)^r}{e^{jur}}e^{jux}(-j)^s j^{r(n-r+1)}\sum_{k=-\infty}^{\infty}e^{2j\pi kx}\Delta_{n+1,k,s}(u)\end{aligned}$$

with $\Delta_{n+1,k,s}(u)$ the Hankel determinant of order r of $\alpha_{n+1}(u)$ with its $(s+1)^{th}$ column replaced by the vector

$$\begin{bmatrix}\frac{1}{(u+2k\pi)^{n+1}} & \frac{1}{(u+2k\pi)^{n}} & \cdots & \frac{1}{(u+2k\pi)^{n-r+2}}\end{bmatrix}^T$$

Similarly, the denominator of $S_{n,r,s}$ can be written as

$$\begin{aligned}A^{(s)}_{n,r,s}(0;e^{ju})&=H_r\left(\frac{A_n(0;\lambda)}{n!}\right)\\ &=\frac{(e^{ju}-1)^r}{e^{jur}}(-j)^{r(n-r+1)}H_r(\alpha_{n+1}(u))\end{aligned}$$

As a consequence,

$$S_{n+1,r,s}(x;e^{ju})=(-j)^s e^{jux}\sum_{k=-\infty}^{\infty}e^{2j\pi kx}\frac{\Delta_{n+1,k,s}(u)}{H_r(\alpha_{n+1}(u))}\tag{8}$$

Let now replace the expression (8) into (5). The details are given for r even as the calculations are similar for r odd.

$$\begin{aligned}I_{2m,r,s}(x)&=\frac{1}{2\pi}\int_0^{2\pi}S_{2m,r,s}(x;e^{ju})du\\ &=\frac{(-j)^s}{2\pi}\int_0^{2\pi}e^{jux}\sum_{k=-\infty}^{\infty}e^{2j\pi kx}\frac{\Delta_{2m,k,s}(u)}{H_r(\alpha_{2m}(u))}du\\ &=\frac{(-j)^s}{2\pi}\sum_{k=-\infty}^{\infty}\int_0^{2\pi}e^{j(u+2k\pi)x}\frac{\Delta_{2m,k,s}(u)}{H_r(\alpha_{2m}(u+2k\pi))}du\quad(\alpha_{2m}\text{is }2\pi\text{-periodic})\\ &=\frac{(-j)^s}{2\pi}\int_{-\infty}^{\infty}e^{jux}\frac{H_{r,s}(\alpha_{2m}(u))}{H_r(\alpha_{2m}(u))}du\end{aligned}$$

$\square$

Theorem 2 (Lee76b, Theorem 2)**.** *The Fourier transform of the Hermite B-splines is*

$$\hat{N}_s(u) = (-j)^s K(m,r)\Big(2\sin\frac{u}{2}\Big)^{2m} H_{r,s}(\alpha_{2m}(u)) \tag{9}$$

with $K(m,r) = (-1)^{m(r+1)} \frac{(2m-1)!(2m-2)!...(2m-r)!}{1!2!...(r-1)!}$

Remark 1. *From Theorem 1 and the definition*

$$N_s = \sum_{k=-(m-r)}^{(m-r)} c_k L_s(\cdot - k) \tag{10}$$

N_s *can be written in the following integral representation*

$$N_s(x) = \frac{(-j)^s}{2\pi}\int_{-\infty}^{\infty} e^{-ju(m-r)}\Pi_{2m-1,r}(e^{ju})e^{jux}\frac{H_{r,s}(\alpha_{2m}(u))}{H_r(\alpha_{2m}(u))}du \tag{11}$$

Example 1. 1. *In the case where* $m = r$*, we know that* $N_0 = (2m-1)!M_{2m}$ *i.e we expect*

$$\hat{N}_0(u) = (2m-1)!\Big(\frac{2\sin(\frac{u}{2})}{u}\Big)^{2m} \tag{12}$$

2. *Assume* $m \geq 1, r = 1$*. Lee's formula is*

$$\hat{N}_0(u) = (2m-1)! \times \Big(2\sin\frac{u}{2}\Big)^{2m}\frac{1}{u}^{2m} \tag{13}$$

3. *Assume* $m = 2, r = 2$*. Lee's formula is*

$$\begin{aligned}\hat{N}_0(u) &= 3!2!\Big(2\sin\frac{u}{2}\Big)^4 \begin{vmatrix} \frac{1}{u^4} & \sum_k \frac{1}{(u+2k\pi)^3} \\ \frac{1}{u^3} & \sum_k \frac{1}{(u+2k\pi)^2}\end{vmatrix} \\ &= 3!2!\Big(2\sin\frac{u}{2}\Big)^4 \frac{1}{u^6}\left[1 + \sum_{k\neq 0}\frac{u^2}{(u+2k\pi)^2} - (1 + \sum_{k\neq 0}\frac{u^3}{(u+2k\pi)^3})\right] \\ &= 3!2!\Big(2\sin\frac{u}{2}\Big)^4 \frac{1}{u^4}\left[\sum_{k\neq 0}\frac{1}{(u+2k\pi)^2} - \sum_{k\neq 0}\frac{u}{(u+2k\pi)^3}\right]\end{aligned}$$

while

$$\begin{aligned}\hat{N}_1(u) &= (-j)3!2!\Big(2\sin\frac{u}{2}\Big)^4 \begin{vmatrix} \sum_k \frac{1}{(u+2k\pi)^4} & \frac{1}{u^4} \\ \sum_k \frac{1}{(u+2k\pi)^3} & \frac{1}{u^3}\end{vmatrix} \\ &= (-j)3!2!\Big(2\sin\frac{u}{2}\Big)^4 \frac{1}{u^7}\left[1 + \sum_{k\neq 0}\frac{u^4}{(u+2k\pi)^4} - (1 + \sum_{k\neq 0}\frac{u^3}{(u+2k\pi)^3})\right] \\ &= (-j)3!2!\Big(2\sin\frac{u}{2}\Big)^4 \frac{1}{u^4}\left[\sum_{k\neq 0}\frac{u}{(u+2k\pi)^4} - \sum_{k\neq 0}\frac{1}{(u+2k\pi)^3}\right]\end{aligned}$$

4. *Assume* $m \geq 2, r = 2$.

   - *Around 0*

$$\begin{aligned}\hat{N}_0(u) &= K(m,2)\Big(2\sin\frac{u}{2}\Big)^{2m} \begin{vmatrix} \frac{1}{u^{2m}} & \sum_k \frac{1}{(u+2k\pi)^{2m-1}} \\ \frac{1}{u^{2m-1}} & \sum_k \frac{1}{(u+2k\pi)^{2m-2}}\end{vmatrix} \\ &= K(m,2)\Big(2\sin\frac{u}{2}\Big)^{2m} \frac{1}{u^{4m-2}}\left[1 + \sum_{k\neq 0}\frac{u^{2m-2}}{(u+2k\pi)^{2m-2}} - (1 + \sum_{k\neq 0}\frac{u^{2m-1}}{(u+2k\pi)^{2m-1}})\right] \\ &= K(m,2)\Big(2\sin\frac{u}{2}\Big)^{2m} \frac{1}{u^{2m}}\left[\beta_{2m-2}(u) - u\beta_{2m-1}(u)\right]\end{aligned}$$

while

$$\begin{aligned}
\hat{N}_1(u) &= (-j)K(m,2)\Big(2\sin\frac{u}{2}\Big)^{2m} \begin{vmatrix} \sum_k \frac{1}{(u+2k\pi)^{2m}} & \frac{1}{u^{2m}} \\ \sum_k \frac{1}{(u+2k\pi)^{2m-1}} & \frac{1}{u^{2m-1}} \end{vmatrix} \\
&= (-j)K(m,2)\Big(2\sin\frac{u}{2}\Big)^{4m-2} \frac{1}{u^{4m-1}} \left[1+\sum_{k\neq 0}\frac{u^{2m}}{(u+2k\pi)^{2m}} - (1+\sum_{k\neq 0}\frac{u^{2m-1}}{(u+2k\pi)^{2m-1}})\right] \\
&= (-j)K(m,2)\Big(2\sin\frac{u}{2}\Big)^{2m} \frac{1}{u^{2m}} \left[u\beta_{2m}(u) - \beta_{2m-1}(u)\right]
\end{aligned}$$

- <u>*Around $2l\pi$*</u>

$$\begin{aligned}
\hat{N}_0(u) &= K(m,2)\Big(2\sin\frac{u}{2}\Big)^{2m} \begin{vmatrix} \frac{1}{u^{2m}} & \sum_k \frac{1}{(u+2k\pi)^{2m-1}} \\ \frac{1}{u^{2m-1}} & \sum_k \frac{1}{(u+2k\pi)^{2m-2}} \end{vmatrix} \\
&= K(m,2)\Big(2\sin\frac{u}{2}\Big)^{2m} \frac{1}{(u-2l\pi)^{2m}} \Big[\frac{(u-2l\pi)^2}{u^{2m}} + \frac{(u-2l\pi)^{2m}}{u^{2m}}\beta_{2m-2}(u-2l\pi) \\
&\quad - \Big(\frac{(u-2l\pi)^1}{u^{2m-1}} + \frac{(u-2l\pi)^{2m}}{u^{2m-1}}\beta_{2m-1}(u-2l\pi)\Big)\Big]
\end{aligned}$$

i.e $\lim_{u\to 2l\pi} \hat{N}_0(u) = 0$ *while*

$$\begin{aligned}
\hat{N}_1(u) &= (-j)K(m,2)\Big(2\sin\frac{u}{2}\Big)^{2m} \begin{vmatrix} \sum_k \frac{1}{(u+2k\pi)^{2m}} & \frac{1}{u^{2m}} \\ \sum_k \frac{1}{(u+2k\pi)^{2m-1}} & \frac{1}{u^{2m-1}} \end{vmatrix} \\
&= K(m,2)\Big(2\sin\frac{u}{2}\Big)^{2m} \frac{1}{(u-2l\pi)^{2m}} \Big[\frac{1}{u^{2m-1}} + \frac{(u-2l\pi)^{2m}}{u^{2m-1}}\beta_{2m}(u-2l\pi) \\
&\quad - \Big(\frac{(u-2l\pi)^1}{u^{2m}} + \frac{(u-2l\pi)^{2m}}{u^{2m}}\beta_{2m-1}(u-2l\pi)\Big)\Big]
\end{aligned}$$

i.e $\lim_{u\to 2l\pi} \hat{N}_1(u) = -\dfrac{jK(m,2)}{(2l\pi)^{2m-1}}$

5. *Assume $m = 3, r = 3$.*

- <u>*Around 0*</u>

$$\hat{N}_0(u) = \frac{5!4!3!}{2!}\Big(2\sin\frac{u}{2}\Big)^6 \begin{vmatrix} \frac{1}{u^6} & \sum_k \frac{1}{(u+2k\pi)^5} & \sum_k \frac{1}{(u+2k\pi)^4} \\ \frac{1}{u^5} & \sum_k \frac{1}{(u+2k\pi)^4} & \sum_k \frac{1}{(u+2k\pi)^3} \\ \frac{1}{u^4} & \sum_k \frac{1}{(u+2k\pi)^3} & \sum_k \frac{1}{(u+2k\pi)^2} \end{vmatrix}$$

$$\begin{aligned} &= \frac{5!4!3!}{2!}\Big(2\sin\frac{u}{2}\Big)^6 \frac{1}{u^{12}}\Big[1 + \sum_{k\neq 0}\frac{u^4}{(u+2k\pi)^4} + \frac{u^2}{(u+2k\pi)^2} + \sum_{k\neq 0}\frac{u^4}{(u+2k\pi)^4}\sum_{k\neq 0}\frac{u^2}{(u+2k\pi)^2} \\ &\quad - (1 + \sum_{k\neq 0}\frac{u^3}{(u+2k\pi)^3} + \frac{u^3}{(u+2k\pi)^3} + \sum_{k\neq 0}\frac{u^3}{(u+2k\pi)^3}\sum_{k\neq 0}\frac{u^3}{(u+2k\pi)^3}) \\ &\quad - (1 + \sum_{k\neq 0}\frac{u^5}{(u+2k\pi)^5} + \frac{u^2}{(u+2k\pi)^2} + \sum_{k\neq 0}\frac{u^5}{(u+2k\pi)^5}\sum_{k\neq 0}\frac{u^2}{(u+2k\pi)^2}) \\ &\quad + (1 + \sum_{k\neq 0}\frac{u^4}{(u+2k\pi)^4} + \frac{u^3}{(u+2k\pi)^3} + \sum_{k\neq 0}\frac{u^3}{(u+2k\pi)^3}\sum_{k\neq 0}\frac{u^4}{(u+2k\pi)^4}) \\ &\quad + 1 + \sum_{k\neq 0}\frac{u^5}{(u+2k\pi)^5} + \frac{u^3}{(u+2k\pi)^3} + \sum_{k\neq 0}\frac{u^5}{(u+2k\pi)^5}\sum_{k\neq 0}\frac{u^3}{(u+2k\pi)^3} \\ &\quad - (1 + \sum_{k\neq 0}\frac{u^4}{(u+2k\pi)^4} + \frac{u^4}{(u+2k\pi)^4} + \sum_{k\neq 0}\frac{u^4}{(u+2k\pi)^4}\sum_{k\neq 0}\frac{u^4}{(u+2k\pi)^4})\Big] \\ &= \frac{5!4!3!}{2!}\Big(2\sin\frac{u}{2}\Big)^6 \frac{1}{u^6}\Big[\beta_4(u)\beta_2(u) - \beta_3(u)\beta_3(u) + \mathcal{O}(u)\Big] \end{aligned}$$

- <u>*Around* $2l\pi$, $v = u - 2l\pi$</u>

$$\hat{N}_0(u) = K(m,3)\Big(2\sin\frac{v}{2}\Big)^{2m} \begin{vmatrix} \frac{1}{u^{2m}} & \frac{1}{v^{2m-1}}(1+v^{2m-1}\beta_{2m-1}(v)) & \frac{1}{v^{2m-2}}(1+v^{2m-2}\beta_{2m-2}(v)) \\ \frac{1}{u^{2m-1}} & \frac{1}{v^{2m-2}}(1+v^{2m-2}\beta_{2m-2}(v)) & \frac{1}{v^{2m-3}}(1+v^{2m-3}\beta_{2m-3}(v)) \\ \frac{1}{u^{2m-2}} & \frac{1}{v^{2m-3}}(1+v^{2m-3}\beta_{2m-3}(v)) & \frac{1}{v^{2m-4}}(1+v^{2m-4}\beta_{2m-4}(v)) \end{vmatrix}$$

$$\begin{aligned} &= K(m,3)\Big(2\sin\frac{u}{2}\Big)^{2m} \frac{1}{v^{12}}\Big[1 + \sum_{k\neq 0}\frac{u^4}{(u+2k\pi)^4} + \frac{u^2}{(u+2k\pi)^2} + \sum_{k\neq 0}\frac{u^4}{(u+2k\pi)^4}\sum_{k\neq 0}\frac{u^2}{(u+2k\pi)^2} \\ &\quad - (1 + \sum_{k\neq 0}\frac{u^3}{(u+2k\pi)^3} + \frac{u^3}{(u+2k\pi)^3} + \sum_{k\neq 0}\frac{u^3}{(u+2k\pi)^3}\sum_{k\neq 0}\frac{u^3}{(u+2k\pi)^3}) \\ &\quad - (1 + \sum_{k\neq 0}\frac{u^5}{(u+2k\pi)^5} + \frac{u^2}{(u+2k\pi)^2} + \sum_{k\neq 0}\frac{u^5}{(u+2k\pi)^5}\sum_{k\neq 0}\frac{u^2}{(u+2k\pi)^2}) \\ &\quad + (1 + \sum_{k\neq 0}\frac{u^4}{(u+2k\pi)^4} + \frac{u^3}{(u+2k\pi)^3} + \sum_{k\neq 0}\frac{u^3}{(u+2k\pi)^3}\sum_{k\neq 0}\frac{u^4}{(u+2k\pi)^4}) \\ &\quad + 1 + \sum_{k\neq 0}\frac{u^5}{(u+2k\pi)^5} + \frac{u^3}{(u+2k\pi)^3} + \sum_{k\neq 0}\frac{u^5}{(u+2k\pi)^5}\sum_{k\neq 0}\frac{u^3}{(u+2k\pi)^3} \\ &\quad - (1 + \sum_{k\neq 0}\frac{u^4}{(u+2k\pi)^4} + \frac{u^4}{(u+2k\pi)^4} + \sum_{k\neq 0}\frac{u^4}{(u+2k\pi)^4}\sum_{k\neq 0}\frac{u^4}{(u+2k\pi)^4})\Big] \\ &= \frac{5!4!3!}{2!}\Big(2\sin\frac{u}{2}\Big)^6 \frac{1}{u^6}\Big[\beta_4(u)\beta_2(u) - \beta_3(u)\beta_3(u)o(u)\Big] \end{aligned}$$

Proof. Using the definition of Hermite B-splines

$$N_s(x) = \sum_{k=-(m-r)}^{m-r} c_k L_{2m,r,s}(x-k)$$

and given the integral representation of $I_{2m,r,s} = L_{2m,r,s}$, the following holds

$$N_s(x) = \frac{(-j)^s}{2\pi}\int_{-\infty}^{\infty} e^{-ju(m-r)}\Pi_{2m-1,r}(e^{ju})e^{jux}\frac{H_{r,s}(\alpha_{2m}(u))}{H_r(\alpha_{2m}(u))}du \tag{14}$$

It was proven by Lee and Sharma in (LeeSh76, Theorem 4) that the following holds

$$H_r\left(\frac{\Pi_n(\lambda)}{n!}\right) = (-1)^{\lfloor \frac{r}{2} \rfloor} C(n,r)(1-\lambda)^{(r-1)(n-r+1)}\Pi_{n,r}(\lambda) \tag{15}$$

with $C(n,r)$ the quantity

$$C(n,r) = \frac{1!2!\dots(r-1)!}{n!(n-1)!\dots(n-r+1)!}$$

Therefore,

$$\Pi_{2m-1,r}(e^{ju}) = (-1)^{\lfloor \frac{r}{2} \rfloor + m(r+1)} K(m,r)(1-e^{ju})^{-(r-1)(2m-r)} H_r\left(\frac{\Pi_{2m-1}(e^{ju})}{(2m-1)!}\right)$$

We have $\Pi_n(\lambda) = A_n(0;\lambda)(1-\lambda)^n$. Using this and (2), we have

$$H_r\left(\frac{\Pi_{2m-1}(e^{ju})}{(2m-1)!}\right) = H_r\left(\frac{A_{2m-1}(0;e^{ju})}{(2m-1)!}\right)(1-e^{ju})^{r(2m-r)}$$

while we previously established

$$H_r\left(\frac{A_{2m-1}(0;e^{ju})}{(2m-1)!}\right) = (e^{ju}-1)^r e^{-jur}(-j)^{r(2m-r)} H_r(\alpha_{2m}(u))$$

Combining the previous relations we have

$$\Pi_{2m-1,r}(e^{ju}) = (-1)^{\lfloor \frac{r}{2} \rfloor + m(r+1) + r}(-j)^{r(2m-r)} \frac{(e^{ju}-1)^{2m}}{e^{jur}} K(m,r) H_r(\alpha_{2m}(u)) \tag{16}$$

Eventually,

$$N_s(x) = \frac{(-j)^s}{2\pi}\int_{-\infty}^{\infty} e^{jux}\left(2\sin\frac{u}{2}\right)^{2m} K(m,r) H_{r,s}(\alpha_{2m}(u))du \tag{17}$$

$\square$