General ideas on interpolation

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V. Uhlmann's lab

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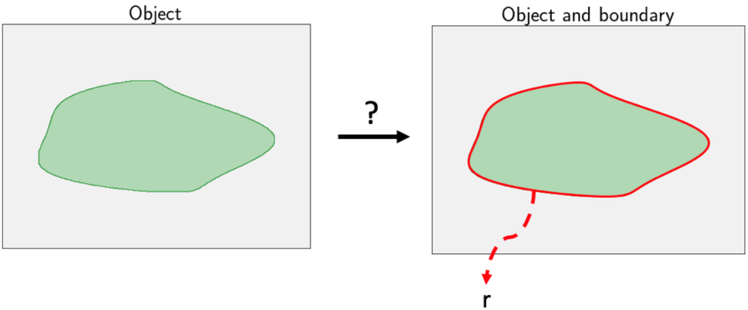
Objective



Objective: find mathematical model for the **boundary**

$$\mathbf{r} = \begin{bmatrix} r_1 \\ \vdots \\ r_d \end{bmatrix} \quad (1)$$

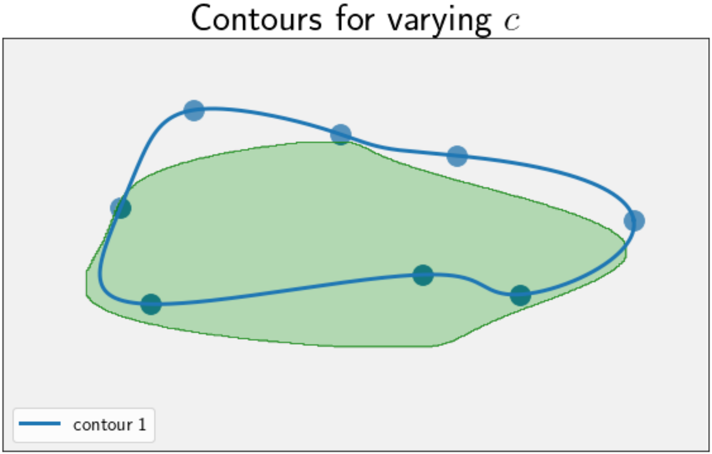
Generic model

$$\mathbf{r}(t) = \begin{bmatrix} r_1 \\ \vdots \\ r_d \end{bmatrix} = \sum_{k\in\mathbb{Z}} \begin{bmatrix} c_1(k) \\ \vdots \\ c_d(k) \end{bmatrix} \phi_k(t) \tag{2}$$

Vocab

- $(t_k = k)_{k\in\mathbb{Z}}$ **knots** locations
- $\mathbf{c}(k) := \begin{bmatrix} c_1(k) \\ \vdots \\ c_d(k) \end{bmatrix}$ are **control points**
- $(\phi_k)_{k\in\mathbb{Z}} :=$ set of **basis functions** or **generators**

Impact of varying $\mathbf{c}(k)$



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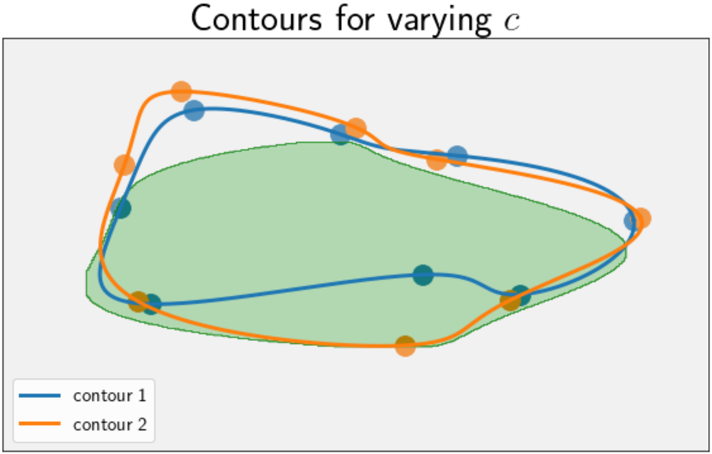

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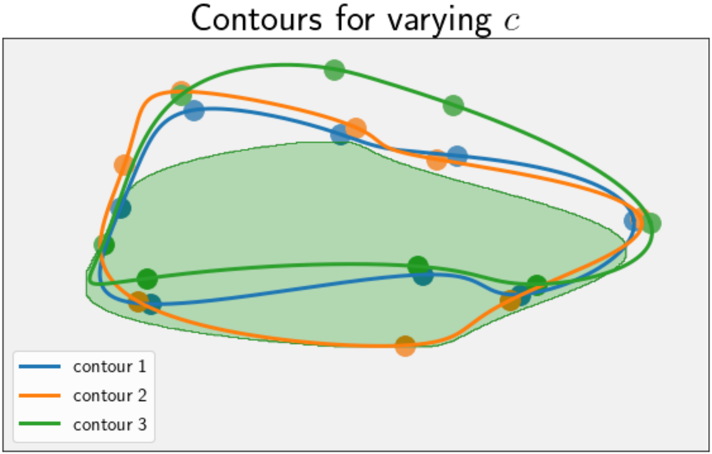

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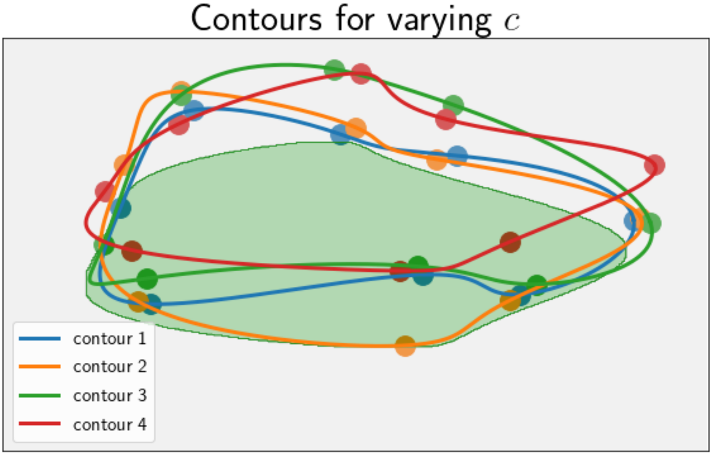
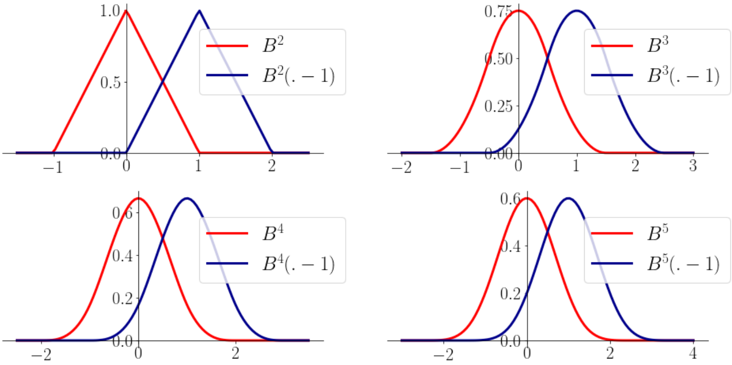


Impact of varying ϕ_k



$$(\phi_k)_{k\in\mathbb{Z}} = (\tau_k B)_{k\in\mathbb{Z}} \quad \text{with} \quad \tau_k f = f(. - k)$$

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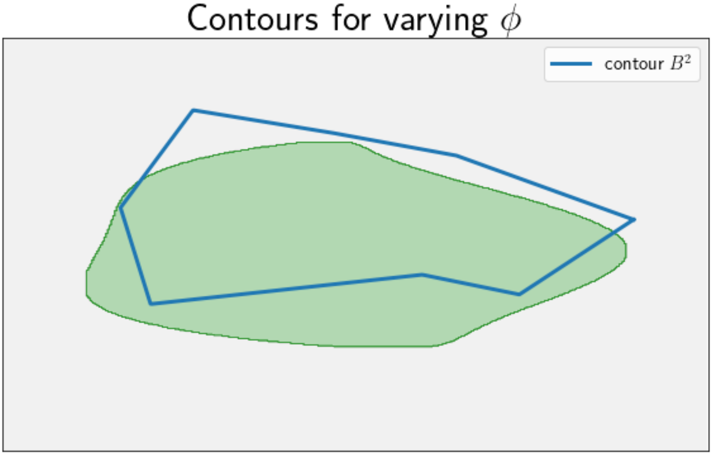

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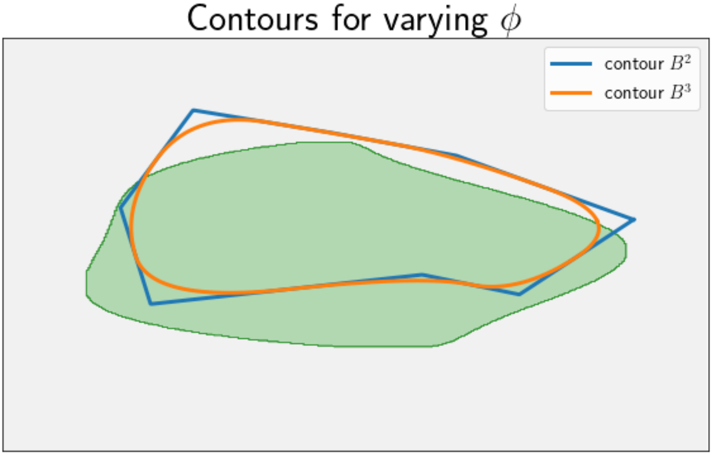

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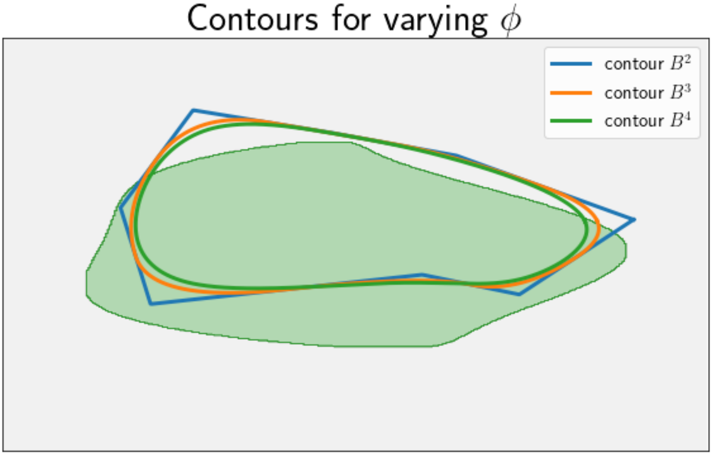

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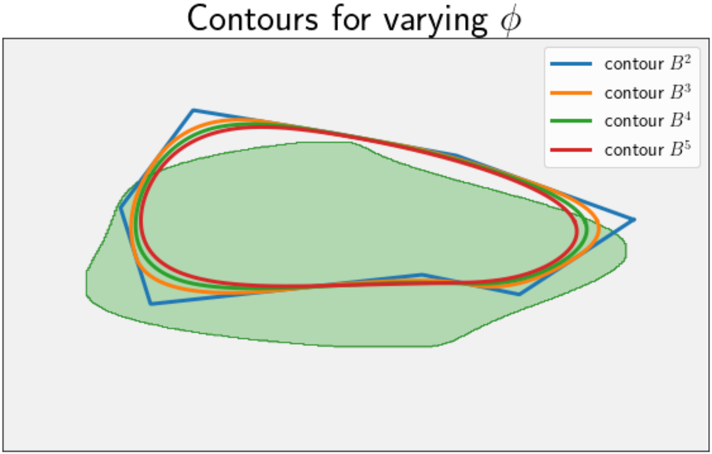
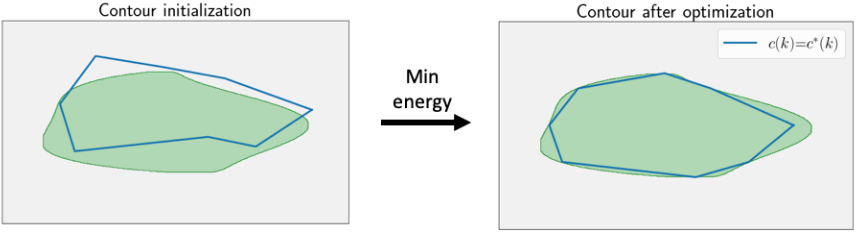


Approximation, interpolation

Boundary approximation Fixed generators ϕ_k, optimize the values of $\mathbf{c}(k)$.



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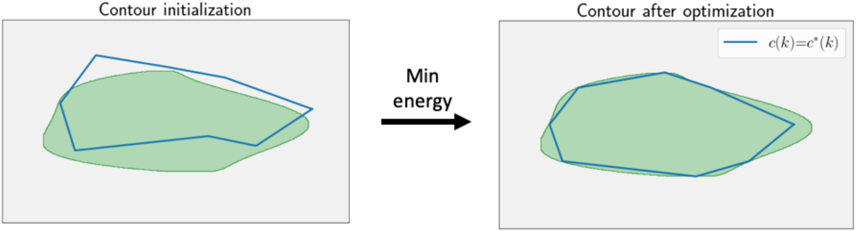
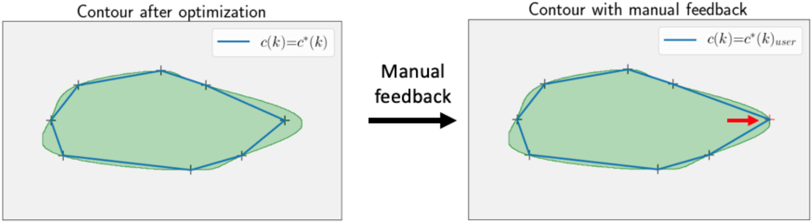


Feedback? Intuitive understanding of control points



Lagrange polynomials

Not.

- $\Pi_{<n} = \{p \in \mathbb{R}[X] |\, \deg(p) \leq n-1\}$
- $t_1 < \cdots < t_n$

Obj: reconstruct $g : \mathbb{R} \to \mathbb{R}$ from $g(t_1), \ldots, g(t_n)$.

Proposition 1 (Lagrange 1795)

$\exists ! f \in \Pi_{<n}$ $s.t$ $\forall i = 1, n,\ f(t_i) = g(t_i)$

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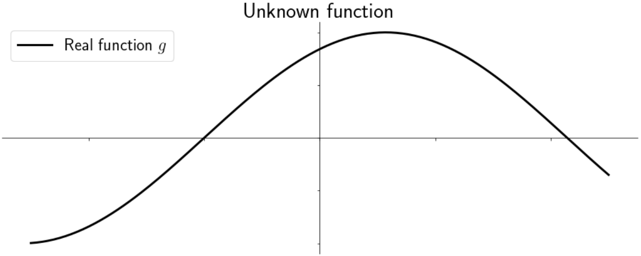

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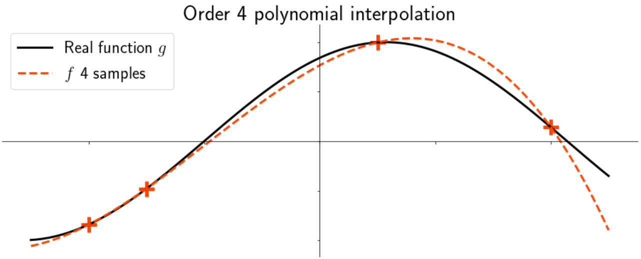

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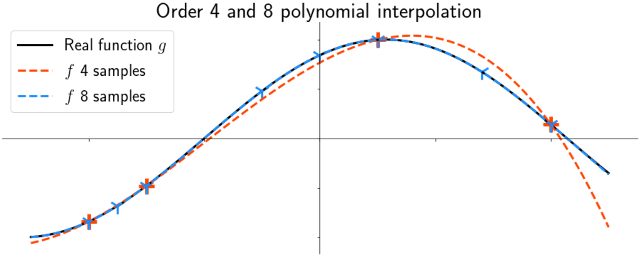

Osculatory interpolation

Not.

- f agrees with g at $\underbrace{(t, \ldots, t)}_{r \text{ times}}$ if $\forall m \in [\![0, r-1]\!], f^{(m)}(t) = g^{(m)}(t)$
- $t_1 = \cdots = t_{r_1} \leq \cdots \leq t_{r_1+\cdots+r_{d-1}+1} = \cdots = t_{r_1+\cdots+r_d}$ with d unique

Proposition 2 (Consequence theorem 1)

$\exists! f \in \Pi_{<n}$ $s.t$ $\forall i \in [\![0, d]\!]$, $f(t_i) = g(t_i), \ldots, f^{(r_i-1)}(t_i) = g^{(r_i-1)}(t_i)$

Divided differences

Not.

- $t_1 \leq \cdots \leq t_n$ with d unique

Definition 1

$[t_1, \ldots, t_n]g$ *leading coeff of* $f \in \Pi_{<n}$ *s.t* $f(t_i) = g(t_i)$

Examples

1. $[t_1]g = g(t_1)$
2. $[t_1, t_2]g = \begin{cases} \frac{g(t_2)-g(t_1)}{t_2-t_1} & \text{if } t_1 \neq t_2 \\ g'(t_1), & \text{otherwise} \end{cases}$

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Proposition 3

1. *if* $g \in \mathcal{C}^{(n-1)}$, $\exists \eta \in [t_1, t_n]$ *s.t* $[t_1, \ldots, t_n]g = \frac{g^{(n-1)}(\eta)}{n!}$
2. $f \in \Pi_{<n}$ *s.t* $f(t_i) = g(t_i)$ *is* $f(t) = \sum_{i=1}^{n} (t - t_1) \ldots (t - t_{i-1})[t_1, \ldots, t_i]g$.

Osculatory theorem

<u>Not.</u>

- $t_1 \leq \cdots \leq t_n$ with d unique

Theorem 1 (C. De Boor 1978)

Let $g \in \mathcal{C}^{(n)}$. *Then* $\forall t$, $g = f_n + (t - t_1)\ldots(t - t_n)[t_1, \ldots, t_n, t]g$.

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Example

- $t_1 = \cdots = t_n$, $g(t) = \sum_{i=0}^{n-1} \frac{g^{(i)}(t_1)}{i!}(t - t_1)^i + \frac{g^{(n)}(\eta_t)}{n!}(t - t_1)^n$

Runge: high-order interpolation can be bad

Not.

- $t_1 < \cdots < t_n$ uniform on $[-1, 1]$
- $g(t) = \frac{1}{1+25t^2}$

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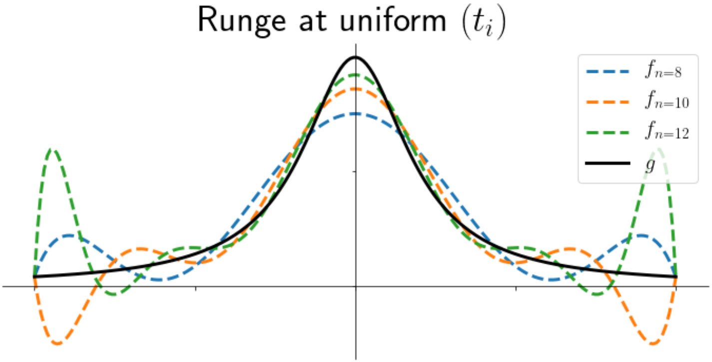

Runge explained

Not.

- $a = t_1 < \cdots < t_n = b$
- $\|f\| = \|f\|_{\infty,[a,b]} = \sup_{a \le t \le b} |f(t)|$
- $f_n = \displaystyle\sum_{i=1}^{n} g(t_i)\phi_i$, ϕ Lagrange pol.
- $\lambda_n(t) = \sum_{i=1}^{n} |\phi_i(t)|$

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Magnitude interpolator

$$\text{For} \quad t_i \le t \le t_{i+1}, |f_n(t)| \le \sup_{1 \le i \le n} |g(t_i)||\lambda_n(t)|$$
$$\le \|g\|\|\lambda_n\|$$
$$\text{hence} \quad \|f_n\| \le \|g\|\|\lambda_n\|$$

- Bound is sharp
- Uniform $(t_i) \implies \|\lambda_n\| \sim \frac{2^n}{en\log(n)}$.

Chebyshev sites are good

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Analysis of the error

$$\text{Osculatory thm} \quad |g(t) - f_n(t)| = |(t - t_1) \ldots (t - t_n)[t_1, \ldots, t_n, t]g|$$
$$\leq \|(. - t_1) \ldots (. - t_n)\| \frac{\|g^{(n)}\|}{n!}$$

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Idea: take $(t_i^c) = \operatorname{argmin}_{a \le t_1 \le \cdots \le t_n \le b} \|(. - t_1)\dots(. - t_n)\|$

1. $t_i^c = [(a+b) - (a-b)\cos(\frac{(2i-1)\pi}{2n})]/2$
2. $\|(. - t_1^c)\dots(. - t_n^c)\| = 2\left(\frac{b-a}{4}\right)^n$

Jackson's theorem

Not.

- $w(g,h) = \sup\{|g(t) - g(s)|, |t-s| \leq h\}$

Theorem 2

If $g \in \mathcal{C}^r([a..b])$ *and* $n > r+1$ *then*

$$\mathrm{dist}(g, \Pi_{<n}) \leq const_r \left(\frac{b-a}{n-1}\right)^r w(g^{(r)}, \frac{b-a}{2n-1-r}) \quad (3)$$

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Idea: break interval $[a, b]$ into pieces.

- Pros low order polynomials
- Cons continuity at break points

Interpolator I_2

Not.

- $a = t_1 < \cdots < t_n = b$, $g : [a, b] \to \mathbb{R}$
- $\Pi_{<n,\mathbf{t}} = \{$pp of order n with breaks at $t_2, \ldots, t_{n-1}\}$
- $\$_{2,\mathbf{t}} = \Pi_{<2,\mathbf{t}} \cap \mathcal{C}^0$

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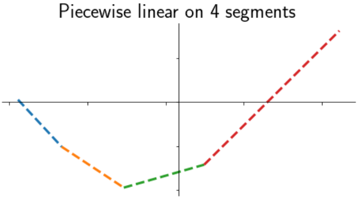

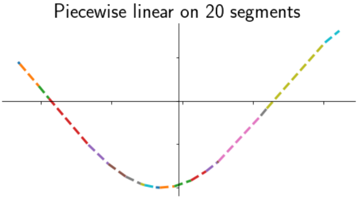

Result: $\exists! f \in \$_{2,\mathbf{t}}$ such that $f(t_i) = g(t_i)$

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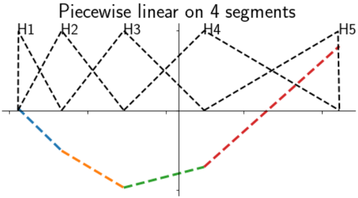

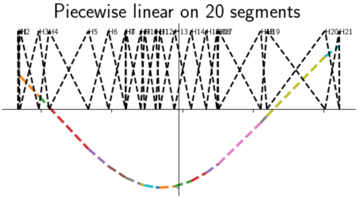

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$$\phi_i(t) = \begin{cases} \dfrac{t - t_{i-1}}{t_i - t_{i-1}} & \text{if } t_{i-1} < t \leq t_i \\ \dfrac{t_{i+1} - t}{t_{i+1} - t_i} & \text{if } t_i \leq t < t_{i+1} \end{cases}, \qquad I_2 g = \sum_{i=1}^{n} g(t_i)\phi_i \tag{4}$$

Approximation power of I_2

Not.

- $a = t_1 < \cdots < t_n = b$, $g : [a, b] \to \mathbb{R}$
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Properties

Proposition 4

1. *For* $f \in \$_{2,\mathbf{t}}$, $I_2 f = f$
2. $\|I_2 g\| \le \|g\|$

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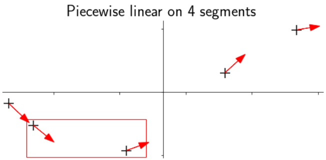
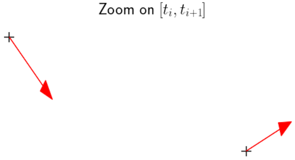
2. $\|I_2 g\| \leq \|g\|$
3. *If* $g \in \mathcal{C}^{(2)}$, $\|g - I_2 g\| \leq \frac{|\mathbf{t}|^2}{8}\|g''\|$
4. *If* $g \in \mathcal{C}^0$, $\|g - I_2 g\| \leq w(g, |\mathbf{t}|)$.

Cubic pieces

Not.

- $a = t_1 < \cdots < t_n = b$, $g : [a, b] \to \mathbb{R}$
- $g(t_1), s_1, \ldots, g(t_n), s_n$





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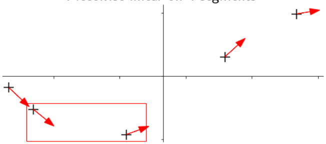

On $[t_i, t_{i+1}]$, 4 conditions for $f_{|[t_i,t_{i+1}]} \implies \exists! f_i \in \Pi_{<4}$ s.t $f_i = f_{|[t_i,t_{i+1}]}$

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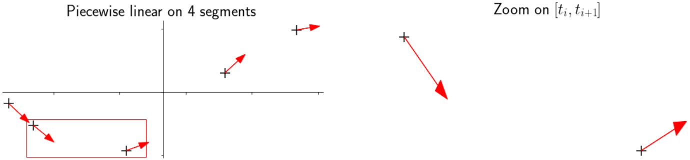

On $[t_i, t_{i+1}]$, 4 conditions for $f_{|[t_i,t_{i+1}]}$ $\implies$ $\exists! f_i \in \Pi_{<4}$ s.t $f_i = f_{|[t_i,t_{i+1}]}$ (Newton form)

$$\begin{aligned} f_{|[t_i,t_{i+1}]} &= [t_i]f_i + (t - t_i)[t_i, t_i]f_i + (t - t_i)^2[t_i, t_i, t_{i+1}]f_i \\ &+ (t - t_i)^2(t - t_{i+1})[t_i, t_i, t_{i+1}, t_{i+1}]f_i \end{aligned}$$

Explicit formula

$$
\begin{aligned}
f_{|[t_i, t_{i+1}]}(t) = g(t_i) + s_i(t - t_i)^2 + (3\frac{\Delta g(t_i)}{\Delta t_i^2} - \frac{s_i + s_{i+1}}{\Delta t_i})(t - t_i)^3 \\
+ (\frac{s_i + s_{i+1}}{\Delta t_i^2} - 2\frac{\Delta g(t_i)}{\Delta t_i^3})(t - t_i)^4
\end{aligned}
$$

4 cases

- case 1 $(s_i) = (g'(t_i))$ i.e **Hermite** interp.
  if $g \in \mathcal{C}^{(4)}, \|g - f\| \leq \dfrac{1}{384}|\mathbf{t}|^4 \|g^{(4)}\|$ De Boor 1978

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- case 2 Enforce $f \in \mathcal{C}^{(2)}$ i.e $f''_{|[t_{i-1},t_i]}(t_i) = f''_{|[t_i,t_{i+1}]}(t_i) \implies$ tridiagonal linear system for $s_2, \ldots, s_{n-1}$.

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  - case 2a $s_1 = g'(t_1), s_n = g'(t_n)$ **Complete** cubic spline.

$$\|g - f\| = \mathcal{O}(|t|^4) \quad \text{De Boor 1978}$$

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Examples

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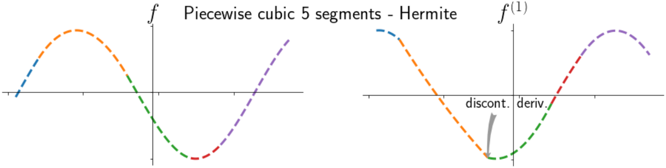

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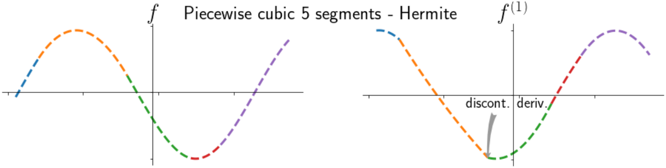
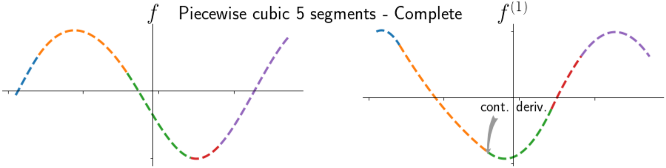


2. $n-2$ equations $f''(t_i) = f''(t_{i+1})$. Cubic spline interpolation.



Formal definition B-splines

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- $\cdots \leq t_i \leq t_{i+1} \leq \cdots$ finite or infinite

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Definition 2 C. De Boor 1978

The j^{th} (normalized B-spline) of order k is

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Definition 3 Schoenberg 1973

The j^{th} B-spline of order k is given by

$$M_{j,k,\boldsymbol{t}}(t) = \frac{1}{k}[t_j, \ldots, t_{j+k}](. - t)_+^{k-1} \tag{6}$$

First properties

Proposition 5

1. (pp function) $B_{j,k,\boldsymbol{t}} \in \Pi_{<k,\boldsymbol{t}}$
2. (rec. relation)

$$B_{j,k} = w_{j,k} B_{j,k-1} + (1 - w_{j+1,k}) B_{j+1,k-1} \qquad w_{jk}(t) = \frac{t - t_j}{t_{j+k} - t_j}$$

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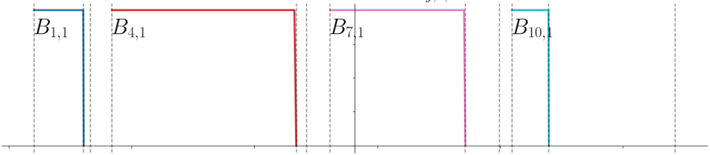
BSPLINE order 1 $B_{j,1,t}$



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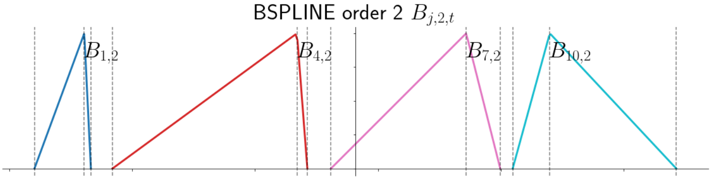

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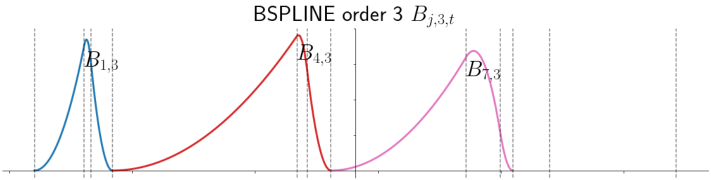

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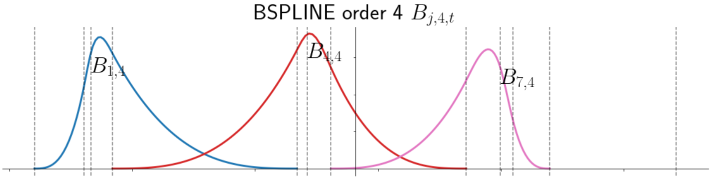

More properties

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A spline is a linear combination of B-splines.

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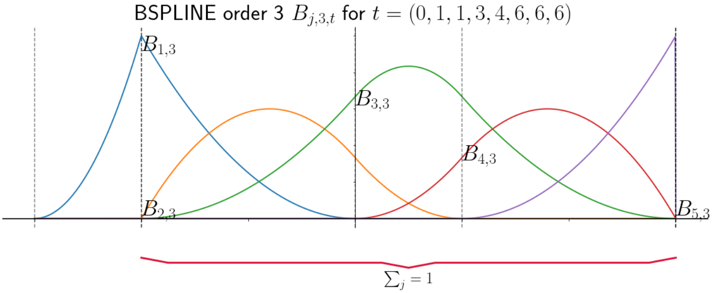
1. (partition unity) $\sum_{j\in\mathbb{Z}} B_{j,k} = 1$
2. (Riesz-basis norm sup) $\exists D_k \in (0,1)$ such that $\forall \boldsymbol{c}$ bounded,
$$D_k \|\boldsymbol{c}\| \leq \|\sum_j c_j B_{j,k}\| \leq \|\boldsymbol{c}\| \quad \text{De Boor 1976}$$
3. (Marsden's identity) For any $\tau \in \mathbb{R}$,
$$(. - \tau)^{k-1} = \sum_j \psi_{j,k}(\tau) B_{j,k} \quad \psi_{j,k}(\tau) = (t_j - \tau)\cdots(t_{j+k-1} - \tau)$$

Multiple knots



Applications

1. In practice, choose $\boldsymbol{t} = \mathbb{Z}_r$ with $r \geq 1$.
   - Landmark paper by Schoenberg
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$$B_{0,1} = \beta_+^0, \quad B_{j,n} = \beta_+^n(-.j)\underbrace{\beta_+^0 * \cdots * \beta_+^0}_{(n-1)\text{times}}(. - j) \tag{7}$$

$$\sum_{j \in \mathbb{Z}} c[j]\beta^n(. - j) \quad \beta^n = \beta_+(. + \frac{n+1}{2}) \tag{8}$$

Allows for efficient algorithms and Fourier analysis,

$$\hat{\beta}_+^n(w) = \left(\frac{1 - e^{-jw}}{jw}\right)^n$$

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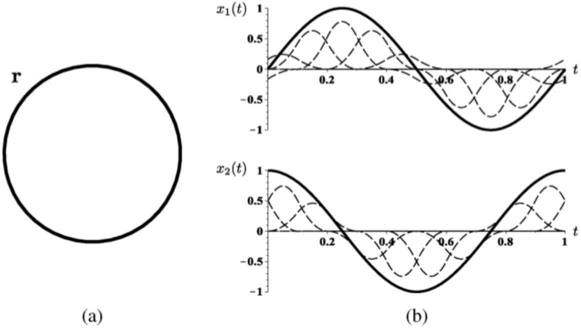

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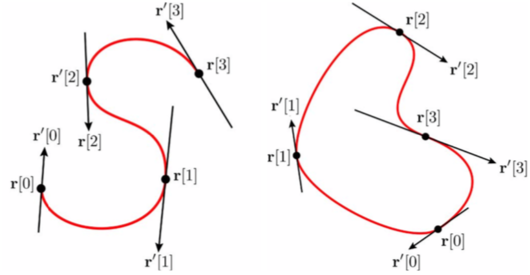

(a) (b)

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   - Space of reproducible functions

$$V = \operatorname{span}(\phi_j) \cap L_2 = \{\sum_{j\in\mathbb{Z}} c[j]\phi_j | c[j] \in \mathbb{R}\} \cap L_2$$

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