General ideas on interpolation (B-splines)

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V. Uhlmann's lab

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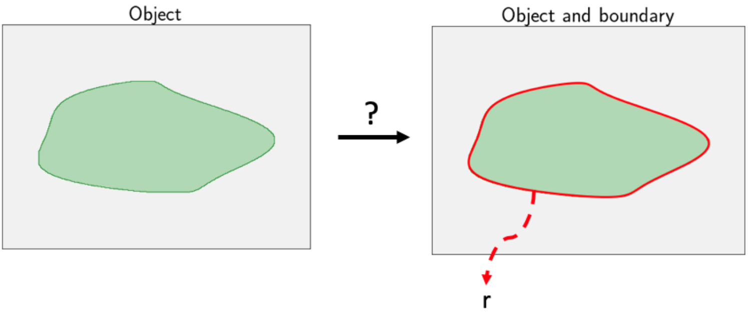
Objective



Objective: find mathematical model for the **boundary**

$$\mathbf{r} = \begin{bmatrix} r_1 \\ \vdots \\ r_d \end{bmatrix} \tag{1}$$

Generic model

$$\mathbf{r}(t) = \begin{bmatrix} r_1 \\ \vdots \\ r_d \end{bmatrix} = \sum_{k\in\mathbb{Z}} \begin{bmatrix} c_1(k) \\ \vdots \\ c_d(k) \end{bmatrix} \phi_k(t) \tag{2}$$

<u>Vocab</u>

- $(t_k = k)_{k\in\mathbb{Z}}$ **knots** locations
- $\mathbf{c}(k) := \begin{bmatrix} c_1(k) \\ \vdots \\ c_d(k) \end{bmatrix}$ are **control points**
- $(\phi_k)_{k\in\mathbb{Z}}$:= set of **basis functions** or **generators**

Summary

Not.

- f agrees with g at t r times if $f^{(j)}(t) = g^{(j)}(t), j = 0, \ldots, r-1$
- $t_1 \leq \cdots \leq t_n$
- $f_n \in \Pi_{<n}$ unique pol. order n that agrees with g at $(t_i)_{i=1}^n$.

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$[t_1, \ldots, t_n]g$ *leading coeff of* f_n

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Examples

1. $[t_1]g = g(t_1)$
2. $[t_1, t_2]g = \begin{cases} \frac{g(t_2)-g(t_1)}{t_2-t_1} & \text{if } t_1 \neq t_2 \\ g'(t_1), & \text{otherwise} \end{cases}$
3. $[t_1, \ldots, t_{n+1}]g = \frac{[t_2,\ldots,t_{n+1}]g-[t_1,\ldots,t_n]g}{t_{n+1}-t_1}$ if $t_1 \neq t_{n+1}$

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Theorem 1 ([Bo01], p7)

Let $g \in \mathcal{C}^n$. *Then*

$$\forall t, \quad g(t) = f_n(t) + (t - t_1) \dots (t - t_n)[t_1, \dots, t_n, t]g \tag{3}$$

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Theorem 2 (Jackson,[Ri69], p23)

Let $g \in \mathcal{C}^r([a..b]), n > r + 1$. *Then*

$$\operatorname{dist}(g, \Pi_{<n}) \leq const_r w(g^{(r)}, \frac{b-a}{2n-1-r})\left(\frac{b-a}{n-1}\right)^r \tag{4}$$

Formal definition B-splines

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Two different normalizations

Definition 2 ([Bo01], p87)

The j^{th} (normalized) B-spline of order k is

$$B_{j,k,\boldsymbol{t}}(t) = (t_{j+k} - t_j)[t_j, \ldots, t_{j+k}](. - t)_+^{k-1} \tag{5}$$

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Definition 3 ([CS66])

The j^{th} B-spline of order k is given by

$$M_{j,k,\boldsymbol{t}}(t) = k[t_j, \ldots, t_{j+k}](. - t)_+^{k-1} \tag{6}$$

First properties

Proposition 1 ([Bo01], p90–91)

1. *(rec. relation)*

$$B_{j,k} = \frac{t - t_j}{t_{j+k-1} - t_j} B_{j,k-1} + \frac{t_{j+k} - t}{t_{j+k} - t_j} B_{j+1,k-1}$$

2. *(pp function)* $B_{j,k,\boldsymbol{t}} \in \Pi_{<k,\boldsymbol{t}}$
3. *(supp. and pos)*
   - $B_{j,k} = 0$ *outside* $[t_j, t_{j+k}]$
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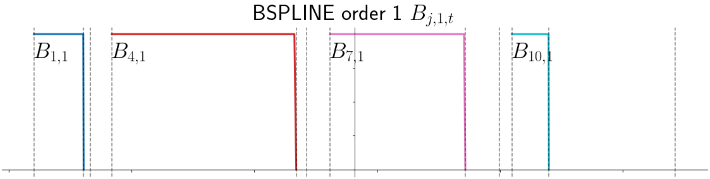

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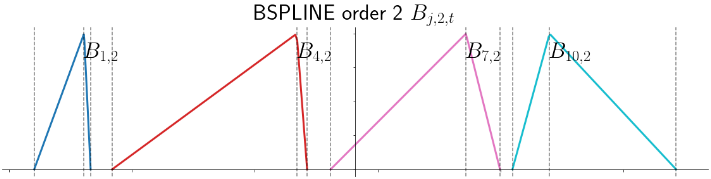

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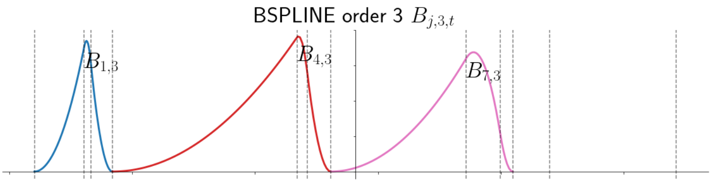

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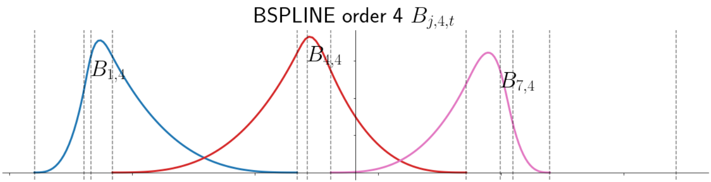

Why B-splines?

Not.

- $\boldsymbol{t}$ nondecreasing sequence, $\boldsymbol{\xi} \subset \boldsymbol{t}$ increasing sequence

Definition 4 ([Bo01], p93)

The collection of polynomial splines of order k with knot sequence $\boldsymbol{t}$ is

$$\mathcal{S}_{k,\boldsymbol{t}} = \{ \sum_{j=-\infty}^{\infty} c(j) B_{j,k,\boldsymbol{t}} | \boldsymbol{c} \in \mathbb{R}^{\mathbb{Z}} \} \subset \Pi_{<k,\boldsymbol{\xi}}$$

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In ([Bo76]) "B-splines are truly basic splines: B-splines express the essentially local, but not completely local, character of splines; certain facts about splines take on their most striking form when put into B-spline terms, and many theorems about splines are most easily proved with the aid of B-splines;"

Basis for polynomials

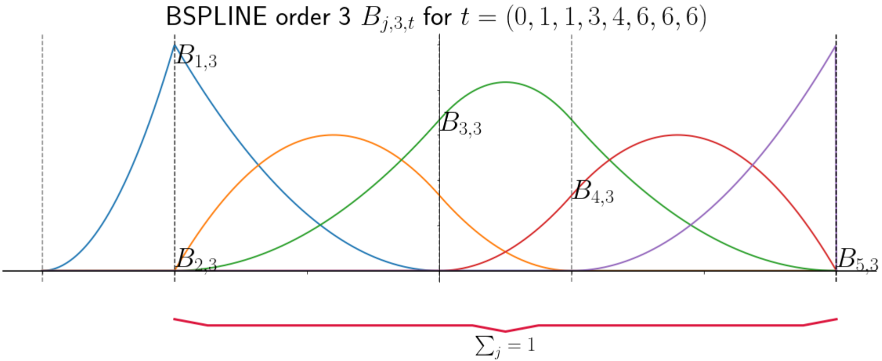
1. ([Ma70],[Bo01] p95) For any $\tau \in \mathbb{R}$,

$$(t-\tau)^{k-1} = \sum_{j=-\infty}^{\infty} \psi_{j,k}(\tau) B_{j,k}(t)$$

$$\text{where} \quad \psi_{j,k}(\tau) = (t_j - \tau) \cdots (t_{j+k-1} - \tau)$$

2. Reproduces $\Pi_{<k}$ i.e $\Pi_{<k} \subset \mathcal{S}_{k,\boldsymbol{t}}$
3. In particular, $\displaystyle\sum_{j=-\infty}^{\infty} B_{j,k} = 1$

Partition unity



Basis for $\Pi_{<k,\boldsymbol{\xi},\boldsymbol{\nu}}$

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- $\boldsymbol{\nu}$ sequence s.t $\nu_j = k - \mathrm{card}\{l | t_l = \xi_j\}$
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$\{B_{j,k,\boldsymbol{t}}\}_{j=-\infty}^{\infty}$ *is a basis for* $\Pi_{<k,\boldsymbol{\xi},\boldsymbol{\nu}}$ *hence* $\mathcal{S}_{k,t} = \Pi_{<k,\boldsymbol{\xi},\boldsymbol{\nu}}$

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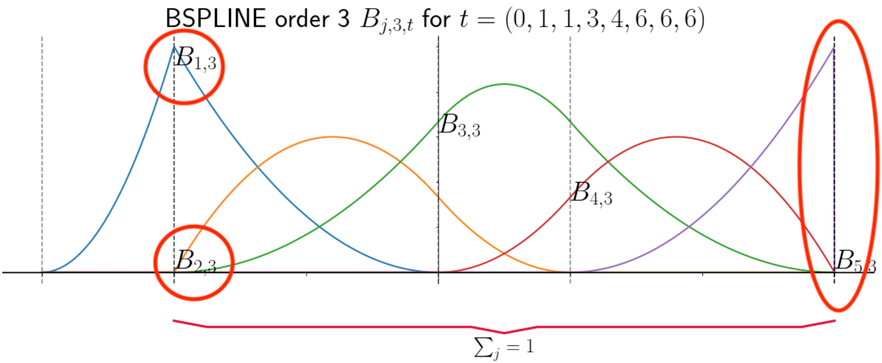
1. $f \in \mathcal{S}_{k,\boldsymbol{t}}$. $f_{|\mathcal{V}(\xi_j)} \in \mathcal{C}^{k-(k-\nu_j)-1}$
2. $B_{j,k,\mathbb{Z}} \in \mathcal{C}^{k-2}$, $B_{j,k,\mathbb{Z}_{\boldsymbol{r}}} \in \mathcal{C}^{k-r-1}$

Knots multiplicity



Good condition for a basis

Riesz-basis $\|\cdot\|_\infty$ ([Bo73],[Bo01] p132)

$\exists D_k > 0$ independent of $\boldsymbol{t}$ such that

$$\forall \boldsymbol{c} \in l^\infty, \quad D_k \|\boldsymbol{c}\|_{l^\infty} \le \left\| \sum_{j=-\infty}^{\infty} c_j B_{j,k,\boldsymbol{t}} \right\|_{L^\infty} \le \|\boldsymbol{c}\|_{l^\infty}$$

Applications

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   - [Un99]

$$\beta^0(t) = B_{0,1}(t - \frac{1}{2}), \quad \beta^n(t) = \underbrace{\beta^0 * \cdots * \beta^0}_{(n-1)\text{times}}(t) \tag{7}$$

Signal represented as

$$s(t) = \sum_{k=-\infty}^{\infty} c(k)\beta^n(t-k) \tag{8}$$

Allows for efficient algorithms and Fourier analysis, with $\hat{\beta}^n(w) = \left(\frac{\sin(\frac{w}{2})}{\frac{w}{2}}\right)^{n+1}$.

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2. Critical choice of **generators** $\{\phi_j\}_{j \in \mathbb{Z}}$
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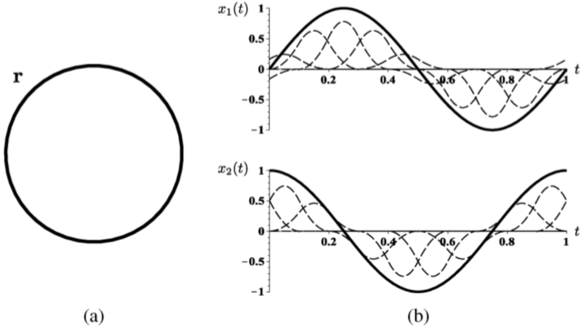
$$r = \sum_i \begin{bmatrix} c_1[j] \\ c_2[j] \end{bmatrix} \phi_j \quad \text{with } \phi_j = \phi(. - j) \quad \text{with} \quad \phi(t) = \lambda_0 \beta_\alpha(t + \frac{3}{2})$$



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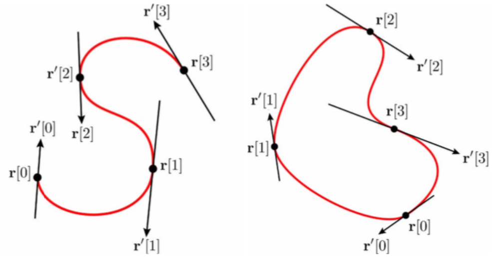
    2. Uhlmann et al. Hermite snakes with control of tangents, *IEEE*. Vol 25, 2016.

$$r = \sum_{j=-\infty}^{\infty} \begin{bmatrix} c_1^1[j] \\ c_2^1[j] \end{bmatrix} \phi_j^1 + \begin{bmatrix} c_1^2[j] \\ c_2^2[j] \end{bmatrix} \phi_j^2 \quad \text{with } \phi_j^1, \phi_j^2 \in \mathcal{S}_{4,\mathbb{Z}_2}$$



(a) (b)

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3. Things to consider
   - Space of reproducible functions

$$V = \mathrm{span}(\phi_j) \cap L_2 = \{\sum_{j\in\mathbb{Z}} c[j]\phi_j | c[j] \in \mathbb{R}\} \cap L_2$$

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   - Desired properties in CAGD
     1. Affine invariance i.e $s \in V \implies As + b \in V$. True if $\sum_j \phi_j = 1$
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   - Approximation power as number of control points increases

References I

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