Cardinal Interpolation and Spline Functions. III. Cardinal Hermite Interpolation*

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ABSTRACT

The results of item [9] in our list of References, concerning cardinal spline interpolation of data of power growth are here extended to the case of Hermite interpolation. Let $m \geqslant 2$, and $1 \leqslant r \leqslant m$. Given are now r sequences $y = (y_\nu)$, $y' = (y_\nu'), \ldots, y^{(r-1)} = (y_\nu^{(r-1)})$ $(-\infty < \nu < \infty)$, and the objective is to find a cardinal spline function $S(x)$ of degree $2m - 1$, having its knots at the integers, and satisfying the conditions $S(\nu) = y_\nu$, $S'(\nu) = y_\nu', \ldots, S^{(r-1)}(\nu) = y_\nu^{(r-1)}$ for all ν. Here we must assume that the knots of $S(x)$ are r-fold, which means that $S(x) \in C^{2m-r-1}(-\infty, \infty)$. If $r = 1$ we have the problem discussed in [9]. In the present paper it is shown that the interpolation problem has a unique solution that grows like a power of $|x|$, provided that all data grow like a power of $|\nu|$. An analogue of the Lagrange formula is also available. Essential use is made of a theorem of Gantmacher and Krein concerning the eigenvalues of oscillation matrices (see [1] and [2; Satz 6, 100]).

INTRODUCTION AND STATEMENT OF RESULTS

The present paper is based on the following apparently new property of the Pascal triangle: We consider the infinite matrix of the binomial coefficients

$$P = \left\| \binom{i}{j} \right\| \qquad (i, j = 0, 1, 2, \ldots)$$

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and form the characteristic matrix

$$P - \lambda I = \left\| \binom{i}{j} - \lambda \delta_{ij} \right\|.$$

Let r and d be natural numbers and consider its square submatrix of order $r + d$

$$\Delta_{r,d}(\lambda) = \left\| \binom{i}{j} - \lambda \delta_{ij} \right\|$$

where $i = r, r + 1, \ldots, d + 2r - 1$, and $j = 0, 1, \ldots, r + d - 1$. *Then the equation* $|\Delta_{r,d}(\lambda)| = 0$ *is a reciprocal equation of degree* d *having only real and simple roots all having the sign of* $(-1)^r$.

For instance

$$|\Delta_{2,3}(\lambda)| = \begin{vmatrix} 1 & 2 & 1-\lambda & 0 & 0 \\ 1 & 3 & 3 & 1-\lambda & 0 \\ 1 & 4 & 6 & 4 & 1-\lambda \\ 1 & 5 & 10 & 10 & 5 \\ 1 & 6 & 15 & 20 & 15 \end{vmatrix} = 0$$

is a reciprocal cubic equation having only simple and positive roots. This theorem and its proof form our main result. Now we proceed to the subject of interpolation.

Let k be a natural number and let S_k denote the class of cardinal spline functions of degree $k - 1$, or order k, having their knots at the integers ν if k is even, or at the halfway points $\nu + \frac{1}{2}$ if k is odd. Here we assume the knots to be simple knots, hence $S_k \subset C^{k-2}$.

Let $y = (y_\nu)$ be a prescribed sequence of numbers, $-\infty < \nu < \infty$. The problem of finding a function $F(x)$ $(-\infty < x < \infty)$ satisfying the relations

$$F(\nu) = y_\nu \quad \text{for all } \nu, \tag{1}$$

and such that $F(x)$ belongs to a preassigned linear space $\mathscr{S}$, is called a *cardinal interpolation problem* and denoted by the symbol

$$\text{C.I.P.}\ (y; \mathscr{S}). \tag{2}$$

In [9] Schoenberg considered, for each $\gamma \geqslant 0$, the class of functions

$$F_\gamma = \{F(x); F(x) \in C \text{ and } F(x) = O(|x|^\gamma) \text{ as } x \to \pm\infty\}, \tag{3}$$

and the class of sequences

$$Y_\gamma = \{y = (y_\nu);\, y_\nu = O(|\nu|^\gamma) \text{ as } \nu \to \pm\infty\}. \tag{4}$$

A main result of [9] is that *the*

$$\text{C.I.P. } (y;\, S_k \cap F_\gamma) \tag{5}$$

has solutions if and only if

$$y \in Y_\gamma \tag{6}$$

in which case the solution of (5) *is unique.*

In the present paper we extend this result to the case of Hermite interpolation, first mentioned in [6, p. 166], assuming throughout that

$$k = 2m \tag{7}$$

is an even number. The basic definition is as follows. Let r be a natural number such that

$$r \leqslant m \tag{8}$$

and let $S_{2m,r}$ denote the class of spline functions of degree $2m - 1$, with their knots at the integers ν and being knots of multiplicity r. This means that we require that

$$S_{2m,r} \subset C^{2m-r-1}. \tag{9}$$

Having at our disposal additional parameters due to the relaxed continuity requirement (9), we now try to solve the following

CARDINAL HERMITE INTERPOLATION PROBLEM. *The r sequences of numbers*

$$y = (y_\nu),\ y' = (y_\nu'),\ldots,\ y^{(r-1)} = (y_\nu^{(r-1)}) \tag{10}$$

being prescribed, we wish to find a function $F(x)$ satisfying the relations

$$F(\nu) = y_\nu,\ F'(\nu) = y_\nu',\ \ldots,\ F^{(r-1)}(\nu) = y_\nu^{(r-1)} \quad \textit{for all } \nu, \tag{11}$$

and such that $F(x)$ belongs to some preassigned linear space $\mathscr{S}$. We denote this problem by the symbol

$$\text{C.H.I.P. } (y, y', \ldots, y^{(r-1)};\, \mathscr{S}). \tag{12}$$

Generalizing the class $F_\gamma = F_{\gamma,1}$ we define the new class

$$F_{\gamma,r} = \{F(x);\, F(x) \in C^{r-1} \text{ and } F^{(s)}(x) = O(|x|^\gamma) \text{ as } x \to \pm\infty$$
$$\text{for} \quad s = 0, 1, \ldots, r-1\}. \tag{13}$$

We may now state

THEOREM 1. *The*

$$\text{C.H.I.P. } (y, \ldots, y^{(r-1)};\, S_{2m,r} \cap F_{\gamma,r}) \tag{14}$$

has solutions if and only if

$$y^{(s)} \in Y_\gamma \quad (s = 0, 1, \ldots, r-1) \tag{15}$$

in which case the problem (14) *has a unique solution.*

Theorem 1 shows that the interpolation relations (11) define a one-to-one correspondence between the elements of the following two classes

$$S_{2m,r} \cap F_{\gamma,r} \leftrightarrow Y_\gamma \times Y_\gamma \times \cdots \times Y_\gamma = (Y_\gamma)^r, \tag{16}$$

and this for every $\gamma \geqslant 0$. Clearly both classes (16) are expanding as γ increases. Between the narrowest classes, for $\gamma = 0$, we have the correspondence

$$S_{2m,r} \cap F_{0,r} \leftrightarrow (Y_0)^r. \tag{17}$$

We now turn to a more precisely structured description of the correspondence (17). For this we need the following new classes

$$\mathscr{L}_{p,r} = \{F(x);\, F \in C^{r-1},\, F^{(s)}(x) \in \mathscr{L}_p \text{ for } s = 0, 1, \ldots, r-1\}, \tag{18}$$

where $1 \leqslant p \leqslant \infty$. This is also an expanding class as p increases, the most inclusive being

$$\mathscr{L}_{\infty,r} = F_{0,r}. \tag{19}$$

The refinement of the mapping (17) is described by our next

THEOREM 2. *If* $1 \leqslant p \leqslant \infty$ *then the*

$$\text{C.H.I.P. } (y, \ldots, y^{(r-1)};\, S_{2m,r} \cap \mathscr{L}_{p,r}) \tag{20}$$

has solutions if and only if

$$y^{(s)} \in l_p \quad (s = 0, 1, \ldots, r-1) \tag{21}$$

and then the problem (20) *has a unique solution.*

In other words: The relations (11) define a 1-1 mapping

$$S_{2m,r} \cap \mathscr{L}_{p,r} \leftrightarrow (l_p)^r \tag{22}$$

which reduces to relation (17) if $p = \infty$.

Let us now consider the following so-called *unit-data*: Let $s = 0, 1, \ldots, r-1$ and let

$$y_\nu^{(\rho)} = \delta_\nu \cdot \delta_{\rho-s} \quad \text{for all } \nu, \quad \rho = 0, \ldots, r-1, \tag{23}$$

where $\delta_\nu = 1$ if $\nu = 0$ and $\delta_\nu = 0$ if $\nu \neq 0$. Explicitly

$$y_\nu^{(\rho)} = \begin{cases} 0 & \text{if } \rho \neq s, \\ \delta_\nu & \text{if } \rho = s. \end{cases} \tag{24}$$

In words: *All* $y_\nu^{(\rho)} = 0$ *except that* $y_0^{(s)} = 1$.

These r sequences are all in l_1, for $s = 0, \ldots, r-1$, and by Theorem 2 we are assured of the existence of a spline function

$$L_{2m,r,s}(x) \in S_{2m,r} \cap \mathscr{L}_{1,r} \tag{25}$$

interpolating the data (23), hence

$$L^{(\rho)}_{2m,r,s}(\nu) = \delta_\nu \cdot \delta_{\rho-s}. \tag{26}$$

THEOREM 3. *There are positive constants A and α, depending on m and r, such that*

$$|L^{(\rho)}_{2m,r,s}(x)| \leqslant A \exp(-\alpha|x|) \quad \textit{for all real } x, \tag{27}$$

while s and ρ are $= 0, 1, \ldots, r-1$.

To simplify notations we write

$$L_s(x) = L_{2m,r,s}(x) \quad (s = 0, \ldots, r-1). \tag{28}$$

From the relations (26) it is clear that the expression

$$F(x) = \sum_{-\infty}^{\infty} y_\nu L_0(x-\nu) + \sum_{-\infty}^{\infty} y_\nu' L_1(x-\nu) + \cdots + \sum_{-\infty}^{\infty} y_\nu^{(r-1)} L_{r-1}(x-\nu) \tag{29}$$

furnishes a *formal* solution of the interpolation problem (11). That it furnishes more than a formal solution is shown by our last

THEOREM 4. *The spline function $S(x)$ which is the unique solution of the problems described by Theorem* 1 *and Theorem* 2, *respectively, is explicitly given by the Lagrange-Hermite expansion*

$$S(x) = \sum_{-\infty}^{\infty} \{y_\nu L_0(x-\nu) + y_\nu' L_1(x-\nu) + \cdots + y_\nu^{(r-1)} L_{r-1}(x-\nu)\}, \tag{30}$$

which converges locally uniformly in x.

We have assumed that (8) holds, or $r \leqslant m$. Why? The reason is a simple one: We know that if $S(x) \in S_{2m,r}$ then $S(x) \in C^{2m-r-1}$. To solve a C.H.I.P. by $S(x)$ evidently requires that $S(x) \in C^{r-1}$. We therefore need the inequality $r - 1 \leqslant 2m - r - 1$ which reduces to (8). Thus for each m we can choose an r in the range

$$1 \leqslant r \leqslant m. \tag{31}$$

Let us briefly examine the extreme cases. The case when $r = 1$ was discussed in [9]. If $r = m$ then something interesting happens: *The*

$$\text{C.H.I.P. } (y, \ldots, y^{(r-1)}; S_{2m,m}) \tag{32}$$

has a unique solution no matter what the nature of the sequences $y, y^1, \ldots, y^{(r-1)}$ might be.

Proof. In the interval $[\nu, \nu+1]$ the spline function $S(x)$ is a polynomial of degree $2m-1$ that is uniquely defined by the 2-point Hermite interpolation problem

$$S^{(s)}(\nu) = y_\nu^{(s)},$$
$$S^{(\nu)}(\nu+1) = y_{\nu+1}^{(s)}, \qquad (s = 0, 1, \ldots, m-1). \tag{33}$$

Conversely, the solutions of this sequence of Hermite interpolation problems furnish the components of the solution which is an element of $S_{2m,m}$.

The simplest example of the problem (32) is for $m = r = 1$ when we have linear interpolation between consecutive data.

Here we do not discuss estimates of the remainder of the Lagrange-Hermite interpolation formula (30), or Eq. (10.3) below. However, a rough quantitative idea of the effect of the value of r on the size of the error, is obtained in Sec. 11, where the interpolation of $f(x) = x^{2m}$ is discussed in some detail. An entirely similar fast decrease of the L_∞-norm of the cardinal monospline (of degree n and class C^k) of least L_∞-norm as k decreases, was observed in [10].

I. THE PROOF OF UNICITY IN THEOREMS 1 AND 2

1. *A Lemma on Cardinal Spline Interpolation*

Let us construct a solution of the

$$\text{C.H.I.P. } (y, \ldots, y^{(r-1)}; S_{2m,r}). \tag{1.1}$$

Let

$$P(x) \in \pi_{2m-1} \quad \text{such that} \quad P^{(s)}(0) = y_0{}^{(s)}, \quad P^{(s)}(1) = y_1{}^{(s)}$$
$$(s = 0, 1, \ldots, r-1). \tag{1.2}$$

Evidently $P(x)$ depends linearly on $2m - 2r$ parameters. If we define

$$S(x) = P(x) \quad \text{in} \quad [0, 1] \tag{1.3}$$

then we claim that the solution $S(x)$ of (1.1) is thereby uniquely defined. Indeed we may write

$$S(x) = P(x) + \sum_{s=0}^{r-1} c_1{}^{(s)}(x-1)_+{}^{2m-s-1} + \sum_0^{r-1} c_2{}^{(s)}(x-2)_+{}^{2m-s-1} + \cdots$$
$$+ \sum_0^{r-1} c_0{}^{(s)}(-x)_+{}^{2m-s-1} + \sum_0^{r-1} c_{-1}^{(s)}(-x-1)_+{}^{2m-s-1} + \cdots, \tag{1.4}$$

with coefficients $c_\nu{}^{(s)}$ yet to be determined. It is readily seen (compare [9, Introduction, Lemma 1]) that the r coefficients $c_1{}^{(s)}$ are uniquely defined by the conditions

$$S^{(s)}(2) = y_2{}^{(s)} \qquad (s = 0, \ldots, r-1).$$

These having been determined, we find the $c_2^{(s)}$ from $S^{(s)}(3) = y_3^{(s)}$ a.s.f. In the opposite direction we determine the $c_0^{(s)}$ from $S^{(s)}(-1) = y_{-1}^{(s)}$, a.s.f.

This establishes

LEMMA 1. *Given r sequences $(y_\nu), (y_\nu'), \ldots, (y_\nu^{(r-1)})$, the problem* (1.1) *always has solutions forming a linear manifold of the dimension $2m - 2r$.*

2. *The Class $S_{2m,r}^0$ and the Eigensplines* (Theorem 5)

We define the class of spline functions

$$S_{2m,r}^0 = \{S(x);\, S(x) \in S_{2m,r},\, S^{(s)}(\nu) = 0 \text{ for all } \nu \text{ and } s = 0, 1, \ldots, r-1\}. \tag{2.1}$$

This is identical with the class of solutions of the problem (1.1) when all data vanish: $y = y' = \cdots = y^{(r-1)} = (0)$. By Lemma 1 we conclude that

$$S_{2m,r}^0 \text{ is a linear space of dimension } \quad d = 2m - 2r. \tag{2.2}$$

We now proceed to find a convenient basis for this space.

DEFINITION 1. *Let $S(x) \in S_{2m,r}^0$. The element $S(x)$ is called an eigenspline provided that it satisfies the relation*

$$S(x+1) = \lambda S(x) \quad \textit{for all real } x, \tag{2.3}$$

for an appropriate constant $\lambda \neq 0$. We refer to λ as the corresponding eigenvalue.

Eigensplines have properties implied by the additive group-structure of the integers. One such property is expressed by

LEMMA 2. *If $S(x)$ is an eigenspline for the eigenvalue λ then*

$$s(x) = S(-x), \tag{2.4}$$

is also an eigenspline for the eigenvalue λ^{-1}.

Proof. Let Eq. (2.3) hold. It is perfectly clear that also $s(x) = S(-x) \in S_{2m,r}^0$. Moreover, by Eq. (2.3)

$$\lambda s(x+1) = \lambda S(-x-1) = S(-x) = s(x).$$

Thus $s(x+1) = \lambda^{-1}s(x)$ which proves the lemma. ∎

In the course of the proof of Lemma 1 we have seen that a spline function $S(x)$ which is a solution of a C.H.I. problem is uniquely defined by its polynomial component $P(x)$ in the interval $[0, 1]$. Let us determine this polynomial component $P(x)$ of an eigenspline $S(x)$. By differentiation of Eq. (2.3) we obtain

$$S^{(s)}(1) = \lambda S^{(s)}(0), \quad (s = 0, 1, \ldots, 2m - r - 1). \tag{2.5}$$

Of these we use only the last $2m - 2r$ relations obtaining

$$P^{(s)}(1) = \lambda P^{(s)}(0), \quad (s = r, r+1, \ldots, 2m - r - 1), \tag{2.6}$$

while we replace the first r relations (2.5) by the stronger information expressed by

$$P^{(s)}(0) = P^{(s)}(1) = 0, \quad (s = 0, 1, \ldots, r - 1) \tag{2.7}$$

and implied by Eq. (2.1).

In Eqs. (2.6) and (2.7) we have a total of $2m - 2r + 2r = 2m$ homogeneous linear equations which will allow to determine $P(x)$ up to a multiplicative factor, provided that λ assumes certain appropriate values. To simplify notations we write

$$n = 2m - 1. \tag{2.8}$$

By Eq. (2.7) we may write

$$P(x) = a_0 x^n + \binom{n}{1} a_1 x^{n-1} + \cdots + \binom{n}{n-r} a_{n-r} x^r, \tag{2.9}$$

where the binomial coefficients are to facilitate the differentiations. Writing the $2m - 2r$ relations (2.6) in the reverse order of *decreasing* s, followed by the relations $P^{(s)}(1) = 0$ $(s = 0, 1, \ldots, r - 1)$ also written in reverse order, we obtain a system of $2m - r$ linear homogeneous equation in the same number $n - r + 1 = 2m - r$ of unknowns $a_0, a_1, \ldots, a_{n-r}$. The matrix of this system is found to be (see Sec. 8 (ii) below for a simple explicit example)

$$\Delta_{r,d}(\lambda) =$$

$$\begin{Vmatrix} 1 & \binom{r}{1} & \binom{r}{2} & \cdots & \binom{r}{r-1} & 1-\lambda & 0 & \cdots & & 0 \\ 1 & \binom{r+1}{1} & \binom{r+1}{2} & \cdots & \binom{r+1}{r-1} & \binom{r+1}{r} & 1-\lambda & \cdots & & 0 \\ \vdots & & & & & & & & & \vdots \\ 1 & \binom{n-r}{1} & & \cdots & & \binom{n-r}{r} & \cdots & \binom{n-r}{n-r-1} & & 1-\lambda \\ 1 & \binom{n-r+1}{1} & & \cdots & & & & & & \binom{n-r+1}{n-r} \\ \vdots & & & & & & & & & \vdots \\ 1 & \binom{n}{1} & & \cdots & & & & & & \binom{n}{n-r} \end{Vmatrix} \tag{2.10}$$

Observe that the element $1 - \lambda$ appears $n - r + 1 - r = 2m - 2r = d$ times. Since $n - r = d + r - 1$ we see that this is precisely the matrix mentioned in the first section of our introduction. It is also the subject of the following

THEOREM 5. *The algebraic equation*

$$|\Delta_{r,d}(\lambda)| = 0 \tag{2.11}$$

is a reciprocal equation of degree $d = 2m - 2r$ and it has all its roots real, simple, and of the sign of $(-1)^r$.

3. *An Application of the Gantmacher-Krein Theorem on Eigenvalues of Oscillation Matrices*

We consider the following two $d \times d$ matrices

$$P_{r,d} = \begin{Vmatrix} 1 & \binom{r}{1} & \binom{r}{2} & \cdots & \binom{r}{d-1} \\ 1 & \binom{r+1}{1} & & \cdots & \binom{r+1}{d-1} \\ \vdots & & & & \\ 1 & \binom{r+d-1}{1} & & \cdots & \binom{r+d-1}{d-1} \end{Vmatrix}, \tag{3.1}$$

and

$$I_d = \left\| \begin{matrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \vdots & & & & \vdots \\ 0 & 0 & & & 1 \end{matrix} \right\| . \tag{3.2}$$

LEMMA 3. *The determinant of the matrix* (2.10) *can be expressed as follows*

$$|\Delta_{r,d}(\lambda)| = (-1)^{rd}|A_d - \lambda I_d|, \tag{3.3}$$

where

$$(-1)^r A_d = (J_d)^r P_{r,d}. \tag{3.4}$$

This lemma will be established in Sec. 4. Let us show here how it implies a proof of Theorem 5. For this purpose we recall the following definitions and theorems (see [1], [2], [3], and [5]).

(A) A matrix A is said to be *totally positive* (T.P.) provided all its minors of all orders $= 1, 2, \ldots,$ are nonnegative. If all these minors are positive then A is called *strictly totally positive* (S.T.P.).

(B) If $AB = C$ and the factors A and B are T.P. (S.T.P.) then also C is T.P. (S.T.P.).

(C) A square matrix A is said to be an *oscillation matrix* provided that A is T.P. and for some natural number k the power A^k is S.T.P.

If $A = \|a_{ij}\|$ $(i, j = 1, \ldots, d)$ *is* T.P. *then* A *is an oscillation matrix if and only if the following two conditions are satisfied*

(i) $|A| \neq 0$.

(ii) $a_{i,i+1} > 0, \quad a_{i+1,i} > 0 \quad (i = 1, \ldots, d-1)$.

(See [1] and [2, Satz 10, 115]).

(D) THEOREM OF GANTMACHER AND KREIN. *If* A *is an oscillation matrix then the roots of the equation*

$$|A - \lambda I| = 0, \tag{3.5}$$

i.e., the eigenvalues of A, *are all positive and simple.*

It is now easy to establish Theorem 5. In [5, p. 550] are described certain "elementary transformations" with the property of transforming a T.P.

matrix into a new matrix that is again T.P. Such transformations are also used in [3, 153–154]. Let us show by induction that the matrix $P_{0,n}$ is T.P. Assuming this true, it is clear that the extended matrix

$$\begin{Vmatrix} 1 & 0 \\ 0 & P_{0,n} \end{Vmatrix}$$

is also T.P. If we now add the first row to the second, the new second row to the third a.s.f., we finally obtain the matrix $P_{0,n+1}$. All these being transformations that preserve total positivity, we conclude that also $P_{0,n+1}$ is T.P. Since $P_{0,1}$ is evidently T.P., we have shown that $P_{0,n}$ is T.P. for all n. By Eq. (3.1) we conclude that the matrix

$$P_{r,d} \quad \text{is T.P.} \tag{3.6}$$

Similar reasonings, again by induction, show that J_d is T.P. and therefore, by the property B, that $(J_d)^r$, and therefore by Eq. (3.4), that

$$(-1)^r A_d \quad \text{is T.P.} \tag{3.7}$$

Let us show that this is also an oscillation matrix. First we easily show by induction that $|P_{r,d}| = 1$; this is a theorem of Zeipel (see [4, 132–134]). It follows from Eq. (3.4) that

$$|(-1)^r A_d| = 1. \tag{3.8}$$

Moreover, since $r \geqslant 1$, we see from Eq. (3.1) that the elements of $P_{r,d}$ that are immediately above or below the main diagonal are positive and evidently the product on the right side of Eq. (3.4) has the same property. This last remark, together with Eqs. (3.7) and (3.8) shows that the matrix $(-1)^r A_d$ satisfies all requirements of property C for an oscillation matrix. Finally, (3.3) implies the identity

$$|\Delta_{r,d}(\lambda)| = |(-1)^r A_d - (-1)^r \lambda I_d|. \tag{3.9}$$

Theorem 5 now follows from the Gantmacher-Krein theorem.

4. Proof of Lemma 3

Our original proof of Lemma 3 is here replaced by a shorter and better proof due to T. N. E. Greville. (With the author's kind permission we reproduce it here.)

In order to prove Lemma 3, we shall need a further definition and another lemma. We write $l = r + d$. It is stated following (3.7) that

$|P_{r,l}| = 1$. Consequently $P_{r,l}$ is nonsingular, and so it has an inverse. Let $Q_{r,l,h}$ $(0 < h < l)$ be defined as the square submatrix of order $l - h$ obtained from $P_{r,l}^{-1}$ by deleting the first h rows and the last h columns. We then have:

LEMMA 4.

$$Q_{r,l,h} = (-1)^h P_{r,l-h}^{-1} J_{l-h}^{-h} \quad (l = 2, 3, \ldots; h = 1, 2, \ldots, l-1). \tag{4.1}$$

Proof. Let $P_{r,l}$ be partitioned in the form

$$P_{r,l} = \left\| \begin{array}{c} P_{r,l-1}u \\ \hdashline v^T \end{array} \right\|.$$

Then

$$v^T P_{r,l}^{-1} = e_l^T. \tag{4.2}$$

Now, note that because of the manner of construction of the Pascal triangle and the definition (3.2) of J_d,

$$P_{r,l} = 1_l v^T - \left\| \begin{matrix} 0 & J_{l-1}P_{r,l-1} \\ 0 & 0 \end{matrix} \right\|, \tag{4.3}$$

where 1_l denotes a column vector of l 1's. Since $P_{r,l}P_{r,l}^{-1} = I_l$, multiplying the first $l-1$ rows of (4.3) by the first $l-1$ columns of $P_{r,l}^{-1}$ and making use of Eq. (4.2) gives

$$-J_{l-1}P_{r,l-1}Q_{r,l,1} = I_{l-1}.$$

Since J_{l-1} and $P_{r,l-1}$ are nonsingular, this implies

$$Q_{r,l,1} = -P_{r,l-1}^{-1}J_{l-1}^{-1},$$

which is Eq. (4.1) for $h = 1$. Now, note that $Q_{r,l,h}$ is a submatrix of $Q_{r,l,1}$ situated in the lower left corner, and that J_{l-1}^{-1} is a square matrix of order $l-1$ having all its diagonal elements equal to 1, all elements immediately above the principal diagonal equal to -1, and all other elements equal to 0. Thus, multiplying the last $l-h$ rows of $P_{r,l-1}^{-1}$ by the first $l-h$ columns of J_{l-1}^{-1} and iterating the same procedure with l replaced by $l-1$, $l-2$, and so on gives

$$Q_{r,l,h} = -Q_{r,l-k,h-1}J_{l-h}^{-1} = (-1)^2 Q_{r,l-2,h-2}J_{l-h}^{-2}$$
$$= \cdots = (-1)^{h-1}Q_{r,l-h+1,1}J_{l-h}^{-h+1} = (-1)^h P_{r,l-h}^{-1}J_{l-h}^{-h}$$

as required.

We now turn to the proof of Lemma 3. Let $P_{r,r+d}$ and $P_{r,r+d}^{-1}$ be partitioned in the form

$$P_{r,r+d} = \begin{Vmatrix} E & F \\ G & H \end{Vmatrix}, \qquad P_{r,r+d}^{-1} = \begin{Vmatrix} K & L \\ Q_{r,r+d,r} & M \end{Vmatrix}$$

where E is $d \times r$ and K is $r \times d$. Then

$$\Delta_{r,d}(\lambda) = \begin{Vmatrix} E & F - \lambda I_d \\ G & H \end{Vmatrix}.$$

Since $|P_{r,r+d}^{-1}| = |P_{r,r+d}|^{-1} = 1$ and $P_{r,r+d}P_{r,r+d}^{-1} = I_{r+d}$ we have

$$|\Delta_{r,d}(\lambda)| = |\Delta_{r,d}(\lambda)P_{r,r+d}^{-1}| = \begin{vmatrix} I_d - \lambda Q_{r,r+d,r} & -\lambda M \\ 0 & I_r \end{vmatrix}$$
$$= |I_d - \lambda Q_{r,r+d,r}|.$$

Application of Lemma 4 now gives

$$|\Delta_{r,d}(\lambda)| = |I_d - \lambda(-1)^r P_{r,d}^{-1}J_d^{-r}| = |I_d - \lambda A_d^{-1}| = |A_d^{-1}|\,|A_d - \lambda I_d|.$$

Since $|P_{r,d}| = |J_d| = 1$, we have

$$|A_d^{-1}| = (-1)^{rd}$$

and Lemma 3 follows. ∎

An Example. In Lemma 3 we assume that $d = 2$, and find easily that

$$(J_2)^r = \begin{Vmatrix} 1 & 1 \\ 0 & 1 \end{Vmatrix}^r = \begin{Vmatrix} 1 & r \\ 0 & 1 \end{Vmatrix}$$

and therefore by Eqs. (3.1) and (3.4) that

$$(-1)^r A_2 = \begin{Vmatrix} 1 & r \\ 0 & 1 \end{Vmatrix}\begin{Vmatrix} 1 & r \\ 1 & r+1 \end{Vmatrix} = \begin{Vmatrix} 1+r & 2r+r^2 \\ 1 & 1+r \end{Vmatrix}.$$

Now Eq. (3.9) shows that the eigenvalues λ_1, λ_2 are equal to the zeros of

$$\begin{vmatrix} 1+r-x & 2r+r^2 \\ 1 & 1+r-x \end{vmatrix} = x^2 - 2(r+1)x + 1 = 0 \tag{4.4}$$

multiplied by $(-1)^r$.

5. Some Properties of the Eigensplines

After the algebraic interlude of Secs. 3 and 4 we return to the eigensplines of Sec. 2. The reciprocal equation

$$|\Delta_{r,d}(\lambda)| = 0 \qquad (d = 2m - 2r), \tag{5.1}$$

having d simple roots, all having the sign of $(-1)^r$, it follows that we can so label the roots

$$\lambda_1, \lambda_2, \ldots, \lambda_d, \tag{5.2}$$

that they satisfy the inequalities

$$0 < |\lambda_1| < |\lambda_2| < \cdots < |\lambda_{m-r}| < 1 < |\lambda_{m-r+1}| < \cdots < |\lambda_{2m-2r}| \tag{5.3}$$

and therefore the relations, by Lemma 2,

$$\lambda_1\lambda_d = \lambda_2\lambda_{d-1} = \cdots = \lambda_{m-r}\lambda_{m-r+1} = 1. \tag{5.4}$$

The quantities (5.2) are the *eigenvalues*. The elements of $S^0_{2m,r}$ that correspond to them according to the construction of Sec. 2 are called the *eigensplines* and are denoted by

$$S_1(x), S_2(x), \ldots, S_d(x), \tag{5.5}$$

respectively. They have the characteristic property

$$S_j(x+1) = \lambda_j S_j(x), \quad \text{for all real } x, \tag{5.6}$$

which immediately implies that

$$S_j(x+n) = \lambda_j{}^n S_j(x) \quad \text{for all integers } n. \tag{5.7}$$

We recall that an eigenspline $S(x)$ is determined in $[0, 1]$ by the polynomial (2.9) whose coefficients satisfy the linear homogeneous system having the matrix (2.10). *We shall see below that if λ is an eigenvalue then the rank of the matrix* (2.10) *must be equal to $d + r - 1$, and therefore the corresponding eigenspline $S(x)$ is determined up to a multiplicative factor.* In order to normalize these functions in a reasonable way we first establish

LEMMA 5. *If $S(x)$ is an eigenspline then*

$$|S(x)| > 0 \quad \textit{if} \quad 0 < x < 1. \tag{5.8}$$

Proof. Observe that $S(x)$ is a cardinal spline function, that

$$S(x) \in C^{2m-r-1}, \tag{5.9}$$

and that it has an r-fold zero at each integer $x = \nu$. From a general theorem concerning the number of possible zeros of such functions counting their multiplicities (see [10, Lemma 2, p. 424]) we conclude the following: If k is a positive integer and if $Z\{S(x); [0, k]\}$ denotes the total number of zeros of $S(x)$ in the interval $[0, k]$, then

$$Z\{S(x); [0, k]\} \leqslant 2m - 1 + (k - 1)r. \tag{5.10}$$

Let us assume that the inequality (5.8) is violated and

$$S(\xi) = 0, \qquad 0 < \xi < 1. \tag{5.11}$$

Now Eq. (5.7) shows that $S(\xi + n) = 0$ for all integer n. It follows that $S(x)$ has at least $r + 1$ zeros in $[\nu, \nu + 1)$, for all ν, and therefore at least $k(r + 1) + r$ zeros in $[0, k]$. We therefore see that

$$Z\{S(x); [0, k]\} \geqslant r + k(r + 1). \tag{5.12}$$

The two results (5.10) and (5.12) are evidently contradictory for sufficiently large k and Lemma 5 is thereby established. ■

The normalization of the eigensplines can be done in a variety of ways. Let us do it by assuming that

$$S_j(x) > 0 \quad \textit{in} \quad 0 < x < 1 \quad \textit{and} \quad S_j^{(r)}(0) = 1. \tag{5.13}$$

A consequence is the *symmetry relation*

$$S_j(- x) = (- 1)^r S_{d-j+1}(x), \tag{5.14}$$

which follows easily from Lemma 2 in Sec. 2.

From Eq. (5.7), or from the fact that the integers are zeros of order r, we conclude that

$$(- 1)^{nr} S_j(x) > 0 \quad \text{if} \quad n < x < n + 1. \tag{5.15}$$

In words: *If r is even then $S_j(x)$ is nonnegative for all s; if r is odd, then $S_j(x)$ changes sign at every integer value of x.*

The behavior of the $S_j(x)$ at infinity follows easily from Eq. (5.7) and the inequalities (5.3): We find that

$$0 < \overline{\lim_{x \to -\infty}} |S_j(x)|/|\lambda_j|^x < \infty, \qquad j = 1, 2, \ldots, m - r, \tag{5.16}$$

$$0 < \overline{\lim_{x \to +\infty}} |S_j(x)|/|\lambda_j|^x < \infty, \qquad j = m - r + 1, \ldots, 2m - 2r, \tag{5.17}$$

while

$$\lim_{x \to +\infty} S_j(x) = 0, \qquad j = 1, \ldots, m - r, \tag{5.18}$$

and

$$\lim_{x \to -\infty} S_j(x) = 0, \qquad j = m - r + 1, \ldots, 2m - 2r. \tag{5.19}$$

The relations (5.16)–(5.19) show that the eigensplines (5.5) are linearly independent and their number d is also the dimension of the space $S^0_{2m,r}$. This settles also the point mentioned in the section preceding Lemma 5: The rank of the matrix $\Delta_{r,d}(\lambda_j)$ cannot be less than $d + r - 1$ because this would produce at least $d + 1$ linearly independent elements of $S^0_{2m,r}$, which is impossible. We have therefore established

LEMMA 6. *If*

$$S(x) \in S^0_{2m,r} \tag{5.20}$$

then there is a unique representation

$$S(x) = \sum_1^d c_j S_j(x) \tag{5.21}$$

with appropriate constants c_j.

6. *Proof of Unicity in Theorems 1 and 2*

The subject of Theorem 1 is the

$$\text{C.H.I.P. } (y, \ldots, y^{(r-1)}; S_{2m,r} \cap F_{\gamma,r}). \tag{6.1}$$

Let us assume for the moment that $\tilde{S}_1(x)$ and $\tilde{S}_2(x)$ are two solutions of Eq. (6.1). It evidently follows that their difference

$$S(x) = \tilde{S}_1(x) - \tilde{S}_2(x) \in S^0_{2m,r} \tag{6.2}$$

and that therefore the function $S(x)$ admits a representation (5.21). On the other hand both $\tilde{S}_1(x)$ and $\tilde{S}_2(x)$ being elements of $F_{\gamma,r}$, we conclude that

$$S(x) \in F_{\gamma,r} \tag{6.3}$$

and in particular that

$$S(x) = O(|x|^\gamma) \quad \text{as} \quad x \to \pm\infty. \tag{6.4}$$

We claim that the relations (5.21) *and* (6.4) *imply that*

$$c_j = 0 \quad (j = 1, \ldots, d), \tag{6.5}$$

in view of the behavior of the $S_j(x)$ at infinity described by the relations (5.16)–(5.19) and the inequalities (5.3). For if Eq. (6.5) were not true, then $S(x)$ would grow to infinity exponentially over an appropriate sequence of values of x converging to $+\infty$, or perhaps to $-\infty$, in contradiction to the restriction (6.4). Therefore $\tilde{S}_1(x) = \tilde{S}_2(x)$ for all x, and the unicity statement of Theorem 1 is established.

We pass now to Theorem 2 and wish to show the unicity of solutions of the

$$\text{C.H.I.P.}\ (y, \ldots, y^{(r-1)};\ S_{2m,r} \cap \mathscr{L}_{p,r}) \tag{6.6}$$

where $1 \leqslant p \leqslant \infty$. Again, considering two solutions $\tilde{S}_1(x)$ and $\tilde{S}_2(x)$ of Eq. (6.6) we find that Eq. (6.2) holds, this time with the restriction

$$S(x) \in \mathscr{L}_{p,r}. \tag{6.7}$$

However, we know that $\mathscr{L}_{p,r} \subset \mathscr{L}_{\infty,r} = F_{0,r}$, and therefore Eq. (6.3) holds with $\gamma = 0$. From our previous section we conclude that $S(x) = 0$, for all x, and the proof is completed.

We have also incidentally established the following lemma which we state for future reference (for $r = 1$ see [8, Theorem 11, p. 195]).

LEMMA 7. *If*

$$S(x) \in S^0_{2m,r} \tag{6.8}$$

and

$$S(x) \in \mathscr{L}_1 \tag{6.9}$$

then

$$S(x) = 0 \quad \textit{for all real } x. \tag{6.10}$$

II. PROOFS OF THEOREMS 1, 2, 3, AND 4

7. *Construction of the Fundamental Functions* $L_{2m,r,s}(x)$

Dropping two of the subscripts we write

$$L_s(x) = L_{2m,r,s}(x), \quad (s = 0, \ldots, r-1), \tag{7.1}$$

and propose to construct these functions in terms of the eigensplines $S_j(x)$ of Sec. 5. They are to have the following properties

$$L_s(x) \in S_{2m,r} \cap \mathscr{L}_{1,r}, \tag{7.2}$$

$$L_s^{(\rho)}(0) = 0 \quad \text{if} \quad \rho \neq s, \quad L_s^{(s)}(0) = 1, \tag{7.3}$$

$$L_s(\nu) = L_s'(\nu) = \cdots = L_s^{(r-1)}(\nu) = 0 \quad \text{for all} \quad \nu \neq 0. \tag{7.4}$$

Moreover the function

$$L_s(x) \textit{ is even if } s \textit{ is even and odd if } s \textit{ is odd.} \tag{7.5}$$

Let us for the moment assume the existence of $L_s(x)$ having these properties and let us analyse its structure. If $L_s(x)$ is known in the intervals $[0, 1]$ and $[1, \infty)$, then (7.5) allows to complete its definition. First we focus our attention on the restriction of $L_s(x)$ to the interval $[1, \infty)$. It is clear from Lemma 1 that we can extend this restriction to a function $\tilde{L}_s(x)$, defined for all real x, and such that

$$\tilde{L}_s(x) \in S^0_{2m,r}. \tag{7.6}$$

In any case

$$\tilde{L}_s(x) = L_s(x) \quad \text{if} \quad x \geqslant 1. \tag{7.7}$$

At this stage we wish to point out that the "decreasing" eigensplines

$$S_j(x), \quad (j = 1, 2, \ldots, m-r), \tag{7.8}$$

have, besides the properties (5.18), the further properties

$$S_j(x) \in \mathscr{L}_{1,r} \quad \text{in the interval } [1, +\infty), \tag{7.9}$$

while nothing of the kinds holds for the others in view of Eq. (5.17). By Eqs. (7.7) and (7.2), and applying Lemma 6, we conclude that only the eigensplines (7.8) may occur in the representation (5.21) of $\tilde{L}_s(x)$. Therefore

$$\tilde{L}_s(x) = \sum_1^{m-r} c_j S_j(x) \quad \text{for all } x.$$

By Eq. (7.7) we conclude that

$$L_s(x) = \sum_1^{m-r} c_j S_j(x) \quad \text{if} \quad x \geqslant 1, \tag{7.10}$$

with coefficients c_j yet to be determined.

Let now

$$P(x) = L_s(x) \quad \text{in} \quad 0 \leqslant x \leqslant 1 \tag{7.11}$$

be the restriction of $L_s(x)$ to the interval $[0, 1]$, where $P(x) \in \pi_{2m-1}$. From

$$L_s(x) \in C^{2m-r-1}, \tag{7.12}$$

Eqs. (7.3) and (7.5) we obtain for $P(x)$ the following information.

1. *If r and s are of the same parity then*

$$P(x) = \frac{1}{s!} x^s + a_1 x^r + a_2 x^{r+2} + \cdots + a_{m-r} x^{2m-r-2} + a_{m-r+1} x^{2m-r}$$
$$+ a_{m-r+2} x^{2m-r+1} + \cdots + a_m x^{2m-1}. \tag{7.13}$$

2. *If r and s are of opposite parities then*

$$P(x) = \frac{1}{s!} x^s + a_1 x^{r+1} + a_2 x^{r+3} + \cdots + a_{m-r} x^{2m-r-1}$$
$$+ a_{m-r+1} x^{2m-r} + a_{m-r+2} x^{2m-r+1} + \cdots + a_m x^{2m-1}. \tag{7.14}$$

Observe that only terms allowed by Eq. (7.12) in conjunction with the requirement (7.5), have been retained. In both cases we are to determine m coefficients a_i and these can be found again from Eqs. (7.12) and (7.7), the latter of which implies the relations

$$P^{(\rho)}(1) = \tilde{L}_s^{(\rho)}(1), \qquad (\rho = 0, \ldots, 2m - r - 1),$$

or, using Eqs. (7.7) and (7.10),

$$P^{(\rho)}(1) = \sum_{j=1}^{m-r} c_j S_j^{(\rho)}(1), \quad (\rho = 0, \ldots, 2m - r - 1). \tag{7.15}$$

These are $2m - r$ nonhomogeneous linear equations to be used to determine the same number $m + (m - r) = 2m - r$ of unknowns a_i and c_j.

We are still to show that the linear system (7.15) is *nonsingular.* Notice that the constant term in Eq. (7.15) is $1/(s - \rho)!$ and that it appears only if $\rho = 0, \ldots, s$. Removing these constant terms we obtain a *homogeneous* system (7.15) that corresponds to a polynomial (7.13), or (7.14), in which the first term $x^s/s!$ has been removed. If the homogeneous system were singular, then our construction would lead to a *nontrivial* spline function $L_*(x)$ having all the properties of $L_s(x)$, except that

$$L_*(x) \in S^0_{2m,r}.$$

However, by Lemma 7 we know that such nontrivial functions do not exist. This completes our construction of the functions $L_s(x)$.

8. Examples: The Fundamental Functions for $m = 2$, $r = 2$, and $m = 3$, $r = 2$

(i) In the case when $m = r = 2$ the construction is trivial as pointed out in the Introduction. We easily find that

$$L_0(x) = \begin{cases} 1 - 3x^2 + 2x^3 & \text{if } 0 \leqslant x \leqslant 1, \\ 0 & \text{if } x \geqslant 1, \\ L_0(-x) & \text{if } x < 0, \end{cases} \tag{8.1}$$

and

$$L_1(x) = \begin{cases} x - 2x^2 + x^3 & \text{if } 0 \leqslant x \leqslant 1, \\ 0 & \text{if } x \geqslant 1, \\ -L_1(-x) & \text{if } x < 0. \end{cases} \tag{8.2}$$

(ii) The case when $m = 3$, $r = 2$ is the simplest nontrivial one. Here $d = 2m - 2r = 2$, $n = 2m - 1 = 5$ and the polynomial (2.9) is

$$P(x) = a_0x^5 + 5a_1x^4 + 10a_2x^3 + 10a_3x^2. \tag{8.3}$$

The conditions which $P(x)$ and λ are to satisfy, namely $P^{(s)}(1) = 0$ ($s = 0, 1$) and $P^{(s)}(1) = \lambda P^{(s)}(0)$ ($s = 2, 3$), become, if written in reverse order,

$$a_0 + 2a_1 + (1 - \lambda)a_2 = 0,$$

$$a_0 + 3a_1 + 3a_2 + (1 - \lambda)a_3 = 0,$$

$$a_0 + 4a_1 + 6a_2 + 4a_3 = 0,$$

$$a_0 + 5a_1 + 10a_2 + 10a_3 = 0. \tag{8.4}$$

Setting the determinant $= 0$ leads to the equation $|\Delta_{2,2}(\lambda)| = 0$, and from Eq. (4.4) we know this to be the equation $\lambda^2 - 6\lambda + 1 = 0$ having the roots

$$\lambda_1 = 3 - 2\sqrt{2}, \qquad \lambda_2 = 3 + 2\sqrt{2}. \tag{8.5}$$

For the construction of the fundamental functions we need the eigenspline $S_1(x)$ corresponding to the eigenvalue λ_1. Setting $\lambda = \lambda_1 = 3 - 2\sqrt{2}$ in Eq. (8.4) and solving for the a_i we obtain for the coefficients of Eq. (8.3) the values

$$a_0 = 20(1 - \sqrt{2}), \quad a_1 = 2(-3 + 4\sqrt{2}), \quad a_2 = -2\sqrt{2}, \quad a_3 = 1. \tag{8.6}$$

The eigenspline $S_1(x)$ is hereby determined from Eq. (5.7) by

$$S_1(x) = \lambda_1{}^n S_1(x - n) = \lambda_1{}^n P(x - n) \quad \text{if} \quad n \leqslant x \leqslant n + 1, \quad (-\infty < n < \infty). \tag{8.7}$$

To determine $L_0(x) = L_{6,2,0}(x)$ we set, by Eq. (7.13),

$$L_0(x) = P_0(x) = 1 + a_1 x^2 + a_2 x^4 + a_3 x^5 \quad \text{in} \quad 0 \leqslant x \leqslant 1, \tag{8.8}$$

where this polynomial and the constant c_1 are to be determined from the equations

$$P_0{}^{(\rho)}(1) = c_1 S_1{}^{(\rho)}(1) \quad (\rho = 0, 1, 2, 3), \tag{8.9}$$

because $L_0(x) \in C^{2m-r-1} = C'''$. We find that

$$a_1 = -5(2 - \sqrt{2}), \quad a_2 = 5(5 - 3\sqrt{2}), \quad a_3 = -2(8 - 5\sqrt{2}), \quad c_1 = \sqrt{2}/2 \tag{8.10}$$

and finally

$$L_0(x) = \begin{cases} P_0(x) & \text{if} \quad 0 \leqslant x \leqslant 1, \\ c_1 S_1(x) & \text{if} \quad x \geqslant 1, \\ L_0(-x) & \text{if} \quad x < 0. \end{cases} \tag{8.11}$$

To determine $L_1(x)$ we use Eq. (7.14) and define

$$L_1(x) = P_1(x) = x + a_1'x^3 + a_2'x^4 + a_3'x^5 \quad \text{in} \quad 0 \leqslant x \leqslant 1, \tag{8.12}$$

and determine the a_i' and c_1' from the 4 equations

$$P_1^{(\rho)}(1) = c_1' S_1^{(\rho)}(1) \qquad (\rho = 0, 1, 2, 3) \tag{8.13}$$

obtaining

$$a_1' = 2(3 - 4\sqrt{2}), \quad a_2' = 16(\sqrt{2} - 1), \quad a_3' = 9 - 8\sqrt{2}, \quad c_1' = 2/5. \tag{8.14}$$

Therefore

$$L_1(x) = \begin{cases} P_1(x) & \text{if} \quad 0 \leqslant x \leqslant 1, \\ c_1' S_1(x) & \text{if} \quad x \geqslant 1, \\ -L_1(-x) & \text{if} \quad x < 0. \end{cases} \tag{8.15}$$

The shapes of the graphs of these functions are given in Fig. 1.

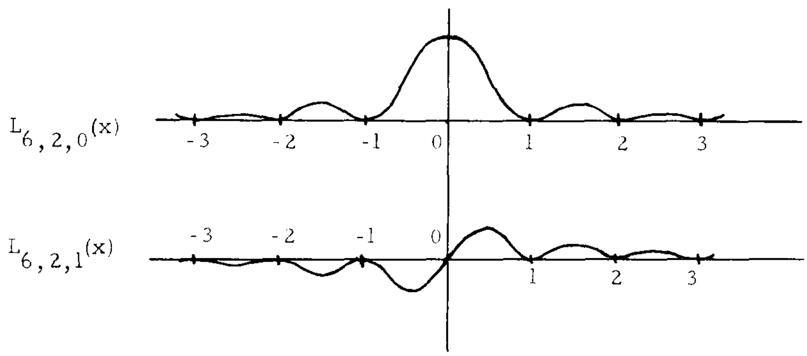

FIG. 1

The corresponding Lagrange-Hermite interpolation formula

$$f(x) = \sum_{-\infty}^{\infty} f(\nu) L_0(x - \nu) + \sum_{-\infty}^{\infty} f'(\nu) L_1(x - \nu) + Rf$$

is exact for the class of cardinal spline functions $f(x)$ of degree 5 and class C''' which are such that

$$f(x) = O(|x|^\gamma), \qquad f'(x) = O(|x|^\gamma) \quad \text{as} \quad x \to \pm\infty$$

for some $\gamma \geqslant 0$, and in particular for polynomials of degree $\leqslant 5$. For the corresponding general statement see Corollary 1 of Sec. 10 below.

9. Proofs of Theorems 1 and 2 Completed and Proofs of Theorems 3 and 4

We begin with the easy proof of Theorem 3, i.e., of the estimate (27). From the representation (7.10) we see that it suffices to establish such estimates for the "decreasing" eigensplines $S_j(x)$ $(j = 1,\ldots, m - r)$ in the interval $[0, \infty)$. Let x be positive and let $n \leqslant x < n + 1$. From Eq. (5.7) we now find that

$$|S_j(x)| = |\lambda_j|^n |S_j(x - n)| \leqslant B|\lambda_j|^{-1}|\lambda_j|^{n+1} < B|\lambda_j|^{-1}|\lambda_j|^x.$$

Hence

$$|S_j(x)| < B|\lambda_j|^{-1} \exp(- x \log|\lambda_j|^{-1}), \tag{9.1}$$

which already shows what we need. Similar estimates hold for $S_j^{(\rho)}(x)$ and *we may therefore consider Theorem* 3 *as established.* ∎

The proofs of the *necessity* of the conditions (15) and (21) in Theorems 1 and 2, respectively, is so obvious that we may omit a further discussion.

We shall now settle the *sufficiency* of the conditions (15) and (21) in Theorems 1 and 2, respectively, by establishing Theorem 4, which states that the Lagrange-Hermite expansion (30) exhibits the solutions. For this we are to establish the following statements:

I. *If*

$$y_\nu^{(\rho)} = O(|\nu|^\gamma) \quad \textit{for} \quad \rho = 0,\ldots, r - 1 \quad (\gamma \geqslant 0), \tag{9.2}$$

then

$$S(x) = \sum_\nu y_\nu L_0(x - \nu) + \cdots + \sum_\nu y_\nu^{(r-1)} L_r(x - \nu) \tag{9.3}$$

converges locally uniformly and its sum satisfies

$$S^{(\rho)}(x) = O(|x|^\gamma) \quad \textit{for} \quad \rho = 0,\ldots, r - 1. \tag{9.4}$$

II. *If*

$$(y_\nu^{(\rho)}) \in l_p \quad \textit{for} \quad \rho = 0,\ldots, r - 1 \quad (1 \leqslant p \leqslant \infty) \tag{9.5}$$

then Eq. (9.3) *converges locally uniformly and its sum satisfies*

$$S^{(\rho)}(x) \in \mathscr{L}_p \quad \textit{for} \quad \rho = 0,\ldots, r - 1. \tag{9.6}$$

Proof of I. This follows from the similar problem for $r = 1$ discussed in [9, Sec. 5]. ■

Proof of II. The locally uniform convergence of Eq. (9.3) follows from the previous case for $\gamma = 0$ because Eq. (9.5) implies that the sequences are bounded. Finally the relations (9.6) follow from an easy adaptation of the reasonings used in [8, Theorem 12, proof of sufficiency, 199–200]. This completes the proofs of the Theorems. ■

10. *The Cardinal Lagrange-Hermite Interpolation Formula*

At this point it is convenient to introduce in terms of Eq. (13) the new function class

$$F_r^* = \bigcup_{\gamma>0} F_{\gamma,r}. \tag{10.1}$$

This class may be described as the class of functions $f \in C^{r-1}$ such that $f, f', \ldots, f^{(r-1)}$ are of *power growth*. From Theorems 1 and 4 we obtain the following.

COROLLARY I. *If*

$$f(x) \in F_r^* \tag{10.2}$$

then we may apply to $f(x)$ *the Lagrange-Hermite formula*

$$f(x) = \sum_\nu f(\nu)L_0(x-\nu) + \cdots + \sum_\nu f^{(r-1)}(\nu)L_{r-1}(x-\nu) + R(x) \tag{10.3}$$

and this formula is exact, i.e., $R(x) = 0$ *for all* x, *whenever* $f(x)$ *satisfies* (10.2) *and is a cardinal spline function of degree* $2m-1$ *and class* C^{2m-r-1}. *In particular Eq.* (10.3) *is exact if*

$$f(x) \in \pi_{2m-1}. \tag{10.4}$$

III. A GENERALIZATION OF THE EVEN-DEGREE BERNOULLI FUNCTIONS

11. *Interpolating the Function* x^{2m}

In view of the exactness of Eq. (10.3) for polynomials of degree $2m-1$, it seems worthwhile to increase the degree by one unit and to interpolate $f(x) = x^{2m}$. We then obtain the identity

$$x^{2m} = S(x) + R(x), \tag{11.1}$$

where $S(x)$ is the interpolating spline function and $R(x)$ is the remainder. $R(x)$ is an element of the class

$$M_{2m,r} = \{r(x); r(x) = x^{2m} - s(x), \text{ where } s(x) \in S_{2m,r}\} \tag{11.2}$$

which is a class of so-called *monosplines*.

THEOREM 6. *The monospline*

$$R(x) = R_{2m,r}(x) \tag{11.3}$$

defined by Eq. (11.1) *is the unique function satisfying the following conditions*

$$R(x) \in M_{2m,r}, \tag{11.4}$$

$$R(\nu) = R'(\nu) = \cdots = R^{(r-1)}(\nu) = 0 \quad \textit{for all integers } \nu, \tag{11.5}$$

$$R(x) \in F_r^*. \tag{11.6}$$

A *proof* is directly implied by Theorems 1 and 4.

Further properties of the function $R(x)$ are as follows.

THEOREM 7. *The monospline $R(x)$ defined by Eq.* (11.1) *is an even periodic function of period* 1, *i.e.*,

$$R(x+1) = R(x), \tag{11.7}$$

$$R(-x) = R(x). \tag{11.8}$$

Proof. Equation (11.1) implies $(x+1)^{2m} = S(x+1) + R(x+1)$. If we expand the binomial and observe that

$$S_1(x) = S(x+1) - \sum_0^{2m-1} \binom{2m}{\nu} x^\nu$$

is also an element of $S_{2m,r}$, we obtain

$$x^{2m} = S_1(x) + R(x+1),$$

which shows that $R(x+1)$ has all properties of $R(x)$ as described by Theorem 6. The unicity in Theorem 6 implies Eq. (11.7).

Similarly $x^{2m} = S(-x) + R(-x)$ implies Eq. (11.8). ■

The next theorem concerns polynomials, and spline functions are not even mentioned.

THEOREM 8. 1. *If* $1 \leqslant r \leqslant m$ *then there is a unique polynomial*

$$P(x) = x^{2m} + \binom{2m}{1} a_1 x^{2m-1} + \cdots + a_{2m} \tag{11.9}$$

satisfying the relations

$$P^{(s)}(0) = P^{(s)}(1) = 0 \qquad (s = 0, 1, \ldots, r-1), \tag{11.10}$$

$$P^{(j)}(0) = P^{(j)}(1) \qquad (j = r, r+1, \ldots, 2m-r-1). \tag{11.11}$$

The polynomial $P(x)$ *has the following additional properties:*

2. $P(x)$ *is even about the point* $x = \frac{1}{2}$, *i.e.*,

$$P(1-x) = P(x). \tag{11.12}$$

3. $$|P(x)| > 0 \quad \textit{if} \quad 0 < x < 1. \tag{11.13}$$

4. $$\operatorname{sgn} P(x) = (-1)^m \quad \textit{if} \quad 0 < x < 1. \tag{11.14}$$

Proof. 1. Let $P(x) \in \pi_{2m}$ such that

$$P(x) = R(x) \quad \text{if} \quad 0 \leqslant x \leqslant 1. \tag{11.15}$$

The properties (11.5) and (11.7) of the monospline $R(x)$ evidently imply the relations (11.10). Moreover, Eq. (11.7) and $R(x) \in C^{2m-r-1}$ also imply the relations (11.11). Therefore existence of $P(x)$ is assured.

Conversely, let now $P(x)$ be a polynomial satisfying Eqs. (11.9), (11.10), and (11.11). If we denote by $\bar{P}(x)$ the periodic extension of $P(x)$, of period 1 from [0, 1], and define

$$R_1(x) = \bar{P}(x),$$

we easily see that $R_1(x)$ has all properties described by Theorem 6. From the unicity in Theorem 6, we conclude that $R_1(x) = R(x)$ and therefore $P(x)$ is uniquely defined.

2. From Eqs. (11.7) and (11.8) we obtain $R(1-x) = R(x-1) = R(x)$ and therefore

$$R(1-x) = R(x). \tag{11.16}$$

Now Eq. (11.15) implies Eq. (11.12).

3. We establish Eq. (11.13) by contradiction. Let us assume that $R(\xi) = P(\xi) = 0$ for some ξ such that $0 < \xi < 1$. From (11.12) we conclude that $R(x)$ has at least 2 zeros in $(0, 1)$, and from (11.10) that $R(x)$ has at least $r + 2$ zeros in $[0, 1)$. We conclude that

$$Z\{R(x); [0, k)\} \geqslant k(r + 2), \tag{11.17}$$

where k is a natural number and $Z\{R(x); [0, k)\}$ denotes the number of zeros of $R(x)$ in $[0, k)$ counting multiplicities.

On the other hand an upper bound for Z is furnished by Lemma 2 of [10, p. 424] already once applied in Sec. 5. This lemma states that if $S(x)$ is a cardinal spline function of degree n (with integer knots) such that $S(x) \not\equiv 0$ in every interval $(\nu, \nu + 1)$, and such that $S(x) \in C^{r'}$, then

$$Z\{S(x); [0, k]\} \leqslant n + (k - 1)(n - r'). \tag{11.18}$$

Applying Eq. (11.18) to $R(x)$ which may be regarded as a cardinal spline function with $n = 2m$ and $r' = 2m - r - 1$, we find that

$$Z\{R(x); [0, k]\} \leqslant 2m + (k - 1)(r + 1). \tag{11.19}$$

Since Eqs. (11.17) and (11.19) are clearly contradictory for sufficiently large values of k, the inequality (11.13) is established.

4. Let $P(x)$ denote the polynomial defined by Eq. (11.15). We use its properties (11.10) in the following integrations by parts:

$$\int_0^1 P(x)P^{(2m)}(x)\,dx$$

$$= \int_0^1 P(x)\,dP^{(2m-1)}(x) = -\int_0^1 P'P^{(2m-1)}\,dx = \cdots$$

$$= (-1)^{r-1}\int_0^1 P^{(r-1)}P^{(2m-r+1)}\,dx = (-1)^{r-1}\int_0^1 P^{(r-1)}\,dP^{(2m-r)},$$

and finally

$$\int_0^1 P(x)P^{(2m)}(x)\,dx = (-1)^r \int_0^1 P^{(r)}P^{(2m-r)}\,dx, \tag{11.20}$$

because of the vanishing boundary values (11.10).

Observe, however, that Eqs. (11.11) imply the relations

$$P^{(j)}(0)P^{(2m-j-1)}(0) = P^{(j)}(1)P^{(2m-j-1)}(1) \quad \text{for} \quad j = r, r+1, \ldots, 2m-r-1.$$

They imply that we may continue the integrations by parts of the right side of Eq. (11.20) and obtain

$$\begin{aligned} \int_0^1 PP^{(2m)}\,dx &= (-1)^r \int_0^1 P^{(r)}\,dP^{(2m-r-1)} \\ &= (-1)^{r+1} \int_0^1 P^{(r+1)}P^{(2m-r-1)}\,dx = \cdots \\ &= (-1)^{m-1} \int_0^1 P^{(m-1)}P^{(m+1)}\,dx = (-1)^{m-1} \int_0^1 P^{(m-1)}\,dP^{(m)}, \end{aligned}$$

and finally

$$\int_0^1 P(x)P^{(2m)}(x)\,dx = (-1)^m \int_0^1 (P^{(m)})^2\,dx. \tag{11.21}$$

The last integration by parts requires that $m \leqslant 2m - r - 1$ or $r \leqslant m - 1$. However, if $r = m$ then clearly Eqs. (11.9) and (11.10) imply that

$$P(x) = x^m(x-1)^m, \tag{11.22}$$

in which case the properties (11.13) and (11.14) are evident anyway. Since $P^{(2m)}(x) = (2m)!$ it is clear that Eq. (11.21) implies Eq. (11.14), in view of Eq. (11.13). This completes our proof of Theorem 8. ■

REMARKS. 1. From their definitions (11.1) and (11.3), there appear to be m distinct monosplines $R_{2m,r}(x)$ $(r = 1, 2, \ldots, m)$. That this is not the case is the subject of

COROLLARY 2. *We have the identity*

$$R_{2m,r}(x) = R_{2m,r+1} \quad \textit{if } r \textit{ is odd and } < m. \tag{11.23}$$

It follows that there are only $[(m+1)/2]$ *distinct among these monosplines.*

Proof. It suffices to count the number of the polynomials $P(x) = P_{2m,r}(x)$ of Theorem 8. Let r be odd and $< m$. From Eqs. (11.8), (11.10), and (11.11) we conclude that $x = 0$ must be an even-order zero of $R_{2m,r}(x)$ and that therefore

$$P^{(r)}(0) = P^{(r)}(1) = 0.$$

The polynomial $P_{2m,r}(x)$ is therefore seen to have all properties required of $P_{2m,r+1}(x)$ and the unicity shows that these two polynomials are identical. ∎

A further consequence of our proof is

COROLLARY 3.

$$R_{2m,r}(x) \in C^{2m-r} \quad \textit{if } r \textit{ is even} \leqslant m. \tag{11.24}$$

Proof. Indeed, assuming r even, Corollary 2 implies that

$$R_{2m,r}(x) = R_{2m,r-1} \in C^{2m-(r-1)-1} = C^{2m-r}. \quad ∎$$

2. In (11.22) we have already mentioned that

$$P_{2m,m}(x) = x^m(x-1)^m \tag{11.25}$$

and therefore

$$R_{2m,m}(x) = \bar{P}_{2m,m}(x). \tag{11.26}$$

For the least value of r we find that

$$P_{2m,1}(x) = P_{2m,2}(x) = B_{2m}(x) - B_{2m}, \tag{11.27}$$

where B_{2m} and $B_{2m}(x)$ denote the Bernoulli numbers and polynomials, respectively. Indeed, the polynomial (11.27) is seen to satisfy the conditions (11.10) for $r = 2$, and (11.11) for $r = 1$. Therefore

$$R_{2m,1}(x) = R_{2m,2}(x) = \bar{B}_{2m}(x) - B_{2m}. \tag{11.28}$$

The polynomials

$$P_{2m,r}(x) \quad (r = 4, 6, 8, \ldots < m) \tag{11.29}$$

bridge the gap between the polynomials (11.27) and (11.25), and in this sense they may be regarded as generalizations of the even-degree Bernoulli polynomials.

It would be of interest to estimate the remainder of the Lagrange-Hermite interpolation formula (10.3) for various subclasses of the class F_r^*. Here we can at least obtain a rough idea by observing that

$$K_{2m,r} = \sup_x |R_{2m,r}(x)| \tag{11.30}$$

represents the error in applying Eq. (10.3) to $f(x) = x^{2m}$. From Eqs. (11.27) and (11.25) we easily find that

$$K_{2m,1} = |B_{2m}|(2 - 2^{-2m+1}) < 2|B_{2m}| \sim 4(2m)!/(2\pi)^{2m} \tag{11.31}$$

and

$$K_{2m,m} = 2^{-2m}. \tag{11.32}$$

Notice the dramatic drop in error from $r = 1$ to $r = m$. We expect a discussion of the remainder of Eq. (10.3) to show this to be a general feature. We also expect this increase in accuracy, for increasing r, to be especially pronounced for low values of r and tapering off as r approaches m.

Elsewhere we might discuss the approximate quadrature formulas corresponding to the monosplines $R_{2m,r}(x)$ (for the connection see, e.g., [7]). For $r = 2$ we obtain, of course, the Euler-Maclaurin quadrature (or summation) formula, while for $r = m$ we have the repeated Hermite 2-point quadrature formula.

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