Summary paper 32: Interpolatory Hermite Spline wavelets

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1 Introduction

This paper by T. Goodman proposes a construction of wavelets from spline functions with multiple knots. These wavelets inherit some properties of the B-splines.

Let ψ a function in $L_2(\mathbb{R})$ and its translated dilates $B = \{2^{k/2}\psi(2^k. - j)\}_{j,k\in\mathbb{Z}}$.

Definition 1. *ψ is said to be an* orthogonal wavelet *if B forms an orthonormal basis of $L^2(\mathbb{R})$.*

Definition 2. *ψ is said to be a* wavelet *if B forms a Riesz basis of $L^2(\mathbb{R})$ and $\psi(2^k. - i)$ orthogonal to $\psi(2^l - j)$ whenever $k \neq l$.*

Idea used in another article by Goodman "Wavelets with multiplicity r" to construct compactly supported spline wavelets $\psi_0, \dots, \psi_{r-1}$ with knots multiplicity r. Here different construction of $\psi_0, \dots, \psi_{r-1}$ related to Schoenberg and Sharma's problem of cardinal Hermite spline interpolation. They satisfy for $s = 0, \dots, r-1$,

$$\psi_s^{(j)}(k) = 0 \quad \text{for} \quad j = 0, \dots, r-1, j \neq s, k \in \mathbb{Z} \tag{1}$$

2 Construction of wavelets

<u>Notations</u>

- $\zeta_{n,r}(S)$ space of spline functions of degree n on $\mathbb{R}$ with knots multiplicity r on set S. Note that $\zeta_{n,r}(\mathbb{Z}) = \$_{n+1,\mathbb{Z}_r}$ in De Boor's notation.
- N_i B-spline in $\zeta_{2r-1,r}(\mathbb{Z})$ with support in $[0,2]$ and knots at $0, 1$ and 2 of multiplicity $r-i, r$ and $i+1$ for $i = 0, \dots, r-1$.
- For $s = 0, \dots, r-1$, B_s Schoenberg spline in $\zeta_{2r-1,r}(\mathbb{Z})$ and support in $[0,2]$ such that for all $j = 0, \dots, r-1$,
$$B_s^{(j)}(1) = \delta_{sj}$$
Note that $B_s = L_s(. - 1) = \phi_{s+1}(. - 1)$ in my notation for order r Hermite interpolation.
- $V_0 = \zeta_{2r-1,r}(\mathbb{Z}) \cap L_2$, $V_1 = \zeta_{2r-1,r}(\frac{1}{2}\mathbb{Z}) \cap L_2$ and W such that $V_1 = V_0 \bigoplus W$.
- $T_s = \{f \in V_1 : f^{(j)}{}_{|\mathbb{Z}} = 0, 0 \leq j \leq r-1, j \neq s\}$
- $U_s = \{f \in \zeta_{4r-1,r}(\frac{1}{2}\mathbb{Z}) : f^{(j)}{}_{|\mathbb{Z}} = 0, 0 \leq j \leq r-1, 2r \leq j \leq 3r-1, j \neq 2r+s\}$
- $U = \{f \in \zeta_{4r-1,r}(\frac{1}{2}\mathbb{Z}) : f^{(j)}{}_{|\mathbb{Z}} = 0, 0 \leq j \leq r-1\}$

$B_0, \dots, B_{r-1}$ forms a basis for $\zeta_{2r-1,r}(\mathbb{Z})_{|[0,2]}$. Any $f \in \zeta_{2r-1,r}(\mathbb{Z})$ with support in $[k, k+N]$ can be written as

$$f = \sum_{i=k}^{k+N-2} \sum_{j=0}^{r-1} f^{(j)}(i+1) B_j(. - i)$$

This is nothing more than Schoenberg's C.H.I.P theorem 4 according to which

$$\begin{aligned} f &= \sum_{i\in\mathbb{Z}} \sum_{j=0}^{r-1} f^{(j)}(i) L_j(.-i) \\ &= \sum_{i=k+1}^{k+N-1} \sum_{j=0}^{r-1} f^{(j)}(i) L_j(.-i) \\ &= \sum_{i=k}^{k+N-2} \sum_{j=0}^{r-1} f^{(j)}(i+1) L_j(.-i-1) \end{aligned}$$

Considering $f = B_s(\frac{\cdot}{2})$ that has support in $[0,4]$ we obtain equation (2.3) in the article. Goodman states that it is only for $m = r$ that Schoenberg splines $L_s = L_{2m,r,s}$ have compact support. Proof of that claim? In his other article [7], Goodman proved that $\{N_j(.-i)\}_{i\in\mathbb{Z}, j=0,\dots,r-1}$ is a Riesz-basis for V_0 which has the consequence that $\{B_j(.-i)\}_{i\in\mathbb{Z}, j=0,\dots,r-1}$ **is also a Riesz basis**.

Objective: For even r, construct $\psi_s \in [0, r+2]$ such that $\{\psi_s(.-i)\}_{i\in\mathbb{Z}, s=0,\dots,r-1}$ is a Riesz-basis for W. Then from [6], $\psi_0, \dots, \psi_{r-1}$ are wavelets of multiplicity r.

Lemma 2.1. *$f \in W \cap T_s$ with support in $[a,b] \implies \exists! g \in U_s$ with support in $[a,b]$ s.t $g^{(2r)} = f$ Conversely, $g \in U_s$ with support in $[a,b] \implies g^{(2r)} \in W \cap T_s$.*

Goodman now constructs $\Psi_s \in U_s$ to define then $\psi_s = \Psi_s^{(2r)}$. Consider

$$S(x) = \sum_{r}^{2r-1} a_j x^j + \sum_{3r}^{4r-1} a_j x^j + \sum_{3r}^{4r-1} b_j (x - \frac{1}{2})_+^j \tag{2}$$

Let $\pi_s(\lambda, t) = \pi_s(t)$ the determinant of the system of $3r+1$ linear equations on

$$T(x) = S(x) + c\frac{x^{2r+s}}{(2r+s)!} \quad 0 \le x \le 1 \tag{3}$$

Then $\pi_s(\lambda, t)$ can be viewed as a polynomial of $r+2$ on λ (degree less than or equal to $r+1$) with coefficients depending on s and t as follows

$$\pi_s(\lambda, t) = \sum_{k=0}^{r+1} \Phi_{s,k}(t) \lambda^{r+1-k} \tag{4}$$

Equations on T translate to equations on π_s as follows

$$\begin{cases} \pi_s^{(j)}(1) = \pi_s^{(j)}(0) = 0 & , 0 \le j \le r-1, 2r \le j \le 3r-1, j \ne 2r+s \\ \pi_s^{(j)}(1) = \lambda \pi_s^{(j)}(0) & , r \le j \le 2r-1, j = 2r+s \\ \pi_s(t+1) = \lambda \pi_s(t) & , t \in \mathbb{R} \end{cases} \tag{5}$$

Define

$$\Psi_s(t) = \Phi_{s,k}(t-k) \quad \text{for} \quad k \le t < k+1, 0 \le k \le r+1$$

Then Ψ_s has support in $[0, r+2]$ and lies in U_s. From lemma, $\psi_s = \Psi_s^{(2r)}$ lies in $W \cap T_s$ and has support in $[0, r+2]$. Note that

$$\pi_s(t) = \sum_{k=-\infty}^{\infty} \Psi_s(t-k) \lambda^{r+1+k}$$

$$\pi(\lambda) = \sum_{k=-\infty}^{\infty} \Psi_s^{(2r+s)}(r+1-k) \lambda^k$$

3 Properties of wavelets

Let's show now that $\{\psi_s(. - i)\}_{i\in\mathbb{Z}, s=0,\dots,r-1}$ is a Riesz-basis for W.

Lemma 3.1. *For $s = 0, \dots, r-1$ and any real number λ, the function $\pi_s = \pi_s(\lambda, .)$ does not vanish identically on $\mathbb{R}$.*

Lemma 3.2. *For $0 \le s \le r-1$, the functions $\Phi_{s,i}$ $(i = 0, \dots, r+1)$ are linearly independent on $[0, \frac{1}{2}]$ and on $[\frac{1}{2}, 1]$.*

Lemma 3.3. *For $0 \le s \le r-1$, any function f in U_s can be written uniquely in the form*

$$f = \sum_{i=-\infty}^{\infty} c_i \psi_s(. - i)$$

for some constants c_i. Moreoever, $\exists K$ such that

$$\forall f, \forall j, \forall i = j - r - 1, \dots, j, \quad |c_i| \le K \|f_{|[j,j+1]}\|_\infty$$

Proof. In the proof Goodman claims that $U_{s|[0,1]}$ has dimension $r+2$ which is true for the reason that the interpolation problem of finding $g \in \zeta_{4r-1,r}(\frac{1}{2}\mathbb{Z})$ for values

$$\begin{cases} g^{(j)}(0) & , j = 0, \dots, 3r-1 \\ g^{(j)}(1) & , j = 0, \dots, r-1, 2r, \dots, 3r-1 \end{cases}$$

has a unique solution while a fonction in U_s already satisfies

$$\begin{cases} g^{(j)}(0) = 0 & , j = 0, \dots, r-1, 2r, \dots, 3r-1, j \neq 2r+s \\ g^{(j)}(1) = 0 & , j = 0, \dots, r-1, 2r, \dots, 3r-1, j \neq 2r+s \end{cases}$$

which leaves $r+2$ free data choices.
Given that $\Phi_{s,i}$ lie in $U_{s|[0,1]}$ and that by lemma 3.2 they are linearly independent, they form a basis for the previous space. As $\Phi_{s,i}(t) = \Psi_s(t+i)$ for $0 \le t \le 1, 0 \le i \le r+1$ we can uniquely write for any $f \in U_s$

$$f(x) = \sum_{i=0}^{r+1} c_i \Psi_s(x+i), \quad 0 \le x \le 1$$

Using the fact $\zeta_s = \mathrm{span}(\Phi_{s,0})$ leads to

$$f(x) = \sum_{i=-1}^{r+1} c_i \Psi_s(x+i), \quad 0 \le x \le 2$$

Regarding the second part of the lemma, the argument on equivalent norms in finite dimension **is not clear**. I do agree that $U_{s|[j,j+1]}$ is finite dimensional (dimension $r+2$) and that therefore norms on it are equivalent but for which norm do we already have

$$\max_{j-r-1 \le i \le j} |c_i| \le C \|f_{|[j,j+1]}\|$$

? □

Theorem 3.1. *Any bounded function f in U can be written uniquely in the form*

$$f = \sum_{s=0}^{r-1} \sum_{i=-\infty}^{\infty} c_i^{(s)} \Psi_s(. - i) \tag{6}$$

for uniformly bounded constants $c_i^{(s)}$. Moreoever, if $f(x)$ decays exponentially as $|x| \to \infty$, then $c_i^{(s)}$ decays exponentially as $|i| \to \infty$.

Remember from lemma 2.1 applied to Ψ_s that ψ_s lies in $W \cap T_s$ and has support in $[0, r+2]$.

Theorem 3.2. *Let $0 \le s \le r-1$. Any element in $W \cap T_s$ with support in $[0, r+2]$ is a constant multiple of ψ_s. The function ψ_s does not have support on any interval $[a,b] \subsetneq [0, r+2]$ and for any $0 \le j \le r+1$ does not vanish identically on $[j, j+1]$. Moreover, ψ_s is symmetric or anti-symmetric about $\frac{r}{2}+1$.*

Proof. We have

$$f = \sum_{i=-\infty}^{\infty} c_i \Psi_s(. - i) = \sum_{i=\infty}^{\infty} c_{-i} \Phi_{s,i}$$

Linear independence of the $\Phi_{s,i}$ from lemma 3.2 is used to deduce that f reduces to $f = c_0 \Psi_s$ as follows

$$f_{|[-1,0]} = \sum_{i=1}^{r+2} c_{-i} \Phi_{s,i} = 0$$

and $\Phi_{s,i}$ $(i = 1, \dots, r+2)$ are linearly independent on $[-1, \frac{-1}{2}]$. Similar argument on $[r+2, r+3]$ leads to $c_1 = \dots = c_{r+2} = 0$. Added to the fact that f is supported in, $[0, r+2]$, only c_0 may be non zero i.e $f = c_0 \Psi_s$ and $g = c_0 \psi_s$. $\square$

Theorem 3.3. *For $0 \le s \le r-1$ and any integer j, the sequence $\{\psi_s(. - i)\}_{i \in \mathbb{Z}}$ is locally linearly independent on $(j, j+1)$.*

Remark 1. *Interestingly $\{\psi_{s,i}\}_i$ are not locally linearly independent on $(0, \frac{1}{2})$. To see this, note that*

$$W \cap T_{s|(0,\frac{1}{2})} = \{p \in \pi_{2r-1|(0,\frac{1}{2})}, p^{(j)}(0) = 0, 0 \le j \le r-1, j \ne s\} \tag{7}$$

and that the latter is a vector space of dimension $r+1$, less than the $r+2$ $\psi_{i,s}$ that have support overlapping $(0, \frac{1}{2})$

Theorem 3.4. *Any function in V_1 can be uniquely written in the form*

$$f = \sum_{s=0}^{r-1} \sum_{i=-\infty}^{\infty} b_i^{(s)} B_s(. - i) + \sum_{s=0}^{r-1} \sum_{i=-\infty}^{\infty} c_i^{(s)} \psi_s(. - i) \tag{8}$$

with sequences $(b_i^{(s)})_i$, $(c_i^{(s)})_i$ in l^2. Moreover if $f(x)$ decays exponentially as $|x| \to \infty$ so do $b_i^{(s)}, c_i^{(s)}$ as $|i| \to \infty$.

Proof. Goodman first considers the case where f has support in $[a,b]$ and claims that there exists then a unique function F in $\zeta_{4r-1,r}(\frac{1}{2}\mathbb{Z})$ that vanishes on $(-\infty, a)$ and satisfies $F^{(2r)} = f$. **Why is that?** Maybe this is simply polynomial interpolation of order $2r$ on each segment $[j, j+\frac{1}{2}]$ and $[j+\frac{1}{2}, j+1]$. As $F^{(2r)} \in \zeta_{2r-1,r}(\mathbb{Z})$ (see p115 De Boor's book), $F^{(2r)}$ is completely determined if we have r interpolation conditions on the half-integer grid $\frac{1}{2}\mathbb{Z}$. Taking as interpolation conditions the first r derivatives (0 to $r-1$) of f at each half-integer leads to $F^{(2r)} = f$ with $F^{(2r)}$ vanishing identically on $(-\infty, a)$ and on (b, ∞).

Then one can write $F = S + \Psi$ with $S \in \zeta_{4r-1.r}(\mathbb{Z}), \Psi \in U$. I am not sure **how Schoenberg's theory applies to prove that $S(x)$ decays exponentially** as $x \to -\infty$?

According to Goodman any function f in V_1 can be written

$$f = \sum_{j=0}^{r-1} \sum_{k=-\infty}^{\infty} a_k^{(j)} B_j(2x - k)$$

with $a_j = (a_k^{(j)})_k$ in l^2 satisfying

$$\|a_j\|_2 \le C \|f\|_2$$

for some C. **Why is that?**. $\square$

Corollary 3.2. *The function $\{\psi_s(. - i)\}_{i \in \mathbb{Z}, 0 \le s \le r-1}$ form a Riesz basis for W.*