Parametrisation of the sphere

Yoann Pradat

May 3, 2019

1 Cardinal Hermite exponential splines

1.1 The parametric model

Conti et al's paper Ellipse-preserving interpolation and subdivision scheme introduces two basis functions from the space $\mathcal{E}_4 =< 1, x, e^{-iw_1 x}, e^{iw_1 x} >$ where $w = \frac{2\pi}{M}$ to reproduce closed curves with M control points. The corresponding parametric representation is

$$r(t) = \sum_{k \in \mathbb{Z}} r(k)\phi_{1,w}(t-k) + r'(k)\phi_{2,w}(t-k) \tag{1}$$

with r and r' assumed to be M-periodic.

The basis functions are **cycloidal splines** (Exponential splines? Exponential B-splines?) given by

$$\phi_{1,w}(x) = \begin{cases} g_{1,w}(x) & \text{for } x \geq 0 \\ g_{1,w}(-x) & \text{for } x < 0 \end{cases} \quad \phi_{2,w}(x) = \begin{cases} g_{2,w}(x) & \text{for } x \geq 0 \\ -g_{2,w}(-x) & \text{for } x < 0 \end{cases} \tag{2}$$

The resulting parametric model has the following properties

1. Unique and stable representation ($\{\phi_{\mathbf{w}}(. - \mathbf{k}) = (\phi_{1,w}(. - k), \phi_{2,w}(. - k))\}_k$ Riesz basis)
2. Affine invariance (partition unity condition on ϕ_1)
3. Perfectly reproduce sinusoids of period M
4. Exact interpolation of points and first derivative
5. Support of ϕ_1, ϕ_2 is $[-1, 1]$
6. Hermite interpolation property of order 1
7. C^1-continuous

1.2 The unit sphere with scaling factors w_1, w_2

The usual continuous representation of the sphere is given by

$$\sigma(u, v) = (\cos(2\pi u)\sin(\pi v), \sin(2\pi u)\sin(\pi v), \cos(\pi v)) \quad (u, v) \in [0, 1]^2 \tag{3}$$

Suppose we have M_1 control points on latitudes, M_2 control points on meridians. The control points are then $c[k, l]_{k \in [0,\dots,M_1-1], l \in [0,\dots,M_2-1]}$. Let $w_1 = \frac{2\pi}{M_1}, w_2 = \frac{\pi}{M_2}$.

From the paper we have (also holds for sin functions)

$$\begin{aligned} \forall u \in [0, M_1] \quad \cos(w_1 u) &= \sum_{k \in \mathbb{Z}} \cos(w_1 k)\phi_{1,w_1}(u-k) - w_1 \sin(w_1 k)\phi_{2,w_1}(u-k) \\ \forall v \in [0, 2M_2] \quad \cos(w_2 v) &= \sum_{l \in \mathbb{Z}} \cos(w_2 l)\phi_{1,w_2}(v-l) - w_2 \sin(w_2 l)\phi_{2,w_2}(v-l) \end{aligned}$$

Normalizing the the continuous parameters leads to

$$\forall u \in [0,1] \quad \cos(2\pi u) = \sum_{k\in\mathbb{Z}} \cos(w_1 k)\phi_{1,w_1}(M_1 u - k) - w_1 \sin(w_1 k)\phi_{2,w_1}(M_1 u - k)$$
$$\forall v \in [0,2] \quad \cos(\pi v) = \sum_{l\in\mathbb{Z}} \cos(w_2 l)\phi_{1,w_2}(M_2 v - l) - w_2 \sin(w_2 l)\phi_{2,w_2}(M_2 v - l)$$

Be aware that in the first representations above $\{\cos(w_1 k), -w_1 \sin(w_1 k)\}$ is (M_1, M_1)-periodic i.e we need point and first derivative values at M_1 control points for a full representation. However in the second representation $\{\cos(w_2 l), -w_2 \sin(w_2 l)\}$ are $(2M_2, 2M_2)$-periodic i.e we need point and first derivative values at $2M_2$ control points for a full representation.

1.3 Representation of the sphere

For all $(u,v) \in [0,1]^2$

$$\begin{aligned}
\sigma(u,v) = &\sum_{(k,l)\in\mathbb{Z}^2} c_1[k,l]\phi_{1,w_1}(M_1 u - k)\phi_{1,w_2}(M_2 v - l)\\
+ &\sum_{(k,l)\in\mathbb{Z}^2} c_2[k,l]\phi_{1,w_1}(M_1 u - k)\phi_{2,w_2}(M_2 v - l)\\
+ &\sum_{(k,l)\in\mathbb{Z}^2} c_3[k,l]\phi_{2,w_1}(M_1 u - k)\phi_{1,w_2}(M_2 v - l)\\
+ &\sum_{(k,l)\in\mathbb{Z}^2} c_4[k,l]\phi_{2,w_1}(M_1 u - k)\phi_{2,w_2}(M_2 v - l)
\end{aligned}$$

Or equivalently for all $(u,v) \in [0,1]^2$

$$\begin{aligned}
\sigma(u,v) = &\sum_{k=0}^{M_1}\sum_{l=0}^{M_2} c_1[k,l]\phi_{1,w_1}(M_1 u - k)\phi_{1,w_2}(M_2 v - l)\\
+ &\sum_{k=0}^{M_1}\sum_{l=0}^{M_2} c_2[k,l]\phi_{1,w_1}(M_1 u - k)\phi_{2,w_2}(M_2 v - l)\\
+ &\sum_{k=0}^{M_1}\sum_{l=0}^{M_2} c_3[k,l]\phi_{2,w_1}(M_1 u - k)\phi_{1,w_2}(M_2 v - l)\\
+ &\sum_{k=0}^{M_1}\sum_{l=0}^{M_2} c_4[k,l]\phi_{2,w_1}(M_1 u - k)\phi_{2,w_2}(M_2 v - l)
\end{aligned}$$

Or equivalently for all $(u,v) \in [0,1]^2$

$$\begin{aligned}
\sigma(u,v) = &\sum_{k=0}^{M_1-1}\sum_{l=0}^{2M_2-1} c_1[k,l]\phi_{1,w_1,per}(M_1 u - k)\phi_{1,w_2,per}(M_2 v - l)\\
+ &\sum_{k=0}^{M_1-1}\sum_{l=0}^{2M_2-1} c_2[k,l]\phi_{1,w_1,per}(M_1 u - k)\phi_{2,w_2,per}(M_2 v - l)\\
+ &\sum_{k=0}^{M_1-1}\sum_{l=0}^{2M_2-1} c_3[k,l]\phi_{2,w_1,per}(M_1 u - k)\phi_{1,w_2,per}(M_2 v - l)\\
+ &\sum_{k=0}^{M_1-1}\sum_{l=0}^{2M_2-1} c_4[k,l]\phi_{2,w_1,per}(M_1 u - k)\phi_{2,w_2,per}(M_2 v - l)
\end{aligned}$$

Or equivalently for all $(u, v) \in [0, 1]^2$r

$$\boxed{\begin{aligned}\sigma(u,v) = &\sum_{k=0}^{M_1-1}\sum_{l=0}^{M_2} c_1[k,l]\phi_{1,w_1,per}(M_1u-k)\phi_{1,w_2}(M_2v-l) \\ &+ \sum_{k=0}^{M_1-1}\sum_{l=0}^{M_2} c_2[k,l]\phi_{1,w_1,per}(M_1u-k)\phi_{2,w_2}(M_2v-l) \\ &+ \sum_{k=0}^{M_1-1}\sum_{l=0}^{M_2} c_3[k,l]\phi_{2,w_1,per}(M_1u-k)\phi_{1,w_2}(M_2v-l) \\ &+ \sum_{k=0}^{M_1-1}\sum_{l=0}^{M_2} c_4[k,l]\phi_{2,w_1,per}(M_1u-k)\phi_{2,w_2}(M_2v-l)\end{aligned}}$$

$$\boxed{\begin{aligned} c_1[k,l] &= \begin{bmatrix}\cos(w_1k)\sin(w_2l)\\ \sin(w_1k)\sin(w_2l)\\ \cos(w_2l)\end{bmatrix} = \sigma(w_1k, w_2l) & c_2[k,l] &= \begin{bmatrix}w_2\cos(w_1k)\cos(w_2l)\\ w_2\sin(w_1k)\cos(w_2l)\\ -w_2\sin(w_2l)\end{bmatrix} = \frac{\partial\sigma}{\partial v}(w_1k, w_2l) \\ c_3[k,l] &= \begin{bmatrix}-w_1\sin(w_1k)\sin(w_2l)\\ w_1\cos(w_1k)\sin(w_2l)\\ 0\end{bmatrix} = \frac{\partial\sigma}{\partial u}(w_1u, w_2v) & c_4[k,l] &= \begin{bmatrix}-w_1w_2\sin(w_1k)\cos(w_2l)\\ w_1w_2\cos(w_1k)\cos(w_2l)\\ 0\end{bmatrix} = \frac{\partial^2\sigma}{\partial u\partial v}(w_1u, w_2v)\end{aligned}}$$

$$\begin{aligned} \phi_{1,w_1,per}(.) &= \sum_{k\in\mathbb{Z}}\phi_{1,w_1}(.-M_1k) & \phi_{1,w_2,per}(.) &= \sum_{k\in\mathbb{Z}}\phi_{1,w_2}(.-2M_2k) \\ \phi_{2,w_1,per}(.) &= \sum_{k\in\mathbb{Z}}\phi_{2,w_1}(.-M_1k) & \phi_{2,w_2,per}(.) &= \sum_{k\in\mathbb{Z}}\phi_{2,w_2}(.-2M_2k)\end{aligned}$$

2 Exponential B-splines in 3D

2.1 The parametric model

Delgado et al's paper Spline-based deforming ellipsoids for 3D bioimage segmentation derive an exponential B-splines-based model that allow to reproduce ellipsoids. The model can well approximate blobs and perfectly spheres and ellipsoids. The corresponding parametric representation is

$$\sigma(u,v) = \sum_{(i,j)\in\mathbb{Z}^2} c[i,j]\phi_1(\frac{u}{T_1}-i)\phi_2(\frac{v}{T_2}-j) \tag{4}$$

with $T_1, T_2 > 0$ sampling steps for each parametric dimension and $\{c[i,j]\}_{(i,j)\in\mathbb{Z}^2}$ are the 3D control points.

The basis functions, reproducing unit period sinusoids with M coefficients, are exponential B-splines given by

$$\varphi_M(.) = \sum_{k=0}^{3}(-1)^k h_M[k]\varsigma_M(.+\frac{3}{2}-k) \tag{5}$$

where $\varsigma_M(.) = \frac{1}{4}sgn(.)\frac{\sin^2(\frac{\pi}{M}.)}{\sin^2(\frac{\pi}{M})}$ and $h_M = [1, 1+2\cos(\frac{2\pi}{M}), 1+2\cos(\frac{2\pi}{M}), 1]$.

Suppose we have M_1 control points on latitudes, M_2 control points on meridians. The resulting parametric model has the following properties

1. Unique and stable representation (sufficient is $\{\phi_1(. - k)\}_k, \{\phi_2(. - k)\}_k$ Riesz basis)
2. Affine invariance (partition unity condition on ϕ_1, ϕ_2)
3. Well-defined Gaussian curvature. ϕ_1, ϕ_2 are twice differentiable with bounded second derivative
4. Perfectly reproduce ellipsoids
5. Support of $\phi_1 = \varphi_{M_1}, \phi_2 = \varphi_{2M_2}$ is $[-\frac{3}{2}, \frac{3}{2}]$

2.2 Conditions for representing the unit sphere

The parametric representation of a closed surface with sphere-like topology, M_1 control points on latitudes and M_2 control points on meridians is

$$\forall (u,v) \in [0,1]^2 \quad \sigma(u,v) = \sum_{k=0}^{M_1-1} \sum_{l=-1}^{M_2+1} c[k,l]\phi_1(M_1 u - k)\phi_2(M_2 v - l) \tag{6}$$

Unlike before, continuity of points and tangents at poles is not guaranteed. Conditions are

$$\forall k = 0, \dots, M_1 - 1 \quad \mathbf{c_N} = c[k,1]\phi_2(-1) + c[k,0]\phi_2(0) + c[k,-1]\phi_2(1) \tag{7}$$

$$\mathbf{c_S} = c[k, M_2+1]\phi_2(-1) + c[k, M_2]\phi_2(0) + c[k, M_2-1]\phi_2(1) \tag{8}$$

$$\mathbf{T_{1,N}} \cos(2\pi u) + \mathbf{T_{2,N}} \sin(2\pi u) = M_2 \sum_{k=0}^{M_1-1} \sum_{l=-1}^{M_2+1} c[k,l]\phi_1(M_1 u - k)\phi_2'(-l) \tag{9}$$

$$\mathbf{T_{1,S}} \cos(2\pi u) + \mathbf{T_{2,S}} \sin(2\pi u) = M_2 \sum_{k=0}^{M_1-1} \sum_{l=-1}^{M_2+1} c[k,l]\phi_1(M_1 u - k)\phi_2'(M_2 - l) \tag{10}$$

$$\tag{11}$$

Incorporating such conditions in the model, a parametric splines-based surface with a sphere-like topology, C^1 continuity and ellipsoid-reproducing capabilities (all positions and orientations) is given by

$$\forall (u,v) \in [0,1]^2 \quad \sigma(u,v) = \sum_{k=0}^{M_1-1} \sum_{l=-1}^{M_2+1} c[k,l]\phi_{1,per}(M_1 u - k)\phi_2(M_2 v - l) \tag{12}$$

where $\{c[i,j]\}_{i \in [0,\dots,M_1-1], j \in [1,\dots,M_2-1]}, \mathbf{T_{1,N}}, \mathbf{T_{2,N}}, \mathbf{T_{1,S}}, \mathbf{T_{1,S}}, \mathbf{c_N}, \mathbf{c_S}$ are free parameters that is $M_1(M_2 - 1) + 6$ control points.

$c[k,-1], c[k,0], c[k,M_2], c[k,M_2+1]$ are constrained by the values of the free parameters.

2.3 Representation of the sphere

The unit sphere is thus represented by

$$\boxed{\forall (u,v) \in [0,1]^2 \quad \sigma(u,v) = \sum_{k=0}^{M_1-1} \sum_{l=-1}^{M_2+1} c[k,l]\phi_{1,per}(M_1 u - k)\phi_2(M_2 v - l)} \tag{13}$$

With coefficients are given by

$$\boxed{c[k,l] = \begin{bmatrix} c_{M_1}[k] s_{2M_2}[l] \\ s_{M_1}[k] s_{2M_2}[l] \\ c_{2M_2}[l] \end{bmatrix} = \begin{bmatrix} \frac{2(1-\cos(\frac{2\pi}{M_1}))}{\cos(\frac{\pi}{M_1})-\cos(\frac{3\pi}{M_1})} \frac{2(1-\cos(\frac{\pi}{M_2}))}{\cos(\frac{\pi}{2M_2})-\cos(\frac{3\pi}{2M_2})} \cos(\frac{2\pi k}{M_1}) \sin(\frac{\pi l}{M_2}) \\ \frac{2(1-\cos(\frac{2\pi}{M_1}))}{\cos(\frac{\pi}{M_1})-\cos(\frac{3\pi}{M_1})} \frac{2(1-\cos(\frac{\pi}{M_2}))}{\cos(\frac{\pi}{2M_2})-\cos(\frac{3\pi}{2M_2})} \sin(\frac{2\pi k}{M_1}) \sin(\frac{\pi l}{M_2}) \\ \frac{2(1-\cos(\frac{\pi}{M_2}))}{\cos(\frac{\pi}{2M_2})-\cos(\frac{3\pi}{2M_2})} \cos(\frac{\pi l}{M_2}) \end{bmatrix}} \tag{14}$$

and

$$
\begin{aligned}
c_M[k] &= \frac{2(1-\cos(\frac{2\pi}{M}))}{\cos(\frac{\pi}{M})-\cos(\frac{3\pi}{M})}\cos(\frac{2\pi k}{M}) \\
s_M[k] &= \frac{2(1-\cos(\frac{2\pi}{M}))}{\cos(\frac{\pi}{M})-\cos(\frac{3\pi}{M})}\sin(\frac{2\pi k}{M})
\end{aligned}
$$

These coefficients satisfy the constraints with

$$
\begin{aligned}
\mathbf{c_N} &= [0\ 0\ 1]^T & \mathbf{c_N} &= [0\ 0\ -1]^T & \mathbf{T_{1,N}} &= [\pi\ 0\ 0]^T \\
\mathbf{T_{2,N}} &= [0\ \pi\ 0]^T & \mathbf{T_{1,S}} &= [-\pi\ 0\ 0]^T & \mathbf{T_{2,S}} &= [0\ -\pi\ 0]^T
\end{aligned}
$$

3 Compactly-supported smooth interpolators for shape modeling

3.1 The parametric model

Schmitter et al's paper Compactly-supported smooth interpolators for shape modeling with varying resolution propose a continuous representation of curves and surfaces with the help of generators that have the advantages of both continuous and discrete schemes. The generator is expressed as a linear combination of half integer shifts of exponential B-spline of vector $\alpha \in \mathbb{C}^n$ i.e

$$\phi_{\lambda,\alpha}(t) = \sum_{k\in\mathbb{Z}} \lambda[k]\beta_\alpha(t - \frac{k}{2}) \tag{15}$$

β_α has support $[-\frac{n}{2}, \frac{n}{2}]$. In what follows we choose to have $\lambda[k] = 0$ for $k \notin [\![-n+2, n-2]\!]$ and $\lambda[-k] = \lambda[k]$. There are therefore $(n-1)$ unknowns $\lambda[0], \dots, \lambda[n-2]$. We also impose that elements in α are 0 or come in complex conjugate pairs and that no pair of purely imaginary elements of α is separated by integer multiple of $2j\pi$ (for Riesz basis property).

This function is interpolatory if and only if $\phi_{\lambda,\alpha}(0) = 1$ and $\phi_{\lambda,\alpha}(1) = \cdots = \phi_{\lambda,\alpha}(n-2) = 0$. This defines a system of $n-1$ equations with $n-1$ unknowns. The system has a solution if the matrix defined by $k, l = 0, \dots, n-2$

$$[A_\alpha]_{k+1,l+1} = \begin{cases} \beta_\alpha(k) & \text{if } l = 0 \\ \beta_\alpha(k - \frac{l}{2}) + \beta_\alpha(k + \frac{l}{2}) & \text{else} \end{cases} \tag{16}$$

is invertible. In that case $\lambda = A_\alpha^{-1}(1, 0, \dots, 0)$ and we define $\phi_\alpha = \phi_{\lambda,\alpha}$. Tensor-product surfaces are represented with the help of two generators in the form

$$\sigma(u, v) = \sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} \sigma[k, l]\phi_{\alpha_1}(u - k)\phi_{\alpha_2}(v - l) \tag{17}$$

The resulting interpolation scheme has the following properties

1. Unique and stable representation ($\alpha_m - \alpha_n \notin 2j\pi\mathbb{Z}$ Riesz basis)
2. Affine invariance ($0 \in \alpha_1, 0 \in \alpha_2$)
3. Perfectly reproduce ellipsoids (conditions on α)
4. ϕ_α is interpolatory
5. ϕ_α is smooth i.e at least $\mathcal{C}^1$
6. Can reproduce the nullspace $\mathcal{N}_\alpha$
7. Can reproduce shapes at various resolutions
8. ϕ_α is compactly supported on $[-n+1, n-1]$

3.2 Conditions for representing the unit sphere

Let M_1 be the number of control points in u and M_2 the number of control points in v. For ϕ_{α_1} to be able to reproduce $\cos(\frac{2\pi.}{M_1}), \sin(\frac{2\pi.}{M_1})$ we need to have $(\frac{-2i\pi}{M_1}, \frac{2i\pi}{M_1}) \in \alpha_1$. Adding affine invariance condition, $\phi_{(0, \frac{-2i\pi}{M_1}, \frac{2i\pi}{M_1})}$ can reproduce constants and M_1-periodic sinusoids with M_1 control points as follows

$$\cos(\frac{2\pi.}{M_1}) = \sum_{k\in\mathbb{Z}} \cos(\frac{2\pi k}{M_1})\phi_{\alpha_1}(. - k) \tag{18}$$

Similarly $\phi_{(0,\frac{-i\pi}{M_2},\frac{i\pi}{M_2})}$ can reproduce constants and $2M_2$-periodic sinusoids with $2M_2$ control points as follows

$$\cos(\frac{\pi .}{M_2}) = \sum_{k\in\mathbb{Z}} \cos(\frac{\pi k}{M_2})\phi_{\alpha_2}(. - k) \tag{19}$$

Generators $\phi_{\alpha_1}, \phi_{\alpha_2}$ both have support of size 4 ($n = 3$) so that they are given by

$$\phi_{\alpha_1}(t) = \lambda_1[0]\beta_{\alpha_1}(t) + \lambda_1[1](\beta_{\alpha_1}(t - 1/2) + \beta_{\alpha_1}(t + 1/2))$$
$$\phi_{\alpha_2}(t) = \lambda_2[0]\beta_{\alpha_2}(t) + \lambda_2[1](\beta_{\alpha_2}(t - 1/2) + \beta_{\alpha_2}(t + 1/2))$$

In order to find $\lambda_1[0], \lambda_1[1]$ one has to solve $\phi_{\alpha_1}(0) = 1, \phi_{\alpha_1}(1) = 0$.

Aparte on tempered distributions

Green function of operator $L_\alpha : \mathcal{S}(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$ is an element ρ_α of $\mathcal{S}(\mathbb{R})$ that satisfies $L\{\rho_\alpha\} = \delta$ where δ is the Dirac tempered distribution. There is a unique such function (to be proved?) that also satisfies $\forall t < 0\ \rho_\alpha(t) < 0$. The tempered distribution $T_{\rho_\alpha} : \phi \mapsto \int_0^\infty e^{\alpha t}\phi(t)dt$ is such that the associated element of $\mathcal{S}(\mathbb{R})$ (bijection $\mathcal{S}(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$? probably not true), ρ_α satisfies $L_\alpha\{\rho_\alpha\} = \delta$.

Consequently β_α^+ is an element of Schwartz space $\mathcal{S}(\mathbb{R})$. **Is that right**? With abuse of notation we write $\beta_\alpha^+(t) = e^{\alpha t}\chi_{[0,1]}(t)$. Using distribution then we would have

$$\beta_\alpha^{+\prime}(t) = \delta(t) + \alpha e^{\alpha t}\chi_{[0,1]}(t)$$

The equality is to be taken in the distribution sense.

3.3 Representation of the sphere

The unit sphere is thus represented by

$$\forall (u, v) \in [0, 1]^2 \quad \sigma(u, v) = \sum_{(k,l)\in\mathbb{Z}^2} c[k, l]\phi_{\alpha_1}(M_1 u - k)\phi_{\alpha_2}(M_2 v - l) \tag{20}$$

Or equivalently

$$\forall (u, v) \in [0, 1]^2 \quad \sigma(u, v) = \sum_{k=0}^{M_1-1}\sum_{l\in\mathbb{Z}} c[k, l]\phi_{\alpha_1,per}(M_1 u - k)\phi_{\alpha_2}(M_2 v - l) \tag{21}$$

Or equivalently

$$\boxed{\forall (u, v) \in [0, 1]^2 \quad \sigma(u, v) = \sum_{k=0}^{M_1-1}\sum_{l=-1}^{M_2+1} c[k, l]\phi_{\alpha_1,per}(M_1 u - k)\phi_{\alpha_2}(M_2 v - l)} \tag{22}$$

Denoting $w_1 = \frac{2\pi}{M_1}, w_2 = \frac{\pi}{M_2}$, the coefficients are given by

$$\boxed{c[k, l] = \begin{bmatrix} \cos(w_1 k)\sin(w_2 l) \\ \sin(w_1 k)\sin(w_2 l) \\ \cos(w_2 l) \end{bmatrix} = \sigma(w_1 k, w_2 l)} \tag{23}$$

4 Smooth shapes with spherical topology

4.1 The parametric model

In 2017 Schmitter et al's paper Smooth shapes with spherical topology derive a parametric model very similar to that presented in Spline-based deforming ellipsoids for 3D bioimage segmentation. In user interactive applications one usually wants a curve/shape reproducing model to have some or all following properties: 1.intuitive manipulation, 2.stable deformation, 3.shape deformation as optimization process requiring fast evaluation of surface and volume integrals, 4.smooth representation. It is usually impossible to find a model optimal w.r.t to all these requirements. In practice a compromise is made with existing models based on polygon meshes, subdivision or NURBS.

Parametric shapes are built as linear combinations of integers shifts of a generator function φ that is to say

$$r(t) = \sum_{k \in \mathbb{Z}} c[k]\varphi(t-k) \tag{24}$$

φ is piecewise exponential. It is the smoothed version of third order exponential B-spline that is $\varphi = \beta * \psi$ with ψ an appropriate smoothing kernel. The model can be extended to tensor-product surfaces $\sigma(u,v)$ as previously done in previous representations.

$$\sigma(u,v) = \sum_{(k,l) \in \mathbb{Z}^2} c[k,l]\varphi_1(u-k)\varphi_2(v-l)$$

Authors define

$$\phi_1(t) = \varphi_{M_1,per}(t) = \sum_{n \in \mathbb{Z}} \varphi_{M_1}(t - M_1 n) \qquad \phi_2(t) = \varphi_{2M_2}(t)$$

$$\forall k \in \mathbb{Z} \quad \phi_{1,k}(t) = \phi_1(M_1 t - k) \qquad \phi_{2,k}(t) = \phi_2(M_2 t - k)$$

The resulting interpolation scheme has the following properties

1. Unique and stable representation ($\{\varphi_M(. - k)\}_k$ Riesz basis)
2. Affine invariance (partition unity condition on φ_M)
3. Well-defined Gaussian curvature. φ_M is twice differentiable with bounded second derivative
4. Perfectly reproduce ellipsoids for $M \geq 3$
5. φ_M is interpolatory
6. Support of φ_M is in $[-2,2]$

4.2 Conditions for representing the unit sphere

The parametric representation of a closed surface with sphere-like topology, M_1 control points on latitudes and M_2 control points on meridians is

$$\forall (u,v) \in [0,1]^2 \quad \sigma(u,v) = \sum_{k=0}^{M_1-1} \sum_{l=-1}^{M_2+1} c[k,l]\phi_{1,k}(u)\phi_{2,k}(v) \tag{25}$$

As for the model from article 7, continuity of points and tangents at poles is not guaranteed.The exact same conditions are used leading to

$$\forall k = 0, \ldots, M_1 - 1 \quad \mathbf{c_N} = c[k, 0] \tag{26}$$

$$\mathbf{c_S} = c[k, M_2] \tag{27}$$

$$\mathbf{T_{1,N}} \cos(2\pi u) + \mathbf{T_{2,N}} \sin(2\pi u) = M_2 \sum_{k=0}^{M_1-1} \sum_{l=-1}^{M_2+1} c[k,l] \varphi_{M_1}(M_1 u - k) \varphi'_{2M_2}(-l) \tag{28}$$

$$\mathbf{T_{1,S}} \cos(2\pi u) + \mathbf{T_{2,S}} \sin(2\pi u) = M_2 \sum_{k=0}^{M_1-1} \sum_{l=-1}^{M_2+1} c[k,l] \varphi_{M_1}(M_1 u - k) \varphi'_{2M_2}(M_2 - l) \tag{29}$$

$$\tag{30}$$

By incorporating conditions in the model to ensure continuity of the surface and of the tangent plane at poles we obtain constraints on $c[k, -1], c[k, 0], c[k, M_2], c[k, M_2 + 1]$. Other values $c[k, l]$ are free parameters as well as $\mathbf{c_N}, \mathbf{c_S}, \mathbf{T_{1,N}}, \mathbf{T_{2,N}}, \mathbf{T_{1,S}}, \mathbf{T_{2,S}}$ describing the poles.

φ_{M_1} can reproduce $\cos(\frac{2\pi \cdot}{M_1})$ with M_1 control points

$$\cos(\frac{2\pi u}{M_1}) = \sum_{k \in \mathbb{Z}} \cos(\frac{2\pi k}{M_1}) \varphi_{M_1}(u - k)$$

$$\cos(2\pi u) = \sum_{k=0}^{M_1-1} \cos(\frac{2\pi k}{M_1}) \phi_{1,k}(u)$$

In a similar fashion φ_{2M_2} can reproduce $\cos(\frac{\pi \cdot}{M_2})$ with $2M_2$ control points i.e

$$\cos(\frac{\pi v}{M_2}) = \sum_{k \in \mathbb{Z}} \cos(\frac{\pi k}{M_2}) \varphi_{2M_2}(v - k)$$

$$\cos(\pi v) = \sum_{k \in \mathbb{Z}} \cos(\frac{\pi k}{M_2}) \phi_{2,k}(v)$$

4.3 Representation of the sphere

Given the usual representation of the unit sphere, it can be represented in our scheme as follows

$$\sigma(u, v) = \sum_{k=0}^{M_1-1} \sum_{l \in \mathbb{Z}} c[k, l] \phi_{1,k}(u) \phi_{2,k}(v) \tag{31}$$

or using the fact φ_{2M_2} has support of size 4

$$\boxed{\forall (u, v) \in [0, 1]^2 \quad \sigma(u, v) = \sum_{k=0}^{M_1-1} \sum_{l=-1}^{M_2+1} c[k, l] \phi_{1,k}(u) \phi_{2,k}(v)} \tag{32}$$

Denoting $w_1 = \frac{2\pi}{M_1}$, $w_2 = \frac{\pi}{M_2}$, the coefficients are given by

$$\boxed{c[k, l] = \begin{bmatrix} \cos(w_1 k) \sin(w_2 l) \\ \sin(w_1 k) \sin(w_2 l) \\ \cos(w_2 l) \end{bmatrix} = \sigma(w_1 k, w_2 l)} \tag{33}$$