Summary paper 44: Exponential Hermite-Euler splines

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This paper by S.Lee extends Schoenberg's exponential splines to the C.H.I.P, defining so-called exponential Euler-Hermite splines.

1 Introduction

Notations

1. n, r positive integers with $n \geq 2r-1$
2. $\mathcal{S}_{n,r}$ cardinal splines of degree n, multiplicity r
3. $\Phi_n(x;t) = \sum_{k=-\infty}^{\infty} t^k Q_{n+1}(x-k)$ exponential Euler spline degree n base t.
4. $\Pi_n(t) = n! \sum_{j=1}^{n} Q_n(j) t^{j-1}$ Euler-Frobenius polynomial. It has $n-1$ simples zeros
$$\lambda_{n-1} < \cdots < \lambda_1 < 0$$
5. $\Pi_{n,r}(t) = (-1)^{m(r-1)} |\Delta_{r,d}(t)|$ Euler-Frobenius polynomial for multiplicity r. It has $2m-2r$ simples zeros
$$\lambda_{2m-2r} < \cdots < \lambda_1 < 0$$
6. $A_n(x;t) = \frac{n!}{(1-t^{-1})^n} \Phi_n(x;t),\ 0 \leq x \leq 1$ exponential Euler polynomial degree n.
$$\frac{t-1}{t-e^z} e^{xz} = \sum_{n=0}^{\infty} \frac{A_n(x;t)}{n!} z^n$$
$$A_n(x;t) = x^n + a_1(t) \binom{n}{1} + \cdots + a_n(t)$$
$$a_n(t) = \frac{\Pi_n(t)}{(t-1)^n}$$
7. $\Phi_n(0;t) = \frac{(1-t^{-1})^n}{n!} A_n(0;t) = \frac{\Pi_n(t)}{n!}$
8. $S_n(x;\lambda) = \frac{\Phi_n(x;\lambda)}{\Phi_n(0;\lambda)}$ for $\lambda \notin \{\lambda_1, \ldots, \lambda_{n-1}\}$ exponential Euler spline.
9. $S_j(x) = \Phi_n(x, \lambda_j)$ for $j = 1, \ldots, n-1$, eigensplines.
$$\forall k \in \mathbb{Z}, S_j(k) = 0$$

As $S_n(x+1,\lambda) = \lambda S_n(x;\lambda)$ and $S_n(0;\lambda) = 1$, $S_n(\cdot,\lambda)$ interpolates $\lambda^{\cdot}$ at integers. As an extensions, Lee proposes to define $S_{n,r}(\cdot,\lambda) \in \mathcal{S}_{n,r}$ that interpolates $\lambda^{\cdot}$ up to $r-1$ at integers i.e $S_{n,r}^{(\rho)}(k) = \lambda^k (\log \lambda)^\rho$.

2 The polynomial $A_{n,r,s}(x;\lambda)$

Let $s = 0, \ldots, r-1$. Define

$$A_{n,r,s}(x;\lambda) = \begin{vmatrix} \frac{A_n(0;\lambda)}{n!} & \cdots & \frac{A_{n-s+1}}{(n-s+1)!} & \frac{A_n(x;\lambda)}{n!} & \cdots & \frac{A_{n-r+1}(0;\lambda)}{(n-r+1)!} \\ \vdots & & & \vdots & & \vdots \\ \frac{A_{n-r+1}(0;\lambda)}{(n-r+1)!} & \cdots & \frac{A_{n-r-s+2}}{(n-r-s+2)!} & \frac{A_n(x;\lambda)}{(n-r+1)!} & \cdots & \frac{A_{n-2r+2}(0;\lambda)}{(n-2r+2)!} \end{vmatrix} \tag{1}$$

As $\frac{A_n'(x;\lambda)}{n!} = \frac{A_{n-1}(x;\lambda)}{(n-1)!}$, one has that

$$A^{(s)}_{n,r,s}(0;\lambda) = H_r(\frac{A_n(0;\lambda)}{n!})$$

with H_r Hankel determinant of order r.

From (LS76, Theorem 4), the following holds

$$H_r(\frac{\Pi_n(\lambda)}{n!}) = (-1)^{[\frac{r}{2}]} C_{n,r} \Pi_{n,r}(\lambda)$$

3 Exponential Euler-Hermite splines

The functions $S_{n,r,s}(x;\lambda)$ defined by

$$\begin{aligned} S_{n,r,s}(x;\lambda) &= \frac{A_{n,r,s}(x;\lambda)}{H_r(\frac{A_n(0;\lambda)}{n!})}, \quad 0 \le x \le 1 \\ S_{n,r,s}(x+1;\lambda) &= \lambda S_{n,r,s}(x;\lambda), \quad \forall x \end{aligned}$$

is such that

$$\begin{aligned} S^{(\rho)}_{n,r,s}(k;\lambda) &= 0, \quad \rho = 0, \ldots, r-1, \rho \neq s \\ S^{(s)}_{n,r,s}(k,\lambda) &= \lambda^k \end{aligned}$$

so that $S_{n,r,s} \in \mathcal{S}^{(s)}_{n,r}$.

Theorem 2.1. *The spline function*

$$S_{n,r}(x;\lambda) = \sum_{s=0}^{r-1} (\log \lambda)^s S_{n,r,s}(x;\lambda)$$

interpolates $\lambda^{\cdot}$ up to $r-1$ at integers and belongs to $S_{n,r}$.

Here is the representation of exponential Euler-Hermite splines in terms of B-splines.

Theorem 3.1. *The exponential Euler-Hermite spline $S_{2m-1,r,s}(x;\lambda)$ is represented in the Hermite B-splines basis as*

$$S_{2m-1,r,s}(x;\lambda) = \frac{1}{\Pi_{2m-1,r}(\lambda)} \sum_{k=-\infty}^{\infty} \lambda^k N_s(x+m-r-k)$$

References

[LS76] S.L. Lee and A. Sharma. "Cardinal Lacunary Interpolation by g-splines. I. The characteristic polynomials." In: *Journal of Approximation theory* 16 (1976), pp. 85–96.

[Sch73] I.J. Schoenberg. "Cardinal Spline Interpolation". In: *CBMS-NSF Regional conference series in applied mathematics* (1973).