Hermite Polynomials snakes order 2

Yoann Pradat

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Contents

I Translation of Schoenberg's 1973 paper

The following is simply a reminder of some of the results found by I.J Schoenberg in his paper *Cardinal Interpolation and Spline Functions. III Cardinal Hermite interpolation.* Let's reintroduce notations of the article and make somehow more explicits what the objects they encode are.

Let r and m be positive integers that satisfy $r \leq m$. The set of cardinal splines of order $2m$ with knot multiplicity r is denoted by $S_{2m,r}$. Note that using De Boor's notations for splines set we have the following

$$S_{2m,r} = \$_{2m,\mathbb{Z}_r} = \Pi_{<2m,\mathbb{Z},2m-r} \tag{1}$$

where $\mathbb{Z}_3$ denotes the sequence of knots $(\dots,-1,\dots,-1,0,\dots,0,1,\dots,1,\dots)$ that is integers with multiplicity r. It is clear from these notations that $S_{2m,r} \subset \mathcal{C}^{2m-r-1}$.

Theorem 1. *Let S be either of the vector spaces $\mathcal{L}_{p,r}, F_{\gamma,r}$ with $\gamma \geq 0$, $p \in \mathbb{N}^*$. Provided a solution to C.H.I.P $\left(y_\nu,\dots,y_\nu^{(r-1)}, S_{2m,r} \cap S\right)$ exists, it is uniquely given by*

$$\forall x \in \mathbb{R} \qquad S(x) = \sum_{\nu=-\infty}^{\infty} y_\nu L_0(x-\nu) + \dots + y_\nu^{(r-1)} L_{r-1}(x-\nu) \tag{2}$$

In order to specify a usable model for active contours it remains to determine explicit expressions for the basis functions $L_0,\dots,L_{r-1}$. In the article they are determined by solving a set of $2m-r$ linear equations. This system is obtained by considering separately the function L_s on $[1,\infty)$ and $[0,1]$. Note that specifying the function on both these intervals completely determine L_s as the latter is even (if s is even) or odd (if s

is odd).

On $[1, \infty)$, L_s can be decomposed into

$$L_s = \sum_{j=1}^{m-r} c_j S_j$$

where $(c_j)_{j=1}^{m-r}$ are $(m-r)$ unknown coefficients to be determined and S_j are the eigensplines for the first $m-r$ "eigenvalues" λ_j, solutions to $|\Delta_{r,d}(\lambda)| = 0$.

On $[0,1]$, L_s is given by a polynomial P of order $2m$ that takes a specific form according to the parities of s and r (we refer to equations (7.13) and (7.14)) in the article. This polynomial introduces m unknown coefficients $(a_j)_{j=1}^{m}$. To determine a total of $m + m - r = 2m - r$ unknown coefficients we make use of the $2m - r$ equality conditions at 1 $P^{(\rho)}(1) = L_s^{(\rho)}(1)$. We end up of a system of $2m - r$ equations for $2m - r$ unknowns that can be solved exactly provided the matrix of the system is non singular. Schoenberg proves with a very nice argument that the matrix of the system is always non singular.

I.1 General case $m = r$

In the case $m = r$, splines L_s $(s = 0, \ldots, r-1)$ vanish outside $[-1, 1]$ and therefore give rise to a local interpolation scheme as given below for the case of periodic sequences. Let renote $\phi_s = L_{s-1}$ those elements of $S_{2r,r}$ with support in $[-1, 1]$ and that together satisfy Hermite interpolation conditions.

Corollary 1. *Given r M-periodic sequences $(r[k], \ldots, r^{(r-1)}[k])_{k\in\mathbb{Z}}$ in $\mathbb{R}^d$, there exists a unique spline curve of order $2r$ whose value and derivatives agree with the sequence of coefficients at integers locations. This spline curve and its derivatives are everywhere bounded and take the form for $t \in \mathbb{R}$*

$$\begin{aligned} r(t) &= \sum_{k\in\mathbb{Z}} \sum_{l=1}^{r} r^{(l-1)}[k]\phi_l(t-k) && (3)\\ &= \sum_{k=0}^{M-1} \sum_{l=1}^{r} r^{(l-1)}[k]\phi_{l,per}(t-k) && (4)\end{aligned}$$

where $\phi_{l,per} = \sum_{k\in\mathbb{Z}} \phi_l(. - Mk)$

In case the sequences are not periodic but have a finite number of non-zero entries, say M, we can similarly restrict the infinite sum to M elements withouth periodizing the basis functions

Corollary 2. *Given r sequences $(r[k], \ldots, r^{(r-1)}[k])_{k\in\mathbb{Z}}$ in $\mathbb{R}^d$ that vanish outside $[\![0, M]\!]$, there exists a unique spline curve of order $2r$ whose value and derivatives agree with the sequence of coefficients at integers locations. This spline curve and its derivatives have compact support and take the form for $t \in \mathbb{R}$*

$$\begin{aligned} r(t) &= \sum_{k\in\mathbb{Z}} \sum_{l=1}^{r} r^{(l-1)}[k]\phi_l(t-k) && (5)\\ &= \sum_{k=0}^{M-1} \sum_{l=1}^{r} r^{(l-1)}[k]\phi_l(t-k) && (6)\end{aligned}$$

I.2 Case $m = r = 3$

In the case $m = r = 3$, L_0, L_1, L_2 are 0 on $[1, \infty)$ and on $[0, 1]$ are given by

$$L_0(x) = 1 + a_1x^3 + a_2x^4 + a_3x^5 \tag{7}$$
$$L_1(x) = x + a_1x^3 + a_2x^4 + a_3x^5 \tag{8}$$
$$L_2(x) = \frac{1}{2}x^2 + a_1x^3 + a_2x^4 + a_3x^5 \tag{9}$$

where the coefficients for each generator are unrelated. Note that L_s have finite support *because* $m = r$. If that was not the case the term $\sum_{j=1}^{m-r} c_jS_j$ may not be 0 and therefore L_s would be non zero on $[1, \infty)$! Can it happen though that $m > r$ and $(c_j)_{j=1}^{m-r}$ are 0? To determine the coefficients above we need to solve independently for each generator the 3 equations $P^{(\rho)}(1) = 0$. This leads to the following systems

$$\begin{cases} a_1 + a_2 + a_3 = -1 \\ 3a_1 + 4a_2 + 5a_3 = 0 \\ 3a_1 + 6a_2 + 10a_3 = 0 \end{cases} \begin{cases} a_1 + a_2 + a_3 = -1 \\ 3a_1 + 4a_2 + 5a_3 = -1 \\ 3a_1 + 6a_2 + 10a_3 = 0 \end{cases} \begin{cases} a_1 + a_2 + a_3 = -\frac{1}{2} \\ 3a_1 + 4a_2 + 5a_3 = -1 \\ 3a_1 + 6a_2 + 10a_3 = -\frac{1}{2} \end{cases}$$

II The resulting snake scheme for $m = r = 3$

II.1 Generating functions

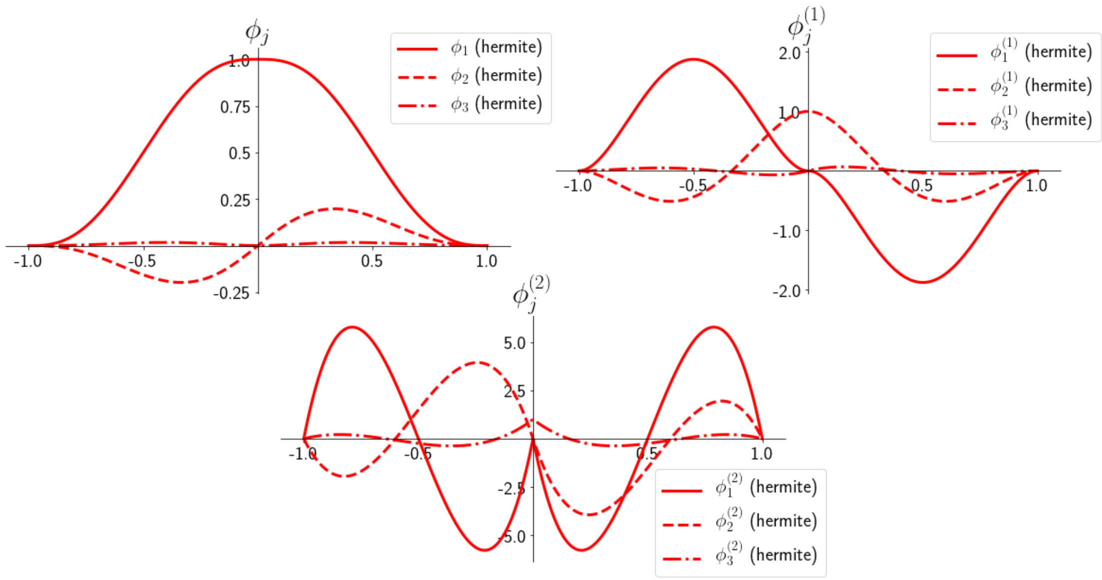

Figure 1: Generators for C.H.I.P with $m = r = 3$

Solving the linear systems written in the previous section yields explicit formulas for the Schoenberg basis generators L_0, L_1, L_2, that we rename ϕ_1, ϕ_2, ϕ_3 in accordance with modern notations (see V. Uhlmann *Hermite Snakes with Controls of Tangents*). The formulas are the following.

$$\phi_1(x) = \begin{cases} 1 - 10x^3 + 15x^4 - 6x^5 & \text{if } 0 \le x \le 1 \\ 1 + 10x^3 + 15x^4 + 6x^5 & \text{if } -1 \le x < 0 \end{cases} \tag{10}$$

$$\phi_2(x) = \begin{cases} x - 6x^3 + 8x^4 - 3x^5 & \text{if } 0 \le x \le 1 \\ x - 6x^3 - 8x^4 - 3x^5 & \text{if } -1 \le x < 0 \end{cases} \tag{11}$$

$$\phi_3(x) = \begin{cases} 0.5x^2 - 1.5x^3 + 1.5x^4 - 0.5x^5 & \text{if } 0 \le x \le 1 \\ 0.5x^2 + 1.5x^3 + 1.5x^4 + 0.5x^5 & \text{if } -1 \le x < 0 \end{cases} \tag{12}$$

In figure 1 are displayed the values of these functions as well as their two first derivatives. As mentioned in the previous section, the generators L_s are elements of $S_{2m,r} = S_{6,3}$ which is a subset of $\mathcal{C}^{2m-r-1} = C^2$. It is apparent in the figure that these functions have continuous derivatives up to order 2 but that higher order derivatives do not exist in neighborhoods of $-1, 0$ and 1.

II.2 Closed planar curves or "contours"

Consider a positive integer M and an M**-periodic parametrized closed curve** $r : \mathbb{R} \to \mathbb{R}^2$ for which we have local derivatives up to order 2 at M location sites regularly spaced that is we know $(r[k], r'[k], r''[k])_{k=0}^{M-1}$.

Corollary 3. *Given M periodic sequences $(r[k], r'[k], r''[k])_{k\in\mathbb{Z}}$ in $\mathbb{R}^2$, there exists a unique spline curve of order 6 whose value and derivatives agree with the sequence of coefficients at integers locations. This spline curve and its derivatives are everywhere bounded and take the form for $t \in \mathbb{R}$*

$$r(t) = \sum_{k\in\mathbb{Z}} r[k]\phi_1(t-k) + r'[k]\phi_2(t-k) + r''[k]\phi_3(t-k) \tag{13}$$

$$= \sum_{k=0}^{M-1} r[k]\phi_{1,per}(t-k) + r'[k]\phi_{2,per}(t-k) + r''[k]\phi_{3,per}(t-k) \tag{14}$$

Proof. As the sequence of coefficients $(r[k], r'[k], r''[k])_{k\in\mathbb{Z}}$ are in $Y_{\gamma,r} = Y_{0,3}$ (i.e they are bounded), application of Schoenberg's theorem 1 yields existence and unicity of an interpolating function in $S_{6,3} \cap F_{0,3}$. Application of theorem 4 then leads to the explicit formulation given above. □

Remark 1. *It is convenient to normalize the continuous parameter to the $[0,1]$ interval as is usual in the implementations. For that let the renormalized curve $s(t) = r(Mt)$ for $t \in [0,1]$. Note that this completely describes the curve as it is enough to describe the curve r on $[0, M]$. Differentiating this equality twice yields $r[k] = s[\frac{k}{M}], r'[k] = \frac{1}{M}s'[\frac{k}{M}], r''[k] = \frac{1}{M^2}s''[\frac{k}{M}]$. Therefore equation (14) is rewritten for $t \in [0,1]$*

$$s(t) = \sum_{k=0}^{M-1} s[\frac{k}{M}]\phi_{1,per}(Mt-k) + \frac{1}{M}s'[\frac{k}{M}]\phi_{2,per}(Mt-k) + \frac{1}{M^2}s''[\frac{k}{M}]\phi_{3,per}(Mt-k) \tag{15}$$

In the rest of this document we will reuse the notation r for the normalized curve and won't make use of the notation s anymore. Equation (15) is the **mathematical representation of a planar curve** and we call it "snake" or "active contour". By playing with the coefficients we can capture a wide variety of contours that arise from closed objects in 2D images like cells membrane in a bioimage.

II.3 Open planar curves

Consider again a positive integer M and a **parametrized open curve** $r : \mathbb{R} \to \mathbb{R}^2$ for which we have local derivatives up to order 2 at M location sites regularly spaced that is we know $(r[k], r'[k], r''[k])_{k=0}^{M-1}$. By "open" we mean a curve that is not periodic.

Corollary 4. *Given biinfinite sequences of coefficients $(\dots, 0, r[0], \dots, r[M-1], 0, \dots)$, $(\dots, 0, r'[0], \dots, r'[M-1], 0, \dots)$, $(\dots, 0, r''[0], \dots, r''[M-1], 0, \dots)$ in $\mathbb{R}^2$, there exists a unique spline curve of order 6 whose value* *and derivatives agree with the sequence of coefficients at integers locations. This spline curve and its derivatives have compact support and take the form for $t \in \mathbb{R}$*

$$\begin{aligned} r(t) &= \sum_{k\in\mathbb{Z}} r[k]\phi_1(t-k) + r'[k]\phi_2(t-k) + r''[k]\phi_3(t-k) \\ &= \sum_{k=0}^{M-1} r[k]\phi_1(t-k) + r'[k]\phi_2(t-k) + r''[k]\phi_3(t-k) \end{aligned}$$

Proof. This result is again a simple application of theorem 1 and 4 given in Schoenberg's paper of 1981. □

Remark 2. *In this setting we are only interested in the curve lying between our coefficients that is the interpolated points with continuous parameter in the interval $[0, M-1]$ The normalization factor is therefore $M-1$ and the renormalized open curve $r(t) = r((M-1)t)$ takes the form*

$$r(t) = \sum_{k=0}^{M-1} r[\frac{k}{M-1}]\phi_1((M-1)t-k) + r'[\frac{k}{M-1}]\frac{\phi_2((M-1)t-k)}{M-1} + r''[\frac{k}{M-1}]\frac{\phi_3((M-1)t-k)}{(M-1)^2} \tag{16}$$

II.4 Closed sphere-like surfaces

In my research project we are interested in developing a mathematical methods for representing a certain type of surfaces with explicit control of local properties including first-order derivatives and curvature. As a consequence extension of the schemes given in equations (15) and (16) to tensor-product surfaces (that is surfaces parametrized by 2 continuous parameters in a way that each continuous parameter appear in separate functions) may be relevant for the questions we have.

Consider positives integers M_1 and M_2 and a **sphere-like parametrized** surface $\sigma : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ with U a subset (closed in our case) of the plane. By "sphere-like" we mean an object that can be described with closed curves on latitudes (u varies while v is fixed) and open curves on longitudes (v varies while u is fixed). Suppose we have local properties of the surface at $M_1 \times (M_2+1)$ locations (counted with multiplicity as some locations may coalesce) on a regular grid.

Corollary 5. *Given the biinfinite sequences of coefficients $(\partial^{i,j}\sigma(k,l))_{k,l\in\mathbb{Z}^2,(i,j)\in\{0,1,2\}^2}$ in $\mathbb{R}^3$, that are M_1 periodic in the first coordinate and vanish when the second coordinate is outside $[\![0, M_2]\!]$, there exists a unique interpolating tensor-product spline curve of order 6 whose value and partial derivatives agree with the sequence of coefficients at the integers grid locations. This tensor-product spline curve and its derivatives are everywhere bounded and take the form for $(u,v) \in [0, M_1] \times [0, M_2]$ (or equiv. $\mathbb{R} \times [0, M_2]$)*

$$\sigma(u,v) = \sum_{k=0}^{M_1-1} \sum_{l=0}^{M_2} \sum_{i,j=0}^{2} \partial^{i,j}\sigma(k,l)\phi_{i+1,per}(u-k)\phi_{j+1}(v-l) \tag{17}$$

Proof. This is a simple application of corollaries 1 and 2 given before. □

Remark 3.

- *Not all surfaces admit a tensor-product representation which limits the range of surfaces one can reach with this kind of interpolation scheme. Tensor-product spline has not yet been defined and we just mention here that we call a tensor-product spline a map $f : U_1 \times U_2 \subseteq \mathbb{R}^2 \to \mathbb{R}$ that can be written as $f_1(u) \times f_2(v)$ with f_1 and f_2 each splines on U_1 and U_2 respectively.*
- *The continuity of the basis functions is automatically transferred to the each coordinate of the spline curve, resulting in surfaces with parametrizations that are twice continuously differentiable. Is this preventing us from representing surfaces with singular points as we would like to have? This question is crucial to our objective and will be addressed later in more details.*

- *Normalizing each continuous parameter to the interval* $[0,1]$ *yields the following representation of the surface (where we note again* σ *the surface with normalized parameters)*

$$\sigma(u,v)=\sum_{k=0}^{M_1-1}\sum_{l=0}^{M_2}\sum_{i,j=0}^{2}\frac{1}{M_1^iM_2^j}\partial^{i,j}\sigma(\frac{k}{M_1},\frac{l}{M_2})\phi_{i+1,per}(M_1u-k)\phi_{j+1}(M_2v-l) \tag{18}$$

III Properties of the interpolation scheme

III.1 The Riesz-Schauder basis property

III.1.1 Definition

Definition 1. *Let* $\mathcal{H}$ *be a Hilbert space over the field* $\mathbb{K}=\mathbb{R}$ *or* $\mathbb{C}$ *of real or complex numbers. A basis* $\{\phi_k\}_{k\in\mathbb{Z}}$ *of* $\mathcal{H}$ *is said to be a Riesz-Schauder basis if there exist positive constants* $0<m\leq M$ *such that*

$$\forall c\in l_2(\mathbb{Z})(\subset\mathbb{K}^{\mathbb{Z}})\qquad m\|c\|_{l_2}^2\leq\Big\|\sum_{k\in\mathbb{Z}}c_k\phi_k\Big\|_{\mathcal{H}}^2\leq M\|c\|_{l_2}^2 \tag{19}$$

Let n be a positive integer, f,g functions $\mathbb{R}^n\to\mathbb{C}$. Define λ the Lebesgue measure on $\mathbb{R}^n$ and the map

$$\langle f,g\rangle=\int_{\mathbb{R}^n}f\bar{g}d\lambda$$

This map is an inner product on the vector space over $\mathbb{C}$ of all measurable functions for which $\langle f,f\rangle$ is finite (to be precise we consider quotient space so that its elements are equivalence classes for the relation that two functions are equivalent if they agree λ-a.e). The latter, endowed with this inner product, is therefore a pre-Hilbert space. It is a known result that this space is complete and therefore a Hilbert space. It is usually denoted by $L_2(\mathbb{R}^n)$. For an element f of that space we denote $\hat{f}$ the extension (by density) to functions in $L_2(\mathbb{R}^n)$ of the Fourier transform as usually defined in mathematics textbooks over functions in $L_1(\mathbb{R}^n)$.

III.1.2 Characterization in the Fourier domain

Theorem 2. *In case the Hilbert space* $\mathcal{H}$ *is* $L_2(\mathbb{R}^n)$ *and* $\phi\in\mathcal{H}$*, the following statements for some* $0<m\leq M$ *are equivalent*

1. $\forall c\in l_2(\mathbb{Z})$, $m\|c\|_{l_2}^2\leq\Big\|\sum_{k\in\mathbb{Z}}c_k\phi(.-k)\Big\|_{L_2}^2\leq M\|c\|_{l_2}^2$
2. $m\leq\sum_{k\in\mathbb{Z}}\big|\hat{\phi}(w+2k\pi)\big|^2\leq M$

with obvious notations. In case any of the two statements holds, the subspace $V=\{\sum_{k\in\mathbb{Z}}c_k\phi(.-k)|c\in l_2(\mathbb{Z})\}$ *of* $L_2(\mathbb{R}^n)$ *is closed and thus a Hilbert space itself, and* $\{\phi(.-k)\}$ *is a Riesz-Schauder basis.*

Proof. This theorem is exactly theorem 2 as given in A. Aldroubi, M. Unser, "Sampling procedures in function spaces and asymptotic equivalence with Shannon's sampling theory," Numer. Funct. Anal. Optim., vol. 15, nos. 1–2, pp. 1–21, May 1994 and is proved there for the case $n=1$ but extension to domain of generic dimension is straightforward. □

For the needs of having this Riesz-Schauder basis property for vector-valued basis functions let's give an immediate corollary for that case. For that let's define the map $\langle\cdot,\cdot\rangle$ over functions $\mathbb{R}^n\to\mathbb{C}^m$ by

$$\langle f,g\rangle=\sum_{r=1}^{m}\int_{\mathbb{R}^n}f_r\bar{g_r}d\lambda$$

with obvious notations. This map defines an inner product and the space $L_2(\mathbb{R}^n,\mathbb{C}^m)$ of functions having finite norm as induced by this inner product is a Hilbert space.

Theorem 3. *In case the Hilbert space $\mathcal{H}$ is $L_2(\mathbb{R}^n,\mathbb{C}^m)$ and $\phi \in \mathcal{H}$, the following statements for some $0 < m \leq M$ are equivalent*

1. $\forall c \in l_2(\mathbb{Z})$, $m\|c\|_{l_2}^2 \leq \|\sum_{k\in\mathbb{Z}} c_k\phi(.-k)\|^2_{L_2(\mathbb{R}^n,\mathbb{C}^m)} \leq M\|c\|_{l_2}^2$

2. $m \leq \sum_{k\in\mathbb{Z}}\sum_{r=1}^{m} |\hat{\phi}_r(w+2k\pi)|^2 = \sum_{k\in\mathbb{Z}} \|\hat{\phi}(w+2k\pi)\|_2^2 \leq M$

with obvious notations. In case any of the two statements holds, the subspace $V = \{\sum_{k\in\mathbb{Z}} c_k\phi(.-k)|c \in l_2(\mathbb{Z})\}$ of $L_2(\mathbb{R}^n,\mathbb{C}^m)$ is closed and thus a Hilbert space itself, and $\{\phi(.-k)\}$ is a Riesz-Schauder basis.

III.1.3 Characterization from Gram matrix

In cases where there are several generators, like the Hermite polynomial interpolation, one can characterize the property of Riesz-Schauder basis from the Gram-matrix of the set of generators $\{\phi_i(.-k)\}$. For that, let $\{\phi^i\}_{i=1}^q$ be functions in a Hilbert space $\mathcal{H}$ and let O be a unitary operator on $\mathcal{H}$ i.e $OO^* = O^*O = I$ (in our case of Hermite interpolation $O\phi = \phi(.-1)$).

Consider the subspace $V = \{\sum_{k\in\mathbb{Z}} C(k)O^k\Phi = \sum_{k\in\mathbb{Z}}\sum_{i=1}^{q} c^i(k)O^k\phi^i|c^1,\ldots,c^q \in l_2(\mathbb{Z})\}$. As seen before V is a well-defined closed subspace of $\mathcal{H}$ if there exists constants $0 < m \leq M$ such that

$$\forall C \in l_2^q(\mathbb{Z}) \quad m\|C\|_{l_2^q}^2 \leq \|\sum_{k\in\mathbb{Z}} C(k)O^k\Phi\|^2_{\mathcal{H}} \leq M\|C\|_{l_2^r}^2 \tag{20}$$

Note that the central term of the inequality above can be rewritten as

$$\begin{aligned}\langle\sum_{k\in\mathbb{Z}} C(k)O^k\Phi, \sum_{l\in\mathbb{Z}} C(l)O^l\Phi\rangle &= \sum_{(k,l)\in\mathbb{Z}^2} C(k)\langle O^k\Phi, O^l\Phi\rangle C(l)^* \\ &= \sum_{(k,l)\in\mathbb{Z}^2} C(k)A(l-k)C(l)^*\end{aligned}$$

where A is a $q\times q$ matrix with elements $A_{i,j}(k) = \langle\phi^i, O^k\phi^j\rangle$. Now the above can be viewed as

$$\sum_{(k,l)\in\mathbb{Z}^2} C(k)A(l-k)C(l)^* = \sum_{i,j=1}^{q}\sum_{l\in\mathbb{Z}} (c^i * A_{i,j})(l)c^{j^*}(l)$$

Using Parseval's theorem for discrete time Fourier transform that is $\hat{c}(w) = \sum_{k\in\mathbb{Z}} c(k)e^{-jwk}$ or in f-notation $\hat{c}(f) = \sum_{k\in\mathbb{Z}} c(k)e^{-j2\pi fk}$ we have that

$$\begin{aligned}\sum_{i,j=1}^{q}\sum_{l\in\mathbb{Z}} (c^i * A_{i,j})(l)c^{j^*}(l) &= \sum_{i,j=1}^{q} \frac{1}{2\pi}\int_0^{2\pi} \hat{c^i}(w)\hat{A}_{i,j}(w)\hat{c^j}^*(w)\frac{dw}{2\pi} \\ &= \frac{1}{2\pi}\int_0^{2\pi} \hat{C}(w)\hat{A}(w)\hat{C}^*(w)\frac{dw}{2\pi} \\ &= \int_0^1 \hat{C}(f)\hat{A}(f)\hat{C}^*(f)df\end{aligned}$$

$\hat{A}(w)$ with elements $A_{i,j}(w) = \sum_{k\in\mathbb{Z}}\langle\phi^i, O^k\phi^j\rangle e^{-jwk}$ is called the **Gram matrix**. Now theorem 2.1 from 1996 paper by Adroubi *Oblique Projections in Atomic Spaces* makes the link between m, M constants characterizing Riesz-Schauder basis property and the minimum and maximum eigenvalue of the Gram matrix as follows

Theorem 4. *The space V is a well-defined closed subspace of $\mathcal{H}$ with Riesz-Schauder basis $\{O^k\phi^i\}_{i=1,\dots,q,k\in\mathbb{Z}}$ if and only if the $q\times q$ Gram matrix $\hat{A}(w)$ is hermitian positive for almost all w and there exist two constants $0<m\leq M$ such that*

$$m \leq \underset{w\in[-\pi,\pi]}{\text{ess inf}}\, \lambda_{min}(\hat{A}(w)) \leq \underset{w\in[-\pi,\pi]}{\text{ess sup}}\, \lambda_{max}(\hat{A}(w)) \leq M \tag{21}$$

Proof. This is exactly theorem 2.1 as stated in the article mentioned and the proof is given there. □

III.2 Riesz-Schauder basis for Hermite interpolation order r

Let r be a positive integer and $\phi_1,\dots,\phi_r$ the elements of $S_{2r,r}\cap\mathcal{L}_{1,r}$ with support in $[-1,1]$ that satisfy Hermite interpolation conditions as introduced in subsection I.1. Consider a positive integer d representing the dimension of the space in which the coefficients of the resulting scheme live i.e

$$V=\{\sum_{k\in\mathbb{Z}} C(k)\Phi(.-k)|C\in l_2^{d\times r}(\mathbb{Z})\} \tag{22}$$

with $\Phi(.-k)=\begin{pmatrix}\phi_1 & \cdots & \phi_r\end{pmatrix}^T$, $C(k)=\begin{pmatrix}c_1(k) & \cdots & c_r(k)\end{pmatrix}$ and $c_l\in l_2^d(\mathbb{Z})$. Note that V is a subspace of $L_2(\mathbb{R},\mathbb{R}^d)$. To state properly the Riesz-basis property we need to exhibit functions that are elements of $L_2(\mathbb{R},\mathbb{R}^d)$. For that consider the functions $\phi_j e_i$ for $j=1,\dots,r$, $i=1,\dots,d$ with $(e_i)_{i=1}^d$ the canonical base of $\mathbb{R}^d$. Then V can be rewritten as

$$V=\{\sum_{k\in\mathbb{Z}}\sum_{j=1}^{r}\sum_{i=1}^{d} c_{j,i}(k)\phi_j(.-k)e_i|c\in l_2(\mathbb{Z})\} \tag{23}$$

which amounts to taking $\Phi(.-k)=\begin{pmatrix}\phi_1 e_1 & \cdots, \phi_1 e_d & \cdots & \phi_r e_1 & \cdots & \phi_r e_d\end{pmatrix}^T$, $C(k)=\begin{pmatrix}c_1(k) & \cdots & c_{rd}(k)\end{pmatrix}$ in notations of (22). Theorem 5 shows that Riesz-Schauder basis property holds at any order r for Hermite polynomial interpolation.

Theorem 5. *Let r be a positive integer, $\phi_1,\dots,\phi_r$ the elements of $S_{2r,r}\cap\mathcal{L}_{1,r}$ with support in $[-1,1]$ that satisfy Hermite interpolation conditions and d the dimension of the coefficients of the induced scheme. Then $\{\phi_j(.-k)e_i\}_{j=1,\dots,r,i=1,\dots,d,k\in\mathbb{Z}}$ is a Riesz-Schauder basis of*

$$V=\{\sum_{k=-\infty}^{\infty}\sum_{j=1}^{r}\sum_{i=1}^{d} c_{k,i}^j\phi_j(.-k)e_i|c\in l_2(\mathbb{Z})\} \tag{24}$$

Remark 4. *Having a Riesz-Schauder basis makes V a closed subspace of $L_2(\mathbb{R},\mathbb{R}^d)$. The latter being a Hilbert space, V is a Hilbert space itself.*

Proof. In order to prove that our set of generators is a Riesz-Schauder basis we are going to use characterization from the Gram matrix $\hat{A}(w)$.

1. **Expression of the Gram matrix**

   The generators are elements of $L_2(\mathbb{R},\mathbb{R}^d)$ with inner product $\langle\phi,\psi\rangle=\sum_{l=1}^{d}\langle\phi_l,\psi_l\rangle_{L_2}$. As a consequence, $\langle\phi_{j_1}e_{i_1},\phi_{j_2}(.-k)e_{i_2}\rangle=0$ whenever $i_1\neq i_2$, and the Gram matrix entries are non-zero only at indices that are multiple integers of d. It easier to express the Gram matrix as the block matrix

$$\hat{A}(w)=\begin{pmatrix}\hat{B}_{1,1}(w) & \hat{B}_{1,2}(w) & \cdots & \hat{B}_{1,r}(w)\\ \vdots & \vdots & \ddots & \vdots\\ \hat{B}_{r,1}(w) & \hat{B}_{r,2}(w) & \cdots & \hat{B}_{r,r}(w)\end{pmatrix} \tag{25}$$

   with the blocks being diagonal matrices given by

$$\hat{B}_{i,j}(w) = \sum_{k\in\mathbb{Z}} \langle \phi_i, \phi_j(.-k)\rangle e^{-jwk} I_d \tag{26}$$

One can show by recurrence (**to be done**) that $rd \times rd$ matrices as in (25) have a determinant which is the d^{th} exponent of the determinant of the submatrix (27) of size $r \times r$

$$\hat{G}(w) = \sum_{k\in\mathbb{Z}} \begin{pmatrix} \langle \phi_1, \phi_1(.-k)\rangle & \cdots & \langle \phi_1, \phi_r(.-k)\rangle \\ \vdots & \ddots & \vdots \\ \langle \phi_r, \phi_1(.-k)\rangle & \cdots & \langle \phi_r, \phi_r(.-k)\rangle \end{pmatrix} e^{-jwk} \tag{27}$$

This is also applies to $\hat{A}(w) - \lambda I$ whose determinant is the characteristic polynomial of $\hat{A}$. Therefore $\chi_{\hat{A}(w)}(\lambda) = \chi_{\hat{G}(w)}(\lambda)^d$ which means that $\hat{A}(w)$ and $\hat{G}(w)$ share the same eigenvalues. The study of eigenvalues of $\hat{A}(w)$ can thus be replaced by that of $\hat{G}(w)$.

2. **Gram matrix is self-adjoint**
To prove this it is enough to prove that $\overline{\hat{B}_{j,i}(w)} = \hat{B}_{i,j}(w)$ which is proven below

$$\begin{aligned}
\overline{\hat{B}_{j,i}(w)} &= \sum_{k\in\mathbb{Z}} \overline{\langle \phi_j, \phi_i(.-k)\rangle e^{-jwk}} I_d \\
&= \sum_{k\in\mathbb{Z}} \langle \phi_i(.-k), \phi_j\rangle e^{jwk} I_d \\
&= \sum_{k\in\mathbb{Z}} \langle \phi_i(.+k), \phi_j\rangle e^{-jwk} I_d \\
&= \sum_{k\in\mathbb{Z}} \langle \phi_i, \phi_j(.-k)\rangle e^{-jwk} I_d \\
&= \hat{B}_{i,j}(w)
\end{aligned}$$

3. **Gram matrix is positive definite a.e**
Suppose on the contrary that $\hat{A}$ is not positive definite a.e. Given that $\hat{A} : w \to \hat{A}(w)$ is 2π-periodic, this means that there exists $E \subseteq [0, 2\pi]$ whose Lebesgue measure $\lambda(E) > 0$ and such that

$$\forall w \in E, \exists \hat{X}(w) \in \mathbb{C}^{rd} \setminus \{0\}, \quad \hat{X}(w)\hat{A}(w)\hat{X}(w)^* \leq 0$$

Let $\hat{C} : \mathbb{R} \to \mathbb{C}^{rd}$ the 2π-periodic function such that $\hat{C}_{|[0,2\pi]} = \hat{X}(w)\mathbb{1}_E(w)$. $\hat{C}$ is the Fourier transform of the discrete function $C : \mathbb{Z} \to \mathbb{C}^{rd}$ with $C(k) = \frac{1}{2\pi}\int_0^{2\pi} \hat{C}(w)e^{-jwk}dw$ and

$$\int_0^{2\pi} \hat{C}(w)\hat{A}(w)\hat{C}(w)^* \leq 0$$

However, from Parseval's theorem and properties of Fourier transform we have

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} \hat{C}(w)\hat{A}(w)\hat{C}^*(w)dw &= \sum_{(k,l)\in\mathbb{Z}^2} C(k)A(l-k)C(l)^* \\
&= \langle \sum_{k\in\mathbb{Z}} C(k)\Phi(.-k), \sum_{l\in\mathbb{Z}} C(l)\Phi(.-l)\rangle \\
&= \| \sum_{k\in\mathbb{Z}} C(k)\Phi(.-k)\|^2_{L_2(\mathbb{R},\mathbb{C}^d)}
\end{aligned}$$

Consequently the norm above is 0 (reminder: it is norm on $L_2(\mathbb{R}, \mathbb{C}^d)$ whose elements are equivalent classes). As our function is continuous this leads to

$$\sum_{k\in\mathbb{Z}} C(k)\Phi(.-k) = 0 \tag{28}$$

We now make use of the Hermite interpolation conditions of $\Phi = \begin{pmatrix}\phi_1 e_1 & \cdots & \phi_1 e_d & \cdots & \phi_r e_1 & \cdots & \phi_r e_d\end{pmatrix}^T$ at integer locations. Let $k_0 \in \mathbb{Z}$. Evaluating (28) and its derivatives up to $(r-1)$ at k_0 leads to $C(k_0) = 0$. k_0 being arbitrary, we have $C \equiv 0$ which in turn leads to $\hat{X} \equiv 0$. This contradicts our initial choice of E and $\hat{X}$. Consequently $\hat{A}$ is positive definite almost everywhere.

4. **Continuity and compacity**

   For the needs of what follows let's prove that the map $T : (\mathbb{C}^{rd})^{\mathbb{Z}} \to L_2(\mathbb{R},\mathbb{C})^d$ such that $T(C) = \sum_{k\in\mathbb{Z}} C(k)\Phi(.-k)$ is continuous and that $\{C \in (\mathbb{C}^{rd})^{\mathbb{Z}} | \|C\|^2_{l_2^{rd}} = 1\}$ is compact (**wrong**). For that, notice that Φ has compact support and thus only a finite number of inner products $\langle \Phi, \Phi(.-k)\rangle$ are non zero. Let k_Φ such that $\langle \Phi, \Phi(.-k)\rangle = 0$ for $|k| > k_\Phi$. Consequently

$$\begin{aligned}
\|\sum_{k\in\mathbb{Z}} C(k)\Phi(.-k)\|^2_{L_2(\mathbb{R},\mathbb{C}^d)} &= \sum_{(k,l)\in\mathbb{Z}^2} C(k)C(l)^* \langle \Phi, \Phi(.-(l-k))\rangle \\
&\leq \sup_{|k|\leq k_\Phi} |\langle \Phi, \Phi(.-k)\rangle| \sum_{|k-l|\leq k_\Phi} |C(k)C(l)^*| \\
&\leq \sup_{|k|\leq k_\Phi} |\langle \Phi, \Phi(.-k)\rangle| \sum_{l=-k_\Phi}^{k_\Phi} \sum_{k\in\mathbb{Z}} |C(k)||C(l+k)|^* \\
&\leq (2k_\Phi + 1) \sup_{|k|\leq k_\Phi} |\langle \Phi, \Phi(.-k)\rangle| \|C\|^2_{l_2^{rd}}
\end{aligned}$$

   using Cauchy-Schwarz inequality for inner product on $(\mathbb{C}^{rd})^{\mathbb{Z}}$ given by $\langle C_1, C_2\rangle = \sum_{k\in\mathbb{Z}} C_1(k)C_2(k)^*$.

   Consider now the map $\psi : (\mathbb{C}^{rd})^{\mathbb{Z}} \to \bar{\mathbb{R}}$ such that $\psi(C) = \|C\|^2_{l_2^{rd}}$. Given that $|\psi(C)| \to \infty$ as $\|C\| \to \infty$, ψ is a proper function. Therefore the preimage of every compact set of $\bar{\mathbb{R}}$ is a compact set hence $\{C \in (\mathbb{C}^{rd})^{\mathbb{Z}} | \|C\|^2_{l_2^{rd}} = 1\} = \psi^{-1}(\{1\})$ is compact. (**wrong**)

5. **Eigenvalues of $\hat{A}$ are essentially bounded**

   Suppose that essential supremum of $\lambda_{max} : w \to \max \mathrm{sp}(\hat{A}(w))$ is infinite. Using 2π-periodicity of $\hat{A}$ it means that

$$\forall p \in \mathbb{N}^*, \exists E_p \subseteq [0, 2\pi], \forall w \in E_p, \quad \lambda_{max}(w) > p$$

   Now for each $w \in E_p$ let $\hat{V}_{max}(w)$ normalized eigenvector i.e $\hat{A}(w)\hat{V}_{max}(w) = \lambda_{max}(w)\hat{V}_{max}(w)$ and $\|\hat{V}_{max}(w)\|^2_{\mathbb{C}^{rd}} = 2\pi$. Let $\hat{C}_p$ the 2π-periodic function such that $\hat{C}_{p|[0,2\pi]}(w) = \lambda(E_p)^{-0.5}\mathbb{1}_{E_p}(w)\hat{V}_{max}(w)$ and let C_p the discrete function that maps $k \in \mathbb{Z}$ to $C_p(k) = \frac{1}{2\pi}\int_0^{2\pi} \hat{C}_p(w)e^{-jwk}dw$. From Parseval's theorem $\|C_p\|^2_{l_2^{rd}} = 1$ and with the same calculations as in the previous point we have

$$\frac{1}{2\pi}\int_0^{2\pi} \hat{C}_p(w)\hat{A}(w)\hat{C}_p(w)^* > p$$

   while

$$
\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} \hat{C_p}(w)\hat{A}(w)\hat{C_p}^*(w)dw &= \sum_{(k,l)\in\mathbb{Z}^2} C_p(k)A(l-k)C_p(l)^* \\
&= \langle \sum_{k\in\mathbb{Z}} C_p(k)\Phi(.-k), \sum_{l\in\mathbb{Z}} C_p(l)\Phi(.-l)\rangle \\
&= \| \sum_{k\in\mathbb{Z}} C_p(k)\Phi(.-k)\|^2_{L_2(\mathbb{R},\mathbb{C}^d)}
\end{aligned}
$$

Therefore

$$
\forall p\in\mathbb{N}^*, \exists C_p \in (\mathbb{C}^{rd})^{\mathbb{Z}}, \quad \|C_p\|^2_{l_2^{rd}} = 1, \| \sum_{k\in\mathbb{Z}} C_p(k)\Phi(.-k)\|^2_{L_2(\mathbb{R},\mathbb{C}^d)} > p \tag{29}
$$

This is absurd.

Similarly suppose that the essential infinimum of $\lambda_{min} : w \to \min \mathrm{sp}(\hat{A}(w))$ is 0. Using 2π-periodicity of $\hat{A}$ it means that

$$
\forall p\in\mathbb{N}^*, \exists E_p \subseteq [0,2\pi], \forall w \in E_p, \quad \lambda_{min}(w) < \frac{1}{p}
$$

Same as previously we then prove that

$$
\forall p\in\mathbb{N}^*, \exists C_p \in (\mathbb{C}^{rd})^{\mathbb{Z}}, \quad \|C_p\|^2_{l_2^{rd}} = 1, \| \sum_{k\in\mathbb{Z}} C_p(k)\Phi(.-k)\|^2_{L_2(\mathbb{R},\mathbb{C}^d)} < \frac{1}{p} \tag{30}
$$

This is also absurd.

$\square$

III.3 Application to the case $r = 3$

The scheme described for $r = 3$ and $d = 2$, i.e 2D vector coefficients, can be rewritten as scalar combinations of integer shifts of functions in $L_2(\mathbb{R},\mathbb{R}^2)$ as follows

$$
r(t) = \sum_{k=-\infty}^{\infty} \sum_{j=1}^{3}\sum_{i=1}^{2} c_{j,i}(k)\phi_j(t-k)e_i \tag{31}
$$

with $c_{j,i}(k) = r^{(j)}[k]_i$, $e_1 = (1,0)$ and $e_2 = (0,1)$ canonical base of $\mathbb{R}^2$. It is thus an element of

$$
\begin{aligned}
V &= \{ \sum_{k=-\infty}^{\infty} c_1(k)\phi_1(t-k) + c_2(j)\phi_2(t-k) + c_3(k)\phi_3(t-k) | c_1, c_2, c_3 \in l_2(\mathbb{Z})^2\} \\
&= \{ \sum_{k=-\infty}^{\infty} \sum_{j=1}^{3}\sum_{i=1}^{2} c_{j,i}(k)\phi_j(t-k)e_i | c \in l_2(\mathbb{Z})\}
\end{aligned}
$$

which is a subspace of $L_2(\mathbb{R},\mathbb{R}^2)$. A direct application of Theorem 5 proves that $\{\phi_j(t-k)e_i\}_{i=1,2,j=1,2,3,k\in\mathbb{Z}}$ is a Riesz-Schauder basis. Therefore V is a Hilbert space itself.

The Gram matrix of $\{\phi_1 e_1(.-k), \phi_1 e_2(.-k), \phi_2 e_1(.-k), \phi_2 e_2(.-k), \phi_3 e_1(.-k), \phi_3 e_2(.-k)\}_{k\in\mathbb{Z}}$ i.e a 6×6 matrix. It is explicitly given by

$$\sum_{k\in\mathbb{Z}}\begin{pmatrix}\langle\phi_1,\phi_1(.-k)\rangle & 0 & \langle\phi_1,\phi_2(.-k)\rangle & 0 & \langle\phi_1,\phi_3(.-k)\rangle & 0\\ 0 & \langle\phi_1,\phi_1(.-k)\rangle & 0 & \langle\phi_1,\phi_2(.-k)\rangle & 0 & \langle\phi_1,\phi_3(.-k)\rangle\\ \langle\phi_2,\phi_1(.-k)\rangle & 0 & \langle\phi_2,\phi_2(.-k)\rangle & 0 & \langle\phi_2,\phi_3(.-k)\rangle & 0\\ 0 & \langle\phi_2,\phi_1(.-k)\rangle & 0 & \langle\phi_2,\phi_2(.-k)\rangle & 0 & \langle\phi_2,\phi_3(.-k)\rangle\\ \langle\phi_3,\phi_1(.-k)\rangle & 0 & \langle\phi_3,\phi_2(.-k)\rangle & 0 & \langle\phi_3,\phi_3(.-k)\rangle & 0\\ 0 & \langle\phi_3,\phi_1(.-k)\rangle & 0 & \langle\phi_3,\phi_2(.-k)\rangle & 0 & \langle\phi_3,\phi_3(.-k)\rangle\end{pmatrix}e^{-jwk} \quad (32)$$

where the inner product is that on $L_2(\mathbb{R},\mathbb{C})$ (even $L_2(\mathbb{R},\mathbb{R})$ as ϕ_j are real-valued).

As explained before, looking into the essential infinimum and supremum eigenvalues of (32) yields the best achievable Riesz-Schauder basis bounds m and M. As mentionned in the proof, the matrix above has a characteristic polynomial that is the square of the characteristic polynomial of the submatrix

$$\hat{G}(w)=\sum_{k\in\mathbb{Z}}\begin{pmatrix}\langle\phi_1,\phi_1(.-k)\rangle & \langle\phi_1,\phi_2(.-k)\rangle & \langle\phi_1,\phi_3(.-k)\rangle\\ \langle\phi_2,\phi_1(.-k)\rangle & \langle\phi_2,\phi_2(.-k)\rangle & \langle\phi_2,\phi_3(.-k)\rangle\\ \langle\phi_3,\phi_1(.-k)\rangle & \langle\phi_3,\phi_2(.-k)\rangle & \langle\phi_3,\phi_3(.-k)\rangle\end{pmatrix}e^{-jwk} \quad (33)$$

Therefore it is equivalent to look into essential infinimum and supremum of the spectrum of $\hat{G}$ if one is interested in knowing the best achievable parameters m and M.