

The Bernstein polynomial basis: A centennial retrospective[☆]

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ABSTRACT

One hundred years after the introduction of the Bernstein polynomial basis, we survey the historical development and current state of theory, algorithms, and applications associated with this remarkable method of representing polynomials over finite domains. Originally introduced by Sergei Natanovich Bernstein to facilitate a constructive proof of the Weierstrass approximation theorem, the leisurely convergence rate of Bernstein polynomial approximations to continuous functions caused them to languish in obscurity, pending the advent of digital computers. With the desire to exploit the power of computers for geometric design applications, however, the Bernstein form began to enjoy widespread use as a versatile means of intuitively constructing and manipulating geometric shapes, spurring further development of basic theory, simple and efficient recursive algorithms, recognition of its excellent numerical stability properties, and an increasing diversification of its repertoire of applications. This survey provides a brief historical perspective on the evolution of the Bernstein polynomial basis, and a synopsis of the current state of associated algorithms and applications.

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1. Introduction

At their inception, it is extremely difficult to predict the subsequent evolution and ultimate significance of novel mathematical ideas. Concepts that at first seem destined to revolutionize the scientific landscape — such as Hamilton's *quaternions* — may gradually lapse into relative obscurity (Crowe, 1967). On the other hand, methods introduced to facilitate theoretical proofs of seemingly limited scope and practical interest may eventually flourish into useful tools that gain widespread acceptance in diverse practical computations.

The latter category undoubtedly includes the *Bernstein polynomial basis*, introduced 100 years ago (Bernstein, 1912) as a means to constructively prove the ability of polynomials to approximate any continuous function, to any desired accuracy, over a prescribed interval. Their slow convergence rate, and the lack of digital computers to efficiently construct them, caused the Bernstein polynomials to lie dormant in the *theory* rather than *practice* of approximation for the better part of a century.¹ Ultimately, the Bernstein basis found its true vocation not in approximation of functions by polynomials, but in exploiting computers to interactively design (vector-valued) polynomial functions — i.e., *parametric curves and surfaces*. In this context, it became apparent that the Bernstein coefficients of a polynomial provide valuable insight into its behavior over a given finite interval, yielding many useful properties and elegant algorithms that are now being increasingly adopted in other application domains.

The centennial anniversary of the introduction of the Bernstein basis is an opportune juncture at which to survey and assess the attractive features and algorithms associated with this remarkable representation of polynomials over finite domains, and its diverse practical applications. It seems probable that Bernstein would be amazed to witness the widespread interest — albeit in rather different contexts — that a simple but powerful idea in the paper (Bernstein, 1912) has elicited, one hundred years after its first appearance.

The intent of this paper is: (1) to provide a historical retrospective on the introduction and evolution of the Bernstein basis as a practical computational tool; (2) to succinctly survey, as a guide to the uninitiated, the key properties and algorithms associated with it; and (3) to briefly enumerate the current variety of applications in which it has found use. The plan for the remainder of the paper is as follows. The academic career of Sergei Natanovich Bernstein is briefly summarized in Section 2, followed by a discussion of his constructive proof of the Weierstrass approximation theorem in Section 3. Section 4 then describes the contributions of the French engineers Pierre Bézier and Paul de Faget de Casteljau, in promoting the use of the Bernstein basis in the context of computer-aided design for the automotive industry during the 1960s and 1970s. Some characteristic features of the Bernstein basis, upon which many useful properties and elegant algorithms are based, are identified in Section 5, while Section 6 describes a fundamental feature of the Bernstein form, that was not fully appreciated until the 1980s: its *numerical stability* with respect to coefficient perturbations or floating-point round-off errors. A synopsis of alternative approaches and interpretations is presented in Section 7 — the *shift operator*; the theory of “blossoming” or *polar forms*; connections with probability theory; and methods based on generating functions and discrete convolutions. Section 8 addresses the central role of the Bernstein form as a cornerstone of *computer-aided geometric design*, while Section 9 summarizes its applications as a basic computational tool in other scientific/engineering fields. Finally, Section 10 assesses the current status and future prospects of the Bernstein representation of polynomials over finite domains.

¹ “In theory, there is no difference between theory and practice. In practice, there is.” Attributed to Yogi Berra.

2. Sergei Natanovich Bernstein

Sergei Natanovich Bernstein² (Fig. 1) was born March 5, 1880 in Odessa, Ukraine. After graduating from high school in 1898, he traveled to Paris to study mathematics at the Sorbonne, where he also developed an interest in engineering, and enrolled in the École d'Électrotechnique Supérieure. During the 1902–1903 academic year, he visited Göttingen and worked under the supervision of David Hilbert. This led to his Sorbonne 1904 doctoral dissertation, *Sur la nature analytique des solutions des équations aux dérivées partielles du second ordre*, concerned with the solution of Hilbert's 19th Problem. The dissertation was enthusiastically approved by a distinguished committee of mathematicians – comprising Émile Picard (chair), Jacques Hadamard, and Henri Poincaré. Bernstein then returned to Russia in 1905, after attending the International Congress of Mathematicians in Heidelberg.

Unfortunately, the academic profession in Russia did not recognize foreign degrees as valid credentials for a university position. Despite the enthusiastic reception of his Sorbonne dissertation, Bernstein was obliged to embark upon a second doctoral program in order to qualify for a research position in Russia. After spending some time in Saint Petersburg, he moved to Kharkov in 1908, where he began teaching and working on his new doctoral research program. Bernstein was apparently offered a position at Harvard University soon after moving to Kharkov, but for unknown reasons did not pursue it.

The subject of Bernstein's Kharkov dissertation was a problem posed by Charles-Jean de La Vallée Poussin in 1908 – namely, can a piecewise-linear function be approximated over a finite interval by a polynomial of degree n , with an $O(1/n)$ approximation error? Bernstein's affirmative solution to this problem was awarded a prize in 1911 from the *Académie Royale des Sciences, des Lettres et des Beaux Arts* of Belgium, and appeared in 1912 as *Sur l'ordre de la meilleure approximation des fonctions continues par les polynômes de degré donné* (On the best approximation of continuous functions by means of polynomials of a given degree) in the *Mémoires des l'Académie royale de Belgique*. Subsequently he defended his doctoral dissertation, *O nailuchshem priblizhenii nepreryvnykh funktsy posredstvom mnogochnenov dannoi stepeni*, based upon this work, at Kharkov University in 1913, and he was appointed professor of mathematics at Kharkov University in 1920.

The short paper *Démonstration du théorème de Weierstrass fondée sur le calcul des probabilités* (Bernstein, 1912), that first introduces the *Bernstein basis*, appeared in the Communications of the Kharkov Mathematical Society in 1912 – see Fig. 2. In this paper, Bernstein proposes an “extremely simple” proof of the Weierstrass theorem based on probability theory.

During the 1920s Bernstein worked on constructive function theory (i.e., approximation theory) and on probability theory. The widespread acclaim for his achievements resulted in many honors, including elections to the Russian Academy of Sciences, the Paris Academy of Sciences, the USSR Academy of Sciences, and Director of the Kharkov Mathematical Institute. However, in 1930 political considerations began to intrude upon academic life in Kharkov. As a result, Bernstein was removed as Director of the Mathematical Institute, and he departed for Leningrad in 1932 – having narrowly escaped a political purge in Kharkov. In Leningrad, Bernstein served as Head of the Department of Probability Theory and Mathematical Statistics in the USSR Academy of Sciences, and also gave lectures at Leningrad University. During the Second World War he

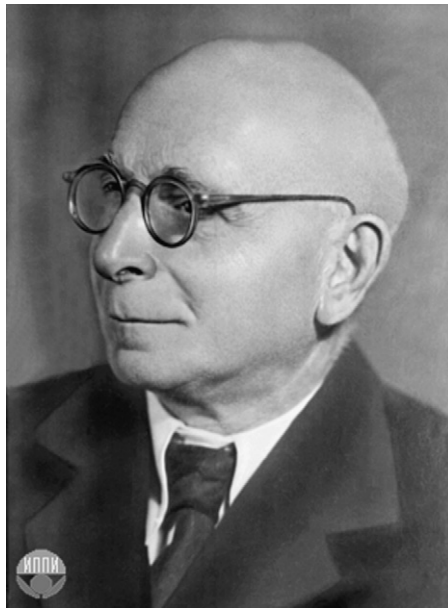


Fig. 1. Sergei Natanovich Bernstein (1880–1968). Photograph reproduced from the Russian Academy of Sciences website — see <http://www.ras.ru>.

² Alternative transliterations of his name use Sergey and Bernshteyn or Bernshteyn.

Démonstration du théorème de Weierstrass fondée sur le calcul des probabilités.

Je me propose d'indiquer une démonstration fort simple du théorème suivant de Weierstrass:

Si $F(x)$ est une fonction continue quelconque dans l'intervalle 01 , il est toujours possible, quel que petit que soit ε , de déterminer un polynôme $E_n(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$ de degré n assez élevé, tel qu'on ait

$$|F(x) - E_n(x)| < \varepsilon$$

en tout point de l'intervalle considéré.

A cet effet, je considère un événement A , dont la probabilité est égale à x . Supposons qu'on effectue n expériences et que l'on convienne de payer à un joueur la somme $F\left(\frac{m}{n}\right)$, si l'événement A se produit m fois. Dans ces conditions, l'espérance mathématique E_n du joueur aura pour valeur

$$E_n = \sum_{m=0}^n F\left(\frac{m}{n}\right) \cdot C_n^m x^m (1-x)^{n-m}. \quad (1)$$

Or, il résulte de la continuité de la fonction $F(x)$ qu'il est possible de fixer un nombre δ , tel que l'inégalité

$$|x - x_0| \leq \delta$$

entraîne

$$|F(x) - F(x_0)| < \frac{\varepsilon}{2};$$

de sorte que, si $\bar{F}(x)$ désigne le maximum et $\underline{F}(x)$ le minimum de $F(x)$ dans l'intervalle $(x - \delta, x + \delta)$, on a

$$\bar{F}(x) - F(x) < \frac{\varepsilon}{2}, \quad F(x) - \underline{F}(x) < \frac{\varepsilon}{2}. \quad (2)$$

Fig. 2. The 1912 paper (Bernstein, 1912) in which the Bernstein basis was introduced.

managed to escape to Kazakhstan before the siege of the city (which lasted from September 1941 to January 1944) but his son — who had remained in Leningrad — was killed while attempting to escape.

Bernstein decided against returning to Leningrad, and instead moved to the University of Moscow, where he embarked on a seven-year project editing the complete works of Pafnuti Lvovich Chebyshev (between 1944 and 1951). However, he was dismissed from his position in 1947, and instead he became Head of the Department of Constructive Function Theory within the Steklov Mathematical Institute, Russian Academy of Sciences, where he remained until his retirement in 1957. Bernstein died on October 26, 1968 in Moscow.

The most authoritative scientific biography of Bernstein was written in 1955 by Naum Ilyich Akhiezer, who became professor at Kharkov University soon after Bernstein left. This has been translated from Russian into German (Akhiezer, 1955), but as yet there is apparently no translation into English. A brief synopsis of his academic career and accomplishments, in English, may be found in the recent *History of Approximation Theory* by Karl-Georg Steffens (2006) — see also the *Dictionary of Scientific Biography* article (Youschkevitch, 1980).

The *Sobranie sochinenii* (collected works) of Bernstein were published in four volumes between 1952 and 1964. His intellectual achievements, during a period of unprecedented upheaval and uncertainty — both World Wars and the Russian Revolution — are remarkable for their fundamental insights and continuing impact in areas remote from their original contexts.

3. Weierstrass approximation theorem

Polynomials are widely used in computational models of scientific/engineering problems, because of their finite evaluation schemes; closure under addition, multiplication, differentiation, integration, and composition; and their ability to approximate functions that have no closed-form expressions.³ It was the need to formulate “well-behaved” polynomial approximations that motivated the introduction of the Bernstein basis.

³ Many of these attributes carry over to *rational functions* (i.e., ratios of polynomials).

3.1. Existence arguments

In 1885 Karl Weierstrass published a proof (Weierstrass, 1885) of what subsequently became known as his *approximation theorem*. This states that, given any continuous function $f(x)$ on an interval $[a, b]$ and a tolerance $\epsilon > 0$, a polynomial $p_n(x)$ of sufficiently high degree n exists, such that

$$|f(x) - p_n(x)| \leq \epsilon \quad \text{for all } x \in [a, b].$$

In other words, polynomials can *uniformly approximate* any function that is merely continuous over a closed interval. This represents a significant advance over using the Taylor series expansion to generate polynomial approximations of a function in two important respects: (a) the function need not be analytic (nor differentiable); and (b) whereas the interval $[a, b]$ can be freely specified, the Taylor series must be confined within its radius of convergence – which can be difficult to compute. Pinkus (2000) describes in detail the contributions of Weierstrass to approximation theory. As a point of departure for his proof, Weierstrass invokes an integral representation for the function,

$$f(x) = \lim_{k \rightarrow 0} \frac{1}{\sqrt{\pi}k} \int_{-\infty}^{+\infty} f(t) \exp\left[-\frac{(t-x)^2}{k^2}\right] dt,$$

which may be regarded as its convolution with a Dirac delta function. Other authors developed alternative proofs of the approximation theorem, including arguments based on Fourier series (Tolstov, 1976), but for the most part these proofs were (a) existential, rather than constructive, in nature; and (b) relied heavily on analytic limit arguments, rather than concrete algebraic processes.

3.2. Bernstein's constructive proof

Steffens (2006) notes that the Russian school of approximation theory, which originated in the 1850s from Chebyshev's interest in the design and analysis of mechanical linkages, regarded such existential arguments as rather suspect, and instead emphasized approaches compatible with practical computations (what we now consider *algorithms*), that can be directly applied to scientific or technical problems. Correspondingly, the distinctive feature of Bernstein's proof of the Weierstrass theorem, compared to its predecessors, is its explicit construction – employing only basic algebraic operations – of a sequence of polynomials $p_n(x)$ approaching the given function $f(x)$ more closely at every point of the interval $x \in [a, b]$ as the degree n increases.

Since the change of variables specified by $t = (x - a)/(b - a)$ maps $x \in [a, b]$ to $t \in [0, 1]$ without changing the max norm of any function, we can restrict our attention to continuous functions $f(t)$ on $t \in [0, 1]$ without loss of generality. The Bernstein basis of degree n on $t \in [0, 1]$ is defined by

$$b_k^n(t) := \binom{n}{k} (1-t)^{n-k} t^k, \quad k = 0, \dots, n, \quad (1)$$

and in terms of it the Bernstein polynomial⁴ associated with any continuous function $f(t)$ is specified as

$$p_n(t) := \sum_{k=0}^n f(k/n) b_k^n(t). \quad (2)$$

Although $p_n(t)$ is nominally of degree n , its actual degree may be less than n (see Section 5.1 below). The uniform convergence of (2) to $f(t)$, as $n \rightarrow \infty$, is predicated on two fundamental properties of the basis functions (1) – they are *non-negative* on $t \in [0, 1]$; and they form a *partition of unity*, i.e.,

$$\sum_{k=0}^n b_k^n(t) = 1. \quad (3)$$

Hence, the value of $p_n(t)$, being a sum of sampled values of $f(t)$ at the $n + 1$ uniformly-spaced ordinates $t = k/n$ weighted by the basis functions $b_k^n(t)$, amounts to a *convex combination* – i.e., a weighted sum, with non-negative weights that sum to unity – of those sampled values.

It should be noted that, in general, the polynomial approximant (2) does not *interpolate* the sampled values, i.e., $p_n(k/n) \neq f(k/n)$. Since the basis functions (1) satisfy

$$b_k^n(0) = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k > 0, \end{cases} \quad b_k^n(1) = \begin{cases} 0 & \text{if } k < n, \\ 1 & \text{if } k = n, \end{cases} \quad (4)$$

⁴ A number of generalizations of the basic Bernstein polynomial approximation (2), that preserve and extend its key properties, have been proposed – see Phillips (2010), Simeonov et al. (2011, 2012), Stancu (1968, 1983).

we always have $p_n(0) = f(0)$ and $p_n(1) = f(1)$, but the intermediate values $f(k/n)$ for $0 < k < n$ are not interpolated. Moreover (2) does not incorporate the “polynomial reproduction” property – i.e., $p_n(t) \neq f(t)$ when $f(t)$ is a polynomial of degree $\leq n$, except in the trivial cases $n = 0$ and 1 .

Specifically, when $f(t) = c$, the fact that $p_n(t) = c$ for all n follows from the partition-of-unity property (3). Similarly, when $f(t) = at + b$, we have

$$p_n(t) = a \sum_{k=0}^n \frac{k}{n} \binom{n}{k} (1-t)^{n-k} t^k + b.$$

Noting that the $k = 0$ term of the sum vanishes, and that

$$\frac{k}{n} \binom{n}{k} = \binom{n-1}{k-1}, \quad (5)$$

a shift of the summation index yields

$$p_n(t) = at \sum_{k=0}^{n-1} \binom{n-1}{k} (1-t)^{n-1-k} t^k + b,$$

and by the partition-of-unity property for the basis of degree $n-1$, this is simply $at + b$. The ability to exactly reproduce linear (or constant) functions is called the *linear precision* property of the Bernstein approximation.

The relation (2) may be regarded as defining an operator that maps any continuous function $f(t)$ on $[0, 1]$ to a polynomial $p_n(t)$ of degree $\leq n$. This operator is *linear*, in the sense that if it maps another continuous function $g(t)$ into the polynomial $q_n(t)$ of degree $\leq n$, then the combination $\lambda f(t) + \mu g(t)$ is mapped into $\lambda p_n(t) + \mu q_n(t)$. Moreover, as a consequence of the fact that the functions (1) are *non-negative* on $[0, 1]$ this operator is *monotone* – i.e., $p_n(t) \geq q_n(t)$ when $f(t) \geq g(t)$ for $t \in [0, 1]$. Equivalently, one may say that it is a *positive* operator – i.e., $p_n(t) \geq 0$ when $f(t) \geq 0$ for $t \in [0, 1]$.

The conventional modern form of Bernstein's proof is based on Korovkin's theorem (Korovkin, 1960) for positive linear operators: if a polynomial approximant $p_n(t)$ to a continuous function $f(t)$ over $t \in [0, 1]$ is specified by a positive linear operator, its uniform convergence to *any* continuous function is guaranteed as $n \rightarrow \infty$, if one can demonstrate such convergence in the “simple” cases $f(t) = 1, t, t^2$. Since the first two cases are covered by the linear precision property, it is only necessary to consider $f(t) = t^2$, in which case we have

$$p_n(t) = \sum_{k=0}^n \left(\frac{k}{n}\right)^2 \binom{n}{k} (1-t)^{n-k} t^k.$$

Noting that the $k = 0$ term vanishes, using the relation (5) again, and shifting the summation index, we obtain

$$p_n(t) = t \sum_{k=0}^{n-1} \frac{k+1}{n} \binom{n-1}{k} (1-t)^{n-1-k} t^k = t \left(\frac{n-1}{n} t + \frac{1}{n} \right),$$

where the final expression follows from the linear precision property. Thus, when $f(t) = t^2$ the approximant $p_n(t)$ is the quadratic polynomial

$$p_n(t) = t^2 + \frac{(1-t)t}{n},$$

and the approximation error is

$$|f(t) - p_n(t)| = \frac{(1-t)t}{n}.$$

Since for each $t \in [0, 1]$ this decreases in proportion to $1/n$ as n increases, $p_n(t)$ converges uniformly to $f(t)$. The maximum error is $1/4n$, and occurs at $t = \frac{1}{2}$. More generally, the asymptotic formula

$$\lim_{n \rightarrow \infty} n[f(t) - p_n(t)] = \frac{1}{2}(1-t)tf''(t) \quad (6)$$

of Voronowskaja (1932) holds for $t \in (0, 1)$ when $f''(t) \neq 0$. Complete details on these proofs can be found in standard introductory texts (Cheney, 1966; Davis, 1975; Powell, 1981; Rivlin, 1981) on approximation theory. Stark (1981) traces the history of Bernstein polynomial approximations, from their introduction in 1912 up to the publication of the classical treatise by Lorentz (1953) in the mid-1950s.

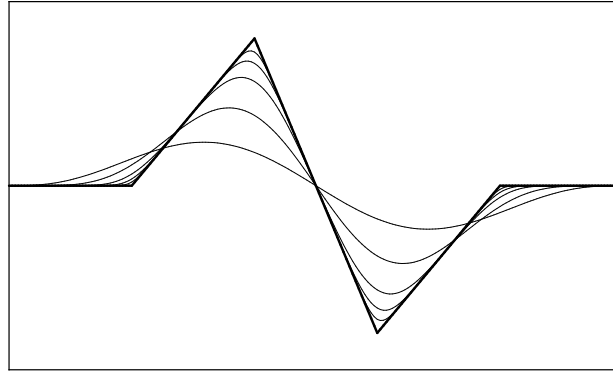


Fig. 3. Bernstein polynomial approximations $p_n(t)$ of degree $n = 10, 30, 100, 300, 1000$ to a continuous piecewise-linear function $f(t)$, defined by (2).

3.3. Properties of the approximant

The Bernstein polynomial approximant (2) to a given function $f(t)$ is always “at least as smooth” as $f(t)$. If $f(t)$ has C^r rather than just C^0 continuity, all derivatives of $p_n(t)$ up to order r converge uniformly to the corresponding derivatives of $f(t)$. A simple demonstration of this convergence may be found in (Floater, 2005). Similarly, if bounds on the derivatives of $f(t)$ of each order over $[0, 1]$ are known, the corresponding derivatives of $p_n(t)$ also satisfy those bounds – this implies, for example, that when $f(t)$ is monotone or convex, $p_n(t)$ is correspondingly monotone or convex (see Davis, 1975 for complete details).

Unfortunately, the slow rate of convergence observed in the case⁵ $f(t) = t^2$ above is typical of Bernstein polynomial approximation. It can be shown (Davis, 1975) that $|f(t) - p_n(t)|$ diminishes in proportion to n^{-1} at any point where $f''(t)$ is defined and non-vanishing (see Fig. 3). This may be compared with, for example, the fast n^{-4} convergence rate (de Boor, 2001) of the cubic spline interpolating $n + 1$ equidistant values of a function with at least four-fold differentiability. The spline increases its interpolatory ability by introducing junctures (knots) between individual cubic segments, rather than increasing the degree. On the other hand, attempting to interpolate more points with a single polynomial by increasing its degree can incur wildly divergent behavior (Epperson, 1987).

In view of these considerations, the Bernstein polynomial approximants (2) remained – prior to the advent of digital computers – very much part of the *theory* rather *practice* of approximations. Notwithstanding their simple construction and orderly convergence properties, it is impractical to employ polynomials with degrees running to hundreds or thousands in “real-world” problems. The eventual widespread adoption of the basis (1) was motivated not by the approximations (2) of a given function $f(t)$, but by a replacement of the values $f(k/n)$ with freely-specified coefficients c_k for $k = 0, \dots, n$ that can be used to intuitively manipulate the behavior of the polynomial

$$p(t) = \sum_{k=0}^n c_k b_k^n(t), \quad t \in [0, 1]. \quad (7)$$

Following past practice (Farouki and Rajan, 1987, 1988) we call expression (2) a *Bernstein polynomial*, while (7) is called a *polynomial in Bernstein form*. Whereas the former refers to a polynomial approximation of a given function $f(t)$, the latter denotes a polynomial with arbitrary coefficients in the Bernstein basis. Compared to the familiar monomial or power form of a polynomial,

$$p(t) = \sum_{k=0}^n a_k t^k, \quad (8)$$

we shall see that the Bernstein form (7) offers many advantages if one wishes to analyze or manipulate polynomials over a finite interval.

4. De Casteljau and Bézier

As noted in Section 3, the orderly convergence of the Bernstein approximation (2) to a continuous function $f(t)$ as $n \rightarrow \infty$ comes at a severe price: as seen in Fig. 3, it proceeds at a very leisurely pace. Philip J. Davis, in his 1963 book *Interpolation and Approximation* (Davis, 1975), remarked on the slow convergence of Bernstein approximations as follows:

⁵ To approximate $f(t) = t^2$ with a maximum error of 10^{-4} over $t \in [0, 1]$ the degree of the Bernstein polynomial (2) must be increased to $n = 2500$.

This fact seems to have precluded any numerical application of Bernstein polynomials from having been made. Perhaps they will find application when the properties of the approximant in the large are of more importance than the closeness of the approximation.

Coincidentally, two engineers employed in the French automotive industry, Paul de Casteljau of Citroën and Pierre Étienne Bézier of Renault, became interested in such an application in the early 1960s.

4.1. A new application emerges

De Casteljau and Bézier were not concerned with the approximation of given functions, but rather with formulating novel mathematical tools that would allow designers to intuitively construct and manipulate complex shapes, such as automobile bodies, using digital computers. This problem was especially critical for “free-form” shapes, that do not admit exact specification through a few simple geometric parameters — centers, axes, angles, dimensions, etc. The motivation was to circumvent the subjective, laborious, and expensive process of sculpting clay models to specify the desired shape.

Although a parametric curve or surface (a vector-valued function of one or two variables) is an infinitude of points, its computer representation must employ just a *finite* data set. The mapping from the finite set of input values to a continuous locus is achieved by interpreting those values as coefficients for certain *basis functions* in the parametric variables. The coefficients must furnish natural “shape handles” that permit intuitive creation or modification of the curve or surface geometry, to satisfy prescribed aesthetic or functional requirements. The choice of basis is thus fundamental to a successful design scheme. Ultimately, the work of de Casteljau and Bézier lead to adoption of the Bernstein form, typified by what is now called a Bézier curve,

$$\mathbf{r}(t) = \sum_{k=0}^n \mathbf{p}_k b_k^n(t), \quad t \in [0, 1], \quad (9)$$

with *control points* $\mathbf{p}_0, \dots, \mathbf{p}_n$ as a propitious design scheme. By connecting the control points we obtain the *control polygon*, which can be used to analyze and manipulate the curve shape in a simple and natural manner.

4.2. Paul de Faget de Casteljau

One must bear in mind that, in the early 1960s, digital computers were still in their infancy, and the goal of exploiting them for shape design would have seemed far-fetched. De Casteljau, for example, described (de Casteljau, 1993) the reaction at Citroën to this goal as follows:

... the designers were astonished and scandalized. Was it some kind of joke? It was considered nonsense to represent a car body mathematically. It was enough to please the eye, the word accuracy had no meaning ...

Citroën's first attempts at digital shape representation employed a Burroughs E101 computer (de Casteljau, 1993) featuring 128 program steps, a 220-word memory, and a 5 kW power consumption! Nevertheless, de Casteljau's “insane” persistence in pursuing this idea led to the increasing adoption of computer-aided design and manufacturing methods within Citroën from 1963 onward.

De Casteljau's approach is based on the use of “pilot points” called *poles* to define curves and surfaces, a term motivated by the syllabic repetition in the phrase “interpolation of polynomials with *polar* forms” (see Section 7.2). Although there is no explicit reference to the Bernstein polynomial basis, key features of de Casteljau's *courbes et surfaces à pôles* are unmistakably linked to it, e.g., the use of barycentric coordinates over intervals and triangles, and the non-negativity and partition-of-unity properties of the basis functions associated with the poles (subsequently identified as Bézier/B-spline control points). However, de Casteljau's ideas were recorded only in Citroën internal documents (de Casteljau, 1963), and remained long unknown to the outside world.

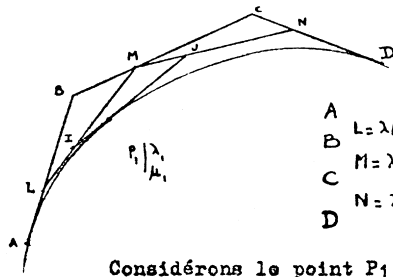
Wolfgang Böhm (1977) was instrumental in ensuring that proper credit was attributed for the eponymous *de Casteljau algorithm*, the most fundamental scheme associated with *courbes à pôles* (now commonly called *Bézier curves*), although it had appeared somewhat earlier (Krautter and Parizot, 1971) in a relatively obscure venue. For a parameter value $\tau \in (0, 1)$ this algorithm *evaluates* and *subdivides* the Bézier curve (9), i.e., it computes the curve point $\mathbf{r}(\tau)$ and the control points defining the “left” and “right” subsegments $t \in [0, \tau]$ and $t \in [\tau, 1]$ of $\mathbf{r}(t)$ as individual Bézier curves, over the parameter interval $[0, 1]$. Setting $\mathbf{p}_j^0 = \mathbf{p}_j$ for $j = 0, \dots, n$, the algorithm computes a triangular array of points

$$\begin{array}{ccccccc} \mathbf{p}_0^0 & \mathbf{p}_1^0 & \mathbf{p}_2^0 & \cdot & \cdot & \cdot & \mathbf{p}_n^0 \\ & \mathbf{p}_1^1 & \mathbf{p}_2^1 & \cdot & \cdot & \cdot & \mathbf{p}_n^1 \\ & & \mathbf{p}_2^2 & \cdot & \cdot & \cdot & \mathbf{p}_n^2 \\ & & & \cdot & \cdot & \cdot & \\ & & & & \mathbf{p}_n^n & & \end{array} \quad (10)$$

defined for $j = r, \dots, n$ and $r = 1, \dots, n$ by the linear interpolations

1.5.- Sous-Pôles d'une courbe

1.5.1.- Définition des sous-pôles.



Considérons une cubique des pôles A, B, C, D. Nous avons vu précédemment que la construction par la méthode des barycentres donnent les différents points L, M, N, I, J, P.

$$\begin{aligned} A & L = \lambda A + \mu B & I &= \lambda^2 A + 2\lambda\mu B + \mu^2 C \\ B & M = \lambda B + \mu C & J &= \lambda^2 B + 2\lambda\mu C + \mu^2 D \\ C & N = \lambda C + \mu D & P &= \lambda^3 A + 3\lambda^2\mu B + 3\lambda\mu^2 C + \mu^3 D \\ D & \end{aligned}$$

Considérons le point P_1 de paramètres λ_1 et μ_1 (avec $\lambda_1 + \mu_1 = 1$) (λ_1 varie de 0 à 1 et μ_1 1 à 0, lorsque P va en D).

Cherchons les pôles de la cubique $P_1 D$. Cette courbe dérive de la cubique initiale AD par changement des paramètres.

Fig. 4. An excerpt from de Casteljau's 1963 SA Citroën notes, showing his algorithm applied to a cubic with poles A, B, C, D. Here λ, μ are barycentric coordinates for the curve parameter domain — i.e., $1 - t$ and t for $t \in [0, 1]$.

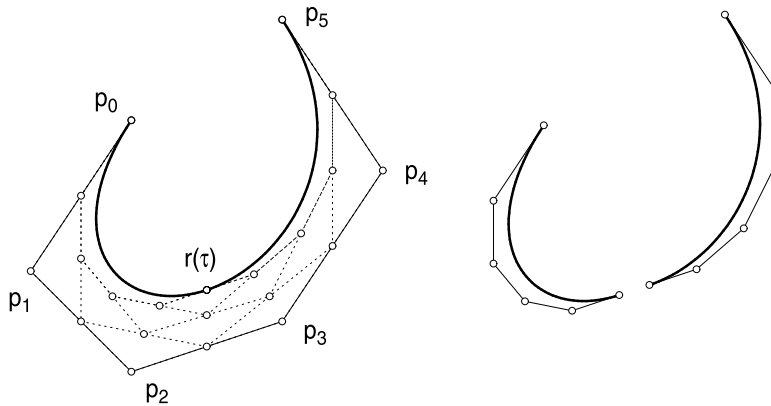


Fig. 5. Subdivision of a quintic Bézier curve $\mathbf{r}(t)$ at $t = \frac{1}{2}$, showing (left) the intermediate points (10) generated by the de Casteljau algorithm (11), and (right) the two individual subsegments, together with their control polygons.

$$\mathbf{p}_j^r = (1 - \tau)\mathbf{p}_{j-1}^{r-1} + \tau\mathbf{p}_j^{r-1}. \quad (11)$$

Beginning with the original control polygon, the algorithm (11) computes a sequence of new polygons for $r = 1, \dots, n$, each with one vertex less than its predecessor — an example of a “corner-cutting” scheme (see Fig. 4).

The vertices of the r -th polygon are points on the legs of the $(r - 1)$ -th polygon that divide them in the ratio $\tau : 1 - \tau$ (see Fig. 5). The final entry in (10) is the curve point that corresponds to the parameter value τ , i.e., $\mathbf{p}_n^r = \mathbf{r}(\tau)$. Furthermore, the entries $\mathbf{p}_0^0, \mathbf{p}_1^1, \mathbf{p}_2^2, \dots, \mathbf{p}_n^n$ and $\mathbf{p}_n^n, \mathbf{p}_{n-1}^{n-1}, \mathbf{p}_{n-2}^{n-2}, \dots, \mathbf{p}_0^0$ on the left- and right-hand sides of (10) are control points for the left and right subsegments $t \in [0, \tau]$ and $t \in [\tau, 1]$ of $\mathbf{r}(t)$, considered as individual Bézier curves over the parameter domain $t \in [0, 1]$. A retrospective on the role of the de Casteljau algorithm in CAGD may be found in Boehm and Müller (1999).

A brief biography of de Casteljau (prepared in connection with the award of an honorary doctorate from the University of Berne in 1997) may be found in the Preface to Volume 16, Number 7 of *Computer Aided Geometric Design*, a special issue dedicated to him. This issue also contains an autobiographical sketch, written in his characteristically humorous, satirical, and self-effacing style⁶ (de Faget de Casteljau, 1999). His approach to curve and surface design using polar forms was described in books (de Casteljau, 1985, 1986) published in the mid-1980s. A further book (de Casteljau, 1987), on the quaternions, illustrates the breadth of his interests.⁷

⁶ He concludes with the observation that “My stay at Citroën was not only an adventure for me, but also an adventure for Citroën.”

⁷ At a 1991 conference in Norway, de Casteljau expressed great interest in a talk by the author on *curves and surfaces in geometrical optics* (Farouki and Chastang, 1991). This was followed by a sequence of elaborate written correspondences, with elegantly drawn multi-colored figures, in which de Casteljau expounded on the properties of reflected and refracted wavefronts.

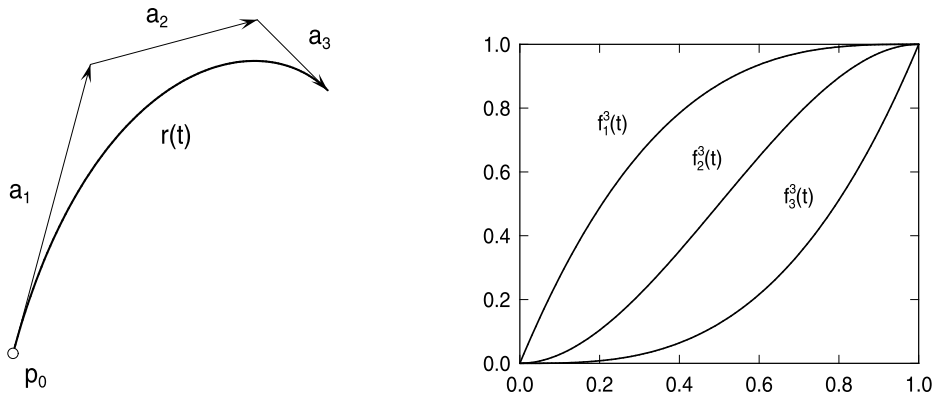


Fig. 6. Left: the Bézier point/vector specification of a cubic curve. Right: the basis functions $f_1^3(t)$, $f_2^3(t)$, $f_3^3(t)$ associated with the vectors \mathbf{a}_1 , \mathbf{a}_2 , \mathbf{a}_3 .

4.3. Pierre Étienne Bézier

Pierre Étienne Bézier joined Renault in 1933, working in the areas of tool design, production engineering, manufacturing automation, and development of NC machines, and serving as the Director of Production Engineering from 1946 (during 1941–1942, he was incarcerated in a prisoner-of-war camp, but even then remained active in his technological interests). His contributions to CAD/CAM technology, for which he is now best known,⁸ began in 1960 when he started work on what eventually became the UNISURF CAD system⁹ (Bézier, 1974). As with de Casteljau, the aim was to develop a quantitative description of the geometry of car bodies, that circumvented the inaccuracies and ambiguities of interpretation associated with paper drawings or sculpted models.

Initially (Bézier, 1966, 1967) Bézier's scheme for curve design, like that of de Casteljau, did not explicitly invoke the Bernstein basis. Instead, he used basis functions $f_1^n(t), \dots, f_n^n(t)$ characterized by the fact that $f_k^n(t)$ increases from 0 to 1 over the interval $t \in [0, 1]$ and has vanishing derivatives to order $k-1$ at $t=0$, and to order $n-k$ at $t=1$. He showed that $f_k^n(t)$ can be expressed as

$$f_k^n(t) = \frac{(-1)^k}{(k-1)!} t^k \frac{d^{k-1}}{dt^{k-1}} \frac{(1-t)^{n-1}}{t} = \sum_{j=k}^n (-1)^{j+k} \binom{n}{j} \binom{j-1}{k-1} t^j$$

for $k=1, \dots, n$ and mischievously attributed this apparently-unknown basis to a fictitious mathematician, *Onésime Durand* (Laurent and Sablonnière, 2001). Bézier specified a curve by an initial point \mathbf{p}_0 and n vectors $\mathbf{a}_1, \dots, \mathbf{a}_n$ through the expression

$$\mathbf{r}(t) = \mathbf{p}_0 + \sum_{k=1}^n \mathbf{a}_k f_k^n(t), \quad (12)$$

with the property that $\mathbf{r}'(0) = n\mathbf{a}_1$, $\mathbf{r}'(1) = n\mathbf{a}_n$, $\mathbf{r}''(0) = n(n-1)(\mathbf{a}_2 - \mathbf{a}_1)$, $\mathbf{r}''(1) = n(n-1)(\mathbf{a}_n - \mathbf{a}_{n-1})$, etc., and $\mathbf{r}(1) = \mathbf{p}_0 + \mathbf{a}_1 + \dots + \mathbf{a}_n$ (see Fig. 6).

Forrest (1972) observed that the expression (12) is actually equivalent to a Bernstein-form polynomial. In particular, one can verify that

$$f_k^n(t) - f_{k+1}^n(t) = b_k^n(t), \quad k=1, \dots, n-1,$$

while $1 - f_1^n(t) = b_0^n(t)$ and $f_n^n(t) = b_n^n(t)$, from which one infers that

$$1 - f_k^n(t) = \sum_{j=0}^{k-1} b_j^n(t) \quad \text{and} \quad f_k^n(t) = \sum_{j=k}^n b_j^n(t)$$

for $k=1, \dots, n$, the partition-of-unity property (3) being invoked to obtain the latter expression. Thus, substituting for $f_k^n(t)$, one finds that the Bézier form (12) is equivalent to the familiar control-point form (9), if we take

⁸ Bézier's contributions to numerical control and manufacturing automation are at least as significant as his methods for geometric design, but beyond the scope of this review.

⁹ See also Chapter 1 of Farin (2002) for Bézier's account of the origins of the UNISURF system.

$$\mathbf{p}_k = \mathbf{p}_0 + \sum_{j=1}^k \mathbf{a}_j, \quad k = 1, \dots, n.$$

Hence, the vectors $\mathbf{a}_1, \dots, \mathbf{a}_n$ in (12) represent the control polygon legs, i.e., $\mathbf{a}_k = \mathbf{p}_k - \mathbf{p}_{k-1}$ for $k = 1, \dots, n$. The control-point form (9) is now universally preferred over the initial-point/vector form (12).

Bézier published his ideas extensively (Bézier, 1966, 1967, 1971, 1972, 1974, 1977, 1986, 1993) during the 1960s and 1970s, and the representation (9) is now conventionally known as a *Bézier curve* – although it is closer in spirit to de Casteljau's formulation than Bézier's original formulation. Bézier curves are now a firmly established and indispensable tool in computer graphics, engineering design, animation, path planning, and related fields. For example, scalable computer fonts such as `PostScript` employ Bézier curves to describe the character shapes.

Bézier retired, after a 42-year career with Renault, in 1975. In addition to degrees in mechanical engineering (1930) and electrical engineering (1931), he received a doctorate in mathematics from the University of Paris (1977). He was also awarded an honorary doctorate from the Technical University of Berlin, and from 1968 to 1979 served as Professor of Production Engineering at the Conservatoire National des Arts et Métiers. Bézier remained active in retirement through correspondence, publications, and presentations until his death in 1999. Correspondence of Christophe Rabut with Bézier, describing the origins of his curve design ideas, shortly before he passed away, may be found in Rabut (2002). Finally, a detailed biography of Bézier by Pierre-Jean Laurent and Paul Sablonnière (2001) observes that “His art of living consisted in doing everything seriously without ever taking himself too seriously.”

New ideas establish a firm foothold only if they are taken up and further developed by others. Among many publications that played a key role in this regard, some noteworthy examples are the papers of Forrest (1972), Gordon and Riesenfeld (1974), Lane and Riesenfeld (1980), and Chang and Wu (1981).

5. Bernstein basis properties

Fig. 7 illustrates the basis functions (1) for $n = 7$. The attractive features of the Bernstein form (7) stem from certain intrinsic properties of these basis functions, and consequent relations among the coefficients c_0, \dots, c_n and the behavior of the graph of $p(t)$ over $[0, 1]$.

5.1. Basic properties and algorithms

We now briefly describe some of the key properties and algorithms associated with the Bernstein form.¹⁰

1. **symmetry.** The basis functions $b_k^n(t)$ and $b_{n-k}^n(t)$ are mirror images of each other about the interval mid-point $t = \frac{1}{2}$ – i.e., $b_{n-k}^n(1-t) \equiv b_k^n(t)$.
2. **recursion.** The basis of degree $n+1$ may be generated from the basis of degree n through the recurrence relation

$$b_k^{n+1}(t) = t b_{k-1}^n(t) + (1-t) b_k^n(t) \quad (13)$$

for $k = 0, \dots, n+1$ where we define $b_k^n(t) \equiv 0$ if $k < 0$ or $k > n$, and initiate the recursion with $b_0^0(t) \equiv 1$.

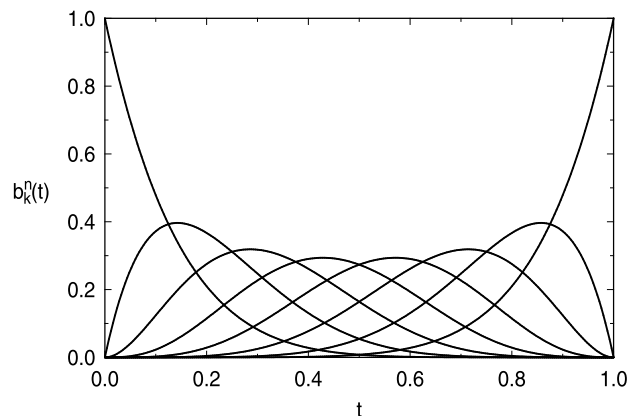


Fig. 7. The Bernstein basis functions (1) of degree 7 on $t \in [0, 1]$.

¹⁰ An object-oriented C++ library that implements many basic functions for univariate Bernstein-form polynomials is described in Tsai and Farouki (2001). See also Berchtold and Bowyer (2000) for a discussion of some of these functions in the tensor-product multivariate case.

3. **non-negativity.** As noted in Section 3, the basis functions (1) satisfy $b_k^n(t) \geq 0$ for $t \in [0, 1]$. This property does not hold outside $[0, 1]$.
4. **partition of unity.** The property (3) results from the fact that the basis functions (1) are simply the $n + 1$ terms in the binomial expansion of $[(1 - t) + t]^n$. This property is not restricted to $[0, 1]$.
5. **unimodality.** $b_k^n(t)$ has a single extremum, at $t = k/n$, on $t \in [0, 1]$. Also, for any fixed value t_* there is a corresponding index k such that

$$b_0^n(t_*) \leq \dots \leq b_{k-1}^n(t_*) \leq b_k^n(t_*) \geq b_{k+1}^n(t_*) \geq \dots \geq b_n^n(t_*),$$

i.e., the values $b_k^n(t_*)$ are unimodal with respect to the index k . Hence, the control point \mathbf{p}_k has the greatest influence on the Bézier curve (9) at $t = t_*$, while the influence of all the other control points diminishes monotonically as their indices decrease or increase away from k .

6. **lower & upper bounds.** Properties 3 and 4 imply that for $t \in [0, 1]$ the polynomial (7) satisfies (Cargo and Shisha, 1966; Rivlin, 1970) the bounds

$$\min_{0 \leq k \leq n} c_k \leq p(t) \leq \max_{0 \leq k \leq n} c_k.$$

7. **end-point values.** By virtue of the property (4) of the basis functions, the polynomial (7) satisfies $p(0) = c_0$ and $p(1) = c_n$.
8. **variation-diminishing property.** The number N of real roots of $p(t)$ on $t \in (0, 1)$ is less than the number $S(c_0, \dots, c_n)$ of sign variations in its Bernstein coefficients by an even amount,¹¹

$$N = S(c_0, \dots, c_n) - 2k, \quad (14)$$

for some integer $k \geq 0$. This is an expression of Descartes' Law of Signs (Uspensky, 1948), since the map $t \in (0, 1) \rightarrow u \in (0, \infty)$ defined by $t(u) = u/(1 + u)$ transforms $p(t)$ into

$$q(u) = p(t(u)) = (1 + u)^{-n} \sum_{k=0}^n a_k u^k, \quad \text{where } a_k = \binom{n}{k} c_k.$$

The coefficients c_k and a_k clearly have the same signs, and roots of $q(u)$ on $(0, \infty)$ are in one-to-one correspondence with roots of $p(t)$ on $(0, 1)$.

9. **relation to monomial basis.** The Bernstein and monomial (power) bases of degree n are related (Farouki and Rajan, 1987) by the expressions

$$t^k = \sum_{j=k}^n \binom{j}{k} \binom{n}{j} b_j^n(t), \quad b_k^n(t) = \sum_{j=k}^n (-1)^{j-k} \binom{n}{j} \binom{j}{k} t^j \quad (15)$$

for $k = 0, \dots, n$. In particular, case $k = 0$ of the first expression yields the partition of unity property, while from case $k = 1$ we obtain

$$t = \frac{b_1^n(t) + 2b_2^n(t) + \dots + nb_n^n(t)}{n},$$

which induces the linear precision property.

10. **scaling the independent variable.** The change of variables $t \rightarrow rt$ maps the interval $[0, 1]$ to $[0, r]$. Setting $1 - rt = (1 - t) + (1 - r)t$ in

$$b_k^n(rt) = \binom{n}{k} (1 - rt)^{n-k} (rt)^k$$

and performing a binomial expansion, one can verify that

$$b_k^n(rt) = \sum_{j=k}^n b_k^j(r) b_j^n(t), \quad j = 0, \dots, n. \quad (16)$$

With $r = \frac{1}{2}$, this allows the Bernstein coefficients of (7) on $[0, \frac{1}{2}]$ – as generated by mid-point subdivision through the de Casteljau algorithm – to be expressed in terms of the coefficients on $[0, 1]$. See (33)–(34) below for the generalization of (16) to an arbitrary interval $[t_1, t_2]$.

¹¹ When counting the number of sign variations in the sequence of Bernstein coefficients, zeros are ignored. Also, each root on $(0, 1)$ contributes to N according to its multiplicity.

11. **derivatives.** One can verify that the basis functions (1) satisfy

$$\frac{d}{dt} b_k^n(t) = n[b_{k-1}^{n-1}(t) - b_k^{n-1}(t)] \quad (17)$$

where $b_{-1}^{n-1}(t) \equiv 0$ and $b_n^{n-1}(t) \equiv 0$. Substituting into the derivative of (7) and combining terms, we obtain

$$p'(t) = \sum_{k=0}^{n-1} n \Delta c_k b_k^{n-1}(t),$$

where $\Delta c_k = c_{k+1} - c_k$. Further derivatives can be written in terms of higher-order differences of the Bernstein coefficients.

12. **integrals.** By setting $n \rightarrow n+1$ in (17), and adding up and integrating instances $k+1, \dots, n+1$ of the resulting equation, we obtain

$$\int b_k^n(t) dt = \frac{1}{n+1} \sum_{j=k+1}^{n+1} b_j^{n+1}(t). \quad (18)$$

Hence, the indefinite integral of (7) may be expressed as

$$\int p(t) dt = \sum_{k=1}^{n+1} \left(\frac{1}{n+1} \sum_{j=0}^{k-1} c_j \right) b_k^{n+1}(t) + \text{constant}.$$

Each of the basis functions (1) has the same definite integral, namely $1/(n+1)$, and hence

$$\int_0^1 p(t) dt = \frac{1}{n+1} \sum_{k=0}^n c_k.$$

13. **degree elevation.** A polynomial $p(t)$ of true degree n can be expressed in the Bernstein basis of degree $n+r$, for all $r > 0$ – i.e., one can find coefficients¹² $c_0^{n+r}, \dots, c_{n+r}^{n+r}$ such that (7) can be written as

$$p(t) = \sum_{k=0}^{n+r} c_k^{n+r} b_k^{n+r}(t).$$

By multiplying (1) with $1 = (1-t) + t$, one can verify that

$$b_k^n(t) = \left(1 - \frac{k}{n+1}\right) b_k^{n+1}(t) + \frac{k+1}{n+1} b_{k+1}^{n+1}(t), \quad k = 0, \dots, n,$$

and hence by substituting into (7) we obtain

$$c_k^{n+1} = \frac{k}{n+1} c_{k-1}^n + \left(1 - \frac{k}{n+1}\right) c_k^n, \quad k = 1, \dots, n,$$

while $c_0^{n+1} = c_0^n$, $c_{n+1}^{n+1} = c_n^n$. This defines a unit degree elevation, which can be applied repeatedly. The outcome of an r -fold degree elevation may be determined by multiplying (1) with $1 = [(1-t) + t]^r$ to obtain

$$b_k^n(t) = \sum_{j=k}^{k+r} \frac{\binom{n}{k} \binom{r}{j-k}}{\binom{n+r}{j}} b_j^{n+r}(t), \quad k = 0, \dots, n, \quad (19)$$

and substituting into (7) then gives (Farouki and Rajan, 1988):

$$c_k^{n+r} = \sum_{j=\max(0, k-r)}^{\min(n, k)} \frac{\binom{r}{k-j} \binom{n}{j}}{\binom{n+r}{k}} c_j^n, \quad k = 0, \dots, n+r.$$

¹² If it is necessary to specify the degree of the Bernstein basis, it appears as a superscript on the coefficients (if no superscript appears, the basis has the default degree n).

14. **degree reduction.** The true degree of a polynomial in Bernstein form is not immediately apparent from its coefficients. The conditions for (7) to be of true degree $n - r$ (with $r \geq 1$) are that the power coefficients in (8) must satisfy $a_n = \dots = a_{n-r+1} = 0 \neq a_{n-r}$. These can be expressed in terms of the Bernstein coefficients by noting from (8) and (15) that

$$a_k = \sum_{j=0}^k (-1)^{k-j} \binom{n}{k} \binom{k}{j} c_j, \quad k = 0, \dots, n.$$

When these conditions hold, the coefficients in the basis of degree $n - r$ can be expressed (Farouki and Rajan, 1988) in terms of those in the degree- n basis as

$$c_k^{n-r} = \sum_{j=0}^k (-1)^{k-j} \frac{\binom{k-j+r-1}{r-1} \binom{n}{j}}{\binom{n-r}{k}} c_j^n, \quad k = 0, \dots, n - r.$$

The term *degree reduction* is also commonly invoked in a looser sense, to connote best approximation (according to a specified error measure) of a given polynomial of degree n on $[0, 1]$ by polynomials of lower degree: see Brunnett et al. (1996), Eck (1993, 1995), Lachance (1988), Szafnicki (2002), Watkins and Worsey (1988). An interesting fact (Lutterkort et al., 1999) in this context is that the polynomial $q(t)$ of degree $< n$ that best approximates (7) – in these sense of the L_2 norm on $[0, 1]$ – can be identified by minimizing the sum of squared differences of Bernstein coefficients of $p(t)$ and $q(t)$, when the degree of the latter is elevated to n .

15. **arithmetic operations.** To add or subtract two polynomials $f(t)$, $g(t)$ of equal degree, one simply adds or subtracts their respective Bernstein coefficients. If they are of unequal degree, the degrees must be matched through a degree elevation before adding/subtracting the coefficients. If $f(t)$ and $g(t)$ are degree m and n with Bernstein coefficients a_0, \dots, a_m and b_0, \dots, b_n their product $f(t)g(t)$ has (Farouki and Rajan, 1988) the Bernstein coefficients

$$c_k = \sum_{j=\max(0, k-n)}^{\min(m, k)} \frac{\binom{m}{j} \binom{n}{k-j}}{\binom{m+n}{k}} a_j b_{k-j}, \quad k = 0, \dots, m + n.$$

In the division $f(t)/g(t)$, the goal is to find the *quotient* and *remainder* polynomials $q(t)$ and $r(t)$ such that $f(t) = q(t)g(t) + r(t)$ with $\deg(q) = m - n$ and $\deg(r) = n - 1$. Since the *long division* process for the power form invokes obvious degree reductions at each step, it does not readily translate to the Bernstein basis. Nevertheless, the Bernstein coefficients q_0, \dots, q_{m-n} and r_0, \dots, r_{n-1} of $q(t)$ and $r(t)$ can be determined by comparing like terms in $f(t) = q(t)g(t) + r(t)$, and solving the resulting system of $m + 1$ linear equations specified (Farouki and Rajan, 1988) for $k = 0, \dots, m$ by

$$\sum_{j=\max(0, k-n)}^{\min(m-n, k)} \frac{\binom{m-n}{j} \binom{n}{k-j}}{\binom{m}{k}} b_{k-j} q_j + \sum_{j=\max(0, k-m+n-1)}^{\min(n-1, k)} \frac{\binom{m-n+1}{k-j} \binom{n-1}{j}}{\binom{m}{k}} r_j = a_k.$$

Minimair (2008) describes a more sophisticated division algorithm, with quadratic worst-case cost dependence on the degrees, compared to the cubic worst-case cost of solving the above linear system. See also Busé and Goldman (2008), who discuss both division and gcd algorithms.

16. **composition.** For polynomials $f(t)$ and $g(u)$ of degree m and n with Bernstein coefficients a_0, \dots, a_m and b_0, \dots, b_n the composition problem is concerned with computing the Bernstein coefficients c_0, \dots, c_{mn} of the polynomial $p(u) = f(g(u))$ defined by substituting $t = g(u)$ in $f(t)$. An elegant recursive algorithm addressing this problem has been described by DeRose (1988). This populates a three-dimensional array of values $a_{i,j}^k$ by first setting $a_{i,0}^0 = a_i$ for $i = 0, \dots, m$. Then the expression

$$a_{i,j}^k = \frac{1}{\binom{kn}{j}} \sum_{l=\max(0, j-n)}^{\min(j, kn-n)} \binom{kn-n}{l} \binom{n}{j-l} [(1 - b_{j-l}) a_{i,l}^{k-1} + b_{j-l} a_{i+1,l}^{k-1}]$$

is used for $k = 1, \dots, m$, $i = 0, \dots, m - k$, and $j = 0, \dots, kn$. Finally, the Bernstein coefficients c_0, \dots, c_{mn} of $p(u) = f(g(u))$ are specified by

$$c_j = a_{0,j}^m, \quad j = 0, \dots, mn.$$

The generalization to multivariate Bernstein-form polynomials, defined over simplex domains (see Section 8.2), is also presented in DeRose (1988).

17. **interpolation & approximation.** There is a unique polynomial $p(t)$ of degree n that interpolates $n + 1$ function values f_0, \dots, f_n associated with nodes t_0, \dots, t_n – i.e., $p(t_k) = f_k$ for $k = 0, \dots, n$. In power form, $p(t)$ can be obtained by solving the *Vandermonde* linear system (which is often ill-conditioned) or by construction of the *Lagrange basis* for the given nodes (Davis, 1975). For nodes $t_0, \dots, t_n \in [0, 1]$, on the other hand, the *Bernstein–Vandermonde system* is typically

well-conditioned, and can be solved by accurate and efficient algorithms (Delgado and Peña, 2009; Marco and Martínez, 2007). Likewise, stable Bernstein-basis algorithms have been developed (Marco and Martínez, 2010) for least-squares approximation of the given data by polynomials of degree $< n$.

18. **resultants.** The *resultant* of two polynomials $f(t)$, $g(t)$ is a polynomial expression in their coefficients, that vanishes if and only if they possess a common root (Uspensky, 1948), i.e., $f(\tau) = g(\tau) = 0$ for some $\tau \in \mathbb{C}$. Resultants are typically formulated as determinants, with entries defined in terms of the power coefficients of $f(t)$ and $g(t)$ – e.g., the *Sylvester* or *Bezout* determinants (Burnside and Panton, 1928). It has been noted (Farouki and Rajan, 1988; Goldman et al., 1984) that these determinants can be expressed in terms of the Bernstein coefficients by replacing each power coefficient by the corresponding *scaled* Bernstein coefficient – i.e., by $\binom{m}{j}a_j$ for $j = 0, \dots, m$ and by $\binom{n}{k}b_k$ for $k = 0, \dots, n$. However, this approach is not optimal in terms of efficiency and stability. Several alternative resultant formulations have recently been developed by Bini and Gemignani (2004), Bini et al. (2005) and Winkler (2000, 2003, 2004, 2008). These methods are also useful in computing *greatest common divisors* of Bernstein-form polynomials.

Products and ratios of binomial coefficients arise in many algorithms for processing Bernstein-form polynomials, and it is desirable to compute their values in exact integer arithmetic, while avoiding the possibility of overflow for large degrees. Algorithms to efficiently compute the prime decompositions (Goetgheluck, 1987, 1988) of binomial coefficients are useful in this regard, since they allow the products and ratios of binomial coefficients to be determined by the addition or subtraction of corresponding prime exponents.

5.2. Shape features of Bézier curves

The interpretation of some of the above properties, in the case of the Bézier curve (9) and its control polygon, is illustrated in Fig. 8 – for complete details, see (Farin, 2002). The control polygon may be viewed as a “caricature” of the curve (9), that exaggerates its shape. However, the correlation between the curve and control polygon diminishes as the degree n increases: see (Nairn et al., 1999) for bounds on the deviation of the curve from its control polygon.

In many applications, such as rendering and intersection computations, the convergent piecewise-linear control polygon approximations to the curve $\mathbf{r}(t)$ generated by subdivision or degree elevation are a valuable feature of the Bézier form, that also generalizes to the B-spline form (see Section 8.3 below). Subdivision and degree elevation are archetypal examples of “corner-cutting” algorithms, whose convergence rates have been studied by many authors – see Cohen and Schumaker (1985), Dahmen (1986), de Boor (1987a), Micchelli and Prautzsch (1989), Prautzsch and Kobbelt (1994). Iterated midpoint subdivision, for example, yields a polygonal approximation converging quadratically to the curve (9).

The *shape-preserving* property of Bézier curves is another a key attribute in geometric design applications. A polynomial basis $\{\phi_0(t), \dots, \phi_n(t)\}$ is said to be *normalized totally positive* on $[0, 1]$ if it satisfies $\phi_0(t) + \dots + \phi_n(t) = 1$ and, for every sequence of points $0 \leq t_1 < \dots < t_m \leq 1$, the $(n+1) \times m$ *collocation matrix* with elements $M_{jk} = \phi_j(t_k)$ is totally positive – i.e., all its minors are non-negative. For such a basis, the number of sign variations of $p(t) = c_0\phi_0(t) + \dots + c_n\phi_n(t)$ on $(0, 1)$ cannot exceed the number of sign changes in the coefficients c_0, \dots, c_n (Goodman, 1989), and hence curves specified in terms of it are shape-preserving. Carnicer and Peña (1993) showed that the Bernstein basis is “optimally” shape-preserving, i.e., the Bézier control polygon gives a better sense of the curve shape than the polygon associated with any other normalized totally positive basis on $[0, 1]$. This arises from the fact that the latter can always be expressed as a linear combination of the Bernstein basis, specified by a stochastic¹³ totally positive matrix. See also Mazure (2004).

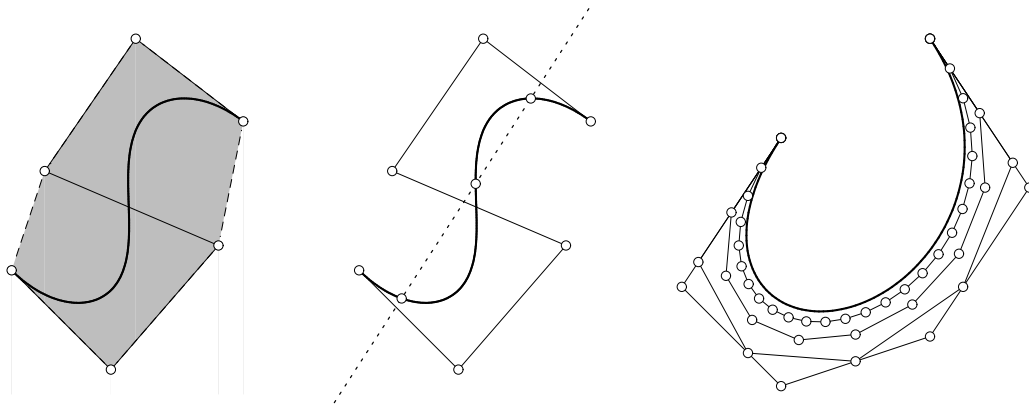


Fig. 8. Left: confinement of a Bézier curve within the convex hull of its control polygon. Center: variation-diminishing property – no straight line may intersect the curve more often than the control polygon. Right: control polygons of a quintic Bézier curve after elevation of the degree to 6, 10, 25.

¹³ A stochastic matrix is determined by elements that are non-negative and sum to unity across each row – see also Section 7.3 below.

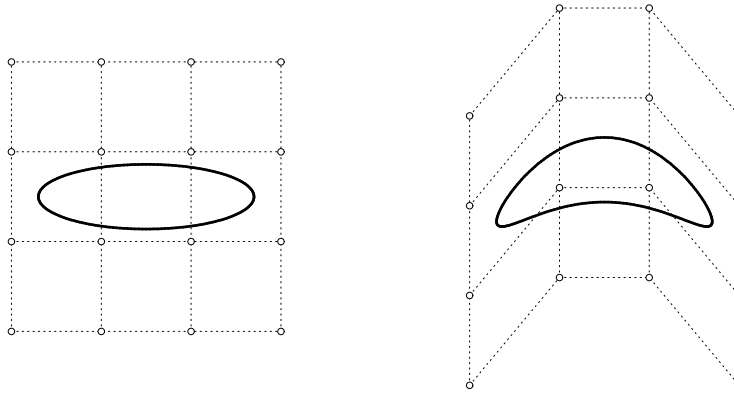


Fig. 9. Free-form deformation of an ellipse (left) by using a tensor-product bicubic polynomial to define a displacement field. The deformed shape (right) is obtained by moving the control points from their original lattice positions.

Although the Bernstein basis is most often used to represent *parametric* curves and surfaces, it also offers an intuitive means to define *implicit* curve or surface segments over finite (simplex or rectangular) domains (Sederberg 1984, 1985). In this context, the implicit curve/surface segment is viewed as the zero set of a multivariate polynomial, determined by Bernstein coefficients associated with each point of a control net, and its geometry can be manipulated by varying these coefficients. A related scheme is concerned with spatial deformations of geometric models (Sederberg and Parry, 1986), using displacement fields specified by alterations to a Bernstein-Bézier control net over a prescribed domain (see Fig. 9).

6. Numerical stability

The control-point paradigm for constructing and manipulating polynomial or rational curves and surfaces, and the many geometrically intuitive algorithms associated with it, were the initial motivations for the enthusiastic adoption of the Bernstein form in computer-aided geometric design in the 60s and 70s. It was not until somewhat later, however, that another key property was fully appreciated – namely, the *numerical stability* of the Bernstein form (Farouki and Rajan, 1987) with respect to perturbations of initial data, or rounding errors that occur during floating-point calculations. The importance of this attribute stems from the high premium placed on the “robustness” (i.e., accuracy and consistency) of the geometrical computations performed in CAD systems. Unlike most other forms of scientific or engineering computation, the output of CAD systems – geometric models – are not ends in themselves. Such models are rather the point of departure for downstream applications (meshing for finite-element analysis, path planning for manufacturing, etc.) that cannot succeed without accurate and consistent geometrical representations.

Any $n + 1$ linearly-independent polynomials $\phi_0(t), \dots, \phi_n(t)$ of degree $\leq n$ constitute a basis for polynomials $p(t)$ of degree n in the variable t , i.e., such a polynomial can be uniquely specified by coefficients c_0, \dots, c_n in the form

$$p(t) = \sum_{k=0}^n c_k \phi_k(t). \quad (20)$$

When choosing a basis for numerical computations, the sensitivity of $p(t)$ to uncertainties in its coefficients is of fundamental concern. Such uncertainties may be regarded as arising from initial measurement error, or – through the method of *backward error analysis* (Wilkinson, 1963) – as representing the accumulation of rounding errors during a floating-point computation.

6.1. Condition numbers

Suppose each coefficient c_k of (20) suffers a random perturbation δc_k and $p(t)$ is consequently perturbed to $p(t) + \delta p(t)$. The magnitude of $\delta p(t)$ depends on (a) the value of the independent variable t ; (b) the statistical distributions of the perturbations $\delta c_0, \dots, \delta c_n$; and (c) the adopted basis $\phi_0(t), \dots, \phi_n(t)$. To focus on the influence of (c), we assume here that (a) t lies on the interval $[0, 1]$; and (b) the coefficient perturbations have uniform distributions, with a fixed maximum relative magnitude ϵ , so that

$$-\epsilon \leq \delta c_k / c_k \leq +\epsilon \quad (21)$$

for $k = 0, \dots, n$. Then the nominal value of (20) is perturbed to

$$p(t) + \delta p(t) := \sum_{k=0}^n c_k \phi_k(t) + \sum_{k=0}^n \delta c_k \phi_k(t),$$

where the perturbation $\delta p(t)$ satisfies the bounds

$$-\sum_{k=0}^n |\delta c_k \phi_k(t)| \leq \delta p(t) \leq \sum_{k=0}^n |\delta c_k \phi_k(t)|. \quad (22)$$

Thus, denoting the basis $\{\phi_0(t), \dots, \phi_n(t)\}$ by Φ and using (21) we may write

$$|\delta p(t)| \leq C_\Phi(p(t)) \epsilon \quad \text{where } C_\Phi(p(t)) := \sum_{k=0}^n |c_k \phi_k(t)|. \quad (23)$$

$C_\Phi(p(t))$ is the condition number for the value of the polynomial (20), at each t , with respect to the uniform relative perturbations (21) of its coefficients in the basis $\Phi = \{\phi_0(t), \dots, \phi_n(t)\}$. It should be noted that $C_\Phi(p(t))$ depends on the choice of basis for the representation of $p(t)$.

A basis $\Phi = \{\phi_0(t), \dots, \phi_n(t)\}$ is said to be non-negative on the interval $t \in [a, b]$ if its components satisfy

$$\phi_k(t) \geq 0 \quad \text{for } t \in [a, b], \quad k = 0, \dots, n.$$

As noted above, the Bernstein basis (1) has this property when $[a, b] = [0, 1]$. Non-negative bases are of interest (Farouki and Goodman, 1996) in the following context.

Theorem 1. Let $\Psi = \{\psi_0(t), \dots, \psi_n(t)\}$ and $\Phi = \{\phi_0(t), \dots, \phi_n(t)\}$ be non-negative bases for polynomials of degree n on $t \in [a, b]$ such that the former can be expressed as a non-negative combination of the latter, i.e.,

$$\psi_j(t) = \sum_{k=0}^n M_{jk} \phi_k(t), \quad j = 0, \dots, n, \quad (24)$$

where $M_{jk} \geq 0$ for $0 \leq j, k \leq n$. Then the condition numbers for the value of **any** degree- n polynomial $p(t)$ at **any** point $t \in [a, b]$ in these bases satisfy

$$C_\Phi(p(t)) \leq C_\Psi(p(t)). \quad (25)$$

Proof. In the given non-negative bases, let $p(t)$ have the representations

$$p(t) = \sum_{j=0}^n a_j \psi_j(t) = \sum_{k=0}^n c_k \phi_k(t). \quad (26)$$

Then, from (24), the coefficients in the two bases must be related by

$$c_k = \sum_{j=0}^n a_j M_{jk} \quad \text{for } k = 0, \dots, n. \quad (27)$$

Since they are both non-negative on $t \in [a, b]$ the condition numbers for the value of $p(t)$ in these bases may be written as

$$C_\Phi(p(t)) = \sum_{k=0}^n |c_k| \phi_k(t) \quad \text{and} \quad C_\Psi(p(t)) = \sum_{j=0}^n |a_j| \psi_j(t). \quad (28)$$

Now substituting (27) into $C_\Phi(p(t))$ and using the triangle inequality

$$\left| \sum_{k=0}^n x_k \right| \leq \sum_{k=0}^n |x_k|, \quad (29)$$

we obtain

$$C_\Phi(p(t)) = \sum_{k=0}^n \left| \sum_{j=0}^n a_j M_{jk} \right| \phi_k(t) \leq \sum_{k=0}^n \left[\sum_{j=0}^n |a_j M_{jk}| \right] \phi_k(t). \quad (30)$$

Then, setting $|a_j M_{jk}| = |a_j| M_{jk}$ (since $M_{jk} \geq 0$ for all j, k) and re-arranging the order of summation on the right-hand side of (30), we have

$$C_\Phi(p(t)) \leq \sum_{j=0}^n |a_j| \sum_{k=0}^n M_{jk} \phi_k(t) = \sum_{j=0}^n |a_j| \psi_j(t) = C_\Psi(p(t)), \quad (31)$$

where we make use of (24) in the second step. \square

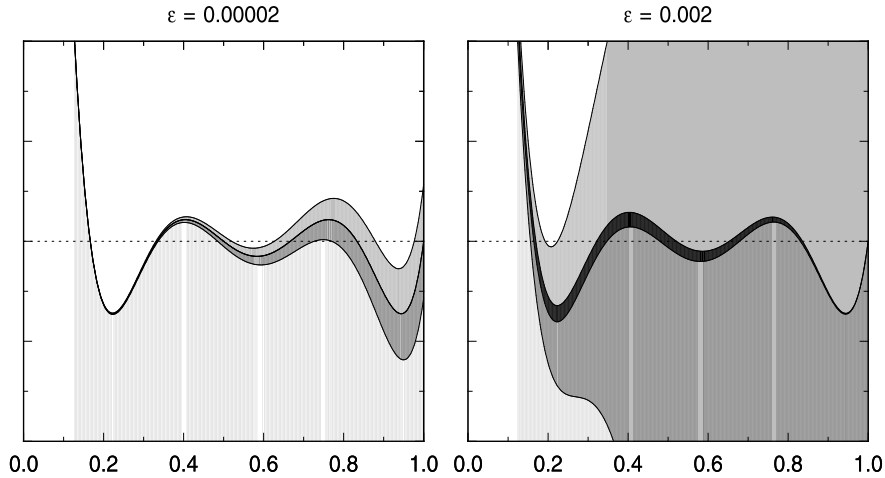


Fig. 10. Perturbation region for the Bernstein (black area) and power (gray area) forms of a degree-6 polynomial, with a given coefficient uncertainty ϵ .

An important instance of Theorem 1 is the case where Φ is the Bernstein basis (1) on $[0, 1]$ and Ψ is the monomial or “power” basis $1, t, \dots, t^n$ (Farouki and Rajan, 1987).

Corollary 1. The condition numbers $C_B(p(t))$ and $C_P(p(t))$ for the value of $p(t)$ in the Bernstein and power representations, (7) and (8), satisfy

$$C_B(p(t)) \leq C_P(p(t))$$

for **any** polynomial $p(t)$ and **any** value $t \in [0, 1]$.

Proof. The Bernstein and power bases are both non-negative on $[0, 1]$ and from the transformations (15) between them we see that the power basis is a non-negative combination of the Bernstein basis, but not vice-versa. Hence, the conditions of Theorem 1 hold. \square

In other words, the Bernstein form is *systematically more stable* than the power form, when evaluating polynomials on $t \in [0, 1]$ whose coefficients are subject to the random perturbations (21) of uniform relative magnitude.

Note that Theorem 1 makes no assumption concerning the magnitude of the relative coefficient error ϵ in deducing the bound (23). The perturbation $\delta p(t)$ of $p(t)$ satisfies this bound for *finite* (not just infinitesimal) errors in the coefficients. The effect of these uncertainties may be regarded as “exploding” the graph of $p(t)$ from a line of zero width to a region of finite (variable) width. At each t , the width of this *perturbation region* depends on the adopted basis. However, in comparing the Bernstein and power forms over $[0, 1]$ we can be sure that, for any polynomial, the perturbation region for the former is always narrower than that for the latter, as illustrated in Fig. 10.

The above arguments can be adapted to describe the sensitivity of the *roots* of a polynomial $p(t)$ to uncertainties in the coefficients. In this context, one must consider the sensitivity of a root to *infinitesimal* perturbations of the coefficients, in accordance with standard theory (Rice, 1966) for the condition of problems with analytic “input–output” relations. If τ is a *simple* root of (20), i.e., $p(\tau) = 0 \neq p'(\tau)$, then in the limit $\epsilon \rightarrow 0$ the perturbation $\delta\tau$ of the root induced by the perturbations (21) satisfies (Gautschi, 1973, 1984) the bound¹⁴

$$|\delta\tau| \leq C_\Phi(\tau)\epsilon \quad \text{where } C_\Phi(\tau) := \frac{1}{|p'(\tau)|} \sum_{k=0}^n |c_k \phi_k(\tau)|. \quad (32)$$

$C_\Phi(\tau)$ is the *condition number for the root τ* of $p(t)$, in the basis Φ . Since it differs from $C_\Phi(p(\tau))$ only by the factor $|p'(\tau)|^{-1}$, which is independent of the basis, the result of Theorem 1 applies also to root condition numbers.

Two more important cases of Theorem 1 are concerned with *subdivision* and *degree elevation* of the Bernstein form (Farouki and Rajan, 1987).

¹⁴ For an m -fold root, $|\delta\tau|$ grows like $\epsilon^{1/m}$ rather than linearly with ϵ , as in (32).

Corollary 2. The condition numbers $\tilde{C}(p(t))$ and $C(p(t))$ for the value of a polynomial $p(t)$ of true degree n in the Bernstein bases of degree $n+r$ and n on $[0, 1]$ satisfy

$$\tilde{C}(p(t)) \leq C(p(t))$$

for **any** polynomial $p(t)$ and **any** value $t \in [0, 1]$ and for all $r \geq 1$.

Corollary 3. Suppose $[t_1, t_2] \subset [0, 1]$. Then the condition numbers $\tilde{C}(p(t))$ and $C(p(t))$ for the value of a degree- n polynomial $p(t)$ in the Bernstein bases of degree n on $[t_1, t_2]$ and $[0, 1]$ satisfy

$$\tilde{C}(p(t)) \leq C(p(t))$$

for **any** polynomial $p(t)$ and **any** value $t \in [t_1, t_2]$.

Corollary 2 follows from the fact that, for all $r \geq 1$, the Bernstein basis of degree n is a non-negative combination of the basis of degree $n+r$ – as expressed by (19). Similarly, Corollary 3 is a consequence of the fact that, whenever $[t_1, t_2] \subset [0, 1]$ the Bernstein basis on $[0, 1]$ can be expressed as a non-negative combination of the basis

$$\tilde{b}_k^n(t) = \binom{n}{k} \frac{(t_2 - t)^{n-k} (t - t_1)^k}{(t_2 - t_1)^n}, \quad k = 0, \dots, n$$

on $[t_1, t_2]$. Specifically (Farouki and Neff, 1990), we have

$$b_j^n(t) = \sum_{k=0}^n M_{jk} \tilde{b}_k^n(t), \quad j = 0, \dots, n, \quad (33)$$

where

$$M_{jk} = \sum_{i=\max(0, j+k-n)}^{\min(j, k)} b_{j-i}^{n-k}(t_1) b_i^k(t_2), \quad 0 \leq j, k \leq n. \quad (34)$$

Corollaries 2 and 3 also apply to the root condition numbers.

6.2. Wilkinson polynomial

A “simple” polynomial whose roots are notoriously difficult to compute (Wilkinson, 1959) is the *Wilkinson polynomial*

$$p(t) = \prod_{k=1}^n (t - k/n), \quad n = 20, \quad (35)$$

with twenty equidistant roots on the interval $t \in [0, 1]$. This polynomial was first employed by the British numerical analyst J.H. Wilkinson¹⁵ in 1959, in the context of testing a software implementation of floating-point arithmetic (only fixed-point arithmetic processors were available then). To compute the roots of $p(t)$, Wilkinson first determined its power coefficients a_0, \dots, a_n by multiplying out (35), and then used the power form to evaluate $p(t)$ and $p'(t)$ for Newton–Raphson iterations. But he found that most of the roots could not be determined with more than just a few accurate digits – if at all. After eliminating the possibility of bugs in the software, Wilkinson found the true source of the problem – the extreme sensitivity of the roots to perturbations in the power coefficients a_0, \dots, a_n . He subsequently called this “the most traumatic experience in my career as a numerical analyst” (Wilkinson, 1984).

A radically different picture emerges (Farouki and Rajan, 1987) if one uses the Bernstein, rather than the power, form of (35). By Theorem 1, we expect the root condition numbers in the Bernstein basis to be systematically smaller than those in the power basis. An explicit computation shows that the difference is substantial: the largest condition numbers are $\sim 10^{13}$ in the power basis, but only $\sim 10^6$ in the Bernstein basis (Farouki and Rajan, 1987) – see Fig. 11. Typically, the base 10 logarithm of a condition number indicates the number of inaccurate significant decimal digits one can expect in the result of a floating-point calculation. Thus, with double-precision arithmetic (~ 15 significant digits) one struggles to secure just a few accurate digits in the roots of $p(t)$ with the power form, but the Bernstein form typically yields at least 9 accurate digits.

The striking difference in accuracy obtained with the power and Bernstein forms has a simple explanation (Farouki, 1991b). For both bases, the coefficients exhibit alternating signs, but the terms in (7) and (8) are of much larger magnitude for the power form than for the Bernstein form. Large cancellations of leading significant digits occur when the terms are summed, incurring amplification of the relative errors inherent in each term (due to the coefficient uncertainties). With the power form, this effect is much more severe than with the Bernstein form, since the individual terms are many orders of magnitude larger.

¹⁵ Wilkinson actually used roots at $t = 1, \dots, 20$. Scaling them down to the unit interval $t \in [0, 1]$ as in (35) does not materially alter the problem.

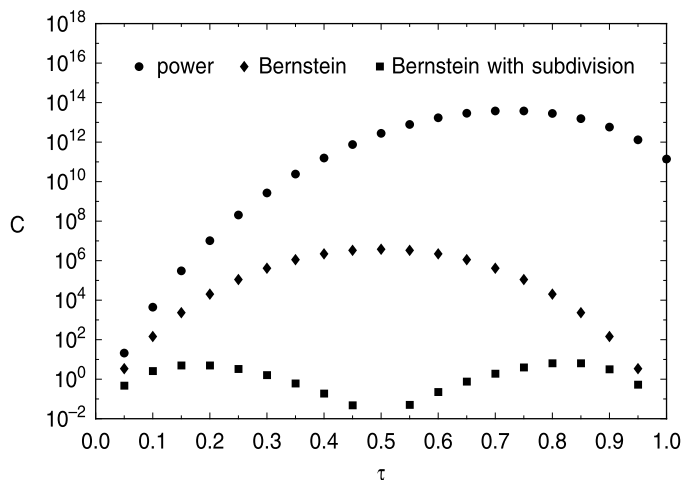


Fig. 11. Root condition numbers for the Wilkinson polynomial (35) in the power basis and Bernstein basis on $[0, 1]$. The Bernstein-basis root condition numbers on the two subintervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ of $[0, 1]$ are also shown.

6.3. Optimal stability

The basic ingredients of Theorem 1 are that (i) $\Psi = \{\psi_0(t), \dots, \psi_n(t)\}$ and $\Phi = \{\phi_0(t), \dots, \phi_n(t)\}$ are both non-negative bases on the specified interval; and (ii) the former is a non-negative combination of the latter, i.e.,

$$\Psi^T = \mathbf{M}\Phi^T \quad (36)$$

for some matrix \mathbf{M} with elements satisfying $M_{jk} \geq 0$ for $0 \leq j, k \leq n$. As noted in Section 6.1, the power and Bernstein bases are archetypal examples satisfying these conditions, and the Bernstein form is therefore systematically more stable than the power form. One is naturally led to ask if other non-negative bases exist, in terms of which the Bernstein basis can be expressed as a non-negative combination, so they are more stable even than the Bernstein basis. This question was studied in Farouki and Goodman (1996) and it was shown that, in fact, the Bernstein basis is “optimally stable” in the sense outlined below.

Let \mathcal{P}_n be the set of all non-negative bases for degree- n polynomials on $[0, 1]$. The transformation (36) between degree- n polynomial bases by means of a non-negative matrix \mathbf{M} establishes a *partial ordering* of \mathcal{P}_n . Specifically, when $\Psi, \Phi \in \mathcal{P}_n$ and (36) holds for some non-negative matrix \mathbf{M} , we write $\Phi \preceq \Psi$. Since the product of two non-negative matrices is non-negative, the transitivity condition $\Psi \preceq \Phi$ and $\Phi \preceq \Theta \implies \Psi \preceq \Theta$ is satisfied.

Now a non-negative matrix has a non-negative inverse if and only if it is the product of a permutation matrix and a positive diagonal matrix (Minc, 1988). Thus, the relations $\Phi \preceq \Psi$ and $\Psi \preceq \Phi$ are *both* satisfied if and only if, with suitable ordering, the elements of Φ are positive multiples of those of Ψ . In that case, we write $\Phi \sim \Psi$. Finally, when $\Phi \preceq \Psi$ but $\Phi \not\sim \Psi$ we write $\Phi < \Psi$. We say that Φ *precedes* Ψ when $\Phi < \Psi$, and Φ is *similar to* Ψ when $\Phi \sim \Psi$.

The relation \preceq determines a *partial ordering* of the set \mathcal{P}_n of non-negative bases on $[0, 1]$. This ordering is only “partial” since bases $\Phi, \Psi \in \mathcal{P}_n$ exist such that neither the matrix \mathbf{M} in (36), nor its inverse, is non-negative – no precedence (or similarity) relation exists between such bases.

A non-negative basis Φ is a *minimal basis* of \mathcal{P}_n if no basis $\Psi \in \mathcal{P}_n$ exists, such that $\Psi < \Phi$. Note that, since \mathcal{P}_n is only partially ordered, there may be – modulo similarity – more than one minimal basis. The following results from (Farouki and Goodman, 1996) show that such minimal bases are “optimally stable” in the sense of the condition numbers defined above (see also Peña 1997, 2002 for extensions and generalizations of these results).

Theorem 2. Any two non-negative bases $\Phi, \Psi \in \mathcal{P}_n$ satisfy

$$\Phi \preceq \Psi \iff C_\Phi(p(t)) \leq C_\Psi(p(t))$$

for every polynomial $p(t)$ and every value t on the unit interval $[0, 1]$.

Theorem 3. The Bernstein basis is a minimal element of \mathcal{P}_n , and it is the only minimal element for which the basis functions have no roots in $(0, 1)$.

At present, the Bernstein basis is the only known optimally-stable basis in common use. Other minimal bases of \mathcal{P}_n can be constructed (see Farouki and Goodman, 1996 for examples in \mathcal{P}_2) but the optimal stability may not suffice for their adoption in applications. A useful basis must admit efficient algorithms for interpolation, approximation, root-finding,

shape manipulation, and similar requirements. The Bernstein basis combines optimal stability with a repertoire of versatile algorithms that address diverse computational requirements.

Remark 1. The optimal numerical stability of the Bernstein basis is closely related, but not identical, to the optimal shape-preserving property discussed in Section 5. The former property is based on a comparison of bases that are *non-negative* on $[0, 1]$ while the latter is concerned with the more restrictive context of *normalized totally positive* bases (i.e., for any set of nodes on $[0, 1]$ the collocation matrices must have minors that are all non-negative).

6.4. The Legendre basis

In the least-squares approximation of a given function $f(t)$ over $t \in [0, 1]$ by degree- n polynomials, i.e., in the construction of the polynomial $p_n(t)$ that minimizes the integral

$$E = \int_0^1 [f(t) - p_n(t)]^2 dt, \quad (37)$$

it is convenient (Rivlin, 1981) to express $p_n(t)$ in terms of an *orthogonal basis*,

$$p_n(t) = \sum_{k=0}^n a_k \phi_k(t), \quad (38)$$

characterized by the property

$$\int_0^1 \phi_j(t) \phi_k(t) dt = \begin{cases} \beta_k & \text{if } j = k, \\ 0 & \text{if } j \neq k, \end{cases} \quad (39)$$

since this allows the coefficients in (38) to be immediately identified as¹⁶

$$a_k = \frac{1}{\beta_k} \int_0^1 f(t) \phi_k(t) dt. \quad (40)$$

Moreover, when $\deg(\phi_k(t)) = k$, the approximant (38) exhibits *permanence of coefficients* with respect to its degree, i.e., the coefficients a_0, \dots, a_n of $p_{n+1}(t)$ agree with those of $p_n(t)$, and on increasing the degree we need only compute the new coefficient a_{n+1} . The relations (37), (39), (40) can be generalized by inserting a non-negative *weight function* $w(t)$ in the integrand.

Now the Bernstein basis (1) is clearly not orthogonal with respect to any non-negative $w(t)$ but it is intimately related to the *Legendre polynomials*, a family of classical orthogonal polynomials with $w(t) = 1$. Key aspects of this relation are (i) the simple and intuitive form of the Legendre polynomials in the Bernstein basis; and (ii) the relative stability of transformations between the Bernstein and Legendre forms (see Section 6.5 below).

To emphasize symmetries, the Legendre polynomials are usually defined (Davis, 1975; Isaacson and Keller, 1994) on $t \in [-1, +1]$. To express them in Bernstein form, however, it is more convenient to use the interval $t \in [0, 1]$. The Legendre polynomials $L_k(t)$ on $[0, 1]$ can be generated through the recurrence relation

$$(k+1)L_{k+1}(t) = (2k+1)(2t-1)L_k(t) - kL_{k-1}(t), \quad (41)$$

for $k = 1, 2, \dots$, commencing with $L_0(t) = 1$ and $L_1(t) = 2t - 1$. This gives

$$L_2(t) = 6t^2 - 6t + 1, \quad L_3(t) = 20t^3 - 30t^2 + 12t - 1, \quad \dots \text{etc.} \quad (42)$$

These polynomials satisfy (39) with $\beta_k = 1/(2k+1)$. Alternatively, they may be defined through Rodrigues' formula

$$L_k(t) = \frac{1}{k!} \frac{d^k}{dt^k} [(t-1)t]^k. \quad (43)$$

From this formula, the following result can be easily proved by induction — see Farouki (2000), Li and Zhang (1998) for complete details of the proof.

¹⁶ See Marco and Martínez (2010) for direct computation of least-squares approximants in the Bernstein basis.

Lemma 1. The Legendre polynomial $L_k(t)$ can be expressed in the Bernstein basis $b_0^k(t), \dots, b_k^k(t)$ of degree k as

$$L_k(t) = \sum_{i=0}^k (-1)^{k+i} \binom{k}{i} b_i^k(t). \quad (44)$$

The Bernstein form (44) offers a simple and intuitive characterization of the Legendre polynomials, easier to remember than the recurrence relation (41) or Rodrigues' formula (43) — namely, the Bernstein coefficients of $L_k(t)$ are simply the ordered sequence of binomial coefficients of order k taken with alternating signs (starting with a + or – sign according to whether k is even or odd). The control polygon associated with these coefficients offers useful insight into the behavior of the graph of $L_k(t)$ for $t \in [0, 1]$.

Another approach to least-squares approximation using polynomials in Bernstein form is to formulate the *dual basis functions* $d_0^n(t), \dots, d_n^n(t)$ which are characterized by the property

$$\int_0^1 b_k^n(t) d_j^n(t) dt = \delta_{jk} = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

Jüttler (1998) has shown that, when the dual basis functions are themselves expressed in Bernstein form as

$$d_j^n(t) = \sum_{k=0}^n a_{jk} b_k^n(t),$$

the coefficients a_{jk} for $0 \leq j, k \leq n$ are given by

$$a_{jk} = \frac{(-1)^{j+k}}{\binom{n}{j} \binom{n}{k}} \sum_{i=0}^{\min(j,k)} (2i+1) \binom{n+i+1}{n-j} \binom{n-i}{n-j} \binom{n+i+1}{n-k} \binom{n-i}{n-k}.$$

The polynomial $p_n(t)$ minimizing (37) then has the Bernstein coefficients

$$c_k = \int_0^1 f(t) d_k^n(t) dt, \quad k = 0, \dots, n.$$

Details on conversions between the Bernstein basis and other orthogonal polynomial bases may be found in Boyd (2008), Coluccio et al. (2008), Rababah (2003, 2004), Rababah and al Natour (2007).

6.5. Basis transformations

In theoretical discussions, the monomial form (8) of a polynomial $p(t)$ is most frequently used. In problems of min–max or least-squares approximation, on the other hand, use of an orthogonal (e.g., Chebyshev or Legendre) basis may be more convenient. As emphasized above, the Bernstein form is best suited to manipulating the graph of a polynomial over a finite domain, to achieve certain desired shape properties. In principle, one may freely switch between alternative representations, since a change of basis for degree- n polynomials corresponds to a linear map, i.e., an $(n+1) \times (n+1)$ matrix, that determines the coefficients in the new basis from those in the old basis.

Such basis transformations should generally be avoided if one wishes to fully exploit the stability of the Bernstein form, since they may themselves incur amplification of relative errors in the coefficients. In other words, the problem should be formulated *ab initio* in the Bernstein form, and subsequent computations should be performed exclusively in that basis. All the familiar polynomial operations, typically performed using the power representation, have straightforward Bernstein-form analogs (Farouki and Rajan, 1988).

The stability of transformations between the Bernstein and other bases has been studied by several authors (Daniel and Daubisse, 1989; Farouki, 1991a, 2000; Hermann, 1996; Li and Zhang, 1998). It may be quantified by a *condition number* for the matrix that relates the coefficients in the two bases. The p -norm of a vector $\mathbf{v} = (v_0, \dots, v_n)^T$ is defined by

$$\|\mathbf{v}\|_p := \left[\sum_{i=0}^n |v_i|^p \right]^{1/p}. \quad (45)$$

In particular, $\|\mathbf{v}\|_1 = |v_0| + \dots + |v_n|$, $\|\mathbf{v}\|_2 = \sqrt{v_0^2 + \dots + v_n^2}$, and $\|\mathbf{v}\|_\infty = \max(|v_0|, \dots, |v_n|)$. The matrix norm *subordinate* to (45) is defined by

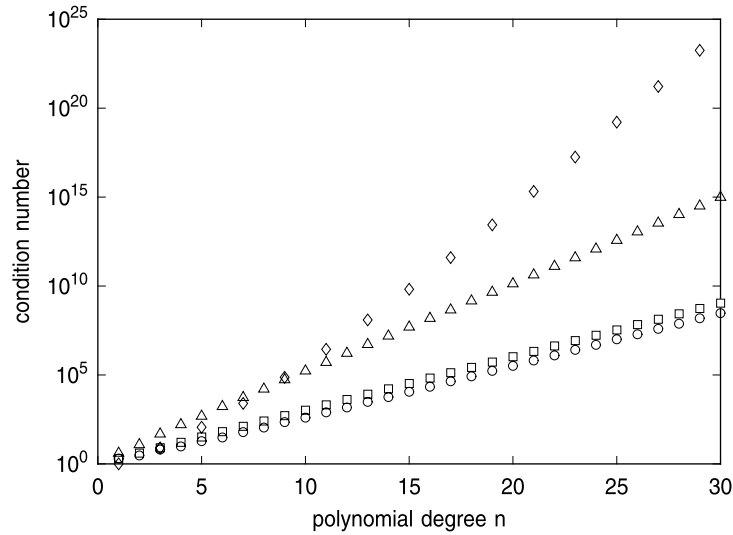


Fig. 12. Condition numbers for transformations between polynomial bases – Bernstein-Legendre (squares $p = 1$, and circles $p = \infty$); Bernstein-power (triangles, both $p = 1$ and ∞); and Bernstein-Hermite (diamonds $p = \infty$).

$$\|\mathbf{M}\|_p := \max_{\mathbf{v} \neq \mathbf{0}} \frac{\|\mathbf{M}\mathbf{v}\|_p}{\|\mathbf{v}\|_p}. \quad (46)$$

Specifically, $\|\mathbf{M}\|_1$ and $\|\mathbf{M}\|_\infty$ are the greatest of the column sums and row sums of absolute values of the matrix elements, respectively, while $\|\mathbf{M}\|_2 = \sqrt{\lambda_{\max}}$, where λ_{\max} is the largest eigenvalue of $\mathbf{M}^T \mathbf{M}$ (Stewart, 1973). Now if the matrix \mathbf{M} maps $\mathbf{x} = (x_0, \dots, x_n)^T$ to $\mathbf{y} = (y_0, \dots, y_n)^T$ according to

$$\mathbf{y} = \mathbf{M}\mathbf{x}, \quad (47)$$

and a perturbation $\delta\mathbf{x} = (\delta x_0, \dots, \delta x_n)$ of the “input” induces a perturbation $\delta\mathbf{y} = (\delta y_0, \dots, \delta y_n)$ of the “output,” the magnitudes of these perturbations may be characterized by the fractional measures

$$\epsilon_{\mathbf{x}} := \frac{\|\delta\mathbf{x}\|_p}{\|\mathbf{x}\|_p} \quad \text{and} \quad \epsilon_{\mathbf{y}} := \frac{\|\delta\mathbf{y}\|_p}{\|\mathbf{y}\|_p}. \quad (48)$$

One can then show (Stewart, 1973) that $\epsilon_{\mathbf{y}}$ is bounded in terms of $\epsilon_{\mathbf{x}}$ by the relation

$$\epsilon_{\mathbf{y}} \leq C_p(\mathbf{M})\epsilon_{\mathbf{x}}, \quad C_p(\mathbf{M}) := \|\mathbf{M}\|_p \|\mathbf{M}^{-1}\|_p, \quad (49)$$

$C_p(\mathbf{M})$ being the p -norm condition number of \mathbf{M} . The bound in (49) is sharp – i.e., a perturbation $\delta\mathbf{x}$ exists for which (49) holds with equality.

For transformations between the Bernstein and power forms, the elements of \mathbf{M} and \mathbf{M}^{-1} are defined by the relations (15) and one can show¹⁷ (Farouki, 1991a) that

$$C_1(\mathbf{M}) = C_\infty(\mathbf{M}) = (n+1) \binom{n}{\nu} 2^\nu, \quad \nu = \left\lfloor \frac{2(n+1)}{3} \right\rfloor. \quad (50)$$

The simpler form $3^{n+1} \sqrt{(n+1)/4\pi}$ is an excellent approximation (Farouki, 1991a).

Fig. 12 illustrates the growth of the condition number with degree n for transformations among the power, Legendre, Hermite, and Bernstein bases, in the $p = 1$ and $p = \infty$ norms. The Legendre–Bernstein transformation (see Section 6.4 above) is comparatively well-conditioned. The power–Bernstein transformation condition number (50) exhibits a faster growth, although not as severe as transformations that involve the Hermite form.

Since subdivision plays a fundamental role in algorithms for Bernstein-form polynomials, it is also important to characterize the stability of the map that relates the coefficients $\tilde{c}_0, \dots, \tilde{c}_n$ in the Bernstein basis on a subinterval $[t_1, t_2]$ to the coefficients c_0, \dots, c_n on the interval $[0, 1]$. In Farouki and Neff (1990) it was shown that this map may be specified by

$$\tilde{c}_j = \sum_{k=0}^n M_{jk} c_k, \quad j = 0, \dots, n,$$

¹⁷ The “floor” function $\lfloor x \rfloor$ indicates the largest integer not exceeding x .

where the elements M_{jk} of the $(n+1) \times (n+1)$ matrix \mathbf{M} are defined by

$$M_{jk} = \sum_{i=\max(0, j+k-n)}^{\min(j, k)} b_{k-i}^{n-j}(t_1) b_i^j(t_2), \quad 0 \leq j, k \leq n.$$

Setting $t_m = \frac{1}{2}(t_1 + t_2)$, the condition number of this matrix in the $p = \infty$ norm may be expressed (Farouki and Neff, 1990) as

$$C_\infty(\mathbf{M}) = \left[\frac{2 \max(t_m, 1 - t_m)}{t_2 - t_1} \right]^n.$$

See also Kiciak and Zidna (2004) for a rounding-error analysis of the de Casteljau algorithm.

7. Alternative approaches

The versatility of the Bernstein form is evident in the wide variety of available approaches for elucidating the algorithms and properties associated with it. Some alternative paradigms are summarized here, based on the *shift operator*, *polar forms* (or *blossoms*), connections to *probability theory*, and the use of *generating functions* and *discrete convolutions*.

7.1. The shift operator

Many properties and algorithms associated with the Bernstein form (7) can be easily derived by invoking a “shift operator” \mathcal{S} , whose action on a sequence of Bernstein coefficients c_0, \dots, c_n is such as to increment each subscript by 1. In this context, we allow the subscripts to assume *all* integer values, i.e., we consider the infinite sequence

$$\dots, c_{-1}, c_0, c_1, \dots, c_{n-1}, c_n, c_{n+1}, \dots,$$

where it is understood that $c_k = 0$ if $k < 0$ or $k > n$. As a short-hand notation, we write

$$\mathcal{S}c_k = c_{k+1}, \quad (51)$$

and correspondingly $\mathcal{S}^r c_k = c_{k+r}$. An inverse \mathcal{S}^{-1} can also be defined, that decrements each subscript by 1, indicated by the notation by $\mathcal{S}^{-1} c_k = c_{k-1}$, so that \mathcal{S}^{-1} acting on $\mathcal{S}c_k$ (or \mathcal{S} on $\mathcal{S}^{-1} c_k$) yields c_k . Finally, we also define the *identity operator* $\mathcal{I} = \mathcal{S}^0$, which leaves subscripts unchanged.

Consider now the operator expression $(1-t)\mathcal{I} + t\mathcal{S}$ for any real value t . By the binomial theorem, its n -th power may be formally expanded as

$$[(1-t)\mathcal{I} + t\mathcal{S}]^n = \sum_{k=0}^n \binom{n}{k} (1-t)^{n-k} t^k \mathcal{S}^k = \sum_{k=0}^n b_k^n(t) \mathcal{S}^k.$$

Hence, the polynomial (7) may be represented by the compact expression

$$p(t) = [(1-t)\mathcal{I} + t\mathcal{S}]^n c_0. \quad (52)$$

Since the first forward differences of the coefficients are $\Delta c_k = c_{k+1} - c_k$, we introduce the operator

$$\Delta = \mathcal{S} - \mathcal{I},$$

and (52) can alternatively be written as

$$p(t) = [\mathcal{I} + t\Delta]^n c_0.$$

By way of example, we illustrate the use of (52) to obtain the derivative and integral of (7). The derivative of $[(1-t)\mathcal{I} + t\mathcal{S}]^n$ with respect to t may be expressed as

$$\frac{d}{dt} [(1-t)\mathcal{I} + t\mathcal{S}]^n = n[(1-t)\mathcal{I} + t\mathcal{S}]^{n-1} \Delta. \quad (53)$$

Thus, by writing

$$p'(t) = n[(1-t)\mathcal{I} + t\mathcal{S}]^{n-1} \Delta c_0, \quad (54)$$

and noting that $\mathcal{S}^k \Delta c_0 = \mathcal{S}^k (c_1 - c_0) = c_{k+1} - c_k = \Delta c_k$, we obtain

$$p'(t) = \sum_{k=0}^{n-1} n \Delta c_k \binom{n-1}{k} (1-t)^{n-1-k} t^k = \sum_{k=0}^{n-1} n \Delta c_k b_k^{n-1}(t)$$

for the derivative of $p(t)$. We can also obtain the indefinite integral of $p(t)$ using the shift operator. Observing that

$$\int [(1-t)\mathcal{I} + t\mathcal{S}]^n dt = \frac{1}{n+1} [(1-t)\mathcal{I} + t\mathcal{S}]^{n+1} \Delta^{-1},$$

the integral of $p(t)$ can be expressed in the form

$$\int p(t) dt = \frac{1}{n+1} [(1-t)\mathcal{I} + t\mathcal{S}]^{n+1} \Delta^{-1} c_0. \quad (55)$$

The inverse Δ^{-1} of the difference operator in (55) is obtained by writing

$$\Delta^{-1} = (\mathcal{S} - \mathcal{I})^{-1} = \mathcal{S}^{-1}(\mathcal{I} - \mathcal{S}^{-1})^{-1}.$$

By analogy with the binomial expansion $(1-x)^{-1} = 1 + x + x^2 + \dots$, we write the above as

$$\Delta^{-1} = \mathcal{S}^{-1}(\mathcal{I} + \mathcal{S}^{-1} + \mathcal{S}^{-2} + \dots) = \mathcal{S}^{-1} + \mathcal{S}^{-2} + \mathcal{S}^{-3} + \dots.$$

Hence, the action of Δ^{-1} on c_k is described by

$$\Delta^{-1} c_k = \sum_{j=-\infty}^{k-1} c_j,$$

which is consistent with the requirement that $\Delta^{-1}(\Delta c_k) = \Delta(\Delta^{-1} c_k) = c_k$. In particular, since $\Delta^{-1} c_0 = \dots + c_{-2} + c_{-1}$, the numerical value of $\mathcal{S}^k \Delta^{-1} c_0$ is zero if $k = 0$, and $c_0 + \dots + c_{k-1}$ if $k \geq 1$. Thus, the indefinite integral of $p(t)$ is given (modulo a constant) by

$$\sum_{k=1}^{n+1} \left[\frac{1}{n+1} \sum_{j=0}^{k-1} c_j \right] \binom{n+1}{k} (1-t)^{n+1-k} t^k = \sum_{k=1}^{n+1} \left[\frac{1}{n+1} \sum_{j=0}^{k-1} c_j \right] b_k^{n+1}(t).$$

Further details on the use of the shift operator may be found in Chang (1984), Hosaka and Kimura (1978), Schwartz (1987).

7.2. Polar forms or blossoms

The *polar form* – or *blossom* – $P(t_1, \dots, t_n)$ of a univariate polynomial $p(t)$ of degree $\leq n$ is a useful tool in the construction and analysis of algorithms for Bézier (and B-spline) curves. $P(t_1, \dots, t_n)$ is the unique polynomial that is linear and symmetric in the n independent variables t_1, \dots, t_n and coincides with $p(t)$ when $t_1 = \dots = t_n = t$. For example, the polar form of the cubic $p(t) = t^3 - 3t^2 + t - 1$ is the symmetric trilinear polynomial

$$P(t_1, t_2, t_3) = t_1 t_2 t_3 - 3 \frac{t_1 t_2 + t_2 t_3 + t_3 t_1}{3} + \frac{t_1 + t_2 + t_3}{3} - 1.$$

There are several different approaches to computing the polar form of a given polynomial (or polynomial curve) – see Section 8 of Ramshaw (1989).

Poles and polar forms feature prominently in the work of de Casteljau, who observed that “the pole, a real pilot point derived from the *polar form* of the *polynomials*, is used for interpolation” (de Casteljau, 1963, 1985, 1986, 1991). In the conclusion of de Casteljau (1986), he also remarks that

It is quite astonishing that the study of polar forms, which was once so highly developed that mathematicians became irritated by being overexposed to it, should now be limited to Cartesian forms. Parametric forms have fallen into disuse, and the general form of interpolation illustrated here could have become standard, but circumstance and fashion decided otherwise.

However, it is mostly the work of Ramshaw (1987, 1989) that earned widespread acceptance of polar forms as tools for analyzing the structure of algorithms (and also established common usage of the term *blossoming*).

As a simple illustration of the use of polar forms, consider the de Casteljau algorithm for evaluating and subdividing a Bézier curve $\mathbf{r}(t)$ at a given point t . Suppose, for example, that the curve (9) is a cubic with polar form

$$\begin{aligned} \mathbf{R}(t_1, t_2, t_3) &= (1-t_1)(1-t_2)(1-t_3)\mathbf{p}_0 \\ &\quad + [(1-t_1)(1-t_2)t_3 + (1-t_1)t_2(1-t_3) + t_1(1-t_2)(1-t_3)]\mathbf{p}_1 \\ &\quad + [(1-t_1)t_2 t_3 + t_1(1-t_2)t_3 + t_1 t_2(1-t_3)]\mathbf{p}_2 \\ &\quad + t_1 t_2 t_3 \mathbf{p}_3, \end{aligned} \quad (56)$$

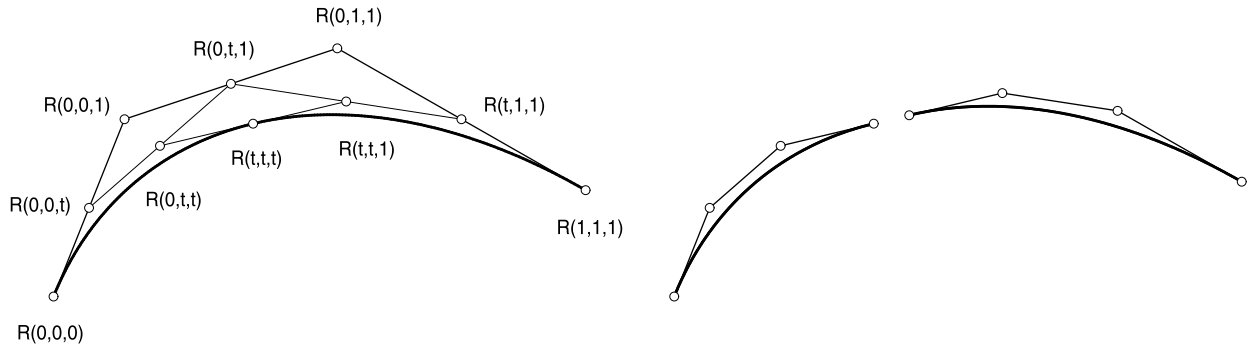


Fig. 13. Use of the polar form to illustrate the de Casteljau algorithm for subdividing a cubic Bézier curve. Left: the intermediate points generated by the algorithm, as values of the polar form (56). Right: the “left” and “right” cubic segments, together with control polygons, after subdivision at $t = \frac{1}{2}$.

the polar values $\mathbf{R}(0, 0, 0)$, $\mathbf{R}(0, 0, 1)$, $\mathbf{R}(0, 1, 1)$, $\mathbf{R}(1, 1, 1)$ corresponding to the control points \mathbf{p}_0 , \mathbf{p}_1 , \mathbf{p}_2 , \mathbf{p}_3 . By the linearity of (56) in t_1, t_2, t_3 we have

$$\mathbf{R}(0, 0, t) = (1 - t)\mathbf{R}(0, 0, 0) + t\mathbf{R}(0, 0, 1),$$

$$\mathbf{R}(0, t, 1) = (1 - t)\mathbf{R}(0, 0, 1) + t\mathbf{R}(0, 1, 1),$$

$$\mathbf{R}(t, 1, 1) = (1 - t)\mathbf{R}(0, 1, 1) + t\mathbf{R}(1, 1, 1).$$

To evaluate $(1 - t)\mathbf{R}(0, 0, t) + t\mathbf{R}(0, t, 1)$ and $(1 - t)\mathbf{R}(0, t, 1) + t\mathbf{R}(t, 1, 1)$ in the second step, we note that $\mathbf{R}(0, 0, t) = \mathbf{R}(0, t, 0)$ and $\mathbf{R}(0, t, 1) = \mathbf{R}(t, 0, 1)$ from the symmetry of (56). By linearity, these expressions reduce to $\mathbf{R}(0, t, t)$ and $\mathbf{R}(t, t, 1)$. Invoking symmetry again to observe that $\mathbf{R}(t, t, 1) = \mathbf{R}(1, t, t)$ the final expression $(1 - t)\mathbf{R}(0, t, t) + t\mathbf{R}(t, t, 1)$ then yields $\mathbf{R}(t, t, t)$.

The polar form interpretation of the de Casteljau algorithm is illustrated in Fig. 13. The symmetry of the computed polar values becomes apparent on arranging them in a triangular array,

$$\begin{array}{cccc} \mathbf{R}(0, 0, 0) & \mathbf{R}(0, 0, 1) & \mathbf{R}(0, 1, 1) & \mathbf{R}(1, 1, 1) \\ & \mathbf{R}(0, 0, t) & \mathbf{R}(0, t, 1) & \mathbf{R}(t, 1, 1) \\ & & \mathbf{R}(0, t, t) & \mathbf{R}(t, t, 1) \\ & & & \mathbf{R}(t, t, t) \end{array}$$

analogous to (10). In fact, the blossoming argument may be regarded as a simple *proof* of the de Casteljau algorithm. The control points of a degree- n Bézier curve appropriate to any parameter interval $t \in [a, b]$ can be easily obtained from its polar form as

$$\mathbf{p}_k = \mathbf{R}(\underbrace{a, \dots, a}_{n-k}, \underbrace{b, \dots, b}_k), \quad k = 0, \dots, n.$$

This is sometimes known (Goldman, 2003) as the *dual functional property* of the blossom. Polar forms prove valuable in many other contexts, e.g., analyzing geometric continuity and generalizing from Bernstein-form polynomials on an interval to B-spline functions on a partitioned domain. More complete details may be found in Boehm (1988), de Boor and Höllig (1987), de Casteljau (1985), Goldman (2003), Ramshaw (1987, 1989), Seidel (1989).

7.3. Connections with probability theory

In addition to approximation theory, Bernstein made important contributions to probability theory. There are, in fact, close connections between these two fields. Considered as a function of an integer variable k that can assume the discrete values $0, \dots, n$, the basis functions $b_k^n(t)$ specified by (1) correspond to the *binomial distribution*, defining the probability of k successes in n trials of a random event, when t is the probability of success in a single trial.

The *non-negativity* and *partition of unity* properties of the basis functions (1) are the key characteristic features of any discrete probability distribution. Similarly, the elements (34) of the matrix that specifies the Bernstein basis on $[0, 1]$ in terms of the basis on any subinterval $[t_1, t_2]$ thereof satisfy

$$M_{jk} \geq 0, \quad 0 \leq j, k \leq n \quad \text{and} \quad \sum_{k=0}^n M_{jk} = 1, \quad j = 0, \dots, n.$$

These are the characteristic features of a (right) *stochastic* or *Markov* matrix (Feller, 1950). A *Markov chain* describes the step-by-step evolution of a probabilistic system that can assume one of several distinct *states*, s_0, \dots, s_n . If the system is

currently in state s_j , the probability that it will transition to state s_k in one step is M_{jk} . The chain commences by assigning initial probabilities p_0, \dots, p_n to the states s_0, \dots, s_n – e.g., $p_i = 1$ and $p_j = 0$ for all $j \neq i$ if the system is known with certainty to begin in the state s_i .

Regarding the curve parameter t as *time*, and the $n + 1$ basis functions (1) as specifying a *time-dependent discrete probability distribution*, one may interpret the Bézier curve (9) as the “expected path” of a point as t increases from 0 to 1, when $b_k^n(t)$ defines the probability at time t that it is coincident with control point \mathbf{p}_k . As an alternative “physical” interpretation of (9), one may consider the basis functions $b_0^n(t), \dots, b_n^n(t)$ as specifying *time-dependent masses*, situated at the control points $\mathbf{p}_0, \dots, \mathbf{p}_n$. The curve (9) then specifies the trajectory of the *center of mass*, as t increases from 0 to 1.

For a comprehensive discussion of the connections between probability theory, Markov chains, and polynomial bases that provide intuitive geometric design schemes, see the articles by Goldman (1984, 1985a, 1985b) and also Gordon and Riesenfeld (1974).

7.4. Generating functions & discrete convolutions

According to Wilf (1990), a *generating function* is “a clothesline on which we hang up a sequence of numbers for display.” More specifically, the sequence of numbers a_0, a_1, a_2, \dots is considered to be the coefficients in the power series expansion $a_0 + a_1x + a_2x^2 + \dots$ of a generating function $f(x)$.

This idea can also be generalized to specify a generating function for a sequence of functions $\phi_0(t), \phi_1(t), \phi_2(t), \dots$ by considering these functions to be the coefficients in the power series expansion of a bivariate function $f(t, x)$ with respect to x , i.e.,

$$f(t, x) = \sum_{k=0}^{\infty} \phi_k(t) x^k.$$

The generating function can prove useful in determining various properties, identities, and recurrence relations for $\phi_0(t), \phi_1(t), \phi_2(t), \dots$. When $f(t, x)$ is a polynomial, a simple generating function (Goldman, 2003) for the Bernstein basis is

$$f(t, x) = (1 - t + tx)^n = \sum_{k=0}^n b_k^n(t) x^k.$$

Simsek (2011a) considers a generalization of the Bernstein basis, defined by

$$b_{k,s}^n(t) = \binom{n}{ks} \frac{(1-t)^{n-ks} t^{ks}}{2^{k(s-1)}},$$

which specializes to (1) when $s = 1$. This basis can be obtained from the generating function expansion

$$f_{k,s}(t, x) = \frac{2^k (\frac{1}{2}tx)^{ks} e^{(1-t)x}}{(ks)!} = \sum_{n=0}^{\infty} b_{k,s}^n(t) \frac{x^n}{n!}, \quad (57)$$

where it is understood that $b_{k,s}^n(t) \equiv 0$ if $ks > n$. For the case $s = 1$ of the “ordinary” Bernstein basis (1), with $b_k^n(t) \equiv 0$ if $k > n$, this reduces to

$$f_k(t, x) = \frac{(tx)^k e^{(1-t)x}}{k!} = \sum_{n=0}^{\infty} b_k^n(t) \frac{x^n}{n!}. \quad (58)$$

Note that this generates basis functions of fixed index k and increasing degree n , rather than vice-versa. The generating functions (57)–(58) are employed in Simsek (2011a, 2011b, 2011c) to derive various basic properties of the Bernstein form.

Another approach to analyzing properties of the Bernstein form is through the interpretation that the basis (1) arises from a *discrete convolution* – see Goldman (2003). The discrete convolution

$$C = A \otimes B = \{c_0(t), \dots, c_{m+n}(t)\}$$

of two function sequences $A = \{a_0(t), \dots, a_m(t)\}$ and $B = \{b_0(t), \dots, b_n(t)\}$ is the function sequence specified by

$$c_k(t) = \sum_{i+j=k} a_i(t) b_j(t).$$

The degree- n Bernstein basis (1) can be regarded as a discrete convolution of bases of degree r and $n - r$, for $1 \leq r \leq n$. For example,

$$\{(1-t), t\} \otimes \{(1-t)^2, 2(1-t)t, t^2\} = \{(1-t)^3, 3(1-t)^2t, 3(1-t)t^2, t^3\}.$$

This interpretation is also useful in analyzing basic properties of the Bernstein form – see Goldman (2003) for further details.

8. Computer aided geometric design

The impetus for revived interest in the Bernstein form as a basis for practical computations arose from the field of *computer aided geometric design*, mainly through the work of Bézier and de Casteljau in the 60s and 70s (see Section 4). This has subsequently evolved into a mature discipline, whose “classical core” is centered on Bézier/B-spline forms. Since there are several excellent sources for this material (Böhm et al., 1984; Farin, 2002; Goldman, 2003; Hoschek and Lasser, 1993; Prautzsch et al., 2002) this section will only highlight a few key concepts involved in generalizing the basic Bézier curve (9).

8.1. Rational Bézier curves

Although the Bézier form (9) is primarily used for “free-form” curve design, it is desirable to have a representation that can also accommodate important “simple” curves (e.g., conic segments), and exhibits closure under projective transformations. This is achieved by introducing scalar “weights” w_0, \dots, w_n that generalize (9) to the *rational Bézier form*

$$\mathbf{r}(t) = \frac{\sum_{k=0}^n w_k \mathbf{p}_k b_k^n(t)}{\sum_{k=0}^n w_k b_k^n(t)}, \quad t \in [0, 1]. \quad (59)$$

To ensure that the curve has no *points at infinity*, it is customary to use only positive weights. Note that only the *ratios* $w_0 : w_1 : \dots : w_n$ matter. Also, for any $\alpha \in (0, 1)$ the rational linear (Möbius) parameter transformation

$$t \mapsto \frac{(1 - \alpha)t}{\alpha(1 - t) + (1 - \alpha)t}$$

preserves the $t \in [0, 1]$ parameter interval, the curve degree n , and the control points $\mathbf{p}_0, \dots, \mathbf{p}_n$ — only the weights w_0, \dots, w_n change. By virtue of these freedoms, a rational Bézier curve may always be expressed in “standard form” with $w_0 = w_n = 1$. For example, the rational quadratic Bézier curves can be specified by three control points $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2$ and a single weight w_1 as

$$\mathbf{r}(t) = \frac{\mathbf{p}_0(1-t)^2 + w_1 \mathbf{p}_1 2(1-t)t + \mathbf{p}_2 t^2}{(1-t)^2 + w_1 2(1-t)t + t^2}, \quad t \in [0, 1]. \quad (60)$$

This determines a segment of an ellipse, parabola, or hyperbola according to whether $w_1 < 1$, $w_1 = 1$, or $w_1 > 1$ (see Fig. 14).

The weights w_0, \dots, w_n in the rational curve (59) offer additional degrees of freedom for manipulating its shape. An intuitive means of exercising these freedoms was proposed by Farin (1989), based on introducing the *weight points*

$$\mathbf{q}_k = \frac{w_{k-1} \mathbf{p}_{k-1} + w_k \mathbf{p}_k}{w_{k-1} + w_k}, \quad k = 1, \dots, n, \quad (61)$$

on each control polygon leg. The location of \mathbf{q}_k between \mathbf{p}_{k-1} and \mathbf{p}_k uniquely fixes the ratio $w_{k-1} : w_k$ so the n points (61), also called *Farin points*, specify the ratios $w_0 : w_1 : \dots : w_n$. Hence, the curve shape may be manipulated by “sliding” the weight points along the control-polygon legs.

If the rational curve (59) is planar, with control points $\mathbf{p}_k = (x_k, y_k)$ for $k = 0, \dots, n$, its *homogeneous coordinates* are specified by the polynomials

$$W(t) = \sum_{k=0}^n w_k b_k^n(t), \quad X(t) = \sum_{k=0}^n w_k x_k b_k^n(t), \quad Y(t) = \sum_{k=0}^n w_k y_k b_k^n(t).$$

The rational planar curve $\mathbf{r}(t)$ may be regarded as the image of a polynomial space curve $\mathbf{R}(t) = (W(t), X(t), Y(t))$ through a *central projection* from the origin $(W, X, Y) = (0, 0, 0)$ onto the plane $W = 1$. A 3×3 matrix acting on the homogeneous coordinate polynomials defines a *projective transformation* of the curve. These ideas are also readily extended to rational space curves.

For a more detailed account of the algorithms and properties associated with rational Bézier curves, see (Farin, 1983). In certain contexts — e.g., the construction of planar Pythagorean-hodograph curves (Farouki, 1994, 2008) — it is advantageous to interpret planar curves as *complex-valued* functions of a real parameter t , i.e., points (x, y) in the plane are

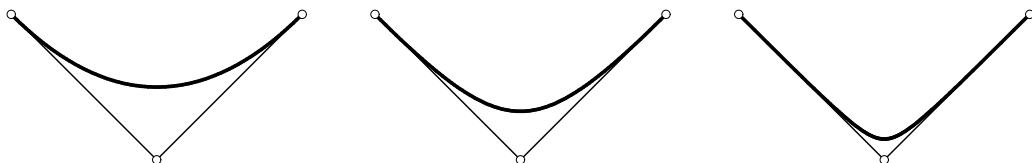


Fig. 14. Left to right: segments of an ellipse ($w_1 < 1$), parabola ($w_1 = 1$), and hyperbola ($w_1 > 1$), defined by the rational quadratic Bézier form (60).

identified with complex values $x + iy$, and the rational Bézier curve (59) then has complex control points $\mathbf{p}_k = x_k + iy_k$. Sánchez-Reyes (2009) describes a generalization of (59) to the case where the numerator and denominator are both complex-valued polynomials. The standard algorithms for rational Bézier curves carry over to such forms, which offer compact representations of circular arcs (by rational linear complex functions), and higher-order curves such as epitrochoids and hypotrochoids. See also Ait-Haddou et al. (2010) for some interesting relations between complex polynomials and the geometry of planar polygons.

8.2. Triangular surface patches

The multivariate extension of the Bernstein basis (1) is important for many applications. The “tensor-product” scheme, in which the multivariate basis functions are simply defined as products of univariate basis functions in each variable, is an obvious approach. For example, a tensor-product surface may be defined over the parameter domain $(s, t) \in [0, 1] \times [0, 1]$ by an array of control points \mathbf{p}_{jk} for $0 \leq j \leq m$ and $0 \leq k \leq n$ by the expression

$$\mathbf{r}(s, t) = \sum_{j=0}^m \sum_{k=0}^n \mathbf{p}_{jk} b_j^m(s) b_k^n(t). \quad (62)$$

A more fundamental approach is to define multivariate bases over *simplex* domains. In the univariate basis (1), the quantities $u = 1 - t$ and $v = t$ are *barycentric coordinates* on a 1-dimensional simplex, the unit interval $[0, 1]$. They are non-negative, sum to unity, and the basis functions arise from the binomial expansion of $1 = (u + v)^n$. To define a bivariate basis, we choose a *reference triangle* T as the 2-dimensional simplex domain. If T has vertices $\mathbf{p}_k = (x_k, y_k)$ for $k = 1, 2, 3$ that are not collinear, the determinant

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \quad (63)$$

is non-zero, and T has *signed area* $A = \frac{1}{2} \Delta$ (positive or negative according to whether the vertices are labeled in a counter-clockwise or clockwise sense).

Given any point $\mathbf{p} = (x, y)$ in the plane, consider the triangles T_1, T_2, T_3 subtended at \mathbf{p} by the sides of the reference triangle (see Fig. 15), with signed areas $A_1 = \frac{1}{2} \Delta_1, A_2 = \frac{1}{2} \Delta_2, A_3 = \frac{1}{2} \Delta_3$, where Δ_k is the determinant defined by replacing (x_k, y_k) in (63) with (x, y) . The barycentric coordinates (u, v, w) of the point $\mathbf{p} = (x, y)$ with respect to the reference triangle T are then defined by the area-ratios

$$u = \frac{\Delta_1}{\Delta}, \quad v = \frac{\Delta_2}{\Delta}, \quad w = \frac{\Delta_3}{\Delta},$$

so that $\mathbf{p} = u\mathbf{p}_1 + v\mathbf{p}_2 + w\mathbf{p}_3$, where $u + v + w = 1$. They are non-negative when \mathbf{p} lies within T (for \mathbf{p} outside T , their signs are as shown in Fig. 15).

Since the barycentric system has no origin, points cannot be treated like vectors, i.e., we cannot add two points (u_1, v_1, w_1) and (u_2, v_2, w_2) to obtain a new point (this would violate $u + v + w = 1$). However, by defining barycentric vectors as triples (λ, μ, ν) that satisfy $\lambda + \mu + \nu = 0$, we can add a vector to a point to obtain a new point, or two vectors to obtain a new vector.

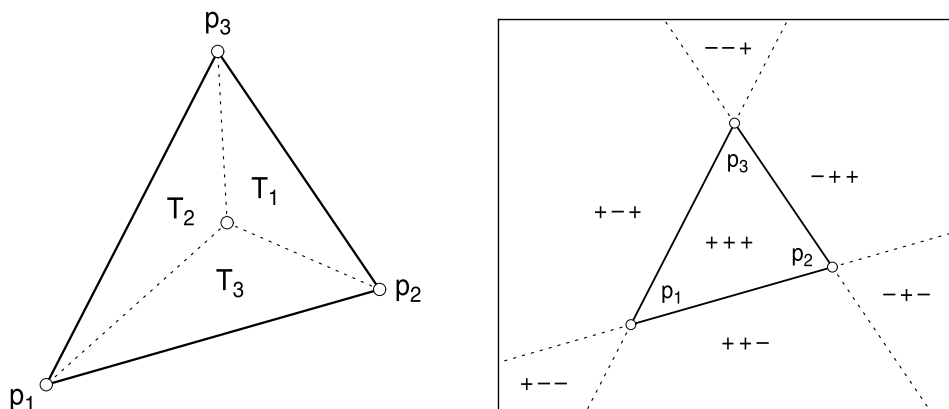


Fig. 15. Left: sub-triangles T_1, T_2, T_3 subtended at a general point \mathbf{p} by the sides of the reference triangle T . Right: signature of barycentric coordinates over the entire plane, subdivided into regions by extension of the sides of T .

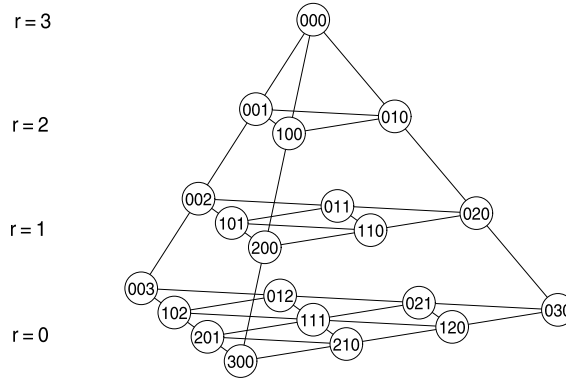


Fig. 16. Labeling of points generated by successive stages $r = 0, 1, 2, 3$ of the bivariate de Casteljau algorithm (65) for the case of a cubic patch, $n = 3$.

From the trinomial expansion of $1 = (u + v + w)^n$ we obtain the degree- n bivariate Bernstein basis defined for $0 \leq i, j, k \leq n$ with $i + j + k = n$ by

$$b_{ijk}^n(u, v, w) := \frac{n!}{i!j!k!} u^i v^j w^k.$$

There are $\binom{n+2}{2} = \frac{1}{2}(n+1)(n+2)$ linearly-independent basis functions. By associating a coefficient c_{ijk} with each basis function, any degree- n bivariate polynomial can be defined over the reference triangle T by the expression

$$f(u, v, w) = \sum_{\substack{0 \leq i, j, k \leq n \\ i+j+k=n}} c_{ijk} b_{ijk}^n(u, v, w).$$

If the scalar values c_{ijk} are replaced by control points \mathbf{p}_{ijk} we obtain a vector map from the parameter domain T to \mathbb{R}^3 , i.e., a triangular surface patch,¹⁸

$$\mathbf{r}(u, v, w) = \sum_{\substack{0 \leq i, j, k \leq n \\ i+j+k=n}} \mathbf{p}_{ijk} b_{ijk}^n(u, v, w). \quad (64)$$

The *control net* for this patch is specified by connecting each point \mathbf{p}_{ijk} with its neighbors $\mathbf{p}_{i-1, j+1, k}$, $\mathbf{p}_{i+1, j-1, k}$, $\mathbf{p}_{i, j-1, k+1}$, $\mathbf{p}_{i, j+1, k-1}$, $\mathbf{p}_{i+1, j, k-1}$, $\mathbf{p}_{i-1, j, k+1}$ (where subscripts remain between 0 and n) – it defines a polyhedral surface with triangular facets. The patch must lie within the convex hull of its control net. It interpolates the corner points \mathbf{p}_{n00} , \mathbf{p}_{0n0} , \mathbf{p}_{00n} and the tangent planes at these points are defined by $(\mathbf{p}_{n00}, \mathbf{p}_{n-1, 1, 0}, \mathbf{p}_{n-1, 0, 1})$, $(\mathbf{p}_{0n0}, \mathbf{p}_{0, n-1, 1}, \mathbf{p}_{1, n-1, 0})$, $(\mathbf{p}_{00n}, \mathbf{p}_{1, 0, n-1}, \mathbf{p}_{0, 1, n-1})$. Furthermore, the peripheral control points define the patch boundaries as individual degree- n Bézier curves.

The triangular patch (64) may be evaluated and subdivided at any chosen point $\mathbf{p}_* = (u_*, v_*, w_*) \in T$ by a bivariate de Casteljau algorithm that is a natural generalization of the univariate algorithm for curves described in Section 4. Setting $\mathbf{p}_{ijk}^0 = \mathbf{p}_{ijk}$ for $0 \leq i, j, k \leq n$ (with $i + j + k = n$), we compute the tetrahedral array of points defined for $r = 1, \dots, n$ by

$$\mathbf{p}_{ijk}^r = u_* \mathbf{p}_{i+1, j, k}^{r-1} + v_* \mathbf{p}_{i, j+1, k}^{r-1} + w_* \mathbf{p}_{i, j, k+1}^{r-1}, \quad (65)$$

where $0 \leq i, j, k \leq n - r$ and $i + j + k = n - r$. Fig. 16 illustrates the arrangement of points generated by this algorithm in a tetrahedral array, for the case $n = 3$. The apex \mathbf{p}_{000}^n of the array identifies the point $\mathbf{r}(u_*, v_*, w_*)$ on the surface (64), while the points on its three faces, specified by

$$\begin{aligned} \mathbf{p}_{0, j, n-r-j}^r & \text{ for } j = 0, \dots, n-r \text{ and } r = 0, \dots, n, \\ \mathbf{p}_{n-r-k, 0, k}^r & \text{ for } k = 0, \dots, n-r \text{ and } r = 0, \dots, n, \\ \mathbf{p}_{i, n-r-i, 0}^r & \text{ for } i = 0, \dots, n-r \text{ and } r = 0, \dots, n, \end{aligned}$$

define control nets for the patches over the subdomains T_1, T_2, T_3 subtended at \mathbf{p}_* by the sides of T . Fig. 17 illustrates the subdivision of a cubic patch.

¹⁸ The form (64) is usually known as a *triangular Bézier patch*. Ironically, it was discussed in some detail in de Casteljau's 1963 Citroën notes (de Casteljau, 1963), but not at all in Bézier's work.

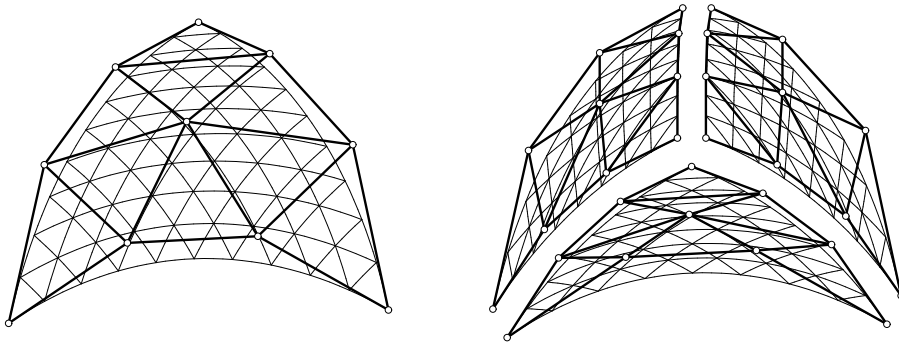


Fig. 17. Left: a cubic triangular Bézier surface patch with its control net. Right: subdivision at an interior point using the de Casteljau algorithm (65).

The degree- n patch (64) can also be *degree-elevated*, i.e., represented in the basis $b_{ijk}^{n+1}(u, v, w)$ on the domain T , by multiplying (64) with $1 = u + v + w$ and collecting like terms. For $0 \leq i, j, k \leq n+1$ with $i + j + k = n+1$, the control points $\hat{\mathbf{p}}_{ijk}$ of the degree-elevated form are given by

$$\hat{\mathbf{p}}_{ijk} = \frac{i\mathbf{p}_{i-1,j,k} + j\mathbf{p}_{i,j-1,k} + k\mathbf{p}_{i,j,k-1}}{n+1},$$

where terms on the right for which any of the indices is < 0 or $> n$ are ignored. The corner control points are unchanged, while those along the sides of the control net are generated by the univariate degree-elevation algorithm.

Under successive degree elevations, the control net of a triangular patch converges to the surface. However, the control nets generated by subdivision of a triangular patch need not converge to the surface, because the boundary curves are not subdivided. In this respect, 3-way subdivision of the triangular patch (64) differs from 4-way subdivision of the tensor-product patch (62) – i.e., subdivision with respect to both s and t .

By associating a scalar weight with each control point, the form (64) may be generalized to *rational* triangular patches, that exactly describe patches on quadric surfaces, tori, surfaces of revolution, etc. An octant of the sphere, for example, can be specified as a rational quartic triangular patch (Farin et al., 1987).

The above ideas also generalize in a natural and straightforward manner to the construction of Bernstein bases in terms of barycentric coordinates over a simplex S in \mathbb{R}^d specified by $d+1$ linearly-independent vertices $\mathbf{p}_0, \dots, \mathbf{p}_d$ for any $d \geq 3$. S may be regarded as the set of points $\mathbf{p} = \mu_0\mathbf{p}_0 + \dots + \mu_d\mathbf{p}_d$ that are *convex combinations* of the vertices, for weights μ_0, \dots, μ_d that are non-negative and sum to unity, and the *faces* $\mathcal{F}_0, \dots, \mathcal{F}_d$ of S are the linear subspaces defined by setting the weights to zero one at a time.

For any point $\mathbf{p} \in S$, there are $d+1$ simplexes S_0, \dots, S_d subtended at \mathbf{p} by each face, forming a *partition* of S . The signed volume of S is defined by a determinant of dimension $d+1$, in which the coordinates of the vertices $\mathbf{p}_0, \dots, \mathbf{p}_d$ appear as columns, analogous to (63) in the case $d=2$. Similarly, the volume of each subsimplex S_k is defined by replacing the column with the coordinates of \mathbf{p}_k by those of \mathbf{p} in this determinant.

The barycentric coordinates (u_0, \dots, u_d) of \mathbf{p} with respect to S are then defined by the volume ratios $u_k = \text{volume}(S_k)/\text{volume}(S)$ for $k=0, \dots, d$. They are non-negative if $\mathbf{p} \in S$, and sum to unity. The degree- n Bernstein basis for the domain S is generated by the terms of the multinomial expansion

$$1 \equiv (u_0 + \dots + u_d)^n = \sum_{\substack{0 \leq e_0, \dots, e_d \leq n \\ e_0 + \dots + e_d = n}} \frac{n!}{e_0! \dots e_d!} u_0^{e_0} \dots u_d^{e_d}.$$

There are $\binom{n+d}{d}$ basis functions, and any degree- n polynomial can be specified by associating coefficients with them. The Bernstein basis thus defined has an elegant hierarchical structure – restricted to a lower-dimension boundary simplex by setting one or more of u_0, \dots, u_d equal to 0 or 1, it specializes to the basis defined over the lower-dimension simplex spanned by the remaining “free” coordinates. In \mathbb{R}^3 , for example, each face of a tetrahedron simplex inherits a bivariate basis, and each edge a univariate basis.

Multivariate Bernstein-form polynomials on simplex domains have a rich repertoire of interesting properties and algorithms, and we have only skimmed the surface here: more comprehensive treatments may be found in de Boor (1987b), Farin (1986).

8.3. The B-spline basis

The Bézier form facilitates smooth connections between curves and surfaces. For example, if $\mathbf{r}(t)$ and $\mathbf{s}(t)$ have control points $\mathbf{p}_0, \dots, \mathbf{p}_m$ and $\mathbf{q}_0, \dots, \mathbf{q}_n$ on $t \in [0, 1]$, a tangent-continuous connection $\mathbf{r}(1) = \mathbf{s}(0)$ can be obtained by taking

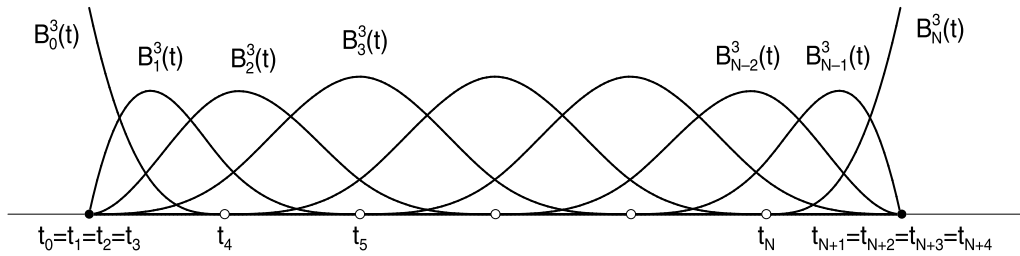


Fig. 18. The cubic B-spline basis functions constructed on a sequence of uniformly-spaced knots, with initial and final knots of multiplicity 4 each.

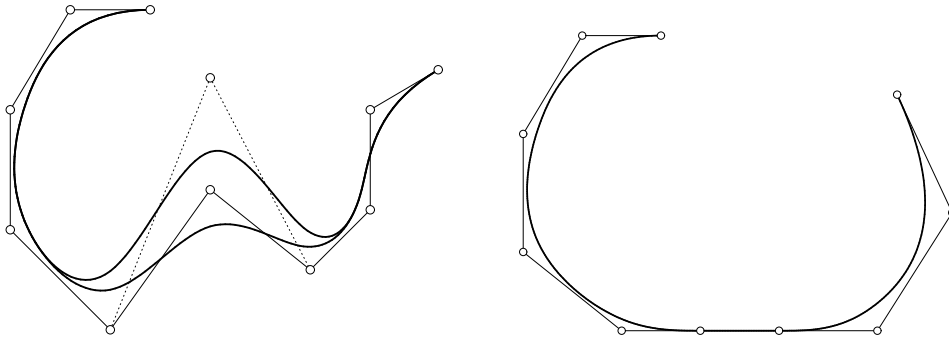


Fig. 19. Left: local modification of a cubic B-spline curve by displacement of a single control point — note that C^2 connections between the modified and unmodified segments are maintained. Right: a line segment embedded in a C^2 cubic B-spline curve (four consecutive control points are collinear).

$\mathbf{p}_m = \mathbf{q}_0$ and requiring \mathbf{p}_{m-1} and \mathbf{q}_1 to be colinear with this common point. However, it is often simpler in such circumstances to directly invoke bases that can describe piecewise-polynomial functions.

The B-spline basis $B_k^n(t)$ may be regarded as an extension of the Bernstein basis $b_k^n(t)$, generalizing the description of a single polynomial on a continuous interval to piecewise-polynomial (*spline*) functions over partitioned domains, specified by a *knot* sequence $\dots, t_{k-1}, t_k, t_{k+1}, \dots$. It retains the non-negativity and partition-of-unity properties of the Bernstein basis. Two new properties, pertinent to the spline context, characterize the degree- n B-spline functions — $B_k^n(t)$ is (usually) of continuity class C^{n-1} and is non-zero over only $n+1$ contiguous subintervals, from t_k to t_{k+n+1} (the *compact support* property). By “usually” we mean that these properties hold for distinct or *simple* knots, but in the presence of *multiple knots*¹⁹ the order of continuity and support of the basis functions influenced by such knots are reduced.

A common approach to specifying a set of linearly-independent B-spline basis functions of degree n is to choose initial/final knots of multiplicity $n+1$. Fig. 18, for example, shows the cubic B-splines over uniform knots, with 4-fold initial and final knots. Note that the number of basis functions depends primarily on the number of knots, rather than the degree n . If there are just two distinct knots, each of multiplicity $n+1$, the B-spline basis specializes to the Bernstein basis of degree n over the interval between them.

The B-spline basis functions can be generated by a simple recursion on the degree n , analogous to (13) for the Bernstein basis, and the Cox–de Boor algorithm is a fundamental recursive scheme for evaluating B-spline functions that essentially generalizes the de Casteljau algorithm. The Bernstein form of the polynomial component of a B-spline function over any subinterval of the knot sequence can be obtained by increasing the multiplicity of the knots delineating it to $n+1$ through a *knot insertion* process.

A B-spline curve may be defined by associating a control point with each B-spline basis function, in a manner analogous to (9). As seen in Fig. 19, such curves offer useful new properties beyond the capability of Bézier curves. For example, the compact support of the basis functions permits strictly local shape changes when the control points are moved. Straight line segments can be smoothly embedded within free-form curves, and the use of multiple knots allows the order of continuity at certain points to be reduced in a controlled manner. By extension to the rational B-spline form, closure under projective transformations is achieved, and simple (e.g., conic) curves may be combined with free-form segments in an integrated representation format. A complete account of the B-spline form is beyond the scope of this survey — the intent here is simply to highlight its intimate connections to the Bernstein basis. Complete details may be found in standard texts, e.g., de Boor (2001), Farin (2002), Schumaker (1981).

¹⁹ If $t_k = t_{k+1} = \dots = t_{k+m-1}$ we have a knot of multiplicity m (or m -fold knot).

8.4. Generalized barycentric coordinates

Although triangular and rectangular surface patches are predominant in most design applications, the need for patches with $N \geq 5$ sides often arises, and a number of schemes to permit their construction subject to given boundary conditions have been formulated (Charrot and Gregory, 1984; Gregory, 1986; Hosaka and Kimura, 1984; Krasauskas, 2002; Loop and DeRose, 1989; Warren, 1992). Such N -sided patches must typically interpolate given boundary curves and cross-tangent data along them, to ensure G^1 continuity with adjacent patches.

Noteworthy among these formulations are the *S-patches* of Loop and DeRose (1989), and the *toric surface patches* introduced by Krasauskas (2002). The former defines an N -sided patch by embedding an N -sided domain polygon P in a simplex S of dimension $N - 1$, such that the sides of P coincide with edges of S . When a vector-valued Bernstein-form polynomial is constructed over the simplex S , its restriction to the polygon P yields an N -sided surface patch. The latter constructs rational N -sided patches over polygon domains with vertices that have integer coordinates. Both schemes subsume triangular and tensor-product Bézier surface patches as special cases.

The study of N -sided patches has also prompted interest in extending the definition of barycentric coordinates from triangular or simplex domains to more general polygons or polytopes. In extending from simplexes to general convex polytopes, one must incorporate certain properties of the barycentric coordinates (Warren, 1996) – if $\lambda_1, \dots, \lambda_N$ are the barycentric coordinates of a point \mathbf{p} with respect to a polytope \mathcal{P} defined by vertices $\mathbf{v}_1, \dots, \mathbf{v}_N$ in \mathbb{R}^d we require:

- (1) *non-negativity*: $\lambda_1, \dots, \lambda_N \geq 0$ for all $\mathbf{p} \in \mathcal{P}$;
- (2) *linear precision*: $\lambda_1 f(\mathbf{v}_1) + \dots + \lambda_N f(\mathbf{v}_N) \equiv f(\mathbf{p})$ for any linear function $f(\mathbf{p})$. Choosing $f(\mathbf{p}) = 1$, this implies the *partition of unity* property, i.e., $\lambda_1 + \dots + \lambda_N = 1$ for all \mathbf{p} .
- (3) *minimal degree*: $\lambda_1, \dots, \lambda_N$ are the “simplest” functions of the location of \mathbf{p} that satisfy (1) and (2).

For a simplex, condition (3) is achieved with a *linear* dependence of $\lambda_1, \dots, \lambda_N$ on the location of \mathbf{p} . For a more general polytope, $\lambda_1, \dots, \lambda_N$ have a *rational* dependence on the location of \mathbf{p} . The first construction of such a system was performed by Wachspress (1975) in the context of constructing a finite element basis for convex polygons – see also Dasgupta (2003), Wachspress (2011). For a convex polygon with m sides, the barycentric coordinates are rational functions with numerators of degree $m - 2$, and a denominator of degree $m - 3$ that defines the *adjoint* of the polygon, i.e., the algebraic curve that passes through the pairwise points of intersection of the linear extensions to the polygon sides (other than the polygon vertices). Meyer et al. (2002) showed that the Wachspress coordinates can be expressed (see Fig. 20) as

$$\lambda_k := \frac{w_k}{w_1 + \dots + w_N}, \quad w_k := \frac{\text{area}(\mathbf{v}_{k-1}, \mathbf{v}_k, \mathbf{v}_{k+1})}{\text{area}(\mathbf{v}_{k-1}, \mathbf{v}_k, \mathbf{p}) \text{area}(\mathbf{p}, \mathbf{v}_k, \mathbf{v}_{k+1})},$$

where vertex indices are interpreted cyclically, and $\text{area}()$ is the *signed* area of the triangle determined by the indicated points. Warren (1996) subsequently generalized the Wachspress scheme to convex polytopes of any dimension; an extension to the case of smooth convex sets may also be found in Warren et al. (2007).

A different approach based on the *mean value theorem* was introduced by Floater (2003). These “mean value coordinates” are defined by

$$\lambda_k := \frac{w_k}{w_1 + \dots + w_N}, \quad w_k := \frac{\tan \frac{1}{2} \alpha_{k-1} + \tan \frac{1}{2} \alpha_k}{r_k},$$

where $r_k = |\mathbf{v}_k - \mathbf{p}|$, and α_k is the angle between the vectors $\mathbf{v}_k - \mathbf{p}$ and $\mathbf{v}_{k+1} - \mathbf{p}$ (again, the indices are interpreted cyclically). The coordinates are non-negative not only for convex polygons, but also within the kernel of any star-shaped polygon. They can also be generalized to non-convex polygons (Hormann and Floater, 2006) although the non-negativity property is then

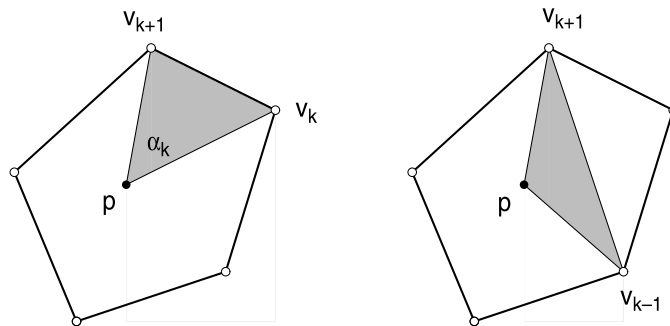


Fig. 20. Left: $\text{area}(\mathbf{p}, \mathbf{v}_k, \mathbf{v}_{k+1})$ is the shaded triangle defined by an interior point \mathbf{p} and consecutive vertices $\mathbf{v}_k, \mathbf{v}_{k+1}$ of the domain polygon, while α_k is the angle subtended at \mathbf{p} by \mathbf{v}_k and \mathbf{v}_{k+1} . Right: $\text{area}(\mathbf{v}_{k-1}, \mathbf{p}, \mathbf{v}_{k+1})$ is the shaded triangle defined by \mathbf{p} and the non-consecutive vertices $\mathbf{v}_{k-1}, \mathbf{v}_{k+1}$.

relinquished. A general approach to constructing barycentric coordinates over convex polygons was presented in Floater et al. (2006). It was shown (see Fig. 20) that the expression

$$w_k := \frac{r_{k+1}^m \text{area}(\mathbf{v}_{k-1}, \mathbf{v}_k, \mathbf{p}) + r_k^m \text{area}(\mathbf{v}_{k-1}, \mathbf{p}, \mathbf{v}_{k+1}) + r_{k-1}^m \text{area}(\mathbf{p}, \mathbf{v}_k, \mathbf{v}_{k+1})}{\text{area}(\mathbf{v}_{k-1}, \mathbf{v}_k, \mathbf{p}) \text{area}(\mathbf{p}, \mathbf{v}_k, \mathbf{v}_{k+1})}$$

defines a family of *three-point coordinates* $\lambda_k := w_k / (w_1 + \dots + w_n)$ subsuming the Wachspress and mean value coordinates as the cases $m = 0$ and $m = 1$. Moreover, it incorporates the *discrete harmonic coordinates* used in Eck et al. (1995), Pinkall and Polthier (1993) as the case $m = 2$ (but these are not barycentric coordinates, since they are not necessarily positive over the interior of the domain polygon). See also Dyken and Floater (2009) and Schaefer et al. (2007) for the case where the polygon is replaced by a smooth closed curve, and data specified along this curve is to be interpolated by a function $f(\mathbf{p})$ that is defined at each point \mathbf{p} of its interior.

Finally, as an extension to non-Euclidean spaces, we mention the case (Alfeld et al., 1996) of barycentric coordinates with respect to spherical triangle domains (in this context, the partition of unity property must be relinquished). As noted by the authors, this construction can be traced to the work of Möbius.

9. Further applications

As the Bernstein form has matured and proved its worth as a core paradigm of CAGD, interest in taking advantage of the useful properties and algorithms developed in that context for other computational disciplines has exhibited increasing momentum in recent years. Some applications that have benefited significantly from this trend are briefly discussed below.

9.1. Equations and inequalities

The ability to compute the real roots of a polynomial or solutions of a system of algebraic equations, to determine the extrema of (or estimate bounds on) polynomials in one or several variables over finite domains, and to optimize polynomial functions under given constraints, are key requirements in many scientific/engineering problems. The subdivision, convex hull, and variation-diminishing features of the Bernstein form are very useful in such contexts.

The basic problem of isolating and approximating the roots of a univariate Bernstein-form polynomial $p(t)$ on the interval $[0, 1]$ was addressed by Lane and Riesenfeld (1981). Their algorithm employs recursive subdivision until the Bernstein coefficients for each subinterval exhibit either no sign changes or just one sign change. By the variation-diminishing property, subintervals in the former category contain no roots, and those in the latter category contain precisely one root.²⁰ Once the real roots are isolated within sufficiently small subintervals, rapidly-convergent approximation methods can be invoked to approximate them to any desired accuracy. For example, simple conditions for guaranteed convergence of the Newton–Raphson method

$$t_{k+1} = t_k - \frac{p(t_k)}{p'(t_k)}, \quad k = 1, 2, \dots \quad (66)$$

can be easily expressed in terms of the Bernstein coefficients c_0, \dots, c_n of $p(t)$ for any subinterval $[a, b]$ on which they have exactly one sign change (with $c_0 c_n < 0$). Namely, the iteration (66) will converge to the unique real root of $p(t)$ on the interval (a, b) from any start point within it, if we have

$$S(\Delta c_0, \dots, \Delta c_{n-1}) = 0, \quad (67)$$

$$S(\Delta^2 c_0, \dots, \Delta^2 c_{n-2}) = 0, \quad (68)$$

$$|c_0| \leq n|\Delta c_0| \quad \text{and} \quad |c_n| \leq n|\Delta c_{n-1}|, \quad (69)$$

where $\Delta c_k = c_{k+1} - c_k$, $\Delta^2 c_k = \Delta c_{k+1} - \Delta c_k = c_{k+2} - 2c_{k+1} + c_k$, and $S()$ indicates the number of sign changes in the ordered sequence of its arguments. Conditions (67) and (68) ensure that $p(t)$ has non-vanishing first and second derivatives for $t \in (a, b)$. The former implies that $p(t)$ is *monotone* over (a, b) and therefore has only one root in that interval. The latter implies that the graph of $p(t)$ is *convex* or *concave* over (a, b) . Finally, condition (69) ensures that the tangent lines to $p(t)$ at $t = a$ and $t = b$ cross the t -axis within the interval (a, b) . A proof of this result may be found in Henri (1964, p. 79).

The above scheme may be modified and improved in a number of ways to exploit further properties of the Bernstein form – the PhD thesis of Spencer (1994), for example, describes a number of interesting variants. The numerical stability of the Bernstein form (see Section 6) is invaluable in the context of root-solvers based on floating-point arithmetic operations.

The extension to systems of polynomial equations in several variables is a non-trivial problem, since the variation-diminishing property of univariate polynomials has no straightforward multivariate generalization. Nonetheless, several methods exploiting the subdivision and convex hull properties have been developed for tensor-product multivariate

²⁰ For subintervals containing multiple roots, the condition of zero or one coefficient sign changes will not be met under repeated subdivision. To accommodate multiple roots, the algorithm must be modified to consider also roots of the derivatives of $p(t)$ – see Zidna and Michel (2008).

Bernstein-form polynomials (Garloff and Smith, 2001; Mourrain and Pavone, 2009; Müller and Otte, 1991; Sherbrooke and Patrikalakis, 1993). Lack of space precludes a thorough description of these methods here: complete details may be found in the cited references.

Closely-related problems are concerned with estimations of the range of multivariate polynomials over given subdomains, in the context of relaxation steps employed in the “branch-and-bound” approach to solving optimization problems (Fünzig et al., 2012; Garloff et al., 2003; Nataraj and Arounassalame, 2011; Ray and Nataraj, 2009; Smith, 2009). The subdivision and convex hull properties of the Bernstein form again play a key role, along with simple monotonicity and convexity tests over subdomains. Most of this work has addressed only tensor-product polynomials defined over rectangular domains in \mathbb{R}^n , but the extension to barycentric forms over simplex domains is straightforward.

9.2. Finite element analysis

Since the geometric models created within CAD systems serve as the point of departure for various physical analyses using the finite element method, one might expect close coordination in the evolution of these fields. For many decades, however, they have evolved in a largely independent manner.

The “isogeometric analysis” paradigm seeks to remedy this disconnect by using NURBS (non-uniform rational B-splines), a commonly-used CAD geometry format, as shape functions in the analysis, bypassing much of the costly and error-prone meshing requirements of traditional FEM approaches (Hughes et al., 2005). Insofar as B-spline forms generalize the Bernstein form to partitioned domains, the method may be considered to fall within the scope of this survey. The key challenge lies in accommodating “trimmed” surface patches – which are invariably present in complex geometric models, but incompatible with standard tensor-product NURBS domains. We focus here on recent methods that directly exploit the Bernstein form in finite element analysis.

Bogdanovich (2000, 2001), Bogdanovich et al. (1995), Bogdanovich and Yushanov (1999) describe the use of tensor-product trivariate Bernstein basis functions to specify three-dimensional displacement fields in hexahedral finite elements, for the analysis of stresses in laminated composite structures with anisotropic material properties. In the context of composites, different continuity properties of the element “shape functions” are desired in directions parallel and perpendicular to interfaces between distinct materials. The Bernstein form facilitates the enforcement of appropriate displacement and stress continuity conditions, and closed-form computation – rather than numerical quadrature – for various integrals that determine the elements of matrices required in the stress analysis. The author argues that the Bernstein basis functions “a rarity in structural analysis, deserve more attention” (Bogdanovich, 2000).

Schumaker (2008) describes the use of bivariate spline functions defined on triangulations, in which the Bernstein polynomial form is used to specify the variation over each triangular subdomain, to solve boundary value problems for elliptic partial differential equations: see also (Lai and Schumaker, 2007). Bernstein-form finite element algorithms for simplicial domains have also recently been studied by Kirby (2011), Kirby and Thinh (2012) with emphasis on efficient algorithms for computing mass and stiffness matrices for high-order polynomials on simplex domains, and on formulation of matrix-free algorithms. A closely-related paper by Ainsworth et al. (2011) shows that finite element algorithms for high-order Bernstein-form polynomials over simplex domains, based upon the Duffy transformation (Duffy, 1982), can achieve computational complexity comparable to that for tensor-product formulations. It is noteworthy (Ainsworth et al., 2011) that, in 2011, use of the Bernstein form “has attracted virtually no attention ... among the finite element community.” An exception is the paper by Scott et al. (2011) on using T-splines (which provide more topological flexibility than tensor-product B-splines) for isogeometric analysis, and extracting Bernstein-form elements from T-spline models.

A generalization of the simplicial Bernstein basis described by Arnold et al. (2009) is concerned with the characterization and geometric decomposition of spaces of finite element differential forms, extending classical finite element vector field spaces in \mathbb{R}^2 and \mathbb{R}^3 to higher-order forms and space dimensions.

The barycentric coordinates on simplex domains that are used to define Bernstein bases also admit useful generalizations. For example, given a set of points $\mathbf{p}_1, \dots, \mathbf{p}_n$ in the plane, their *Voronoi diagram* or *Dirichlet tessellation* (Preparata and Shamos, 1985) comprises a set of contiguous polygonal tiles T_1, \dots, T_n such that $\mathbf{p}_k \in T_k$ and every point of T_k is closer to \mathbf{p}_k than to all the other points \mathbf{p}_j for $j \neq k$. When a new point \mathbf{p} is introduced,²¹ a new tile T will be associated with it in the updated tessellation, overlapping certain tiles T_{k_1}, \dots, T_{k_m} of the original tessellation. The location of \mathbf{p} can be specified in terms of its *neighbor points* $\mathbf{p}_{k_1}, \dots, \mathbf{p}_{k_m}$ in the form

$$\mathbf{p} = \sum_{j=1}^m u_j \mathbf{p}_{k_j},$$

where u_j is the fractional area of T that overlaps T_{k_j} . The *Sibson coordinates* (Sibson, 1980) u_1, \dots, u_m of \mathbf{p} are non-negative and form a partition of unity – i.e., $u_1, \dots, u_m \geq 0$ and $u_1 + \dots + u_m = 1$. Furthermore, when there are just three neighbor points ($m = 3$), these Sibson coordinates specialize to barycentric coordinates over the triangle defined by those points.

²¹ The new point \mathbf{p} must lie in the *convex hull* of the given points $\mathbf{p}_1, \dots, \mathbf{p}_n$.

The Sibson (or *natural neighbor*) coordinates are unique, and continuous with respect to the location of \mathbf{p} . Furthermore, they can be generalized from \mathbb{R}^2 to \mathbb{R}^n for $n \geq 3$, since the Dirichlet tessellation is defined in any number of Euclidean dimensions. Sibson (1981) defined an interpolant to data values f_1, \dots, f_n associated with the data sites $\mathbf{p}_1, \dots, \mathbf{p}_n$ by the expression

$$f(u_1, \dots, u_m) = \sum_{j=1}^m f_j u_j. \quad (70)$$

Farin (1990) showed that Sibson's interpolant exhibits C^1 continuity and has linear precision (see Section 8.4), and interpreted (70) as the projection of a linear Bernstein-form polynomial, defined on a simplex of dimension $m - 1$. Using the Bernstein form, he also proposed a generalized interpolant with quadratic precision (completeness) – i.e., when f_1, \dots, f_n are sampled from a quadratic function, it exactly recovers that function.

In the finite element context, the generalized Sibson interpolant has been adopted (Braun and Sambridge, 1995; Sukumar and Moran, 1999; Sukumar et al., 1998) as the basis for an approach called the *natural element method*, which may be considered “meshfree” or “meshless” since it does not require any element connectivity information. The methods for generalizing barycentric coordinates from simplex domains to general polygons/polytopes discussed in Section 8.4 have also attracted interest within the finite element research community. As noted above, these ideas originated with the rational finite element basis of Wachspress (1975). More recent results can be found in Dasgupta (2003), Malsch and Dasgupta (2004), Malsch et al. (2005), Sukumar and Malsch (2006), Sukumar and Tabarraei (2004). An interesting new method, the “maximum entropy” approach, interprets the construction of barycentric coordinates over general polygons/polytopes as a least-biased solution of an underdetermined system of linear equations (Hormann and Sukumar, 2008; Sukumar, 2004). Closed-form expressions for the coordinates are no longer available in this context, but they can be efficiently computed through a constrained optimization problem.

9.3. Robust control of dynamic systems

A key consideration in designing controllers for linear dynamic systems is the selection of controller “gains” that are consistent with desired performance and *stability* of the controller, i.e., for any bounded input, the system output should remain bounded. The stability of a linear system can be characterized by the complex-plane distribution of the roots of its *characteristic polynomial* (Dorf and Bishop, 2005) – for analog systems, the roots must all lie to the left of the imaginary axis (*Hurwitz condition*); for digital systems, the roots must all lie within the unit disk centered on the origin of the complex plane (*Schur condition*).

The problem of *robust control* is concerned with systems whose physical parameters are not precisely known, but may vary within prescribed ranges. Thus, instead of a unique characteristic polynomial, the stability analysis of such systems incurs *parameter-dependent* characteristic polynomials. To be *robustly stable*, each instance of the system – corresponding to each possible combination of the parameter values – must be stable (Ackermann, 1993; Barmish, 1994).

To guarantee certain performance criteria beyond nominal stability, it is often desirable to restrict the characteristic polynomial roots to some subset Γ of the left half-plane, or unit disk – the system is then said to be Γ -stable. Suppose the characteristic polynomial has the form

$$p(t) = \sum_{k=0}^n a_k(q_1, \dots, q_m) t^k \quad (71)$$

where q_1, \dots, q_m are the system parameters, each confined to a given interval, $q_k \in [q_k, \bar{q}_k]$. The *zero exclusion principle* is a key tool for testing Γ -stability. For the characteristic polynomial (71), the *value set* $\mathcal{V}(\mathbf{z})$ is defined as the set of all its complex values when the independent variable has a fixed complex value \mathbf{z} , and the parameters q_1, \dots, q_m all vary over their respective intervals. According to the zero exclusion principle (Barmish, 1994) the system is Γ -stable if $0 \notin \mathcal{V}(\mathbf{z})$ as \mathbf{z} traverses the stability-region boundary, $\partial\Gamma$. Fig. 21, for example, shows a sampling of the value set for the cubic interval polynomial

$$p(t) = [0.9, 1.1]t^3 + [-1.2, -0.8]t^2 + [2.3, 2.7]t + [-1.2, -0.8] \quad (72)$$

as t moves on the imaginary axis between $-1.4i$ and $+1.4i$.

For systems in which the coefficients a_k have a polynomial dependence on the parameters q_1, \dots, q_m the testing of the zero exclusion principle is greatly facilitated by expressing the coefficients $a_k(q_1, \dots, q_m)$ in the tensor-product Bernstein basis over the m -dimensional volume $[q_1, \bar{q}_1] \times \dots \times [q_m, \bar{q}_m]$ and exploiting the convex hull, variation-diminishing, and subdivision properties. For more comprehensive details on this approach, the reader may consult the papers Akyar et al. (2010), Dzhaifarov et al. (2010), Garloff (2000), Garloff and Graf (1999), Malan et al. (1997), Šiljak and Stipanović (1999), Zettler and Garloff (1998) and references therein.

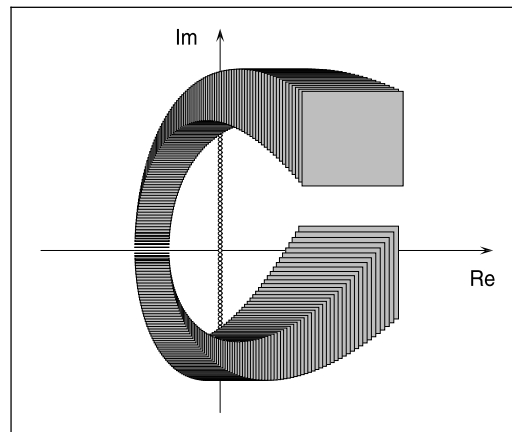


Fig. 21. Sampling of the value set for the interval polynomial (72) as the independent variable t traverses the imaginary-axis interval $[-1.4i, +1.4i]$.

9.4. Other problems

Applications of the Bernstein basis have also elicited interest in many other computational problems, a few of which are briefly mentioned here. In Ho and Rabitz (2000) the Bernstein basis is employed in modeling inter-molecular potential energy surfaces through the reproducing kernel Hilbert space interpolation method. Algorithms for neurofuzzy networks modeling non-linear dynamical systems have invoked the bivariate Bernstein basis on triangular domains to represent the system input (Hong and Harris, 2000). Tensor-product Bernstein basis neural networks have also been used in reconstructing 3D models from measured data and in the calibration of optical range sensors (Knopf and Kofman, 1999, 2002). Degree elevation of univariate Bernstein-form polynomials has been employed in Samadi (2000) to facilitate design of filter sharpening functions for signal processing applications. Convergent piecewise-linear approximations obtained from the Bernstein form have been invoked (Ernesto and Farouki, 2012; Farouki et al., 2000) in computing time-optimal feedrates along curved paths for CNC machines with prescribed axis acceleration bounds, which amounts to a calculus of variations problem with polynomial point-wise constraints.

10. Closure

This survey has endeavored to describe the historical origins, current status, and diverse applications of the Bernstein polynomial basis, on the centennial anniversary of its introduction. The adoption of the Bernstein representation as a medium for practical computations has been an evolutionary process, in which three distinct phases may be discerned.

First, beginning with Bernstein's 1912 paper, it served as a fundamental theoretical tool for analyzing the approximative capability of polynomials on finite domains. However, the necessity of proceeding to very high degrees to enforce tight error bounds precluded widespread use of Bernstein polynomial approximations in practical computations. Second, the work of de Casteljau and Bézier in the 1960s revealed that the Bernstein form provides an excellent medium for the specification/modification of (vector-valued) polynomials on finite domains, although at first their ideas were not explicitly identified with the Bernstein basis. The ensuing elaboration of algorithms and properties for the "Bernstein-Bézier form" in the field of *computer aided geometric design* has led to a high degree of maturity and utility for this representation. Third, drawing on the extensive body of methods developed in the CAGD context, the attractive features of the Bernstein form have been increasingly exploited in other scientific and engineering applications over the past few decades (see Section 9). It can be expected that a further development and diversification of this latter phase may continue for many years to come.

The volume of papers concerned with properties, algorithms, applications, and extensions of the Bernstein polynomial representation has grown to vast proportions over the past century. Although it has not been possible to cite every relevant work herein, it is nevertheless hoped that this survey offers a balanced perspective on the principal research themes and trends, and thus helps encourage further useful applications of the Bernstein form.

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