

# Summary paper 6: Efficient prior shapes for spline-based snakes Delgado

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Active detouring and outlining object is now popularly done with the help of spline-based schemes but the efficiency of is dependent on the choice of a good basis function for the type of shapes we want to detect as well as on the choice of a energy adapted to the image we are looking at. This paper introduces a new scheme based on exponential B-spline basis functions that incorporates a prior shape in the energy of the snakes. The four main contributions of the paper are

1. Presentation of a shape projector that maps any continuously-defined closed spline curve to a space shape
2. Derivation of an analytical form for a measure of distance to the shape space
3. A spline-based solution at no additional computational cost with a finite number of control-points.
4. Ready-to-use plugins on Icy.

## Parametric snake model

We look for a basis-function  $\phi$  that can represent 2D parametric closed curves  $r(t) = (r_1(t), r_2(t))$   $t \in \mathbb{R}$  with the help of  $M$  control points in the following form

$$\forall t \in [0, M] \quad r(t) = \sum_{k \in \mathbb{Z}} c[k] \phi(t - k) \quad (1)$$

with  $(r[k])_{k \in \mathbb{Z}}$  an M-periodic sequence. The domain of  $t$  can be remapped to  $[0, 1]$  while the sum can be limited to  $M$  elements by rewriting as

$$\forall t \in [0, 1] \quad r(t) = \sum_{k=0}^{M-1} c[k] \phi_M(Mt - k) \quad (2)$$

As usual we require that this spline scheme has the following properties: uniqueness of representation and stability w.r.t to the control parameters (Riesz-basis); affine invariance (partition of unity). Additionally we require the scheme to be able to reproduce the curve to be segmented.

The Riesz-basis property is proved in the Fourier domain. Note that for any function  $\phi : \mathbb{R} \rightarrow \mathbb{C}$  in  $\mathcal{L}_1 \cup \mathcal{L}_2$ ,  $\forall w \in \mathbb{R} \quad \sum_{k \in \mathbb{Z}} \hat{\phi}(w + 2k\pi) = \sum_{k \in \mathbb{Z}} \phi(k) e^{-ikw}$ . Indeed let  $\psi$  the  $2\pi$ -periodized version of  $\hat{\phi}$ . Then  $\hat{\psi} : \mathbb{Z} \rightarrow \mathbb{R}$

with  $\hat{\psi}[k] = \int_{-\pi}^{\pi} \psi(w) e^{-i w k} dw$ . Just notice then that  $\hat{\psi}[k] = 2\pi \phi(-k)$  and that  $\forall w \quad \psi(w) = \sum_{k \in \mathbb{Z}} \frac{\hat{\psi}[k]}{2\pi} e^{ikw}$  to have the result.

Any 2D affine transformation is dependent on a  $2 \times 2$  matrix  $\mathbf{A}$  and a  $2 \times 1$  translation vector  $\mathbf{b}$ . The transformation can be rewritten in matrix form using homogeneous coordinates that is to the 2D vector  $r = [r_1, r_2]^T$  we associate the 3D vector  $r_h = [r_1, r_2, 1]^T$ . The transformed point in homogeneous coordinates is then  $\mathbf{H}r_h$  with

$$\mathbf{H} = \begin{bmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{0} & 1 \end{bmatrix} \quad (3)$$

Then affine invariance simply translates into  $\forall t \in \mathbb{R} \quad \sum_{k \in \mathbb{Z}} \phi(Mt - k) = 1$ .

### Efficient shape prior

The prior knowledge of the shape is encoded in  $r^{ref}$ , the reference curve. The set of all affine transformations of  $r^{ref}$ ,  $S_{ref}$ , is a vector space of dimension 6 given the matrix form of an affine transform in homogeneous coordinates. Let  $S$  the vector space of all splines that takes the form of (2). We endow this space with the scalar product

$$\forall r, s \in S \quad \langle r, s \rangle = \int_0^1 r^T s$$

making  $(S_{ref}, \langle \cdot, \cdot \rangle)$  a finite dimension vector subspace on which we can define an orthogonal projector.

For  $r \in S$  we denote  $r^p$  its orthogonal projection on  $S_{ref}$ . By property of orthogonal projection we know that  $r^p$  is the sole element in  $S_{ref}$  that minimizes  $\|r - s\|^2 = \int_0^1 \|r(t) - s(t)\|^2 dt$  over  $s \in S_{ref}$ . Note that the norms on the left and on the right of the equation are different, first one is derived from the scalar product defined above while second one is canonical norm on  $\mathbb{R}^2$ .

To any  $r \in S$  we associate  $C$  the  $M \times 2$  matrix of coordinates of its control points and  $C_h$  the  $M \times 3$  matrix of coordinates of  $r_h$  at its control points. Note that the functional relation  $r_h = \mathbf{H}r_h^{ref}$  can equivalently be written as the matrix equality  $C_h = C_h^{ref} \mathbf{H}^T$ .

We can now write explicitly the expression of the projection of any  $r \in S$  through the following theorem

**Theorem 1.** *Let  $r \in S$ ,  $C_h$  the coordinates matrix of its homogeneous form  $r_h$ . Let  $r_h^p$  its projection onto  $S_{ref}$  and  $C_h^p$  the associated matrix. Then*

$$C_h^p = P^{ref} C_h \quad (4)$$

with  $P^{ref} = C_h^{ref} \left( C_h^{refT} \Phi C_h^{ref} \right)^{-1} C_h^{refT} \Phi$  and  $\Phi$  is the autocorrelation matrix that is

$$\Phi_{i,j} = \int_0^1 \phi_M(Mt - i) \phi_M(Mt - j) dt$$

This theorem is proved naturally by rewriting the minimization of the norm as the minimization of a matrix form over the matrix variable  $\mathbf{H}$ . In the proof we make use of  $\nabla_H \text{Tr}(AH^T) = A$  and  $\nabla_H \text{Tr}(HAH^T) = H(A + A^T)$ .

### Active contours with prior shapes

The coefficients of the snake to be found are parameters in the problem of minimizing the snake energy which in its general form can be decomposed as

$$E_{snake}(\Omega) = E_{image}(\Omega) + E_{internal}(\Omega) + E_{constraint}(\Omega) \quad (5)$$

The image term makes the snake converge to boundaries in the images which is what you want to do to outline shapes. The internal energy is a regularization term that enforces smoothness constraints into the snake and the constraint term allows for user interaction by forcing some values of coefficients where the user put a control point for example.

In practice the image term is a balance between an “edge” term and a “region” term that is  $E_{image} = \alpha E_{edge} + (1 - \alpha) E_{region}$ . Be aware that the region term is generally computationally much more expensive

than the edge term as it looks at all pixels in the region surrounding the region of interest to compute the associated energy.

### Internal energy with shape prior

The authors introduce a new energy that encourages the snake to stay close to  $S_{ref}$ . The energy is given with a weight  $\beta$

$$E_{internal} = \beta \|(I - P^{ref})C_h\|_F \quad (6)$$

The proposed prior shape model can also be transposed into discrete setting using the conventional discrete shape-space formalism. If we consider a pixel-based approach where we uniformly sample the parameter space of a continuous curve that is  $\{r(\frac{n}{N})\}_{n=0,\dots,N-1}$ , such pointwise representation converges to the curve when  $N \rightarrow \infty$  if it does not contain discontinuities. The discrete mean-squared error is then  $\frac{1}{N} \sum_{n=0}^{N-1} \|r(\frac{n}{N}) - r^p(\frac{n}{N})\|^2$ . An equivalent projection operation would be defined as an  $N \times N$  matrix which is usually much larger than  $M \times M$  in the continuous case.

Optimization is carried out with a Powell-like line search method ([34]). The method requires the partial derivatives of the energy function w.r.t to the parameters that is the coordinates of the control points. Note that the contribution to the gradient by the internal energy is

$$2\beta(I - P^{ref})^T (I - P^{ref}) C_h \quad (7)$$

### Validation and experiments

In a first experiment, authors tested sensitivity of the snake to initialization. They then investigated the effect of  $\beta$  before analyzing the performance of the scheme on real biological data.

Sensitivity to initialization is tested with synthetic 8-bit image of mitotic cell in fluorescence. The ground truth is given by a 9-point spline curve that perfectly match the edges. The experience was repeated 50 times with different initialization and two different values of  $\beta$ , 0 and 10. The basin of attraction (where Jaccard index is above 0.9) is twice as large with  $\beta = 10$ !

Then sensitivity to different level of noise was tested with one initialization and 50 different noise realizations. With  $\beta = 10$  the average Jaccard index remains the same despite increase level of noise and even the variance remains the same. With  $\beta = 0$ , the snake always fail to converge to a satisfying contour with average Jaccard indices at 0.75 only.

In the end a qualitative example is shown with body tracking of a *Drosophila* fly along frame sequences without prior and with prior shape correspond to the fly body shape. With the latter the results are much more satisfactory. The result also proved really good on segmenting yeast cells with a yeast cell shape prior.