

General ideas on interpolation

Yoann Pradat

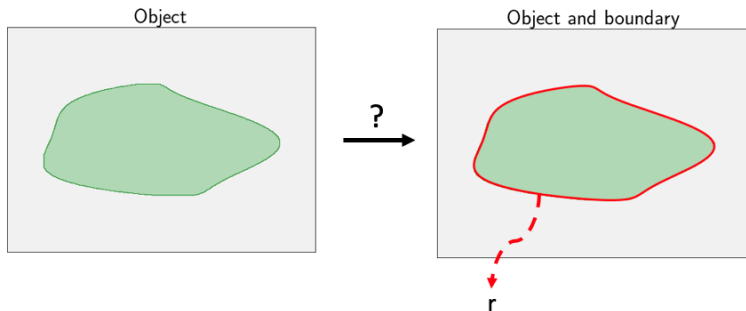
V. Uhlmann's lab

July 4, 2019



- 1 Introduction
- 2 I/ Polynomial interpolation
 - I/1 Principles
 - I/2 Limits
- 3 II/ Piecewise-polynomial interpolation
 - II/1 Piecewise-linear
 - II/2 Piecewise cubic interpolation
- 4 III/ B-splines and Splines
 - III/1 Definition and fundamental properties
 - III/2 Applications

Objective



Objective: find mathematical model for the **boundary**

$$\mathbf{r} = \begin{bmatrix} r_1 \\ \vdots \\ r_d \end{bmatrix} \quad (1)$$

Generic model

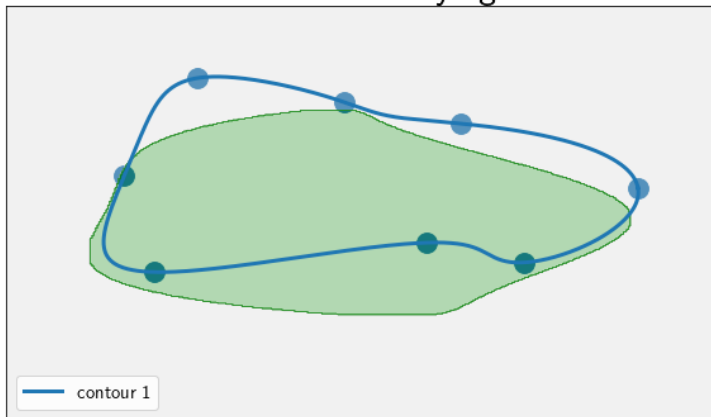
$$\mathbf{r}(t) = \begin{bmatrix} r_1 \\ \vdots \\ r_d \end{bmatrix} = \sum_{k \in \mathbb{Z}} \begin{bmatrix} c_1(k) \\ \vdots \\ c_d(k) \end{bmatrix} \phi_k(t) \quad (2)$$

Vocab

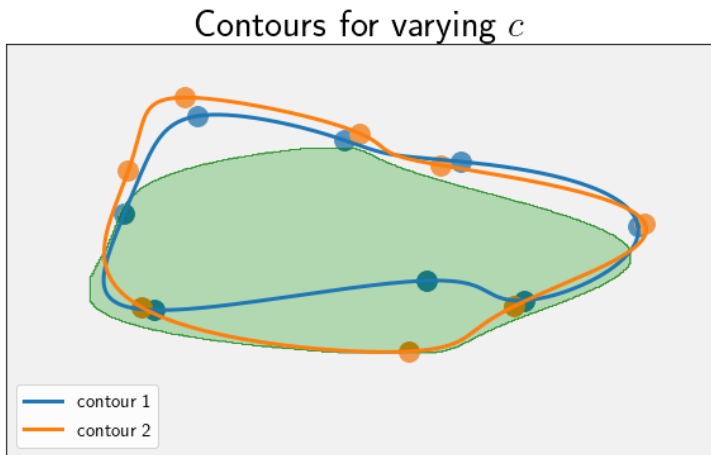
- $(t_k = k)_{k \in \mathbb{Z}}$ **knots** locations
- $\mathbf{c}(k) := \begin{bmatrix} c_1(k) \\ \vdots \\ c_d(k) \end{bmatrix}$ are **control points**
- $(\phi_k)_{k \in \mathbb{Z}} :=$ set of **basis functions** or **generators**

Impact of varying $c(k)$

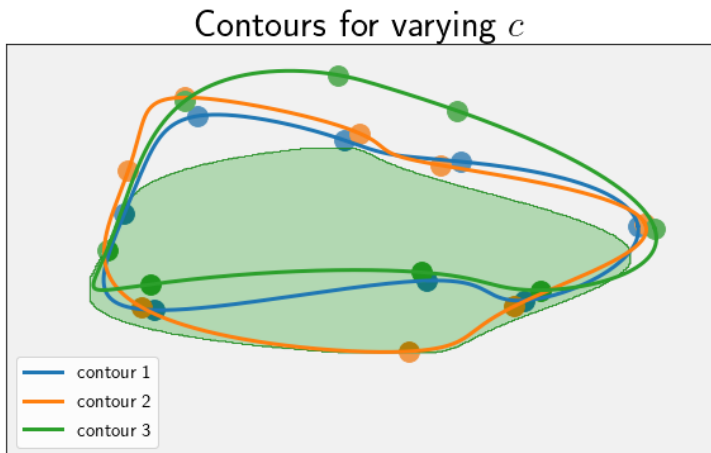
Contours for varying c



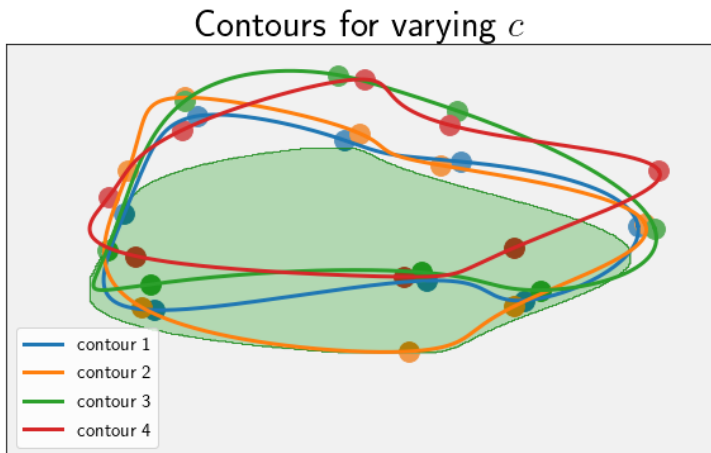
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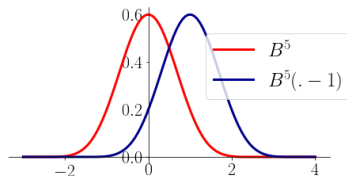
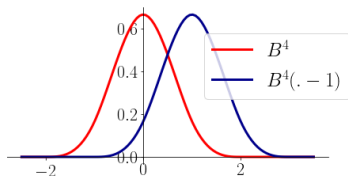
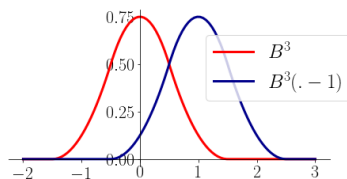
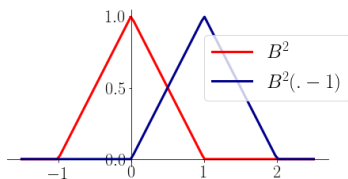


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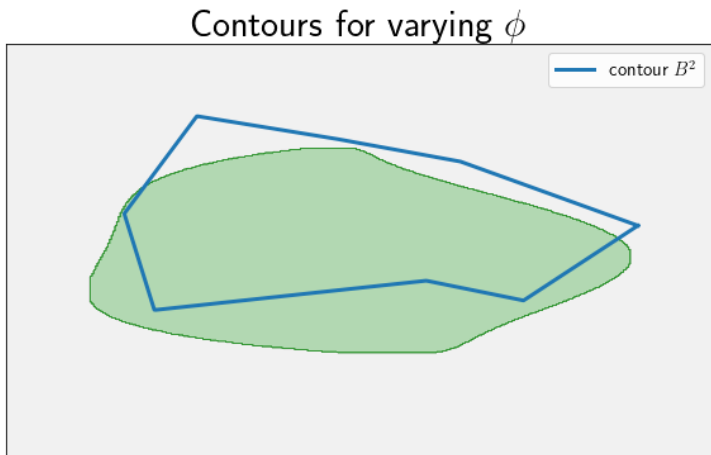


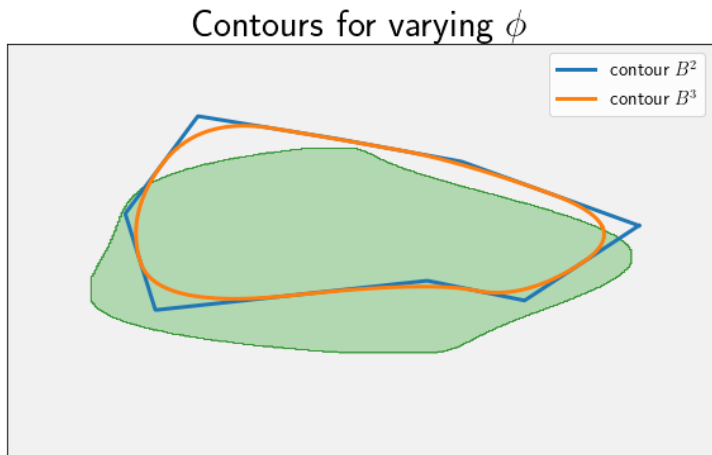
Impact of varying $c(k)$

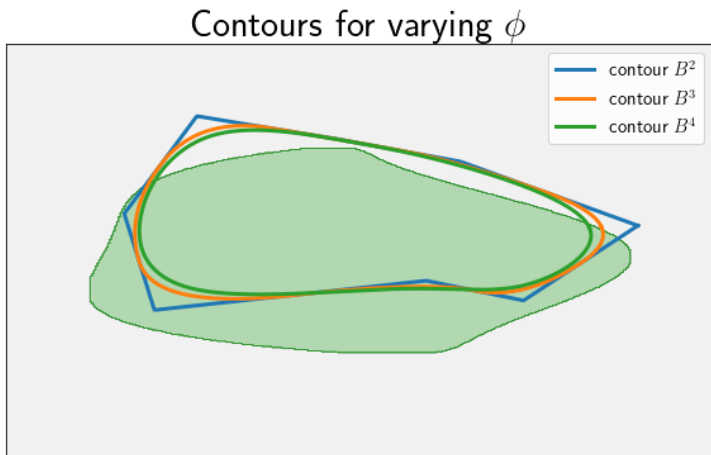


Impact of varying ϕ_k 

$$(\phi_k)_{k \in \mathbb{Z}} = (\tau_k B)_{k \in \mathbb{Z}} \quad \text{with} \quad \tau_k f = f(\cdot - k)$$

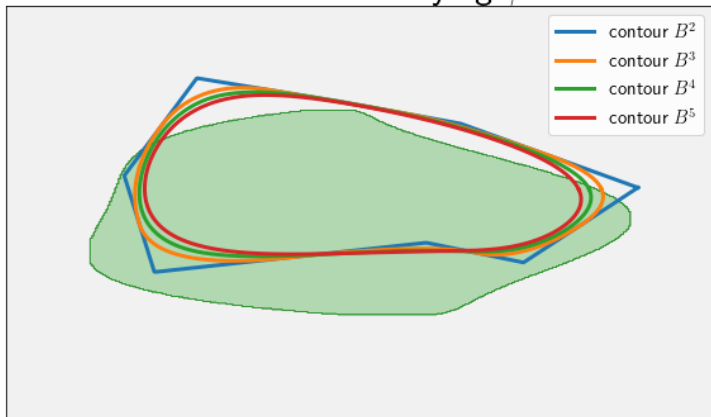
Impact of varying ϕ_k 

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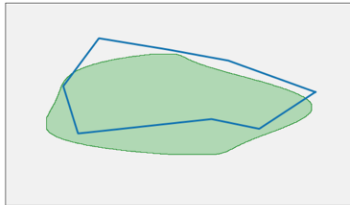
Contours for varying ϕ



Approximation, interpolation

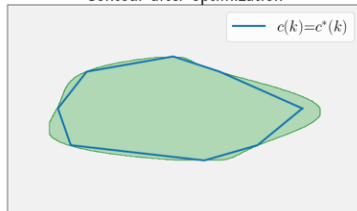
Boundary approximation Fixed generators ϕ_k , optimize the values of $c(k)$.

Contour initialization



Min
energy
→

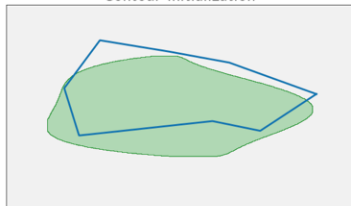
Contour after optimization



Approximation, interpolation

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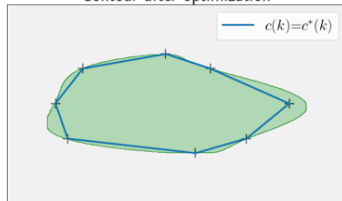
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Contour after optimization



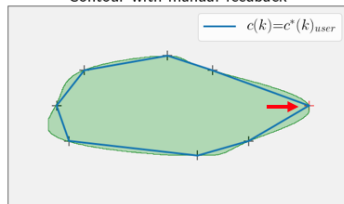
Feedback? Intuitive understanding of control points

Contour after optimization



Manual
feedback
→

Contour with manual feedback



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Lagrange polynomials

Not.

- $\Pi_{<n} = \{p \in \mathbb{R}[X] \mid \deg(p) \leq n-1\}$
- $t_1 < \dots < t_n$

Obj: reconstruct $g : \mathbb{R} \rightarrow \mathbb{R}$ from $g(t_1), \dots, g(t_n)$.

Proposition 1 (Lagrange 1795)

$\exists! f \in \Pi_{<n}$ s.t. $\forall i = 1, n, f(t_i) = g(t_i)$

Lagrange polynomials

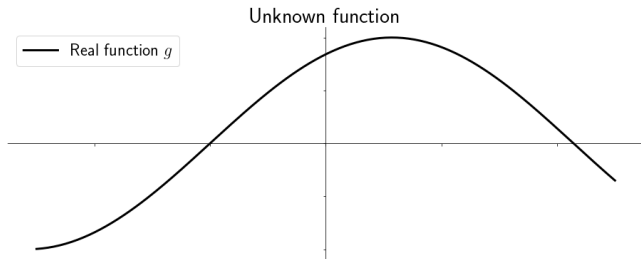
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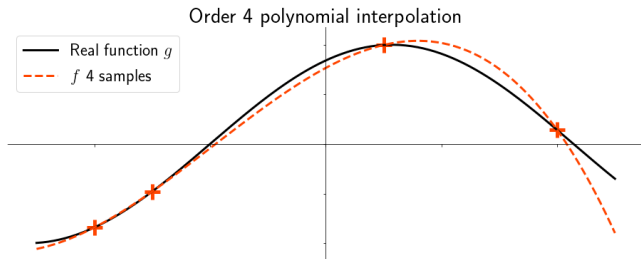
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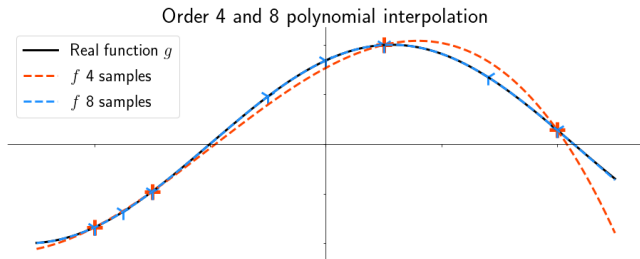
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Osculatory interpolation

Not.

- f agrees with g at $\underbrace{(t, \dots, t)}_{r \text{ times}}$ if $\forall m \in \llbracket 0, r-1 \rrbracket, f^{(m)}(t) = g^{(m)}(t)$
- $t_1 = \dots = t_{r_1} \leq \dots \leq t_{r_1+\dots+r_{d-1}+1} = \dots = t_{r_1+\dots+r_d}$ with d unique

Proposition 2 (Consequence theorem 1)

$$\exists! f \in \Pi_{<n} \text{ s.t. } \forall i \in \llbracket 0, d \rrbracket, f(t_i) = g(t_i), \dots, f^{(r_i-1)}(t_i) = g^{(r_i-1)}(t_i)$$

Divided differences

Not.

- $t_1 \leq \dots \leq t_n$ with d unique

Definition 1

$[t_1, \dots, t_n]g$ leading coeff of $f \in \Pi_{<n}$ s.t $f(t_i) = g(t_i)$

Examples

- 1 $[t_1]g = g(t_1)$
- 2 $[t_1, t_2]g = \begin{cases} \frac{g(t_2) - g(t_1)}{t_2 - t_1} & \text{if } t_1 \neq t_2 \\ g'(t_1), & \text{otherwise} \end{cases}$

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Proposition 3

- 1 if $g \in \mathcal{C}^{(n-1)}$, $\exists \eta \in [t_1, t_n]$ s.t $[t_1, \dots, t_n]g = \frac{g^{(n-1)}(\eta)}{n!}$
- 2 $f \in \Pi_{<n}$ s.t $f(t_i) = g(t_i)$ is $f(t) = \sum_{i=1}^n (t - t_1) \dots (t - t_{i-1}) [t_1, \dots, t_i]g$.

Osculatory theorem

Not.

- $t_1 \leq \dots \leq t_n$ with d unique

Theorem 1 (C. De Boor 1978)

Let $g \in \mathcal{C}^{(n)}$. Then $\forall t, g = f_n + (t - t_1) \dots (t - t_n)[t_1, \dots, t_n, t]g$.

Osculatory theorem

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Theorem 1 (C. De Boor 1978)

Let $g \in \mathcal{C}^{(n)}$. Then $\forall t, g = f_n + (t - t_1) \dots (t - t_n)[t_1, \dots, t_n, t]g$.

Example

- $t_1 = \dots = t_n, g(t) = \sum_{i=0}^{n-1} \frac{g^{(i)}(t_1)}{i!} (t - t_1)^i + \frac{g^{(n)}(\eta_t)}{n!} (t - t_1)^n$

Runge: high-order interpolation can be bad

Not.

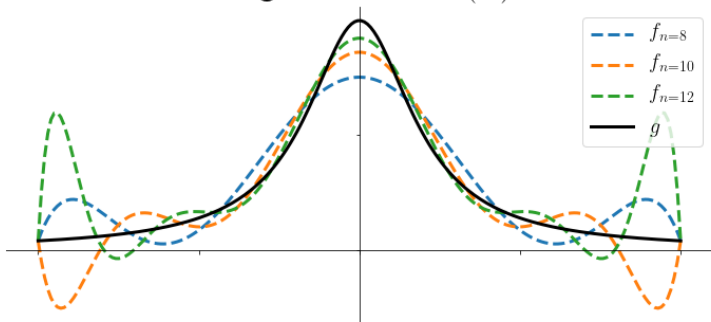
- $t_1 < \cdots < t_n$ uniform on $[-1, 1]$
- $g(t) = \frac{1}{1+25t^2}$

Runge: high-order interpolation can be bad

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Runge at uniform (t_i)



Runge explained

Not.

- $a = t_1 < \dots < t_n = b$
- $\|f\| = \|f\|_{\infty, [a, b]} = \sup_{a \leq t \leq b} |f(t)|$
- $f_n = \sum_{i=1}^n g(t_i) \phi_i$, ϕ Lagrange pol.
- $\lambda_n(t) = \sum_{i=1}^n |\phi_i(t)|$

Runge explained

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Magnitude interpolator

$$\begin{aligned} \text{For } t_i \leq t \leq t_{i+1}, |f_n(t)| &\leq \sup_{1 \leq i \leq n} |g(t_i)| \lambda_n(t) \\ &\leq \|g\| \lambda_n \\ \text{hence } \|f_n\| &\leq \|g\| \lambda_n \end{aligned}$$

- Bound is sharp
- Uniform $(t_i) \implies \|\lambda_n\| \sim \frac{2^n}{en \log(n)}$.

Chebyshev sites are good

Not.

- $a = t_1 < \dots < t_n = b$
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Analysis of the error

$$\begin{aligned} \text{Osculatory thm} \quad |g(t) - f_n(t)| &= |(t - t_1) \dots (t - t_n)[t_1, \dots, t_n, t]g| \\ &\leq \|(\cdot - t_1) \dots (\cdot - t_n)\| \frac{\|g^{(n)}\|}{n!} \end{aligned}$$

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Idea: take $(t_i^c) = \operatorname{argmin}_{a \leq t_1 \leq \dots \leq t_n \leq b} \|(\cdot - t_1) \dots (\cdot - t_n)\|$

- 1 $t_i^c = [(a + b) - (a - b) \cos(\frac{(2i-1)\pi}{2n})]/2$
- 2 $\|(\cdot - t_1^c) \dots (\cdot - t_n^c)\| = 2(\frac{b-a}{4})^n$

Jackson's theorem

Not.

- $w(g, h) = \sup\{|g(t) - g(s)|, |t - s| \leq h\}$

Theorem 2

If $g \in C^r([a..b])$ and $n > r + 1$ then

$$\text{dist}(g, \Pi_{<n}) \leq \text{const}_r \left(\frac{b-a}{n-1} \right)^r w(g^{(r)}, \frac{b-a}{2n-1-r}) \quad (3)$$

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Idea: break interval $[a, b]$ into pieces.

- Pros low order polynomials
- Cons continuity at break points

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Interpolator I_2

Not.

- $a = t_1 < \dots < t_n = b, g : [a, b] \rightarrow \mathbb{R}$
- $\Pi_{<n,\mathbf{t}} = \{\text{pp of order } n \text{ with breaks at } t_2, \dots, t_{n-1}\}$
- $\mathcal{S}_{2,\mathbf{t}} = \Pi_{<2,\mathbf{t}} \cap \mathcal{C}^0$

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Piecewise linear on 4 segments



Piecewise linear on 20 segments

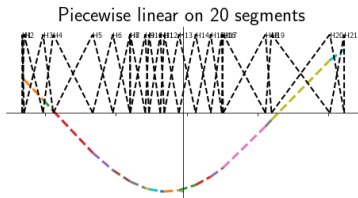
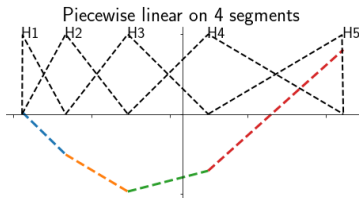


Result: $\exists! f \in \mathcal{S}_{2,\mathbf{t}}$ such that $f(t_i) = g(t_i)$

Interpolator l_2

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- $\mathcal{S}_{2,t} = \Pi_{<2,t} \cap \mathcal{C}^0$



Result: $\exists! f \in \mathcal{S}_{2,t}$ such that $f(t_i) = g(t_i)$

$$\phi_i(t) = \begin{cases} \frac{t - t_{i-1}}{t_i - t_{i-1}} & \text{if } t_{i-1} < t \leq t_i \\ \frac{t_{i+1} - t}{t_{i+1} - t_i} & \text{if } t_i \leq t < t_{i+1} \end{cases}, \quad l_2 g = \sum_{i=1}^n g(t_i) \phi_i \quad (4)$$

Approximation power of l_2

Not.

- $a = t_1 < \dots < t_n = b, g : [a, b] \rightarrow \mathbb{R}$
- $l_2 g = \sum_{i=1}^n g(t_i) \phi_i \in \mathcal{S}_{2,t}$

Properties

Proposition 4

- 1 For $f \in \mathcal{S}_{2,t}, l_2 f = f$
- 2 $\|l_2 g\| \leq \|g\|$

Approximation power of l_2

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Properties

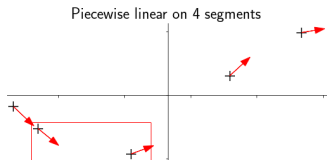
Proposition 4

- 1 For $f \in \mathcal{S}_{2,\mathbf{t}}, l_2 f = f$
- 2 $\|l_2 g\| \leq \|g\|$
- 3 If $g \in \mathcal{C}^{(2)}, \|g - l_2 g\| \leq \frac{|\mathbf{t}|^2}{8} \|g''\|$
- 4 If $g \in \mathcal{C}^0, \|g - l_2 g\| \leq w(g, |\mathbf{t}|)$.

Cubic pieces

Not.

- $a = t_1 < \dots < t_n = b, g : [a, b] \rightarrow \mathbb{R}$
- $g(t_1), s_1, \dots, g(t_n), s_n$



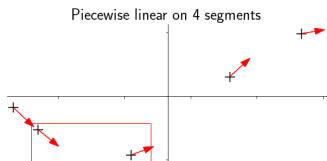
Zoom on $[t_i, t_{i+1}]$



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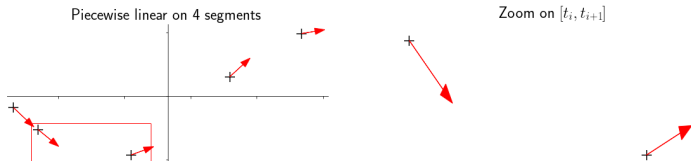


On $[t_i, t_{i+1}]$, 4 conditions for $f|_{[t_i, t_{i+1}]}$ $\implies \exists! f_i \in \Pi_{<4}$ s.t $f_i = f|_{[t_i, t_{i+1}]}$

Cubic pieces

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On $[t_i, t_{i+1}]$, 4 conditions for $f_{|[t_i, t_{i+1}]}$ $\implies \exists! f_i \in \Pi_{<4}$ s.t $f_i = f_{|[t_i, t_{i+1}]}$
(Newton form)

$$\begin{aligned} f_{|[t_i, t_{i+1}]} &= [t_i]f_i + (t - t_i)[t_i, t_i]f_i + (t - t_i)^2[t_i, t_i, t_{i+1}]f_i \\ &\quad + (t - t_i)^2(t - t_{i+1})[t_i, t_i, t_{i+1}, t_{i+1}]f_i \end{aligned}$$

Explicit formula

$$\begin{aligned} f_{[t_i, t_{i+1}]}(t) = & g(t_i) + s_i(t - t_i)^2 + \left(3\frac{\Delta g(t_i)}{\Delta t_i^2} - \frac{s_i + s_{i+1}}{\Delta t_i}\right)(t - t_i)^3 \\ & + \left(\frac{s_i + s_{i+1}}{\Delta t_i^2} - 2\frac{\Delta g(t_i)}{\Delta t_i^3}\right)(t - t_i)^4 \end{aligned}$$

4 cases

- case 1 $(s_i) = (g'(t_i))$ i.e **Hermite** interp.
if $g \in \mathcal{C}^{(4)}, \|g - f\| \leq \frac{1}{384} |t|^4 \|g^{(4)}\|$ De Boor 1978

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- case 2 Enforce $f \in \mathcal{C}^{(2)}$ i.e $f''_{|[t_{i-1}, t_i]}(t_i) = f''_{|[t_i, t_{i+1}]}(t_i) \implies$ tridiagonal linear system for s_2, \dots, s_{n-1} .

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 - case 2a $s_1 = g'(t_1), s_n = g'(t_n)$ **Complete** cubic spline.
 $\|g - f\| = \mathcal{O}(|t|^4)$ De Boor 1978

4 cases

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 if $g \in \mathcal{C}^{(4)}, \|g - f\| \leq \frac{1}{384}|t|^4 \|g^{(4)}\|$ De Boor 1978
- case 2 Enforce $f \in \mathcal{C}^{(2)}$ i.e $f''_{|[t_{i-1}, t_i]}(t_i) = f''_{|[t_i, t_{i+1}]}(t_i) \implies$ tridiagonal linear system for s_2, \dots, s_{n-1} .
 - case 2a $s_1 = g'(t_1), s_n = g'(t_n)$ **Complete** cubic spline.
 $\|g - f\| = \mathcal{O}(|t|^4)$ De Boor 1978
 - case 2b s_1, s_n unknown, enforce $f''(t_1) = f''(t_n) = 0$ **Natural** cubic spline
 $\|g - f\| = \mathcal{O}(|t|^2)$ De Boor 1978

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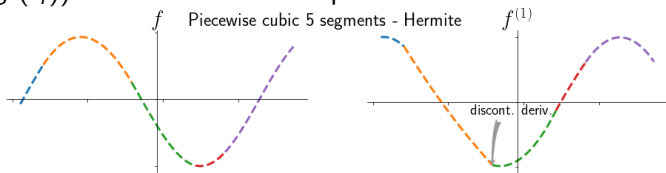
$$\|g - f\| = \mathcal{O}(|t|^2) \quad \text{De Boor 1978}$$

- case 2c s_1, s_n unknown, enforce continuity of $f^{(3)}$ at t_2 and t_n **Not-a-knot** condition.

$$\|g - f\| = \mathcal{O}(|t|^4) \quad \text{Schwartz \& Vozga 1972}$$

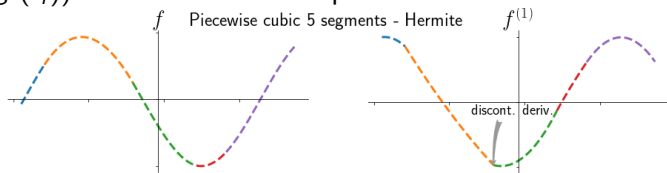
Examples

- ① $(s_i) = (g'(t_i))$. Cubic Hermite interpolation.

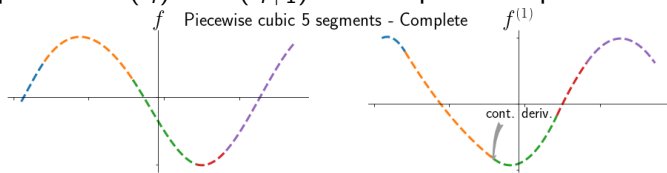


Examples

- ① $(s_i) = (g'(t_i))$. Cubic Hermite interpolation.



- ② $n - 2$ equations $f''(t_i) = f''(t_{i+1})$. Cubic spline interpolation.



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