Estimation of the twist vector

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1 Estimation of twist vector at corners of a single patch

Suppose we would like to represent a surface over the patch $[0,1]^2$ from data at each of the 4 corners. Depending on the available data, different interpolation techniques (with different properties) can be used. In case of bicubic polynomial interpolation ("bicubic patch") one can represent a surface from knowledge of coordinates of the surface value and it's first-order derivatives (that is 16 vectors in total) as $\sigma : [0,1]^2 \to \mathbb{R}^3$ given by

$$\sigma(u,v) = \begin{bmatrix} f_1(u) \\ f_1(1-u) \\ f_2(u) \\ -f_2(1-u) \end{bmatrix}^T \begin{bmatrix} \sigma(0,0) & \sigma(0,1) & \sigma_2(0,0) & \sigma_2(0,1) \\ \sigma(1,0) & \sigma(1,1) & \sigma_2(1,0) & \sigma_2(1,1) \\ \sigma_1(0,0) & \sigma_1(0,1) & \sigma_{12}(0,0) & \sigma_{12}(0,1) \\ \sigma_1(1,0) & \sigma_1(1,1) & \sigma_{12}(1,0) & \sigma_{12}(1,1) \end{bmatrix} \begin{bmatrix} f_1(v) \\ f_1(1-v) \\ f_2(v) \\ -f_2(1-v) \end{bmatrix} \tag{1}$$

With f_1 and f_2 being the cubic Hermite polynomials over $[0,1]$

$$f_1(u) = 1 - 3u^2 + 2u^3$$
$$f_2(u) = u - 2u^2 + u^3$$

The parameters σ_{12} are the cross-derivatives according to each direction and as such do not lend itself to a consistent interpretation. As a consequence, techniques have been developed to estimate the "optimal" twist vector with different optimality criterions.

1.1 Selesnick's method

In his paper of 1980 Selesnick proposes to estimate separately the normal and tangential component of the twist vector **from the rest of the parameters and the Gaussian curvature K.**

Let's denote $n = \sigma_1 \wedge \sigma_2$ the normal vector and $\hat{n} = \frac{n}{|n|}$ it's normalized version. As for the surface with denote we subscript 1 and 2 the derivative with respect to u and v in that order. Then the normal component of the twist vector can be obtained from

$$\hat{n}.\sigma_{12} = \pm\sqrt{\hat{n}.\sigma_{11}\hat{n}.\sigma_{22} - K\left(\sigma_1^2\sigma_2^2 - \sigma_1.\sigma_2\right)} \tag{2}$$

How to get the values for σ_{11}, σ_{22}? Supposing $\sigma, \sigma_1, \sigma_2$ are known at the corners, one can readily compute the values of σ_{11} and σ_{22}. This is a consequence of having interpolators that are *twice differentiable* and *satisfy Hermite interpolation properties* that is to say

$$f_1(\nu) = \delta_\nu \quad f_1^{(1)}(\nu) = 0 \quad f_2(\nu) = 0 \quad f_2^{(1)}(\nu) = \delta_\nu \tag{3}$$

as the terms corresponding to the twist vectors will appear with factor $f_2^{(2)}(u)f_2(v)$, which is equal to 0 when u and v are 0 or 1. Knowing all quantities on the right-hand side of the equation above, we can estimate the normal component of the twist vector. As for the sign, one can choose say positive sign for the first corner and deduce the signs for all other corners under the assumption that the variation of the interpolated surface between the data points should less than the variation implied by the points themselves.

To get a complete description of the twist vector it is enough to compute projection of the vector on two additional vectors that are each orthogonal to the normal and between themselves. One is naturally led towards computing tangential components of the twist vector that is $\sigma_1.\sigma_{12}$ and $\sigma_2.\sigma_{12}$. Introducing the parametrisation by arclength s for say u at fixed v, it is straightforward to show that

$$\sigma_1.\sigma_{12} = \left[\frac{ds}{du}\right] \frac{\partial}{\partial v} \left[\frac{ds}{du}\right] \tag{4}$$

with $\frac{ds}{du} = |\sigma_1|$.
In a similar fashion the other tangential component is obtained by

$$\sigma_2.\sigma_{12} = \left[\frac{ds}{dv}\right] \frac{\partial}{\partial u} \left[\frac{ds}{dv}\right] \tag{5}$$

with $\frac{ds}{du} = |\sigma_2|$.

1.2 Minimum quadratic oscillation

A piecewise Coon's surface is described on a square $[a, b] \times [c, d]$ by $M_1 \times M_2$ patches of the form $I_{k,l} = [u_k, u_{k+1}] \times [v_l, v_{l+1}]$.

In case of a *bilinear interpolating patch*, the surface on $I_{k,l}$ is given by

$$\sigma(u, v) = \begin{bmatrix} f_1(u) \\ f_1(1-u) \end{bmatrix}^T \begin{bmatrix} \sigma(0,0) & \sigma(0,1) \\ \sigma(1,0) & \sigma(1,1) \end{bmatrix} \begin{bmatrix} f_1(v) \\ f_1(1-v) \end{bmatrix} \tag{6}$$

wiht $f_1(u) = u$.

In case of *bicubic Coon's patch*, the surface on $I_{k,l}$ is given by

$$\sigma(u, v) = \begin{bmatrix} f_1(u) \\ f_1(1-u) \\ f_2(u) \\ -f_2(1-u) \end{bmatrix}^T \begin{bmatrix} \sigma(0,0) & \sigma(0,1) & \sigma_2(0,0) & \sigma_2(0,1) \\ \sigma(1,0) & \sigma(1,1) & \sigma_2(1,0) & \sigma_2(1,1) \\ \sigma_1(0,0) & \sigma_1(0,1) & \sigma_{12}(0,0) & \sigma_{12}(0,1) \\ \sigma_1(1,0) & \sigma_1(1,1) & \sigma_{12}(1,0) & \sigma_{12}(1,1) \end{bmatrix} \begin{bmatrix} f_1(v) \\ f_1(1-v) \\ f_2(v) \\ -f_2(1-v) \end{bmatrix} \tag{7}$$

with renoting $u \leftarrow \frac{u-u_k}{\Delta u_k}$, $v \leftarrow \frac{v-v_l}{\Delta v_l}$, $\sigma(u, v) \leftarrow \sigma(\frac{u-u_k}{\Delta u_k}, \frac{v-v_l}{\Delta v_l})$ and f_1 and f_2 being the cubic Hermite polynomials over $[0, 1]$

$$f_1(u) = 1 - 3u^2 + 2u^3$$
$$f_2(u) = u - 2u^2 + u^3$$

<u>Remark</u> Renoting $\sigma(u, v) \leftarrow \sigma(\frac{u-u_k}{\Delta u_k}, \frac{v-v_l}{\Delta v_l})$ means we note again σ the function on $[0, 1]^2$ that is equal to σ on $I_{k,l}$ mapped to $[0, 1]^2$ that is $\sigma(u, v) = \sigma(\Delta u_k u + u_k, \Delta v_l v + v_l)$. As a consequence, $\sigma_1(0, 0)$ in new notation corresponds to $\Delta u_k \sigma_1(u_k, v_l)$ in old notation. In coherence with previous usage we would write $\sigma_1(u, v) \leftarrow \Delta u_k \sigma_1(\frac{u-u_k}{\Delta u_k}, \frac{v-v_l}{\Delta v_l})$.

In their paper of 2017, X. Guo et X. Han derives a method to determine twist vectors that are optimal in the sense that they minimize the quadratic oscillation in average that is to say they minimize the distance between the bicubic and bilinear interpolant

$$\int_a^b \int_c^d \|Q(x, y) - L(x, y)\|^2 dxdy \tag{8}$$

2 Estimation of twist vector over the whole surface

2.1 Bicubic Coon's is cubic Hermite spline interpolation

Despite having a different name, bicubic Coon's patches are exactly the same as a cubic Hermite spline interpolation for tensor-product surface. Indeed, for the latter case the surface is given by

$$
\begin{aligned}
\sigma(u,v) = & \sum_{k=0}^{M_1}\sum_{l=0}^{M_2} c_1[k,l]\phi_1(M_1u-k)\phi_1(M_2v-l) \\
& + \sum_{k=0}^{M_1}\sum_{l=0}^{M_2} c_2[k,l]\phi_1(M_1u-k)\phi_2(M_2v-l) \\
& + \sum_{k=0}^{M_1}\sum_{l=0}^{M_2} c_3[k,l]\phi_2(M_1u-k)\phi_1(M_2v-l) \\
& + \sum_{k=0}^{M_1}\sum_{l=0}^{M_2} c_4[k,l]\phi_2(M_1u-k)\phi_2(M_2v-l)
\end{aligned}
$$

With

$$
\phi_1(x) = \begin{cases} f_1(x) & \text{for } 0 \le x \le 1 \\ f_1(-x) & \text{for } -1 \le x < 0 \end{cases} \qquad \phi_{2,w}(x) = \begin{cases} f_2(x) & \text{for } 0 \le x \le 1 \\ -f_2(-x) & \text{for } -1 \le x < 0 \end{cases}
$$

and f_1, f_2 are as previously the cubic Hermite polynomials.

Restricting our attention to a single patch $I_{k,l} = [\frac{k}{M_1}, \frac{k+1}{M_1}] \times [\frac{l}{M_2}, \frac{l+1}{M_2}]$, and *because* ϕ_1, ϕ_2 have support $[-1,1]$, the expression above boils down to

$$
\boxed{\sigma(u,v) = \begin{bmatrix} f_1(u) \\ f_1(1-u) \\ f_2(u) \\ -f_2(1-u) \end{bmatrix}^T \begin{bmatrix} \sigma(0,0) & \sigma(0,1) & \sigma_2(0,0) & \sigma_2(0,1) \\ \sigma(1,0) & \sigma(1,1) & \sigma_2(1,0) & \sigma_2(1,1) \\ \sigma_1(0,0) & \sigma_1(0,1) & \sigma_{12}(0,0) & \sigma_{12}(0,1) \\ \sigma_1(1,0) & \sigma_1(1,1) & \sigma_{12}(1,0) & \sigma_{12}(1,1) \end{bmatrix} \begin{bmatrix} f_1(v) \\ f_1(1-v) \\ f_2(v) \\ -f_2(1-v) \end{bmatrix}} \tag{9}
$$

with renoting $u \leftarrow M_1u-k, v \leftarrow M_2v-l$ and $\sigma(u,v) \leftarrow \sigma(M_1u-k, M_1v-l)$.

2.2 Romani Conti's ellipse-reproducing splines

Conti et al's paper Ellipse-preserving interpolation and subdivision scheme introduces two basis functions from the space $\mathcal{E}_4 =< 1, x, e^{-iw_1x}, e^{iw_1x} >$ where $w = \frac{2\pi}{M}$ to reproduce closed curves with M control points. The basis functions are **cycloidal splines** (Exponential splines? Exponential B-splines?) given by

$$
\phi_{1,w}(x) = \begin{cases} g_{1,w}(x) & \text{for } x \ge 0 \\ g_{1,w}(-x) & \text{for } x < 0 \end{cases} \qquad \phi_{2,w}(x) = \begin{cases} g_{2,w}(x) & \text{for } x \ge 0 \\ -g_{2,w}(-x) & \text{for } x < 0 \end{cases} \tag{10}
$$

The surface with $M_1 \times (M_2+1)$ (M_1 because of periodicity over u) control points is given by

For all $(u,v) \in [0,1]^2$

$$\begin{aligned}\sigma(u,v) = &\sum_{k=0}^{M_1}\sum_{l=0}^{M_2} c_1[k,l]\phi_{1,w_1}(M_1u-k)\phi_{1,w_2}(M_2v-l)\\ &+\sum_{k=0}^{M_1}\sum_{l=0}^{M_2} c_2[k,l]\phi_{1,w_1}(M_1u-k)\phi_{2,w_2}(M_2v-l)\\ &+\sum_{k=0}^{M_1}\sum_{l=0}^{M_2} c_3[k,l]\phi_{2,w_1}(M_1u-k)\phi_{1,w_2}(M_2v-l)\\ &+\sum_{k=0}^{M_1}\sum_{l=0}^{M_2} c_4[k,l]\phi_{2,w_1}(M_1u-k)\phi_{2,w_2}(M_2v-l)\end{aligned}$$

$$c_1[k,l] = \begin{bmatrix}\cos(w_1k)\sin(w_2l)\\ \sin(w_1k)\sin(w_2l)\\ \cos(w_2l)\end{bmatrix} = \sigma(\frac{k}{M_1},\frac{l}{M_2}) \qquad c_2[k,l] = \begin{bmatrix}w_2\cos(w_1k)\cos(w_2l)\\ w_2\sin(w_1k)\cos(w_2l)\\ -w_2\sin(w_2l)\end{bmatrix} = \frac{1}{M_2}\frac{\partial\sigma}{\partial v}(\frac{k}{M_1},\frac{l}{M_2})$$

$$c_3[k,l] = \begin{bmatrix}-w_1\sin(w_1k)\sin(w_2l)\\ w_1\cos(w_1k)\sin(w_2l)\\ 0\end{bmatrix} = \frac{1}{M_1}\frac{\partial\sigma}{\partial u}(\frac{k}{M_1},\frac{l}{M_2}) \quad c_4[k,l] = \begin{bmatrix}-w_1w_2\sin(w_1k)\cos(w_2l)\\ w_1w_2\cos(w_1k)\cos(w_2l)\\ 0\end{bmatrix} = \frac{1}{M_1M_2}\frac{\partial^2\sigma}{\partial u\partial v}(\frac{k}{M_1},\frac{l}{M_2})$$

The parameters $c_4[k,l]$ are the twist vectors at each location of the control points.

Again restricting our attention to a single patch $I_{k,l} = [\frac{k}{M_1},\frac{k+1}{M_1}]\times[\frac{l}{M_2},\frac{l+1}{M_2}]$, and *because* $\phi_{1,w_i},\phi_{2,w_i}$ have support $[-1,1]$, the expression above boils down to

$$\boxed{\sigma(u,v) = \begin{bmatrix}g_{1,w_1}(u)\\ g_{1,w_1}(1-u)\\ g_{2,w_1}(u)\\ -g_{2,w_1}(1-u)\end{bmatrix}^T \begin{bmatrix}\sigma(0,0) & \sigma(0,1) & \sigma_2(0,0) & \sigma_2(0,1)\\ \sigma(1,0) & \sigma(1,1) & \sigma_2(1,0) & \sigma_2(1,1)\\ \sigma_1(0,0) & \sigma_1(0,1) & \sigma_{12}(0,0) & \sigma_{12}(0,1)\\ \sigma_1(1,0) & \sigma_1(1,1) & \sigma_{12}(1,0) & \sigma_{12}(1,1)\end{bmatrix}\begin{bmatrix}g_{1,w_2}(v)\\ g_{1,w_2}(1-v)\\ g_{2,w_2}(v)\\ -g_{2,w_2}(1-v)\end{bmatrix}} \tag{11}$$

with renoting $u \leftarrow M_1u-k$, $v \leftarrow M_2v-l$ and $\sigma(u,v) \leftarrow \sigma(M_1u-k, M_1v-l)$.

The ressemblance with Hermite cubic interpolation (or bicubic Coon's patch equivalently) is striking. The only difference is the use of different basis functions on each continuous direction, the difference disappearing in case $M_1 = 2M_2$. Observe also that g is not polynomial but an exponential polynomial. However the values taken by g and its derivative over $[0,1]$ are very close to that taken by f and its derivative over $[0,1]$.