Cardinal Spline Interpolation Operators on l^p Data

M. J. MARSDEN, F. B. RICHARDS
& S. D. RIEMENSCHNEIDER

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Introduction. For a natural number n, the space $\mathcal{S}_n = \{S(x)\}$ of bounded cardinal splines of degree $n - 1$ is taken to consist of those functions satisfying:

(i) $S \varepsilon C^{n-2}(-\infty, +\infty)$

(ii) $||S|| = \sup_{-\infty<x<+\infty} |S(x)| < +\infty$

(iii) $S(x)$ reduces to a polynomial of degree at most $n - 1$ on each of the intervals $[\nu + \frac{1}{2}n, \nu + \frac{1}{2}n + 1]$, $\nu = 0, \pm 1, \pm 2, \cdots$, *i.e.* $S(x)$ has knots at the integers or half-integers if n is respectively even or odd.

For a sequence $y = \{y_\nu\}_{\nu=-\infty}^{+\infty} \varepsilon \ell^p(\mathbf{Z})$, the Banach space of doubly infinite bounded sequences with the usual norm $||y||_{\ell^p}^p = \sum_{\nu=-\infty}^{+\infty} |y_\nu|^p$, there is a unique element $\mathcal{L}_n y \varepsilon \mathcal{S}_n \cap L^p(\mathbf{R})$ interpolating the given data at the integers, *i.e.* $\mathcal{L}_n y(\nu) = y_\nu$, $\nu \varepsilon \mathbf{Z}$ (Schoenberg [9]). The operator $\mathcal{L}_n : \ell^p(\mathbf{Z}) \to \mathcal{S}_n \cap L^p(\mathbf{R})$ is called the cardinal spline interpolation operator of order n, and its norm is

$$||\mathcal{L}_n||_p = \sup_{||y||_{\ell^p} \leq 1} ||\mathcal{L}_n y||_{L^p} .$$

It is the purpose of the present paper to investigate the behaviour of $||\mathcal{L}_n||_p$, p-fixed, as n tends to infinity. For the case $p = +\infty$, Richards [5] has shown that $||\mathcal{L}_n||_\infty$ is asymptotic to $(2/\pi) \log n$; a result which is similar to the "best" rates obtainable for polynomial interpolation on $[-1, 1]$ (Rivlin [7] and Erdös [2]). For $p = 1$, in section 2 we show that $||\mathcal{L}_n||_1$ is asymptotic to $(4/\pi^2) \log n$ which is of the same order as the L^1-norm of the Dirichlet kernel that relates to the "best" rate obtainable for projection operators from $C[-\pi, \pi]$ to trigonometric polynomials of degree n.

When $p = 2$, Richards [4], for n-even, and Marsden and Mureika [3], for n-odd, have shown that $||\mathcal{L}_n||_2 = 1$. The behaviour of $||\mathcal{L}_n||_p$ for the cases $p = 1, 2$, and $+\infty$ suggested the Hilbert transform. In section 3, we obtain the uniform boundedness of $||\mathcal{L}_n||_p$, $1 < p < +\infty$, in terms of the norm of a Hilbert trans-

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form. This in turn allows us to prove that for $y \in \ell^p(\mathbf{Z})$, $\mathcal{L}_n y$ converges in L^p-norm to the Whittaker cardinal series. For $p = +\infty$, Schoenberg [11] has shown that if $f \in L^p(\mathrm{R})$ and $f(x) = \int_{-\pi}^{\pi} e^{ixt}\, d\beta(t)$, $\beta \in$ B.V. $[-\pi, \pi]$, where $\beta(\pi - 0) - \beta(\pi) = \beta(-\pi + 0) - \beta(-\pi)$, then for $y_\nu = f(\nu)$, $\nu \in \mathbf{Z}$, $\mathcal{L}_n y$ converges uniformly to $f(x)$. In the case $1 < p < +\infty$, we show that this condition is both necessary and sufficient for L^p-convergence (Theorem 3.4), and thus we characterize the function class in L^p for which the n^{th} order cardinal spline interpolants converge.

In section 1 we establish notation and some necessary preliminary results.

§1. Preliminaries.

1.1. The fundamental cardinal spline function. There is a particular representation for the operator $\mathcal{L}_n y$ which is of interest to us. Consider the functions

$$\psi_n(u) = [(2/u) \sin u/2]^n \tag{1.1}$$

$$\phi_n(u) = \sum_{j=-\infty}^{+\infty} \psi_n(u + 2\pi j) \tag{1.2}$$

and

$$\gamma_n(u) = \sum_{j=-\infty}^{+\infty} (-1)^j \psi_n(u + 2\pi j). \tag{1.3}$$

These functions were first investigated by Schoenberg [8].

The importance of the functions (1.1) and (1.2) lies in the fact that the Fourier transform

$$L_n(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} [\psi_n(u)/\phi_n(u)] e^{iux}\, du \tag{1.4}$$

is characterized by the properties

(i) $L_n \in \mathcal{S}_n$

(ii) $L_n(\nu) = \begin{cases} 1 & \nu = 0 \\ 0 & \nu = \pm 1, \pm 2, \pm 3, \cdots \end{cases}$

(iii) $|L_n(x)| \to 0$ exponentially as $|x| \to +\infty$ (see Schoenberg [8], [9], and [10]). The function $L_n(x)$ is called the fundamental cardinal spline of degree $n - 1$. Moreover, the representation

$$\mathcal{L}_n y(x) = \sum_{\nu=-\infty}^{+\infty} y_\nu L_n(x - \nu), \qquad -\infty < x < +\infty \tag{1.5}$$

holds.

For the case $p = 1$, we need the following lemma relating the three functions ψ_n, ϕ_n, and γ_n.

Lemma 1.1. [5]. *If $n \geqq 3$ and $0 < u < \pi$, then*

$$\left| \frac{\phi_n(u)}{\psi_n(u) + \psi_n(u - 2\pi)} - 1 \right| < 2^{-n} \tag{1.6}$$

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and

$$\left|\frac{\gamma_n(u)}{\psi_n(u) - \psi_n(u - 2\pi)} - 1\right| < 2^{-n}. \tag{1.7}$$

When $1 < p < +\infty$, we need to use the Fourier transform of ψ_n/ϕ_n effectively. This requires a knowledge of the variational properties of ψ_n/ϕ_n.

Proposition 1.2. *The function $\psi_n(u)/\phi_n(u)$ is of bounded variation on $(-\infty, +\infty)$. In particular,*

(a) *ψ_n/ϕ_n is an even function which strictly decreases from 1 to 0 on $[0, 2\pi]$,*

(b) $$\mathrm{Var}_{(-\infty,+\infty)}\{\psi_n(u)/\phi_n(u)\} = 2 + \mathcal{O}(2^{-n}), \tag{1.8}$$

and

(c) $$\mathrm{Var}_{(0,\pi-\epsilon_n)\cup(\pi+\epsilon_n,2\pi)}\{\psi_n(u)/\phi_n(u)\} = \mathcal{O}(n^{-2}) \tag{1.9}$$

where $\epsilon_n = (2\pi \log n)/n$.

Proof. We have

$$\psi_n(u)/\phi_n(u) = 1/u^n \sum_{j=-\infty}^{+\infty} (-1)^{jn}(u + 2\pi j)^{-n} = 1/u^n f(u)$$

where

$$\begin{aligned} f(u) &= u^{-n} + \sum_{j=1}^{\infty} \{[2\pi(2j-1) - u]^{-n} + (-1)^n[2\pi(2j-1) + u]^{-n} \\ &\qquad + (-1)^n[4\pi j - u]^{-n} + [4\pi j + u]^{-n}\} \\ &= u^{-n} + \sum_{j=1}^{\infty} C_{j,n}(u). \end{aligned}$$

The following properties follow easily: (i) $f(-u) = (-1)^n f(u)$; (ii) $f(u + 2\pi) = (-1)^n f(u)$; (iii) $f(2\pi - u) = f(u)$; (iv) $f''(u) = f''(u; n) = n(n+1)f(u; n+2)$; and (v) $f(2\pi j)$ is undefined.

If n is even, then $C_{j,n}(u) \geqq 0$ for $0 < u \leqq \pi$, and so, $C_{j,n+2}(u) \geqq 0$. Hence $f''(u) \geqq 0$ and $f(u)$ is convex. Being symmetric (on $(0, 2\pi)$) about π, this gives us that $f(\pi)$ is a minimum.

If n is odd, then $C_{j,n}(u) = h(2j-1) - h(2j)$ where $h(a) = (2\pi a - u)^{-n} - (2\pi a + u)^{-n}$. Since $h'(a) = -2\pi n[(2\pi a - u)^{-n-1} - (2\pi a + u)^{-n-1}] \leqq 0$ for $0 < u \leqq \pi$, $h(a)$ is decreasing and so $C_{j,n}(u) \geqq 0$. Thus, $C_{j,n+2}(u) \geqq 0$, $f''(u) \geqq 0$, and $f(u)$ is convex on $(0, 2\pi)$. Again this results in $f(\pi)$ being a minimum.

In either case, $f(\pi)$ is the minimum on $(0, 2\pi)$ and

$$f(\pi) = 2\pi^{-n} + \sum_{j=1}^{\infty} \{2[(4j+1)\pi]^{-n} + (-1)^n 2[(4j-1)\pi]^{-n}\} > \pi^{-n}.$$

Thus,

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$$\begin{aligned}
\mathrm{Var}_{[2\pi j, 2\pi j+2\pi]}[\psi_n(u)/\phi_n(u)] &= \mathrm{Var}_{[2\pi j, 2\pi j+2\pi]}[(u^n f(u))^{-1}] \\
&= \int_{2\pi j}^{2\pi j+2\pi} |d/du[(u^n f(u))^{-1}]|\, du \\
&\leqq \int_{2\pi j}^{2\pi j+2\pi} \{[1/f(u)]n\, |u|^{-n-1} + |u|^{-n}\, |d/du(1/f(u))|\}\, du \\
&\leqq \pi^n(2\pi)^{-n}\, |\min(j, j+1)|^{-n} + (2\pi)^{-n}\, |\min(j, j+1)|^{-n}\, 2\pi^n \\
&= 3\cdot 2^{-n}\, |\min(j, j+1)|^{-n}, \qquad j \neq 0 \quad \text{or} \quad -1. \qquad (1.10)
\end{aligned}$$

Hence,

$$\mathrm{Var}_{(-\infty, -2\pi]\cup[2\pi, +\infty)}\, [\psi_n(u)/\phi_n(u)] \leqq 6\cdot 2^{-n} \sum_{j=1}^{\infty} j^{-n}.$$

We next show that $\psi_n(u)/\phi_n(u)$ is decreasing in $(0, 2\pi)$. Since $f(u)$ decreases on $(0, \pi)$ and increases on $(\pi, 2\pi)$, we only need to show that $u^n f(u)$ increases on $(0, \pi)$. Now, $u^n f(u) = 1 + u^n \sum_{j=1}^{\infty} C_{j,n}(u) \geqq 1$ for $0 < u < \pi$. Setting $D_{j,n}(u) = u^n C_{j,n}(u)$, we obtain

$$\begin{aligned}
D_{j,n}'(u) = 2\pi n u^{-2}\{&(2j-1)[(2\pi/u)(2j-1) - 1]^{-n-1} \\
&+ (-1)^n(2j-1)[(2\pi/u)(2j-1) + 1]^{-n-1} + (-1)^n 2j[(4\pi j/u) - 1]^{-n-1} \\
&+ 2j[(4\pi j/u) + 1]^{-n-1}\}.
\end{aligned}$$

In order to show $D_{j,n}'(u) > 0$, we set $h(a) = a[(2\pi/u)a - 1]^{-n-1} + (-1)^n a[(2\pi/u)a + 1]^{-n-1}$. Then $D_{j,n}'(u) = 2\pi n u^{-2}\{h(2j-1) + (-1)^n h(2j)\}$. Easy estimates using $0 < u < 2\pi$, $a \geqq 1$, show that $h'(a) < 0$, and so $h(a)$ decreases. Since $h(a)$ decreases, we have $D_{j,n}'(u) > 0$ and so $u^n f(u)$ increases. Thus, $\psi_n(u)/\phi_n(u) = 1/u^n f(u)$ decreases on $(0, 2\pi)$.

Combining the above, we have $\mathrm{Var}_{(-\infty,+\infty)}\, [\psi_n(u)/\phi_n(u)] = 2 + \mathcal{O}(2^{-n})$. However, we desire to show that most of this variation occurs near $u = \pm\pi$.

Let $u_n = \pi - \epsilon_n$ and $v_n = \pi + \epsilon_n$ where $\epsilon_n = 2\pi \log n/n$. For $0 < u < \pi$,

$$\begin{aligned}
u^n f(u) &= 1 + \sum_{j=1}^{\infty} D_{j,n}(u) \\
&\leqq 1 + [(2\pi/u) - 1]^{-n} + [(2\pi/u) + 1]^{-n} + [(4\pi/u) - 1]^{-n} \\
&\quad + [(4\pi/u) + 1]^{-n} + \sum_{j=2}^{\infty} D_{j,n}(u) \\
&\leqq 1 + [(2\pi/u) - 1]^{-n} + 3^{3-n}.
\end{aligned}$$

Hence,

$$\begin{aligned}
(u_n)^n f(u_n) &\leqq 1 + 3^{3-n} + \left(\frac{n - 2\log n}{n + 2\log n}\right)^n \\
&\leqq 1 + 3^{3-n} + n^{-2} \\
&\leqq 1/(1 - 10n^{-2}) \quad (\text{for} \quad n \geqq 4).
\end{aligned}$$

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Thus, $\psi_n(u_n)/\phi_n(u_n) \geqq 1 - 10n^{-2}$ and $\mathrm{Var}_{[0,u_n]}\,[\psi_n(u)/\phi_n(u)] \leqq 10/n^2$ for $n \geqq 4$.
For $\pi < u < 2\pi$,

$$\begin{aligned} u^n f(u) &= 1 + \sum_{j=1}^{\infty} D_{j,n}(u) \geqq 1 + D_{1,n}(u) \\ &\geqq 1 + [(2\pi/u) - 1]^{-1} - [(2\pi/u) + 1]^{-n} - [(4\pi/u) - 1]^{-n} \\ &\geqq [(2\pi/u) - 1]^{-n} - 2^{-n} \\ &\geqq [(2\pi/u) - 1]^{-n} - 1. \end{aligned}$$

Hence,

$$(v_n)^n f(v_n) \geqq \left(\frac{n + 2\log n}{n - 2\log n}\right)^n - 1 \geqq n^2 - 1 \geqq \frac{n^2}{2} \qquad (n > 1).$$

Thus, $\psi_n(v_n)/\phi_n(v_n) = \mathrm{Var}_{[v_n, 2\pi]}\,\{\psi_n(u)/\phi_n(u)\} \leqq 2/n^2$.
The proposition follows by combining the above.

1.2. The Operators H_u. The proof of the ℓ^p case, $1 < p < +\infty$, involves some operators related to the Hilbert transform. Since $\mathfrak{L}_n$ maps a discrete space to a continuous one, we need to define a mixed Hilbert transform

$$\mathcal{H}y(x) = \sum{}' \, y_\nu/(x - \nu)$$

where $\sum'$ denotes that the sum is taken over those $\nu \in \mathbf{Z}$ satisfying $x \notin [\nu - 1/2, \nu + 1/2)$.

Proposition 1.3. *The mixed Hilbert transform,* $\mathcal{H}y(x) = \sum' y_\nu/(x - \nu)$, *is a bounded linear transformation from* $\ell^p(\mathbf{Z})$ *into* $L^p(\mathbf{R})$ *for* $1 < p < +\infty$ *i.e.* $\|\mathcal{H}\|_p = B_p < +\infty$.

Proof. Let $\nu(x)$ be the unique integer satisfying $\nu(x) - 1/2 \leqq x < \nu(x) + 1/2$. Then for $\nu \neq \nu(x)$,

$$1/(x - \nu) = 1/(\nu(x) - \nu) + (\nu(x) - \nu)^{-2}[(\nu(x) - x)(\nu(x) - \nu)/(x - \nu)].$$

Hence,

$$\left|\sum{}' y_\nu/(x - \nu)\right| \leqq \left|\sum_{\nu \neq \nu(x)} y_\nu/(\nu(x) - \nu)\right| + C \sum_{\nu \neq \nu(x)} |y_\nu|/(\nu(x) - \nu)^2$$

where C is a constant independent of x and ν. Thus,

$$\begin{aligned} &\left\|\sum{}' y_\nu/(x - \nu)\right\|_{L^p} \\ &\leqq \left\{\int_{-\infty}^{+\infty} \Big(\Big|\sum_{\nu \neq \nu(x)} y_\nu/(\nu(x) - \nu)\Big| + C \sum_{\nu \neq \nu(x)} |y_\nu|/(\nu(x) - \nu)^2\Big)^p \, dx\right\}^{1/p} \\ &\leqq \left\{\sum_{j=-\infty}^{+\infty} \int_{j-1/2}^{j+1/2} \Big(\Big|\sum_{\nu \neq j} y_\nu/(j - \nu)\Big| + C \sum_{\nu \neq j} |y_\nu|/(j - \nu)^2\Big)^p \, dx\right\}^{1/p} \\ &\leq \Big\|\sum_{\nu \neq j} y_\nu/(j - \nu)\Big\|_{\ell^p} + C\Big\|\sum_{\nu \neq j} |y_\nu|/(j - \nu)^2\Big\|_{\ell^p} \\ &\leqq M\, \|y\|_{\ell^p} \end{aligned}$$

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since the discrete Hilbert transform, $\sum_{j\neq\nu} y_\nu/(j-\nu)$, and convolution with $1/j^2$ are bounded linear transformations from $\ell^p(\mathbf{Z})$ to $\ell^p(\mathbf{Z})$, $1 < p < +\infty$.

We also need a family of operators H_u, $0 < u < +\infty$, defined by

$$H_u y(x) = \sum_{\nu=-\infty}^{+\infty} y_\nu[\sin u(x-\nu)]/(x-\nu).$$

The operator $(1/\pi)H_\pi = W$ is the well-known Whittaker operator and $Wy(x) = \sum_{\nu=-\infty}^{+\infty} y_\nu \sin \pi(x-\nu)/\pi(x-\nu)$ is the Whittaker cardinal series. The Whittaker operator enters into our $\ell^p(\mathbf{Z})$ discussion in a fundamental way, and, in fact, Wy becomes the limiting function for the functions $\mathfrak{L}_n y$ in certain instances.

The norms of the operators H_u are related in the following way.

Lemma 1.4.

(a) *Let* $0 < u < +\infty$ *and* k *be an integer such that* $u = k\beta(u)$ *with* $0 < \beta(u) \leqq \pi$. *Then*

$$||H_u||_p \leqq k^{1/p'}\,||H_{\beta(u)}||_p$$

where $1/p + 1/p' = 1$.

(b) $\sup_{0<u\leqq\pi} ||H_u||_p < +\infty$.

Proof. We first prove part (b). Let $0 < u \leqq \pi$, then $|(x-\nu)^{-1} \sin u(x-\nu)| \leqq u$ if $0 < |x-\nu| \leqq \pi/2u$, and, in particular, if $0 < |x-\nu| \leqq 1/2$. This yields

$$\begin{aligned}\left|\sum_{-\infty}^{+\infty} y_\nu[\sin u(x-\nu)]/(x-\nu)\right| \\ &\leqq \left|\sum{}' y_\nu[\sin u(x-\nu)]/(x-\nu)\right| \\ &\quad + |y_{\nu(x)}[\sin u(x-\nu(x))]/(x-\nu(x))| \\ &\leqq \left|\sum{}' y_\nu \cos u\nu/(x-\nu)\right| + \left|\sum{}' y_\nu \sin u\nu/(x-\nu)\right| + u\,|y_{\nu(x)}|\end{aligned}$$

where $\nu(x) - 1/2 \leqq x < \nu(x) + 1/2$. Hence,

$$||H_u y||_{L^p} \leqq 2\,||\mathfrak{H}||_p\,||y||_{\ell^p} + u\,||y||_{\ell^p}$$

where $\mathfrak{H}$ is the mixed Hilbert transform. Part (b) follows immediately.

For the proof of part (a), let $y = \{y_\nu\} \in \ell^p(\mathbf{Z})$ and $g(x) \in L^{p'}(\mathbf{R})$, $1/p + 1/p' = 1$. Then

$$\begin{aligned}\left|\int_{-\infty}^{+\infty} \sum_{\nu=-\infty}^{+\infty} y_\nu[\sin u(x-\nu)/(x-\nu)]g(x)\,dx\right| \\ &= \left|\int_{-\infty}^{+\infty} \sum_{\nu=-\infty}^{+\infty} y_\nu[\sin \beta(u)(kx-k\nu)/(kx-k\nu)]kg(x)\,dx\right| \\ &= \left|\int_{-\infty}^{+\infty} \sum_{\nu=-\infty}^{+\infty} y_\nu[\sin \beta(u)(t-k\nu)/(t-k\nu)]g(t/k)\,dt\right| \\ &= \left|\int_{-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} y_n'[\sin \beta(u)(t-n)/(t-n)]g(t/k)\,dt\right| \\ &\leqq ||H_{\beta(u)}||_p\,||y'||_{\ell^p}\,||g(\cdot/k)||_{L^{p'}},\end{aligned}$$

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where $y' = \{y_n'\}$, $y_n' = y_\nu$ when $k\nu = n$ for some ν and $y_n' = 0$ otherwise. Since $||y'||_{\ell^p} = ||y||_{\ell^p}$ and $||g(\cdot/k)||_{L^{p'}} = k^{1/p'}\, ||g||_{L^p}$ this yields $||H_u||_p \leqq ||H_{\beta(u)}||_p k^{1/p'}$.

§2. $p = 1$. The norms of the cardinal spline interpolation operators as mappings from $\ell^1(\mathbf{Z})$ to $L^1(\mathbf{R})$ satisfy the following asymptotic relation

Theorem 2.1. *The norms of the n^{th} order cardinal spline interpolation operators from $\ell^1(\mathbf{Z})$ to $L^1(\mathbf{R})$ satisfy*

$$\lim_{n\to+\infty} (||\mathfrak{L}_n||_1 - (4/\pi^2)\log n) = (2A/\pi) + (4/\pi^2)[\log(4/\pi) + \gamma] \tag{2.1}$$

where γ is the Euler–Mascheroni constant and

$$A = \int_0^\pi [t^{-1}\tan(t/2) - 2/\pi(\pi - t)]\, dt. \tag{2.2}$$

Proof. The proof closely parallels the proof of Theorem 1 in [5], and, for brevity, we shall refer to that proof whenever possible.

The first step is to obtain a formula for $||\mathfrak{L}_n||_1$. For any $y = \{y_\nu\} \in \ell^1(\mathbf{Z})$,

$$\int_{-\infty}^{+\infty} \left|\sum_{\nu=-\infty}^{+\infty} y_\nu L_n(x-\nu)\right| dx \leqq \int_{-\infty}^{+\infty} |L_n(x)|\, dx \sum_{-\infty}^{+\infty} |y_\nu|.$$

Hence, $||\mathfrak{L}_n||_1 \leqq \int_{-\infty}^{+\infty} |L_n(x)|\, dx$. To obtain equality, consider the sequence $y_0 = 1$, $y_\nu = 0$ $\nu \neq 0$.

A more useful formula involving the functions γ_{n+1} and ϕ_n is the next objective. Define the sequence $z_\nu = \operatorname{sgn}_{0<x<1} L_n(x-\nu)$, $\nu \in \mathbf{Z}$. Then $z_\nu = (-1)^{\nu+1}$ for $\nu \geqq 1$, $z_\nu = (-1)^\nu$ for $\nu \leqq 0$, and $\int_0^1 |L_n(x-\nu)|\, dx = \int_0^1 z_\nu L_n(x-\nu)\, dx$. Using the Fourier transform representation for $L_n(x-\nu)$, we obtain

$$\begin{aligned}\int_0^1 |L_n(x-\nu)|\, dx &= (1/2\pi)\int_0^1 z_\nu \int_{-\infty}^{+\infty} [\psi_n(u)/\phi_n(u)]e^{iux}e^{-iu\nu}\, du\, dx \\ &= (1/2\pi)\int_{-\infty}^{+\infty} [\psi_n(u)/\phi_n(u)][(e^{iu}-1)/iu]z_\nu e^{-iu\nu}\, du \end{aligned} \tag{2.3}$$

where the interchange of order in the integration is valid since $|\psi_n(u)/\phi_n(u)| \leqq \min(1, \pi u^{-n})$. Summing the geometric series to obtain

$$\begin{aligned}\sum_{\nu=-N+1}^{N} z_\nu e^{-iu\nu} &= \sum_{\nu=1}^{N} (-1)^{\nu+1}e^{-iu\nu} + \sum_{\nu=0}^{N+1} (-1)^\nu e^{iu\nu} \\ &= (-1)^{N+1}\{[e^{-iuN} + e^{iuN}] + 2\}/(e^{iu}+1) \\ &= [2/(e^{iu}+1)][1 + (-1)^{N+1}\cos Nu],\end{aligned}$$

and then using this together with (2.3) yields

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$$\sum_{\nu=-N+1}^{N} \int_0^1 |L_n(x-\nu)|\,dx$$

$$= \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\psi_n(u)}{\phi_n(u)}\left(\frac{e^{iu}-1}{e^{iu}+1}\right)\left(\frac{1+(-1)^{N+1}\cos Nu}{iu}\right)du$$

$$= \frac{1}{\pi}\int_{-\infty}^{+\infty} \{\psi_n(u)/\phi_n(u)\}u^{-1}\tan u/2[1+(-1)^{N+1}\cos Nu]\,du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \{\psi_{n+1}(u)/\phi_n(u)\}\sec u/2[1+(-1)^{N+1}\cos Nu]\,du.$$

Breaking this last integration over the intervals $[2\pi j-\pi, 2\pi j+\pi]$, changing variables, and invoking periodicity, we obtain

$$\sum_{\nu=-N+1}^{N} \int_0^1 |L_n(x-\nu)|\,dx$$

$$= \frac{1}{2\pi}\sum_{j=-\infty}^{+\infty}\int_{-\pi}^{\pi} \frac{\psi_{n+1}(u+2\pi j)}{\phi_n(u)}(-1)^j\sec u/2[1+(-1)^{N+1}\cos Nu]\,du$$

$$= (1/2\pi)\int_{-\pi}^{+\pi}[\gamma_{n+1}(u)/\phi_n(u)]\sec u/2\,du$$

$$+ (-1)^{N+1}/2\pi\int_{-\pi}^{+\pi}[\gamma_{n+1}(u)/\phi_n(u)]\sec u/2\cos Nu\,du. \tag{2.4}$$

Since ϕ_n and γ_{n+1} are trigonometric polynomials in the variable $u/2$, $\phi_n(u) > 0$ and $\gamma_{n+1}(\pi) = \gamma_{n+1}(-\pi) = 0$ (see Schoenberg [9] and Richards [5; section 3]), $(\sec u/2)\gamma_{n+1}(u)/\phi_n(u)$ is continuous on $[-\pi, \pi]$. Consequently, the second term on the right in (2.4) tends to zero as $N \to +\infty$, by the Riemann–Lebesgue lemma. Hence, letting $N \to +\infty$,

$$||\mathfrak{L}_n||_1 = \int_{-\infty}^{+\infty}|L_n(x)|\,dx = 1/\pi\int_0^{\pi}[\gamma_{n+1}(u)/\phi_n(u)]\sec u/2\,du. \tag{2.5}$$

As in section 3 of Richards [5], we use Lemma 1.1 to obtain a quantity $||\mathfrak{L}_n||_1^*$, differing very little from $||\mathfrak{L}_n||_1$, but which is easier to estimate. Indeed, let

$$R_n(u) = \frac{\gamma_{n+1}(u)[\psi_n(u)+\psi_n(u-2\pi)]}{\phi_n(u)[\psi_{n+1}(u)-\psi_{n+1}(u-2\pi)]}, \qquad 0 < u < \pi. \tag{2.6}$$

Then, for $n \geqq 3$, there exist points $\xi_n \in (0, \pi)$ such that

$$||\mathfrak{L}_n||_1 = \frac{R_n(\xi_n)}{\pi}\int_0^{\pi}\frac{\psi_{n+1}(u)-\psi_{n+1}(u-2\pi)}{\psi_n(u)+\psi_n(u-2\pi)}\sec u/2\,du$$

where by Lemma 1.1, $|R_n(\xi_n) - 1| < 2^{-n+2}$. Thus, setting

$$||\mathfrak{L}_n||_1^* = (1/\pi)\int_0^{\pi}\frac{\psi_{n+1}(u)-\psi_{n+1}(u-2\pi)}{\psi_n(u)+\psi_n(u-2\pi)}\sec u/2\,du, \tag{2.7}$$

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we have $||\mathfrak{L}_n||_1 - ||\mathfrak{L}_n||_1^* \to 0$ if $||\mathfrak{L}_n||_1^* = o(2^n)$. Hence, we need an estimate for $||\mathfrak{L}_n||_1^*$.

Using the definition of the function ψ_n, equation (1.1), (2.7) becomes

$$||\mathfrak{L}_n||_1^* = (2/\pi) \int_0^\pi \left[\frac{u^{-n-1} - (2\pi - u)^{-n-1}}{u^{-n} + (2\pi - u)^{-n}}\right] \tan u/2 \, du. \tag{2.8}$$

Let $f(u) = \tan u/2 - 2u/\pi(\pi - u)$. Then, (i) $\lim_{u\to\pi} f(u)$ exists and $f \in C[0, \pi]$, and (ii) $f(u) = ug(u)$ where $g \in C[0, \pi]$. Indeed, (i) follows since $\lim_{u\to\pi} (\pi - u) \tan u/2 = \lim_{u\to\pi} 2(\pi - u)u/\pi(\pi - u) = 2$, and (ii) follows since f is holomorphic on $(-\pi, \pi)$, $f(0) = 0$, and $f \in C[0, \pi]$.

Setting

$$I_{1,n} = (2/\pi) \int_0^\pi \left[\frac{1 - [u/(2\pi - u)]^{n+1}}{1 + [u/(2\pi - u)]^n}\right] g(u) \, du = (2/\pi) \int_0^\pi h_n(u) g(u) \, du \tag{2.9}$$

$$I_{2,n} = (4/\pi^2) \int_0^\pi \left[\frac{(1 - 2\pi/u)^{n+1} + (-1)^n}{(1 - 2\pi/u)^n + (-1)^n}\right] \frac{u}{(u - 2\pi)(\pi - u)} \, du, \tag{2.10}$$

we have $||\mathfrak{L}_n||_1^* = I_{1,n} + I_{2,n}$ from (2.8).

Since $|h_n(u)| \leqq 1$, and $h_n(u)$ converges to 1 as $n \to +\infty$ for each t in $[0, \pi)$, we can apply the bounded convergence theorem to obtain

$$\lim_{n\to+\infty} I_{1,n} = (2/\pi) \int_0^\pi g(u) \, du = 2A/\pi \tag{2.11}$$

from (2.9) and (2.2).

In order to estimate $I_{2,n}$, we perform the change of variable, $u \to x = (2\pi/u) - 1$. Then (2.10) becomes

$$I_{2,n} = (8/\pi^2) \int_1^\infty \frac{x^{n+1} - 1}{x^n + 1} \frac{dx}{(x^2 - 1)x}. \tag{2.12}$$

Assume for the time being that $n = 2m$ is even. Simple algebra transforms (2.12) into

$$I_{2,2m} = (8/\pi^2) \int_1^\infty \frac{x^{2m-2} + x^{2m-4} + \cdots + 1}{x^{2m} + 1} \, dx + (8/\pi^2) \int_1^\infty \frac{dx}{(x^{2m} + 1)(x + 1)x}. \tag{2.13}$$

The last integral in (2.13) is clearly $o(1)$ as $m \to +\infty$. To estimate the first integral in (2.13), observe that integrating the same integrand on [0, 1] yields the same value as the integral on $[1, \infty)$, and that the integrand is even. Thus,

$$\begin{aligned} I_{2,2m} &= (2/\pi^2) \int_{-\infty}^{+\infty} \left[\frac{x^{2m-2} + x^{2m+4} + \cdots + 1}{x^{2m}}\right] dx + o(1) \\ &= \frac{2}{\pi} \left\{\sum_{\nu=1}^{m} (1/\pi) \int_{-\infty}^{+\infty} \frac{x^{2m-2\nu}}{x^{2m} + 1} \, dx\right\} + o(1). \end{aligned} \tag{2.14}$$

The integrals in (2.14) were estimated in [5; eq. (4.4) and (4.7)], and applying

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these estimates gives

$$I_{2,2m} = (4/\pi^2) \log 2m + (4/\pi^2) (\log (4/\pi) + 2\gamma_{2m} - \gamma_m) + o(1) \tag{2.15}$$

where $\gamma_n = \sum_{\nu=1}^{n} (1/\nu) - \log n$.

Returning to (2.12), we observe that the integrand is increasing in n. Since $I_{2,2m+2} - I_{2,2m} = o(1)$, equation (2.15) implies the relation

$$I_{2,n} = (4/\pi^2) \log n + (4/\pi^2) (\log (4/\pi) + \gamma) + o(1) \tag{2.16}$$

for all n.

Equations (2.16) and (2.11) imply $||\mathcal{L}_n||_1^* = (2A/\pi) + (4/\pi^2) \log n + (4/\pi^2) [\log (4/\pi) + \gamma] + o(1)$. Consequently, since $||\mathcal{L}_n||_1^* = \mathcal{O} (\log n) = o(2^n)$, the relation also holds for $||\mathcal{L}_n||_1$. The theorem is proved.

§3. $1 < p < +\infty$. The first result of this section is the uniform boundedness of the sequence $||\mathcal{L}_n||_p$.

Theorem 3.1. *The n^{th} order cardinal spline interpolation operator satisfies*

$$||\mathcal{L}_n||_p \leqq 1 + B_p/\pi + \mathcal{O}(2^{-n})$$

$1 < p < +\infty$, *where B_p is the norm of the mixed Hilbert transform.*

Proof. We shall use the converse of Hölder's inequality. Let $y = \{y_\nu\} \in \ell^p(\mathbf{Z})$, $1 < p < +\infty$, with $y_\nu = 0$ for $|\nu| > N_1$, and $g \in L^{p'}(\mathbf{R})$, $1/p + 1/p' = 1$, with the support of g contained in $[-N_2, N_2]$, where N_1, N_2 are arbitrary positive integers. Let $\nu(x)$ be the unique integer such that $\nu(x) - 1/2 \leqq x < \nu(x) + 1/2$. Then

$$\begin{aligned}
&\left| \int_{-\infty}^{+\infty} \mathcal{L}_n y(x) g(x)\, dx \right| \\
&\quad = \left| \int_{-N_2}^{N_2} \sum_{\nu=-N_1}^{N_1} y_\nu L_n(x - \nu) g(x)\, dx \right| \\
&\quad = \left| \int_{-N_2}^{N_2} [y_{\nu(x)} L_n(x - \nu(x)) + \textstyle\sum' y_\nu L_n(x - \nu)] g(x)\, dx \right| \\
&\quad \leqq \left| \int_{-N_2}^{N_2} y_{\nu(x)} L_n(x - \nu(x)) g(x)\, dx \right| + \left| \int_{-N_2}^{N_2} \textstyle\sum' y_\nu L_n(x - \nu) g(x)\, dx \right| \\
&\quad \leqq \left\{ \sum_{-N_1}^{N_1} |y_\nu|^p \right\}^{1/p} ||g||_{L^{p'}} + \left| \int_{-N_2}^{N_2} \textstyle\sum' y_\nu L_n(x - \nu) g(x)\, dx \right|,
\end{aligned} \tag{3.1}$$

since $|L_n(x - \nu(x))| \leqq 1$ on $[\nu(x) - 1/2, \nu(x) + 1/2]$. Using the Fourier transform representation for $L_n(x - \nu)$ and integrating by parts, the second term on the right in (3.1) becomes

$$\frac{1}{2\pi} \left| \int_{-N_2}^{N_2} \int_{-\infty}^{+\infty} \textstyle\sum' [y_\nu/(x - \nu)] e^{i(x-\nu)u}\, d(\psi_n(u)/\phi_n(u)) g(x)\, dx \right| \cdot$$

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Applying Fubini's theorem to this integral yields

$$\left| \int_{-N_2}^{N_2} \sum{}' \, y_\nu L_n(x - \nu) g(x) \, dx \right|$$

$$\leqq \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left| \int_{-N_2}^{N_2} \sum{}' \, [y_\nu/(x - \nu)] e^{-iu\nu} g(x) e^{ixu} \, dx \right| \, |d(\psi_n(u)/\phi_n(u))|$$

$$\leqq \frac{1}{2\pi} \int_{-\infty}^{+\infty} ||\mathcal{H}||_p \, ||\{y_\nu e^{-iu\nu}\}||_{\ell^p} \, ||e^{iu\cdot} g(\cdot)||_{L^{p'}} \, |d(\psi_n(u)/\phi_n(u))|$$

$$\leqq (2\pi)^{-1} B_p \, ||y||_{\ell^p} \, ||g||_{L^{p'}} \, \mathrm{Var}_{(-\infty, +\infty)} \{\psi_n(u)/\phi_n(u)\}. \tag{3.2}$$

Letting $N_2 \to +\infty$, we get (3.1) and (3.2) for every g with compact support. Then letting $N_1 \to +\infty$ gives (3.1) and (3.2) for any sequence of finite length. Combining (3.1), (3.2), the converse of Hölder's inequality and Proposition 1.2(b), we obtain the theorem.

The question arises as to whether $||\mathcal{L}_n||_p$ converges to a limit. Our next theorems suggest that the limit should be $(1/\pi) \, ||H_\pi||_p = ||W||_p$, the norm of the Whittaker operator, although our methods do not give this.

Theorem 3.2. *The sequence of norms of the n^{th} order cardinal spline interpolation operators satisfy*

$$\limsup_{n \to +\infty} ||\mathcal{L}_n||_p \leqq \lim_{n \to +\infty} \sup_{\pi - \epsilon_n \leqq u \leqq \pi + \epsilon_n} (1/\pi) \, ||H_u||_p \, , \tag{3.3}$$

$1 < p < +\infty$, *where* $\epsilon_n = 2\pi \log n/n$.

Proof. Again let $y = \{y_\nu\} \in \ell^p(\mathbf{Z})$, $y_\nu = 0$ for $|\nu| > N_1$, and $g(x) \in L^{p'}(\mathbf{R})$, $1/p + 1/p' = 1$, with the support of g contained in $[-N_2, N_2]$. Then, using the evenness of $\psi_n(u)/\phi_n(u)$, and the Fourier transform representation of $L_n(x - \nu)$, we obtain

$$\left| \int_{-\infty}^{+\infty} \mathcal{L}_n y(x) g(x) \, dx \right|$$

$$= \left| \int_{-N_2}^{N_2} \sum_{\nu=-N_1}^{N_1} y_\nu (1/2\pi) \int_{-\infty}^{+\infty} [\psi_n(u)/\phi_n(u)] \cos u(x - \nu) \, du \; g(x) \, dx \right| \cdot$$

Proceeding as in the previous proof, this leads to

$$\left| \int_{-\infty}^{+\infty} \mathcal{L}_n y(x) g(x) \, dx \right| \leqq (1/\pi) \int_0^\infty ||H_u||_p \, |d(\psi_n(u)/\phi_n(u))| \, ||y||_{\ell^p} \, ||g||_{L^{p'}} \, .$$

Thus,

$$||\mathcal{L}_n||_p \leqq (1/\pi) \sum_{j=0}^{\infty} \int_{2\pi j}^{2\pi j + 2\pi} ||H_u||_p \, |d(\psi_n(u)/\phi_n(u))|. \tag{3.4}$$

Applying Lemma 1.4 with $k = 2(j + 1)$; equation (1.10) and Proposition 1.2(c) to the right side of (3.4), we obtain

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$$
\begin{aligned}
(1/\pi) \sum_{j=0}^{\infty} \int_{2\pi j}^{2\pi j+2\pi} & ||H_u||_p \, |d(\psi_n(u)/\phi_n(u))| \\
& \leqq (1/\pi) \int_0^{2\pi} ||H_u||_p \, |d(\psi_n(u)/\phi_n(u))| \\
& \quad + (1/\pi) \sum_{j=1}^{\infty} \int_{2\pi j}^{2\pi j+2\pi} (2j+2)^{1/p'} \, ||H_{\beta(u)}||_p \, |d(\psi_n(u)/\phi_n(u))| \\
& \leqq (1/\pi) \int_0^{2\pi} ||H_u||_p \, |d(\psi_n(u)/\phi_n(u))| \\
& \quad + \sup_{0 \leqq u \leqq \pi} ||H_u||_p (1/\pi) \sum_{j=1}^{\infty} (2j+2)^{1/p'} \operatorname{Var}_{(2\pi j, 2\pi j+2\pi)} [\psi_n(u)/\phi_n(u)] \\
& = (1/\pi) \int_0^{2\pi} ||H_u||_p \, |d(\psi_n(u)/\phi_n(u))| + O(2^{-n}), \qquad n \geqq 2 \\
& \leqq (1/\pi) \int_{\pi-\epsilon_n}^{\pi+\epsilon_n} ||H_u||_p \, |d(\psi_n(u)/\phi_n(u))| + O(n^{-2}) \\
& \leqq \sup_{\pi-\epsilon_n \leqq u \leqq \pi+\epsilon_n} (1/\pi) \, ||H_u||_p + O(n^{-2}).
\end{aligned}
$$

This together with (3.4) gives the desired conclusion.

It seems reasonable that the right side of (3.3) should be replaced by $(1/\pi) \, ||H_\pi||_p = ||W||_p$. The following convergence result further reinforces the conjecture that $||\mathcal{L}_n||_p$ should converge to $||W||_p$ since an easy argument from it gives $||W||_p \leqq \liminf_{n\to+\infty} ||\mathcal{L}_n||_p$.

Theorem 3.3. *Let* $y = \{y_\nu\} \in \ell^p(\mathbf{Z})$, $1 < p < +\infty$, *then*

$$||\mathcal{L}_n y - Wy||_{L^p} \to 0 \quad \text{as} \quad n \to +\infty.$$

Proof. Since the norms of $\mathcal{L}_n - W$ are uniformly bounded by Theorem 3.1, the Banach–Steinhaus Theorem proclaims that we only need to prove the result for a fundamental set in $\ell^p(\mathbf{Z})$. The sequences $y^j = \{y_\nu{}^j : y_\nu{}^j = 1 \text{ if } \nu = j, y_\nu{}^j = 0 \text{ otherwise}\}$, $j \in \mathbf{Z}$, form such a fundamental system. Thus, we have to show that $||L_n(\cdot - j) - \sin \pi(\cdot - j)/\pi(\cdot - j)||_{L^p} = ||L_n(\cdot) - \sin \pi(\cdot)/\pi(\cdot)||_{L^p} \to 0$ as $n \to +\infty$.

Now $L_n(x) - (\sin \pi x)/\pi x$ converges to zero uniformly (as can easily be seen from their Fourier representations). Moreover, $|L_n(x) - (\sin \pi x)/\pi x| = \mathcal{O}(\min\{1, 1/|x|\})$ independently of n. In fact, integrating the Fourier transform of $L_n(x)$ by parts and using the variation of $\psi_n(u)/\phi_n(u)$, we obtain

$$|L_n(x)| \leqq (1/2\pi \, |x|) \int_{-\infty}^{+\infty} |d(\psi_n(u)/\phi_n(u))| \leqq C/2\pi \, |x|.$$

Therefore, an application of Lebesgue's dominated convergence theorem yields $||L_n(\cdot) - \sin \pi(\cdot)/\pi(\cdot)||_{L^p} \to 0$ as $n \to +\infty$. The proof is complete.

We now turn to the question of characterizing the class of functions obtainable as L^p-limits of the cardinal spline interpolation operators on ℓ^p data. For the case $p = +\infty$, the sufficient condition of Schoenberg, stated in the introduction, has been complemented by a necessary result (Richards and Schoenberg [6])

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but a sizeable gap remains. For $1 < p < +\infty$, our previous theorem and known theory allow us to characterize this class completely.

Theorem 3.4. *The following classes of functions are equivalent when* $1 < p < +\infty$:

(i) $L^p(\mathbf{R}) \cap \{f: f(x) = \int_{-\pi}^{\pi} e^{ixt}\, d\beta(t),\ \beta \in \text{B.V.}\ [-\pi, \pi]\}$

(ii) $\{f: f(x) = Wy(x),\ y = \{y_\nu\} \in \ell^p(\mathbf{Z})\}$

(iii) $\{f: f(x) = L^p - \lim_{n\to+\infty} \mathcal{L}_n y(x),\ y = \{y_\nu\} \in \ell^p(\mathbf{Z})\}$.

Proof. The equivalence of (ii) and (iii) follows from Theorem 3.3. The fact that the class (ii) is contained in the class (i) follows from a result of Whittaker [12, p. 68].

Suppose that $f \in L^p(\mathbf{R})$ and has the representation in (i). Then Theorem 6.7.15 of [1, p. 101] shows that $\sum_{\nu=-\infty}^{+\infty} |f(\nu)|^p < +\infty$. If $Wf(x) = \sum_{\nu=-\infty}^{+\infty} f(\nu) \sin \pi(x - \nu)/\pi(x - \nu)$, then we know that Wf also belongs to the class (i). Since $Wf(\nu) = f(\nu) = o(1)$, Theorem 10.11 of [13, p. 145] gives that $f(x) - Wf(x) = o(1)$. Finally, Corollary 9.4.2 in [1, p. 156] implies $f(x) = Wf(x)$, and the proof is complete.

Corollary 3.5. *If* $f \in L^p(\mathbf{R})$, $1 < p < +\infty$, $f(x) = \int_{-\pi}^{\pi} e^{ixt} d\beta(t)$, $\beta \in$ B.V. $[-\pi, \pi]$, *and* $y_f = \{f(\nu)\}$, *then*

$$\|f - \mathcal{L}_n y_f\|_{L^p} \to 0 \quad as \quad n \to +\infty.$$

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University of Pittsburgh
University of Alberta

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