B-splines

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I B-splines

I.1 Motivation

In the problem of interpolating data, may it be values or derivatives of some unknown function, the intuitive method consisting in interpolating all the data at once is prone to large errors when the data exceeds a few points. This is the consequence of resorting to high order polynomials which tend to oscillate a lot when forced to perfectly interpolate the data. A *natural* way of approaching problems deemed too hard consist in splitting them into several subproblems of lesser complexity then solving each of these before putting solutions of all subproblems together. In our case, the less difficult problems are that of interpolating a subset of the data and then joining interpolating pieces together so that the global solution interpolates all the data. The price to pay for doing so is the decrease in the smoothness of the interpolating function at the break points we will have chosen. As we will see later, the maximum degree of smoothness achievable is a decreasing function of the number of derivatives we try to interpolate. In polynomial interpolation, the subproblems are that of interpolating on subintervals with small degree polynomials so that the general solution belongs to the space of piecewise polynomial functions of given order defined as follows

Definition 1. *Let k an integer and $\boldsymbol{\xi}$ a strictly increasing sequence of real numbers. The set of all piecewise polynomials of order k with breaks at $\boldsymbol{\xi}$ is denoted $\Pi_{<k,\boldsymbol{\xi}}$. It consists in all functions that are polynomials of order k on all intervals (ξ_i, ξ_{i+1}). The elements of $\boldsymbol{\xi}$ are called* knots.

For the needs of further results, let's introduce here the subspaces where a certain number of derivatives are made be continuous at the knots.

Definition 2. *Let k an integer, $\boldsymbol{\xi}$ a strictly increasing sequence of real numbers and $\boldsymbol{\nu}$ a sequence of nonnegative integers. Define*

$$\Pi_{<k,\boldsymbol{\xi},\boldsymbol{\nu}} = \{f \in \Pi_{<k,\boldsymbol{\xi}} \mid jump_{\xi_i} D^{j-1} f = 0, j = 1, \ldots, \nu_i\} \tag{1}$$

The maximum degree of continuity achievable at a knot is the order of the polynomials on each side of the knots, k in our notation. Indeed, in the case $\nu_i = k$, writing the polynomials in their Taylor expansion at ξ_i up to order k we see that both polynomials share the same coefficients in the expansion. Consequently polynomials on each side of the knot join *perfectly* in the sense that they are subparts of the same polynomial of order k. The value $\nu_i = 0$ means we impose no continuity condition at the knot ξ_i. If $\boldsymbol{\nu_1}, \boldsymbol{\nu_2}$ are two sequences such that $\boldsymbol{\nu_1} \leq \boldsymbol{\nu_2}$ it follows from the definition that $\Pi_{<k,\boldsymbol{\xi},\boldsymbol{\nu_2}} \subset \Pi_{<k,\boldsymbol{\xi},\boldsymbol{\nu_1}}$.

I.2 Definitions

Let $\boldsymbol{t}$ a sequence of nondecreasing real numbers. This sequence can be finite on both sides or only one side or it can be biinfinite. In his book ([Bo01]), De Boor defines *normalized* B-splines using the divided difference operator as follows

Definition 3. *The j^{th}* normalized *B-spline of order k is*

$$B_{j,k,\boldsymbol{t}}(t) = (t_{j+k} - t_j)[t_j, \ldots, t_{j+k}](\cdot - t)_+^{k-1} \tag{2}$$

In order to ligthen notations we will usually drop the dependence in the knots sequence $\boldsymbol{t}$ when it is clear from the context what these knots are. In the definition above we adopt the convention that $0^0 = 0$ which has the consequence of making our B-splines right-continuous.

Example 1. • $\underline{k = 1}$

$$\begin{aligned} B_{j,1}(t) &= (t_{j+1} - t_j)[t_j, t_{j+1}](\cdot - t)^0_+ \\ &= (t_{j+1} - t)^0_+ - (t_j - t)^0_+ \\ &= \begin{cases} 1 & t_j \leq t < t_{j+1} \\ 0 & \textit{elsewhere} \end{cases} \end{aligned}$$

• $\underline{k = 2}$

$$\begin{aligned} B_{j,2}(t) &= (t_{j+2} - t_j)\frac{[t_{j+1}, t_{j+2}](\cdot - t)^1_+ - [t_j, t_{j+1}](\cdot - t)^1_+}{t_{j+2} - t_j} \\ &= \frac{(t_{j+2} - t)^1_+ - (t_{j+1} - t)^1_+}{t_{j+2} - t_{j+1}} - \frac{(t_{j+1} - t)^1_+ - (t_j - t)^1_+}{t_{j+1} - t_j} \\ &= \begin{cases} \dfrac{t - t_j}{t_{j+1} - t_j} & t_j \leq t < t_{j+1} \\ \dfrac{t_{j+2} - t}{t_{j+2} - t_{j+1}} & t_{j+1} \leq t < t_{j+2} \\ 0 & \textit{elsewhere} \end{cases} \end{aligned}$$

For the needs of incoming properties we need to define the **basic interval** $I_{k,\boldsymbol{t}}$ as follows

$$I_{k,\boldsymbol{t}} = (t_-, t_+), \quad t_- := \begin{cases} t_k & \text{if } \boldsymbol{t} = (t_1, \ldots) \\ \inf t_j & \text{otherwise} \end{cases}, \quad t_+ := \begin{cases} t_{n+1} & \text{if } \boldsymbol{t} = (\ldots, t_{n+k}) \\ \sup t_j & \text{otherwise} \end{cases}, \tag{3}$$

These B-splines are normalized in the sense that they satisfy $\sum_j B_{j,k,\boldsymbol{t}} = 0$ on $I_{k,\boldsymbol{t}}$. The definition De Boor uses is different from that previously used by Curry and Schoenberg in ([CS66]) which is the following

Definition 4. *The j^{th} B-spline of Curry and Schoenberg is*

$$M_{j,k,\boldsymbol{t}}(t) = k[t_j, \ldots, t_{j+k}](\cdot - t)^{k-1}_+ \tag{4}$$

These B-splines are closely related to the previously defined B-splines as $B_{j,k,\boldsymbol{t}} = \frac{t_{j+k} - t_j}{k} M_{j,k,\boldsymbol{t}}$. In the following paragraphs we will focus on the properties of $B_{j,k,\boldsymbol{t}}$ as they can be written more elegantly than with $M_{j,k,\boldsymbol{t}}$. However they can easily be translated in terms $M_{j,k,\boldsymbol{t}}$ given the relation above. For the sake of completeness let at least mention that $M_{j,k,\boldsymbol{t}}$ have the property of being kernels for the divided difference operator ([CS66], eq(1.5)) in the sense that for any function $f \in \mathcal{C}^{(k)}$ we have

$$[t_j, \ldots, t_{j+k}]f = \frac{1}{k!}\int_{t_j}^{t_{j+k}} M_{j,k,\boldsymbol{t}} f^{(k)}(t)dt$$

I.3 Properties

References

[Bo01] Carl de Boor. *A practical guide to splines*. Revised edition. Springer, 2001.

[CS66] H.B Curry and I.J Schoenberg. “On Pólya frequency functions IV: the fundamental spline functions and their limits”. In: *J. Analyse Math.* 17 (1966), pp. 71–107.