Summary paper 14: Cardinal exponential splines: Part I

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This paper by M. Unser and T. Blu is part of a larger research program that is intent on bridging the gap between continuous and discrete signal processing. In that matter, polynomial splines are of high interest although the amount of research for that specific issue remains limited as polynomial splines found to be advantageous elsewhere, notably in the context of high-quality interpolation. In the paper authors consider a larger class of splines, namely **exponential splines** with a restriction to the case of cardinal splines as these are the most suited for signal processing, lending themselves to very efficient digital filtering techniques while using concepts and algorithms familiar to other communities as the "signal-and-systems" community.

The mathematical concepts involved in the paper include Fourier analysis, generalized functions, Green's function of ordinary differential operators and z-transforms. As I am not so familiar with the latter, here is a short topo on the subject.

1 Preliminaries

Topo on Z-transform

The Z-transform converts a discrete-time signal into a complex frequency-domain representation. Can be viewed discrete-time equivalent of (unilateral) Laplace transform $\mathcal{L}f(s) = F(s) = \int_0^\infty f(t)e^{-st}dt$. The (bilateral) Z-transform of a discrete time signal $(a[k])_k$ is

$$A(z) = \sum_{k=-\infty}^{\infty} a[k]z^{-k}$$

The region of convergence (ROC) is the set of points where the summation converges i.e

$$\text{ROC} = \left\{ z : \left| \sum_{k=-\infty}^{\infty} a[k]z^{-k} \right| < \infty \right\}$$

The inverse Z-transform is

$$a[k] = \frac{1}{2\pi j} \int_C A(z)z^{k-1}dz$$

with C a counterclockwise path encircling the origin and entirely in the ROC. In case the ROC is causal, C must enclose all poles of $A(z)$. In case where C is the unit circle, the inverse Z-transform simplifies into the inverse discrete-time Fourier transform i.e

$$a[k] = \frac{1}{2\pi} \int_0^{2\pi} A(e^{jw})e^{jwk}dw$$

<u>Examples</u>

1. $a[k] = \frac{1}{2^k}$ for $k \in \mathbb{Z}$. Then ROC$=\emptyset$.

2. $a[k] = \frac{1}{2^k}$ for $k \in \mathbb{Z}_+$. Then $\sum_{k=0}^{\infty} a[k]z^{-k} = \frac{1}{1-\frac{1}{2z}}$ and ROC$=\{z : |z| > \frac{1}{2}\}$. The ROC is causal.

3. $a[k] = \frac{1}{2^k}$ for $k \in \mathbb{Z}_-$. Then $\sum_{k=0}^{\infty} a[k]^{-1} z^k = \frac{1}{1-\frac{1}{2z}}$ and ROC=$\{z : |z| < \frac{1}{2}\}$. The ROC is anti-causal.

In case the ROC includes neither $z = 0$ nor $|z| = \infty$, we say that the system is mixed-causality. The Z-transform of δ is 1, that of $a[k - k_0]$ is $A(z)z^{-k_0}$. A lot more properties can be found on wiki's page.

Preliminaries of article

Generic differential operator of order N

$$Lf = D^N f + a_{N-1} D^{N-1} f + \cdots + a_0 I f$$

$L = L_\alpha$ with α the roots of the characteristic polynomial. The "Fourier transform" of the operator L_α is

$$L_\alpha(jw) = \prod_{n=1}^{N} (jw - \alpha_n)$$

Meaning that $\widehat{L_\alpha f}(w) = L_\alpha(jw)\hat{f}(w)$. Let's try to clarify that. Let L be a continuous LSI operator from $\mathcal{S} \to \mathcal{S}'$. From Schwartz kernel theorem we know there exists a (generalized) function $h_L \in \mathcal{S}'$ such that

$$Lf = h_L * f \quad \text{meaning} \quad Lf(t) = \langle f(t - .), h_L \rangle$$

h_L is related to L by $h_L = L\delta$. In case $L = D$ for instance, $h_L = D\delta$ i.e it is the distribution that associates to any test function ψ the scalar $-\psi'(0)$. We can now use the properties of the Fourier transform directly on this convolution as follows

$$\widehat{Lf}(w) = \hat{h}_L(w)\hat{f}(w)$$

Are we sure though that $\hat{h}_L(w)$ can always be interpreted as a usual function? In case $L = D$ note for instance that $\hat{h}_L$ is the distribution that associates to each ψ the quantity $\int jw\psi(w)dw = -\hat{\psi}'(0)$ that is $\hat{h}_L$ can be intepreted as the conventional function $\hat{h}_L(w) = jw$.

Why is the nullspace of L_α of dimension N. For that note the following Cauchy theorem

Theorem Cauchy. *Let E a finite-dimensional vector space on $\mathbb{K}$ real or complex field. The differential system*

$$X'(t) = A(t)X(t) + B(t)$$

with $A \in \mathcal{C}(I, \mathcal{L}(E)), B \in \mathcal{C}(I, E)$ is such that

1. *for any $(t_0, X_0) \in I \times E$, $\exists!$ solution to the system that satisfies $X(t_0) = X_0$.*
2. *for any $t_0 \in I$, the map $\phi : X \in S_0 \mapsto X(t_0) \in E$ is an isormorphism between $\mathbb{K}$-vector spaces.*

$$\begin{aligned} y \in S &\iff y \text{ solution to } y^{(N)} + a_{N-1}y^{(N-1)} + \cdots + a_0 y = 0 \\ &\iff Y = \begin{bmatrix} y \\ \vdots \\ y^{(N-1)} \end{bmatrix} \text{ solution to } Y' = AY \end{aligned}$$

with $A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -a_{N-1} & -a_{N-2} & -a_{N-3} & \cdots & -a_0 \end{bmatrix}$

In link with the formulation in the theorem, $A \in \mathcal{C}(\mathbb{R}, \mathcal{L}(\mathbb{C}^N))$ and therefore the space of solutions to $Y' = AY$, T, is of dimension N. However S and T are isomorphic given the equivalence above, hence $\dim S = N$.

Definition 1. *An exponential spline with parameter α and knots $-\infty < \cdots < t_k < t_{k+1} < \cdots < \infty$ is a function $s(t)$ such that*

$$L_\alpha\{s(t)\} = \sum_k a_k \delta(t - t_k)$$

The space spanned by exponential polynomials is shift-invariance. Specifically, for any shift τ, we have

$$(t-\tau)^n e^{\alpha(t-\tau)} = \sum_{m=1}^{n} a_{\tau,m} t^m e^{\alpha t}$$

2 Cardinal exponential splines

Exponential B-splines are localized, shortest-possible version of Green's functions that generate the exponential splines. The first-order B-splines with parameter α is

$$\beta_\alpha(t) = \rho_\alpha(t) - e^\alpha \rho_\alpha(t-1) \tag{1}$$

B-splines are always well-defined and compactly supported. Changing the sign of α has the effect

$$\beta_{-\alpha}(t) = \left(\prod_{i=1}^{N} e^{-\alpha_i}\right) \beta_\alpha(-t+N)$$

In case component of α can be grouped into opposite signs pair, β_α is symmetric wrt its center line.

Theorem 1. *The set of functions $\{\beta_\alpha(t-k)\}_{k\in\mathbb{Z}}$ provides a Riesz-basis of V_α iif $\alpha_n - \alpha_m \notin 2\pi\mathbb{Z}$ for all pairs of distinct, purely imaginary roots.*

There are two important aspects to this theorem: completeness and stability. Formally we can reconstruction the Green function by inverting the relation $\beta_\alpha = \Delta_\alpha\{\rho_\alpha\}$ leading to

$$\rho_\alpha(t) = \sum_{k=0}^{\infty} p_\alpha[k]\beta_\alpha(t-k)$$

with $p_\alpha[k]$ explicitly computable. In the first-order case extended to the negative domain we have

$$e^{\alpha t} = \sum_{k=-\infty}^{\infty} e^{\alpha k}\beta_\alpha(t-k)$$

Proposition 2. *Let φ_α a function that reproduces exponential polynomials in $\mathcal{N}_{(\alpha,\ldots,\alpha)}$. Then for any function φ such that $\int \varphi(t)e^{-\alpha t}dt \neq 0$, $\varphi * \varphi_\alpha$ also reproduce these exponential polynomials.*

Proof. (Theorem 1) The completeness is a consequence of the reproduction properties discussed above. The stability result is more tricky. The upper Riesz bound is easily obtained for compactly supported propototype as we have

$$\begin{aligned}\sum_k |\beta_\alpha(w+2k\pi)|^2 &= \sum_k \langle \beta_\alpha, \beta_\alpha(.-k)\rangle e^{-jwk} \\ &\leq \sum_{k\in\mathbb{Z}} |\langle \beta_\alpha, \beta_\alpha(.-k)\rangle| \\ &< \infty\end{aligned}$$

More difficult is to prove the existence of a lower Riesz bound. Since $A_\alpha(w)$ is continuous and 2π-periodic, we have a lower Riesz bound if and only if $A_\alpha(w) > 0$ on $[0, 2\pi]$ (in that particular case our vector space is generated by a single prototype which makes the Gram matrix a scalar) that is $\bigcap_{k\in\mathbb{Z}} Z_{k,\alpha} = \emptyset$. The author says when $\Re(\alpha_n) = 0$ there is always one $Z_{k,\alpha}$ that is empty **which leads to an empty intersection**?? □

Proposition 3. *The upper Riesz bound can be estimated by*

$$\sup_{w\in\mathbb{R}} |\hat{\beta}_\alpha(w)| \leq R_\alpha \leq \frac{M_\alpha}{\sqrt{\max_{1\leq n\leq N} M_{-|\alpha_n|}}}$$

Proposition 4. *If the roots are such that $\Im\alpha_n - w_0 \in (-\pi, \pi)$ for all $1 \leq n \leq N$, then we have the following lower Riesz bound estimate*

$$r_\alpha \geq M_\alpha \prod_{n=1}^{N} \frac{2\cos(\frac{\Im\alpha_n - w_0}{2})}{\pi + |\Im\alpha_n - w_0|}$$

Theorem 2. *Let $f \in L_2$ be a function such that $D^N f \in L_2$ and let $P_T f$ denote its orthogonal projection onto the spline space $V_{\alpha,T}$ at scale T. Then,*

$$\|f - P_T f\|_{L_2} = C_N T^N \|L_\alpha f\|_{L_2} \text{ as } T \to 0$$