Parametrisation of the sphere

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1 Cardinal Hermite exponential splines

1.1 The parametric model

Conti et al's paper Ellipse-preserving interpolation and subdivision scheme introduces two basis functions from the space $\mathcal{E}_4 =< 1, x, e^{-iw_1 x}, e^{iw_1 x} >$ where $w = \frac{2\pi}{M}$ to reproduce closed curves with M control points. The corresponding parametric representation is

$$r(t) = \sum_{k\in\mathbb{Z}} r(k)\phi_{1,w}(t-k) + r'(k)\phi_{2,w}(t-k) \tag{1}$$

with r and r' assumed to be M-periodic.

The basis functions are **cycloidal splines** (Exponential splines? Exponential B-splines?) given by

$$\phi_{1,w}(x) = \begin{cases} g_{1,w}(x) & \text{for } x \geq 0 \\ g_{1,w}(-x) & \text{for } x < 0 \end{cases} \quad \phi_{2,w}(x) = \begin{cases} g_{2,w}(x) & \text{for } x \geq 0 \\ -g_{2,w}(-x) & \text{for } x < 0 \end{cases} \tag{2}$$

The resulting parametric model has the following properties

1. Unique and stable representation ($\{\phi_{\mathbf{w}}(. - \mathbf{k}) = (\phi_{1,w}(. - k), \phi_{2,w}(. - k))\}_k$ Riesz basis)
2. Affine invariance (partition unity condition on ϕ_1)
3. Perfectly reproduce sinusoids of period M
4. Exact interpolation of points and first derivative
5. Support of ϕ_1, ϕ_2 is $[-1, 1]$
6. Hermite interpolation property of order 1
7. C^1-continuous

1.2 The unit sphere with scaling factors w_1, w_2

The usual continuous representation of the sphere is given by

$$\sigma(u, v) = (\cos(2\pi u)\sin(\pi v), \sin(2\pi u)\sin(\pi v), \cos(\pi v)) \quad (u, v) \in [0, 1]^2 \tag{3}$$

Suppose we have M_1 control points on latitudes, M_2 control points on meridians. The control points are then $c[k, l]_{k\in[0,\ldots,M_1-1], l\in[0,\ldots,M_2-1]}$. Let $w_1 = \frac{2\pi}{M_1}, w_2 = \frac{\pi}{M_2}$.

From the paper we have (also holds for sin functions)

$$\forall u \in [0, M_1] \quad \cos(w_1 u) = \sum_{k\in\mathbb{Z}} \cos(w_1 k)\phi_{1,w_1}(u-k) - w_1 \sin(w_1 k)\phi_{2,w_1}(u-k)$$
$$\forall v \in [0, 2M_2] \quad \cos(w_2 v) = \sum_{l\in\mathbb{Z}} \cos(w_2 l)\phi_{1,w_2}(v-l) - w_2 \sin(w_2 l)\phi_{2,w_2}(v-l)$$

Normalizing the the continuous parameters leads to

$$\forall u \in [0,1] \quad \cos(2\pi u) = \sum_{k\in\mathbb{Z}} \cos(w_1 k)\phi_{1,w_1}(M_1 u - k) - w_1 \sin(w_1 k)\phi_{2,w_1}(M_1 u - k)$$
$$\forall v \in [0,2] \quad \cos(\pi v) = \sum_{l\in\mathbb{Z}} \cos(w_2 l)\phi_{1,w_2}(M_2 v - l) - w_2 \sin(w_2 l)\phi_{2,w_2}(M_2 v - l)$$

Be aware that in the first representations above $\{\cos(w_1 k), -w_1 \sin(w_1 k)\}$ is (M_1, M_1)-periodic i.e we need point and first derivative values at M_1 control points for a full representation. However in the second representation $\{\cos(w_2 l), -w_2 \sin(w_2 l)\}$ are $(2M_2, 2M_2)$-periodic i.e we need point and first derivative values at $2M_2$ control points for a full representation.

1.3 Representation of the sphere

For all $(u,v) \in [0,1]^2$

$$\begin{aligned}
\sigma(u,v) = & \sum_{(k,l)\in\mathbb{Z}^2} c_1[k,l]\phi_{1,w_1}(M_1 u - k)\phi_{1,w_2}(M_2 v - l) \\
& + \sum_{(k,l)\in\mathbb{Z}^2} c_2[k,l]\phi_{1,w_1}(M_1 u - k)\phi_{2,w_2}(M_2 v - l) \\
& + \sum_{(k,l)\in\mathbb{Z}^2} c_3[k,l]\phi_{2,w_1}(M_1 u - k)\phi_{1,w_2}(M_2 v - l) \\
& + \sum_{(k,l)\in\mathbb{Z}^2} c_4[k,l]\phi_{2,w_1}(M_1 u - k)\phi_{2,w_2}(M_2 v - l)
\end{aligned}$$

Or equivalently for all $(u,v) \in [0,1]^2$

$$\begin{aligned}
\sigma(u,v) = & \sum_{k=0}^{M_1}\sum_{l=0}^{M_2} c_1[k,l]\phi_{1,w_1}(M_1 u - k)\phi_{1,w_2}(M_2 v - l) \\
& + \sum_{k=0}^{M_1}\sum_{l=0}^{M_2} c_2[k,l]\phi_{1,w_1}(M_1 u - k)\phi_{2,w_2}(M_2 v - l) \\
& + \sum_{k=0}^{M_1}\sum_{l=0}^{M_2} c_3[k,l]\phi_{2,w_1}(M_1 u - k)\phi_{1,w_2}(M_2 v - l) \\
& + \sum_{k=0}^{M_1}\sum_{l=0}^{M_2} c_4[k,l]\phi_{2,w_1}(M_1 u - k)\phi_{2,w_2}(M_2 v - l)
\end{aligned}$$

Or equivalently for all $(u,v) \in [0,1]^2$

$$\begin{aligned}
\sigma(u,v) = & \sum_{k=0}^{M_1-1}\sum_{l=0}^{M_2} c_1[k,l]\phi_{1,w_1,per}(M_1 u - k)\phi_{1,w_2}(M_2 v - l) \\
& + \sum_{k=0}^{M_1-1}\sum_{l=0}^{M_2} c_2[k,l]\phi_{1,w_1,per}(M_1 u - k)\phi_{2,w_2}(M_2 v - l) \\
& + \sum_{k=0}^{M_1-1}\sum_{l=0}^{M_2} c_3[k,l]\phi_{2,w_1,per}(M_1 u - k)\phi_{1,w_2}(M_2 v - l) \\
& + \sum_{k=0}^{M_1-1}\sum_{l=0}^{M_2} c_4[k,l]\phi_{2,w_1,per}(M_1 u - k)\phi_{2,w_2}(M_2 v - l)
\end{aligned}$$

Or equivalently for all $(u,v) \in [0,1]^2$

$$\begin{aligned}
\sigma(u,v) = & \sum_{(k,l)\in[0,\dots,M_1-1]\times[0,\dots,2M_2-1]} c_1[k,l]\phi_{1,w_1,per}(M_1u-k)\phi_{1,w_2,per}(M_2v-l) \\
+ & \sum_{(k,l)\in[0,\dots,M_1-1]\times[0,\dots,2M_2-1]} c_2[k,l]\phi_{1,w_1,per}(M_1u-k)\phi_{2,w_2,per}(M_2v-l) \\
+ & \sum_{(k,l)\in[0,\dots,M_1-1]\times[0,\dots,2M_2-1]} c_3[k,l]\phi_{2,w_1,per}(M_1u-k)\phi_{1,w_2,per}(M_2v-l) \\
+ & \sum_{(k,l)\in[0,\dots,M_1-1]\times[0,\dots,2M_2-1]} c_4[k,l]\phi_{2,w_1,per}(M_1u-k)\phi_{2,w_2,per}(M_2v-l)
\end{aligned}$$

$$c_1[k,l] = \begin{bmatrix} \cos(w_1k)\sin(w_2l) \\ \sin(w_1k)\sin(w_2l) \\ \cos(w_2l) \end{bmatrix} = \sigma(w_1k, w_2l) \qquad c_2[k,l] = \begin{bmatrix} w_2\cos(w_1k)\cos(w_2l) \\ w_2\sin(w_1k)\cos(w_2l) \\ -w_2\sin(w_2l) \end{bmatrix} = \frac{\partial\sigma}{\partial v}(w_1k, w_2l)$$

$$c_3[k,l] = \begin{bmatrix} -w_1\sin(w_1k)\sin(w_2l) \\ w_1\cos(w_1k)\sin(w_2l) \\ 0 \end{bmatrix} = \frac{\partial\sigma}{\partial u}(w_1u, w_2v) \quad c_4[k,l] = \begin{bmatrix} -w_1w_2\sin(w_1k)\cos(w_2l) \\ w_1w_2\cos(w_1k)\cos(w_2l) \\ 0 \end{bmatrix} = \frac{\partial^2\sigma}{\partial u\partial v}(w_1u, w_2v)$$

$$\phi_{1,w_1,per}(.) = \sum_{k\in\mathbb{Z}} \phi_{1,w_1}(.-M_1k) \qquad \phi_{1,w_2,per}(.) = \sum_{k\in\mathbb{Z}} \phi_{1,w_2}(.-2M_2k)$$

$$\phi_{2,w_1,per}(.) = \sum_{k\in\mathbb{Z}} \phi_{2,w_1}(.-M_1k) \qquad \phi_{2,w_2,per}(.) = \sum_{k\in\mathbb{Z}} \phi_{2,w_2}(.-2M_2k)$$

2 Exponential B-splines in 3D

2.1 The parametric model

Delgado et al's paper Spline-based deforming ellipsoids for 3D bioimage segmentation derive an exponential B-splines-based model that allow to reproduce ellipsoids. The model can well approximate blobs and perfectly spheres and ellipsoids. The corresponding parametric representation is

$$\sigma(u,v) = \sum_{(i,j)\in\mathbb{Z}^2} c[i,j]\phi_1(\frac{u}{T_1}-i)\phi_2(\frac{v}{T_2}-j) \tag{4}$$

with $T_1, T_2 > 0$ sampling steps for each parametric dimension and $\{c[i,j]\}_{(i,j)\in\mathbb{Z}^2}$ are the 3D control points.

The basis functions, reproducing unit period sinusoids with M coefficients, are exponential B-splines given by

$$\varphi_M(.) = \sum_{k=0}^{3} (-1)^k h_M[k]\varsigma_M(.+\frac{3}{2}-k) \tag{5}$$

where $\varsigma_M(.) = \frac{1}{4}sgn(.)\frac{\sin^2(\frac{\pi}{M}.)}{\sin^2(\frac{\pi}{M})}$ and $h_M = [1, 1+2\cos(\frac{2\pi}{M}), 1+2\cos(\frac{2\pi}{M}), 1]$.

Suppose we have M_1 control points on latitudes, M_2 control points on meridians. The resulting parametric model has the following properties

1. Unique and stable representation (sufficient is $\{\phi_1(.-k)\}_k, \{\phi_2(.-k)\}_k$ Riesz basis)
2. Affine invariance (partition unity condition on ϕ_1, ϕ_2)

3. Well-defined Gaussian curvature. ϕ_1, ϕ_2 are twice differentiable with bounded second derivative

4. Perfectly reproduce ellipsoids

5. Support of $\phi_1 = \varphi_{M_1}, \phi_2 = \varphi_{2M_2}$ is $[-\frac{3}{2}, \frac{3}{2}]$

2.2 Conditions for representing the unit sphere

The parametric representation of a closed surface with sphere-like topology, M_1 control points on latitudes and M_2 control points on meridians is

$$\forall(u,v) \in [0,1]^2 \quad \sigma(u,v) = \sum_{(i,j)\in\mathbb{Z}^2} c[i,j]\phi_1(M_1u-i)\phi_2(M_2v-j) \tag{6}$$

Unlike before, continuity of points and tangents at poles is not guaranteed. Incorporating such conditions in the model, a parametric splines-based surface with a sphere-like topology, C^1 continuity and ellipsoid-reproducing capabilities (all positions and orientations) is given by

$$\forall(u,v) \in [0,1]^2 \quad \sigma(u,v) = \sum_{i=0}^{M_1-1}\sum_{j=-1}^{M_2+1} c[i,j]\phi_{1,per}(M_1u-i)\phi_2(M_2v-j) \tag{7}$$

where $\{c[i,j]\}_{i\in[0,\ldots,M_1-1],j\in[1,\ldots,M_2-1]}, \mathbf{T_{1,N}}, \mathbf{T_{2,N}}, \mathbf{T_{1,S}}, \mathbf{T_{1,S}}, \mathbf{c_N}, \mathbf{c_S}$ are free parameters that is $M_1(M_2-1)+6$ control points.

$c[i,-1], c[i,0], c[i,M_2], c[i,M_2+1]$ are constrained by the values of the free parameters.

2.3 Representation of the sphere

The unit sphere is thus represented by

$$\forall(u,v) \in [0,1]^2 \quad \sigma(u,v) = \sum_{i=0}^{M_1-1}\sum_{j=-1}^{M_2+1} c[i,j]\phi_{1,per}(M_1u-i)\phi_2(M_2v-j) \tag{8}$$

With coefficients are given by

$$c[i,j] = \begin{bmatrix} c_{M_1}[i]s_{2M_2}[j] \\ s_{M_1}[i]s_{2M_2}[j] \\ c_{2M_2}[j] \end{bmatrix} = \begin{bmatrix} \frac{2(1-\cos(\frac{2\pi}{M_1}))}{\cos(\frac{\pi}{M_1})-\cos(\frac{3\pi}{M_1})}\frac{2(1-\cos(\frac{\pi}{M_2}))}{\cos(\frac{\pi}{2M_2})-\cos(\frac{3\pi}{2M_2})}\cos(\frac{2\pi i}{M_1})\sin(\frac{\pi j}{M_2}) \\ \frac{2(1-\cos(\frac{2\pi}{M_1}))}{\cos(\frac{\pi}{M_1})-\cos(\frac{3\pi}{M_1})}\frac{2(1-\cos(\frac{\pi}{M_2}))}{\cos(\frac{\pi}{2M_2})-\cos(\frac{3\pi}{2M_2})}\sin(\frac{2\pi i}{M_1})\sin(\frac{\pi j}{M_2}) \\ \frac{2(1-\cos(\frac{\pi}{M_2}))}{\cos(\frac{\pi}{2M_2})-\cos(\frac{3\pi}{2M_2})}\cos(\frac{\pi j}{M_2}) \end{bmatrix} \tag{9}$$

and

$$c_M[k] = \frac{2(1-\cos(\frac{2\pi}{M}))}{\cos(\frac{\pi}{M})-\cos(\frac{3\pi}{M})}\cos(\frac{2\pi k}{M})$$
$$s_M[k] = \frac{2(1-\cos(\frac{2\pi}{M}))}{\cos(\frac{\pi}{M})-\cos(\frac{3\pi}{M})}\sin(\frac{2\pi k}{M})$$

These coefficients satisfy the constraints with

$$\mathbf{c_N} = [0\ 0\ 1]^T \qquad \mathbf{c_N} = [0\ 0\ -1]^T \qquad \mathbf{T_{1,N}} = [\pi\ 0\ 0]^T$$
$$\mathbf{T_{2,N}} = [0\ \pi\ 0]^T \qquad \mathbf{T_{1,S}} = [-\pi\ 0\ 0]^T \qquad \mathbf{T_{2,S}} = [0\ -\pi\ 0]^T$$