# Approximation properties of Hermite B-splines

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## I Optimally supported

Notations

- $r \in \mathbb{N}^*$ is the number of derivatives interpolated
- $\phi_s = L_{s-1},\ s = 0, \dots, r-1$ are Schoenberg *fundamental* splines that serve to expand solutions of C.H.I.P $(\boldsymbol{y}^{(0)}, \dots, \boldsymbol{y}^{(r-1)}, \mathcal{S}_{2r,r} \cap \mathcal{S})$ with $\mathcal{S}$ a linear space, either $S^{\gamma}_{2r,r}$ or $L^p_r$. $\phi_s$ is also equal to $\frac{1}{c_0} N_{s-1}$ where $N_s$ are the Hermite B-splines as defined by Schoenberg.

Following Theorem 3 of J.Fageot et al's paper Support and Approximation properties of Hermite splines, we hypothesize the following theorem

**Theorem 1.** *Let $\varphi_1, \dots, \varphi_r$ be $r$ compactly supported functions. We assume that*

$$\begin{cases} \beta^{r+1}(t) = \sum_{i \in \mathbb{Z}} a_1^{(r+1)}(i)\varphi_1(t-i) + \cdots + a_r^{(r+1)}(i)\varphi_r(t-i) \\ \vdots \\ \beta^{2r}(t) = \sum_{i \in \mathbb{Z}} a_1^{(2r)}(i)\varphi_1(t-i) + \cdots + a_r^{(2r)}(i)\varphi_r(t-i) \end{cases} \tag{1}$$

*with reproduction sequences satisfying*

$$\sum_{i \in \mathbb{Z}} |i|^{r+1}(|a_1^{(r+1)}(i)| + \cdots + |a_r^{(2r)}(i)|) < \infty \tag{2}$$

*Then,*

$$|\operatorname{Supp} \varphi_1| + \cdots |\operatorname{Supp} \varphi_r| \geq 2r \tag{3}$$

*Proof.* In the Fourier domain, (1) becomes

$$\begin{cases} \hat{\beta}^{r+1}(w) = \left(\dfrac{1-e^{-jw}}{jw}\right)^{r+1} = A_1^{(r+1)}(w)\hat{\varphi}_1(w) + \cdots + A_r^{(r+1)}(w)\hat{\varphi}_r(w) \\ \vdots \\ \hat{\beta}^{2r}(w) = \left(\dfrac{1-e^{-jw}}{jw}\right)^{2r} = A_1^{(2r)}(w)\hat{\varphi}_1(w) + \cdots + A_r^{(2r)}(w)\hat{\varphi}_r(w) \end{cases} \tag{4}$$

Now let

$$\det(w) = \det A(w) = \begin{vmatrix} A_1^{(r+1)}(w) & \dots & A_r^{(r+1)}(w) \\ \vdots & \ddots & \vdots \\ A_1^{(2r)}(w) & \dots & A_r^{(2r)}(w) \end{vmatrix} \tag{5}$$

Letting $B = \operatorname{com} A^T$ the transpose of the comatrix, the system (4) can be reversed as follows

$$\begin{cases} \det(w)\hat{\varphi}_1(w) = B_1^{(r+1)}(w)\hat{\beta}^{r+1}(w) + \cdots + B_r^{(r+1)}(w)\hat{\beta}^{2r}(w) \\ \vdots \\ \det(w)\hat{\varphi}_r(w) = B_1^{(2r)}(w)\hat{\beta}^{r+1}(w) + \cdots + B_r^{(2r)}(w)\hat{\beta}^{2r}(w) \end{cases} \tag{6}$$

Let's first show that $\det(w) \neq 0$ when $w \notin 2\pi\mathbb{Z}$. Suppose by contradiction that there exists $w_0$ not a multiple of $2\pi$ such that $\det(w_0) = 0$. By $2\pi$-periodicity we also have

$$\det(w_0 + 2\pi m) = 0 \text{ for } m = 0, \dots, r-1$$

Denote $\alpha_1, \dots, \alpha_r$, the $r$ mutually distinct quantities $\frac{1-e^{-jw_0}}{jw_0}, \dots, \frac{1-e^{-jw_0}}{j(w_0+2\pi(r-1))}$. From the first equation of the system (6) evaluated at $w_0$ and subsequent translates by multiples of $2\pi$ and the $2\pi$-periodicity of the discrete Fourier transforms $B_1^{(r+1)}, \dots, B_r^{(r+1)}$ we have

$$\begin{bmatrix} \alpha_1^{r+1} & \alpha_1^{r+2} & \dots & \alpha_1^{2r} \\ \alpha_2^{r+1} & \alpha_2^{r+2} & \dots & \alpha_2^{2r} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_r^{r+1} & \alpha_r^{r+2} & \dots & \alpha_r^{2r} \end{bmatrix} \begin{bmatrix} B_1^{(r+1)}(w_0) \\ B_2^{(r+1)}(w_0) \\ \vdots \\ B_r^{(r+1)}(w_0) \end{bmatrix} = 0 \tag{7}$$

The matrix on the left is Vandermondian and therefore has non-zero determinant. As a consequence $B_1^{(r+1)}, \dots, B_r^{(r+1)}$ all vanish at $w_0$. A similar reasoning shows that $B_1^{(r+2)}, \dots, B_r^{(r+2)}, \dots, B_1^{(2r)}, \dots, B_r^{(2r)}$ also vanish at $w_0$ which means that the matrix $B = \operatorname{com} A^T$ vanishes at $w_0$. It is a classical exercise to show that when the comatrix vanishes, the matrix has rank less than $r-2$. Looking into the system (1) at $w_0$ and subsequent translates by $2\pi$ leads to

$$A(w_0) \begin{bmatrix} \hat{\varphi}_1(w_0) & \hat{\varphi}_1(w_0+2\pi) & \dots & \hat{\varphi}_1(w_0+2\pi(r-1)) \\ \hat{\varphi}_2(w_0) & \hat{\varphi}_2(w_0+2\pi) & \dots & \hat{\varphi}_2(w_0+2\pi(r-1)) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\varphi}_r(w_0) & \hat{\varphi}_r(w_0+2\pi) & \dots & \hat{\varphi}_r(w_0+2\pi(r-1)) \end{bmatrix} = \begin{bmatrix} \alpha_1^{r+1} & \alpha_1^{r+2} & \dots & \alpha_1^{2r} \\ \alpha_2^{r+1} & \alpha_2^{r+2} & \dots & \alpha_2^{2r} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_r^{r+1} & \alpha_r^{r+2} & \dots & \alpha_r^{2r} \end{bmatrix}^T \tag{8}$$

Looking at the ranks shows the contradiction. Indeed the matrix-product on the left has rank less than the rank of $A(w_0)$ that is to say less than $r-2$ while the matrix on the right has full rank $r$.

Let now look at the behaviour of $\det(w)$ around 0. In virtue of condition (2), the discrete Fourier transforms $A_1^{(r+1)}, \dots, A_r^{(2r)}$ are all $r+1$ times differentiable. The determinant is then also $r+1$ times differentiable as it is polynomial in the coefficients of $A$. Expanding it around 0 leads to

$$\det(w) = \det(0) + \det(0)^{(1)}w + \cdots + \frac{1}{(r+1)!}\det(0)^{(r+1)}w^{r+1} + o(w^{r+1}) \tag{9}$$

Suppose by contradiction that $\det(0) = \cdots = \det(0)^{(r+1)} = 0$. Let $F(w) = \frac{1-e^{-jw}}{jw}$. Expanding $F$ in Taylor series at 0 and $2\pi m$ for non-zero integer $m$ proves

$$F(w) = 1 - \frac{j}{2}w + o_{w\to 0}(w)$$
$$F(w) = \frac{1}{2\pi m}(w - 2\pi m) + o_{w\to 2\pi m}((w-2\pi m))$$

Letting $k \geq 1$ a positive integer and expanding $F^k$ in its Taylor series as we did for $F$ leads to

$$\begin{aligned} F^k(w) &= 1 - \frac{jk}{2} w + o_{w\to 0}(w) \\ F^k(w) &= \frac{(w-2\pi m)^k}{(2\pi m)^k} + o_{w\to 2\pi m}\left((w-2\pi m)^k\right) \end{aligned}$$

Given our assumption about $\det(w)$ and given its $2\pi$-periodicity we also have

$$\begin{aligned} \det(w) &= o_{w\to 0}(w^{r+1}) \\ \det(w) &= o_{w\to 2\pi m}\left((w-2\pi m)^{r+1}\right) \end{aligned}$$

.

Rewrite then the first equation of system (6) as follows

$$\frac{\det(w)}{F^{r+1}(w)} = B_1^{(r+1)}(w) + B_2^{(r+1)}(w)F(w) + \cdots + B_r^{(r+1)}F^{r-1}(w) \tag{10}$$

For any integer $m$ non-zero, equation (10) can also be written as

$$\begin{aligned} o(1) = &\left(B_1^{(r+1)}(0) + B_1^{(r+1)^{(1)}}(0)(w-2\pi m) + \dots\right) \\ &+ \left(B_2^{(r+1)}(0) + B_2^{(r+1)^{(1)}}(0)(w-2\pi m) + \dots\right)\left(\frac{(w-2\pi m)}{2\pi m} + o((w-2\pi m))\right) \\ &\vdots \\ &+ \left(B_r^{(r+1)}(0) + B_r^{(r+1)^{(1)}}(0)(w-2\pi m) + \dots\right)\left(\frac{(w-2\pi m)^{r-1}}{(2\pi m)^{r-1}} + o((w-2\pi m)^{r-1})\right) \end{aligned}$$

which can only happen if

$$B_1^{(r+1)}(0) = 0 \tag{11}$$

Evaluating (10) at 0 adds the following equation

$$B_2^{(r+1)}(0) + \cdots + B_r^{(r+1)}(0) = 0 \tag{12}$$

Repeating the reasoning for the other lines in the system (6) gives the same results on $B^{(r+2)}, \dots, B^{(2r)}$. As a consequence the first column of $B$ vanish at 0 and has rows summing to 0. However $B(0)$ is a comatrix and as such can only have rank $r$, 1 or 0 if $A(0)$ has rank $r$, $r-1$, $\leq r-2$ respectively.

$\square$

# II Other ideas for the proof

Let's expand into higher orders. For that develop the exponential into its series allow to identify the derivatives of $F$ in 0. Then, define $G(w) = \frac{w}{w+2\pi}$ so that

$$F(w+2\pi) = F(w)G(w) \tag{13}$$

It is easy to show that for $p \geq 1$ one has $G^{(p)}(w) = \frac{2\pi p!(-1)^{p+1}}{(w+2\pi)^{p+1}}$. Applying Leibniz's formula to (13) and evaluating in 0 provides the derivatives of $F$ at $2\pi$. All calculations done,

$$\begin{aligned} F^{(p)}(0) &= \frac{(-1)^p j^p}{p+1} \\ F^{(p)}(2\pi) &= \sum_{q=1}^{p} \frac{p!\, j^p(-1)^{p+1}}{(p-q+1)!} \frac{1}{(2\pi j)^q} \end{aligned}$$

Let the sequences $(a_{p,s})_{p\in\mathbb{N}}, (b_{p,s})_{p\in\mathbb{N}}$ defined by

$$
\begin{aligned}
a_{p,0} &= \frac{(-1)^p j^p}{(p+1)!} \\
a_{p,s+1} &= \sum_{q=0}^{p} \binom{p}{q} a_{p-q,s} a_{q,0} \\
b_{p,0} &= \sum_{q=1}^{p} \frac{j^p (-1)^{p+1}}{(p-q+1)!} \frac{1}{(2\pi j)^q} \\
b_{p,s+1} &= \sum_{q=1}^{p} \binom{p}{q} b_{p-q,s} b_{q,0}
\end{aligned}
$$

Then, the following Taylor expansions hold for any integers $s$ and $n$

$$F^s(w) = \sum_{p=0}^{n} a_{p,s} w^p + o_{w\to 0}(w^n) \tag{14}$$

$$F^s(w) = \sum_{p=0}^{n} b_{p,s} (w-2\pi)^p + o_{w\to 2\pi}((w-2\pi)^n) \tag{15}$$

Note that $\nabla \det(A) = \mathrm{com}(A)$ so that $\det(w)' = \mathrm{Tr}(A'(w)B(w))$.