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Master Thesis

A unified view of Hermite splines for the practitioner

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Chapter 1

From Newton's formulae to B-splines

1.1 "Classical" interpolation theory

Excellente intro !!

C'est perturbant qu'on appelle "fonction" sans nom de variable ce qu'on cherche à approximer et f son approximation !

The age of scientific revolution, spanning from early 17th to late 19th century, brought invaluable scientific knowledge in all domains including the mathematics with the contributions of the likes of Descartes, Leibniz, Newton, Euler, Lagrange, Gauss and many others. This era saw the development of new theories and tools that laid the foundations for modern mathematics as we know them. In particular, new techniques were devised for interpolating between a given set of points, a problem that can be traced back to ancient times. As a matter of fact, interpolation was already in use in ancient Babylon where farmers were concerned with predictions about astronomical events as the positions of the sun, moon and known planets. Lists recording these however contained unavoidable gaps that needed to be filled somehow or in other words *interpolated*. As E. Meijering mentioned in his very instructive chronology of interpolation (Me02), linear interpolation but also higher-order interpolation methods were used during these times. The latter are now seen as subcases of far more general interpolation formulae as that of Newton in his celebrated *Principia* (1687) and *Methodus Differentialus* (1711) manuscripts. Readers interested in the history of interpolation are urged to consult ~~the work of E. Meijering that covers ancient times up to early twenty-first century~~ [Me02], as well as references therein.

↳ c'est un peu redondant avec ce que tu dis déjà au dessus

1.1.1 Distinct locations

Suggestion : tu considères une fonction f, que tu connais aux entiers f(ℤ), et que tu modèles avec un polynôme f_ξ

In the first place, mathematicians were concerned with the problem of interpolating equally-spaced data with polynomials. For that, let $n \in \mathbb{N}^*$ to be an integer and Π_{n-1} the set of all polynomials of degree up to $n-1$. Later on, we will also consider the *order* instead of the *degree* to define polynomials and de note as $\Pi_{<n}$ the set of all polynomials of order n. Obviously, $\Pi_{<n} = \Pi_{n-1}$. Let's first consider measurements of some function at $\mathbb{Z}$ and be interested in computing its value at intermediate points ξ for $\xi \in \mathbb{R}$. For that, we locally model it as a polynomial f_ξ of degree $n-1$. An obvious basis for the n-dimensional vector space $\Pi_{<n}$ is the family of n monomials $(1, \xi, \dots, \xi^{n-1})$. For interpolation, though, it is more convenient to consider the basis $(1, [\xi], \dots, [\xi]^{n-1})$ formed ~~of~~ by the polynomials $[\xi]^k = \xi(\xi-1)\dots(\xi-k+1)$ for $k = 0, \dots, n-1$, yielding

tu veux dire "on the integer grid ℤ" je pense

$$f(x) = c_0 + c_1[\xi] + \dots + c_{n-1}[\xi]^{n-1}. \tag{1.1}$$

To see why this basis is more convenient, let's introduce the p^{th}-order difference operator Δ^p defined recursively for any function g as

↳ si tu veux tu peux le mettre en length lors de sa 1ère occurrence ici

$$\Delta^p g(\xi) = \begin{cases} g(\xi) & \text{if } p = 0, \\ \Delta^{p-1} g(\xi+1) - \Delta^{p-1} g(\xi) & \text{otherwise}. \end{cases} \tag{1.2}$$

The coefficients in the decomposition (1.1) are now easily related to f with the help of ~~this operator~~ Δ^p. ~~For that,~~ We indeed notice that $\Delta[\xi]^k = k[\xi]^{k-1}$ which, when applied recursively, leads to $k!c_k = \Delta^k f(0)$ for $k = 0, \dots, n-1$. The quantities $\Delta^k f(0)$ are readily computed from the known samples and therefore so are the coefficients c_k.

This formulation of interpolation can then be extended to equally-spaced measurements $t_0 + \mathbb{Z}\tau$ where we are now interested in computing the value at $t_0 + \xi\tau$ for $\xi \in \mathbb{R}$. Letting $f \in \Pi_{<n}$ be the local interpolant and $\tilde{f}$ the function $\tilde{f}(\xi) = f(t_0 + \xi\tau)$, it amounts again to interpolating locally a function with known samples on the integer grid, leading to

$$f(t_0 + \xi\tau) = f(t_0) + \xi\Delta_\tau f(t_0) + \cdots + \xi(\xi-1)\dots(\xi-n+2)\frac{\Delta_\tau^{n-1} f(t_0)}{(n-1)!}, \tag{1.3}$$

no indent

with Δ_τ the difference operator with spacing τ ($\Delta_1 = \Delta$).

exactly corresponds to ... of ... → une ref serait bien

Equation (1.3) ~~is exactly~~ the formula ~~written down by~~ Gregory in 1670 but also by Newton in Lemma V, ref ~~in book III of *Principia*~~ which he seems to have discovered independently of Gregory. Notwithstanding this, Newton's contributions in the field go far beyond the case of equally-spaced data as illustrated by his general formula of interpolation at arbitrary distinct locations $(t_i)_{i\in\mathbb{Z}}$. The formula makes use of the *divided* difference operator that is recursively defined for any function g as

$$[t_0,\dots,t_p]g = \begin{cases} g(t_0) & \text{if } p = 0, \\ \dfrac{[t_1,\dots,t_p]g - [t_0,\dots,t_{p-1}]g}{t_p - t_0} & \text{otherwise}. \end{cases} \tag{1.4}$$

no indent

Newton's formula for the local polynomial interpolant at $t_0,\dots,t_{n-1}$ then reads

$$f(t) = [t_0]f + (t-t_0)[t_0,t_1]f + \cdots + (t-t_0)\cdots(t-t_{n-2})[t_0,\dots,t_{n-1}]f. \tag{1.5}$$

The coefficients $[t_0,\dots,t_k]f$ for $k = 0,\dots,n-1$ are readily computed from the known samples using (1.4). This is known as the *Newton form* and will be extended in the next subsection to the case of arbitrary locations, where several locations may coalesce.

1.1.2 Arbitrary locations

For Sections ~~this part~~ 1.1.2 and ~~the next one~~ 1.1.3, we adopt the following notations

- $r, n \in \mathbb{N}^*$;
- $\boldsymbol{t} = (t_i)_{i=0}^{n-1}$ is a sequence of arbitrary locations;
- $\tilde{\boldsymbol{t}} = (\tilde{t}_i)_{i=0}^{d-1}$ are unique elements of the sequence $\boldsymbol{t}$;
- $(\tilde{r}_i = \#\{l | t_l = \tilde{t}_i\})_{i=0}^{d-1}$ are the multiplicities of each unique location, hence ~~Note that~~ $\sum_{i=0}^{d-1} \tilde{r}_i = n$;
- $g : \mathbb{R} \to \mathbb{R}$ is a function that is $\tilde{r}_i - 1$ times differentiable in $\mathcal{V}(\tilde{t}_i)$, for $i = 0,\dots,d-1$ (unless stated otherwise).

Most of these notations ~~here~~ are the ones used by C. De Boor ~~throughout his publications~~ as the following elements ~~presentation~~ heavily relies on his work, especially his exhaustive treatment of splines for the practitioner (Bo01). In order to consider repeated locations, we first have to define what it means to interpolate a function more than once at a location.

→ ça a pas un nom de définition ?

new paragraph

Definition 1.1 (Osculatory interpolation, (Bo01, p6)). *Let $f, g : I \to \mathbb{R}$ be functions that are $r-1$ times differentiable on some open interval I and let $t \in I$. We say that f and g agree at t with* multiplicity *r if*

$$\forall j = 0,\dots,r-1,\ f^{(j)}(t) = g^{(j)}(t). \tag{1.6}$$

The case $r = 1$ is the standard concept of interpolation, while ~~the case~~ $r > 1$ is referred to as *osculatory* interpolation. Based on this definition, we say that functions f and g *agree* at $\boldsymbol{t}$ if they agree at each ~~of~~ $\tilde{t}_j$ with multiplicity $\tilde{r}_j$ for $j = 0, \ldots, d-1$.

when

~~In the case where~~ the locations are all distinct, it is easily proved that there exists a unique polynomial of order n that interpolates g at $\boldsymbol{t}$. A Polynomial in Newton form (1.5) is in $\Pi_{<n}$ and exactly interpolat~~ory~~es at $\boldsymbol{t}$, which proves ~~the~~ existence ~~of such a polynomial~~. Considering $f, \tilde{f} \in \Pi_{<n}$ two such polynomials, their difference can be divided by the degree n polynomial $(t - t_0) \ldots (t - t_n)$, which can only occur if $f - \tilde{f} = 0$, proving unicity. More interestingly, this result also holds for arbitrary locations as expressed in the following theorem.

→ si dans De Boor, citer une ref (sinon, c'est que c'est de toi)

Theorem 1.1. *There exists a unique polynomial of order n that agrees with g at $\boldsymbol{t}$.*

Proof. (Existence). For $i = 0, \ldots, d-1$ define the set of $\tilde{r}_i$ polynomials for $k = 0, \ldots, \tilde{r}_i - 1$

$$P_{i,k}(t) = \frac{1}{k! \prod_{j=0, j \neq i}^{d-1} (\tilde{t}_i - \tilde{t}_j)^{\tilde{r}_j}} (t - \tilde{t}_i)^k \prod_{j=0, j\neq i}^{d-1} (t - \tilde{t}_j)^{\tilde{r}_j}, \tag{1.7}$$

and the set

$$Q_{i,k}(t) = \begin{cases} P_{i,\tilde{r}_i - 1} & \text{if } k = \tilde{r}_i - 1, \\ P_{i,k} - \sum_{l=k+1}^{\tilde{r}_i - 1} \alpha_j^{i,k} P_{i,l}(t) & \text{otherwise}, \end{cases} \tag{1.8}$$

no indent

where $\alpha_l^{i,k}$ are uniquely chosen so that $Q_{i,k}$ has vanishing l^{th}-derivative at $\tilde{t}_i$ for $l \neq k$. To see why such a choice is possible and unique, observe that, by construction, $P_{i,k}$ has vanishing derivatives up to $k-1$ and unit k^{th} derivative at $\tilde{t}_i$, ~~while~~ but also vanishing derivatives up to $\tilde{r}_j - 1$ at other locations $\tilde{t}_j$.

As $P_{i,k} \in \Pi_{<n-\tilde{r}_i+1+k} \subset \Pi_{<n}$, each $Q_{i,k}$ is also in $\Pi_{<n}$ ~~and thefore so is any combination of these polynomials~~. The ~~following~~ order n polynomial provides us with the existence result.

$$f = \sum_{i=0}^{d-1} \sum_{k=0}^{\tilde{r}_i - 1} g^{(k)}(\tilde{t}_i) Q_{i,k} \tag{1.9}$$

Note how (1.8) defines a total of $\tilde{r}_0 + \cdots + \tilde{r}_{d-1} = n$ polynomials that form a basis ~~for~~ of the n-dimensional vector space $\Pi_{<n}$.

(Unicity). Let $f, \tilde{f}$ be two polynomials in $\Pi_{<n}$ that agree with g at $\boldsymbol{t}$. The difference polynomial $f - \tilde{f}$ vanishes up to order $\tilde{r}_i - 1$ at $\tilde{t}_i$ for $i = 0, \ldots, d-1$. Let $\tilde{r}_{\boldsymbol{t}} = \max_{i=0,\ldots,d-1} \tilde{r}_i$. Repeated application of Rolle's theorem shows that $(f - \tilde{f})^{(\tilde{r}_{\boldsymbol{t}})}$ vanishes at $n - \tilde{r}_{\boldsymbol{t}}$ locations while being in $\Pi_{<n-\tilde{r}_{\boldsymbol{t}}}$, ~~so~~ implying that the derivative is 0. Constants from successive integrations all vanish, leading to $f - \tilde{f} \equiv 0$. $\square$

Including repeated locations in our treatment of interpolation calls for an extended definition of the *divided* difference operator mentioned earlier ~~at~~ in (1.4). De Boor defines this operator in a somewhat elegant manner as he does not make use of a recursive definition as is usually done.

→ num d'équation/de definition?

Definition 1.2 (Extended divided difference, (Bo01, p3)). *The n-th divided difference at $t_0, \ldots, t_n$, $[t_0, \ldots, t_n]g$, is defined as the leading coefficient of the unique $f \in \Pi_{<n}$ that agrees with g at $\boldsymbol{t}$.*

C'est effectivement élégant :)

This extended divided difference operator allows ~~us to~~ extending the *Newton form* (1.5) of the interpolant to the general case of arbitrary locations. To see this, let $f_k(t) \in \Pi_{<k}$ be the function that uniquely

agrees with g at $t_0,\dots,t_{k-1}$. Clearly $f_1(t) = g(t_0) + (t-t_0)[t_0,t_1]g$. Assume now that

$$f_k(t) = \sum_{i=0}^{k-1}(t-t_0)\dots(t-t_{i-1})[t_0,\dots,t_i]g .$$

As $f_{k+1}-f_k$ vanishes at $t_0,\dots,t_k$, it can be divided by $p_{k+1}(t) := (t-t_0)\dots(t-t_k)$. As f_k is of order k while f_{k+1} and p_{k+1} are of order $k+1$, there exists some $c_{k+1}\in\mathbb{C}$ such that

$$f_{k+1}-f_k = c_{k+1}p_{k+1} .$$

The leading coefficient of f_{k+1} being $[t_0,\dots,t_k]g$ by definition, we have that $c_{k+1} = [t_0,\dots,t_k]g$ which completes the induction. Newton's interpolant at $t_0,\dots,t_{n-1}$ then reads, as in (1.5),

$$f(t) = [t_0]f + (t-t_0)[t_0,t_1]f + \cdots + (t-t_0)\cdots(t-t_{n-2})[t_0,\dots,t_{n-1}]f, \tag{1.10}$$

where ~~now~~ the locations $t_0,\dots,t_n$ are now completely arbitrary.

The divided difference operator defined above has a number of properties that will ~~come~~ be useful in understanding the properties of B-splines and splines in general. ~~Let mention some of them here in the~~

Proposition 1.1. *1. $[t_0,\dots,t_n]g$ is a symmetric function of $t_0,\dots,t_n$* ~~i.e.~~, meaning that *it is not affected by permutations of the order of the locations.*

2. $[t_0,\dots,t_n]g$ is linear in g.

3. Suppose that *$t_0\le\cdots\le t_n$.* Then,

$$[t_0,\dots,t_n]g = \begin{cases} \frac{[t_1,\dots,t_n]g-[t_0,\dots,t_{n-1}]g}{t_n-t_0} & \text{if } t_0 < t_n, \\ \frac{g^{(n)}}{n!} & \text{if } t_0 = t_n . \end{cases}$$

4. If $g\in\Pi_{<n}$ on $I\supset \boldsymbol{t}$, $[t_0,\dots,t_n]g = 0$.

5. If $g\in C^n$, $\exists\eta\in[t_0,\dots t_n]$ such that $[t_0,\dots,t_n]$ space *$g = \frac{g^{(n)}(\eta)}{n!}$.*

6. (Leibniz) If $f = gh$, then

?

add ref or convention

$$[t_0,\dots,t_n]f = \sum_{k=0}^{n}[t_0,\dots,t_k]g[t_k,\dots,t_n]h . \tag{1.11}$$

In (Bo01, chapter I), ~~De Boor gives proofs or~~ direct or references, ~~where the~~ proofs ~~is indicated~~ are given for each of these properties. ~~However~~ Moreover most of these claims can also be easily verified by the reader and will not be detailed here.

le fameux "proof is left for the reader as exercise" ... :-)

1.1.3 From polynomial to piecewise-polynomial

It ~~looks~~ may be tempting to stop here as the problem of interpolating a function at arbitrary locations, including repeated locations, is completely solved by polynomials. However, we need to ask ourselves how good our interpolant is, that is, how closely it reproduces the underlying function ~~for which we only have measurements~~ accessed through its measurements. For that, we look at the error between the interpolant and the interpolated function at all points of some interval of interest $[a,b]$, usually the interval $[t_0,t_{n-1}]$ (assuming $t_0\le\cdots\le t_{n-1}$) or ~~some~~ a larger ~~interval~~ one. The norm that we will use to quantify the quality of the interpolation is the supremum norm, that is,

$$\|f-g\|_{\infty,[a,b]} = \|f-g\| = \sup_{a\le t\le b} |f(t)-g(t)| .$$

A very convenient result for characterizing the norm of the error is the following *osculatory theorem* that links any function $g \in \mathcal{C}^n$ to its interpolant at the locations $t_0, \ldots, t_n$. ~~The theorem~~ IT is proved by induction in the reference mentioned: thm ref ?

Theorem 1.2 (Osculatory theorem, (Bo01, p7))**.** *Suppose $g \in \mathcal{C}^n$. Then, for all $t \in [a,b]$,*

$$g(t) = f_n(t) + (t-t_0)\ldots(t-t_{n-1})[t_0,\ldots,t_{n-1},t]g, \tag{1.12}$$

with $f_n \in \Pi_{<n}$ the unique polynomial interpolant to g at $\boldsymbol{t}$.

To understand why (1.12) is very useful, consider the case where the interpolated function is in C^n and let $a=t_0, b=t_{n-1}$. Then, using (1.12), we can exactly bound the error as ~~follows~~

$$\|f_n-g\| \le \|(\cdot-t_0)\ldots(\cdot-t_{n-1})\|\|[t_0,\ldots,t_{n-1},\cdot]g\| . \tag{1.13}$$

Now, from item 5 of Proposition 1.1, we have that $\|[t_0,\ldots,t_{n-1},\cdot]g\| \le \frac{\|g^{(n)}\|}{n!}$ so that (1.13) becomes

$$\|f_n-g\| \le \|(\cdot-t_0)\ldots(\cdot-t_{n-1})\|\frac{\|g^{(n)}\|}{n!} . \tag{1.14}$$

To understand if this bound can be useful, we have to bound $\|(\cdot-t_0)\ldots(\cdot-t_{n-1})\|$ ~~somehow~~. Unfortunately, this quantity can grow quite ~~large~~ big depending on the distribution of the locations $t_0,\ldots,t_n$ as n increases. Consider for instance the Runge example, where the function

$$g(x) = \frac{1}{1+25x^2}$$

is approximated on the interval $[-1,1]$ by interpolating ~~of~~ n uniformly spaced locations.

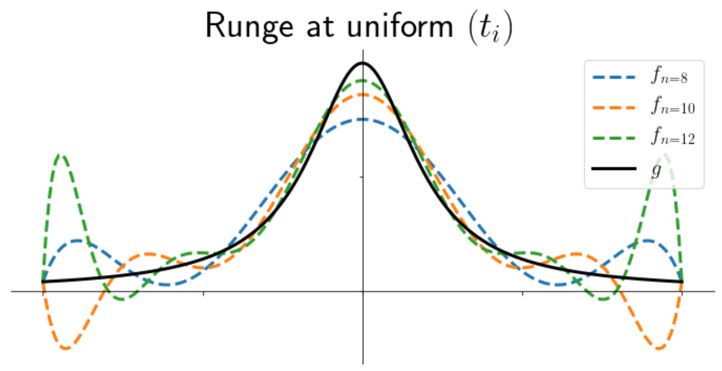

FIGURE 1.1: Runge function in thick black line and interpolant f_n for different values of n in dotted lines

As ~~can be~~ seen in Figure 1.1, the interpolant magnitude increases so much with the number of measurements that the approximation error *grows* as n grows, ~~contrary~~ in contradiction to what one would expect. ~~Now it so happens that~~ If $\boldsymbol{t}$ is chosen as the zeroes of the Chebyshev polynomials of degree n on the interval $[a,b]$, however, such behaviours do not occur. To understand why, one should observe that the zeroes of the Chebyshev polynomial minimize the quantity $\|(\cdot-t_0)\ldots(\cdot-t_{n-1})\|$ over all possible locations with minimal value $\frac{2(b-a)^n}{4^n}$ (Bo01, p23).

↳ même remarque que les autres => on préfère mettre un num d'équation/ définition/ thm que de page si possible

This provides us with ~~an~~ the upper bound on the minimal error achievable by a function in $\Pi_{<n}$ (not necessarily interpolatory) approximating $g \in \mathcal{C}^n$,

$$\operatorname{dist}(g, \Pi_{<n}) \leq 2 \frac{(b-a)^n}{4^n} \frac{\|g^{(n)}\|}{n!} \tag{1.15}$$

There is hope therefore ~~that we can~~ to find a polynomial that is interpolant *and* that tightly reproduces g as the number of measurements grows. Note, however, that the bound (1.15) holds only for ran n-times continously derivable function on (a, b). ~~and that~~ It may also happen that the supremum norm of this derivative is not finite, as is the case for $g : x \to \sqrt{1+x}$ on $[-1, 1]$, where the derivatives grow to be infinity close to -1. Fortunately, ~~there is~~ a more precise result by Jackson ~~that~~ bounds the distance of g to $\Pi_{<n}$ for larger classes of functions.

Theorem 1.3 (Jackson, (Bo01, p26)). *Suppose that $g \in \mathcal{C}^r[a, b]$ and $n > r + 1$. Then,* we have that

$$\operatorname{dist}(g, \Pi_{<n}) \leq const_r \left(\frac{b-a}{n-1}\right)^r w\left(g^{(r)}, \frac{b-a}{2(n-1-r)}\right) \tag{1.16}$$

\left(\right)

where $w(\phi, \epsilon) = \sup_{|x-y| \leq \epsilon}\{|\phi(x) - \phi(y)|\}$ is the modulus of ϕ continuity at ϵ.

As mentioned by De Boor, the bound (1.16) is sharp. ~~We~~ One can thus find functions g for which the bound is reached. Therefore, the only way ~~for~~ to making the error small is to ~~make~~ reduce $\frac{b-a}{n-1}$ ~~small~~. ~~For~~ To do so, ~~that two things come to mind:~~ one can either increase n or decrease $b - a$. Increasing n leads to using high-order polynomials, whose evaluations require n operations and are prone to errors. ~~On the other hand,~~ Breaking the segment $[a, b]$ into k smaller intervals on which a separate interpolation at lower order is performed is with contrast more stable and less computationally demanding, while retaining the same approximation power as ~~that of a~~ high-order polynomial. However, interpolating independently ~~on~~ each subsegment raises the question of the smoothness of the interpolant. ~~If we are not careful (or do not mind),~~ It may indeed happen that the interpolant is ~~not even~~ discontinuous where pieces of polynomials meet. ~~Fortunately,~~ Smoothness is hence a question of vital importance in spline theory ~~and~~ that ~~even~~ actually motivated their development.

1.2 Introduction to splines

~~For this section 1.2~~ We adopt the following notations in this section:

- $k \in \mathbb{N}^*$;
- $\boldsymbol{t}$ is a sequence of nondecreasing real numbers, finite or infinite;
- $\boldsymbol{\xi}$ is a sequence of increasing real numbers, finite or infinite;
- $\boldsymbol{\nu}$ is a sequence of nonnegative integers, finite of infinite.

~~In the problem of~~ When interpolating data, may it be values or derivatives of some unknown function, the intuitive method consisting in interpolating all ~~the data~~ known values at once is prone to large errors when the data exceeds a few points, as ~~we saw previously~~ extensively discussed in 1.1.3. ~~This is the consequence of resorting to high-order polynomials which tend to oscillate a lot when forced to perfectly interpolate the data~~. A more *natural* ~~and promising~~ approach to the problem consists in splitting it into subproblems of lesser complexity. The price to pay for doing so is the decrease in the smoothness of the interpolating function at the break points ~~we will have chosen~~. As a matter of fact, the maximum degree of smoothness achievable is a decreasing function of the number of derivatives ~~we try~~ to be interpolated, as we shall see.

to les explique juste avant

1.2.1 Definitions

We are now going to ~~replace~~ switch from polynomial interpolation as detailed in the previous section ~~by~~ to piecewise-polynomial interpolation, which allows ~~us to keep~~ using low-order degrees while retaining good approximation properties. The resulting interpolant ~~made of all the arcs of polynomials~~ is a piecewise polynomial function. ~~as defined in~~

Definition 1.3. *The set of all piecewise polynomials of order k with breaks at $\boldsymbol{\xi}$ is denoted* by *$\Pi_{<k,\boldsymbol{\xi}}$. It consists ~~in~~ of all functions that are polynomials of order k on all intervals (ξ_i, ξ_{i+1}). The elements of $\boldsymbol{\xi}$ are called* knots.

For the needs of further results, let's introduce ~~here~~ the subspaces of piecewise polynomials with specified degrees of continuity.

Definition 1.4. *The set of all piecewise polynomials at order k with knots $\boldsymbol{\xi}$ and continuity degrees $\boldsymbol{\nu}$ is, by definition,* expressed as

$$\Pi_{<k,\boldsymbol{\xi},\boldsymbol{\nu}} = \{f \in \Pi_{<k,\boldsymbol{\xi}} \mid jump_{\xi_i} D^{j-1} f = 0, j = 1, \ldots, \nu_i, (\xi_i, \nu_i) \in \boldsymbol{\xi} \times \boldsymbol{\nu}\}, \tag{1.17}$$

where $jump_{\xi_i} \phi = \phi(\xi_i^+) - \phi(\xi_i^-)$.

The maximum degree of continuity achievable at a knot is the order of the polynomials on each side of the knots, corresponding to k in our notation. ~~Indeed, in the case~~ When $\nu_i = k$, writing ~~the polynomials in their~~ Taylor expansion at ξ_i up to order k ~~we see that both polynomials~~ of the polynomials shows that they share the same coefficients ~~in the expansion~~. Consequently, polynomials on each side of the knot join *perfectly*, in the sense that they are subparts of the same polynomial of order k. The value $\nu_i = 0$ ~~means~~ implies that ~~we impose~~ no continuity condition is imposed at the knot ξ_i. If $\boldsymbol{\nu_1}, \boldsymbol{\nu_2}$ are two sequences such that $\boldsymbol{\nu_1} \leq \boldsymbol{\nu_2}$, it follows from the definition that $\Pi_{<k,\boldsymbol{\xi},\boldsymbol{\nu_2}} \subset \Pi_{<k,\boldsymbol{\xi},\boldsymbol{\nu_1}}$.

1.2.2 Schoenberg's cardinal splines

In 1946, Schoenberg noted in his landmark paper (Sch46) that, for every osculatory interpolation formula to equidistant data, there exists an even function $L : \mathbb{R} \to \mathbb{R}$ ~~in terms of which~~ such that

$$f(t) = \sum_{i=-\infty}^{\infty} y_i L(t - i) . \tag{1.18}$$

This formula depends only one ref? on the function L, which he termed the *basis* function. For ~~example~~ instance, $L(t) = \frac{\sin(\pi t)}{\pi t}$ was known to Whittaker and, by analogy, Schoenberg refers to (1.18) as a formula of *cardinal type*. Using (1.18), Schoenberg obtains ~~(see~~ (Sch46, Theorem 5) a general parametric representation of functions made of individual ~~arcs~~ pieces of degree $k-1$ joined together with $k-2$ degrees of continuity, which he defines as *splines of order k*. An elegant compilation of Schoenberg's works in spline theory can be found in the form of lectures (Sch73), in which ~~he defines~~ cardinal splines, as follows, are defined:

Definition 1.5 (Cardinal splines, (Sch73, Lecture 1))**.** *The set $\mathcal{S}_k$ of cardinal splines of* order *k denotes all functions S such that*

1. *$S \in \Pi_{<k}$ on $(i, i+1)$ for $i \in \mathbb{Z}$;*
2. *$S \in \mathcal{C}^{k-2}$.*

At times, it is convenient to consider the splines halfway between the integers, that is,

$$\mathcal{S}_k^* = \{S | S(\cdot + \frac{1}{2}) \in \mathcal{S}_k\} .$$

Remark 1.1. *~~Once again be aware of~~ We recall the difference between* degree *and* order*. ~~In his work~~ Schoenberg originally defined spline spaces using the notion of degree, but we chose to use the notion of order so as to ~~more easily~~ relate to the notations of De Boor* more easily.

It is interesting to note that differentiation reduces $\boxed{\text{by one unit}}$ the degree of continuity ~~as~~ and ~~well as~~ the order of the polynomial, so that, for any $j = 0, \dots, k$,

$$S \in \mathcal{S}_k \iff S^{(j)} \in \mathcal{S}_{k-j} . \tag{1.19}$$

~~In the following lecture Schoenberg goes on to define~~ *B-splines* are then defined for equidistant knots as those elementary function whose combinations allow to represent the most general cardinal spline.

Definition 1.6 (B-splines equidistant knots, (Sch73, Lecture 2))**.** *The forward B-spline of degree $k-1$ is* given by

$$Q_k(t) = k[0, 1, \dots, k](\cdot - t)_+^{k-1} , \tag{1.20}$$

and the central B-spline of degree $k-1$ as

$$M_k(t) = k\left[\frac{-k}{2}, \frac{-k}{2} + 1, \dots, \frac{k}{2}\right](\cdot - t)_+^{k-1} = Q_k(t + \frac{k}{2}) . \tag{1.21}$$

Clearly,

$$Q_k \in \mathcal{S}_k \text{ for all } k, \quad M_k \in \begin{cases} \mathcal{S}_k & \text{if } k \text{ is even} , \\ \mathcal{S}_k^* & \text{if } k \text{ is odd} . \end{cases}$$

These functions have a number of properties detailed (Sch73, Lectures 1 and 2) ~~by Schoenberg in his first two lectures~~. We summarize them in the following proposition ~~these properties~~ and refer the reader to (Sch73) for more details. Note that the properties are connected in the sense that the first ones can be used to proved the following ones.

Proposition 1.2. *1.*

$$\Delta^k f(0) = \int_0^k Q_k(t) f^{(k)}(t)dt, \quad \delta^k f(0) = \int_{-\frac{k}{2}}^{\frac{k}{2}} M_k(t) f^{(k)}(t)dt ,$$

with Δ as defined in (1.2), i.e, $\Delta f(t) = f(t+1) - f(t)$ and $\delta f(t) = f(t + \frac{1}{2}) - f(t - \frac{1}{2})$;

2.

$$Q_k(t) = \frac{1}{(k-1)!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(x-i)_+^{k-1}, M_k(t) = \frac{1}{(k-1)!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(x + \frac{k}{2} - i)_+^{k-1} ;$$

3.

$$\int_{-\infty}^{\infty} Q_k(t)e^{-jwt}dt = \left(\frac{1 - e^{-jw}}{jw}\right)^k, \quad \int_{-\infty}^{\infty} M_k(t)e^{-jwt}dt = \left(\frac{w\sin(\frac{w}{2})}{w}\right)^k ;$$

4.

$$Q_k'(t) = Q_{k-1}(t) - Q_{k-1}(t-1), \quad M_k'(t) = M_{k-1}(t + \frac{1}{2}) - M_{k-1}(t - \frac{1}{2}) ;$$

5.

$$1 = \sum_{i=-\infty}^{\infty} M(t-i), \quad t = \sum_{i=-\infty}^{\infty} iM(t-i) \quad \forall t .$$

Schoenberg defined cardinal splines independently of ~~his~~ the *B*-splines (~~see~~ see Definition 1.5) and relate ~~both~~ them through the following theorem.

Theorem 1.4 (Cardinal B-splines expansion, (Sch73, Lecture 2)). *If $S \in \mathcal{S}_k$, S admits a unique representation of the form*

$$S(t) = \sum_{i=-\infty}^{\infty} c_i Q_k(t-i) , \tag{1.22}$$

with $\boldsymbol{c} = (c_i)_{i\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$.

As Q and M are related by $M_k(t) = Q_k(t + \frac{k}{2})$, it is also true that

$$S(t) = \sum_{i=-\infty}^{\infty} c_i M_k(t-i) \tag{1.23}$$

represents uniquely any $S \in \begin{cases} \mathcal{S}_k & \text{if } k \text{ even} \\ \mathcal{S}_k^* & \text{if } k \text{ odd} \end{cases}$.

1.2.3 De Boor's reversed definition

\textit{i.e.},

~~In his Book,~~ De Boor defines *normalized* B-splines and, for the case of general knots (i.e not only equally-spaced knots), using the divided difference operator, ~~as follows~~

Definition 1.7 ((Bo01, p87)). *The j^{th}* normalized *B-spline of order k is* given by

$$B_{j,k,\boldsymbol{t}}(t) = (t_{j+k} - t_j)[t_j, \dots, t_{j+k}](\cdot - t)_+^{k-1} . \tag{1.24}$$

In order to lighten notations, we will usually drop the dependence in the knots sequence $\boldsymbol{t}$ when it is clear from the context what these knots are. In the definition above, we adopt the convention that $0^0 = 0$, which ~~has the consequence of~~ mak~~ing~~es our B-splines right-continuous.

Example 1.1. • $\underline{k = 1}$

$$\begin{aligned} B_{j,1}(t) &= (t_{j+1} - t_j)[t_j, t_{j+1}](\cdot - t)_+^0 \\ &= (t_{j+1} - t)_+^0 - (t_j - t)_+^0 \\ &= \begin{cases} 1 & t_j \le t < t_{j+1} , \\ 0 & \textit{elsewhere} . \end{cases} \end{aligned}$$

upright

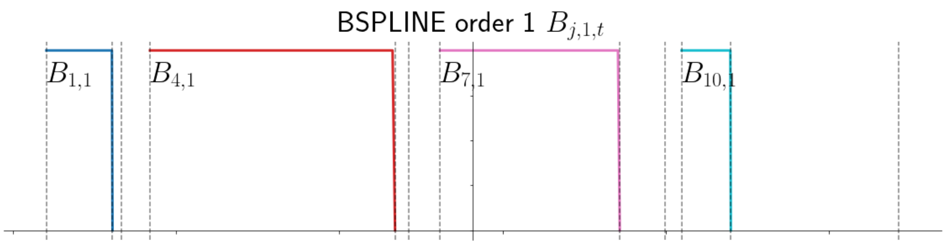

FIGURE 1.2: Some B-splines of order 1

- $\underline{k=2}$

$$\begin{aligned}
B_{j,2}(t) &= (t_{j+2}-t_j)\frac{[t_{j+1},t_{j+2}](\cdot - t)_+^1 - [t_j,t_{j+1}](\cdot - t)_+^1}{t_{j+2}-t_j} \\
&= \frac{(t_{j+2}-t)_+^1 - (t_{j+1}-t)_+^1}{t_{j+2}-t_{j+1}} - \frac{(t_{j+1}-t)_+^1 - (t_j-t)_+^1}{t_{j+1}-t_j} \\
&= \begin{cases} \dfrac{t-t_j}{t_{j+1}-t_j} & t_j \le t < t_{j+1}, \\ \dfrac{t_{j+2}-t}{t_{j+2}-t_{j+1}} & t_{j+1} \le t < t_{j+2}, \\ 0 & \textit{elsewhere}. \end{cases}
\end{aligned}$$

upright

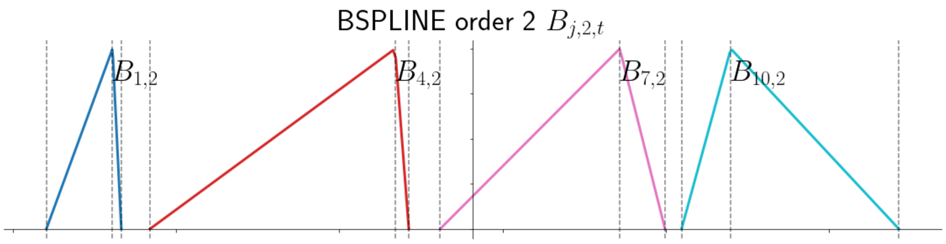

FIGURE 1.3: Some B-splines of order 2

Remark 1.2. *1. Schoenberg also defined a general B-spline ~~is his lecture 1 of~~* , lecture 1) *(Sch73) as* ~~follows~~

$$M_{j,k,\boldsymbol{t}}(t) = k[t_j,\dots,t_{j+k}](\cdot - t)_+^{k-1}.$$

It is related to De Boor's general B-spline ~~by~~ as

$$M_{j,k,t}(t) = \frac{k}{t_{j+k}-t_j}B_{j,k,\boldsymbol{t}},$$

which ~~amounts to~~ yield *the same definition provided* that $t_{j+k}-t_j = k$.

2. ~~*These*~~ De Boor's *B-splines are the same as* the *forward B-splines (1.20) of Schoenberg* ~~in the case~~ when $\boldsymbol{t} = \mathbb{Z}$, *i.e*,

$$B_{j,k,\mathbb{Z}}(t) = Q_k(t-j).$$

~~After having defined what~~ from the above definition of B-splines ~~are~~ in the most general case, ~~De Boor defines what~~ a *spline* is defined as follows.

Definition 1.8 (Bo01, p93). *A spline function of order k with knot sequence $\boldsymbol{t}$ is a linear combination of B-splines. $\mathcal{S}_{k,\boldsymbol{t}}$ denotes the collection of all such splines, i.e,*

$$\mathcal{S}_{k,\boldsymbol{t}} := \{\sum_{i=-\infty}^{\infty} c_i B_{i,k,\boldsymbol{t}} | \boldsymbol{c} \in \mathbb{R}^{\mathbb{Z}}\}. \tag{1.25}$$

As suggested in the ~~the title of this~~ subsection title, ~~means to convey,~~ the approach used by De Boor mirrors ~~is the reversed of~~ that of Schoenberg. Schoenberg first defines cardinal splines and then goes on to prove that cardinal B-splines form a basis for the collection formed by the cardinal splines. De Boor chose to start by defining B-splines and later on defined splines as those functions that are written as linear combinations of B-splines. In the end, ~~it sums up to the same~~ both definition, are equivalent given ~~the incoming~~ Theorem 1.5, ~~proved~~ proposed by Curry and Schoenberg.

~~For the needs of incoming properties we need to~~ Let us define the **basic interval** $I_{k,\boldsymbol{t}}$ as follows

$$I_{k,\boldsymbol{t}} = (t_-, t_+), \quad t_- := \begin{cases} t_k & \text{if } \boldsymbol{t} = (t_1, \dots) \\ \inf t_j & \text{otherwise} \end{cases}, \quad t_+ := \begin{cases} t_{n+1} & \text{if } \boldsymbol{t} = (\dots, t_{n+k}) \\ \sup t_j & \text{otherwise} \end{cases}. \tag{1.26}$$

In the following proposition, we summarize the most useful properties of B-splines.

Proposition 1.3 (Bo01, chapters IX, X, XI)**.** *1. (Recurrence relation)*

2. (Marsden's identity)

3. (Reproduction capabilities)

4. (Uniqueness and stability)

5. (Local linear independence)

Proof. □

~~We would like to finish this section with a powerful result by Curry and Schoenberg that completely characterize~~ The spline space $\mathcal{S}_{k,\boldsymbol{t}}$ ~~and~~ is finally completely characterized by the Curry-Schoenberg theorem, which also unveils its connection with piecewise-polynomial functions.

Theorem 1.5 (CS66,Bo01 p97)**.**

Chapter 2

The Hermite interpolation problem

the interpolation problem was

~~In Chapter 1 interpolation was introduced~~ from a historical perspective, ~~which in the first~~ initially stated as ~~place was interested in~~ finding an interpolating function at equally-spaced locations, as described in Chapter 1. Later on, as the interest in interpolation grew among mathematicians, a number of more general formulas were ~~found that each encompassed~~ identified to cover fairly arbitrary point configurations. However, configurations ~~in which locations could be~~ with repeated interpolation point locations remained largely unexplored until the general ~~interpolation~~ problem formulated by Hermite (Her78) in 1878. ~~In this paper~~ There, Hermite sets ~~himself~~ the task of finding a polynomial of degree $n-1$ that ~~would~~ satisfies a total of n interpolating conditions in the form of consecutive derivatives at distinct locations. Theorem 1.1 ~~shows~~ states that there exists a unique such polynomial, and we discussed ~~in the paragraph following the proof of the theorem~~ how ~~one could write~~ the interpolating polynomial can be written in *Newton form* using the extended divided difference operator. However, the result does not extend to the problem of interpolating at an infinite number of locations, even when only values and not derivatives are to be reproduced, as the degree of the interpolating polynomial would then be infinite. Another approach to the problem, as ~~already~~ previously mentioned at the end of Section 1.2, is to break it down ~~in~~to an infinite number of finite interpolation problems. ~~where~~ In that setting, the focus is restricted to a subset of the conditions, defining a *piece* of the interpolant, before joining ~~the~~ pieces in a "smooth" fashion. This is the essence of splines, which ~~will be~~ are fundamental in our formulation of solutions to the so-called *cardinal Hermite interpolation problem*.

2.1 Schoenberg's theorems

~~This section~~ We here introduces the fundamental results found ~~by~~ of I.J Schoenberg ~~about~~ on the general problem of interpolating a function and a certain number of consecutive derivatives on the integer grid. Schoenberg dedicated a good part of his life to his work on splines, starting from his landmark paper ~~in 1946~~ (Sch46), and continued with a series of papers ~~published at the beginning of the 70's~~ among which (Sch69, Sch72a, Sch72b, LS73, LS73, SS73) are ~~especially~~ of particular relevance to ~~the subject of~~ our work. This ~~part~~ section is highly inspired by his work. ~~to which we will refer a lot.~~ An elegant compilation of ~~his work~~ Schoenberg's work on splines can be found ~~in the book~~ (Sch73), ~~that~~ which contains 10 lectures, ~~where~~ each ~~lecture~~ presenting one specific aspect of splines while laying the foundations for the lectures that follows. ~~To begin with, let's remind us of some~~ We start by stating the notations ~~that~~ we will use through this chapter.

- $r, m, n \in \mathbb{N}^*$, $r \geq m$;
- $\mathbb{Z}_r = \{t_j = k | kr \leq j < k(r+1)\}_{j\in\mathbb{Z}}$ is the set of integers repeated r times;
- $\mathcal{S}_n$ are the (Schoenberg) cardinal splines of *order* n ~~as defined in~~ (Definition 1.5);
- $\mathcal{S}_{n,\mathbb{Z}_r}$ are the (De Boor) splines of order n with knots on $\mathbb{Z}_r$ ~~as defined in~~ (Definition 1.8);
- $\boldsymbol{y}^{(0)}, \ldots, \boldsymbol{y}^{(r-1)}$ are sequences of real numbers.

Cet exemple (bien que je l'aime beaucoup) est un peu surprenamment spécifique étant donné le ton du reste de la section. Suggestion : donner l'exemple des pixels de façon très générale (pas bio) et connecter au bio à la fin de l'exemple

2.1.1 The cardinal Hermite interpolation problem

Definition 2.1 (C.H.I.P, LS73, (10)-(12))**.** *The r sequences of real number $\boldsymbol{y}^{(0)}, \ldots, \boldsymbol{y}^{(r-1)}$ being prescribed, the cardinal Hermite interpolation problem C.H.I.P $(\boldsymbol{y}^{(0)}, \ldots, \boldsymbol{y}^{(r-1)}, \mathcal{V})$ for the vector space $\mathcal{V}$ is the problem of finding a function $S \in \mathcal{V}$ that agrees ~~with the sequences~~ $\boldsymbol{y}^{(0)}, \ldots, \boldsymbol{y}^{(r-1)}$ in the sense that*

$$\forall s = 0, \ldots, r-1, \forall k \in \mathbb{Z}, \quad S^{(s)}(k) = y_k^{(s)} \, . \tag{2.1}$$

In non-mathematical words, a C.H.I.P ~~is the problem of~~ aims at interpolating an unknown function for which we only have samples of its values and possibly its derivatives on a uniformly-spaced grid. Such questions arise, for instance in signal processing, in ~~the~~ image analysis, where the uniform grid is the ~~focal plane~~ array detector of ~~the~~ a camera and the samples are the pixels, all of which form ~~the~~ an image. ~~Therefore~~ An image is ~~only~~ thus a discretized version of an underlying continuous reality that we would like to ~~discover~~ approximate from the pixels. One such continuous reality ~~is~~ could for instance be the surface of a cell ~~that~~ which in a microscopy image, ~~if~~ successfully ~~reconstructed~~ modelled, ~~would~~ can increase our understanding of ~~the~~ its mechanical and biological properties ~~of the cells~~.

In Chapter 1, ~~we defined~~ the space of cardinal splines of order n, $\mathcal{S}_n$ was defined as those functions that are polynomials of order n on all intervals $(k, k+1)$ for $k \in \mathbb{Z}$ and that belong to the class $\mathcal{C}^{n-2}$. It is possible and relevant for the Hermite interpolation problem to consider similar functions ~~but~~ with less degrees of continuity at the joining points (or *knots*).

Definition 2.2 (Sch73, Lecture 5)**.** *The set $\mathcal{S}_{n,r}$ of cardinal splines of* order n *and* multiplicity *r denotes all functions S such that*

1. *$S \in \Pi_{<n}$ on $(k, k+1)$ for $k \in \mathbb{Z}$*;
2. *$S \in \mathcal{C}^{n-r-1}$*.

At times, it is convenient to consider the splines halfway between the integers, that is,

$$\mathcal{S}^*_{n,r} = \{S | S(\cdot + \frac{1}{2}) \in \mathcal{S}_{n,r}\} \, .$$

Remark 2.1. *This new set of splines connects to the sets defined in ~~the first part~~ Section XX, Definition Y as ~~follows~~,*

$$\mathcal{S}_{n,r} = \mathcal{S}_{n,\mathbb{Z}_r} = \Pi_{<n,\mathbb{Z},\boldsymbol{\nu}} \, ,$$

where $\boldsymbol{\nu}$ is the constant sequence with value $n-r$. Also, ~~note that~~ splines of order n are splines for the particular case of multiplicity $r = 1$, i.e.,

$$\mathcal{S}_n = \mathcal{S}_{n,1} \, .$$

Solutions ~~for~~ of the C.H.I.P 2.1 are readily obtained in the form of functions in $S_{n,r}$ or $S^*_{n,r}$ as expressed in the following lemma.

Lemma 2.1. *1. The Hermite interpolation problem 2.1 with $\mathcal{V} = \mathcal{S}_{n,r}$ has infinitely many solutions that form a linear manifold of dimension $n - 2r$.*

2. The Hermite interpolation problem 2.1 with $\mathcal{V} = \mathcal{S}^*_{n,r}$ has infinitely many solutions that form a linear manifold of dimension $n - r$.

The proof is given in ~~annex~~ Appendix A, extending Schoenberg's proof of (Sch73, Lemma 1.1, Lecture 4) for cardinal interpolation, which is none other than cardinal Hermite interpolation for $r = 1$. ~~The reader is encouraged to read it as we believe it is instructive. Let's give~~ An immediate corollary ~~to~~ of this lemma ~~in the special case where the~~ for sequences to be interpolated that vanish identically is obtained as follows.

Lemma 2.2. *Define*

$$\mathring{\mathcal{S}}_{n,r} = \{S \in \mathcal{S}_{n,r} | S^{(\rho)}(k) = 0, \rho = 0, \dots, r-1, k \in \mathbb{Z}\} , \tag{2.2}$$
$$\mathring{\mathcal{S}}^*_{n,r} = \{S \in \mathcal{S}^*_{n,r} | S^{(\rho)}(k) = 0, \rho = 0, \dots, r-1, k \in \mathbb{Z}\} . \tag{2.3}$$

Then, $\mathring{\mathcal{S}}_{n,r}, \mathring{\mathcal{S}}^*_{n,r}$ *are linear spaces with*

$$\dim \mathring{\mathcal{S}}_{n,r} = n - r, \quad \dim \mathring{\mathcal{S}}^*_{n,r} = n - 2r . \tag{2.4}$$

Proof. The proof is immediate using Lemma 2.1. Indeed, $\mathring{\mathcal{S}}_{n,r}$ is exactly the set of solutions to the C.H.I.P $(0, \dots, 0, \mathcal{S}_{n,r})$, which is not only a linear manifold but also a linear space as it contains the trivial spline. Furthermore, it has dimension $n - r$. A similar reasoning applies to $\mathring{\mathcal{S}}^*_{n,r}$. □

2.1.2 Spline interpolant to sequences of power growth

Let $\gamma \geq 0$ be a nonnegative real number and let

$$\mathcal{Y}^\gamma = \{\boldsymbol{y} \in \mathbb{R}^{\mathbb{Z}} | y_k = \mathcal{O}_{|k|\to\infty}(|k|^\gamma)\}, \tag{2.5}$$
$$\mathcal{S}^\gamma_{n,r} = \{S \in \mathcal{S}_{n,r} | S(t) = \mathcal{O}_{|t|\to\infty}(|t|^\gamma)\} \tag{2.6}$$

be respectively the spaces of power growth sequences and power growth splines with power γ. As mentioned ~~by Schoenberg~~ in (Sch73, p44), application of Markov's theorem on bounds shows that all derivatives of $S \in \mathcal{S}^\gamma_{n,r}$ satisfy the same decay condition. This is something Schoenberg most probably noticed after publishing (LS73) as, in the latter, he defines $\mathcal{S}^\gamma_{n,r}$ as the set of splines whose derivatives up to $r-1$ are of power growth γ.

To super interessant d'avoir repéré ça!

pourquoi bold?

From now, on we assume that $n = 2m$ **is an even number** with $m \geq r$. This is the choice made by Lipow and Schoenberg ~~in their article~~ (LS73) in order to avoid ~~doubling the size of their article~~ rewriting ~~similar~~ known results with slightly different notations. Indeed, all results ~~found~~ for ~~an~~ even values of n are easily extended to the case ~~n odd~~ of odd values of n using the spline space $S^*_{n,r}$ and all subsequently defined sets as we did in Lemma 2.1. ~~Lemma 2.1~~ This lemma also shows that ~~there is~~ the C.H.I.P 2.1 has an infinite number of solutions in the form of splines ~~to the C.H.I.P 2.1~~. However, if we consider functions in the set $\mathcal{S}^\gamma_{2m,r}$ of splines with power growth γ, the set of solutions to ~~the C.H.I.P~~ $(\boldsymbol{y}^{(0)}, \dots, \boldsymbol{y}^{(r-1)}, \mathcal{S}^\gamma_{2m,r})$ reduces to a unique element provided ~~the~~ of the sequences satisfy the same power growth. This is the ~~object~~ topic of the following theorem, which is the main result and is central to the theory of Hermite interpolation.

Theorem 2.1 (LS73, Theorems 1, 4)**.** *The C.H.I.P* $(\boldsymbol{y}^{(0)}, \dots, \boldsymbol{y}^{(r-1)}, \mathcal{S}^\gamma_{2m,r})$ *has a unique solution if and only if* $\boldsymbol{y}^{(0)}, \dots, \boldsymbol{y}^{(r-1)}$ *are in* $\mathcal{Y}^\gamma$*. In that case, the solution is explicitly given by the Lagrange-Hermite expansion*

$$S(t) = \sum_{k=-\infty}^{\infty} \sum_{s=0}^{r-1} y_k^{(s)} L_s(t-k), \tag{2.7}$$

where $L_0, \dots, L_{r-1}$ *are the fundamental splines as defined in Definition B.10. This expansion converges absolutely, and locally uniformly.*

~~This important theorem~~ The full proof is ~~proved~~ provided in (LS73) and (Sch73, Lecture 5). ~~Let's~~ We hereafter give ~~a here~~ a sketch of the proof and refer the reader to ~~the references for complete details~~ (LS73) for details.

Proof sketch. **(Unicity)** ~~Lipow and Schoenberg begin with the unicity part of the theorem. For that,~~ we start by noticing that the difference of two solutions S belongs to $\mathring{\mathcal{S}}^\gamma_{2m,r}$. From Lemma 2.2, this set is a linear space of dimension $2m-2r$. The $2m-2r$ "eigensplines" $\{S_1,\dots,S_{2m-2r}\}$ ~~(see~~ Definition B.9) form a basis ~~for~~ of this linear space (Sch73, Lemma 3, Lecture space 5). As a consequence, there exists coefficients $c_1,\dots,c_{2m-2r}$ such that

$$S=\sum_{j=1}^{2m-2r} c_j S_j \; . \tag{2.8}$$

Eigensplines ~~having the following behaviour at~~ behaving towards infinity as (LS73, (5.16), (5.17))

$$0<\overline{\lim}_{x\to-\infty}\frac{|S_j(x)|}{|\lambda_j|^x}<\infty \quad j=1,\dots,m-r \, , \tag{2.9}$$

$$0<\overline{\lim}_{x\to\infty}\frac{|S_j(x)|}{|\lambda_j|^x}<\infty \quad j=m-r+1,\dots,2m-2r \, , \tag{2.10}$$

~~While~~ and $S(t)=\mathcal{O}_{|t|\to\infty}(|t|^\gamma)$ having power growth γ at infinity, all ~~the~~ coefficients must vanish and so does S.

(Existence) An explicit solution is constructed using an expansion in terms of *fundamental* splines $L_s := L_{2m,r,s}$ for $s=0,\dots,r-1$ ~~that are defined~~ (~~see also~~ Definition B.10) They are defined as as even (if s is even) or odd (if s is odd) bounded functions such that

$$L_s(t)=\begin{cases} P_s(t) & \text{if } 0\le t\le 1 \, ,\\ \displaystyle\sum_{j=1}^{m-r} c_{j,s}S_j(t) & \text{if } t\ge 1 \, ,\end{cases} \tag{2.11}$$

with

C'est quoi cette notation?

$$P_s(t)=\begin{cases} \frac{1}{s!}t^s+a_{1,s}t^r+a_{2,s}t^{r+2}+\dots+a_{m-r+1,s}t^{2m-r}+a_{m-r+2,s}t^{2m-r+1}+\dots+a_{m,s}t^{2m-r} & \text{if } r-s\equiv 0[2] \, ,\\ \frac{1}{s!}t^s+a_{1,s}t^{r+1}+a_{2,s}t^{r+3}+\dots+a_{m-r,s}t^{2m-r-1}+a_{m-r+1,s}t^{2m-r}+\dots+a_{m,s}t^{2m-r} & \text{otherwise} \, ,\end{cases} \tag{2.12}$$

where the $2m-r$ unknowns $a_{1,s},\dots,a_{m,s},c_{1,s},\dots,c_{m-r,s}$ defining L_s are obtained as the unique solution ~~to~~ of the linear system of $2m-r$ equations given by

$$\forall\rho=0,\dots,2m-r-1,\quad P_s^{(\rho)}(1)=\sum_{j=1}^{m-r} c_{j,s}S_j^{(\rho)}(1) \; . \tag{2.13}$$

The solution is unique because the associated homogeneous system (removing the $\frac{1}{s!}t^s$) is non-singular. Indeed, if it were to be singular, there would exist a non trivial bounded spline in $\mathcal{S}^0_{2m,r}$ that vanishes on the integer grid, along with all its derivatives up to order $r-1$ ~~on the integer grid~~. However, from the proof of unicity we know that there can be at most one such spline. ~~and~~ The trivial spline ~~is~~ being one of them ~~hence the~~ leads to a contradiction.

A solution to the C.H.I.P $(\boldsymbol{y}^{(0)},\dots,\boldsymbol{y}^{(r-1)},\mathcal{S}^\gamma_{2m,r})$ is then given by

$$S(t)=\sum_{k=-\infty}^{\infty}\sum_{s=0}^{r-1} y_k^{(s)}L_s(t-k)$$

~~as~~ since, by construction, the *fundamental* splines L_s are in $\mathcal{S}^0_{2m,r}\subset\mathcal{S}^\gamma_{2m,r}$ and satisfy

$$\forall\rho=0,\dots,r-1,\forall k\in\mathbb{Z},\quad L_s^{(\rho)}(k)=\delta_k\delta_{s-\rho} \; . \tag{2.14}$$

This completes the proof. □

have been exted

~~Lipow and Schoenberg extended~~ These results to the cases of sequences in l^p for $1 \le p \le \infty$, ~~with~~ where the interpolating spline is a function in the set

$$L_r^p = \{F : \mathbb{R} \to \mathbb{C} | F^{(\rho)} \in L^p(\mathbb{R}, \mathbb{C}), \rho = 0, \dots, r-1\} . \tag{2.15}$$

~~It is the object of the following theorem.~~

Theorem 2.2 (LS73, Theorems 2, 4)**.** *Let $1 \le p \le \infty$. The C.H.I.P $(\boldsymbol{y}^{(0)}, \dots, \boldsymbol{y}^{(r-1)}, \mathcal{S}_{2m,r} \cap L_r^p)$ has a unique solution if and only if $\boldsymbol{y}^{(0)}, \dots, \boldsymbol{y}^{(r-1)}$ are in l^p. In that case, the solution is explicitly given by the Lagrange-Hermite expansion*

$$S(t) = \sum_{k=-\infty}^{\infty} \sum_{s=0}^{r-1} y_k^{(s)} L_s(t-k) .$$

This expansion converges absolutely and locally uniformly.

2.2 The Hermite generators

~~By~~ We refer to as *generators* ~~we mean~~ a set of functions that spans a given linear space in the usual algebraic sense. This set of functions can be infinite as we shall see. ~~but~~ In some cases, it is itself generated as all ~~possible~~ integer translates of a finite set of functions, ~~to~~ which we shall also refer to as *generators* by extension. In that sense, the set of functions arising from the Whittaker-Shannon interpolation formula, ~~for instance, that is to say~~ expressed as

$$SW = \{y(t) = \sum_{k=-\infty}^{\infty} y_k \operatorname{sinc}(t-k) | y_k \in \mathcal{Y}\}$$

with $\mathcal{Y}$ all real or complex sequences satisfying $\sum_{k=-\infty, k\neq 0}^{\infty} |\frac{y_k}{k}| < \infty$, is generated by the infinite set of functions $\{\operatorname{sinc}(\cdot - k)\}_{k\in\mathbb{Z}}$, which is itself generated by the single generator ~~{~~sinc~~}~~. We shall thus refer to sinc as a generator for the space SW.

In that sense, ~~also~~ the set of r fundamental splines $\{L_0, \dots, L_{r-1}\}$ is also a set of generators for the linear space

$$V = \{S(t) = \sum_{k=-\infty}^{\infty} \sum_{s=0}^{r-1} c_{k,s} L_s(t-k) | c_0, \dots, c_{r-1} \in \mathbb{R}^{\mathbb{Z}}\} . \tag{2.16}$$

~~Note that~~ The functions appearing as infinite sums in (2.16) ~~are~~ may not always be properly defined, in which case, conditions must be set on sequences for the series to converge. From Theorem 2.1 and Theorem 2.2, we know that, if the sequences of coefficients are in $\mathcal{Y}^\gamma$ or in l^p, then the series converges locally uniformly to a function in $S_{2m,r}^\gamma$ or $S_{2m,r} \cap L_r^p$ respectively.

~~From this writing,~~ Many questions naturally arise about the properties of the set of generators and the nature of the functions in V. ~~This sections aims at~~ We proceed to addressing these questions for $\{L_0, \dots, L_{r-1}\}$ that we call (polynomial) *Hermite generators* in the following section.

2.2.1 Hermite B-splines

Definition from fundamental splines

The Hermite B-splines ~~are functions that~~ were first introduced ~~by Schoenberg and Sharma~~ in an attempt to extend the B-spline representation existing in the set of splines of order n, $\mathcal{S}_n$, to the set of splines of order n and multiplicity r, $\mathcal{S}_{2m,r}$.

Definition 2.3 (Hermite B-splines, SS73, Definition 1)**.** *The Hermite B-splines of $S_{2m,r}$ are the r elements of $S_{2m,r}$*

$$N_0, \dots, N_{r-1},$$

defined by

$$N_s(t) = \sum_{k=-(m-r)}^{m-r} c_k L_s(t-k), \tag{2.17}$$

with c_k the coefficients of the Euler-Frobenius polynomial ~~for~~ of multiplicity r (B.7, ~~Annex~~ Appendix B).

~~Let's~~ Consider the case of multiplicity 1 (i.e $r = 1$) (i.e. 3) in order to see how ~~they~~ Hermite B-splines relate to the B-splines of S_{2m}, ~~namely M_{2m}~~ M_{2m}. As ~~the function~~ M_{2m} is supported in $(-m, m)$, the sequence $\boldsymbol{y} = (M_{2m}(k))_{k\in\mathbb{Z}}$ has at most $2m-1$ non zero elements and hence is bounded. From Theorem 2.1, the Lagrange-Hermite formula for $r = 1$ and the bounded sequence $\boldsymbol{y}$ defines the unique bounded element in $\mathcal{S}_{2m,1} = \mathcal{S}_{2m}$ that interpolates $\boldsymbol{y}$. As M_{2m} itself is one such element, unicity implies that the Lagrange-Hermite expansion for $\boldsymbol{y}$ equals M_{2m} everywhere ~~as~~, meaning that

$$\forall t \in \mathbb{R}, \quad M_{2m}(t) = \sum_{k=-(m-1)}^{m-1} M_{2m}(k) L_0(t-k). \tag{2.18}$$

From Definition B.11 of the Euler-Frobenius polynomial with running index shifted by $m-1$, one has that

$$\Pi_{2m}(t) = (2m-1)! \sum_{k=-(m-1)}^{m-1} Q_{2m}(k+m) t^{k+m-1}.$$

However, from (B.7), $(2m-1)!Q_{2m}(k+m) = c_k$ ~~and~~. From (1.21), $Q_{2m}(k+m) = M_{2m}(k)$. Consequently, multiplying (2.18) by $(2m-1)!$ and replacing $(2m-1)!M_{2m}(k)$ by c_k leads to

$$\forall t \in \mathbb{R}, \quad (2m-1)!M_{2m}(t) = N_0(t).$$

The central B-spline M_{2m} of order $2m$ is localized in the sense that it is supported within the compact set $[-m, m]$. ~~and~~ Therefore, so is N_0 in the case $r = 1$. The following lemma extends this observation to all Hermite B-splines ~~for~~ of general multiplicity r.

Lemma 2.3 (SS73, Lemma 2)**.** *The B-splines $N_0, \dots, N_{r-1}$ ~~have their~~ are supported in*

$$[-(m-r+1), m-r+1].$$

Moreover, N_s has the same parity as s.

The proof can be found in (SS73) and is reproduced in ~~Annex~~ Appendix A.

Fais bien la distinction entre recopier une preuve et ajouter des éléments (= la développer plus ou identifier des étapes "cachées"). "Reproduced" = pas clair dans quel cas on est

Hermite B-splines are a basis ~~for~~ of $\mathcal{S}^{(s)}_{2m,r}$

The "B" in B-splines stands for basis. Therefore, the names of $N_0, \dots, N_{r-1}$ suggest that they, with their integer translates, form a basis ~~for~~ of some spline spaces. This result is true ~~as we~~ and

together with ...

we shall prove ~~in this subsection in the sense~~ that, for fixed $s \in \{0,\dots,r-1\}$, N_s ~~with~~ and its translates ~~is~~ forms a basis for the subspace

$$\mathcal{S}^{(s)}_{2m,r} = \{S \in \mathcal{S}_{2m,r} | S^{(\rho)}(k) = 0, \rho \in \{0,\dots,r-1\} \setminus \{s\}, k \in \mathbb{Z}\} \; . \tag{2.19}$$

Observe first that $N_s \in \mathcal{S}^{(s)}_{2m,r}$. Indeed, combining (2.14) and (2.17) shows that

$$\forall \rho \in \{0,\dots,r-1\}, \forall k \in \mathbb{Z}, \quad N_s^{(\rho)}(k) = \delta_{s-\rho} \sum_{k=-(m-r)}^{m-r} c_k \delta_{l-k} \; . \tag{2.20}$$

~~We~~ Thus, only ~~the~~ derivatives of order s of N_s may be non-zero on the integer grid. The definition (2.19) indicates that $\mathcal{S}^{(s)}_{2m,r}$ is invariant by integer shift ~~and~~. Therefore, all integer translates of N_s are also in that subspace. The set $\mathcal{S}^{(s)}_{2m,r}$ being a linear space, any function S of the form

$$S = \sum_{k=-\infty}^{\infty} c_k N_s(\cdot - k) \tag{2.21}$$

is also in $\mathcal{S}^{(s)}_{2m,r}$, so that

$$\mathrm{span}\{N_s(\cdot - k) | k \in \mathbb{Z}\} \subseteq \mathcal{S}^{(s)}_{2m,r} \; . \tag{2.22}$$

~~Note that~~ The function ~~given by~~ (2.21) is well-defined for any sequence~~s~~ of coefficients ~~as~~ since, at any real t, the quantity $N_s(t-k)$ does not vanish only for a finite number of integers k.

It remains to be shown that any spline $S \in \mathcal{S}^{(s)}_{2m,r}$ can also be written as (2.21) for a unique sequence $\boldsymbol{c}$. If that is the case, then $\{N_s(\cdot - k) | k \in \mathbb{Z}\}$ ~~will be~~ is a basis ~~for~~ of $\mathcal{S}^{(s)}_{2m,r}$ and the name~~s~~ "Hermite B-splines" ~~will be~~ is justified. In order to be able to prove that, Schoenberg and Sharma ~~had to use the~~ assum~~ption~~e (SS73, Assumption 1) that the polynomial $\Pi_{2m,r}$ is irreducible over the rational field. ~~As mentioned by the authors,~~ This assumption is most probably true but is too ~~deep a theorem~~ complex compared to the arguments used so far and ~~so~~ therefore is slightly unsatisfying. ~~Having made this assumption though, authors were then able to~~ From these, one can then show that the $\{N_s(\cdot - k) | k \in \mathbb{Z}\}$ are locally linearly independent, ~~in the sense~~ meaning that the relation

$$\sum_{k=-(2m-2r+1)}^{0} c_k N_s(t-k) = 0, \quad -(m-r+1) \le t \le -(m-r)$$

can only hold if all the coefficients vanish ~~i.e.~~ as

$$c_{-(2m-2r+1)} = \dots = c_0 = 0 \; .$$

However, it was however shown ~~by S. Lee in his paper~~ (Lee73) that this assumption is not ~~needs not be~~ needed and that this local linear independence always holds.

Lemma 2.4 (Lee73, Lemma 1)**.** *For every $s \in \{0,\dots,r-1\}$, the $2m-2r+2$ polynomials*

$$N_s, N_s(\cdot + 1), \dots, N_s(\cdot + 2m - 2r + 1)$$

are linearly independent over $(-(m-r+1), -(m-r))$.

The proof consists in establishing that the determinant of the matrix of a homogeneous system of equations is non-zero, which is quite technical. ~~and explains why an entire paper is dedicated to that task.~~ ~~The~~ We refer the reader ~~is referred~~ to ~~the reference~~ (Lee73) for ~~all the~~ details.

The following theorem can now be established and concludes our discussion ~~and therefore this paragraph~~.

Theorem 2.3 (SS73, Theorem 3). *Every $S \in S_{2m,r}^{(s)}$ admits a unique representation of the form*

$$S = \sum_{k=-\infty}^{\infty} a_k N_s(\cdot - k) \quad (2.23)$$

2.2.2 The Riesz-Schauder basis property

2.2.3 Support and approximation properties

2.2.4 Multiresolution and mutliwavelets

2.3 The Hermite exponential splines

Chapter 3

Application to the modeling of surfaces

Appendix A

~~Annex for~~ Proofs

Proof of Lemma 2.1 → très clair !

Proof. We rewrite ~~reproduce~~ here the proof given by Schoenberg with ~~slightly~~ more details. The main observation for the proof is that any $S \in \mathcal{S}_{n,r}$ can be uniquely written in the form

$$S(t) = P(t) + \sum_{i=1}^{\infty} \sum_{s=0}^{r-1} c_i^{(s)} (t-i)_+^{n-1-s} + \sum_{i=-\infty}^{0} \sum_{s=0}^{r-1} c_i^{(s)} (-t+i)_+^{n-1-s},$$

with $P \in \Pi_{<n}$. For that, we observe first that $S_{|(0,1)}$ is a polynomial of order n and ~~so~~ $P = S_{|(0,1)}$ is thus uniquely defined. Then, looking at decreasing derivatives from $n-1$ to $n-r$ at a point in $(1,2)$, (for instance 1.5) ~~say 1.5 for instance,~~ determines uniquely the constants $c_1^{(s)}$ for $s = 0, \dots, r-1$ as they satisfy

$$c_1^{(0)}(n-1)! = S^{(n-1)}(1.5) - P^{(n-1)}(1.5),$$

$$c_1^{(0)} \frac{1}{2} \frac{(n-1)!}{1!} + c_1^{(1)}(n-2)! = S^{(n-2)}(1.5) - P^{(n-2)}(1.5),$$

$$\vdots$$

$$\sum_{s=0}^{r-1} c_1^{(s)} \frac{1}{2^{r-1-s}} \frac{(n-s-1)!}{(r-1-s)!} = S^{(n-r)}(1.5) - P^{(n-r)}(1.5).$$

Conversely, choosing $\{c_1(s)\}_{s=0,\dots,r-1}$ as the unique solution of ~~to~~ the system above provides us with the ~~a~~ polynomial on $(1,2)$

$$P(t) + \sum_{s=0}^{r-1} c_1^{(s)} (t-1)^{n-1-s}$$

that agrees with S at $(\underbrace{1,\dots,1}_{n-r}, \underbrace{1.5,\dots,1.5}_{r})$. As $S_{|(1,2)}$ is also a polynomial that satisfies the same conditions, unicity leads to

$$S_{|(1,2)} = P + \sum_{s=0}^{r-1} c_1^{(s)} (\cdot - 1)^{n-1-s}.$$

The same reasoning applies to $c_2^{(s)}, \dots$ and $c_{-1}^{(s)}, \dots$. Consider now that P is chosen so that

$$\forall s = 0, \dots, r-1, \quad P^{(s)}(0) = y_0^{(s)}, \quad P^{(s)}(1) = y_1^{(s)}.$$

The constants $c_i^{(s)}$ are then uniquely determined by the interpolation conditions at $y_i^{(s)}$. This leaves $n - 2r$ ~~free~~ degrees of freedom for P, hence the dimension of the manifold of solutions.

Similarly, any $S \in \mathcal{S}^*_{n,r}$ can be uniquely written in the form

$$S(t) = Q(t) + \sum_{i=1}^{\infty} \sum_{s=0}^{r-1} c_i^{(s)} \left(t - \frac{2i-1}{2}\right)_+^{n-1-s} + \sum_{i=-\infty}^{0} \sum_{s=0}^{r-1} c_i^{(s)} \left(-t + \frac{2i-1}{2}\right)_+^{n-1-s},$$

with Q a polynomial of order n on $(-\frac{1}{2}, \frac{1}{2})$ ~~and~~. The coefficients $c_i^{(s)}$ are determined by the interpolation conditions once Q is chosen, so that

$$\forall s = 0, \ldots, r-1, \quad Q^{(s)}(0) = y_0^{(s)}.$$

This leaves $n - r$ ~~free~~ degrees of freedom for Q, hence the dimension of the manifold of solutions. □

Proof of Lemma 2.3

Proof. Let $s \in \{0, \ldots, r-1\}$ be fixed. The Hermite B-spline N_s is, by definition, given as

$$N_s = \sum_{k=-(m-r)}^{m-r} c_k L_s(\cdot - k),$$

where $c_{-(m-r)}, \ldots, c_{m-r}$ are the coefficients of the Euler-Frobenius polynomial from Proposition B.6 ~~as~~,

$$\Pi_{2m,r}(t) = \sum_{k=-(m-r)}^{m-r} c_k t^{k+m-r}.$$

Let $t > m - r + 1$. Then, $t - k > 1$ for all $k = -(m-r), \ldots, m-r$. However, on $(1, \infty)$, the fundamental splines are expressed as linear combination of the $m - r$ decreasing eigensplines ~~as~~ as

$$L_s = \sum_{l=1}^{m-r} d_l S_l, \quad \text{on} \quad (1, \infty).$$

Therefore,

$$\begin{aligned}
N_s(t) &= \sum_{l=1}^{m-r} d_l \sum_{k=-(m-r)}^{m-r} c_k S_l(t-k) \\
&= \sum_{l=1}^{m-r} d_l S_l(t) \sum_{k=-(m-r)}^{m-r} c_k \lambda_l^{-k} && (S_l(t+1) = \lambda_l S(t)) \\
&= \sum_{l=1}^{m-r} d_l S_l(t) \lambda^{-(m-r)} \sum_{k=-(m-r)}^{m-r} c_k \lambda_l^{k+m-r} && (c_k = c_{-k}) \\
&= 0. && (\Pi_{2m,r}(\lambda_l) = 0)
\end{aligned}$$

We have proved that N_s vanishes on $(1, \infty)$. Let's now prove that $N_s(-t) = (-1)^s N_s(t)$ for all t ~~and~~ the proof ~~of the Lemma will be complete~~. For $t \in \mathbb{R}$,

to complete

$$\begin{aligned}
N_s(-t) &= \sum_{k=-(m-r)}^{m-r} c_k L_s(-t-k) \\
&= \sum_{k=-(m-r)}^{m-r} (-1)^s c_k L_s(t+k) && (L_s \text{ has same parity as } s) \\
&= \sum_{k=-(m-r)}^{m-r} (-1)^s c_k L_s(t-k) && (c_k = c_{-k}) \\
&= (-1)^s N_s(t) .
\end{aligned}$$

$\square$

Appendix B

The spline~~s~~ zoo

This appendix aims at ~~putting in one place~~ gathering all ~~the~~ splines that we have encountered in the litterature in order to catch a glimpse of their broad diversity and ~~in order to~~ facilitate comparisons. If relevant, connections will be made between the different notations and definitions. In order to ~~have a presentation as~~ be as coherent as possible, we will stick to the following notation rules.

1. n or m related quantities denote the *order* of the spline, not to be confused with the *degree*;
2. r denotes the multiplicity of the interpolation or, in other words, the order up to which derivatives are reproduced;
3. j, s are used ~~for the~~ as running indices from 0 to $r-1$;
4. i, k, l are used ~~for~~ as indices for running integers;
5. j is used for the purely imaginary number $j = \sqrt{-1}$;
6. calligraphed letters such as $\mathcal{S}$ or $\mathcal{C}$ denote functional sets;
7. bold lowercase letters such as $\boldsymbol{c}$ denote sequences or vectors;
8. bold uppercase letters such as $\boldsymbol{M}$ denote matrices.

To avoid confusion with the index, use \mathrm{j}

B.1 Theoretical papers

B.1.1 General B-splines

Definition B.1 (Sch73, Lecture 1~~/2~~)**.** *The general B-spline of order n with knots $t_0 < \cdots < t_n$ is* given by

$$M(t; t_0, \ldots, t_n) = n[t_0, \ldots, t_n](\cdot - t)_+^{n-1} . \tag{B.1}$$

Proposition B.1. ~~:~~ *(Peano's theorem) For any $f \in \mathcal{C}^n$,* we have that

$$[t_0, \ldots, t_n]f = \frac{1}{n!}\int_{t_0}^{t_n} M(t; t_0, \ldots, t_n) f^{(n)}(t)dt .$$

B.1.2 Cardinal splines

Definition B.2 (B-splines equidistant knots, Sch73, Lecture 2)**.** *The forward B-spline of order n is* given by

$$Q_n(t) = n[0, 1, \ldots, n](\cdot - t)_+^{n-1} ,$$

and the central B-spline of order n by

$$M_n(t) = n\left[\frac{-n}{2}, \frac{-n}{2}+1, \ldots, \frac{n}{2}\right](\cdot - t)_+^{n-1} = Q_n(t + \frac{n}{2}) .$$

pourquoi ces equations ne sont-elles pas numerotees?

B.1.3 Exponential splines

Definition B.3 (Exponential spline of degree n for the base t, Sch73, p17)**.** *The exponential spline of degree n for the base t is the function in $\mathcal{S}_{n+1}$ defined by*

$$\Phi_n(x;t) = \sum_{k=-\infty}^{\infty} t^k Q_{n+1}(x-k)\ . \tag{B.2}$$

Proposition B.2. *The exponential Euler spline satisfies*

1. $\Phi_n(x+1;t) = t\Phi_n(x;t)$, $\forall x, t$;
2. $\Phi_n^{(n)}(x;t) = (1-t^{-1})^n$, *for* $0 < x < 1$.

Definition B.4 (Euler-Frobenius polynomial, Sch73, p22)**.** *The Euler-Frobenius polynomial of order n is* given by

$$\Pi_n(t) = n! \sum_{j=1}^{n} Q_n(j) t^{j-1} \quad . \tag{B.3}$$

It has $n-1$ simple zeros in reciprocal pairs .

$$\lambda_{n-1} < \cdots < \lambda_1 < 0$$

Definition B.5 (Exponential Euler polynomial, Sch73, p21)**.** *The exponential Euler polynomial of degree n is* given by

$$A_n(x;t) = \frac{n!}{(1-t^{-1})^n}\Phi_n(x;t),\ \ 0 \le x \le 1 \quad . \tag{B.4}$$

Proposition B.3. *The exponential Euler polynomial satisfies*

$$\frac{t-1}{t-e^z}e^{xz} = \sum_{n=0}^{\infty} \frac{A_n(x;t)}{n!} z^n \ ;$$

$$A_n(x;t) = x^n + a_1(t)\binom{n}{1} + \cdots + a_n(t), \quad a_n(t) = \frac{\Pi_n(t)}{(t-1)^n}\ .$$

Definition B.6 (Exponential Euler spline, Sch73, p26)**.** *The (normalized) exponential Euler spline of degree n for base λ is* given by

$$S_n(x;\lambda) = \frac{\Phi_n(x;\lambda)}{\Phi_n(0;\lambda)} \text{ for } \lambda \notin \{\lambda_1, \ldots, \lambda_{n-1}\}\ . \tag{B.5}$$

*It is in $\mathcal{S}_{n+1}$ and ~~has the property of~~ interpolat~~ing~~*es *λ^x at* the *integers.*

Definition B.7 (Eigensplines, Sch73, p26)**.** *The eigenspline for* the *base λ_j is* given by

$$S_j(x) = \Phi_n(x;\lambda_j)\ . \tag{B.6}$$

*It is in $\mathcal{S}_{n+1}$ and ~~has the property of~~ vanish~~ing~~*es *at* the *integers.*

B.1.4 Eigensplines $S_1, \ldots, S_{2m-2r}$ ~~&~~ and fundamental splines $L_0, \ldots, L_{r-1}$

Definition B.8 (Euler-Frobenius polynomial of multiplicity r, Sch73, p47)**.** *The Euler-Frobenius polynomial ~~for~~ of multiplicity r is the degree $2m-2r$ polynomial* given by

$$\Pi_{2m,r}(\lambda) = (-1)^{m(r-1)} \begin{vmatrix} 1 & \binom{r}{1} & \ldots & \binom{r}{r-1} & 1-\lambda & 0 & \ldots & 0 \\ 1 & \binom{r+1}{1} & \ldots & \binom{r+1}{r-1} & \binom{r+1}{r} & 1-\lambda & \ldots & 0 \\ \vdots & & & & & & & \vdots \\ 1 & \binom{2m-r-1}{1} & & \ldots & & & \binom{2m-r-1}{2m-r-2} & 1-\lambda \\ 1 & \binom{2m-r}{1} & & \ldots & & & \binom{2m-r}{2m-r-2} & \binom{2m-r}{2m-r-1} \\ \vdots & & & & & & & \vdots \\ 1 & \binom{2m-1}{1} & & \ldots & & & \binom{2m-1}{2m-r-2} & \binom{2m-1}{2m-r-1} \end{vmatrix}. \tag{B.7}$$

Proposition B.4. *The Euler-Froebenius polynomial has integer coefficients* given by

$$\Pi_{2m,r}(\lambda) = \sum_{k=0}^{2m-2r} c_{k-(m-r)}\lambda^k, \quad c_0 > 0, c_{-k} = c_k, c_{-(m-r)} = c_{m-r} = \pm 1. \tag{B.8}$$

It has $2m-2r$ real simple zeros of sign $(-1)^r$.

$$\boxed{0 < |\lambda_1| < \cdots < |\lambda_{m-r}| < 1 < |\lambda_{m-r+1}| < \cdots < |\lambda_{2m-2r}|}$$

Definition B.9 (Eigensplines ~~for~~ of muliplicity r, Sch73, Lecture 5)**.** *The eigensplines ~~for~~ of multiplicity r are the $2m-2r$ functions that have polynomial form on $[0,1]$*

$$S_j(x) = a_{0,j}x^{2m-1} + \binom{2m-1}{1}a_{1,j}x^{2m-2} + \cdots + \binom{2m-1}{2m-r-1}a_{2m-r-1,j}x^r, \quad 0 \leq x \leq 1,$$

where ~~with~~ $a_{0,j}, \ldots, a_{2m-r-1,j}$ is *the unique solution ~~to~~ of the system formed ~~of~~ by the homogeneous equations (B.7) at $\lambda = \lambda_j$ and the equation $S_j^{(r)}(0) = 1$. Elsewhere, the eigenspline is defined by the periodic relation*

$$S_j(x+1) = \lambda_j S_j(x), \quad x \in \mathbb{R}.$$

These eigensplines are in $\mathcal{S}_{2m,r}$ and ~~have the property of~~ vanish~~ing~~ up to $r-1$ at the *integers, more compactly ~~summarized by saying that~~* expressed as *$S_j \in \mathring{\mathcal{S}}_{2m,r}$.*

Examples

1. Let $r = 1$. If $m = 1$, $\dim \mathring{\mathcal{S}}_{2m,r} = 0$ and the only eigenspline is the trivial function. If $m = 2$, $P = S_{|[0,1]}$ takes the form

$$P = a_0x^3 + 3a_1x^2 + 3a_2x,$$

with homogeneous system

$$\begin{aligned} a_0 + 3a_1 + 3a_2 &= 0, \\ 3a_0 + 6a_1 + 3(1-\lambda)a_2 &= 0, \\ 6a_0 + 6a_1(1-\lambda) &= 0. \end{aligned}$$

the parameter λ is chosen so that the matrix of the system in singular in order for the eigenspline not to be trivial. Coefficients are then determined up to a constant, which is fixed by the ~~following~~ constraint $S^{(1)}(0) = 1$, adding the equation

$$3a_2 = 1.$$

The matrix of the homogeneous system is singular if and only if λ is a zero of

$$\Pi_{2m,r}(\lambda) = 1 + 4\lambda + \lambda^2,$$

yielding ~~that is~~ $\lambda_1 = -2 + \sqrt{3}, \lambda_2 = -2 - \sqrt{3}$.

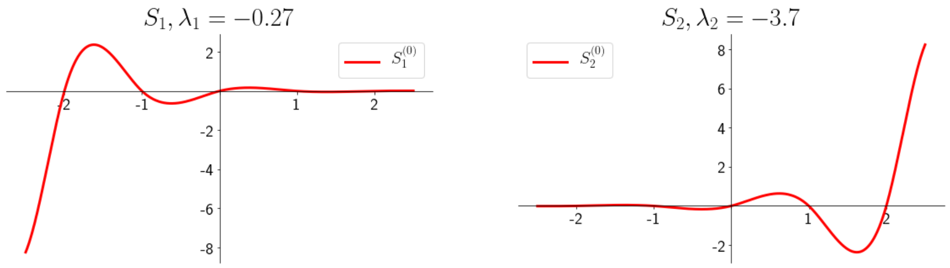

FIGURE B.1: Eigensplines for $r = 1$, $m = 2$

2. Let $r = 2$. If $m = 2$, $\dim \mathring{\mathcal{S}}_{2m,r} = 0$ and the only eigenspline is the trivial function. If $m = 3$, $P = S_{|[0,1]}$ takes the form

$$P = a_0x^5 + 5a_1x^4 + 10a_2x^3 + 10a_3x^2,$$

with homogeneous system

$$\begin{aligned} a_0 + 5a_1 + 10a_2 + 10a_3 &= 0, \\ 5a_0 + 20a_1 + 30a_2 + 20a_3 &= 0, \\ 20a_0 + 60a_1 + 60a_2 + 20a_3(1-\lambda) &= 0, \\ 60a_0 + 120a_1 + 60a_2(1-\lambda) &= 0. \end{aligned}$$

the parameter λ is chosen so that the matrix of the system in singular in order for the eigenspline not to be trivial. Coefficients are then determined up to a constant, which is fixed by the ~~following~~ constraint $S^{(2)}(0) = 1$, adding the equation

$$20a_3 = 1.$$

The matrix of the homogeneous system is singular if and only if f λ is a zero of

$$\Pi_{2m,r}(\lambda) = 1 - 6\lambda + \lambda^2,$$

yielding ~~that is~~ $\lambda_1 = 3 - 2\sqrt{2}, \lambda_2 = 3 + 2\sqrt{2}$.

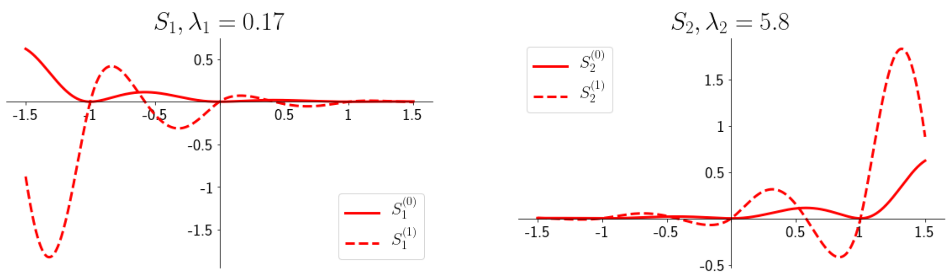

FIGURE B.2: Eigensplines for $r = 2$, $m = 3$

Proposition B.5 (Sch73, p49)**.** *To every $S \in \mathring{\mathcal{S}}_{2m,r}$ corresponds a unique sequence $(c_1, \dots, c_{2m-2r})$ such that*

$$S = \sum_{j=1}^{2m-2r} c_j S_j \ . \tag{B.9}$$

Definition B.10 (Fundamental splines, Sch73, (5.1)-(5.5), Lecture 5)**.** *The r fundamental splines $L_s := L_{2m,r,s}$ for $s = 0, \dots, r-1$ are defined as the even (if s is even) or odd (if s is odd) functions such that*

$$L_s(t) = \begin{cases} P_s(t) & \text{if } 0 \le t \le 1 \, ,\\ \displaystyle\sum_{j=1}^{m-r} c_{j,s} S_j(t) & \text{if } t \ge 1 \, , \end{cases} \tag{B.10}$$

with (LS73, p20)

$$P_s(t) = \begin{cases} \frac{1}{s!} t^s + a_{1,s} t^r + a_{2,s} t^{r+2} + \dots + a_{m-r+1,s} t^{2m-r} & \\ \quad + a_{m-r+2,s} t^{2m-r+1} + \dots + a_{m,s} t^{2m-r} & \text{if } r - s \equiv 0[2] \, ,\\ \frac{1}{s!} t^s + a_{1,s} t^{r+1} + a_{2,s} t^{r+3} + \dots + a_{m-r,s} t^{2m-r-1} & \\ \quad + a_{m-r+1,s} t^{2m-r} + \dots + a_{m,s} t^{2m-r} & \text{otherwise} \ . \end{cases} \tag{B.11}$$

~~and the~~ The $2m - r$ *unknowns $a_{1,s}, \dots, a_{m,s}, c_{1,s}, \dots, c_{m-r,s}$ defining L_s are obtained as the unique solution to the linear system of $2m - r$ equations*

$$\forall \rho = 0, \dots, 2m-r-1, \quad P_s^{(\rho)}(1) = \sum_{j=1}^{m-r} c_{j,s} S_j^{(\rho)}(1) \ . \tag{B.12}$$

Examples

1. If $r = 1, m = 1$, L_0 is even, and has compact support in $[-1, 1]$ with

$$L_0(t) = \begin{cases} 1 + a_{1,0} t & \text{if } 0 \le t \le 1 \, ,\\ 0 & \text{if } t \ge 1 \ . \end{cases} \tag{B.13}$$

~~and the~~ The coefficient $a_{1,0}$ satisfies

$$1 + a_{1,0} = 0 \, , \tag{B.14}$$

~~that is~~ and L_0 is, therefore the hat function.

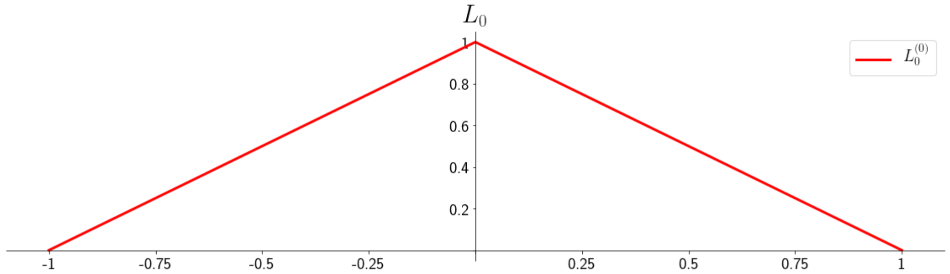

FIGURE B.3: Fundamental spline for $r = 1$, $m = 1$

2. If $r = 1, m = 2$, L_0 is even, and infinitely supported, with

$$L_0(t) = \begin{cases} 1 + a_{1,0}t^2 + a_{2,0}t^3 & \text{if } 0 \le t \le 1, \\ c_{1,0}S_1(t) & \text{if } t \ge 1 . \end{cases}$$

~~and~~ The coefficients satisfy

$$\begin{aligned} a_{1,0} + a_{2,0} - c_{1,0}S_1(1) &= -1, \\ 2a_{1,0} + 3a_{2,0} - c_{1,0}S_1^{(1)}(1) &= 0, \\ 2a_{1,0} + 6a_{2,0} - c_{1,0}S_1^{(2)}(1) &= 0 . \end{aligned} \tag{B.15}$$

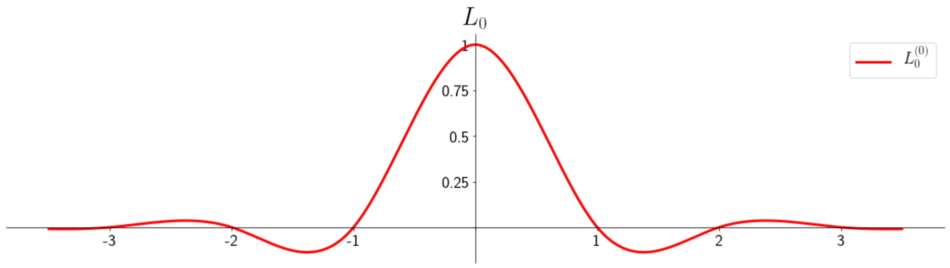

FIGURE B.4: Fundamental spline for $r = 1$, $m = 2$

3. If $r = 2, m = 2$, L_0 and L_1 are compactly supported in $[-1, 1]$ and expressed as

$$L_0(t) = \begin{cases} 1 + a_{1,0}t^2 + a_{2,0}t^3 & \text{if } 0 \le t \le 1, \\ 0 & \text{if } t \ge 1, \end{cases}$$

$$L_1(t) = \begin{cases} t + a_{1,1}t^2 + a_{2,1}t^3 & \text{if } 0 \le t \le 1, \\ 0 & \text{if } t \ge 1 . \end{cases}$$

~~and~~ The coefficients satisfy

$$\begin{aligned} a_{1,0} + a_{2,0} &= -1, \\ 2a_{1,0} + 3a_{2,0} &= 0, \end{aligned}$$

and

$$\begin{aligned} a_{1,1} + a_{2,1} &= -1, \\ 2a_{1,1} + 3a_{2,1} &= -1 . \end{aligned}$$

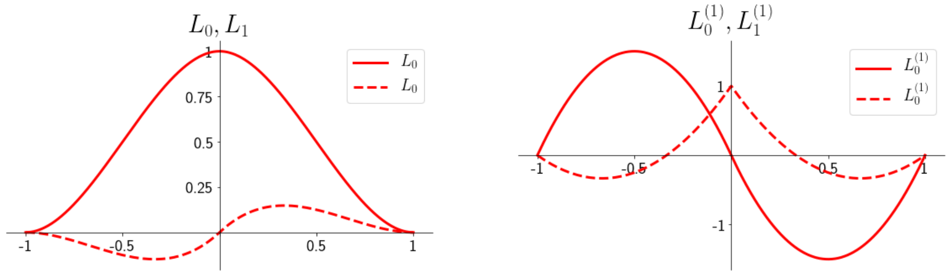

FIGURE B.5: Fundamental splines for $r = 2$, $m = 2$

4. If $r=2, m=3$, L_0 and L_1 are infinitely supported ~~with~~ and given by

$$L_0(t)=\begin{cases}1+a_{1,0}t^2+a_{2,0}t^4+a_{3,0}t^5 & \text{if } 0\le t\le 1,\\ c_{1,0}S_1 & \text{if } t\ge 1,\end{cases}$$

$$L_1(t)=\begin{cases}t+a_{1,1}t^3+a_{2,1}t^4+a_{3,1}t^5 & \text{if } 0\le t\le 1,\\ c_{1,1}S_1 & \text{if } t\ge 1.\end{cases}$$

~~and~~ The coefficients satisfy

$$\begin{aligned}a_{1,0}+a_{2,0}+a_{3,0}-c_{1,0}S_1(1) &= -1,\\ 2a_{1,0}+4a_{2,0}+5a_{3,0}-c_{1,0}S_1^{(1)}(1) &= 0,\\ 2a_{1,0}+12a_{2,0}+20a_{3,0}-c_{1,0}S_1^{(2)}(1) &= 0,\\ 24a_{2,0}+60a_{3,0}-c_{1,0}S_1^{(3)}(1) &= 0,\end{aligned}$$

and

$$\begin{aligned}a_{1,1}+a_{2,1}+a_{3,1}-c_{1,1}S_1(1) &= -1,\\ 3a_{1,1}+4a_{2,1}+5a_{3,1}-c_{1,1}S_1^{(1)}(1) &= -1,\\ 6a_{1,1}+12a_{2,1}+20a_{3,1}-c_{1,1}S_1^{(2)}(1) &= 0,\\ a_{1,1}+24a_{2,1}+60a_{3,1}-c_{1,1}S_1^{(3)}(1) &= 0.\end{aligned}$$

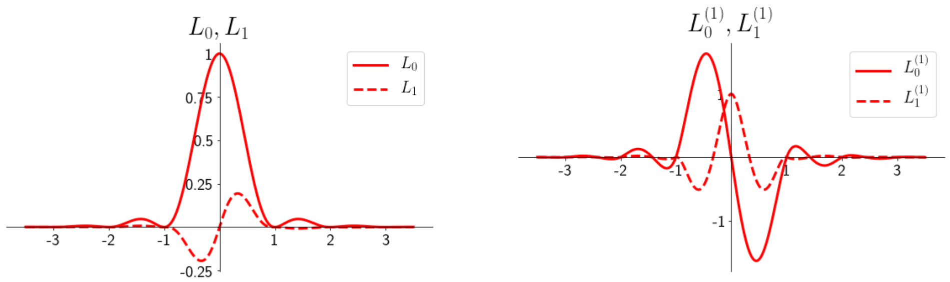

FIGURE B.6: Fundamental splines for $r=2$, $m=3$

B.1.5 Hermite B-splines

B.1.6 Exponential Euler splines

Definition B.11 (Euler-Frobenius polynomial, Sch73, Lecture 3, p22)**.** *The Euler-Frobenius polynomial of order n is* given by

$$\Pi_n(t)=n!\sum_{k=0}^{n-1}Q_{n+1}(k+1)t^k. \tag{B.16}$$

Proposition B.6. *The Euler-Frobenius polynomial is such that*

1. $\dfrac{\Pi_n(t)}{(1-t)^{n+1}}=\displaystyle\sum_{k=1}^{\infty}k^n t^{k-1}$;
2. *It has $n-1$ simple zeros* that are *reciprocal in pairs* .

$$\lambda_{n-1}<\lambda_{n-2}<\cdots<\lambda_2<\lambda_1<0$$

Definition B.12 (Exponential Euler spline, Sch73, Lecture 2, p17). *The exponential Euler spline of degree n ~~to the~~ n base t is* given by

$$\Phi_n(x;t) = \sum_{k=-\infty}^{\infty} t^k Q_{n+1}(x-k) \quad . \tag{B.17}$$

Definition B.13 (Exponential Euler polynomial, Sch73, Lecture 3, p21). *The exponential Euler polynomial of degree n for base t is* expressed as

$$A_n(x;t) = \frac{n!}{(1-t^{-1})^n}\Phi_n(x;t) \quad . \tag{B.18}$$

B.1.7 Exponential Euler-Hermite splines

B.1.8 L-splines

B.2 Applied splines

↳ qu'est ce qu'une spline theorique ? :)

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