Summary paper 11: Snakes with an ellipse-reproducing property Delgado et al

Yoann Pradat

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[3] Reviews of snakes energies and algorithms Jacob, Blu, Unser "Efficients energies and algorithms for parametric snakes". Snakes are popular tools for outlining in all sorts of images. They vary from one another by the type of snake used and, in the case on continuous snakes, by the basis generators of the representation. Snakes may be classified into

1. discrete snakes that make use of a large number of control points
2. continuous snakes
3. implicit snakes where representation of the curve is described at level set of a surface.

Continous snakes are parametrized by only a few points which, along with small support of the basis generators, makes computations very fast. The drawback from having only a few anchor points is of course that not every curve can be represented, only those in the span of integer shifts of the generators are perfectly reproduced. However good approximation can be obtained by simply increasing the number of control points. In this paper authors propose generators that have minimal support, an important feature for fast and efficient algorithms and that have the property of perfectly reproducing ellipses.

<u>Remark</u> In the parametric representation of a curve we always consider the space spanned by integer shifts of the basis generators as this allows to take advantage of fast and stable algorithms.

Desired properties of the snake

In here we are interested in reproducing closed curves from a M-periodic sequence $\{c[k]\}_{k\in\mathbb{Z}}$ of control points in the following form

$$r(t) = \sum_{k\in\mathbb{Z}} c[k]\varphi(Mt-k) \tag{1}$$

There are some properties we would like φ to satisfy in order to have a good snake model. As usual the properties are

1. Representation should be unique and numerically stable. Verified if φ satisfies Riesz-basis property.
2. Affine invariance i.e representation of affinely transformed curve is the affine transformation of the representation. Verified iif φ satisfies partition of unity condition
3. Well-defined curvature that is $\kappa(x_1, x_2) = \frac{\dot{x_1}\ddot{x_2} - \ddot{x_1}\dot{x_2}}{(\dot{x_1}+\dot{x_2})^{\frac{3}{2}}}$ defined everywhere. For this φ must be twice-diff with bounded second derivative.

Reproduction of ellipses

Given affine invariance it is enough to reproduce the unit sphere in order to reproduce any ellipse. A parametric snake is said to reproduce the unit sphere with M anchor points if there exists two M-periodic sequences $\{c_1[k]\}_{k\in\mathbb{Z}}$, $\{c_2[k]\}_{k\in\mathbb{Z}}$ such that

$$\cos(2\pi t) = \sum_{k\in\mathbb{Z}} c_1[k]\varphi(Mt-k) \tag{2}$$

$$\sin(2\pi t) = \sum_{k\in\mathbb{Z}} c_2[k]\varphi(Mt-k) \tag{3}$$

Here is now the main result of the paper that explicitly give the expression of the generate φ with all properties stated above.

Theorem 1. *The centered generating function with minimal support that has the Riesz-basis, partition of unity, twice differentiable with bounded second derivative and ellipse-reproducing with M anchor points properties is given by*

$$\varphi(t) = \begin{cases} \dfrac{\cos(\frac{2\pi|t|}{M})\cos(\frac{\pi}{M}) - \cos(\frac{2\pi}{M})}{1-\cos(\frac{2\pi}{M})} & \text{if } 0 \leq |t| < \dfrac{1}{2} \\ \dfrac{1-\cos(\frac{2\pi(\frac{3}{2}-|t|)}{M})}{2(1-\cos(\frac{2\pi}{M}))} & \text{if } \dfrac{1}{2} \leq |t| < \dfrac{3}{2} \\ 0 & \text{if } |t| \geq \dfrac{3}{2} \end{cases} \tag{4}$$

This is proved using a theorem from paper 10-"Exponential splines and minimal-support bases for curve representation" that will be read after this one. The theorem states that every minimal support function that reproduces $e^{\alpha_n t}$ for $n = 0, \ldots, N-1$, with α such that there it has no pair of purely imaginary numbers separated by a distance that is a multiple of 2π, can be written as

$$\varphi(t) = \sum_{n=0}^{N-1} \lambda_n \frac{d^n t}{dt^n} \beta_\alpha(t-a)$$

with a is an arbitrary shift that correspond to the lower extremity of the support of φ. In order to satisfy partition of unity and reproduce sinusoids we need to choose $\alpha = (\frac{-2j\pi}{M}, 0, \frac{2j\pi}{M})$. For such an α, β_α has a discontinuous second derivative. This leads us to choosing $\lambda_1 = \lambda_2 = 0$. In the end, given the property of exponential B-splines, we get

$$\lambda_0 = \frac{(\frac{2\pi}{M})^2}{2(1-\cos(\frac{2\pi}{M}))} \tag{5}$$

Approximation properties of φ

Noticing that φ converges to the quadratic B-spline as M grows to infinity, we can expect φ to have approximately the same approximation properties as that of a quadractic B-spline. The space spanned by φ is not shift-invariance in general. Therefore the approximation error using M coefficients is dependent upon a shift in the continous parameter t of the function of unit period s. The minimum mean-square approximation error for a shift function is

$$\gamma(\tau, M) = \int_0^1 \|s(.-\tau) - r(.)\|^2 = \|s(.-\tau) - r(.)\|^2_{L_2([0,1])} \tag{6}$$

where r is the best approximation from the span $\{\varphi(M.-k)\}_{k\in\mathbb{Z}}$. Since τ is usually unknown, the measure error is average over all possible shift that is

$$\eta(M) = \left(\int_0^1 \gamma(\tau, M) d\tau\right)^{\frac{1}{2}} \tag{7}$$

It is shown in the paper, based on the main result of [30] M. Jacob, T. Blu, and M. Unser, "Sampling of periodic signals: A quantitative error analysis" that $\eta(M) = \mathcal{O}(M^{-3})$.

Explicit coefficients for best ellipse fitting and sinusoids

Since the snake can reproduce curves, it is natural to wonder what is the best ellipse that approximates a curve r defined by the M-periodic sequence $\{c[k]\}_{k\in\mathbb{Z}}$ that is find r_e that minimizes

$$\int_0^1 \|r(t) - r_e(t)\|^2 dt$$

Aparte. For a function $f : \mathbb{R} \to \mathbb{C}$ that is 2π-periodic one can represent f in its Fourier series as

$$f(t) = \sum_{k\in\mathbb{Z}} \frac{\hat{f}[k]}{2\pi} e^{jkt}$$

with $\hat{f}[k] = \int_{-\pi}^{\pi} f(t)e^{-jkt}dt$. What about a function g that is $\frac{2\pi}{a}$-periodic? Let $g(t) = f(at)$. We have

$$\hat{f}[k] = a \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} f(at)e^{-jkat}dt$$

and

$$g(t) = \sum_{k\in\mathbb{Z}} \frac{\hat{f}[k]}{2\pi} e^{jkat}$$

For example for $a = 2\pi$ i.e g is 1-periodic

$$g(t) = \sum_{k\in\mathbb{Z}} \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} g(u)e^{-2\pi jku}du\right) e^{j2\pi kt}$$

The continuous 1-periodic function r can be represented as

$$r(t) = \sum_{k\in\mathbb{Z}} R[k]e^{j2\pi kt} \tag{8}$$

with $R[k] = \left(\int_0^1 g(u)e^{-2\pi jku}du\right)$. From classical result in harmonic analysis, the best approximating ellispe r_e is the first-order truncation of the series above i.e $R[0], R[1], R[-1]$.

Finally we derive explicit formulas for the coefficient of the sinusoids of unit period with M anchor points. Recall the exponential-reproducing property of exponential B-spline

$$e^{\alpha t} = \sum_{k\in\mathbb{Z}} e^{\alpha k}\beta_\alpha(t-k) \tag{9}$$

Taking $\alpha = \frac{2j\pi}{M}$ and convolving both sides above with $\beta_{(0, \frac{-2j\pi}{M})}$ allow us to find the coefficients for representing $\cos(2\pi(t + \frac{3}{2}))$ and the equivalent sin function in the span of $\{\varphi(M. - k)\}$

$$c_1[k] = \frac{2(1-\cos(\frac{2\pi}{M}))\cos(\frac{\pi(2k+3)}{M})}{\cos(\frac{\pi}{M}) - \cos(\frac{3\pi}{M})} \tag{10}$$

$$c_2[k] = \frac{2(1-\cos(\frac{2\pi}{M}))\sin(\frac{\pi(2k+3)}{M})}{\cos(\frac{\pi}{M}) - \cos(\frac{3\pi}{M})} \tag{11}$$