Parametrisation of the sphere

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1 Cardinal Hermite exponential splines

1.1 The parametric model

Conti et al's paper Ellipse-preserving interpolation and subdivision scheme introduces two basis functions from the space $\mathcal{E}_4 =< 1, x, e^{-iw_1 x}, e^{iw_1 x} >$ where $w = \frac{2\pi}{M}$ to reproduce closed curves with M control points. The corresponding parametric representation is

$$r(t) = \sum_{k\in\mathbb{Z}} r(k)\phi_{1,w}(t-k) + r'(k)\phi_{2,w}(t-k) \tag{1}$$

with r and r' assumed to be M-periodic.

The basis functions are **cycloidal splines** (Exponential splines? Exponential B-splines?) given by

$$\phi_{1,w}(x) = \begin{cases} g_{1,w}(x) & \text{for } x \geq 0 \\ g_{1,w}(-x) & \text{for } x < 0 \end{cases} \quad \phi_{2,w}(x) = \begin{cases} g_{2,w}(x) & \text{for } x \geq 0 \\ -g_{2,w}(-x) & \text{for } x < 0 \end{cases} \tag{2}$$

The resulting parametric model has the following properties

1. Unique and stable representation ($\{\phi_{\mathbf{w}}(. - \mathbf{k}) = (\phi_{1,w}(. - k), \phi_{2,w}(. - k))\}_k$ Riesz basis)
2. Affine invariance (partition unity condition on ϕ_1)
3. Perfectly reproduce sinusoids of period M
4. Exact interpolation of points and first derivative
5. Support of ϕ_1, ϕ_2 is $[-1, 1]$
6. Hermite interpolation property of order 1
7. C^1-continuous

1.2 The unit sphere with scaling factors w_1, w_2

The usual continuous representation of the sphere is given by

$$\sigma(u, v) = (\cos(2\pi u)\sin(\pi v), \sin(2\pi u)\sin(\pi v), \cos(\pi v)) \quad (u, v) \in [0, 1]^2 \tag{3}$$

Suppose we have M_1 control points on latitudes, M_2 control points on meridians. The control points are then $c[k, l]_{k\in[0,\ldots,M_1-1], l\in[0,\ldots,M_2-1]}$. Let $w_1 = \frac{2\pi}{M_1}, w_2 = \frac{\pi}{M_2}$.

From the paper we have (also holds for sin functions)

$$\forall u \in [0, M_1] \quad \cos(w_1 u) = \sum_{k\in\mathbb{Z}} \cos(w_1 k)\phi_{1,w_1}(u-k) - w_1 \sin(w_1 k)\phi_{2,w_1}(u-k)$$
$$\forall v \in [0, 2M_2] \quad \cos(w_2 v) = \sum_{l\in\mathbb{Z}} \cos(w_2 l)\phi_{1,w_2}(v-l) - w_2 \sin(w_2 l)\phi_{2,w_2}(v-l)$$

Normalizing the the continuous parameters leads to

$$\forall u \in [0,1] \quad \cos(2\pi u) = \sum_{k\in\mathbb{Z}} \cos(w_1 k)\phi_{1,w_1}(M_1 u - k) - w_1 \sin(w_1 k)\phi_{2,w_1}(M_1 u - k)$$
$$\forall v \in [0,2] \quad \cos(\pi v) = \sum_{l\in\mathbb{Z}} \cos(w_2 l)\phi_{1,w_2}(M_2 v - l) - w_2 \sin(w_2 l)\phi_{2,w_2}(M_2 v - l)$$

Be aware that in the first representations above $\{\cos(w_1 k), -w_1 \sin(w_1 k)\}$ is (M_1, M_1)-periodic i.e we need point and first derivative values at M_1 control points for a full representation. However in the second representation $\{\cos(w_2 l), -w_2 \sin(w_2 l)\}$ are $(2M_2, 2M_2)$-periodic i.e we need point and first derivative values at $2M_2$ control points for a full representation.

1.3 Representation of the sphere

For all $(u,v) \in [0,1]^2$

$$\begin{aligned}
\sigma(u,v) = &\sum_{(k,l)\in\mathbb{Z}^2} c_1[k,l]\phi_{1,w_1}(M_1 u - k)\phi_{1,w_2}(M_2 v - l)\\
&+ \sum_{(k,l)\in\mathbb{Z}^2} c_2[k,l]\phi_{1,w_1}(M_1 u - k)\phi_{2,w_2}(M_2 v - l)\\
&+ \sum_{(k,l)\in\mathbb{Z}^2} c_3[k,l]\phi_{2,w_1}(M_1 u - k)\phi_{1,w_2}(M_2 v - l)\\
&+ \sum_{(k,l)\in\mathbb{Z}^2} c_4[k,l]\phi_{2,w_1}(M_1 u - k)\phi_{2,w_2}(M_2 v - l)
\end{aligned}$$

Or equivalently for all $(u,v) \in [0,1]^2$

$$\begin{aligned}
\sigma(u,v) = &\sum_{k=0}^{M_1}\sum_{l=0}^{M_2} c_1[k,l]\phi_{1,w_1}(M_1 u - k)\phi_{1,w_2}(M_2 v - l)\\
&+ \sum_{k=0}^{M_1}\sum_{l=0}^{M_2} c_2[k,l]\phi_{1,w_1}(M_1 u - k)\phi_{2,w_2}(M_2 v - l)\\
&+ \sum_{k=0}^{M_1}\sum_{l=0}^{M_2} c_3[k,l]\phi_{2,w_1}(M_1 u - k)\phi_{1,w_2}(M_2 v - l)\\
&+ \sum_{k=0}^{M_1}\sum_{l=0}^{M_2} c_4[k,l]\phi_{2,w_1}(M_1 u - k)\phi_{2,w_2}(M_2 v - l)
\end{aligned}$$

Or equivalently for all $(u,v) \in [0,1]^2$

$$\begin{aligned}
\sigma(u,v) = &\sum_{k=0}^{M_1-1}\sum_{l=0}^{2M_2-1} c_1[k,l]\phi_{1,w_1,per}(M_1 u - k)\phi_{1,w_2,per}(M_2 v - l)\\
&+ \sum_{k=0}^{M_1-1}\sum_{l=0}^{2M_2-1} c_2[k,l]\phi_{1,w_1,per}(M_1 u - k)\phi_{2,w_2,per}(M_2 v - l)\\
&+ \sum_{k=0}^{M_1-1}\sum_{l=0}^{2M_2-1} c_3[k,l]\phi_{2,w_1,per}(M_1 u - k)\phi_{1,w_2,per}(M_2 v - l)\\
&+ \sum_{k=0}^{M_1-1}\sum_{l=0}^{2M_2-1} c_4[k,l]\phi_{2,w_1,per}(M_1 u - k)\phi_{2,w_2,per}(M_2 v - l)
\end{aligned}$$

Or equivalently for all $(u,v) \in [0,1]^2$r

$$\boxed{\begin{aligned}\sigma(u,v) = &\sum_{k=0}^{M_1-1}\sum_{l=0}^{M_2} c_1[k,l]\phi_{1,w_1,per}(M_1u-k)\phi_{1,w_2}(M_2v-l)\\ &+\sum_{k=0}^{M_1-1}\sum_{l=0}^{M_2} c_2[k,l]\phi_{1,w_1,per}(M_1u-k)\phi_{2,w_2}(M_2v-l)\\ &+\sum_{k=0}^{M_1-1}\sum_{l=0}^{M_2} c_3[k,l]\phi_{2,w_1,per}(M_1u-k)\phi_{1,w_2}(M_2v-l)\\ &+\sum_{k=0}^{M_1-1}\sum_{l=0}^{M_2} c_4[k,l]\phi_{2,w_1,per}(M_1u-k)\phi_{2,w_2}(M_2v-l)\end{aligned}}$$

$$\boxed{\begin{aligned}c_1[k,l] &= \begin{bmatrix}\cos(w_1k)\sin(w_2l)\\ \sin(w_1k)\sin(w_2l)\\ \cos(w_2l)\end{bmatrix} = \sigma(w_1k,w_2l) & c_2[k,l] &= \begin{bmatrix}w_2\cos(w_1k)\cos(w_2l)\\ w_2\sin(w_1k)\cos(w_2l)\\ -w_2\sin(w_2l)\end{bmatrix} = \frac{\partial\sigma}{\partial v}(w_1k,w_2l)\\ c_3[k,l] &= \begin{bmatrix}-w_1\sin(w_1k)\sin(w_2l)\\ w_1\cos(w_1k)\sin(w_2l)\\ 0\end{bmatrix} = \frac{\partial\sigma}{\partial u}(w_1u,w_2v) & c_4[k,l] &= \begin{bmatrix}-w_1w_2\sin(w_1k)\cos(w_2l)\\ w_1w_2\cos(w_1k)\cos(w_2l)\\ 0\end{bmatrix} = \frac{\partial^2\sigma}{\partial u\partial v}(w_1u,w_2v)\end{aligned}}$$

$$\begin{aligned}\phi_{1,w_1,per}(.) &= \sum_{k\in\mathbb{Z}}\phi_{1,w_1}(.-M_1k) & \phi_{1,w_2,per}(.) &= \sum_{k\in\mathbb{Z}}\phi_{1,w_2}(.-2M_2k)\\ \phi_{2,w_1,per}(.) &= \sum_{k\in\mathbb{Z}}\phi_{2,w_1}(.-M_1k) & \phi_{2,w_2,per}(.) &= \sum_{k\in\mathbb{Z}}\phi_{2,w_2}(.-2M_2k)\end{aligned}$$

2 Exponential B-splines in 3D

2.1 The parametric model

Delgado et al's paper Spline-based deforming ellipsoids for 3D bioimage segmentation derive an exponential B-splines-based model that allow to reproduce ellipsoids. The model can well approximate blobs and perfectly spheres and ellipsoids. The corresponding parametric representation is

$$\sigma(u,v) = \sum_{(i,j)\in\mathbb{Z}^2} c[i,j]\phi_1(\frac{u}{T_1}-i)\phi_2(\frac{v}{T_2}-j) \tag{4}$$

with $T_1, T_2 > 0$ sampling steps for each parametric dimension and $\{c[i,j]\}_{(i,j)\in\mathbb{Z}^2}$ are the 3D control points.

The basis functions, reproducing unit period sinusoids with M coefficients, are exponential B-splines given by

$$\varphi_M(.) = \sum_{k=0}^{3}(-1)^k h_M[k]\varsigma_M(.+\frac{3}{2}-k) \tag{5}$$

where $\varsigma_M(.) = \frac{1}{4}sgn(.)\frac{\sin^2(\frac{\pi}{M}.)}{\sin^2(\frac{\pi}{M})}$ and $h_M = [1, 1+2\cos(\frac{2\pi}{M}), 1+2\cos(\frac{2\pi}{M}), 1]$.

Suppose we have M_1 control points on latitudes, M_2 control points on meridians. The resulting parametric model has the following properties

1. Unique and stable representation (sufficient is $\{\phi_1(.-k)\}_k, \{\phi_2(.-k)\}_k$ Riesz basis)
2. Affine invariance (partition unity condition on ϕ_1, ϕ_2)
3. Well-defined Gaussian curvature. ϕ_1, ϕ_2 are twice differentiable with bounded second derivative
4. Perfectly reproduce ellipsoids
5. Support of $\phi_1 = \varphi_{M_1}, \phi_2 = \varphi_{2M_2}$ is $[-\frac{3}{2}, \frac{3}{2}]$

2.2 Conditions for representing the unit sphere

The parametric representation of a closed surface with sphere-like topology, M_1 control points on latitudes and M_2 control points on meridians is

$$\forall (u,v) \in [0,1]^2 \quad \sigma(u,v) = \sum_{k=0}^{M_1-1} \sum_{l=-1}^{M_2+1} c[k,l]\phi_1(M_1 u - k)\phi_2(M_2 v - l) \tag{6}$$

Unlike before, continuity of points and tangents at poles is not guaranteed. Conditions are

$$\forall k = 0, \dots, M_1 - 1 \quad \mathbf{c_N} = c[k,1]\phi_2(-1) + c[k,0]\phi_2(0) + c[k,-1]\phi_2(1) \tag{7}$$

$$\mathbf{c_S} = c[k, M_2+1]\phi_2(-1) + c[k, M_2]\phi_2(0) + c[k, M_2-1]\phi_2(1) \tag{8}$$

$$\mathbf{T_{1,N}} \cos(2\pi u) + \mathbf{T_{2,N}} \sin(2\pi u) = M_2 \sum_{k=0}^{M_1-1} \sum_{l=-1}^{M_2+1} c[k,l]\phi_1(M_1 u - k)\phi_2'(-l) \tag{9}$$

$$\mathbf{T_{1,S}} \cos(2\pi u) + \mathbf{T_{2,S}} \sin(2\pi u) = M_2 \sum_{k=0}^{M_1-1} \sum_{l=-1}^{M_2+1} c[k,l]\phi_1(M_1 u - k)\phi_2'(M_2 - l) \tag{10}$$

$$\tag{11}$$

Incorporating such conditions in the model, a parametric splines-based surface with a sphere-like topology, C^1 continuity and ellipsoid-reproducing capabilities (all positions and orientations) is given by

$$\forall (u,v) \in [0,1]^2 \quad \sigma(u,v) = \sum_{k=0}^{M_1-1} \sum_{l=-1}^{M_2+1} c[k,l]\phi_{1,per}(M_1 u - k)\phi_2(M_2 v - l) \tag{12}$$

where $\{c[i,j]\}_{i\in[0,\dots,M_1-1], j\in[1,\dots,M_2-1]}, \mathbf{T_{1,N}}, \mathbf{T_{2,N}}, \mathbf{T_{1,S}}, \mathbf{T_{1,S}}, \mathbf{c_N}, \mathbf{c_S}$ are free parameters that is $M_1(M_2 - 1) + 6$ control points.

$c[k,-1], c[k,0], c[k,M_2], c[k,M_2+1]$ are constrained by the values of the free parameters.

2.3 Representation of the sphere

The unit sphere is thus represented by

$$\boxed{\forall (u,v) \in [0,1]^2 \quad \sigma(u,v) = \sum_{k=0}^{M_1-1} \sum_{l=-1}^{M_2+1} c[k,l]\phi_{1,per}(M_1 u - k)\phi_2(M_2 v - l)} \tag{13}$$

With coefficients are given by

$$\boxed{c[k,l] = \begin{bmatrix} c_{M_1}[k] s_{2M_2}[l] \\ s_{M_1}[k] s_{2M_2}[l] \\ c_{2M_2}[l] \end{bmatrix} = \begin{bmatrix} \frac{2(1-\cos(\frac{2\pi}{M_1}))}{\cos(\frac{\pi}{M_1})-\cos(\frac{3\pi}{M_1})} \frac{2(1-\cos(\frac{\pi}{M_2}))}{\cos(\frac{\pi}{2M_2})-\cos(\frac{3\pi}{2M_2})} \cos(\frac{2\pi k}{M_1}) \sin(\frac{\pi l}{M_2}) \\ \frac{2(1-\cos(\frac{2\pi}{M_1}))}{\cos(\frac{\pi}{M_1})-\cos(\frac{3\pi}{M_1})} \frac{2(1-\cos(\frac{\pi}{M_2}))}{\cos(\frac{\pi}{2M_2})-\cos(\frac{3\pi}{2M_2})} \sin(\frac{2\pi k}{M_1}) \sin(\frac{\pi l}{M_2}) \\ \frac{2(1-\cos(\frac{\pi}{M_2}))}{\cos(\frac{\pi}{2M_2})-\cos(\frac{3\pi}{2M_2})} \cos(\frac{\pi l}{M_2}) \end{bmatrix}} \tag{14}$$

and

$$
\begin{aligned}
c_M[k] &= \frac{2(1-\cos(\frac{2\pi}{M}))}{\cos(\frac{\pi}{M})-\cos(\frac{3\pi}{M})}\cos(\frac{2\pi k}{M}) \\
s_M[k] &= \frac{2(1-\cos(\frac{2\pi}{M}))}{\cos(\frac{\pi}{M})-\cos(\frac{3\pi}{M})}\sin(\frac{2\pi k}{M})
\end{aligned}
$$

These coefficients satisfy the constraints with

$$
\begin{aligned}
\mathbf{c_N} &= [0\ 0\ 1]^T & \mathbf{c_N} &= [0\ 0\ -1]^T & \mathbf{T_{1,N}} &= [\pi\ 0\ 0]^T \\
\mathbf{T_{2,N}} &= [0\ \pi\ 0]^T & \mathbf{T_{1,S}} &= [-\pi\ 0\ 0]^T & \mathbf{T_{2,S}} &= [0\ -\pi\ 0]^T
\end{aligned}
$$

3 Compactly-supported smooth interpolators for shape modeling

3.1 The parametric model

Schmitter et al's paper Compactly-supported smooth interpolators for shape modeling with varying resolution propose a continuous representation of curves and surfaces with the help of generators that have the advantages of both continuous and discrete schemes. The generator is expressed as a linear combination of half integer shifts of exponential B-spline of vector $\alpha \in \mathbb{C}^n$ i.e

$$\phi_{\lambda,\alpha}(t) = \sum_{k\in\mathbb{Z}} \lambda[k]\beta_\alpha(t - \frac{k}{2}) \tag{15}$$

β_α has support $[-\frac{n}{2}, \frac{n}{2}]$. In what follows we choose to have $\lambda[k] = 0$ for $k \notin [\![-n+2, n-2]\!]$ and $\lambda[-k] = \lambda[k]$. There are therefore $(n-1)$ unknowns $\lambda[0], \dots, \lambda[n-2]$. We also impose that elements in α are 0 or come in complex conjugate pairs and that no pair of purely imaginary elements of α is separated by integer multiple of $2j\pi$ (for Riesz basis property).

This function is interpolatory if and only if $\phi_{\lambda,\alpha}(0) = 1$ and $\phi_{\lambda,\alpha}(1) = \dots = \phi_{\lambda,\alpha}(n-2) = 0$. This defines a system of $n-1$ equations with $n-1$ unknowns. The system has a solution if the matrix defined by $k, l = 0, \dots, n-2$

$$[A_\alpha]_{k+1,l+1} = \begin{cases} \beta_\alpha(k) & \text{if } l = 0 \\ \beta_\alpha(k - \frac{l}{2}) + \beta_\alpha(k + \frac{l}{2}) & \text{else} \end{cases} \tag{16}$$

is invertible. In that case $\lambda = A_\alpha^{-1}(1, 0, \dots, 0)$ and we define $\phi_\alpha = \phi_{\lambda,\alpha}$. Tensor-product surfaces are represented with the help of two generators in the form

$$\sigma(u,v) = \sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} \sigma[k,l]\phi_{\alpha_1}(u-k)\phi_{\alpha_2}(v-l) \tag{17}$$

The resulting interpolation scheme has the following properties

1. Unique and stable representation ($\alpha_m - \alpha_n \notin 2j\pi\mathbb{Z}$ Riesz basis)
2. Affine invariance ($0 \in \alpha_1, 0 \in \alpha_2$)
3. Perfectly reproduce ellipsoids (conditions on α)
4. ϕ_α is interpolatory
5. ϕ_α is smooth i.e at least $\mathcal{C}^1$
6. Can reproduce the nullspace $\mathcal{N}_\alpha$
7. Can reproduce shapes at various resolutions
8. ϕ_α is compactly supported on $[-n+1, n-1]$

3.2 Conditions for representing the unit sphere

Let M_1 be the number of control points in u and M_2 the number of control points in v. For ϕ_{α_1} to be able to reproduce $\cos(\frac{2\pi.}{M_1}), \sin(\frac{2\pi.}{M_1})$ we need to have $(\frac{-2i\pi}{M_1}, \frac{2i\pi}{M_1}) \in \alpha_1$. Adding affine invariance condition, $\phi_{(0,\frac{-2i\pi}{M_1},\frac{2i\pi}{M_1})}$ can reproduce constants and M_1-periodic sinusoids with M_1 control points as follows

$$\cos(\frac{2\pi.}{M_1}) = \sum_{k\in\mathbb{Z}} \cos(\frac{2\pi k}{M_1})\phi_{\alpha_1}(. - k) \tag{18}$$

Similarly $\phi_{(0,\frac{-i\pi}{M_2},\frac{i\pi}{M_2})}$ can reproduce constants and $2M_2$-periodic sinusoids with $2M_2$ control points as follows

$$\cos(\frac{\pi .}{M_2}) = \sum_{k\in\mathbb{Z}} \cos(\frac{\pi k}{M_2})\phi_{\alpha_2}(. - k) \tag{19}$$

Generators $\phi_{\alpha_1}, \phi_{\alpha_2}$ both have support of size 4 ($n = 3$) so that they are given by

$$\begin{aligned}
\phi_{\alpha_1}(t) &= \lambda_1[0]\beta_{\alpha_1}(t) + \lambda_1[1](\beta_{\alpha_1}(t - 1/2) + \beta_{\alpha_1}(t + 1/2)) \\
\phi_{\alpha_2}(t) &= \lambda_2[0]\beta_{\alpha_2}(t) + \lambda_2[1](\beta_{\alpha_2}(t - 1/2) + \beta_{\alpha_2}(t + 1/2))
\end{aligned}$$

In order to find $\lambda_1[0], \lambda_1[1]$ one has to solve $\phi_{\alpha_1}(0) = 1, \phi_{\alpha_1}(1) = 0$.

Aparte on tempered distributions

Green function of operator $L_\alpha : \mathcal{S}(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$ is an element ρ_α of $\mathcal{S}(\mathbb{R})$ that satisfies $L\{\rho_\alpha\} = \delta$ where δ is the Dirac tempered distribution. There is a unique such function (to be proved?) that also satisfies $\forall t < 0\ \rho_\alpha(t) < 0$. The tempered distribution $T_{\rho_\alpha} : \phi \mapsto \int_0^\infty e^{\alpha t}\phi(t)dt$ is such that the associated element of $\mathcal{S}(\mathbb{R})$ (bijection $\mathcal{S}(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$? probably not true), ρ_α satisfies $L_\alpha\{\rho_\alpha\} = \delta$.

Consequently β_α^+ is an element of Schwartz space $\mathcal{S}(\mathbb{R})$. **Is that right**? With abuse of notation we write $\beta_\alpha^+(t) = e^{\alpha t}\chi_{[0,1]}(t)$. Using distribution then we would have

$$\beta_\alpha^{+\prime}(t) = \delta(t) + \alpha e^{\alpha t}\chi_{[0,1]}(t)$$

The equality is to be taken in the distribution sense.

3.3 Representation of the sphere

The unit sphere is thus represented by

$$\forall (u,v) \in [0,1]^2 \quad \sigma(u,v) = \sum_{(k,l)\in\mathbb{Z}^2} c[k,l]\phi_{\alpha_1}(M_1 u - k)\phi_{\alpha_2}(M_2 v - l) \tag{20}$$

Or equivalently

$$\forall (u,v) \in [0,1]^2 \quad \sigma(u,v) = \sum_{k=0}^{M_1-1}\sum_{l\in\mathbb{Z}} c[k,l]\phi_{\alpha_1,per}(M_1 u - k)\phi_{\alpha_2}(M_2 v - l) \tag{21}$$

Or equivalently

$$\boxed{\forall (u,v) \in [0,1]^2 \quad \sigma(u,v) = \sum_{k=0}^{M_1-1}\sum_{l=-1}^{M_2+1} c[k,l]\phi_{\alpha_1,per}(M_1 u - k)\phi_{\alpha_2}(M_2 v - l)} \tag{22}$$

Denoting $w_1 = \frac{2\pi}{M_1}, w_2 = \frac{\pi}{M_2}$, the coefficients are given by

$$\boxed{c[k,l] = \begin{bmatrix} \cos(w_1 k)\sin(w_2 l) \\ \sin(w_1 k)\sin(w_2 l) \\ \cos(w_2 l) \end{bmatrix} = \sigma(w_1 k, w_2 l)} \tag{23}$$

4 Smooth shapes with spherical topology

4.1 The parametric model

In 2017 Schmitter et al's paper Smooth shapes with spherical topology derive a parametric model very similar to that presented in Spline-based deforming ellipsoids for 3D bioimage segmentation. In user interactive applications one usually wants a curve/shape reproducing model to have some or all following properties: 1.intuitive manipulation, 2.stable deformation, 3.shape deformation as optimization process requiring fast evaluation of surface and volume integrals, 4.smooth representation. It is usually impossible to find a model optimal w.r.t to all these requirements. In practice a compromise is made with existing models based on polygon meshes, subdivision or NURBS.

Parametric shapes are built as linear combinations of integers shifts of a generator function φ that is to say

$$r(t) = \sum_{k \in \mathbb{Z}} c[k]\varphi(t-k) \tag{24}$$

φ is piecewise exponential. It is the smoothed version of third order exponential B-spline that is $\varphi = \beta * \psi$ with ψ an appropriate smoothing kernel. The model can be extended to tensor-product surfaces $\sigma(u, v)$ as previously done in previous representations.

$$\sigma(u,v) = \sum_{(k,l)\in\mathbb{Z}^2} c[k,l]\varphi_1(u-k)\varphi_2(v-l)$$

Authors define

$$\phi_1(t) = \varphi_{M_1,per}(t) = \sum_{n\in\mathbb{Z}} \varphi_{M_1}(t - M_1 n) \qquad \phi_2(t) = \varphi_{2M_2}(t)$$

$$\forall k \in \mathbb{Z} \quad \phi_{1,k}(t) = \phi_1(M_1 t - k) \qquad \phi_{2,k}(t) = \phi_2(M_2 t - k)$$

The resulting interpolation scheme has the following properties

1. Unique and stable representation ($\{\varphi_M(. - k)\}_k$ Riesz basis)
2. Affine invariance (partition unity condition on φ_M)
3. Well-defined Gaussian curvature. φ_M is twice differentiable with bounded second derivative
4. Perfectly reproduce ellipsoids for $M \geq 3$
5. φ_M is interpolatory
6. Support of φ_M is in $[-2, 2]$

4.2 Conditions for representing the unit sphere

The parametric representation of a closed surface with sphere-like topology, M_1 control points on latitudes and M_2 control points on meridians is

$$\forall (u,v) \in [0,1]^2 \quad \sigma(u,v) = \sum_{k=0}^{M_1-1} \sum_{l=-1}^{M_2+1} c[k,l]\phi_{1,k}(u)\phi_{2,k}(v) \tag{25}$$

As for the model from article 7, continuity of points and tangents at poles is not guaranteed.The exact same conditions are used and by incorporating conditions in the model to ensure continuity of the surface and of the tangent plane at poles we obtain constraints on $c[k,-1], c[k,0], c[k,M_2], c[k,M_2+1]$. Other values $c[k,l]$ are free parameters as well as 6 the 6 $c_N, c_S, T_{1,N}, T_{2,N}, T_{1,S}, T_{2,S}$ describing the poles.

φ_{M_1} can reproduce $\cos(\frac{2\pi .}{M_1})$ with M_1 control points

$$\cos(\frac{2\pi u}{M_1}) = \sum_{k\in\mathbb{Z}} \cos(\frac{2\pi k}{M_1})\varphi_{M_1}(u-k)$$
$$\cos(2\pi u) = \sum_{k=0}^{M_1-1} \cos(\frac{2\pi k}{M_1})\phi_{1,k}(u)$$

In a similar fashion φ_{2M_2} can reproduce $\cos(\frac{\pi .}{M_2})$ with $2M_2$ control points i.e

$$\cos(\frac{\pi v}{M_2}) = \sum_{k\in\mathbb{Z}} \cos(\frac{\pi k}{M_2})\varphi_{2M_2}(v-k)$$
$$\cos(\pi v) = \sum_{k\in\mathbb{Z}} \cos(\frac{\pi k}{M_2})\phi_{2,k}(v)$$

4.3 Representation of the sphere

Given the usual representation of the unit sphere, it can be represented in our scheme as follows

$$\sigma(u,v) = \sum_{k=0}^{M_1-1} \sum_{l\in\mathbb{Z}} c[k,l]\phi_{1,k}(u)\phi_{2,k}(v) \tag{26}$$

or using the fact φ_{2M_2} has support of size 4

$$\boxed{\forall (u,v) \in [0,1]^2 \quad \sigma(u,v) = \sum_{k=0}^{M_1-1} \sum_{l=-1}^{M_2+1} c[k,l]\phi_{1,k}(u)\phi_{2,k}(v)} \tag{27}$$

Denoting $w_1 = \frac{2\pi}{M_1}$, $w_2 = \frac{\pi}{M_2}$, the coefficients are given by

$$\boxed{c[k,l] = \begin{bmatrix} \cos(w_1 k)\sin(w_2 l) \\ \sin(w_1 k)\sin(w_2 l) \\ \cos(w_2 l) \end{bmatrix} = \sigma(w_1 k, w_2 l)} \tag{28}$$