Summary paper 1: Cardinal Hermite Interpolation Schoenberg

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Given a sequence of numbers y and a linear space $\mathcal{S}$ we denote as C.I.P $(y, \mathcal{S})$ the problem of finding $F \in \mathcal{S}$ that satisfies $F(\nu) = y_\nu$ for all ν.

Given r sequences of numbers $y, \ldots, y^{(r-1)}$ and a linear space $\mathcal{S}$ we denote as C.H.I.P $(y, \mathcal{S})$ the problem of finding $F \in \mathcal{S}$ that satisfies $F^{(s)}(\nu) = y_\nu^{(s)}$ for all $s = 0, \ldots, r-1$ for all ν.

Notations

$$S_{2m,r} = \{\text{Cardinal splines of order } 2m-1 \text{ with knots at integers of multiplicity r}\} \tag{1}$$

$$F_{\gamma,r} = \{F | F^{(s)}(x) = \mathcal{O}(|x|^\gamma) \quad \text{as } x \to \pm\infty \quad \forall s = 0, \ldots, r-1\} \tag{2}$$

$$\mathcal{L}_{p,r} = \{F | F^{(s)} \in \mathcal{L}_p \quad \forall s = 0, \ldots, r-1\} \tag{3}$$

$$S^0_{2m,r} = \{S \in S_{2m,r} | S^{(s)}(\nu) = 0 \quad \forall s = 0, \ldots, r-1 \quad \forall \nu\} \tag{4}$$

Theorem 1. *C.H.I.P $(y, S_{2m,r} \cap F_{\gamma,r})$ has a solution iif $y_\nu^{(s)} = \mathcal{O}(|\nu|^\gamma)$ as $\nu \to \pm\infty$. If it exists the solution is unique.*

Theorem 2. *C.H.I.P $(y, S_{2m,r} \cap \mathcal{L}_{p,r})$ has a solution iif $y^{(s)} \in l_p$. If it exists the solution is unique.*

For $s = 0, \ldots, r-1$, let the r sequences $y_\nu^{(\rho)} = \delta_\nu \delta_{\rho-s}$ for $\rho = 0, \ldots, r-1$. As these are in l_1, there exists a unique $L_{2m,r,s} \in S_{2m,r} \cap \mathcal{L}_{1,r}$ such that for all $s = 0, \ldots, r-1$, $L^{(\rho)}_{2m,r,s}(\nu) = \delta_\nu \delta_{\rho-s}$ for all $\rho = 0, \ldots, r-1$ and ν. Let $L_s = L_{2m,r,s}$.

Theorem 3. *There exists $A(m,r)$, $\alpha(m,r)$ such that*

$$\forall s, \rho = 0, \ldots, r-1 \quad \forall x \quad |L^{(\rho)}_{2m,r,s}(x)| \le A(m,r) \exp(-\alpha(m,r)x) \tag{5}$$

Theorem 4. *The spline function unique solution to theorems 1 and 2 is given by Lagrange-Hermite interpolation*

$$\forall x \quad S(x) = \sum_{-\infty}^{\infty} y_\nu L_0(x-\nu) + \cdots + y_\nu^{(r-1)} L_{r-1}(x-\nu) \tag{6}$$

I. Proof of unicity in theorems 1 and 2

Note that a spline is uniquely defined by $P(x) \in \pi_{2m-1}$ that satisfies $\forall x \in [0,1] \quad S(x) = P(x)$. Indeed, with $n = 2m-1$, then $S(x) = P(x) + \sum_{s=0}^{r-1} c_1^{(s)} (x-1)_+^{2m-1-s} + \cdots + \sum_{s=0}^{r-1} c_0^{(s)} (-x)_+^{2m-1-s} + \sum_{s=0}^{r-1} c_{-1}^{(s)} (-x-1)_+^{2m-1-s} + \cdots$ and $c_2^{(s)}$ are uniquely defined by $S^{(s)}(2) = y^{(s)}(2)$, etc.

$S^0_{2m,r}$ is a linear space of dimension $d = 2m - 2r$. $S \in S^0_{2m,r}$ that satisfies $\forall x \quad S(x+1) = \lambda S(x)$ is an eigenspline for eigenvalue λ. Let P be the polynomial component of S on $[0,1]$. The conditions $P^{(s)}(1) = P^{(s)}(0) = 0$ for $s = 0, \ldots, r-1$ allows to write

$$P(x) = a_0 x^n + a_1 \binom{n}{1} x^{n-1} + \cdots + a_{n-r} \binom{n}{n-r} x^r \tag{7}$$

The conditions $P^{(s)}(1) = \lambda P^{(s)}(1) = 0$ for $s = r, \ldots, 2m-r-1$ (from $S_{2m,r} \subseteq \mathcal{C}^{2m-r-1}$) transform into

$$\Delta_{r,d}(\lambda)[a_0, \ldots, a_{n-r}]^T = 0 \tag{8}$$

Theorem 5. $|\Delta_{r,d}(\lambda)| = 0$ *is a reciprocal equation of degree* $d = 2m - 2r$ *and has all its roots real, simple and of sign* $(-1)^r$.

Lemma 3. $|\Delta_{r,d}(\lambda)| = (-1)^{rd}|A_d - \lambda I_d|$ *with* $(-1)^r A_d = (J_d)^r P_{r,d}$

This lemma is proved by proving that $(-1)^r A_d$ is an oscillation matrix and then using the Gantmatcher-Krein theorem.

$|\Delta_{r,d}(\lambda)| = 0$ have d simple roots all of sign $(-1)^r$ and they are such that

$$0 < |\lambda_1| < \cdots < |\lambda_{m-r}| < 1 < |\lambda_{m-r+1}| < \cdots < |\lambda_{2m-2r}| \tag{9}$$

Let $S_1, \ldots, S_d$ the associated eigenspline. These are defined up to a factor (as the kernel of $\Delta_{r,d}(\lambda_i)$ is a line) and we choose to have $\forall 0 < x < 1 \quad 0 < S(x) < 1$ and $S^{(r)} = 0$.

As a consequence, $\forall x \quad S_j(x) = (-1)^r S_{d-j+1}(x)$ and $\forall n < x < n+1 \quad (-1)^{nr} S_j(x) > 0$.

Lemma 6. *If* $S \in S^0_{2m,r}$*, there exists a unique decomposition*

$$\forall x \quad S(x) = \sum_{j=1}^{d} c_j S_j(x) \tag{10}$$

II. Proof of theorems 1,2,3 and 4

$\forall s = 0, \ldots, r-1 \quad \forall x \quad L_s(x) = L_{2m,r,s}(x)$ and $L_s \in S_{2m,r} \cap \mathcal{L}_{1,r}$. Note that L_s is even if s is even, odd if s is odd. To construct L_s we first look at extension of the restriction to $[1, \infty)$ i.e $\tilde{L}_s(x) = L_s(x)$ for $x \geq 1$ and $\tilde{L}_s \in S^0_{2m,r}$. Applying lemma 6 and as $S_j \in \mathcal{L}_{1,r}$ we can write

$$\tilde{L}_s(x) = \sum_{j=1}^{m-r} c_j S_j(x) \tag{11}$$

Let $P(x) = L_s(x) \forall x \in [0, 1]$. Then we can write

- if r and s have same parity

$$P(x) = \frac{x^s}{s!} + a_1 x^r + a_2 x^{r+2} + \cdots + a_{m-r+1} x^{2m-r} + a_{m-r+2} x^{2m-r+1} + \cdots + a_m x^{2m-1} \tag{12}$$

- if r and s have opposite parity

$$P(x) = \frac{x^s}{s!} + a_1 x^{r+1} + a_2 x^{r+3} + \cdots + a_{m-r} x^{2m-r-1} + a_{m-r+1} x^{2m-r} + a_{m-r+2} x^{2m-r+1} + \cdots + a_m x^{2m-1} \tag{13}$$

The conditions $P^{(\rho)}(1) = \tilde{L}^{(\rho)}(1)$ for $\rho = 0, \ldots, 2m-r-1$ yields $2m-r$ equations for the $m+m-r = 2m-r$ unknowns c_j and a_j. We can then explicitly compute the expressions of the L_s. The system resulting from these equations is non singular as there is no non-trivial spline in $S_{2m,r} \cap \mathcal{L}_1$.

Corollary Cardinal Lagrange-Hermite interpolation.

$$f(x) = \sum_{-\infty}^{\infty} f(\nu) L_0(x - \nu) + \cdots + f^{(r-1)}(\nu) L_{r-1}(x - \nu) + R(x) \tag{14}$$

is exact (i.e $R = 0$*) if* $f \in F^*_r$ *and is a cardinal spline function of degree* $2m-1$ *and class* $\mathcal{C}^{2m-r-1}$*. It is exact also for* $f \in \pi_{2m-1}$.