Summary paper 28: Riesz bases of splines and regularized splines with multiple knots

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This paper is interested in characterizing L_2-norm and Sobolev norm stability of polynomial splines with multiple knots, that is to say osculatory interpolation. Stability estimates are reformulated in terms of Riesz basis property. Authors said their work was motivated by talk paper of C. de Boor, *The quasi-interpolant as a tool in elementary polynomial spline theory* that seemingly has the same interrogation as I have for a lower bound on the Gram matrix. In their previous paper published in 1995, *A generalization of de Boor's stability result and symmetric preconditioning*, authors developed a generalization for regularized splines with simples knots by tempered distributions.

This paper achieves three different results

1. Extends L_2 stability estimates to Sobolev norms
2. Extends stability estimates to regularized splines based on tempered distributions
3. Prove that scattered Hermite interpolation can be regular provided a given condition is satisfied.

1 Notations

$X = \{x_i\}$ with $x_i < x_{i+m}$ and $\lim_{i\to-\infty} x_i = -\infty, \lim_{i\to\infty} x_i = \infty$ denotes a knot sequence of order m. Divided differences are renoted $\nu_{i,k}(f) = [x_i, \dots, x_{i+k}]f$ and the B-splines are defined with Schoenberg's normalization that is $B_{i,m}(t) = m\nu_{i,m}((. - t)_+^{m-1})$. Let $\beta_{i,m}$ the i^{th} B-spline with de Boor's normalization. Then

$$\beta_{i,m}(t) = (x_{i+m} - x_i)[x_i, \dots, x_{i+m}](. - t)_+^{m-1} \quad \text{while} \quad B_{i,m}(t) = m[x_i, \cdots, x_{i+m}](. - t)_+^{m-1}$$

$$\sum_i \beta_{i,m}(t) = 1 \quad \text{while} \quad \int_{\mathbb{R}} B_{i,m}(t) = 1$$

Note that for Schoenberg's normalization, we have the following property for $f \in \mathcal{C}^{m-1}$

$$[x_i, \dots, x_{i+m}]f = \int B_{i,m} \frac{f^{(m)}}{m!} \tag{1}$$

Let $f(t) = \mathbb{1}_+(t)$ the Heaviside function. Then for any test function ψ, $\langle f', \psi\rangle = \psi(0) = \langle \delta, \psi\rangle$. Therefore $\hat{f}'(w) = 1 = (jw)\hat{f}(w)$ i.e $\hat{f}(w) = \frac{1}{jw}$. Note now that $\frac{t_+^{m-1}}{(m-1)!} = f * \cdots * f$ with f appearing m times. Consequently the function $h(t) = \frac{t_+^{m-1}}{(m-1)!}$ is the fundamental solution to the differential operator D^m that is $D^m h = \delta$.

The equality $\nu_{i,m}((. - t)_+^{m-1}) = (-1)^m \nu_{i,m}((t - .)_+^{m-1})$ is not so straightforward to me but can be verified manually for $m = 2$. Maybe using $t_+^m = t^{m-1}t_+$? $\nu_{i,m}$ is a **functional** operator that can be viewed as a tempered distribution. Indeed, $\nu_{i,m}$ takes values in $\mathbb{C}$, $\nu_{i,m}$ is $\mathbb{C}$-linear and $\nu_{i,m}$ is continuous. For the latter, notice that from properties of divided differences we have

$$\forall \phi, \psi \in \mathcal{S}(\mathbb{R}), \quad |\nu_{i,m}(\phi - \psi)| \leq \frac{1}{(m-1)!} \|\phi - \psi\|_{0,m-1} \tag{2}$$

where $\|\phi\|_{a,b}$ is the semi-norm $\sup_{t\in\mathbb{R}} |t^a \partial^b \phi(t)|$. As convergence in the Schwartz space is defined as the convergence in all such semi-norms, continuity of our operator follows. Consequently, $\nu_{i,m} \in \mathcal{S}'(\mathbb{R})$.

For any function ϕ in the Schwartz space, convolution with the tempered distribution $\nu_{i,m}$ is defined as $\phi * \nu_{i,m}(t) = \langle \phi(t - .), \nu_{i,m} \rangle := \nu_{i,m}(\phi(t - .))$. Now our function h is not strictly speaking an element of the Schwartz space but can be viewed as the tempered distribution that associates to any test function ϕ the scalar quantity $\langle \phi, h \rangle = \int_0^{+\infty} \frac{t^{m-1}}{(m-1)!} \phi(t) dt$. **How to define then convolution between two tempered distributions? And how to prove that** $h * \nu_{i,m} = \nu_{i,m}(\frac{(t-.)_+^{m-1}}{(m-1)!})$?

Distribution with compact support. A distribution T is said to vanish on an open set V if for every test function ϕ with compact support $\operatorname{supp} \phi \subseteq V$, $\langle \phi, T \rangle = 0$. Then the support of T is the complement of the maximal open set on which the distribution vanishes. For example, the support of the Dirac distribution is $\{0\}$.

It is possible to define the convolution of two distributions provided one has compact support. From wikipedia, if T, S are distributions with T compactly supported, then for any test function ϕ, $\langle \phi, S * T \rangle = \langle \psi, S \rangle$ with $\psi(t) = \langle \phi(t + .), T \rangle$ a test function itself. Applying this to the distributions h and $\nu_{i,m}$ leads to $\langle \phi, h * \nu_{i,m} \rangle = \int_0^{\infty} \frac{t^{m-1}}{(m-1)!} \nu_{i,m}(\phi(t + .)) dt$ which I cannot relate to the formula in the article.

Lemma 1. *$\exists D_m > 0$ such that for any sequence $v = (v_j)_{j\in\mathbb{Z}}$*

$$D_m \|v\|_{l_2} \leq \left\| \sum_{j\in\mathbb{Z}} v_j \sqrt{d_{j,m}} B_{j,m} \right\|_{L_2} \leq \|v\|_{l_2}$$

Proof. In de Boor's talk paper, the property holds with the basis $N_{im2} = d_{j,m}^{-\frac{1}{2}} \beta_{i,m}$. Now note that $d_{j,m}^{\frac{1}{2}} B_{i,m} = d_{j,m}^{\frac{1}{2}} d_{j,m}^{-1} \beta_{i,m} = N_{im2}$. □

Let $\sigma_{v,m} = \sum_{j\in\mathbb{Z}} v_j \sqrt{d_{j,m}} B_{j,m}$. Another result proven again by de Boor in his book *A Practical Guide to splines*, explicitly gives the derivative of a spline as

$$\sigma'_{v,m} = \sigma_{w,m-1}$$

with $w = D_m v$ and D_m the lower two-banded matrix $d_{j,j}^m = (d_{j,m-1}, d_{j,m})^{-1/2}$, $d_{j,j-1}^m = -(d_{j,m-1}, d_{j-1,m})^{-1/2}$.

2 Sobolev norm stability of splines

Let $q_k := \inf_j d_{j,k}$. The Sobolev norm in $H^s = W_2^s$ is defined from the Fourier transform as

$$\|f\|_{H^s} = \left(\frac{1}{2\pi} \int_{\mathbb{R}} (1 + |w|^2)^s |\hat{f}(w)|^2 dw \right)^{1/2} \tag{3}$$

In proof of lemma 2.1, where does x come from when bounding integral over $[\xi_j, \xi_{j+1} - x]$? From Cauchy-Schwarz!

Lemma 2.1. *Let $q_m > 0$. Then for any $v \in l_2(\mathbb{Z})$ and the corresponding spline*

$$f = \sigma_v = \sum_{j\in\mathbb{Z}} v_j \sqrt{d_{j,m}} B_{j,m},$$

we have

$$\|f(. + x) - f\|_{L_2}^2 \leq const_m \frac{|x|}{q_m} \|v\|_{l_2}^2 \quad for \quad |x| < q_m \tag{4}$$

Lemma 2.2. *Let $q_m > 0$. Then for any $v \in l_2(\mathbb{Z})$ and the corresponding spline*

$$f = \sigma_v = \sum_{j \in \mathbb{Z}} v_j \sqrt{d_{j,m}} B_{j,m},$$

we have

$$\int\int_{\mathbb{R}^2} \frac{|f(x) - f(y)|^2}{|x-y|^{1+2t}} dxdy \le const_{m,q_m} \frac{1}{t(1-2t)} \|v\|_{l_2}^2 \tag{5}$$

Proof. Change of variable x becomes $x + y$ followed by splitting of the integral on x according to absolute value relatively to q. Application of lemma 2.1 completes the proof. □

Theorem 1. *Let $\mu \in \{1, \ldots, m\}$ with $q_\mu > 0$ and let $0 \le t < \frac{1}{2}$. Then for any $v \in l_2$ and the corresponding spline $f = \sigma_v$, we have*

$$D_m \|v\|_{l_2} \le \left\| \sum_{j \in \mathbb{Z}} v_j \sqrt{d_{j,m}} B_{j,m} \right\|_{H^{m-\mu+t}} \le const_{m,t,q_\mu} \|v\|_{l_2} \tag{6}$$

Proof. Only the upper bound has to be verified as for any $s \ge 0$,

$$\begin{aligned} \|f\|_{H^s}^2 &= \frac{1}{2\pi} \int (1 + |w|^2)^s |\hat{f}(w)|^2 dw \\ &\ge \frac{1}{2\pi} \int |\hat{f}(w)|^2 dw \\ &= \|f\|_{L_2}^2 \end{aligned}$$

□

3 Riesz bases of regularized splines

X is minimally separated of order m if $q_m > 0$. As before we define functions $B_{i,m}$ as

$$B_{i,m}(t) = m!(-1)^m h * \nu_{i,m}(t)$$

with h a **tempered distribution** satisfying for some integer m

1. $\hat{h} \in \mathcal{C}^0(\mathbb{R}^*)$
2. $\hat{h} \ne 0$ on $\mathbb{R}^*$
3. $(jw)^m \hat{h}(w) \in L_\infty(\mathbb{R})$
4. $\left((jw)^m \hat{h}(w)\right)^{-1} \in L_\infty(U_0)$ for some $U_0 \in \mathcal{V}(0)$.

In that case let $G(w) = |w|^{2m} |\hat{h}(w)|^2$ and define the matrix $B = (b_{i,j})_{(i,j) \in \mathbb{Z}^2}$ by

$$b_{i,j} = \frac{\sqrt{d_{i,m} d_{j,m}}}{2\pi} \int G(w) \hat{B}_{i,m}(w) \overline{\hat{B}_{j,m}(w)} dw \tag{7}$$

Clearly B is Hermitian and allows us to define the quadratic form $\langle By, y \rangle = y^* By$ for any $y \in l_2^0$. In fact it is possible to extend to all of l_2 given that

$$|\langle By, y \rangle| \le \Gamma \|y\|_{l_2}^2$$

with $\Gamma = \|G\|_{L_\infty}$.

Theorem 2. *Let $q_m > 0$. Then $\exists \gamma > 0$ such that $\forall y \in l_2$,*

$$\gamma \|y\|_{l_2}^2 \le |\langle By, y \rangle| \tag{8}$$

Proof. The proof relies on the fact that ess $\inf_{|w|\leq n} G(w) := \gamma_n > 0$ for any $n > 0$. For that let $\epsilon > 0$, $\epsilon < n$ such that $|\left((jw)^m \hat{h}(w)\right)|^{-1}$ is bounded above on $[-\epsilon, \epsilon]$ by some $\gamma_\epsilon > 0$. Then, forall $w \in [-\epsilon, \epsilon]$,

$$|w|^{2m} |\hat{h}(w)|^2 \geq \frac{1}{\gamma_\epsilon^2}$$

The function $w \to |w|^{2m} |\hat{h}(w)|^2$ is continuous on $[\epsilon, n]$ and does not vanish there. Thefore there exists $\gamma_+ > 0$ such that $|w|^{2m} |\hat{h}(w)|^2 \geq \gamma_+$ on $[\epsilon, n]$. The same holds for some γ_- on $[-n, -\epsilon]$ and $\gamma_n \geq \min\{\frac{1}{\gamma_\epsilon^2}, \gamma_+, \gamma_-\}$. □

This result can be reformulated in terms of generalized splines. For that let $\mu_j = \mu_{j,m} = m!\sqrt{d_{j,m}}\nu_{j,m}$ the modified divided difference and $\phi_j = h * \mu_j$ ($(-1)^m$, disappeared but it is not important?). This convolution is well-defined given that μ_j has compact support. Then,

Theorem 3. *Suppose $q_m > 0$, assumptions on h are satisfied. Let for $i, j \in \mathbb{Z}$*

$$b_{i,j} = \langle \phi_i, \phi_j \rangle_{L_2}$$

i.e B is the Gramian matrix of (ϕ_i). Let

$$V_X := clos_{L_2} span\{\phi_i, i \in \mathbb{Z}\}$$

Then (ϕ_i) is a Riesz basis for V_X with

$$\gamma \|a\|_{l_2}^2 \leq \left\| \sum_{j \in \mathbb{Z}} a_j \phi_j \right\|_{L_2}^2 \leq \Gamma \|a\|_{l_2}^2 \tag{9}$$