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Approximation of Minimum Energy Surfaces Using Optimal Twists

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Abstract—This paper gives a method for specifying the optimal 'twist vectors' at grid points for an interpolating surface to a rectangular network of curves. These twists are uniquely determined by minimizing an approximate energy form of the surface and can be obtained by solving a well-defined linear system.

Keywords—Coons patch, Twist, Interpolation, Approximation, Surface patch.

1. INTRODUCTION

Given a mesh of C^1 spline curves, $C_j(u)$ over the knots $u_1 < \cdots < u_m$ and $D_i(v)$ over $v_1 < \cdots < v_n$, with intersections at $C_j(u_i) = D_i(v_j)$, we seek an interpolating piecewise-bicubic blended Coons patch $X(u,v)$ which matches the curves, that is $X(u,v_j) = C_j(u)$ and $X(u_i,v) = D_i(v)$. These requirements do not determine the surface uniquely, and some surfaces are 'better' than others in some sense. The $(i,j)^{\text{th}}$ patch of the surface is uniquely defined by the points X and partial derivatives X_u, X_v, X_{uv} at the four corners of the patch. Matching the curves at all patches is achieved if $X(u_i,v_j) = C_j(u_i)$, $X_u(u_i,v_j) = C_j'(u_i)$, and $X_v(u_i,v_j) = D_i'(v_j)$, for all $1 \le i \le m$ and $1 \le j \le n$. However, the so called twists, $X_{uv}(u_i,v_j)$, are determined by some smoothing criteria or just remain at the designers' disposal.

So far many methods for the creation of twists have been given, such as Zero twists [1,2], Adini's twists [1,3], Bessel's twists [1], Selesnick's method [1,4] and Brunet's method [1,5], etc. In this paper, we give a natural alternative approach to compute the optimum twist vectors that minimize an energy integral over the entire piecewise bicubically blended Coons patch complex. This work is partially an extension of the results obtained on the computation of minimum energy curves in [6]. A more detailed investigation on the approximation of interpolatory minimum energy surfaces is being carried out.

The a exact energy functional $\int_s (k_1^2 + k_2^2)\,ds$ is a standard fairness criterion for surfaces in engineering, where k_1, k_2 are the principal curvatures of the surface and ds is the surface area measure [7,8]. Minimizing the exact energy is too complicated and difficult to be used efficiently as a design tool. Several approximation methods have been investigated. These include the thin plate approach in [9]

$$E = \iint_p \left(X_{uu}^2 + 2X_{uv}^2 + X_{vv}^2\right) du\,dv, \tag{1.1}$$

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which assumes small deflections of the surface curvatures; the membrane model in [10]

$$E = \iint_D \left(|X_u|^2 + |X_v|^2\right) du\, dv, \tag{1.2}$$

which approximates small variations of the surface area; and a generalized form of the thin plate model in [8]

$$E = \iint_D \left(\alpha^\top \cdot M \cdot \alpha\right) du\, dv, \tag{1.3}$$

where M is a 9×9 positive definite matrix and α is the second-order partial derivative vector

$$\alpha = \left(x_{uu}, x_{uv}, x_{vv}, y_{uu}, y_{uv}, y_{vv}, z_{uu}, z_{uv}, z_{vv}\right)^\top,$$

and x, y, z are the $x-, y-, z-$ components of $X(u,v)$. Different choices of the quadratic form M define different approximate energy forms. In this paper, we use the generalized form of the thin plate model as our fairness criterion for optimizing the surfaces defined by piecewise-bicubic blended Coons patches.

2. ONE PATCH CASE

We consider first, the case of one bicubic blended Coons patch $X(u,v)$, $u, v \in [0,1]$. Let us proceed with minimizing the energy monomial

$$E = \int_0^1 \int_0^1 (X_{uu}, X_{uv}, X_{vv}) \cdot Q \cdot \begin{pmatrix} X_{uu} \\ X_{uv} \\ X_{vv} \end{pmatrix} du\, dv, \tag{2.1}$$

where $X(k,l), X_u(k,l), X_v(k,l)$ are prescribed at the corners $k, l \in \{0,1\}$ of the unit square. $X(u,v)$ can be expressed in terms of its corner values as [1,11]

$$\begin{aligned} X(u,v) = {} & \alpha_0(u)X(0,v) + \alpha_1(u)X(1,v) + \beta_0(u)X_u(0,v) + \beta_1(u)X_u(1,v) \\ & + \alpha_0(v)X(u,0) + \alpha_1(v)X(u,1) + \beta_0(v)X_v(u,0) + \beta_1(v)X_v(u,1) \\ & - (\alpha_0(u), \alpha_1(u), \beta_0(u), \beta_1(u)) \\ & \begin{pmatrix} X(0,0) & X(0,1) & X_v(0,0) & X_v(0,1) \\ X(1,0) & X(1,1) & X_v(1,0) & X_v(1,1) \\ X_u(0,0) & X_u(0,1) & X_{uv}(0,0) & X_{uv}(0,1) \\ X_u(1,0) & X_u(1,1) & X_{uv}(1,0) & X_{uv}(1,1) \end{pmatrix} \begin{pmatrix} \alpha_0(v) \\ \alpha_1(v) \\ \beta_0(v) \\ \beta_1(v) \end{pmatrix}, \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} \alpha_0(t) &= 1 - 3t^2 + 2t^3, & \alpha_1(t) &= 3t^2 - 2t^3, \\ \beta_0(t) &= t - 2t^2 + t^3, & \beta_1(t) &= -t^2 + t^3, \end{aligned} \tag{2.3}$$

are the cubic blending functions over $[0,1]$.

By Farin's method, the following four tangent ribbons can be defined: $X_v(u,0)$, $X_v(u,1)$, $X_u(0,v)$, $X_u(1,v)$ as in [11]

$$\begin{aligned} X_v(u,0) &= X_v(0,0)\alpha_0(u) + X_v(1,0)\alpha_1(u) + X_{uv}(0,0)\beta_0(u) + X_{uv}(1,0)\beta_1(u), \\ X_v(u,1) &= X_v(0,1)\alpha_0(u) + X_v(1,1)\alpha_1(u) + X_{uv}(0,1)\beta_0(u) + X_{uv}(1,1)\beta_1(u), \\ X_u(0,v) &= X_u(0,0)\alpha_0(v) + X_u(0,1)\alpha_1(v) + X_{uv}(0,0)\beta_0(v) + X_{uv}(0,1)\beta_1(v), \\ X_u(1,v) &= X_u(1,0)\alpha_0(v) + X_u(1,1)\alpha_1(v) + X_{uv}(1,0)\beta_0(v) + X_{uv}(1,1)\beta_1(v). \end{aligned}$$

Since

$$
\begin{aligned}
F(u,v) &= \alpha_0(u)X(0,v) + \alpha_1(u)X(1,v) + \alpha_0(v)X(u,0) + \alpha_1(v)X(u,1) \\
&\quad + \beta_0(u)(X_u(0,0)\alpha_0(v) + X_u(0,1)\alpha_1(v)) + \beta_1(u)(X_u(1,0)\alpha_0(v) \\
&\quad + X_u(1,1)\alpha_1(v)) + \beta_0(v)(X_v(0,0)\alpha_0(u) + X_v(1,0)\alpha_1(u)) \\
&\quad + \beta_1(v)(X_v(0,1)\alpha_0(u) + X_v(1,1)\alpha_1(u)) \\
&\quad - (\alpha_0(u), \alpha_1(u), \beta_0(u), \beta_1(u)) \\
&\quad \begin{pmatrix} X(0,0) & X(0,1) & X_v(0,0) & X_v(0,1) \\ X(1,0) & X(1,1) & X_v(1,0) & X_v(1,1) \\ X_u(0,0) & X_u(0,1) & 0 & 0 \\ X_u(1,0) & X_u(1,1) & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_0(v) \\ \alpha_1(v) \\ \beta_0(v) \\ \beta_1(v) \end{pmatrix} \qquad (2.4) \\
&= \alpha_0(u)X(0,v) + \alpha_1(u)X(1,v) + \alpha_0(v)X(u,0) + \alpha_1(v)X(u,1) \\
&\quad - (\,\alpha_0(u)\alpha_1(u)\,) \begin{pmatrix} X(0,0) & X(0,1) \\ X(1,0) & X(1,1) \end{pmatrix} \begin{pmatrix} \alpha_0(v) \\ \alpha_1(v) \end{pmatrix},
\end{aligned}
$$

we can rewrite (2.2) as

$$
X(u,v) = (\beta_0(u), \beta_1(u)) \begin{pmatrix} X_{uv}(0,0) & X_{uv}(0,1) \\ X_{uv}(1,0) & X_{uv}(1,1) \end{pmatrix} \begin{pmatrix} \beta_0(v) \\ \beta_1(v) \end{pmatrix} + F(u,v). \tag{2.5}
$$

Therefore, we have

$$
X_{uu}(u,v) = (\beta_0''(u), \beta_1''(u)) \begin{pmatrix} X_{uv}(0,0) & X_{uv}(0,1) \\ X_{uv}(1,0) & X_{uv}(1,1) \end{pmatrix} \begin{pmatrix} \beta_0(v) \\ \beta_1(v) \end{pmatrix} + F_{uu}(u,v), \tag{2.6}
$$

$$
X_{uv}(u,v) = (\beta_0'(u), \beta_1'(u)) \begin{pmatrix} X_{uv}(0,0) & X_{uv}(0,1) \\ X_{uv}(1,0) & X_{uv}(1,1) \end{pmatrix} \begin{pmatrix} \beta_0'(v) \\ \beta_1'(v) \end{pmatrix} + F_{uv}(u,v), \tag{2.7}
$$

$$
X_{vv}(u,v) = (\beta_0(u), s\beta_1(u)) \begin{pmatrix} X_{uv}(0,0) & X_{uv}(0,1) \\ X_{uv}(1,0) & X_{uv}(1,1) \end{pmatrix} \begin{pmatrix} \beta_0''(v), \\ \beta_1''(v) \end{pmatrix} + F_{vv}(u,v). \tag{2.8}
$$

The minimum of (2.1) can be determined by using the necessary conditions

$$
\frac{\partial E}{\partial X_{uv}(s,t)} = 0, \qquad \text{for } s,t \in \{0,1\}. \tag{2.9}
$$

From (2.5)–(2.8) we obtain, for $s,t \in \{0,1\}$,

$$
\int_0^1 \int_0^1 (\beta_s''(u)\beta_t(v), \beta_s'(u)\beta_t'(v), \beta_s(u)\beta_t''(v)) \cdot Q \cdot \begin{pmatrix} X_{uu}(u,v) \\ X_{uv}(u,v) \\ X_{vv}(u,v) \end{pmatrix} du\, dv = 0. \tag{2.10}
$$

Let $A_{s,t} = (\beta_s''(u)\beta_t(v), \beta_s'(u)\beta_t'(v), \beta_s(u)\beta_t''(v))^{\mathsf{T}}$, and

$$
h(s,t) = -\int_0^1 \int_0^1 A_{s,t}^{\mathsf{T}} \cdot Q \cdot \begin{pmatrix} F_{uu}(u,v) \\ F_{uv}(u,v) \\ F_{vv}(u,v) \end{pmatrix} du\, dv, \tag{2.11}
$$

$$
g(k,l,s,t) = \int_0^1 \int_0^1 A_{s,t}^{\mathsf{T}} \cdot Q \cdot A_{k,l}\, du\, dv. \tag{2.12}
$$

Then the conditions in (2.10) become

$$
\sum_{k,l \in \{0,1\}} g(k,l,s,t) X_{uv}(k,l) = h(s,t), \qquad s,t \in \{0,1\}. \tag{2.13}
$$

The matrix form of (2.13) is given by

$$GX_{uv} = H, \tag{2.14}$$

where $X_{uv} = (X_{uv}(0,0), X_{uv}(0,1), X_{uv}(1,0), X_{uv}(1,1))^\top$, $H = (h(0,0), h(0,1), h(1,0), h(1,1))^\top$, and

$$G = \begin{pmatrix} g(0,0,0,0) & g(0,1,0,0) & g(1,0,0,0) & g(1,1,0,0) \\ g(0,0,0,1) & g(0,1,0,1) & g(1,0,0,1) & g(1,1,0,1) \\ g(0,0,1,0) & g(0,1,1,0) & g(1,0,1,0) & g(1,1,1,0) \\ g(0,0,1,1) & g(0,1,1,1) & g(1,0,1,1) & g(1,1,1,1) \end{pmatrix}. \tag{2.15}$$

The regularity of the linear system (2.14) will be discussed in Section 4.

3. GENERAL CASE

We now consider the general situation with $m\times n$ rectangular patches, $X^{[s,t]}(u,v)$, $0 \le s \le m-1$ and $0 \le t \le n-1$. As in (2.3) and (2.4), for all s and t, we define $X^{[s,t]}(u,v)$ as

$$X^{[s,t]}(u,v) = F^{[s,t]}(u,v) + (\beta_0(u), \beta_1(u)) \begin{pmatrix} X_{uv}^{[s,t]}(0,0) & X_{uv}^{[s,t]}(0,1) \\ X_{uv}^{[s,t]}(1,0) & X_{uv}^{[s,t]}(1,1) \end{pmatrix} \begin{pmatrix} \beta_0(v) \\ \beta_1(v) \end{pmatrix},$$

where

$$\begin{aligned} F^{[s,t]}(u,v) = &-(\alpha_0(u), \alpha_1(u)) \begin{pmatrix} X^{[s,t]}(0,0) & X^{[s,t]}(0,1) \\ X^{[s,t]}(1,0) & X^{[s,t]}(1,1) \end{pmatrix} \begin{pmatrix} \alpha_0(v) \\ \alpha_1(v) \end{pmatrix} \\ &+ \alpha_0(u) X^{[s,t]}(0,v) + \alpha_1(u) X^{[s,t]}(1,v) \\ &+ \alpha_0(v) X^{[s,t]}(u,0) + \alpha_1(v) X^{[s,t]}(u,1). \end{aligned} \tag{3.1}$$

Each patch $X^{[s,t]}(u,v)$ has four corners $X^{[s,t]}(0,0)$, $X^{[s,t]}(0,1)$, $X^{[s,t]}(1,0)$, and $X^{[s,t]}(1,1)$. For C^1 continuity it is required that $X^{[s_1,t_1]}(k_1,l_1) = X^{[s_2,t_2]}(k_2,l_2)$, whenever $s_1+k_1 = s_2+k_2$ and $t_1+l_1 = t_2+l_2$, where $s_1, s_2 \in [0, m-1]$; $t_1, t_2 \in [0, n-1]$ and $k_1, k_2, l_1, l_2 \in \{0,1\}$. The twist vectors must satisfy the C^1 compatibility conditions $X_{uv}^{[s_1,t_1]}(k_1,l_1) = X_{uv}^{[s_2,t_2]}(k_2,l_2)$, whenever $s_1+k_1 = s_2+k_2$ and $t_1+l_1 = t_2+l_2$.

(i) Corner point. For the point $X^{[s,t]}(s/m-1, t/n-1)$ of the patch $X^{[s,t]}(u,v)$, where $s = 0$ or $m-1$, $t = 0$ or $n-1$, no other patch interpolates this point. The necessary condition $\frac{\partial E}{\partial X_{uv}^{[s,t]}(s/(m-1), t/(n-1))} = 0$ for minimizing E becomes

$$\frac{\partial E^{[s,t]}}{\partial X_{uv}^{[s,t]}(s/(m-1), t/(n-1))} = 0. \tag{3.2}$$

Equation (3.2) can be written as

$$\begin{gathered} \sum_{k,l\in\{0,1\}} g\left(k, l, \frac{s}{m-1}, \frac{t}{n-1}\right) X_{uv}^{[s,t]}(k,l) = h^{[s/(m-1), t/(n-1)]}(s,t), \\ s \in \{0, m-1\}, \quad t \in \{0, n-1\}, \end{gathered} \tag{3.3}$$

where $g(k,l,s,t)$ is defined by (2.11) and $h^{[s,t]}(s_1,t_1)$ is defined by

$$h^{[s,t]}(s_1,t_1) = -\int_0^1 \int_0^1 A_{s,t}^\top \cdot Q \cdot \begin{pmatrix} F_{uu}^{[s_1,t_1]}(u,v) \\ F_{uv}^{[s_1,t_1]}(u,v) \\ F_{vv}^{[s_1,t_1]}(u,v) \end{pmatrix} du\, dv. \tag{3.4}$$

(ii) Interior point. For the point $X^{[s,t]}(0,0)$, where $s \in [1, m-1]$, $t \in [1, n-1]$, four patches $X^{[s,t-1]}$, $X^{[s-1,t-1]}$, $X^{[s-1,t]}$, and $X^{[s,t]}$ all interpolate this point. The necessary condition $\frac{\partial E}{\partial X_{uv}^{[s,t]}(0,0)} = 0$ for minimizing the energy E is

$$\sum_{s_1,t_1\in\{0,1\}} \frac{\partial E^{[s-s_1,t-t_1]}}{\partial X_{uv}^{[s-s_1,t-t_1]}(s_1,t_1)} = 0, \qquad 0 < s < m, \quad 0 < t < n. \tag{3.5}$$

Equation (3.5) is equivalent to

$$\begin{aligned}&\sum_{s_1,t_1\in\{0,1\}}\ \sum_{k,l\in\{0,1\}} g(k,l,s_1,t_1)X_{uv}^{[s-s_1,t-t_1]}(k,l)\\&\quad=\sum_{s_1,t_1\in\{0,1\}} h^{[s_1,t_1]}(s-s_1,t-t_1),\qquad s\in[1,m-1],\quad t\in[1,n-1],\end{aligned}\tag{3.6}$$

where g is defined by (2.11) and $h^{[s,t]}(s_1,t_1)$ is defined by (3.4).

(iii) Boundary edge point. Points $X^{[s,(n-1)t]}(0,t)$ for $1\le s\le m-1$ and $t\in\{0,1\}$ and $X^{[(m-1)s,t]}(s,0)$ for $1\le t\le n-1$ and $s\in\{0,1\}$. Each of these is shared by two patches. The necessary condition for minimizing E is given by two equations

$$\sum_{s_1=0}^{1}\frac{\partial E^{[s-s_1,(n-1)t]}}{\partial X_{uv}^{[s-s_1,(n-1)t]}(s_1,t)}=0,\qquad t\in\{0,1\},\quad 1\le s\le m-1,\tag{3.7}$$

$$\sum_{t_1=0}^{1}\frac{\partial E^{[(m-1)s,t-t_1]}}{\partial X_{uv}^{[(m-1)s,t-t_1]}(s,t_1)}=0,\qquad s\in\{0,1\},\quad 1\le t\le n-1.\tag{3.8}$$

Equations (3.7) and (3.8) can be simplified, respectively, to

$$\begin{aligned}&\sum_{s_1=0}^{1}\ \sum_{k,l\in\{0,1\}} g(k,l,s_1,t)X_{uv}^{[s-s_1,(n-1)t]}(k,l)\\&\quad=\sum_{s_1=0}^{1} h^{[s_1,t]}(s-s_1,(n-1)t),\qquad t\in\{0,1\},\quad 1\le s\le m-1,\end{aligned}\tag{3.9}$$

$$\begin{aligned}&\sum_{t_1=0}^{1}\ \sum_{k,l\in\{0,1\}} g(k,l,s,t_1)X_{uv}^{[(m-1)s,t-t_1]}(k,l)\\&\quad=\sum_{t_1=0}^{1} h^{[s,t_1]}((m-1)s,t-t_1),\qquad s\in\{0,1\},\quad 1\le t\le n-1.\end{aligned}\tag{3.10}$$

The optimal twist vectors must satisfy all the equations (3.3), (3.6), (3.9), and (3.10).

4. UNIQUENESS OF THE OPTIMAL TWISTS

In previous sections, we derive the linear system that the optimal twist vectors must satisfy. But we must make sure that the system is nonsingular, otherwise, we still cannot solve the twist vectors from the equations. In Kallay and Ravani's paper (cf. [8]), they also obtained some similar equations. Since they considered the general quadratic energy form and the uniqueness of twist vectors cannot be proven. In the following, we assume that the quadratic energy form Q is nontrivial, i.e., Q is semipositive definite and $Q\neq 0$, which is quite natural, because in practice only positive energy norms are considered. We show that the corresponding linear systems obtained is well defined under this mild assumption.

LEMMA 4.1. *The matrix in (2.14) has the following properties.* $g(s,t,s,t)\equiv$ *constant, for* $\forall\, s,t\in\{0,1\}$;

$$\begin{aligned}g(0,1,0,0)&=g(0,0,0,1)=g(1,1,1,0)=g(1,0,1,1),\\ g(1,0,0,0)&=g(0,0,1,0)=g(0,1,1,1)=g(1,1,0,1),\\ g(1,1,0,0)&=g(0,0,1,1);\qquad g(1,0,0,1)=g(0,1,1,0).\end{aligned}$$

PROOF. This lemma can be proven by direct evaluations.

From Lemma 4.1, by defining $a = g(0,0,0,0)$, $b = g(0,1,0,0)$, $c = g(1,0,0,0)$, $d_2 = g(1,1,0,0)$, and $d_1 = g(1,0,0,1)$, we obtain

$$G = \begin{pmatrix} a & b & c & d_2 \\ b & a & d_1 & c \\ c & d_1 & a & b \\ d_2 & c & b & a \end{pmatrix}. \tag{4.1}$$

By arranging equations (3.3), (3.6), (3.9), (3.10), and twist vectors properly, we can obtain the following coefficient matrix of the resulting linear system

$$H(m,n) = \begin{pmatrix} A_m & C_m & 0 & \cdots & 0 & 0 \\ C_m^\top & 2A_m & C_m & \cdots & 0 & 0 \\ 0 & C_m^\top & 2A_m & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2A_m & C_m \\ 0 & 0 & 0 & \cdots & C_m^\top & A_m \end{pmatrix}, \text{and there are } (n-1) \text{ matrix } 2A_m$$

on the diagonal of the matrix between the first and the last A_m, and A_m and C_m are $(m+1) \times (m+1)$ matrices defined as

$$A_m = \begin{pmatrix} a & b & 0 & \cdots & 0 & 0 \\ b & 2a & b & \cdots & 0 & 0 \\ 0 & b & 2a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2a & b \\ 0 & 0 & 0 & \cdots & b & a \end{pmatrix} \text{ and } C_m = \begin{pmatrix} c & d_2 & 0 & \cdots & 0 & 0 \\ d_1 & 2c & d_2 & \cdots & 0 & 0 \\ 0 & d_1 & 2c & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2c & d_2 \\ 0 & 0 & 0 & \cdots & d_1 & c \end{pmatrix}.$$

To show that the coefficient matrix $H(m,n)$ is positive definite, we introduce two lemmas here first.

LEMMA 4.2. *The matrix G in (2.14) is positive definite.*

PROOF. By (2.11), $g(s_1,t_1,s,t) = \int_0^1\int_0^1 A_{s_1,t_1}^\top \cdot Q \cdot A_{s,t}\,du\,dv$, where

$$A_{k,l} = \left(\beta_k''(u)\beta_l(v), \beta_k'(u)\beta_l'(v), \beta_k(u)\beta_l''(v)\right)^\top$$

and Q is the quadratic energy form. Since by assumption Q is semipositive definite, we can get that

$$\begin{aligned} G &= \begin{pmatrix} g(0,0,0,0) & g(0,1,0,0) & g(1,0,0,0) & g(1,1,0,0) \\ g(0,0,0,1) & g(0,1,0,1) & g(1,0,0,1) & g(1,1,0,1) \\ g(0,0,1,0) & g(0,1,1,0) & g(1,0,1,0) & g(1,1,1,0) \\ g(0,0,1,1) & g(0,1,1,1) & g(1,0,1,1) & g(1,1,1,1) \end{pmatrix} \\ &= \int_0^1\int_0^1 \begin{pmatrix} A_{00}^\top \\ A_{01}^\top \\ A_{10}^\top \\ A_{11}^\top \end{pmatrix} \cdot Q \cdot (A_{00}, A_{01}, A_{10}, A_{11})\,du\,dv. \end{aligned} \tag{4.2}$$

Suppose $X = (x_1,x_2,x_3,x_4)^\top \neq 0$ and $W = (A_{00}, A_{01}, A_{10}, A_{11})$. We have $X^\top \cdot G \cdot X = \int_0^1\int_0^1 (WX)^\top \cdot Q \cdot (WX)\,du\,dv \geq 0$, and the equality holds if and only if

$$X^\top \cdot \begin{pmatrix} A_{00}^\top \\ A_{01}^\top \\ A_{10}^\top \\ A_{11}^\top \end{pmatrix} \cdot Q \cdot (A_{00}, A_{01}, A_{10}, A_{11}) \cdot X = 0, \qquad \text{for } \forall\, u,v \in [0,1]. \tag{4.3}$$

Since Q is semipositive definite and $Q \neq 0$, there exists a nonzero matrix

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix},$$

such that $Q = M \cdot M^\top$. Therefore, we have

$$\begin{aligned} X^\top \cdot W^\top &= (x_1, x_2, x_3, x_4) \cdot \begin{pmatrix} \beta_0''(u)\beta_0(v) & \beta_0'(u)\beta_0'(v) & \beta_0(u)\beta_0''(v) \\ \beta_0''(u)\beta_1(v) & \beta_0'(u)\beta_1'(v) & \beta_0(u)\beta_1''(v) \\ \beta_1''(u)\beta_0(v) & \beta_1'(u)\beta_0'(v) & \beta_1(u)\beta_0''(v) \\ \beta_1''(u)\beta_1(v) & \beta_1'(u)\beta_1'(v) & \beta_1(u)\beta_1''(v) \end{pmatrix} \\ &= (k_{uu}(u,v), k_{uv}(u,v), k_{vv}(u,v)), \end{aligned} \tag{4.4}$$

where $k(u,v) = \beta_0(u)(x_1\beta_0(v) + x_2\beta_1(v)) + \beta_1(u)(x_3\beta_0(v) + x_4\beta_1(v))$. Because the condition in (4.3) becomes

$$X^\top W^\top Q W X = (k_{uu}(u,v), k_{uv}(u,v), k_{vv}(u,v))\, M M^\top \begin{pmatrix} k_{uu}(u,v) \\ k_{uv}(u,v) \\ k_{vv}(u,v) \end{pmatrix} = 0, \qquad \text{for } \forall\, u, v \in [0,1] \tag{4.5}$$

it suffices to show that the functions k_{uu}, k_{uv}, k_{vv} are linearly independent over $[0,1]^2$, which can be proven directly. Therefore, we can conclude that G is positive definite.

COROLLARY 4.1.

$$G_0 = \begin{pmatrix} a & c & b & d_2 \\ c & a & d_1 & b \\ b & d_1 & a & c \\ d_2 & b & c & a \end{pmatrix}$$

is positive definite.

LEMMA 4.3. *Suppose A and C are square matrices of size N and A is symmetric. Let*

$$k_n = \begin{pmatrix} A & C & 0 & \cdots & 0 & 0 \\ C^\top & 2A & C & \cdots & 0 & 0 \\ 0 & C^\top & 2A & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2A & C \\ 0 & 0 & 0 & \cdots & C^\top & A \end{pmatrix}$$

be a square matrix of size $N \times (N+2)$. If $k_0 = \left(\begin{smallmatrix} A & C \\ C^\top & A \end{smallmatrix}\right)$ is positive definite, then k_n is also positive definite for all $n \geq 1$.

PROOF. By assumption k_0 is obviously positive. We use induction to complete the proof. Suppose k_r is positive definite for some $r \geq 0$. We show now that k_{r+1} is also positive definite. Note that $k_{r+1} = \left(\begin{smallmatrix} k_r & 0 \\ 0 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 0 \\ 0 & k_0 \end{smallmatrix}\right)$ and define $X = (X_1, X_2, \dots, X_r, X_{r+1})^\top$, then $X^\top \cdot k_{r+1} \cdot X = X^\top \cdot \left(\begin{smallmatrix} k_r & 0 \\ 0 & 0 \end{smallmatrix}\right) \cdot X + X^\top \cdot \left(\begin{smallmatrix} 0 & 0 \\ 0 & k_0 \end{smallmatrix}\right) \cdot X \geq 0$, and the equality holds if and only if

$$(X_1, X_2, \dots, X_r) \cdot k_r \cdot \begin{pmatrix} X_1 \\ X_2 \\ \cdots \\ X_r \end{pmatrix} = 0 \text{ and } (X_r, X_{r+1}) \cdot k_0 \cdot \begin{pmatrix} X_r \\ X_{r+1} \end{pmatrix} = 0.$$

Since k_r and k_0 are all positive definite, we have $(X_1, X_2, \dots, X_r) = 0$ and $(X_r, X_{r+1}) = 0$, that is, $X = 0$. Therefore, k_{r+1} is also positive definite. Thus we conclude that k_n is positive definite for all $n \geq 1$.

By Lemma 4.3, to prove that $H(m,n)$ is positive definite, we only need to show $H(m,1) = \left(\begin{smallmatrix} A_m & C_m \\ C_m^\top & A_m \end{smallmatrix}\right)$ is positive definite. Because of the symmetry of m and n, we have that there exists a nonsingular matrix W such that $H(m,1) = W^\top \cdot H_0(1,m) \cdot W$, where

$$H_0(1,m) = \begin{pmatrix} A_1 & C_1 & 0 & \cdots & 0 & 0 \\ C_1^\top & 2A_1 & C_1 & \cdots & 0 & 0 \\ 0 & C_1^\top & 2A_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2A_1 & C_1 \\ 0 & 0 & 0 & \cdots & C_1^\top & A_1 \end{pmatrix}$$

and $A_1 = \left(\begin{smallmatrix} a & c \\ c & a \end{smallmatrix}\right)$ and $C_1 = \left(\begin{smallmatrix} b & d_2 \\ d_1 & b \end{smallmatrix}\right)$. By Lemma 4.3 to prove that $H_0(1,m)$ is positive definite, we only need to prove that

$$H_0(1,1) = \begin{pmatrix} a & c & b & d_2 \\ c & a & d_1 & b \\ b & d_1 & a & c \\ d_2 & b & c & a \end{pmatrix}$$

is positive definite. From Corollary 4.1 of Lemma 4.2, we know that $H_0(1,1) = G_0$ is positive definite. This can be summarized in following theorem.

THEOREM 4.1. *If the quadratic form Q is semipositive definite and $Q \neq 0$, then the coefficient matrix of the linear system derived from minimizing the energy is positive definite. Thus the optimal twists are determined uniquely.*

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