On the representation of sphere-like surfaces with splines

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Framework and motivation

Unkown function $g : \mathbb{R}^d \to \mathbb{R}^d, \mathbb{R}^{d+1}$ to be reconstructed from **finite** set of values

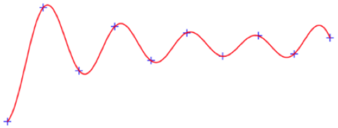
1. $g(\mathbf{t_1}), \ldots, g(\mathbf{t_n})$
2. Possibly $\partial^{\boldsymbol{\alpha}} g(\mathbf{t_1}), \ldots, \partial^{\boldsymbol{\alpha}} g(\mathbf{t_n})$ with $\boldsymbol{\alpha} \in \mathbb{N}^d$

General model for the approximation f to g

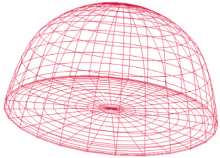
$$f(t) = \sum_k c_k^T \Phi(t - t_k) \tag{1}$$

Example applications

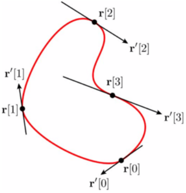
1D $g : \mathbb{R} \to \mathbb{R}$. Known values $g(0), \ldots, g(9)$.
2D $r = g : \mathbb{R} \to \mathbb{R}^2$. Known values $r(0), \ldots, r(3)$, $\partial^1 r(0), \ldots, \partial^1 r(3)$
3D $\sigma = g : \mathbb{R}^2 \to \mathbb{R}^3$. Known values $\sigma(k,l), \partial^{1,0}\sigma(k,l), \partial^{0,1}\sigma(k,l), \partial^{1,1}\sigma(k,l)$.



Example 1D



Example 3D



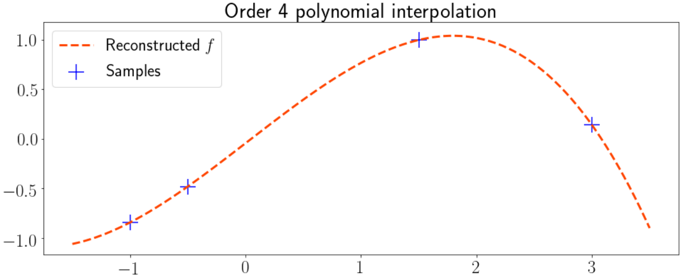
Example 2D

Lagrange polynomials

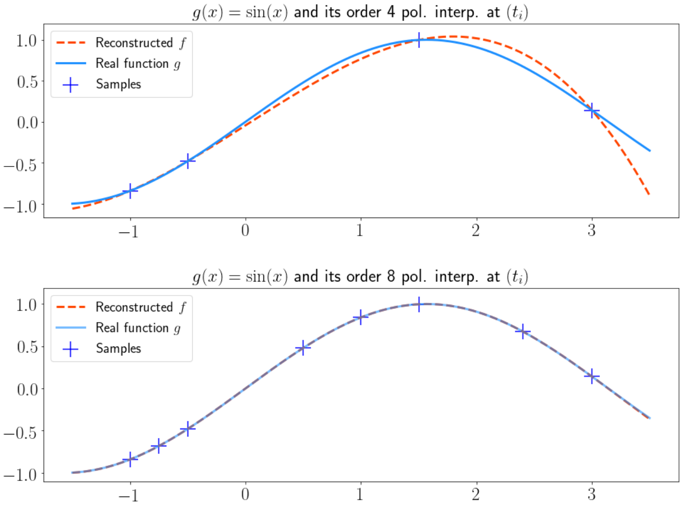
Denote $\Pi_{<k}$ set of polynomials of order k (vector space).
Obj: reconstruct $g : \mathbb{R} \to \mathbb{R}$ from $g(t_1), \ldots, g(t_n)$.

Theorem

Given n sites (not necessarily distinct), there exists a unique f in $\Pi_{<n}$ that agrees with g at $(t_i)_1^n$.

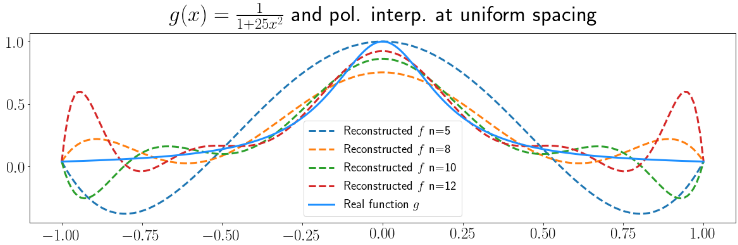


Simply choose n big enough?



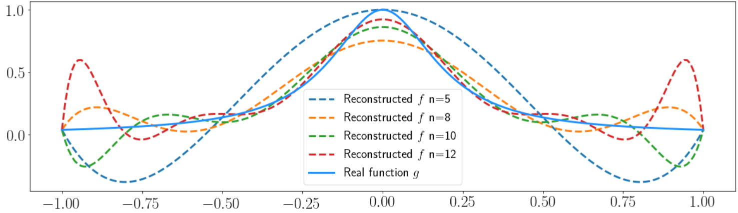
Take sufficiently many points and all is well?

Runge example



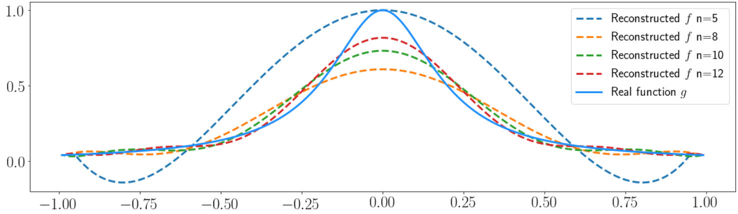
Runge example

$g(x) = \frac{1}{1+25x^2}$ and pol. interp. at uniform spacing



Chebychev sites are good. $[a..b]$ choose $t_i = [a + b - (a - b)\cos\left(\frac{\pi(2i-1)}{2n}\right)]/2$

$g(x) = \frac{1}{1+25x^2}$ and pol. interp. at Chebychev sites



Jackson's theorem

Theorem

If $g \in \mathcal{C}^r([a..b])$ and $n > r+1$ then

$$\mathrm{dist}(g, \Pi_{<n}) \leq const_r \left(\frac{b-a}{n-1}\right)^r w(g^{(r)}, \frac{b-a}{2n-1-r}) \qquad (2)$$

Idea: break $b-a$ into l pieces $x_{k+1} - x_k$ and do piecewise pol. approximation.

Pros: multiple small polynomials instead of 1 big polynomial.
Cons: junction at breaks?

Piecewise-linear

Let $t = (t_i)_1^n$.

- $\Pi_{<k,\boldsymbol{t}}$ set of pp. of order k with breaks at $t_2, \ldots, t_{n-1}$.
- $S_{2,\boldsymbol{t}} = \Pi_{<2,\boldsymbol{t}} \cap \mathcal{C}^0$.

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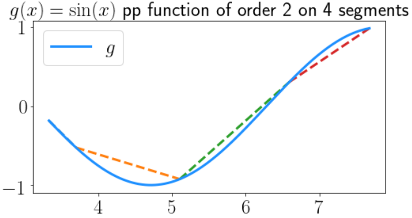
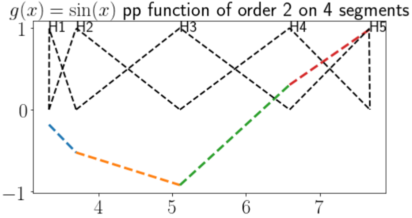
Order 2 interpolant operator $I_2 : g \mapsto I_2 g \in S_{2,\boldsymbol{t}}$





$$I_2 g(t) = \sum_{i=1}^{n} g(t_i) H_i(t) \tag{3}$$

$(H_1, \ldots, H_n)$ is basis for $\$_{2,\boldsymbol{t}}$

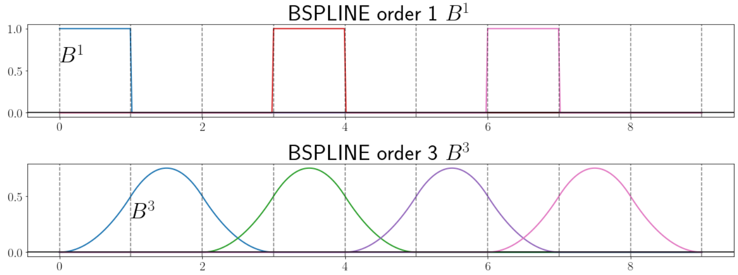
The B-spline basis

Definition

The cardinal B-spline of order n (and knot multiplicity 1) is given by recursive relation

$$B^n = B^1 * B^1 * \ldots * B^1 \quad (n \text{ times}) \tag{4}$$

with $B^1(t) = \begin{cases} 1 & \text{if } 0 \leq 1 < 1 \\ 0 & \text{else} \end{cases}$



Properties

Good properties

1. $B^n \in \Pi_{<n,\mathbb{Z}} \cap \mathcal{C}^{n-2}$
2. B^n is zero outside $[0, n]$
3. (partition unity) $\forall t \quad \sum_{k\in\mathbb{Z}} B^n(t-k) = 1$
4. (Riesz-basis) $\exists D_n \in (0,1)$ such that $\forall \boldsymbol{c}$ bounded,
$$D_n\|\boldsymbol{c}\| \leq \|\sum_{k\in\mathbb{Z}} c[k]B^n(.-k)\| \leq \|\boldsymbol{c}\|$$
5. $\hat{B}^n(w) = \left(\frac{1-e^{-jw}}{jw}\right)^n$

Spline curve

Definition

A (cardinal) ***spline*** *or order n (and knot multiplicity 1) is a linear combination of* $\{B^n(. - k)\}$

$$f(t) = \sum_{k \in \mathbb{Z}} c[k] B^n(t-k) \tag{5}$$

with $c[k] \in \mathbb{R}$*. We denote* $S_{n,1}$ *the set of all such functions.*

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Remarks

1. $(c[k])$ is the sequence of **control points**.
2. $S_{n,1} \subset \Pi_{<n,\mathbb{Z}} \cap \mathcal{C}^{n-2}$ (in fact these sets are equal!)
3. $r_k = \#\{l|t_l = k\}$ (multiplicity), f is $\mathcal{C}^{n-r_k-1}$ in $\mathcal{V}(k)$.
4. Denote $S_{n,r}$ splines of order n with knots of multiplicity r.
5. A **spline curve** is a vector-valued spline i.e $c[k] \in \mathbb{R}^d$ (d=2,3 usually)

Dream pursued

Find approximation scheme f interpolating $(\boldsymbol{y}_k)$

$$f(t) = \sum_k \boldsymbol{y}_k^T \phi(t - t_k) \tag{6}$$

and ϕ retains good properties of splines (partition-unity, Riesz-basis, compact support, subdivision).

Hermite interpolation

Let $\mathcal{S}$ a vector space. C.H.I.P $(y, \ldots, y^{(r-1)}, \mathcal{S})$ is solved if $\exists f \in \mathcal{S}$ such that

$$\forall s = 0, \ldots, r-1, \forall k \in \mathbb{Z} \qquad f^{(s)}(k) = y_k^{(s)} \tag{7}$$

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Theorem

n even, $1 \leq r \leq n/2$. The C.H.I.P $(y, \ldots, y^{(r-1)}, S_{n,r} \cap F_{\gamma,r}$ (resp. $\mathcal{L}_{p,r}$)) has a unique solution iif $y^{(s)} \in Y_\gamma$ (resp. l_p). In that case

$$f(t) = \sum_{k \in \mathbb{Z}} y_k L_{n,r,0}(t-k) + \ldots, y_k^{(r-1)} L_{n,r-1}(t-k) \tag{8}$$

Remarks

1. $L_{n,r,s}$ are elements of $S_{n,r}$. Calculated by solving system of $n-r$ equations.
2. $f \in S_{n,r}$ is a **spline** of order n

Hermite snakes (Uhlmann 2016)

Data: at M sites $\{r[k]\}_{k=0}^{M-1}, \{r'[k]\}_{k=0}^{M-1}$.
Obj: Continuous representation of the curve $r(t) = (r_1(t), r_2(t))$

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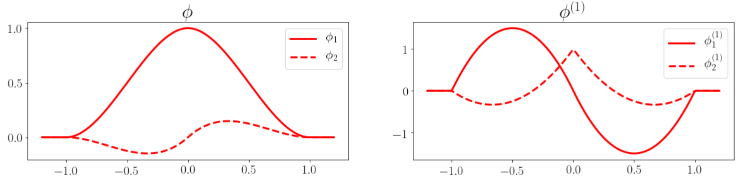
Schoenberg for $n = 4$ and $r = 2$

$$r(t) = \sum_{k\in\mathbb{Z}} r[k]\phi_1(t-k) + r'[k]\phi_2(t-k) \tag{9}$$

$\phi_1 = L_{4,2,0}, \phi_2 = L_{4,2,1}$ (polynomials order 4). $r_1, r_2 \in \mathcal{C}^{n-r-1} = \mathcal{C}^1$.



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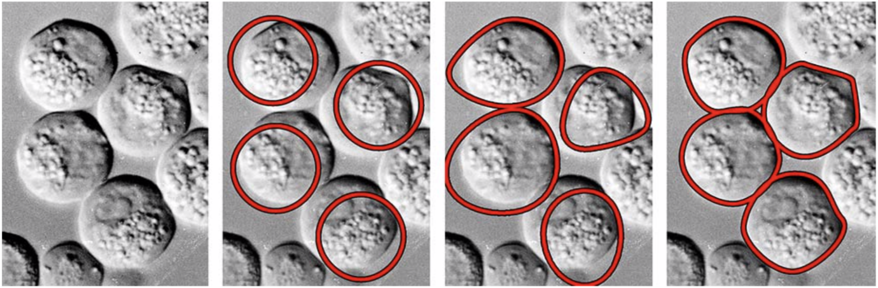

Fig. 11. Segmentation of differential-interference-contrast images of animal pancreatic acinar cells using closed Hermite snakes. From left to right: original image, snake initialization, automated segmentation result using a classical spline-snake [14], automated segmentation result using the Hermite snake.

Ellipse-preserving Hermite (Conti 2015)

Data: at M sites $\{r[k]\}_{k=0}^{M-1}, \{r'[k]\}_{k=0}^{M-1}$.
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$$r(t) = \sum_{k \in \mathbb{Z}} r[k]\phi_1(t-k) + r'[k]\phi_2(t-k) \tag{10}$$

with $\phi_i(t) = \left(a_i(t) + b_i(t)t + c_i(t)e^{jwx} + d_i(t)e^{-jwx}\right)\chi_{[-1,1]}$ and $w = \frac{2\pi}{M}$ (exponential polynomials).

Remark: here the curve is said to be an **exponential spline curve**

Ellipse-preserving Hermite (Conti 2015)

Reproducing sinusoids with M coefficients

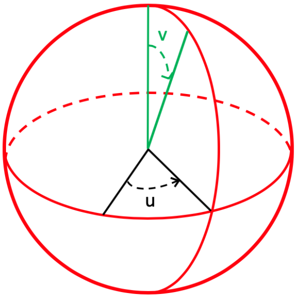
$$\cos(2\pi t) = \sum_{k\in\mathbb{Z}} \cos(\frac{2\pi k}{M})\phi_1(Mt-k) - \frac{2\pi}{M}\sin(\frac{2\pi k}{M})\phi_2(Mt-k) \quad (11)$$

$$\sin(2\pi t) = \sum_{k\in\mathbb{Z}} \sin(\frac{2\pi k}{M})\phi_1(Mt-k) + \frac{2\pi k}{M}\cos(\frac{2\pi k}{M})\phi_2(Mt-k) \quad (12)$$

The case of the unit sphere

The sphere surface parametrized by $g = \sigma : \mathbb{R}^2 \to \mathbb{R}^3$

$$\sigma(u,v) = \begin{bmatrix} \cos(2\pi u)\sin(\pi v) \\ \sin(2\pi u)\sin(\pi v) \\ \cos(\pi v) \end{bmatrix} \quad (u,v) \in [0,1]^2 \tag{13}$$



From 2D to 3D for tensor-product surfaces

Data: M_1 control points on latitudes, $M_2 + 1$ control points on longitudes.
At each point 4 parameters $\sigma, \partial^{1,0}\sigma, \partial^{0,1}\sigma, \partial^{1,1}\sigma$.
Obj: Continuous representation of $g : \mathbb{R}^2 \to \mathbb{R}^3$

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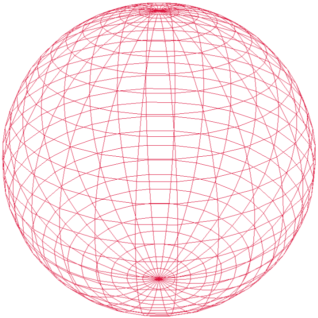
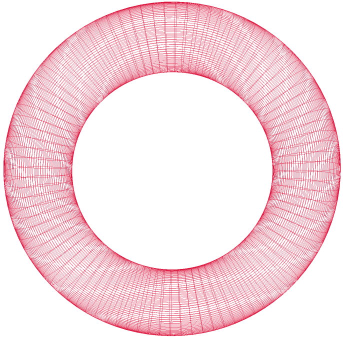
$$\begin{aligned}
\sigma(u,v) = & \sum_{k=0}^{M_1-1} \sum_{l=0}^{M_2} c_1[k,l]\phi_{1,w_1,per}(M_1 u - k)\phi_{1,w_2}(M_2 v - l) \\
&+ \sum_{k=0}^{M_1-1} \sum_{l=0}^{M_2} c_2[k,l]\phi_{1,w_1,per}(M_1 u - k)\phi_{2,w_2}(M_2 v - l) \\
&+ \sum_{k=0}^{M_1-1} \sum_{l=0}^{M_2} c_3[k,l]\phi_{2,w_1,per}(M_1 u - k)\phi_{1,w_2}(M_2 v - l) \\
&+ \sum_{k=0}^{M_1-1} \sum_{l=0}^{M_2} c_4[k,l]\phi_{2,w_1,per}(M_1 u - k)\phi_{2,w_2}(M_2 v - l)
\end{aligned}$$

From 2D to 3D for tensor-product surfaces

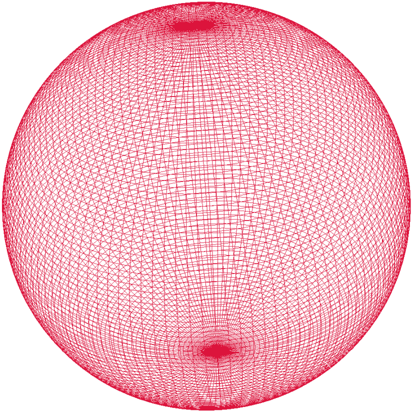
$$c_1[k,l] = \sigma(\frac{k}{M_1}, \frac{l}{M_2}) \qquad c_2[k,l] = \frac{1}{M_2}\frac{\partial \sigma}{\partial v}(\frac{k}{M_1}, \frac{l}{M_2})$$

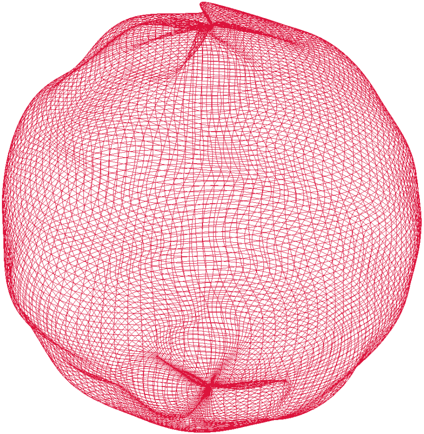
$$c_3[k,l] = \frac{1}{M_1}\frac{\partial \sigma}{\partial u}(\frac{k}{M_1}, \frac{l}{M_2}) \qquad c_4[k,l] = \frac{1}{M_1 M_2}\frac{\partial^2 \sigma}{\partial u \partial v}(\frac{k}{M_1}, \frac{l}{M_2})$$





Mysterious twist vector





Summary and what next

Take-away

1. How to go from discrete to continuous
2. What are spline curves
3. What is a good basis

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2. What are spline curves
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Ongoing

1. Estimation of not intuitive parameters
2. Second-order Hermite interpolation
3. What basis for what purpose