$$\sum_{k\in\mathbb{Z}} C(k)\Phi(.-k) = 0 \tag{28}$$

We now make use of the Hermite interpolation conditions of $\Phi = \begin{pmatrix}\phi_1 e_1 & \cdots & \phi_1 e_d & \cdots & \phi_r e_1 & \cdots & \phi_r e_d\end{pmatrix}^T$ at integer locations. Let $k_0 \in \mathbb{Z}$. Evaluating (28) and its derivatives up to $(r-1)$ at k_0 leads to $C(k_0) = 0$. k_0 being arbitrary, we have $C \equiv 0$ which in turn leads to $\hat{X} \equiv 0$. This contradicts our initial choice of E and $\hat{X}$. Consequently $\hat{A}$ is positive definite almost everywhere.

4. **Continuity and compacity**

For the needs of what follows let's prove that the map $T : (\mathbb{C}^{rd})^{\mathbb{Z}} \to L_2(\mathbb{R},\mathbb{C})^d$ such that $T(C) = \sum_{k\in\mathbb{Z}} C(k)\Phi(.-k)$ is continuous and that $\{C \in (\mathbb{C}^{rd})^{\mathbb{Z}} | \|C\|^2_{l_2^{rd}} = 1\}$ is compact (**wrong**). For that, notice that Φ has compact support and thus only a finite number of inner products $\langle \Phi, \Phi(.-k)\rangle$ are non zero. Let k_Φ such that $\langle \Phi, \Phi(.-k)\rangle = 0$ for $|k| > k_\Phi$. Consequently

$$\begin{aligned}
\Big\|\sum_{k\in\mathbb{Z}} C(k)\Phi(.-k)\Big\|^2_{L_2(\mathbb{R},\mathbb{C}^d)} &= \sum_{(k,l)\in\mathbb{Z}^2} C(k)C(l)^*\langle \Phi, \Phi(.-(l-k))\rangle \\
&\le \sup_{|k|\le k_\Phi} |\langle \Phi, \Phi(.-k)\rangle| \sum_{|k-l|\le k_\Phi} |C(k)C(l)^*| \\
&\le \sup_{|k|\le k_\Phi} |\langle \Phi, \Phi(.-k)\rangle| \sum_{l=-k_\Phi}^{k_\Phi} \sum_{k\in\mathbb{Z}} |C(k)||C(l+k)|^* \\
&\le (2k_\Phi + 1) \sup_{|k|\le k_\Phi} |\langle \Phi, \Phi(.-k)\rangle| \|C\|^2_{l_2^{rd}}
\end{aligned}$$

using Cauchy-Schwarz inequality for inner product on $(\mathbb{C}^{rd})^{\mathbb{Z}}$ given by $\langle C_1, C_2\rangle = \sum_{k\in\mathbb{Z}} C_1(k)C_2(k)^*$.

Consider now the map $\psi : (\mathbb{C}^{rd})^{\mathbb{Z}} \to \bar{\mathbb{R}}$ such that $\psi(C) = \|C\|^2_{l_2^{rd}}$. Given that $|\psi(C)| \to \infty$ as $\|C\| \to \infty$, ψ is a proper function. Therefore the preimage of every compact set of $\bar{\mathbb{R}}$ is a compact set hence $\{C \in (\mathbb{C}^{rd})^{\mathbb{Z}} | \|C\|^2_{l_2^{rd}} = 1\} = \psi^{-1}(\{1\})$ is compact. (**wrong**)

5. **Eigenvalues of $\hat{A}$ are essentially bounded**

Suppose that the essential supremum of $\lambda_{max} : w \to \max \mathrm{sp}(\hat{A}(w))$ is infinite. Using the 2π-periodicity of $\hat{A}$, it means that

$$\forall p \in \mathbb{N}^*, \exists E_p \subseteq [0, 2\pi], \forall w \in E_p, \quad \lambda_{max}(w) > p.$$

Now for each $w \in E_p$ let $\hat{V}_{max}(w)$ be a normalized eigenvector, i.e, $\hat{A}(w)\hat{V}_{max}(w) = \lambda_{max}(w)\hat{V}_{max}(w)$ and $\|\hat{V}_{max}(w)\|^2_{\mathbb{C}^{rd}} = 2\pi$. Let $\hat{C}_p$ be the 2π-periodic function such that $\hat{C}_{p|[0,2\pi]}(w) = \lambda(E_p)^{-0.5}\mathbb{1}_{E_p}(w)\hat{V}_{max}(w)$, and let C_p be the discrete function that maps $k \in \mathbb{Z}$ to $C_p(k) = \frac{1}{2\pi}\int_0^{2\pi} \hat{C}_p(w)e^{-jwk}dw$. From Parseval's theorem, $\|C_p\|^2_{l_2^{rd}} = 1$. ~~and~~ With the same calculations as in ~~the previous point~~ 4.) we have

$$\frac{1}{2\pi}\int_0^{2\pi} \hat{C}_p(w)\hat{A}(w)\hat{C}_p(w)^* > p,$$

while

$$\frac{1}{2\pi}\int_0^{2\pi}\hat{C_p}(w)\hat{A}(w)\hat{C_p}^*(w)dw = \sum_{(k,l)\in\mathbb{Z}^2} C_p(k)A(l-k)C_p(l)^*$$
$$= \langle\sum_{k\in\mathbb{Z}} C_p(k)\Phi(.-k), \sum_{l\in\mathbb{Z}} C_p(l)\Phi(.-l)\rangle$$
$$= \|\sum_{k\in\mathbb{Z}} C_p(k)\Phi(.-k)\|^2_{L_2(\mathbb{R},\mathbb{C}^d)} .$$

Therefore,

$$\forall p\in\mathbb{N}^*, \exists C_p\in(\mathbb{C}^{rd})^{\mathbb{Z}}, \quad \|C_p\|^2_{l_2^{rd}} = 1, \|\sum_{k\in\mathbb{Z}} C_p(k)\Phi(.-k)\|^2_{L_2(\mathbb{R},\mathbb{C}^d)} > p, \tag{29}$$

~~This~~ which is absurd. ? on est d'accord qu'il manque un argument ici ?!

Similarly, suppose that the essential infinimum of $\lambda_{min} : w \to \min \mathrm{sp}(\hat{A}(w))$ is 0. Using the 2π-periodicity of $\hat{A}$, it means that

$$\forall p\in\mathbb{N}^*, \exists E_p\subseteq[0,2\pi], \forall w\in E_p, \quad \lambda_{min}(w) < \frac{1}{p} .$$

~~Same as~~ As previously, we then prove that

$$\forall p\in\mathbb{N}^*, \exists C_p\in(\mathbb{C}^{rd})^{\mathbb{Z}}, \quad \|C_p\|^2_{l_2^{rd}} = 1, \|\sum_{k\in\mathbb{Z}} C_p(k)\Phi(.-k)\|^2_{L_2(\mathbb{R},\mathbb{C}^d)} < \frac{1}{p}, \tag{30}$$

~~This~~ which is also absurd.

$\square$

III.3 Application to the case $r = 3$

The scheme described for $r = 3$ and $d = 2$, i.e, 2D vector coefficients, can be rewritten as scalar combinations of integer shifts of functions in $L_2(\mathbb{R},\mathbb{R}^2)$ as ~~follows~~

$$r(t) = \sum_{k=-\infty}^{\infty}\sum_{j=1}^{3}\sum_{i=1}^{2} c_{j,i}(k)\phi_j(t-k)e_i, \tag{31}$$

Je serais en faveur de mettre tous ces gars en \mathbf pour signifier qu'il s'agit de vecteurs (pareil avec les $c_{i,j}$)

with $c_{j,i}(k) = r^{(j)}[k]_i$ and $e_1 = (1,0)$ and $e_2 = (0,1)$ the canonical ~~base~~ basis of $\mathbb{R}^2$. It is thus an element of

$$V = \{\sum_{k=-\infty}^{\infty} c_1(k)\phi_1(t-k) + c_2(j)\phi_2(t-k) + c_3(k)\phi_3(t-k) | c_1, c_2, c_3 \in l_2(\mathbb{Z})^2\}$$
$$= \{\sum_{k=-\infty}^{\infty}\sum_{j=1}^{3}\sum_{i=1}^{2} c_{j,i}(k)\phi_j(t-k)e_i | c\in l_2(\mathbb{Z})\},$$

which is a subspace of $L_2(\mathbb{R},\mathbb{R}^2)$. A direct application of Theorem 5 proves that $\{\phi_j(t-k)e_i\}_{i=1,2,j=1,2,3,k\in\mathbb{Z}}$ is a Riesz-Schauder basis. Therefore, V is a Hilbert space ~~itself~~.

The Gram matrix of $\{\phi_1e_1(.-k), \phi_1e_2(.-k), \phi_2e_1(.-k), \phi_2e_2(.-k), \phi_3e_1(.-k), \phi_3e_2(.-k)\}_{k\in\mathbb{Z}}$ ~~be~~ is a 6×6 matrix, ~~It is~~ explicitly given by

$$\sum_{k\in\mathbb{Z}} \begin{pmatrix} \langle\phi_1,\phi_1(.-k)\rangle & 0 & \langle\phi_1,\phi_2(.-k)\rangle & 0 & \langle\phi_1,\phi_3(.-k)\rangle & 0 \\ 0 & \langle\phi_1,\phi_1(.-k)\rangle & 0 & \langle\phi_1,\phi_2(.-k)\rangle & 0 & \langle\phi_1,\phi_3(.-k)\rangle \\ \langle\phi_2,\phi_1(.-k)\rangle & 0 & \langle\phi_2,\phi_2(.-k)\rangle & 0 & \langle\phi_2,\phi_3(.-k)\rangle & 0 \\ 0 & \langle\phi_2,\phi_1(.-k)\rangle & 0 & \langle\phi_2,\phi_2(.-k)\rangle & 0 & \langle\phi_2,\phi_3(.-k)\rangle \\ \langle\phi_3,\phi_1(.-k)\rangle & 0 & \langle\phi_3,\phi_2(.-k)\rangle & 0 & \langle\phi_3,\phi_3(.-k)\rangle & 0 \\ 0 & \langle\phi_3,\phi_1(.-k)\rangle & 0 & \langle\phi_3,\phi_2(.-k)\rangle & 0 & \langle\phi_3,\phi_3(.-k)\rangle \end{pmatrix} e^{-jwk}, \quad (32)$$

where the inner product is that on $L_2(\mathbb{R},\mathbb{C})$ (and actually even $L_2(\mathbb{R},\mathbb{R})$ as ϕ_j are real-valued).

As explained before, looking into the essential infinimum and supremum eigenvalues of (32) yields the best achievable Riesz-Schauder basis bounds m and M. As mentionned in the proof, the matrix above has a characteristic polynomial that is the square of the characteristic polynomial of the submatrix

$$\hat{G}(w) = \sum_{k\in\mathbb{Z}} \begin{pmatrix} \langle\phi_1,\phi_1(.-k)\rangle & \langle\phi_1,\phi_2(.-k)\rangle & \langle\phi_1,\phi_3(.-k)\rangle \\ \langle\phi_2,\phi_1(.-k)\rangle & \langle\phi_2,\phi_2(.-k)\rangle & \langle\phi_2,\phi_3(.-k)\rangle \\ \langle\phi_3,\phi_1(.-k)\rangle & \langle\phi_3,\phi_2(.-k)\rangle & \langle\phi_3,\phi_3(.-k)\rangle \end{pmatrix} e^{-jwk}. \quad (33)$$

Therefore, it is equivalent to look into essential infinimum and supremum of the spectrum of $\hat{G}$ if one is interested in knowing the best achievable parameters m and M.

IV Fourier transforms

IV.1 First fourier transforms

ce n'est pas clair que y est la sequence de coefs et pas la fonction que l'on interpole!

Case $r = 1$

~~In case~~ For $m = r = 1$, Schoenberg's solution to the C.H.I.P $(y,\ S_{2,1} \cap S)$ reduces to piecewise linear interpolation with basis function ϕ_1 given by

$$\phi_1(x) = \begin{cases} 1-x & \text{if } 0 \le x < 1 \\ 1+x & \text{if } -1 < x \le 0 \end{cases}.$$

Its Fourier transform is given by

$$\begin{aligned} \hat{\phi}_1(w) &= \int_0^1 (1-x)e^{-jwx} + \int_{-1}^0 (1+x)e^{-jwx} \\ &= \frac{-2(\cos(w)-1)}{w^2}. \end{aligned}$$

Même remarque qu'au dessus: on interpole y avec y[k] et y'[k], pas clair ici

Case $r = 2$

~~In case~~ For $m = r = 2$, Schoenberg's solution to the C.H.I.P $(y,\ y^{(1)},\ S_{4,2} \cap S)$ are linear combinations of ϕ_1, ϕ_2 given by

$$\phi_1(x) = \begin{cases} 1-3x^2+2x^3 & \text{if } 0 \le x < 1 \\ 1-3x^2-2x^3 & \text{if } -1 < x \le 0 \end{cases}, \qquad \phi_2(x) = \begin{cases} x-2x^2+x^3 & \text{if } 0 \le x < 1 \\ x+2x^2+x^3 & \text{if } -1 < x \le 0 \end{cases}.$$

For computing Fourier transforms ~~let's~~ let us first ~~give~~ consider the following generic formulas:

$$\forall n \ge 2, \quad \int_0^1 x^n \cos(wx)dx = \frac{\sin(w)}{w} + \frac{n\cos(w)}{w^2} - \frac{n(n-1)}{w^2}\int_0^1 x^{n-2}\cos(wx)dx,$$

$$\text{while,} \quad \int_0^1 \cos(wx)dx = \frac{\sin(w)}{w} \quad \text{and} \quad \int_0^1 x\cos(wx)dx = \frac{\sin(w)}{w} + \frac{\cos(w)-1}{w^2};$$

$$\forall n \ge 2, \quad \int_0^1 x^n \sin(wx)dx = \frac{-\cos(w)}{w} + \frac{n\sin(w)}{w^2} - \frac{n(n-1)}{w^2}\int_0^1 x^{n-2}\sin(wx)dx,$$

$$\text{while,} \quad \int_0^1 \sin(wx)dx = \frac{1-\cos(w)}{w} \quad \text{and} \quad \int_0^1 x\sin(wx)dx = \frac{-\cos(w)}{w} + \frac{\sin(w)}{w^2}.$$

This leads us to

$$\hat{\phi}_1(w) = \frac{-12(w\sin(w) + 2\cos(w) - 2)}{w^4}, \quad \hat{\phi}_2(w) = \frac{-4j(w\cos(w) + 2w - 3\sin(w))}{w^4}. \tag{34}$$

Case $r = 3$

~~In case~~ For $m = r = 3$, Schoenberg's solution to the C.H.I.P $(y,\ y^{(1)}, y^{(2)},\ S_{6,3} \cap S)$ are linear combinations of ϕ_1, ϕ_2, ϕ_2 given by

$$\phi_1(x) = \begin{cases} 1 - 10x^3 + 15x^4 - 6x^5 & \text{if } 0 \leq x \leq 1 \\ 1 + 10x^3 + 15x^4 + 6x^5 & \text{if } -1 \leq x < 0 \end{cases},$$

$$\phi_2(x) = \begin{cases} x - 6x^3 + 8x^4 - 3x^5 & \text{if } 0 \leq x \leq 1 \\ x - 6x^3 - 8x^4 - 3x^5 & \text{if } -1 \leq x < 0 \end{cases},$$

$$\phi_3(x) = \begin{cases} 0.5x^2 - 1.5x^3 + 1.5x^4 - 0.5x^5 & \text{if } 0 \leq x \leq 1 \\ 0.5x^2 + 1.5x^3 + 1.5x^4 + 0.5x^5 & \text{if } -1 \leq x < 0 \end{cases}.$$

Their Fourier transforms are given by

$$\hat{\phi}_1(w) = \frac{120(w^2\cos(w) - w^2 - 6w\sin(w) - 12\cos(w) + 12)}{w^6}, \tag{35}$$

$$\hat{\phi}_2(w) = \frac{-48j(w^2\sin(w) + 7w\cos(w) + 8w - 15\sin(w))}{w^6}, \tag{36}$$

$$\hat{\phi}_3(w) = \frac{6(w^2\cos(w) - 3w^2 - 8w\sin(w) - 20\cos(w) + 20)}{w^6}. \tag{37}$$