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A New Computational Approach to the Twists of Bicubic Coons Surfaces

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Abstract In this paper, we present a new optimal approach to the calculation of the twists for constructing bicubic Coons interpolating surfaces from given points. A new objective function called minimal oscillation is proposed to generate the bicubic Coons surface to approximate bilinear interpolation by optimizing twists of Coons surfaces. A linear system with block tridiagonal coefficient matrix is established to solve the optimal problem. The efficiency of the proposed method is illustrated by some numerical examples.

Keywords Coons surface · Twists · Interpolation · Approximation

1 Introduction

Coons surface is the most foundational method of surface interpolation in computer-aided geometric design. A bicubic blended Coons patch is defined by four points, their partial derivatives and mixed partial derivatives, i.e., twists. The twist vectors need to be determined appropriately to produce Coons surfaces that meet the requirements

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in many applications, including the design of cars, airplanes, ship bodies, modeling robots [5,7], and image processing [13,17]. In the past, the twists have been viewed as inconvenient evaluation to control the surface shape. So far, many methods for twist estimators have been given, such as zero twists [12], Adini twists [2], Bessel twists [3], Hagen twists [9], Brunet's method [4], and Selesnick's method [16]. Most of these methods are constructed from the point of view of geometry, while the alternative method is to seek the twists that minimize a user-defined energy integral over the surface. Kallay and Ravani [15] provided an optimal method to determine twists with minimizing the integral in the second partial derivatives over the entire surface. Qu [14] presented a method for finding the optimal twist by optimizing an approximate energy form of the surface.

In recent years, the variational principle has been applied to produce Coons patch with better shape. Farin and Hansford [8] constructed discrete Coons patches, which minimize the twist with the integral in mixed partial derivatives being taken over the unit square. They applied this principle to the discrete Coons patch that satisfies the Euler–Lagrange PDE. Han [10] presented an optimal method for constructing bicubic Coons surfaces based upon twist vectors at rectangular grid points. These twists were determined by using an optimal criterion derived from biharmonic PDE. While the surfaces obtained in [10] only approximated biharmonic PDE better, Han [11] presents a construction method of biquadratic non-uniform B-spline surface which piecewise accurately satisfies biharmonic PDE.

We propose a new optimal principle, namely the minimal oscillation over all the surface, which is applied to determine the twists. This paper is organized as follows: Some preliminaries are shown in Sect. 2. In Sect. 3, we define the optimal problem which is based on approximating bilinear interpolation on all patches. We establish a linear system to solve this optimal problem. The coefficient matrix is block tridiagonal which is diagonally dominant. In Sect. 4, some numeric examples are given, and comparisons among our method, zero twists, and Adini twists are presented.

2 Preliminary

In this section, for given $n \times m$ points $p_{i,j}$ ($1 \le i \le n, 1 \le j \le m$), we describe piecewise Coons surface $Q(x, y)$ defined on a square domain $a = x_1 \le x_2 \le \cdots \le x_n \le b$ and $c \le y_1 \le y_2 \le \cdots \le y_m \le d$. For clarity of exposition, we set $h_i = x_{i+1} - x_i$, $i = 1, 2, \ldots, n-1$ and $l_j = y_{j+1} - y_j$, $j = 1, 2, \ldots, m-1$.

2.1 Bilinear Interpolation Surface

For $(x, y) \in I_{i,j} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$, let $u = (x - x_i)/h_i$, $v = (y - y_j)/l_j$ then a bilinear interpolating patch $L_{i,j}(u, v)$ defined over one rectangle $I_{i,j}$ is given by

$$L_{i,j}(u, v) = \begin{bmatrix} 1-u & u \end{bmatrix} \begin{bmatrix} p_{i,j} & p_{i,j+1} \\ p_{i+1,j} & p_{i+1,j+1} \end{bmatrix} \begin{bmatrix} 1-v \\ v \end{bmatrix}. \tag{1}$$

Then the piecewise bilinear surface can be defined as $L(x, y) = L_{i,j}(u, v)$ for $(x, y) \in I_{i,j}$, $i = 1, 2, \ldots, n-1$, $j = 1, 2, \ldots, m-1$. Whereas linear interpolation fits the "simplest" curve between two points, bilinear interpolation fits the "simplest" surface between four points [6]. Although the bilinear interpolation is easy to obtain, the main drawback of it is that in general two adjacent patches will be C^0 continuous only.

2.2 Bicubic Coons Surface

For $(x, y) \in [x_i, x_{i+1}] \times [y_j, y_{j+1}]$, we set $u = (x - x_i)/h_i$, $v = (y - y_i)/l_j$, and blending functions of the cubic Coons patch are as follows:

$$\begin{aligned} f_0(u) &= 2u^3 - 3u^2 + 1, & f_1(u) &= -2u^3 + 3u^2, \\ f_0(v) &= 2v^3 - 3v^2 + 1, & f_1(v) &= -2v^3 + 3v^2, \\ g_0(u) &= h_i u(u-1)^2, & g_1(u) &= h_i u^2(u-1), \\ g_0(v) &= l_j v(v-1)^2, & g_1(v) &= l_j v^2(v-1). \end{aligned}$$

When the partial derivatives $f_{i,j}$, $g_{i,j}$ and twist $r_{i,j}$ of bicubic Coons patch at each corner are provided, then an interpolation bicubic Coons patch $Q_{i,j}(u, v)$ can be obtained by

$$Q_{i,j}(u,v) = \begin{bmatrix} f_0(u) \\ f_1(u) \\ g_0(u) \\ g_1(u) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} p_{i,j} & p_{i,j+1} & g_{i,j} & g_{i,j+1} \\ p_{i+1,j} & p_{i+1,j+1} & g_{i+1,j} & g_{i+1,j+1} \\ f_{i,j} & f_{i,j+1} & r_{i,j} & r_{i,j+1} \\ f_{i+1,j} & f_{i+1,j+1} & r_{i+1,j} & r_{i+1,j+1} \end{bmatrix} \begin{bmatrix} f_0(v) \\ f_1(v) \\ g_0(v) \\ g_1(v) \end{bmatrix}. \tag{2}$$

Therefore, the piecewise bicubic Coons surface can be represented by $Q(x, y) = Q_{i,j}(u, v)$ for $(x, y) \in [x_i, x_{i+1}] \times [y_j, y_{j+1}]$, $i = 1, 2, \ldots, n-1$, $j = 1, 2, \ldots, m-1$.

There are many methods [1] to obtain the partial derivatives $f_{i,j}$ and $g_{i,j}$, which are estimated by Bessel method [6] in our work. However, twists $r_{i,j}$ are difficult to evaluate. Twist estimations have been discussed in [3,7]. In this paper, our purpose is to determine twist by minimizing the oscillation in average (see Sect. 3) so as to make the shape of surface well.

3 Twist Optimization

3.1 The Minimum Principle

Except for the slackness of continuity, the bilinear interpolant to the data points preserves the shape well. Therefore, we introduce a definition to measure the deviation, namely the quadratic oscillation in average, between bicubic Coons surfaces and bilinear interpolation surfaces. The quadratic oscillation of a bicubic Coons surface $Q(x, y)$ is the value of the functional $E : C^1([a, b] \times [c, d]) \to \mathbb{R}$ defined by

$$E = \int_c^d \int_a^b \|Q(x, y) - L(x, y)\|^2 \mathrm{d}x\mathrm{d}y. \tag{3}$$

We propose a new optimal method to determine twist vectors by minimization of the quadratic oscillation in average of $Q(x, y)$ on the domain $U = [a, b] \times [c, d]$. That is, the twist of a surface $Q(x, y)$ is optimized by

$$\min_{r_{i,j}} E = \min_{r_{i,j}} \int_U \|Q(x, y) - L(x, y)\|^2 \mathrm{d}S. \tag{4}$$

The main idea of our work is that we prove the existence and uniqueness of the bicubic Coons surfaces with minimal quadratic oscillation over the entire patches. Let

$$E_{i,j} = h_i l_j \int_0^1 \int_0^1 \left\| Q_{i,j}(u, v) - L_{i,j}(u, v) \right\|^2 \mathrm{d}u\mathrm{d}v, \tag{5}$$

then the minimization problem can be represented by

$$\min_{r_{i,j}} \sum_{i=1}^{n} \sum_{j=1}^{m} E_{i,j}. \tag{6}$$

3.2 Twists Computation

For the purposes of calculating the difference between (1) and (2), we elevate the bilinear patch (1) into cubic form as (2). Since

$$[1-u \quad u] = [f_0(u) \quad f_1(u) \quad g_0(u) \quad g_1(u)] \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -\frac{1}{h_i} & \frac{1}{h_i} \\ -\frac{1}{h_i} & \frac{1}{h_i} \end{bmatrix},$$

and

$$\begin{bmatrix} 1-v \\ v \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\frac{1}{l_j} & -\frac{1}{l_j} \\ 0 & 1 & \frac{1}{l_j} & \frac{1}{l_j} \end{bmatrix} \begin{bmatrix} f_0(v) \\ f_1(v) \\ g_0(v) \\ g_1(v) \end{bmatrix},$$

then the bilinear interpolation patch (1) can be reformulated as

$$L_{i,j}(u, v) = \begin{bmatrix} f_0(u) \\ f_1(u) \\ g_0(u) \\ g_1(u) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} p_{i,j} & p_{i,j+1} & \bar{g}_{i,j} & \bar{g}_{i,j+1} \\ p_{i+1,j} & p_{i+1,j+1} & \bar{g}_{i+1,j} & \bar{g}_{i+1,j+1} \\ \bar{f}_{i,j} & \bar{f}_{i,j+1} & \bar{r}_{i,j} & \bar{r}_{i,j+1} \\ \bar{f}_{i+1,j} & \bar{f}_{i+1,j+1} & \bar{r}_{i+1,j} & \bar{r}_{i+1,j+1} \end{bmatrix} \begin{bmatrix} f_0(v) \\ f_1(v) \\ g_0(v) \\ g_1(v) \end{bmatrix}, \tag{7}$$

where

$$\bar{f}_{i,k} = \bar{f}_{i+1,k} = \frac{p_{i+1,k} - p_{i,k}}{h_i}, k = j, j+1,$$

$$\bar{g}_{l,j} = \bar{g}_{l,j+1} = \frac{p_{l,j+1} - p_{l,j}}{l_j}, \ l = i, i+1,$$

$$\bar{r}_{i,j} = \bar{r}_{i+1,j} = \bar{r}_{i,j+1} = \bar{r}_{i+1,j+1} = \frac{p_{i,j} - p_{i,j+1} - p_{i+1,j} + p_{i+1,j+1}}{h_i l_j}.$$

Now, let $a_{i,j}(u, v)$ denote the twist-independent part in Eq. (2)

$$a_{i,j}(u,v) = \begin{bmatrix} f_0(u) \\ f_1(u) \\ g_0(u) \\ g_1(u) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} p_{i,j} & p_{i,j+1} & g_{i,j} & g_{i,j+1} \\ p_{i+1,j} & p_{i+1,j+1} & g_{i+1,j} & g_{i+1,j+1} \\ f_{i,j} & f_{i,j+1} & 0 & 0 \\ f_{i+1,j} & f_{i+1,j+1} & 0 & 0 \end{bmatrix} \begin{bmatrix} f_0(v) \\ f_1(v) \\ g_0(v) \\ g_1(v) \end{bmatrix},$$

then $Q_{i,j}(u, v)$ can be expressed as

$$Q_{i,j}(u,v) = a_{i,j}(u,v) + \sum_{s,t\in\{0,1\}} g_s(u) g_t(v) r_{i+s,j+t}. \tag{8}$$

Stepping further, we introduce Frobenius inner product between two matrices defined as

$$A : B = \sum_i \sum_j a_{i,j} b_{i,j},$$

for $A = (a_{i,j})$, $B = (b_{i,j})$.

We define

$$H(u,v) = \begin{bmatrix} f_0(u) \\ f_1(u) \\ g_0(u) \\ g_1(u) \end{bmatrix} [f_0(v) \quad f_1(v) \quad g_0(v) \quad g_1(v)],$$

and

$$B_{i,j} = \begin{bmatrix} 0 & 0 & \tilde{g}_{i,j} & \tilde{g}_{i,j+1} \\ 0 & 0 & \tilde{g}_{i+1,j} & \tilde{g}_{i+1,j+1} \\ \tilde{f}_{i,j} & \tilde{f}_{i,j+1} & \tilde{r}_{i,j} & \tilde{r}_{i,j+1} \\ \tilde{f}_{i+1,j} & \tilde{f}_{i+1,j+1} & \tilde{r}_{i+1,j} & \tilde{r}_{i+1,j+1} \end{bmatrix},$$

where $\tilde{g}_{i+k,j+l} = g_{i+k,j+l} - \bar{g}_{i+k,j+l}$, $\tilde{f}_{i+k,j+l} = f_{i+k,j+l} - \bar{f}_{i+k,j+l}$, $\tilde{r}_{i+k,j+l} = r_{i+k,j+l} - \bar{r}_{i+k,j+l}$, for $k, l \in \{0, 1\}$. By subtracting (7) from (8), we have the following equation

$$Q_{i,j}(u,v) - L_{i,j}(u,v) = H(u,v) : B_{i,j} + \sum_{s,t\in\{0,1\}} g_s(u) g_t(v) r_{i+s,j+t}. \tag{9}$$

The quadratic oscillation of each Coons patch $E_{i,j}$ then can be expressed as

$$E_{i,j} = h_i l_j \int_0^1 \int_0^1 \|H(u,v) : B_{i,j} + \sum_{s,t\in\{0,1\}} g_s(u) g_t(v) r_{i+s,j+t}\|^2 \mathrm{d}u\mathrm{d}v. \tag{10}$$

For all of the patches, we seek the minimum of $E = \sum_i \sum_j E_{i,j}$. The necessary conditions are

$$\frac{\partial E}{\partial r_{i,j}} = 0, \text{ for } 1 \le i \le n, 1 \le j \le m. \tag{11}$$

Stepping into each patch, we have

$$\begin{aligned}
\frac{\partial E_{i,j}}{\partial r_{i,j}} &= 2h_i l_j \int_0^1 \int_0^1 \Bigg[(H(u,v) : B_{i,j}) \\
&\quad + \sum_{s,t\in\{0,1\}} g_s(u) g_t(v) r_{i+s,j+t}\Bigg] g_0(u) g_0(v) \mathrm{d}u\mathrm{d}v, \\
\frac{\partial E_{i,j}}{\partial r_{i,j+1}} &= 2h_i l_{j+1} \int_0^1 \int_0^1 \Bigg[(H(u,v) : B_{i,j}) \\
&\quad + \sum_{s,t\in\{0,1\}} g_s(u) g_t(v) r_{i+s,j+t}\Bigg] g_0(u) g_1(v) \mathrm{d}u\mathrm{d}v, \\
\frac{\partial E_{i,j}}{\partial r_{i+1,j}} &= 2h_{i+1} l_j \int_0^1 \int_0^1 \Bigg[(H(u,v) : B_{i,j}) \\
&\quad + \sum_{s,t\in\{0,1\}} g_s(u) g_t(v) r_{i+s,j+t}\Bigg] g_1(u) g_0(v) \mathrm{d}u\mathrm{d}v, \\
\frac{\partial E_{i,j}}{\partial r_{i+1,j+1}} &= 2h_{i+1} l_{j+1} \int_0^1 \int_0^1 \Bigg[(H(u,v) : B_{i,j}) \\
&\quad + \sum_{s,t\in\{0,1\}} g_s(u) g_t(v) r_{i+s,j+t}\Bigg] g_1(u) g_1(v) \mathrm{d}u\mathrm{d}v.
\end{aligned}$$

This indicates that the twist $r_{i,j}$ of surface $P(x, y)$ at the knots (x_i, y_j) involved in the patches which intersect at this corner. This implies that $r_{i,j}$ appears on four Coons patches for $2 \le i \le n-1, 2 \le j \le m-1$, one Coons patch for four boundary corners and two Coons patches for the else points. Substituting Eq. (10) into Eq.(11), the optimality conditions yield a set of linear equations as

$$\begin{cases} \dfrac{\partial E_{1,d}}{\partial r_{1,d}} = 0, \quad d = 1, \\ \dfrac{\partial E_{1,d-1}}{\partial r_{1,d}} + \dfrac{\partial E_{1,d}}{\partial r_{1,d}} = 0, \quad d = 2, \dots, m-1, \\ \dfrac{\partial E_{1,d-1}}{\partial r_{1,d}} = 0, \quad d = m, \end{cases} \tag{12}$$

$$\begin{cases} \dfrac{\partial E_{q,d}}{\partial r_{q,d}} + \dfrac{\partial E_{q+1,d}}{\partial r_{q,d}} = 0, \quad d = 1, \\ \dfrac{\partial E_{q-1,d-1}}{\partial r_{q,d}} + \dfrac{\partial E_{q-1,d}}{\partial r_{q,d}} + \dfrac{\partial E_{q,d-1}}{\partial r_{q,d}} + \dfrac{\partial E_{q,d}}{\partial r_{q,d}} = 0, \quad d = 2, \dots, m-1, \\ \dfrac{\partial E_{q-1,d-1}}{\partial r_{q,d}} + \dfrac{\partial E_{q,d-1}}{\partial r_{q,d}} = 0, \quad d = m, \end{cases} \tag{13}$$

for $q = 2, \dots, n-1$, and

$$\begin{cases} \dfrac{\partial E_{n-1,d}}{\partial r_{n,d}} = 0, \quad d = 1, \\ \dfrac{\partial E_{n-1,d-1}}{\partial r_{n,d}} + \dfrac{\partial E_{n-1,d}}{\partial r_{n,d}} = 0, \quad d = 2, \dots, m-1, \\ \dfrac{\partial E_{n-1,d-1}}{\partial r_{n,d}} = 0, \quad d = m. \end{cases} \tag{14}$$

There are $n \times m$ equations and $n \times m$ variables in linear system (12–14). Let $R = (r_{1,1}, r_{2,1}, \dots, r_{n,1}, r_{1,2}, \dots, r_{n,2}, \dots, r_{1,m}, \dots, r_{n-1,m}, r_{n,m})^{\mathrm{T}}$,

$$W = \begin{bmatrix} 4 & -3 & & & \\ -3 & 4 & -3 & & \\ & \ddots & \ddots & \ddots & \\ & & -3 & 4 & -3 \\ & & & -3 & 4 \end{bmatrix}, D = \begin{bmatrix} h_1^3 & h_1^3 & & & \\ h_1^3 & h_1^3 + h_2^3 & h_2^3 & & \\ & \ddots & \ddots & \ddots & \\ & & h_{n-2}^3 & h_{n-1}^3 + h_{n-2}^3 & h_{n-1}^3 \\ & & & h_{n-1}^3 & h_{n-1}^3 \end{bmatrix},$$

$$K = \begin{bmatrix} 4l_1^3 & -3l_1^3 & & & \\ -3l_1^3 & 4(l_1^3 + l_2^3) & -3l_2^3 & & \\ & \ddots & \ddots & \ddots & \\ & & -3l_{m-2}^3 & 4(l_{m-2}^3 + l_{m-1}^3) & -3l_{m-1}^3 \\ & & & -3l_{m-1}^3 & 4l_{m-1}^3 \end{bmatrix},$$

then system (12–14) becomes

$$\frac{1}{176400} M \cdot R = C, \tag{15}$$

where $C = (c_{11}, c_{21}, \ldots, c_{n1}, c_{12}, c_{22}, \ldots, c_{n2}, \ldots, c_{1m}, \ldots, c_{nm})^{\mathrm{T}}$ is defined as

$$\begin{cases} c_{11} = h_1 l_d G_1 : B^{1,1}, \quad c_{n1} = h_{n-1} l_d G_2 : B^{n-1,1}, \\ c_{q1} = h_{q-1} l_d G_2 : B^{q-1,1} + h_q l_d G_1 : B^{q,1}, \quad q = 2, \ldots, n-1, \\ c_{1d} = h_1 l_{d-1} G_3 : B^{1,d-1} + h_1 l_d G_1 : B^{1,d}, \quad d = 2, \ldots, m-1, \\ c_{qd} = h_{q-1} l_{d-1} G_4 : B^{q-1,d-1} + h_q l_{d-1} G_3 : B^{q,d-1} + \\ \qquad h_{q-1} l_d G_2 : B^{q-1,d} + h_q l_d G_1 : B^{q,d}, \quad d=2, \ldots, m-1, \; q=2, \ldots, n-1 \\ c_{nd} = h_{n-1} l_{d-1} G_4 : B^{n-1,d-1} + h_{n-1} l_d G_3 : B^{n-1,d}, \quad d = 2, \ldots, m-1, \\ c_{1m} = h_1 l_{d-1} G_3 : B^{1,m-1}, \quad c_{nm} = h_{n-1} l_{d-1} G_4 : B^{n-1,m-1}, \\ c_{qm} = h_{q-1} l_{d-1} G_4 : B^{q-1,m-1} + h_q l_{d-1} G_3 : B^{q,m-1}, \quad q = 2, \ldots, n-1, \end{cases}$$

and $M = K \bigotimes (W * D)$. Here $\bigotimes$ is Kronecker product, denoted by

$$A \bigotimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B \\ a_{21}B & a_{22}B & \cdots & a_{2m}B \\ \vdots & \vdots & & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nm}B \end{bmatrix}$$

for any $A = (a_{i,j})_{n \times m}$, $B = (b_{i,j})_{r \times s}$. And notation $*$ is Hadamard product, denoted by

$$A * B = \begin{bmatrix} a_{11}b_{11} & a_{12}b_{11} & \cdots & a_{1m}b_{1m} \\ a_{21}b_{11} & a_{22}b_{11} & \cdots & a_{2m}b_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1}b_{n1} & a_{n2}b_{n2} & \cdots & a_{nm}b_{nm} \end{bmatrix}$$

for any $A = (a_{i,j})_{n \times m}$, $B = (b_{i,j})_{n \times m}$.

The $W * D$ and K are diagonally dominant, according to the property of Kronecker product, then the coefficient matrix M is non-singular, and therefore, the linear system (15) has a unique solution $R = M^{-1}C$ and each Coons patch of (2) can be determined. Finally, the resulting piecewise bicubic Coons surface with C^1 continuity has minimal oscillation to the bilinear interpolation.

4 Numerical Examples

In the following examples we present a comparative study of the twists computed by the quadratic oscillating minimization and other classical twists, including zero twist and Adini twist. The differences between the surfaces in some pictures are not quite pronounced. Since the color of patch is proportional to surface height, noting color change on each patch helps compare the surfaces.

Example 1 Figure 1 illustrates the comparison between different interpolants on mesh points (see Fig. 1a): (0,1,7.64), (1,1,1.64), (0,2,1.64), (1,2.5,1.64), (0,3,5.64),

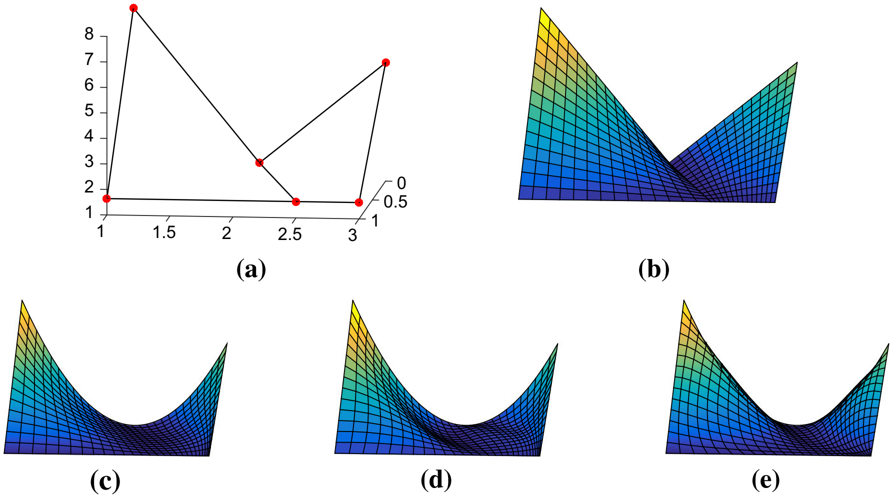

Fig. 1 Surfaces in Example 1. **a** Mesh grid. **b** Bilinear interpolation. **c** Coons surface with zero twist. **d** Coons surface with Adini twist. **e** Coons surface with minimal oscillation

(1,3,1.64). Our method estimates the twist of the bicubic Coons surface by solving the following linear equations

$$\begin{aligned}
&2r^s_{1,1} - 1.5r^s_{1,2} - 1.5r^s_{2,1} + 1.125r^s_{2,2} = C^s_1,\\
&-1.5r^s_{1,1} + 4r^s_{1,2} - 1.5r^s_{1,3} + 1.125r^s_{2,1} - 3r^s_{2,2} + 1.125r^s_{2,3} = C^s_2,\\
&-1.5r^s_{1,2} + 2r^s_{1,3} + 1.125r^s_{2,2} - 1.5r^s_{2,3} = C^s_3,\\
&-1.5r^s_{1,1} + 1.125r^s_{1,2} + 2r^s_{2,1} - 1.5r^s_{2,2} = C^s_4,\\
&1.125r^s_{1,1} - 3r^s_{1,2} + 1.125r^s_{1,3} - 1.5r^s_{2,1} + 4r^s_{2,2} - 1.5r^s_{2,3} = C^s_5,\\
&1.125r^s_{1,2} - 1.5r^s_{1,3} - 1.5r^s_{2,2} + 2r^s_{2,3} = C^s_6,
\end{aligned} \tag{16}$$

where $s \in \{x, y, z\}$, $C^x = [0, 0, 0, 0, 0, 0]$, $C^y = [-11.25, 0, 11.25, 19.375, 0, -19.375]$, $C^z = [194, 0.5, -193.5, -112.25, 0.5, 112.75]$.

The oscillation of bicubic Coons surface with twist computed by Eq. (16) is 0.1683, while the oscillation corresponding to zero twist and Adini twist is 0.6264 and 0.7009. Although the visual differences are not obvious, we still find that the color variation in Fig. 1e is more like Fig. 1b . That means that the twist estimated by our method keeps fluctuation features of control mesh more effectively than both zero twist and Adini twist.

Example 2 In this example the boundary control points mesh with noticeable shift is shown in Fig. 2a. The remarkable improvement of surface at four corner patches is achieved by solving minimization problem (6). The feature of tensile and bend at corner patches in Fig. 2e is similar to Fig. 2b. Furthermore, the oscillation of cubic Coons surface we constructed is 10.836, whereas the oscillation corresponding to zero

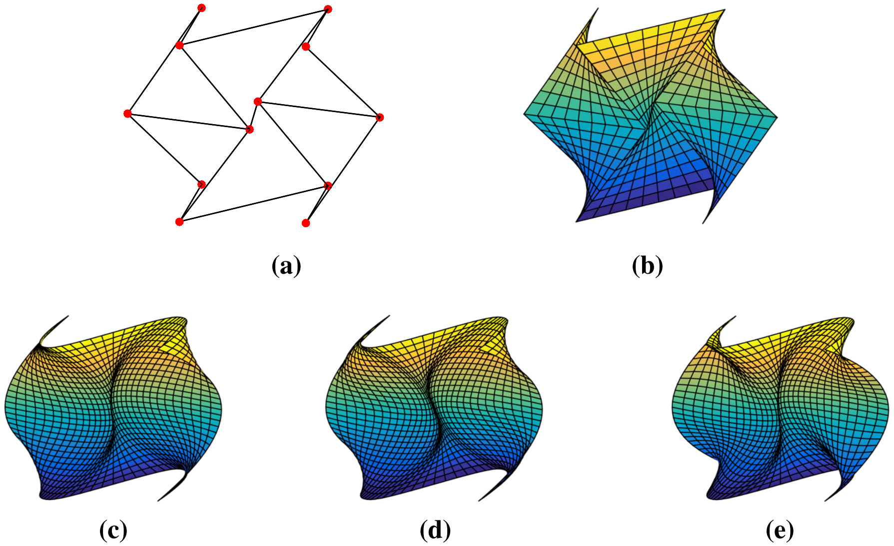

Fig. 2 Surfaces in Example 2. **a** Mesh grid. **b** Bilinear interpolation. **c** Coons surface with zero twist. **d** Coons surface with Adini twist. **e** Coons surface with minimal oscillation

twist and Adini twist is 14.394 and 16.88, respectively. It is clear that bicubic Coons surface generated by minimizing quadric oscillation is actually closer to the bilinear interpolation.

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