Hermite Polynomials snakes order 2

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I Translation of Schoenberg's 1973 paper for the case $r = 3, m = 3$

The following is simply a reminder of some of the results found by I.J Schoenberg in his paper *Cardinal Interpolation and Spline Functions. III Cardinal Hermite interpolation.* Let's reintroduce notations of the article and make somehow more explicits what the objects they encode are.

Let r and m be positive integers that satisfy $r \leq m$. The set of cardinal splines of order $2m$ with knot multiplicity r is denoted by $S_{2m,r}$. Note that using De Boor's notations for splines set we have the following

$$S_{2m,r} = \$_{2m,\mathbb{Z}_3} = \Pi_{<2m,\mathbb{Z},2m-r} \tag{1}$$

where $\mathbb{Z}_3$ denotes the sequence of knots $(\dots, -1, -1, -1, 0, 0, 0, 1, 1, 1, \dots)$. It is clear from these notations that $S_{2m,r} \subset \mathcal{C}^{2m-r-1}$.

Theorem 1. *Let S be either of the vector spaces $\mathcal{L}_{p,r}, F_{\gamma,r}$ with $\gamma \geq 0$, $p \in \mathbb{N}^*$. Provided a solution to C.H.I.P $\left(y_\nu, \dots, y_\nu^{(r-1)}, S_{2m,r} \cap S\right)$ exists, it is uniquely given by*

$$\forall x \in \mathbb{R} \qquad S(x) = \sum_{\nu=-\infty}^{\infty} y_\nu L_0(x-\nu) + \dots + y_\nu^{(r-1)} L_{r-1}(x-\nu) \tag{2}$$

In order to specify a usable model for active contours it remains to determine explicit expressions for the basis functions $L_0, \dots, L_{r-1}$. In the article they are determined by solving a set of $2m - r$ linear equations. This system is obtained by considering separately the function L_s on $[1, \infty)$ and $[0, 1]$. Note that specifying the function on both these intervals completely determine L_s as the latter is even (if s is even) or odd (if s is odd).

On $[1, \infty)$, L_s can be decomposed into

$$L_s = \sum_{j=1}^{m-r} c_j S_j$$

where $(c_j)_{j=1}^{m-r}$ are $(m - r)$ unknown coefficients to be determined and S_j are the eigensplines for the first $m - r$ "eigenvalues" λ_j, solutions to $|\Delta_{r,d}(\lambda)| = 0$.

On $[0, 1]$, L_s is given by a polynomial P of order $2m$ that takes a specific form according to the parities of s and r (we refer to equations (7.13) and (7.14)) in the article. This polynomial introduces m unknown coefficients $(a_j)_{j=1}^m$. To determine a total of $m + m - r = 2m - r$ unknown coefficients we make use of the $2m - r$ equality conditions at 1 $P^{(\rho)}(1) = L_s^{(\rho)}(1)$. We end up of a system of $2m - r$ equations for $2m - r$ unknowns that can be solved exactly provided the matrix of the system is non singular. Schoenberg proves with a very nice argument that the matrix of the system is always non singular.

In the case $m = r = 3$, L_0, L_1, L_2 are 0 on $[1, \infty)$ and on $[0, 1]$ are given by

$$L_0(x) = 1 + a_1 x^3 + a_2 x^4 + a_3 x^5 \tag{3}$$
$$L_1(x) = x + a_1 x^3 + a_2 x^4 + a_3 x^5 \tag{4}$$
$$L_2(x) = \frac{1}{2}x^2 + a_1 x^3 + a_2 x^4 + a_3 x^5 \tag{5}$$
$$\tag{6}$$

where the coefficients for each generator are unrelated. Note that L_s have finite support *because* $m = r$. If that was not the case the term $\sum_{j=1}^{m-r} c_j S_j$ may not be 0 and therefore L_s would be non zero on $[1, \infty)$! Can it happen though that $m > r$ and $(c_j)_{j=1}^{m-r}$ are 0? To determine the coefficients above we need to solve independently for each generator the 3 equations $P^{(\rho)}(1) = 0$. This leads to the following systems

$$\begin{cases} a_1 + a_2 + a_3 = -1 \\ 3a_1 + 4a_2 + 5a_3 = 0 \\ 3a_1 + 6a_2 + 10a_3 = 0 \end{cases} \begin{cases} a_1 + a_2 + a_3 = -1 \\ 3a_1 + 4a_2 + 5a_3 = -1 \\ 3a_1 + 6a_2 + 10a_3 = 0 \end{cases} \begin{cases} a_1 + a_2 + a_3 = -\frac{1}{2} \\ 3a_1 + 4a_2 + 5a_3 = -1 \\ 3a_1 + 6a_2 + 10a_3 = -\frac{1}{2} \end{cases}$$

II Properties of the resulting snake scheme

Solving the linear systems written in the previous section yields explicit formulas for the Schoenberg basis generators L_0, L_1, L_2, that we rename ϕ_1, ϕ_2, ϕ_3 in accordance with modern notations (see V. Uhlmann *Hermite Snakes with Controls of Tangents*) as follows

$$\phi_1(x) = \begin{cases} 1 - 10x^3 + 15x^4 - 6x^5 & \text{if } 0 \le x \le 1 \\ 1 + 10x^3 + 15x^4 + 6x^5 & \text{if } -1 \le x < 0 \end{cases} \tag{7}$$
$$\phi_2(x) = \begin{cases} x - 6x^3 + 8x^4 - 3x^5 & \text{if } 0 \le x \le 1 \\ x - 6x^3 - 15x^4 - 3x^5 & \text{if } -1 \le x < 0 \end{cases} \tag{8}$$
$$\phi_3(x) = \begin{cases} 0.5x^2 - 1.5x^3 + 1.5x^4 - 0.5x^5 & \text{if } 0 \le x \le 1 \\ 0.5x^2 + 1.5x^3 + 1.5x^4 + 0.5x^5 & \text{if } -1 \le x < 0 \end{cases} \tag{9}$$

In figure 1 are display the values of these functions as well as their two first derivatives.

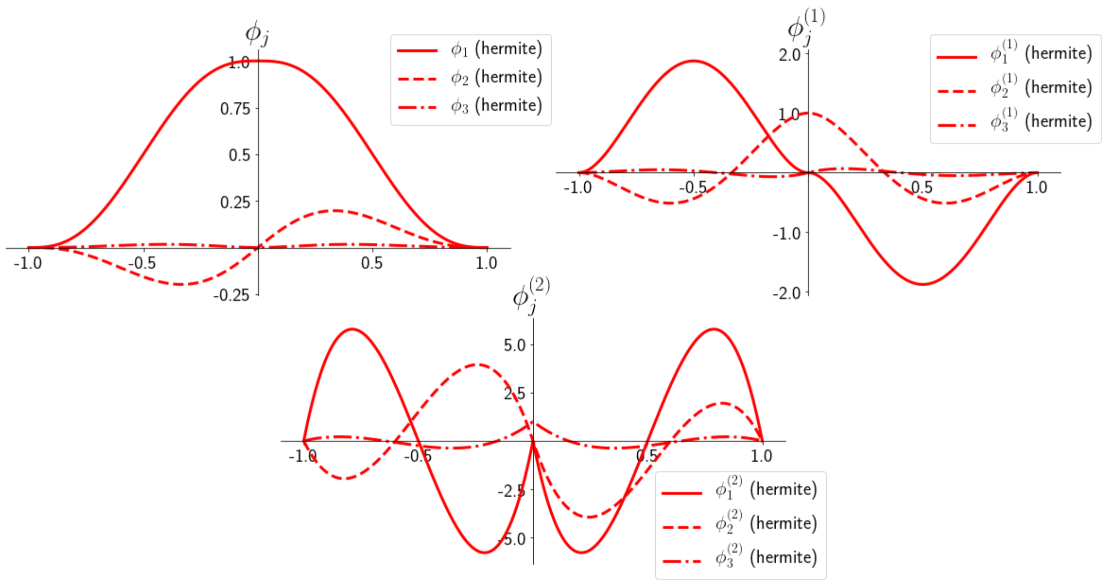

Figure 1: Generators for C.H.I.P with $m = r = 3$