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Cardinal Spline Interpolation

I. J. SCHOENBERG
The Mathematics Research Center
The University of Wisconsin-Madison

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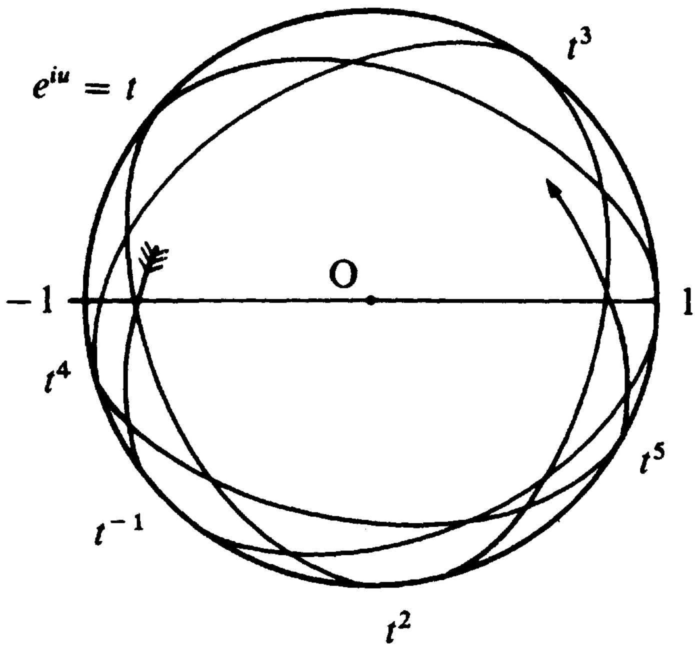
An Homage to Leonhard Euler



A bounded exponential Euler spline
(*page* 29).

"Spline approximation contains the delicious paradox of Prokofieff's Classical Symphony: it seems as though it might have been written several centuries ago, but of course it could not have been."

Philip J. Davis
Symposium on Approximation of Functions
at General Motors, 1964

Preface

In the paper [63], written during 1944–45 at the Ballistics Research Laboratories, Aberdeen, Maryland, the author developed certain methods for the smooth approximation of empirical tables. He would have profited greatly from the results of the paper [57] by Quade and Collatz, but heard of its existence only as late as the Fall of 1964. Rather, the paper [63] was strongly suggested by the actuarial work on *Osculatory Interpolation* so aptly summarized and extended in Greville's paper [33]. In 1944–45 applicable methods were needed, and there was no time to develop their mathematical properties. The present monograph aims at doing that. It is based on a number of papers in our list of references, but mainly on the eight papers [1]–[8]. The opportunity of describing the whole animal in one place, rather than limb by limb, leads to economy; moreover, the assembling of the limbs becomes more functional and biologically viable.

This monograph is an homage to Leonhard Euler. I quote from Stečkin's Preface to Subbotin's Russian translation [11] of the Ahlberg–Nilson–Walsh book [10]: " ... Splines entered numerical mathematics and approximation theory gradually, over an extended period of time. One will have to mention here Euler's method for the numerical solution of ordinary differential equations as the "early bird" ... ". The present monograph will show that Euler contributed to the central parts of spline theory. The role played by the exponential function in calculus is taken over in equidistant spline theory by the *exponential spline*

$$\Phi_n(x;t) = \sum_{-\infty}^{\infty} t^j Q_{n+1}(x-j) \qquad (t \text{ is a parameter } \neq 0)$$

of the last sections of Lecture 2. This is a piecewise polynomial function of degree n in each interval between consecutive integers. The polynomial representing $\Phi_n(x;t)$ in the interval [0, 1] is, up to a constant factor, identical with the exponential Euler polynomial $A_n(x;t)$ of Lecture 3. This is the seed provided by Euler, and the entire exponential spline grows out of it by means of the functional equation $f(x+1) = tf(x)$. See also [15], where Birkhoff and de Boor state that Euler was probably aware of the optimal property of the mechanical spline for small deflections. Last, but not least, the mathematics that we use is Eulerian in character.

Related in subject matter to the present work are the Regional Conference monographs by Birkhoff [16] and Varga [88]. While these authors cover wide fields of theory and applications, we restrict ourselves to the special problem of interpolation at all integers. Let (y_ν), $\nu = 0, \pm 1, \pm 2, \cdots$, be a doubly-infinite sequence of data. The analogue of the Lagrange interpolation formula for the problem of finding $f(x)$ such that $f(\nu) = y_\nu$, for all integers ν, is the so-called *cardinal series*

$$f(x) = \sum_{-\infty}^{\infty} y_\nu \frac{\sin \pi(x - \nu)}{\pi(x - \nu)} \tag{1}$$

(see [90, § 11]). The piecewise linear analogue of (1) is well known: If $M_2(x)$ is the so-called "roof-function" such that $M_2(x) = x + 1$ in $[-1, 0]$, $M_2(x) = 1 - x$ in $[0, 1]$, and $M_2(x) = 0$ elsewhere, then

$$S_1(x) = \sum_{-\infty}^{\infty} y_\nu M_2(x - \nu) \tag{2}$$

is clearly the piecewise linear interpolant. The purpose of *cardinal spline interpolation* is to bridge the gap between the linear spline (2) and the cardinal series (1). It aims at retaining some of the sturdiness and simplicity of (2), at the same time capturing some of the smoothness and sophistication of (1).

The present work is based on the ten lectures delivered at Northwestern University during the week of June 14 to 18, 1971, at the Regional Conference on "Analysis with special emphasis on approximation theory". The author thanks the organizer and director of the conference, Joseph W. Jerome, for asking him to serve as principal lecturer. The support of this conference by the National Science Foundation through a grant to the Conference Board of the Mathematical Sciences is gratefully acknowledged. Much of the work was done during the years since 1964 which the author spent at the Mathematics Research Center of the University of Wisconsin—Madison; the author is grateful to its director, J. Barkley Rosser, for the privilege of working in its inspiring scientific ambience. Thanks are due to Mmes. Dorothy Bowar, Carol Chase, and Doris Whitmore, of the MRC Staff, for the fine typing of the manuscript.

I. J. SCHOENBERG

LECTURE 1

Introduction, Background and Examples

The cardinal spline functions are defined in § 1, these being the main objects of discussion. In § 2 we describe the B-splines that were probably known to Hermite and certainly to Peano. New is the name and the point of view of regarding them as spline functions. The reader may safely pass from just before Theorem 1 (§ 2) to Lecture 2, without loss of continuity. The remainder of this lecture sketches briefly background material in § 3. Spline functions are frequently the solutions of appropriate extremum problems, especially if the Chebyshev-norm is involved. Particularly attractive examples due to Glaeser and Louboutin [28], [29], [49], [76], are discussed in § 4.

1. Cardinal spline functions. Throughout these lectures we shall use the following notations. Let n be an integer, $n \geqq 0$. The symbol

$$\mathscr{S}_n = \{S(x)\} \tag{1.1}$$

denotes the class of functions $S(x)$ satisfying the following two conditions

$$S(x) \in C^{n-1}(\mathbb{R}), \tag{1.2}$$

$$S(x) \in \pi_n \quad \text{in each interval} \quad (\nu, \nu + 1),\ \nu = 0, \pm 1, \pm 2, \cdots, \tag{1.3}$$

where π_n stands for the class of polynomials of degree not exceeding n, over the field $\mathbb{C}$ of complex numbers.

Clearly such $S(x)$ are n-times differentiable, and $S^{(n)}(x)$ are step-functions that may be discontinuous at the integers. Since differentiation reduces by one unit the order of continuity as well as the degree of a polynomial, while integration produces a similar increase, we see that

$$S(x) \in \mathscr{S}_n \quad \text{holds if and only if} \quad S^{(k)}(x) \in \mathscr{S}_{n-k}, \qquad 1 \leqq k \leqq n. \tag{1.4}$$

In particular, we may think of $\mathscr{S}_n$ as the class of n-fold integrals of the step-functions of the class $\mathscr{S}_0$. The elements of $\mathscr{S}_n$ are called *cardinal spline functions* of degree n. The integers are called the *knots* of the spline $\mathscr{S}(x)$.

At times it is convenient to place the knots of the splines halfway between the integers. Accordingly, we define the class

$$\mathscr{S}_n^* = \left\{S(x);\, S\left(x + \frac{1}{2}\right) \in \mathscr{S}_n\right\}. \tag{1.5}$$

This work was sponsored by the United States Army under Contract DA-31-124-ARO-D-462.

Its elements are called midpoint cardinal splines. Thus $\mathscr{S}_0^*$ denotes the class of step-functions having possible discontinuities halfway between the integers, while $\mathscr{S}_1$ is a continuous piecewise linear function with vertices at the integers. The possible points of discontinuity need not actually be such; in fact we have the obvious inclusions

$$\pi_n \subset \mathscr{S}_n, \qquad \pi_n \subset \mathscr{S}_n^*, \qquad n = 0, 1, \cdots. \tag{1.6}$$

The properties of the classes $\mathscr{S}_n$ and $\mathscr{S}_n^*$ are the main topics of the present lectures, in particular their interpolatory properties at the integers.

2. The B-splines. It was already pointed out in the Preface that

$$S(x) = \sum_{-\infty}^{\infty} y_\nu M_2(x - \nu)$$

represents the most general element of $\mathscr{S}_1$. We need a similar representation for the elements of $\mathscr{S}_n$, to be discussed in Lecture 2. Here we describe some of the properties of the general B-spline

$$M(x) = M(x; x_0, x_1, \cdots, x_n), \qquad x_0 < x_1 < \cdots < x_n, \tag{2.1}$$

of degree $n - 1$, having the knots $x_0, \cdots, x_n$.

Let

$$x_+ = \max(0, x), \qquad -\infty < x < \infty. \tag{2.2}$$

Let $f(x) \in C^n(\mathbb{R})$ and let $f(x_0, x_1, \cdots, x_n)$ denote its divided difference of order n, based on the $n + 1$ points

$$x_0 < x_1 < \cdots < x_n. \tag{2.3}$$

By Peano's theorem (see, for example, [24, § 7.3]) we have the identity

$$f(x_0, x_1, \cdots, x_n) = \frac{1}{n!}\int_{x_0}^{x_n} M(x; x_0, \cdots, x_n) f^{(n)}(x)\, dx, \tag{2.4}$$

where the kernel (2.1) is obtained by forming the divided difference of the function

$$M(x; t) = n(t - x)_+^{n-1} \tag{2.5}$$

with respect to the variable t and based on the points (2.3). We obtain

$$M(x; x_0, \cdots, x_n) = \sum_0^n \frac{n(x_\nu - x)_+^{n-1}}{\omega'(x_\nu)}, \qquad x \in \mathbb{R}, \tag{2.6}$$

where $\omega(x) = (x - x_0) \cdots (x - x_n)$. In view of the identity $x_+^{n-1} = x^{n-1} + (-1)^n(-x)_+^{n-1}$, we may also write

$$M(x; x_0, \cdots, x_n) = (-1)^n \sum_0^n \frac{n(x - x_\nu)_+^{n-1}}{\omega'(x_\nu)}, \qquad x \in \mathbb{R}. \tag{2.7}$$

In various ways (see [23, § 1]) we find that the function (2.1) has the properties

(2.8) $M(x) > 0$ if $x_0 < x < x_n$, $M(x) = 0$ if $x < x_0$ or $x > x_n$.

Finally, choosing $f(x) = x^n$ in (2.4), we obtain

$$\int_{-\infty}^{\infty} M(x; x_0, \cdots, x_n)\, dx = 1, \tag{2.9}$$

showing that (2.1) may be thought of as a frequency function.

We also mention the following geometric interpretation of the B-spline (2.1).

THEOREM 1. *The B-spline $M(x; x_0, \cdots, x_n)$ is identical with the linear density function obtained by projecting orthogonally on the x-axis the volume of an n-dimensional simplex σ_n, of volume unity, that is so located that its $n + 1$ vertices project orthogonally into the points $x_0, x_1, \cdots, x_n$ of the x-axis, respectively.*

In other words, $M(x; x_0, \cdots, x_n)$ represents the value of the $(n - 1)$-dimensional measure of the intersection of σ_n with a hyper-plane orthogonal to the x-axis at the point x. An application of the Brunn–Minkowski theorem to σ_n and its cross-sections shows that

$$\log M(x; x_0, \cdots, x_n) \quad \textit{is a concave function in} \quad x_0 < x < x_n.$$

3. The origin of the Pólya frequency functions. It followed from (2.8), (2.9) that $M(x; x_0, \cdots, x_n)$ is a frequency function. We may even consider the more general case of coalescent knots by requiring only that

$$x_0 \leqq x_1 \leqq \cdots \leqq x_n, \quad x_0 < x_n. \tag{3.1}$$

Suppose now that these knots also depend on n; hence,

$$x_{0,n} \leqq x_{1,n} \leqq \cdots \leqq x_{n,n}, \quad x_{0,n} < x_{n,n}, \tag{3.2}$$

and let us also consider the corresponding distribution functions

$$F_n(x) = F_n(x; x_{0,n}, \cdots, x_{n,n}) = \int_{-\infty}^{x} M(t; x_{0,n}, \cdots, x_{n,n})\, dt, \qquad n \geqq 1. \tag{3.3}$$

In 1947 the author proposed and solved the following problem.

PROBLEM 1. *Let $F(x)$ be a distribution function* (i.e., $F(x)$ is nondecreasing, $F(-\infty) = 0$, $F(+\infty) = 1$) *with the property that a sequence of distribution functions* (3.3) *exists such that the relation*

$$\lim_{n \to \infty} F_n(x; x_{0,n}, \cdots, x_{n,n}) = F(x) \tag{3.4}$$

holds at all continuity points of $F(x)$. What is the nature of the limit function $F(x)$?

The class of distribution functions so obtained may be characterized in four essentially different ways that ultimately prove to be equivalent.

(i) *Let $\mathscr{F}_1$ denote the class of distribution functions $F(x)$ with the limit property* (3.4), *excluding from this class the distribution function having its entire unit mass concentrated at one single point.*

(ii) Let $\Lambda(x)$ be a frequency function, hence $\Lambda(x) \geqq 0, \int \Lambda(x)\,dx = 1$. We assume that $\Lambda(x)$ is *totally positive*. This means that if

$$x_1 < x_2 < \cdots < x_n, \qquad y_1 < y_2 < \cdots < y_n, \tag{3.5}$$

then

$$\det \|\Lambda(x_i - y_j)\|_{i,j=1,\cdots,n} \geqq 0, \tag{3.6}$$

and this for all choices of the reals (3.5) and for $n = 1, 2, \cdots$.

Let $\mathscr{F}_{\mathrm{II}}$ denote the class of the corresponding distribution functions

$$F(x) = \int_{-\infty}^{x} \Lambda(t)\,dt. \tag{3.7}$$

(iii) Again, let $\Lambda(x)$ be a frequency function, which we assume to have the *variation diminishing property*. By this we mean the following: We begin with a definition. If $f(x) \in C(\mathbb{R})$ and $x_1 < x_2 < \cdots < x_n$, we define the number $v(f)$ of variations of sign of $f(x)$ by

$$v(f) = \sup S^{-}(f(x_1), f(x_2), \cdots, f(x_n)),$$

the supremum being formed for all possible increasing x_i and for all $n = 2, 3, \cdots$. Here $S^{-}(f(x_1), \cdots, f(x_n))$ stands for the number of changes of sign in the sequence $f(x_1), \cdots, f(x_n)$. If $f(x)$ is bounded and continuous on $\mathbb{R}$, we define its convolution transform by

$$g(x) = \int_{-\infty}^{\infty} \Lambda(x - t) f(t)\,dt. \tag{3.8}$$

We say that $\Lambda(x)$ is *variation diminishing*, provided that (3.8) implies the inequality

$$v(g) \leqq v(f). \tag{3.9}$$

Let $\mathscr{F}_{\mathrm{III}}$ denote the class of the corresponding distribution functions

$$F(x) = \int_{-\infty}^{x} \Lambda(t)\,dt. \tag{3.10}$$

(iv) We consider the so-called Laguerre–Pólya class of entire functions $\Psi(s)$ of the form

$$\Psi(s) = e^{-\gamma s^2 + \delta s} \prod_{\nu=1}^{\infty} (1 + \delta_\nu s)\, e^{-\delta_\nu s}, \tag{3.11}$$

where the real constants $\gamma, \delta, \delta_\nu$, are subject to the conditions

$$\gamma \geqq 0, \qquad 0 < \gamma + \sum_{1}^{\infty} \delta_\nu^2 < \infty. \tag{3.12}$$

Observe that the first strict inequality sign in (3.12) *excludes* the possibility that $\Psi(s) = e^{\delta s}$. The reciprocal $1/\Psi(s)$ is clearly a meromorphic function which is regular in an open vertical strip R of the complex s-plane that contains the origin 0.

It is not hard to show that there is a representation

(3.13) $$\frac{1}{\Psi(s)} = \int_{-\infty}^{\infty} e^{-sx}\Lambda(x)\,dx \quad \text{if} \quad s \in R,$$

by a convergent bilateral Laplace transform, where $\Lambda(x)$ is a frequency function. In view of (3.11), (3.12) and (3.13), we see that $\Lambda(x)$ is the convolution of a normal frequency function with a possibly infinite sequence of one-sided exponential frequency functions having a finite total variance. *We denote by $\mathscr{F}_{\mathrm{IV}}$ the class of the corresponding distribution functions*

(3.14) $$F(x) = \int_{-\infty}^{x} \Lambda(t)\,dt.$$

The following theorem holds.

THEOREM 2. *The four classes of distribution functions $\mathscr{F}_{\mathrm{I}}$, $\mathscr{F}_{\mathrm{II}}$, $\mathscr{F}_{\mathrm{III}}$, $\mathscr{F}_{\mathrm{IV}}$, are identical; hence,*

(3.15) $$\mathscr{F}_{\mathrm{I}} = \mathscr{F}_{\mathrm{II}} = \mathscr{F}_{\mathrm{III}} = \mathscr{F}_{\mathrm{IV}}.$$

Their elements are called *Pólya distribution functions* and their derivatives are the Pólya frequency functions $\Lambda(x)$. The relations (3.15) imply their many properties. Among them are the normal and the one-sided exponential frequency functions, the latter being the only ones having a jump discontinuity, all other $\Lambda(x)$ being continuous. If the relations (3.3), (3.4) and (3.7) hold, then we can differentiate the relation (3.4) to obtain $\lim M(x; x_{0n}, \cdots, x_{nn}) = \Lambda(x)$ as $n \to \infty$, and this limit relation holds for all x, except when $\Lambda(x)$ is the one-sided exponential and x is its point of discontinuity. (For a proof of $\mathscr{F}_{\mathrm{II}} = \mathscr{F}_{\mathrm{IV}}$, see [64], while $\mathscr{F}_{\mathrm{III}} = \mathscr{F}_{\mathrm{IV}}$ is shown in [65]. Finally, $\mathscr{F}_{\mathrm{I}} = \mathscr{F}_{\mathrm{IV}}$ was shown in [23, Part II].)

A word about the genesis of these results: They are based on the fundamental papers [53], [54], of Pólya and Schur. The author's contributions, starting in 1947, were obtained by following Jacobi's advice "One must always invert", and thereby replacing the quest for *zero-increasing* differential operators, by the inverse problem of characterizing the *variation diminishing* convolution transformations. These could be described by means of the notion of total positivity. For this subject, in the wider setting of total positivity, see the superb book of Karlin [38]; also for the beautiful work of A. Edrei on the related discrete problems, whose results the author conjectured but was unable to establish. See also [92]. This brief sketch of Pólya frequency functions leads to the last matters to be touched upon in the present section.

A. THE ORIGIN OF SPLINE INTERPOLATION. We return to the Laplace transform (3.13) of the Pólya frequency function $\Lambda(x)$. By Theorem 2 we know that $\Lambda(x)$ is totally positive; hence, that the inequality (3.6) is a consequence of the inequalities (3.5). In [66] the following question was answered: *The function $\Psi(s)$ being known in its explicit form* (3.11), *what are the conditions on the quantities x_i and y_j that will ensure that the nonnegative determinant* (3.6) *will actually be positive*?

We need not describe the answer here. We only wish to point out a corollary of the general results of [66], if applied to the special function

$$\Lambda(x) = \frac{1}{n!}x_+^n\, e^{-x}.$$

That this is indeed a Pólya frequency function we see if we observe that its distribution function belongs to $\mathscr{F}_{\mathrm{IV}}$: its transform

$$\frac{1}{n!}\int_0^\infty x^n\, e^{-x}\, e^{-sx}\, dx = \frac{1}{(1+s)^{n+1}}$$

is evidently of the form $1/\Psi(s)$. The corollary is as follows: let

$$S(x) = P(x) + \sum_{\nu=1}^{k} c_\nu (x - \xi_\nu)_+^n, \qquad P(x) \in \pi_n, \tag{3.16}$$

be a spline function of degree n having the k prescribed knots

$$\xi_1 < \xi_2 < \cdots < \xi_k. \tag{3.17}$$

Observe that $S(x)$ is an element of a linear family depending on $n + k + 1$ parameters. Let

$$x_1 < x_2 < \cdots < x_{n+k+1} \tag{3.18}$$

be prescribed abscissae. *When does the interpolation problem*

$$S(x_i) = y_i, \qquad i = 1, \cdots, n+k+1, \tag{3.19}$$

have a unique solution (3.16) *for arbitrarily given* y_i?

The answer is provided by the following theorem.

THEOREM 3. *The interpolation problem* (3.19) *has a unique solution for any choice of* y_i *if and only if the knots* ξ_j *and the nodes* x_i *satisfy the inequalities*

$$\begin{aligned} x_1 &< \xi_1 < x_{n+2}, \\ x_2 &< \xi_2 < x_{n+3}, \\ &\ \vdots \\ x_k &< \xi_k < x_{n+k+1}. \end{aligned} \tag{3.20}$$

This result was used by de Boor in [19]. A simple application of Theorem 3 is the following. Let

$$(x_i, y_i), \qquad i = 1, 2, \cdots, m+1, \qquad x_1 < x_2 < \cdots < x_{m+1}, \tag{3.21}$$

be $m + 1$ given points. There is a unique π_m interpolating them. However, we can also proceed as follows. Select any k, $k \leqq m - 1$, among the "inside" points $x_2, \cdots, x_m$, and call them $\xi_1, \cdots, \xi_k$. I claim that the points (3.21) can be uniquely interpolated by a spline of degree $n = m - k$ and having the ξ_i as knots. Indeed, notice that the number of nodes is $m + 1 = n + k + 1$ as it should be. Secondly, by our choice of the knots ξ_i, the conditions (3.20) are automatically satisfied.

In the extreme case when $k = m - 1$ we see that the interpolating spline has the degree $n = m - k = m - (m - 1) = 1$. This, of course, is the only trivial case.

B. AN OPEN QUESTION. This paragraph will be rather vague, as I know nothing about an answer. The B-splines of § 2 have been generalized from the present polynomial splines to Chebyshevian splines. Essentially equivalent to the latter are the so-called L-splines (see [36], [88]). For these we may also consider splines of minimal support obtaining the corresponding B-splines. Which of the properties of polynomial B-splines described in § 2 will carry over to these more general B-splines? In particular, what are the analogues of the Pólya frequency functions that were the answer to Problem 1?

4. The perfect splines of Glaeser and Louboutin. The well-known results on interpolation by splines of degree $2m - 1$ show that the spline solutions minimize the L_2-norm of the mth derivative among all interpolating functions. However, even more frequently spline functions arise if we seek to minimize the Chebyshev-norm of a certain higher derivative of a function subject to appropriate conditions. We shall find such results in §§ 1–3 of Lecture 9. Here we discuss a beautiful result due to G. Glaeser [28], [29], and obtained in continuing some work of J. Favard.

Let $f(x)$ be a spline of degree n, defined in $[a, b]$, and having as knots the interior points of the sequence $a = x_0 < x_1 < \cdots < x_k < x_{k+1} = b$. As should be clear from previous context, this means

$$1.\ f(x) \in C^{n-1}[a, b], \qquad 2.\ f(x) \in \pi_n \quad \text{in} \quad (x_i, x_{i+1}),\ i = 0, \cdots, k.$$

Glaeser calls $f(x)$ a *perfect spline*, provided that

$$f(x) = (-1)^i C x^n + \text{lower degree terms}, \quad \text{in} \quad (x_i, x_{i+1}), i = 0, \cdots, k, \tag{4.1}$$

where the constant C does not depend on i. Thus $|f^{(n)}(x)| = \text{constant}$.

THEOREM 4 (G. Glaeser). *If the* $2n$ *reals*

$$y_a^{(\nu)}, \quad y_b^{(\nu)}, \qquad \nu = 0, \cdots, n - 1, \tag{4.2}$$

are preassigned, then the 2*-point Hermite interpolation problem*

$$f^{(\nu)}(a) = y_a^{(\nu)}, \qquad f^{(\nu)}(b) = y_b^{(\nu)}, \qquad \nu = 0, \cdots, n - 1, \tag{4.3}$$

has a unique solution $f_0(x)$ *which is a perfect spline of degree* n *having in* (a, b) k *knots* $x_1, \cdots, x_k$ *such that*

$$0 \leqq k \leqq n - 1. \tag{4.4}$$

THEOREM 5 (G. Glaeser). *The perfect spline interpolant* $f_0(x)$ *of Theorem* 4 *is the unique function that minimizes the sup-norm* $\|f^{(n)}\|_\infty$ *among all functions* $f(x)$, *such that* $f^{(\nu)}(x)$, $\nu = 0, \cdots, n - 1$, *are absolutely continuous in* $[a, b]$ *and satisfy the boundary conditions* (4.3).

These are pure existence theorems. It seems remarkable that in the special case when all $y_a^{(\nu)}$ and $y_b^{(\nu)}$ vanish, except that $y_b \neq 0$, explicit results were obtained by

Louboutin [49]. We may as well let $a = -1$, $b = +1$, and consider the interpolation problem

$$
\begin{aligned}
&f^{(\nu)}(-1) = 0, && \nu = 0, \cdots, n-1, \\
f(1) = 1, \quad &f^{(\nu)}(1) = 0, && \nu = 1, \cdots, n-1.
\end{aligned}
\tag{4.5}
$$

THEOREM 6 (R. Louboutin). *Let*

$$x_\nu = -\cos\frac{\pi\nu}{n}, \qquad \nu = 1, \cdots, n-1. \tag{4.6}$$

The optimal perfect spline interpolant $f_0(x)$ of Theorem 4 for the problem (4.5) *has the $n-1$ knots* (4.6), *and*

$$|f_0^{(n)}(x)| = \text{const.} = \|f_0^{(n)}\|_\infty = 2^{n-2}(n-1)!. \tag{4.7}$$

Moreover, $f_0(x)$ may be expressed by

$$f_0(x) = (-1)^{n-1}2^{n-2}\int_{-1}^{x}(x-t)^{n-1}\operatorname{sgn} T_n'(t)\,dt, \qquad -1 \leqq x \leqq 1, \tag{4.8}$$

where $T_n(x)$ is the Chebyshev polynomial.

In fact for Louboutin's perfect spline $f_0(x)$ we find that

$$f_0'(x) = (-1)^{n-1}2^{n-2}(n-1)\int_{-1}^{x}(x-t)^{n-2}\operatorname{sgn} T_n'(t)\,dt \tag{4.9}$$

is identical with the perfect B-spline of degree $n-1$

$$f_0'(x) = M(x; x_0, x_1, \cdots, x_n), \quad \text{where} \quad x_\nu = -\cos\frac{\pi\nu}{n}, \ \nu = 0, \cdots, n. \tag{4.10}$$

For these particular nodes we find for the divided difference the expression

$$
\begin{aligned}
g(x_0, x_1, \cdots, x_n) = (-1)^n\frac{2^{n-2}}{n}(g(x_0) - 2g(x_1) + 2g(x_2) - \cdots \\
+ (-1)^{n-1}2g(x_{n-1}) + (-1)^n g(x_n)),
\end{aligned}
\tag{4.11}
$$

which seems little known.

See [76] for a direct derivation of Louboutin's Theorem 6, and of the least-norm properties of $f_0(x)$, a derivation that is independent of Glaeser's Theorems 4 and 5. There the reader will also find an application to the following problem of time-optimal control.

PROBLEM 2. *A particle F moves on the y-axis such that $y = F(t)$ and that the velocities of higher orders*

$$F^{(\nu)}(t), \qquad \nu = 0, \cdots, n-1, \tag{4.12}$$

are all absolutely continuous, while the nth *velocity satisfies at all times the inequality*

$$|F^{(n)}(t)| \leqq A \qquad (A \text{ is prescribed}). \tag{4.13}$$

We assume that the particle F starts from rest at $y = 0$ *at the time* $t = 0$, *i.e.*,

$$F^{(\nu)}(0) = 0, \qquad \nu = 0, \cdots, n-1, \tag{4.14}$$

and that F reaches the point $y = l$, *also at rest, at the time* $t = T\,(>0)$, *i.e.*,

$$F(T) = l, \qquad F^{(\nu)}(T) = 0, \qquad \nu = 1, \cdots, n-1. \tag{4.15}$$

We are to find the shortest time T_0 *during which this motion can be performed, and are to describe the nature of this optimal motion.*

As an application of Theorem 6 and of the least-norm properties of (4.8), we find in [76, p. 271] the following theorem.

THEOREM 7. *Let* $f_0(t)$ *denote the perfect spline* (4.8) *of Theorem* 6. *The shortest time during which a motion satisfying* (4.14), (4.15), *subject to the restriction* (4.13), *can occur is* T_0, *where*

$$T_0 = 2(2^{n-2}(n-1)!\,lA^{-1})^{1/n}, \tag{4.16}$$

and the corresponding motion is described by the function

$$F_0(t) = lf_0\left(\frac{2t}{T_0} - 1\right), \qquad 0 \leqq t \leqq T_0. \tag{4.17}$$

The solution for the case when $n = 2$ was well known (see [76] for the references to books on Optimal Control). The perfect B-spline $f_0'(x)$ reduces in this case to a "roof-function" with $f_0''(x) = +1$ if $-1 \leqq x \leqq 0$, $f_0''(x) = -1$ if $0 \leqq x \leqq 1$.

On the right side of (4.11) we see appearing coefficients proportional to 1, -2, 2, $\cdots$, $(-1)^{n-1}2$, $(-1)^n$. S. Karlin (oral communication) has recently solved the problem of characterizing the coefficients A_ν that may appear in the explicit expression

$$f(x_0, x_1, \cdots, x_n) = \sum_0^n A_\nu f(x_\nu), \qquad x_0 < \cdots < x_n,$$

of a divided difference.

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LECTURE 2

The Basis Property of B-Splines

We specialize the B-spline (2.1) of Lecture 1 to the case when its knots are consecutive integers. In this case $Q_n(x) = M(x; 0, 1, \cdots, n)$ was already known to Laplace as the frequency function of a sum of n independent random variables, each of which is uniformly distributed in the interval $(0, 1)$. Also its Fourier transform (1.7) below was known to Laplace (see also [17, pp. 37–38], where Bochner notices the increasing order of smoothness of $Q_n(x)$ for increasing n). It was shown in [63] that the shifted B-splines $Q_n(x - j)$, for integer j, form a basis of $\mathscr{S}_{n-1}$. This property is indeed fundamental and is established in § 2. In § 3 we show how the coefficients of the B-spline expansion of a spline $S(x)$ may be expressed in terms of S and its derivatives, and we discuss some applications. The so-called exponential splines appear in § 4.

1. B-splines with equidistant knots. From (2.1) of Lecture 1 we know that $M(x; a, a + 1, \cdots, a + n)$ is the B-spline of degree $n - 1$ having the knots at a, $a + 1, \cdots, a + n$. These are clearly translates of one and the same function. Nevertheless, it is convenient to select *two* standard forms: For $a = 0$ we obtain

$$Q_n(x) = M(x; 0, 1, \cdots, n), \tag{1.1}$$

called the *forward B-spline*, while on choosing $a = -n/2$ we have

$$M_n(x) = M\left(x; -\frac{n}{2}, -\frac{n}{2} + 1, \cdots, \frac{n}{2}\right), \tag{1.2}$$

called the *central B-spline*. The reasons for these names are the relations

$$\Delta^n f(0) = \int_0^n Q_n(x) f^{(n)}(x)\, dx, \qquad \delta^n f(0) = \int_{-n/2}^{n/2} M_n(x) f^{(n)}(x)\, dx, \tag{1.3}$$

which are special cases of (2.4) of Lecture 1. Here Δ and δ are the usual symbols for the *forward* and *central* difference operator of step 1, respectively. From (2.7) and (2.6) of Lecture 1 we find the explicit expressions

$$\begin{aligned} Q_n(x) &= \frac{1}{(n-1)!} \sum_0^n (-1)^\nu \binom{n}{\nu} (x - \nu)_+^{n-1}, \\ Q_n(x) &= \frac{(-1)^n}{(n-1)!} \sum_0^n (-1)^\nu \binom{n}{\nu} (\nu - x)_+^{n-1}, \end{aligned} \tag{1.4}$$

respectively, valid for all real x. Similar expressions for $M_n(x)$ follow from the identity

$$M_n(x) = Q_n\left(x + \frac{n}{2}\right). \tag{1.5}$$

Clearly (1.1) and (1.2) are cardinal splines. More precisely,

$$Q_n(x) \in \mathcal{S}_{n-1} \quad \text{for all } n, \quad M_n(x) \in \begin{cases} \mathcal{S}_{n-1} & \text{if } n \text{ is even}, \\ \mathcal{S}^*_{n-1} & \text{if } n \text{ is odd}. \end{cases} \tag{1.6}$$

If $f(x) = e^{ixt}$, then $\delta f(x) = 2i \sin(t/2)\, e^{ixt}$ and therefore $\delta^n f(x) = (2i \sin t/2)^n e^{ixt}$, and in particular $\delta^n f(0) = (2i \sin t/2)^n$, while $f^{(n)}(x) = (it)^n e^{ixt}$. Substituting these values into the second relation (1.3) we find that

$$\int_{-\infty}^{\infty} M_n(x)\, e^{ixt}\, dx = \left\{\frac{2 \sin(t/2)}{t}\right\}^n \tag{1.7}$$

is the Fourier transform of $M_n(x)$, already known to Laplace. Using the notation

$$\psi_n(t) = \left\{\frac{2 \sin(t/2)}{t}\right\}^n, \tag{1.8}$$

and observing that $\psi_{n-m}(t)\psi_m(t) = \psi_n(t)$, we obtain by a familiar property of Fourier transforms, that

$$\int_{-\infty}^{\infty} M_{n-m}(x - t) M_m(t)\, dt = M_n(x), \qquad 0 < m < n. \tag{1.9}$$

Observing that $M_1(t) = 1$ in $(-\frac{1}{2}, \frac{1}{2})$, while $M_1(t) = 0$ elsewhere, we obtain for $m = 1$,

$$M_n(x) = \int_{-1/2}^{1/2} M_{n-1}(x - t)\, dt = \int_{x-1/2}^{x+1/2} M_{n-1}(t)\, dt$$

and by differentiation

$$M_n'(x) = M_{n-1}(x + \tfrac{1}{2}) - M_{n-1}(x - \tfrac{1}{2}). \tag{1.10}$$

Further relations will appear later and will be developed when needed. Let us rather turn to an important property of B-splines that also explains their name ("B" stands for "basis").

2. The representation of cardinal splines in terms of B-splines.

THEOREM 1. *If*

$$S(x) \in \mathcal{S}_{n-1}, \tag{2.1}$$

then $S(x)$ admits a unique representation of the form

$$S(x) = \sum_{-\infty}^{\infty} c_\nu Q_n(x - \nu) \tag{2.2}$$

with appropriate constant coefficients. Conversely, every such expansion represents an element of $\mathcal{S}_{n-1}$.

Proof. For simplicity we write $Q_n(x) = Q(x)$. A proof will follow from several statements the first of which is as follows.

1°. *The support of* $S(x)$ *cannot be shorter than n consecutive unit intervals, unless* $S(x)$ *vanishes for all real* x.

Observe that $Q(x)$ has the support $(0, n)$. Let $S(x)$ be such that

$$S(x) = 0 \quad \text{if} \quad x < 0 \quad \text{or if} \quad x > n - 1. \tag{2.3}$$

It follows that the knots of $S(x)$ can only be $0, 1, \cdots, n - 1$. From (2.3) we conclude that $S(x)$ admits a representation of the form

$$S(x) = c_0 x_+^{n-1} + c_1(x-1)_+^{n-1} + \cdots + c_{n-1}(x-n+1)_+^{n-1} \text{ for all real } x. \tag{2.4}$$

However, if $x > n - 1$, then the subscript "+" in (2.4) can be dropped and then (2.3) implies the identity

$$c_0 x^{n-1} + c_1(x-1)^{n-1} + \cdots + c_{n-1}(x-n+1)^{n-1} = 0.$$

Expanding the binomials and equating to zero the coefficients of the powers of x, we find that c_ν are solutions of a nonsingular homogeneous system and must therefore all vanish. Now (2.4) shows that $S(x) = 0$ for all real x.

2°. *If q is an integer and*

$$S(x) = 0 \quad in \quad (q-1, q), \quad and\ also\ in \quad (q+n-1, q+n), \tag{2.5}$$

then

$$S(x) = 0 \quad in \quad (q-1, q+n). \tag{2.6}$$

For if we define a new spline $S_1(x)$ by

$$S_1(x) = \begin{cases} S(x) & \text{if} \quad q \leqq x \leqq q + n - 1, \\ 0 & \text{elsewhere,} \end{cases}$$

then clearly $S_1(x) \in \mathcal{S}_{n-1}$ because of (2.5). From Statement 1° we conclude that (2.6) holds, because $S_1(x) = 0$ for all real x.

3°. *The n functions*

$$Q(x), Q(x-1), \cdots, Q(x-n+1) \tag{2.7}$$

are linearly independent in the interval $n - 1 \leqq x \leqq n$.

Let us assume that

$$\sum_0^{n-1} c_\nu Q(x-\nu) = 0 \quad \text{in} \quad (n-1, n) \tag{2.8}$$

and let us show that *all* c_ν *must vanish.* To see this we consider the spline

$$S(x) = \sum_0^{n-1} c_\nu Q(x-\nu), \qquad x \in \mathbb{R}. \tag{2.9}$$

Clearly $S(x) \in \mathcal{S}_{n-1}$, and by (2.8) we have

$$S(x) = 0 \quad \text{in the interval} \quad (n-1, n). \tag{2.10}$$

On the other hand, (2.9) clearly implies that

$$S(x) = 0 \quad \text{if} \quad x < 0. \tag{2.11}$$

If we apply Statement 2° with $q = 0$, we conclude from (2.10), (2.11) that $S(x) = 0$ in the interval $(0, n)$. Now (2.11) shows that $S(x) = 0$ in $(-\infty, n)$. But then from (2.9) we easily conclude that $c_0 = 0$, and then $c_1 = 0$, and so on, until we find that also $c_{n-1} = 0$.

4°. *If*

$$S(x) = 0 \quad \textit{if} \quad x < 0, \tag{2.12}$$

then $S(x)$ *admits a unique representation*

$$S(x) = \sum_0^\infty c_\nu Q(x - \nu) \quad \textit{for all} \quad x. \tag{2.13}$$

From (2.12) we conclude that $S(x) = a_0 x_+^{n-1}$ in $(0, 1)$. The first expression (1.4) of $Q_n(x)$ shows that, for a uniquely determined c_0, the difference $S(x) - c_0 Q(x)$ vanishes if $x < 1$. But then, with a uniquely defined c_1, $S(x) - c_0 Q(x) - c_1 Q(x-1)$ will vanish for $x < 2$. This procedure may be continued indefinitely and establishes the unique expansion (2.13).

5°. *If*

$$S(x) = 0 \quad \textit{for} \quad x > n - 1, \tag{2.14}$$

then $S(x)$ *admits a unique representation of the form*

$$S(x) = \sum_{-\infty}^{-1} c_\nu Q(x - \nu), \qquad x \in \mathbb{R}. \tag{2.15}$$

This follows from Statement 4° if applied to the new spline function $S_1(x) = S(n - 1 - x)$.

Let us now assume (2.1) and establish the expansion (2.2). By Statement 3° we can determine constants c_ν such that the new spline

$$S_0(x) = S(x) - \sum_0^{n-1} c_\nu Q(x - \nu) \tag{2.16}$$

has the property

$$S_0(x) = 0 \quad \text{if} \quad n - 1 < x < n. \tag{2.17}$$

We now define two new splines by

$$S_1(x) = \begin{cases} S_0(x) & \text{if} \quad x < n-1, \\ 0 & \text{if} \quad x \geqq n-1, \end{cases} \qquad S_2(x) = \begin{cases} 0 & \text{if} \quad x \leqq n, \\ S_0(x) & \text{if} \quad x > n. \end{cases} \tag{2.18}$$

Since both these as well as $S_0(x)$ vanish in $(n - 1, n)$, we find that

$$S_1(x) + S_2(x) = S_0(x), \qquad x \in \mathbb{R}. \tag{2.19}$$

By Statements 5° and 4° we can write

$$S_1(x) = \sum_{-\infty}^{-1} c_\nu Q(x - \nu), \qquad S_2(x) = \sum_{n}^{\infty} c_\nu Q(x - \nu). \tag{2.20}$$

Combining (2.16), (2.19) and (2.20), we find that all expansions fit nicely together to yield (2.2). Also unicity is thereby established.

Theorem 1 was established in a different way in [63, § 3.15]. A more general result can be found in [23, § 4].

3. The coefficients of the B-spline expansion. We consider the expansion

$$S(x) = \sum_{-\infty}^{\infty} c_\nu M_n(x - \nu) \tag{3.1}$$

in terms of the central B-splines. By (1.5), (1.6) and Theorem 1, (3.1) is the most general element of its class ($\mathscr{S}^*_{n-1}$ if n is odd). *How are the coefficients c_ν expressed in terms of $S(x)$?*

The answer is described by the following theorem.

THEOREM 2. *Let $(\gamma_{2r}^{(n)})$ be the sequence of rationals generated by the expansion*

$$\left\{\frac{u}{2\sin(u/2)}\right\}^n = \sum_{r=0}^{\infty} \gamma_{2r}^{(n)} u^{2r}. \tag{3.2}$$

If (3.1) *holds, then*

$$c_\nu = \sum_{r=0}^{s} (-1)^r \gamma_{2r}^{(n)} S^{(2r)}(\nu), \quad \textit{where} \quad s = [(n-1)/2]. \tag{3.3}$$

For a proof see [2, § 6]. Observe that if n is odd, $n = 2m - 1$ say, then the highest order of derivatives in (3.3) is $2s = 2m - 2 = n - 1$, while $S^{(n-1)}(x)$ is a discontinuous step-function. However, its knots are at $\nu + \frac{1}{2}$ (halfway between integers) and $S^{(n-1)}(\nu)$ is well defined. If n is even, $n = 2m$, then $2s = 2(m - 1) = n - 2$ and again $S^{(n-2)}(\nu)$ is well defined.

If we write

$$\gamma_{2\nu}^{(n)} = (-1)^\nu \frac{1}{2^{2\nu}(2\nu)!} D_{2\nu}^{(n)},$$

then $D_{2\nu}^{(n)}$ is a polynomial in n of degree ν. In Nörlund's Table 6 [52, p. 460] we find these polynomials listed for $\nu = 0, 1, \cdots, 6$. These allow us to write (3.3) explicitly for all $n \leqq 13$. The first few coefficients are

$$\gamma_0^{(n)} = 1, \qquad \gamma_2^{(n)} = n/24, \qquad \gamma_4^{(n)} = n(5n + 2)/5760. \tag{3.4}$$

Using these we obtain

$$n = 2: c_\nu = S(\nu),$$
$$n = 3: c_\nu = S(\nu) - \tfrac{1}{8}S''(\nu),$$
$$n = 4: c_\nu = S(\nu) - \tfrac{1}{6}S''(\nu),$$
$$n = 5: c_\nu = S(\nu) - \tfrac{5}{24}S''(\nu) + \tfrac{3}{128}S^{(4)}(\nu),$$
$$n = 6: c_\nu = S(\nu) - \tfrac{1}{4}S''(\nu) + \tfrac{1}{30}S^{(4)}(\nu).$$

Since x^ν, $\nu = 0, 1, \cdots, n-1$, admits expansions of the form (3.1), we obtain the corresponding coefficients from (3.3). In this way we obtain the identities

$$1 = \sum_{-\infty}^{\infty} M_n(x-\nu),$$

$$x = \sum_{-\infty}^{\infty} \nu M_n(x-\nu), \qquad n > 1,$$

$$x^2 = \sum_{-\infty}^{\infty} \left(\nu^2 - \frac{n}{12}\right) M_n(x-\nu), \qquad n > 2,$$

$$x^3 = \sum_{-\infty}^{\infty} \left(\nu^3 - \frac{n}{4}\nu\right) M_n(x-\nu), \qquad n > 3,$$

$$x^4 = \sum_{-\infty}^{\infty} \left(\nu^4 - \frac{n}{2}\nu^2 + \frac{n(5n+2)}{240}\right) M_n(x-\nu), \qquad n > 4,$$

$$x^5 = \sum_{-\infty}^{\infty} \left(\nu^5 - \frac{5}{6}n\nu^3 + \frac{n(5n+2)}{480}\nu\right) M_n(x-\nu), \qquad n > 5.$$

A worthwhile application of Theorem 2 is as follows.

THEOREM 3. *Let* $1 \leqq p \leqq \infty$. *If*

$$S(x) = \sum_{-\infty}^{\infty} c_\nu M_n(x-\nu), \tag{3.5}$$

then

$$S(x) \in L_p(\mathbb{R}) \quad \textit{if and only if} \quad (c_\nu) \in l_p. \tag{3.6}$$

This was first established in [1, § 10], and [2, § 7] gives a simpler proof. For its details we refer to [2]. However, the basic idea is simple and runs as follows. We discuss only the harder "only if" implication. From $S(x) \in L_p$ we conclude easily that also its derivatives $S^{(2r)}(x)$ are in L_p. From this we conclude that all sequences $(S^{(2r)}(\nu))$ are in l_p, and now (3.3) shows that $(c_\nu) \in l_p$.

Theorem 3 is of theoretical as well as computational relevance. It shows that $S(\nu)$ is roughly of the same order of magnitude as c_ν, and that the representation (3.5) does not allow the dreaded loss of accuracy that occurs when two large quantities of the same order of magnitude are subtracted, at least for well-behaved splines.

4. The exponential splines. Our Theorem 2 shows how cardinal splines, and in particular polynomials, can be explicitly expanded in terms of B-splines. However, we certainly do not need B-splines in order to study polynomials. The immediate task is to discover special cardinal splines having interesting properties—properties, moreover, that only splines can have and are denied to polynomials.

Some such splines are already in the literature, in fact have been there for a long time. One such is the periodic extension $\bar{B}_n(x)$, of period 1, of the Bernoulli polynomial $B_n(x)$ in $[0, 1]$. This extension was invented by Poisson for the purpose of providing the Euler–Maclaurin summation formula with a usable remainder term. It will be discussed later in these lectures under the name of Bernoulli monospline, for the reason that $\bar{B}_n(x) - x^n \in \mathcal{S}_{n-1}$. Outstanding elements of $\mathcal{S}_n$ are the so-called *Euler splines* (see Lectures 4 and 9 below). These were invented by Hermite and Sonin about 80 years ago [52, Chap. 2, § 3]. Here we discuss a more general class of cardinal splines which we call *exponential splines.*

Let t be a constant, $t \neq 0$, $t \neq 1$, otherwise real or complex. We consider the exponential function $f(x) = t^x$ (any fixed branch thereof) and observe that it satisfies the functional equation

$$f(x + 1) = tf(x). \tag{4.1}$$

We ask: *What are the elements of $\mathcal{S}_n$ satisfying this functional equation*? An answer is given by the following lemma.

LEMMA 1. *The most general element $S(x)$ of $\mathcal{S}_n$ satisfying the functional equation*

$$S(x + 1) = tS(x), \qquad x \in \mathbb{R}, \tag{4.2}$$

is

$$S(x) = c_0 \sum_{-\infty}^{\infty} t^j Q_{n+1}(x - j) \qquad (c_0 \textit{ is a constant}). \tag{4.3}$$

Proof. Starting with the representation

$$S(x) = \sum_{-\infty}^{\infty} c_j Q(x - j), \qquad Q = Q_{n+1}, \tag{4.4}$$

of Theorem 1, we find that

$$S(x + 1) = \sum c_j Q(x + 1 - j) = \sum c_{j+1} Q(x - j)$$

while $tS(x) = \sum tc_j Q(x - j)$. The identity of the two right sides required by (4.2) and the unicity of the representation in Theorem 1, shows that we must have $c_{j+1} = tc_j$, for all j. Therefore, $c_j = c_0 t^j$, for all j, and (4.4) reduces to (4.3). Conversely, it is clear that the function (4.3) satisfies (4.2).

DEFINITION 1. We define the function

$$\Phi_n(x; t) = \sum_{-\infty}^{\infty} t^j Q_{n+1}(x - j), \qquad t \neq 0, t \neq 1, \tag{4.5}$$

and call it the *exponential spline of degree n to the base t.*

Let us differentiate (4.5). From (1.5) and (1.10) we find that $Q'_{n+1}(x) = Q_n(x) - Q_n(x-1)$, and therefore,

$$\Phi'_n(x;t) = \sum t^j Q'_{n+1}(x-j) = \sum t^j(Q_n(x-j) - Q_n(x-j-1))$$
$$= \sum t^j Q_n(x-j) - \sum t^{j-1} Q_n(x-j),$$

whence

$$\Phi'_n(x;t) = (1-t^{-1})\Phi_{n-1}(x;t). \tag{4.6}$$

Repeating the operation ν times we find

$$\Phi_n^{(\nu)}(x;t) = (1-t^{-1})^\nu \Phi_{n-\nu}(x;t), \tag{4.7}$$

the last of which is

$$\Phi_n^{(n)}(x;t) = (1-t^{-1})^n \Phi_0(x;t) = (1-t^{-1})^n \sum_j t^j Q_1(x-j).$$

Since $Q_1(x) = 1$ in $(0, 1)$, $Q_1(x) = 0$ elsewhere, we find that

$$\Phi_n^{(n)}(x;t) = (1-t^{-1})^n \quad \text{if} \quad 0 < x < 1. \tag{4.8}$$

For use in our next lecture we record some of these properties in the next lemma.

LEMMA 2. *The exponential spline* (4.5) *satisfies*

$$\Phi_n(x+1;t) = t\Phi_n(x;t), \tag{4.9}$$

and its polynomial component in the interval $0 < x < 1$ *has the form*

$$\Phi_n(x;t) = \frac{1}{n!}(1-t^{-1})^n x^n + \text{lower degree terms}, \qquad 0 < x < 1. \tag{4.10}$$

5. The geometric behavior of the exponential splines. As in § 4 we assume that t is a real or complex constant, $t \neq 0$, $t \neq 1$, and want to get an idea of the shape of the curve

$$\Gamma_{n,t} : z = \Phi_n(x;t), \qquad -\infty < x < \infty. \tag{5.1}$$

To obtain it we use the relations (4.6) and (4.9). The relation (4.6) shows that, except for the nonvanishing constant factor

$$\gamma = 1 - t^{-1}, \tag{5.2}$$

the curve $\Gamma_{n-1,t}$ is the *hodograph* of the motion (5.1), if we think of x as the time variable. Let $\Gamma_{n,t}[\nu, \nu+1]$ denote the arc of $\Gamma_{n,t}$ obtained if we confine x to the interval $[\nu, \nu+1]$. The relation (4.9) shows that (in an obvious symbolism) $\Gamma_{n,t}[1, 2] = t\Gamma_{n,t}[0, 1]$, and generally

$$\Gamma_{n,t}[\nu, \nu+1] = t^\nu \Gamma_{n,t}[0, 1], \qquad \text{for integers } \nu. \tag{5.3}$$

To fix the ideas, let us assume that $t = |t|\, e^{iu}$, $|t| > 1$, $0 < u < \pi$. I claim that $\Gamma_{n,t}$ has the shape of the logarithmic spiral

$$z = |t|^x e^{iux}, \qquad -\infty < x < \infty, \tag{5.4}$$

i.e., $\Gamma_{n,t}$ is a locally convex curve turning about the origin counterclockwise and winding down to the origin 0 as $x \to -\infty$, and spiralling out to ∞ as $x \to +\infty$. From (4.6) we see that this will be the case, if we can show $\Gamma_{n-1,t}$ has all these characteristics. In other words, we apply induction in n. And induction works, because $\Gamma_{1,t}$ is the path of $z = \Phi_1(x;t) = \sum t^j Q_2(x-j) = \sum t^{j+1} M_2(x-j)$, and this is indeed a spiral-shaped polygonal line with vertices at the points of the sequence (t^ν).

There are obvious changes if $|t| \leqq 1$, (or if $u = 0$ or π). If $|t| = 1$, the curves $\Gamma_{n,t}$, $n = 1, 2, \cdots$, are seen to be *bounded*. If $|t| < 1$, then the curves spiral down from ∞ to 0 as x varies from $-\infty$ to $+\infty$, while turning counterclock for all x.

If $u = 0$ or π, then the shapes change drastically. Indeed, then $\Phi_n(x;t)$ is a real function, and $\Gamma_{n,t}$ has the shape of

$$z = |t|^x \cos ux. \tag{5.5}$$

Let us record our findings in the following way.

THEOREM 4. *If* $t = |t|\, e^{iu}$, $-\pi < u < \pi$, $t \neq 0$, $t \neq 1$, *and* $u \neq 0$, *then the curve*

$$\Gamma_{n,t}: z = \Phi_n(x;t) = \sum_{-\infty}^{\infty} t^j Q_{n+1}(x-j), \qquad -\infty < x < \infty, \tag{5.6}$$

is of class $C^{n-1}(\mathbb{R})$, *it is locally convex and has the same shape as the curve*

$$z = |t|^x e^{iux}, \qquad -\infty < x < \infty. \tag{5.7}$$

If t is real, $t \neq 0$, $t \neq 1$, hence $u = 0$, or $u = \pi$, then $\Gamma_{n,t}$ is on the real axis. The graph of the real-valued function

$$y = \Phi_n(x;t) \tag{5.8}$$

has the same shape as the graph of the function

$$y = |t|^x \cos ux. \tag{5.9}$$

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LECTURE 3

The Exponential Euler Splines

We come to what is probably the central part of the lectures. The exponential splines $\Phi_n(x;t)$, defined by (4.5) of Lecture 2, will be used as thoroughly as the American Indians utilized the buffalo, to the last bone. For all but certain $n-1$ negative values of t, called the *eigenvalues of* $\mathscr{S}_n$, $\Phi_n(x;t)$ is (up to a constant factor) identical with the *exponential Euler spline* $S_n(x;t)$ of § 5 below. If t does assume one of the eigenvalues, then $\Phi_n(x;t)$ becomes a so-called *eigenspline of* $\mathscr{S}_n$ (§ 5). The $n-1$ eigensplines will help to solve the problem of cardinal spline interpolation in Lecture 4. Here, and later on, our developments are largely based on Euler's generating functions (1.4) and (1.6) below.

1. The exponential Euler polynomials. We return to the definition (4.5) of Lecture 2 of the exponential spline $\Phi_n(x;t)$. Since $\Phi_n(x;t) \in \mathscr{S}_n$, we know that in the interval $0 < x < 1$, $\Phi_n(x;t)$ is identical with an element of π_n. From (4.10) of Lecture 2 we also know that the coefficient of x^n in this polynomial is $=(1-t^{-1})^n/n!$. It follows that $n!(1-t^{-1})^{-n}\Phi_n(x;t)$ is identical in $(0, 1)$ with a certain monic polynomial.

DEFINITION 1. We define the monic polynomial $A_n(x;t) = x^n +$ (lower degree terms) by

$$A_n(x;t) = n!(1-t^{-1})^{-n}\Phi_n(x;t) \quad \text{in} \quad 0 \leqq x \leqq 1, \quad t \neq 0, t \neq 1, \tag{1.1}$$

and call it the *exponential Euler polynomial of degree n for the base t*.

These polynomials enjoy a number of remarkable properties, mostly due to Euler, some of which we state as follows.

THEOREM 1. *The coefficients of the exponential Euler polynomial*

$$A_n(x;t) = x^n + \binom{n}{1}a_1(t)x^{n-1} + \binom{n}{2}a_2(t)x^{n-2} + \cdots + + a_n(t) \tag{1.2}$$

depend, aside from the binomial coefficients, on t but not on n, as indicated in (1.2). *The* $A_n(x;t)$ *therefore form an Appell sequence of polynomial, i.e.,* $A_n'(x;t) = nA_{n-1}(x;t)$. *The coefficients* $a_\nu(t)$ *are generated by the expansion*

$$\frac{t-1}{t-e^z} = \sum_0^\infty \frac{a_\nu(t)}{\nu!}z^\nu. \tag{1.3}$$

It follows from (1.2) *that the* $A_n(x;t)$ *are generated by*

$$\frac{t-1}{t-e^z}e^{xz} = \sum_0^\infty \frac{A_n(x;t)}{n!}z^n. \tag{1.4}$$

To investigate the $a_n(t)$ further, we set

$$a_n(t) = \frac{\Pi_n(t)}{(t-1)^n}, \tag{1.5}$$

whence, by (1.3),

$$\frac{t-1}{t-e^z} = \sum_0^\infty \frac{\Pi_n(t)}{(t-1)^n}\frac{z^n}{n!}. \tag{1.6}$$

THEOREM 2. 1. *$\Pi_n(t)$ is a monic reciprocal polynomial in t of degree $n-1$ having only integer coefficients. It is related to the forward B-spline $Q_{n+1}(x)$, defined by* (1.1) *of Lecture 2, by the identity*

$$\Pi_n(t) = n! \sum_{j=0}^{n-1} Q_{n+1}(j+1)t^j. \tag{1.7}$$

2. *The polynomial $\Pi_n(t)$ can be independently defined by the expansion*

$$\frac{\Pi_n(t)}{(1-t)^{n+1}} = \sum_0^\infty (\nu+1)^n t^\nu. \tag{1.8}$$

3. *It satisfies the recurrence relation*

$$\Pi_{n+1}(t) = (1+nt)\Pi_n(t) + t(1-t)\Pi_n'(t), \qquad \Pi_0(t) = 1. \tag{1.9}$$

4. *The zeros λ_ν of $\Pi_n(t)$ are all simple and negative. We label them so that*

$$\lambda_{n-1} < \lambda_{n-2} < \cdots < \lambda_2 < \lambda_1 < 0. \tag{1.10}$$

They are reciprocal in pairs; hence,

$$\lambda_1\lambda_{n-1} = \lambda_2\lambda_{n-2} = \cdots = 1. \tag{1.11}$$

Notice that we have written λ_ν instead of $\lambda_\nu^{(n)}$. From the recurrence relation (1.9) we easily find that

$$\begin{aligned} &\Pi_0(t) = 1, && \Pi_3(t) = t^2 + 4t + 1,\\ &\Pi_1(t) = 1, && \Pi_4(t) = t^3 + 11t^2 + 11t + 1,\\ &\Pi_2(t) = t + 1, && \Pi_5(t) = t^4 + 26t^3 + 66t^2 + 26t + 1. \end{aligned} \tag{1.12}$$

The $\Pi_n(t)$ are remarkable polynomials with a long history. They were discovered by Euler, who found (1.8) and also their generating function (1.6) (see [26, Chap. VIII, § 178]). They were studied by several mathematicians prior to Frobenius [27] who found (1.9) and derived from it in a few lines by induction in n the nature of the zeros described by (1.10). For this reason we call $\Pi_n(t)$ the *Euler–Frobenius polynomials.* They were rediscovered more recently by nearly everyone working on spline interpolation, the first among these being Quade and Collatz [57]. We retained the notation $\Pi_n(t)$ used by Quade and Collatz in appreciation of their outstanding paper [57]. See [4, § 3] for one way of establishing Theorems 1 and 2.

The polynomials $A_n(x;t)$ are very well known for the special case $t = -1$ when (1.4) becomes

$$\frac{2}{1+e^z}e^{xz} = \sum_0^\infty \frac{A_n(x;-1)}{n!}z^n. \tag{1.13}$$

This shows that $A_n(x;-1) = E_n(x)$ are the well known *Euler polynomials* (see [52, p. 35]).

2. The construction of the exponential splines. The definition (4.5) of Lecture 2 is well suited for the evaluation of $\Phi_n(x;t)$ because the defining series has at most $n+1$ nonvanishing and consecutive terms for each value of x. However, a far simpler way is afforded by (1.1) and the use of the exponential Euler polynomial $A_n(x;t)$.

We begin with the evaluation of the $a_\nu(t)$ by (1.3). Multiplying (1.3) by $t - e^z$ and comparing the coefficients of the powers of x, we obtain the relations

$$1 + \binom{n}{1}a_1(t) + \binom{n}{2}a_2(t) + \cdots + a_n(t) = ta_n(t), \qquad n = 1, 2, \cdots, \tag{2.1}$$

giving the $a_n(t)$ recursively. The coefficients $a_n(2)$ are particularly easy to find, for if $t = 2$, then (2.1) becomes

$$1 + \binom{n}{1}a_1(2) + \cdots + \binom{n}{n-1}a_{n-1}(2) = a_n(2), \tag{2.2}$$

showing that $A_n(x;2)$ has integer coefficients. From (2.2) we obtain the table

n	0	1	2	3	4	5	6	7
$a_n(2)$	1	1	3	13	75	541	4683	47293

Thus

$$\begin{aligned} A_7(x;2) = x^7 + 7x^6 + 63x^5 + 455x^4 + 2625x^3 + 11361x^2 \\ + 32781x + 47293. \end{aligned} \tag{2.3}$$

Having determined the $a_\nu(t)$ by (2.1), and therefore $A_n(x;t)$ by (1.2), the very definition of $A_n(x;t)$ yields

$$\Phi_n(x;t) = \frac{1}{n!}(1 - t^{-1})^n A_n(x;t) \quad \text{in} \quad 0 \leqq x \leqq 1. \tag{2.4}$$

Finally, we extend this function to all real x by the functional equation

$$\Phi_n(x+1;t) = t\Phi_n(x;t) \tag{2.5}$$

of Lemma 2 of Lecture 2.

It is clear that $A_n(x; t)$ must be a very particular polynomial to give rise to an element of $\mathcal{S}_n$ by this construction. For (2.5) combined with the condition that $\Phi_n(x; t) \in \mathscr{C}^{n-1}(\mathbb{R})$ require $A_n(x; t)$ to satisfy in $[0, 1]$ the boundary conditions

$$A_n^{(\nu)}(1; t) = tA_n^{(\nu)}(0; t) \quad \text{for} \quad \nu = 0, 1, \cdots, n-1. \tag{2.6}$$

These n relations determine the *monic* polynomial $A_n(x; t)$ uniquely. The relations (2.6) are easily obtained from Euler's generating relation (1.4.).

3. An extremum property of the exponential Euler polynomial. For value $t > 1$ we consider the polynomial

$$P(x) = A_n(x; t)/a_n(t) \quad \text{in the interval} \quad 0 \leqq x \leqq 1. \tag{3.1}$$

By (1.2) and (2.6), it satisfies the conditions:

1. $P(0) = 1$.
2. $P^{(\nu)}(1) = tP^{(\nu)}(0)$, $\nu = 0, 1, \cdots, n-1$.

Let us define $\mathscr{F}_n$ as the class of real-valued functions defined in $[0, 1]$ and satisfying the following conditions:

0. $f(x) \in \mathscr{C}^{n-1}[0, 1]$, $f^{(n-1)}(x)$ satisfies a Lip. 1 condition.
1. $f(0) \geqq 1$.
2. $f^{(\nu)}(1) \geqq tf^{(\nu)}(0)$, $\nu = 0, \cdots, n-1$.

The following theorem holds.

THEOREM 3. *The polynomial* $P(x)$, *defined by* (3.1), *is the unique element of* $\mathscr{F}_n$ *that minimizes the norm*

$$\|f^{(n)}\| = \operatorname*{ess.\,sup}_{0 \leqq x \leqq 1} |f^{(n)}(x)| \quad \textit{for} \quad f(x) \in \mathscr{F}_n, \tag{3.2}$$

giving it its least value

$$\|P^{(n)}\| = n!(t-1)^n/\Pi_n(t). \tag{3.3}$$

For details see [4, § 6].

4. The exponential Euler polynomials of the class $\mathcal{S}_n^*$. In § 1 of Lecture 1 we have defined the class $\mathcal{S}_n^*$ of cardinal midpoint splines. The analogues for this class of the results of §§ 1 and 2 require no new ideas. We define its exponential splines by

$$\Phi_n^*(x; t) = \Phi_n(x + \tfrac{1}{2}; t). \tag{4.1}$$

Its polynomial component within the interval $-\frac{1}{2} \leqq x \leqq \frac{1}{2}$ is up to a constant factor the monic polynomial $A_n(x + \frac{1}{2}; t)$. However, it is of interest to obtain its expansion in powers of x. Accordingly we define the *midpoint exponential Euler polynomial by*

$$B_n(x; t) = A_n(x + \tfrac{1}{2}; t). \tag{4.2}$$

Substituting x by $x + \frac{1}{2}$ in (1.4) we obtain the generating function

$$\frac{t-1}{t-e^z} e^{(x+1/2)z} = \sum_0^\infty \frac{B_n(x; t)}{n!} z^n. \tag{4.3}$$

Writing

$$B_n(x;t) = x^n + \binom{n}{1} b_1(t)x^{n-1} + \cdots + b_n(t) \tag{4.4}$$

we obtain from (4.3), for $x = 0$, the expansion

$$\frac{t-1}{t-e^z} e^{z/2} = \sum_0^\infty \frac{b_n(t)}{n!} z^n. \tag{4.5}$$

It is convenient to define a new sequence of polynomials $\rho_n(t)$ by setting

$$b_n(t) = \frac{\rho_n(t)}{2^n(t-1)^n}. \tag{4.6}$$

The analogue of Theorem 2 is as follows.

THEOREM 4. *The $\rho_n(t)$ defined by the generating function*

$$\frac{t-1}{t-e^z} e^{z/2} = \sum_0^\infty \frac{\rho_n(t)}{2^n(t-1)^n} \frac{z^n}{n!} \tag{4.7}$$

are monic reciprocal polynomials of degree n having integer coefficients. In terms of the B-spline $Q_{n+1}(x)$ we may write

$$\rho_n(t) = 2^n n! \sum_{j=0}^n Q_{n+1}(j + \tfrac{1}{2}) t^j. \tag{4.8}$$

2. *$\rho_n(t)$ can be defined independently by the expansion*

$$\frac{\rho_n(t)}{(1-t)^{n+1}} = \sum_{\nu=0}^\infty (2\nu + 1)^n t^\nu. \tag{4.9}$$

3. *$\rho_n(t)$ satisfies the recurrence relation*

$$\rho_{n+1}(t) = (1 + (2n+1)t)\rho_n(t) + 2t(1-t)\rho_n'(t), \qquad \rho_0 = 1. \tag{4.10}$$

4. *The zeros of $\rho_n(t)$ are simple and negative. We label them so that*

$$\mu_n < \mu_{n-1} < \cdots < \mu_2 < \mu_1 < 0, \tag{4.11}$$

and therefore,

$$\mu_n \mu_1 = \mu_{n-1}\mu_2 = \cdots = 1. \tag{4.12}$$

From (4.10) we find

$$\begin{aligned} &\rho_1(t) = t + 1, && \rho_3(t) = t^3 + 23t^2 + 23t + 1, \\ &\rho_2(t) = t^2 + 6t + 1, && \rho_4(t) = t^4 + 76t^3 + 230t^2 + 76t + 1. \end{aligned} \tag{4.13}$$

The definition (4.1) shows that $\Phi_n^*(x;t)$ is a shifted version of $\Phi_n(x;t)$. However, it is clear that $\Phi_n^*(x,t)$ can also be constructed in $(-1/2, 1/2)$ in terms of $B_n(x;t)$, afterwards extending its definition by the functional equation

$$\Phi_n^*(x+1;t) = t\Phi_n^*(x;t).$$

5. The exponential Euler splines and the eigensplines. We return to the exponential splines $\Phi_n(x;t)$ of § 2. From (2.5) we find that

$$\Phi_n(\nu;t) = t^\nu \Phi_n(0;t) \quad \text{for all integers} \quad \nu. \tag{5.1}$$

If we try to find an appropriate constant C such that $f(x) = C\Phi_n(x;t)$ satisfies

$$f(\nu) = t^\nu \quad \text{for all integers} \quad \nu, \tag{5.2}$$

we must distinguish two cases.

If t is such that

$$\Phi_n(0;t) \neq 0, \tag{5.3}$$

then

$$S_n(x;t) = \frac{\Phi_n(x;t)}{\Phi_n(0;t)} \tag{5.4}$$

is an element of $\mathscr{S}_n$ having the interpolatory property (5.2), because of (5.1). However, if $\Phi_n(0;t) = 0$, then $\Phi_n(\nu;t) = 0$ for all integers ν, by (5.1), and there is no multiple of $\Phi_n(x;t)$ with the property (5.2).

As to when (5.3) holds, or fails, is easily decided. By (1.1), (1.2), (1.5) and Part 4 of Theorem 2 we may state the following corollaries.

COROLLARY 1. *The condition* (5.3) *is equivalent to the requirement that*

$$t \neq \lambda_\nu, \qquad \nu = 1, \cdots, n-1, \tag{5.5}$$

where λ_ν are the $n-1$ zeros of the Euler–Frobenius polynomial $\Pi_n(t)$.

COROLLARY 2. *The $n-1$ functions*

$$S_\nu(x) = \Phi_n(x;\lambda_\nu), \qquad \nu = 1, \cdots, n-1, \tag{5.6}$$

are elements of $\mathscr{S}_n$ having the property that

$$S_\nu(j) = 0 \quad \textit{for all integer} \quad j. \tag{5.7}$$

Moreover, they satisfy the functional equations

$$S_\nu(x+1) = \lambda_\nu S_\nu(x), \qquad \nu = 1, \cdots, n-1. \tag{5.8}$$

DEFINITION 2. 1. The function $S_n(x;t)$, defined by (5.4), is called the *exponential Euler spline of degree n for the base t.* It exists provided that (5.5) holds, in particular for all real or complex t, $t \neq 0$, $t \neq 1$, provided that t is not a negative real. Of course,

$$S_n(x;t) \in \mathscr{S}_n \quad \text{and} \quad S_n(\nu;t) = t^\nu \quad \text{for all integers} \quad \nu. \tag{5.9}$$

2. The $n-1$ functions (5.6) are called the *eigensplines* of the class $\mathscr{S}_n$, while λ_ν are the corresponding *eigenvalues.*

An entirely similar discussion concerns the $\mathscr{S}_n^*$ of midpoint splines. In this case we use the relations

$$n!(1-t^{-1})^{-n}\Phi_n(x+\tfrac{1}{2};t) = B_n(x;t) = x^n + \cdots + b_n(t), \quad -1/2 < x < 1/2, \tag{5.10}$$

$$b_n(t) = \rho_n(t)2^{-n}(t-1)^{-n}, \tag{5.11}$$

and Part 4 of Theorem 4. The results are as follows.

COROLLARY 3. *Let*

$$t \neq \mu_\nu, \qquad \nu = 1, \cdots, n, \tag{5.12}$$

where the μ_ν *are the zeros of* $\rho_n(t)$; *then*

$$S_n^*(x;t) = \Phi_n(x + \tfrac{1}{2}; t)/\Phi_n(\tfrac{1}{2}; t) \tag{5.13}$$

(called the midpoint exponential Euler spline of degree n for the base t) *has the property*

$$S_n^*(j;t) = t^j \quad \textit{for all integers} \quad j. \tag{5.14}$$

COROLLARY 4. *The n functions*

$$S_\nu^*(x) = \Phi_n(x + \tfrac{1}{2}; \mu_\nu), \qquad \nu = 1, \cdots, n, \tag{5.15}$$

are elements of $\mathscr{S}_n^*$ *such that*

$$S_\nu^*(j) = 0 \quad \textit{for all integers} \quad j. \tag{5.16}$$

They satisfy the functional equations

$$S_\nu^*(x+1) = \mu_\nu S_\nu^*(x), \qquad \nu = 1, \cdots, n. \tag{5.17}$$

The functions (5.15) are called the *eigensplines of* $\mathscr{S}_n^*$, and μ_ν are the *eigenvalues*.

6. Properties of the exponential Euler splines. Let

$$t = \tau e^{iu}, \quad \tau > 0, \quad -\pi < u \leqq \pi, \quad \text{hence}, \quad \tau = |t|, \qquad t \neq 1, \tag{6.1}$$

and let t^x denote the exponential function defined by

$$t^x = \tau^x e^{iux} \quad \text{for all real} \quad x. \tag{6.2}$$

We also assume that t is not an eigenvalue of $\mathscr{S}_n$; hence, $S_n(x;t)$ exists.

LEMMA 1. *The function* $\Omega_n(x)$, *defined by the relation*

$$S_n(x;t) = t^x \Omega_n(x) \quad \textit{for all real} \quad x, \tag{6.3}$$

is a periodic function of period 1.

Proof. This is evident, as both $S_n(x;t)$ and t^x satisfy the same relation $f(x+1) = tf(x)$. Explicitly,

$$\Omega_n(x+1) = S_n(x+1;t)t^{-x-1} = tS_n(x;t)t^{-1}t^{-x} = \Omega_n(x).$$

Let us find the Fourier series expansion of $\Omega_n(x)$. For this purpose we assume that

$$0 \leqq x \leqq 1, \tag{6.4}$$

so that

$$S_n(x;t) = A_n(x;t)/A_n(0;t). \tag{6.5}$$

Our main tool is Euler's generating function (1.4). Dividing it by z^{n+1} we obtain

$$F(z) = \frac{t-1}{t-e^z}\frac{e^{xz}}{z^{n+1}} = \sum_{\nu=0}^{\infty} \frac{A_\nu(x;t)}{\nu!} z^{\nu-n-1}. \tag{6.6}$$

The left is a meromorphic function having a pole of order $n + 1$ at the origin and simple poles at the zeros of $e^z - t$. By (6.1) these poles are at

$$t_k = \log \tau + iu + 2\pi ki, \qquad k = 0, \pm 1, \pm 2, \cdots. \tag{6.7}$$

Observe that all $t_k \neq 0$ because we have assumed that $t \neq 1$. By Cauchy's residue theorem and our assumption (6.4), we conclude that the sum of all the residues of $F(z)$ in the finite plane vanishes (for details, see [4, § 7]). The result is the following theorem.

THEOREM 5. *The periodic function $\Omega_n(x)$ of Lemma 1 admits the Fourier series expansion*

$$\Omega_n(x) = \sum_{-\infty}^{\infty} \frac{1}{t_k^{n+1}} e^{2\pi ikx} \Big/ \sum_{-\infty}^{\infty} \frac{1}{t_k^{n+1}}, \qquad x \in \mathbb{R}. \tag{6.8}$$

This result has a number of corollaries (see [4, §§ 8, 9]). One such is the following.

COROLLARY 5. *If t is not a negative real, then*

$$\lim_{n\to\infty} S_n(x; t) = t^x, \qquad x \in \mathbb{R}. \tag{6.9}$$

The question of convergence for negative t, in which case we must assume that $t \neq \lambda_\nu$, $\nu = 1, \cdots, n-1$, is more delicate. If n is odd, we may choose $t = -1$ (which in this case is not an eigenvalue) and find that $S_n(x; -1) \to \cos \pi x$, *which limit is certainly different from $t^x = e^{i\pi x}$*. However, if we choose $t = -e$ (e is the base of natural logarithms), then surely $t \neq \lambda_\nu$, because e is transcendental, while all λ_ν are algebraic numbers. Even so $S_n(x; -e)$ *does not converge to a limit* (see [4, relation (10.16)] for the family of limit functions that are possible).

Two classes of exponential Euler splines should be singled out for special study. The first corresponds to a *positive* t, $t \neq 1$. As already pointed out by (2.2), the nicest case is when $t = 2$. Actually [4, relations (8.8), (8.9)],

$$|S_n(x; 2)2^{-x} - 1| = O(\gamma^n) \quad \text{as} \quad n \to \infty,$$

where $\gamma = \{1 + 4\pi^2(\log 2)^{-2}\}^{-1/2} = .1096$. This shows that $S_n(x; 2)$ approximates to 2^x with nearly n significant figures for all real values of x. This could be used on a computer where *very smooth approximations of 2^x are wanted.*

More interesting seems the second class, the class of *bounded* exponential Euler splines. It is clear from (5.9) that $S_n(x; t)$ can be a bounded function only if $|t| = 1$, $t = e^{iu}$, say. That this condition is also sufficient is seen from (6.3) which shows that

$$S_n(x; e^{iu}) = e^{iux}\Omega_n(x), \qquad -\pi < u < \pi. \tag{6.10}$$

Notice, however, that $u = \pi$, hence $t = -1$, is possible only if n is odd, $n = 2m - 1$ say, for otherwise -1 is an eigenvalue. For this reason *we restrict further discussion to the case when*

$$n = 2m - 1. \tag{6.11}$$

Also the Fourier series (6.8) simplifies greatly in this case. Indeed, notice that (6.7) becomes $t_k = (u + 2\pi k)i$, since $\log \tau = 0$, while $n + 1 = 2m$ is even. Thus (6.8) becomes

$$\Omega_n(x) = \sum_{-\infty}^{\infty} \frac{1}{(u + 2\pi k)^{2m}} e^{2\pi i k x} \Big/ \sum_{-\infty}^{\infty} \frac{1}{(u + 2\pi k)^{2m}}. \tag{6.12}$$

Observing that the coefficients of this expansion are positive we conclude from (6.10) that

$$|S_{2m-1}(x; e^{iu})| \leqq 1 \quad \text{for all real} \quad x. \tag{6.13}$$

Let us mark the points $t^v = e^{iuv}$, $v = 0, \pm 1, \cdots$, on the unit circle $|z| = 1$ assuming that $0 < u < \pi$ (Fig. 1). From (6.13) we conclude that the curve Γ traced out by

$$\Gamma: z = S_{2m-1}(x; e^{iu}), \qquad -\infty < x < \infty, \tag{6.14}$$

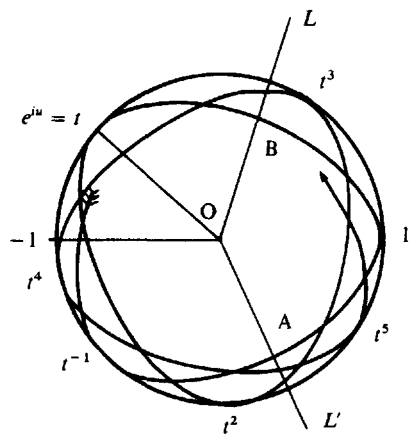

FIG. 1.

is entirely contained within the circle $|z| \leqq 1$, only touching it at the points t^v. Morever, the relation

$$S_{2m-1}(x + 1; e^{iu}) = e^{iu} S_{2m-1}(x; e^{iu}) \tag{6.15}$$

shows that all arches of the curve are obtained from the arc joining $z = 1$ to $z = e^{iu}$ by rotations through angles which are multiples of u. However, these remarks are already contained in the case $|t| = 1$ of Theorem 4 of Lecture 2.

Actually much more follows from that Theorem 4 (Lecture 2), in particular the following.

THEOREM 6. *Let* $-\pi < u < \pi$, $u \neq 0$. *The arcs* $\Gamma[v, v + 1]$ *of the curve* Γ, *shown in Fig.* 1, *are all convex.* Γ *is traced out by z counterclockwise if* $0 < u < \pi$, *and* *clockwise if* $-\pi < u < 0$. *If* $u = \pi$, *then we have a to-and-fro motion on the segment* $[-1, 1]$.

If $m = 1$, when S_{2m-1} is a linear spline, then Γ reduces to the infinite polygonal line with vertices (t^ν). Let us also record a further theorem.

THEOREM 7. *The curve* Γ *is a closed curve if and only if* t *is a root of unity, or equivalently, if* u *is a rational multiple of* π. *If*

$$u = M\pi/N, \quad \textit{where} \quad (M, N) = 1, \tag{6.16}$$

then $S_{2m-1}(x; e^{iu})$ *is a periodic spline of period* N *if* M *is even, and of period* $2N$, *if* M *is odd.*

If (6.16) holds and $m = 1$ we obtain a (perhaps star-shaped) regular polygon of N, or $2N$, sides. The periodic splines of Theorem 7 were studied by Golomb in [31].

If in Figure 1 t is fixed and m increases, then the arc $\Gamma\,[0, 1]$ bulges out and tends to the circumference $|z| = 1$ as $m \to \infty$, by (6.9). However, if m is fixed and u increases to π, then the arc $\Gamma\,[0, 1]$ flattens out and converges to the diameter $[-1, 1]$.

For applications in Lecture 10 to the approximation of Fourier transforms we establish here the following.

THEOREM 8. *If*

$$-\pi \leqq u \leqq \pi, \tag{6.17}$$

then

$$|e^{iux} - S_{2m-1}(x; e^{iu})| \leqq C_m \left(\frac{|u|}{\pi}\right)^{2m} \quad \textit{for all real} \quad x. \tag{6.18}$$

Here C_m *is a constant which is* <3 *for all* m. *Some best values of* C_m *are*

$$C_1 = \frac{\pi^2}{8}, \qquad C_2 = 1, \qquad C_3 = 1. \tag{6.19}$$

It is conjectured that the best value is $C_m = 1$ for all $m > 3$.

For the case of periodic splines, hence when

$$u = \pi r, \qquad \text{where } r \text{ is a rational number}, \tag{6.20}$$

the inequality (6.18), with the constant C_m replaced by 4, was established by Golomb in [31]. In [31] the restriction (6.20) is essential, since Golomb does not consider the cardinal spline $S_{2m-1}(x; e^{iu})$. A proof of (6.18) for the present cardinal case seems to be simpler, as will now be shown.

PROOF. With $t = e^{iu}$, (6.3) and (6.8) imply that

$$|e^{iux} - S_{2m-1}(x; e^{iu})| = \left| \sum_{-\infty}^{\infty} \frac{1}{t_k^{2m}} (e^{2\pi ikx} - 1) \Big/ \sum_{-\infty}^{\infty} \frac{1}{t_k^{2m}} \right|. \tag{6.21}$$

As in the derivation of (6.12), we have $t_k = (u + 2\pi k)i$. Observing that $|e^{2\pi ikx} - 1|$ is $=0$ if $k = 0$, and $\leqq 2$ otherwise, we obtain from (6.21) that

$$|e^{iux} - S_{2m-1}(x; e^{iu})| \leqq 2\left\{\sum_{k\neq 0} \frac{1}{(u + 2\pi k)^{2m}}\right\} \Big/ \left\{\frac{1}{u^{2m}} + \sum_{k\neq 0} \frac{1}{(u + 2\pi k)^{2m}}\right\}.$$

Writing

$$\Phi(u) = 2\frac{\displaystyle\sum_{k=1}^{\infty}\left\{\frac{1}{(2\pi k + u)^{2m}} + \frac{1}{(2\pi k - u)^{2m}}\right\}}{\displaystyle\left\{1 + \sum_{k=1}^{\infty}\left[\left(\frac{u}{2k\pi + u}\right)^{2m} + \left(\frac{u}{2k\pi - u}\right)^{2m}\right]\right\}} \tag{6.22}$$

the last inequality yields

$$|e^{iux} - S_{2m-1}(x; e^{iu})| \leqq \Phi(u) \cdot |u|^{2m}. \tag{6.23}$$

We easily obtain an upper bound of $\Phi(u)$ as follows. Since $\Phi(u)$ is an even function, we may assume that $0 \leqq u \leqq \pi$. As the numerator of (6.22) is an even *convex* function of u in the interval $(-2\pi, 2\pi)$ (being a sum of convex functions), we conclude that it assumes its *largest value* in $[0, \pi]$ for $u = \pi$. On the other hand the denominator of (6.22) is *least* if $u = 0$. The ratio of these two extreme values gives an upper bound for $\Phi(u)$ and we obtain that

$$\Phi(u) \leqq A_m \pi^{-2m}, \qquad -\pi \leqq u \leqq \pi, \tag{6.24}$$

where

$$A_m = 2\left(1 + 2\sum_{\nu=1}^{\infty} \frac{1}{(2\nu + 1)^{2m}}\right) < A_1 = (\pi^2 - 4)2^{-1} < 3. \tag{6.25}$$

The inequalities (6.23), (6.24), and (6.25), establish (6.18) with $C_m < 3$. The values (6.19) require an additional discussion which we omit.

We have assumed above that $n = 2m - 1$ is odd. However, the developments of this section up to and including (6.10) remain valid also for even values of n, provided that we assume that $t \neq -1$, since -1 is now an eigenvalue. Let therefore

$$n = 2m, \qquad t = e^{iu}, \qquad 0 < u < \pi. \tag{6.26}$$

Without proof we add the information that the graph of the curve

$$\Gamma_{2m}: z = S_{2m}(x; e^{iu}), \qquad -\infty < x < \infty, \tag{6.27}$$

differs from the graph shown in Figure 1 as follows: Γ_{2m} still touches the circle $|z| = 1$ at $z = t^\nu$ if $x = \nu$. However, the arcs $\Gamma_{2m}[\nu, \nu + 1]$ are now *outside* the circle. In order to obtain results that are valid for all t such that $|t| = 1$, we must pass to the *midpoint* exponential Euler spline $S^*_{2m}(x; e^{iu})$ defined by (5.13). However, this spline can be derived from (6.27) by writing

$$S^*_{2m}(x; e^{iu}) = S_{2m}(x + \tfrac{1}{2}; e^{iu})/S_{2m}(\tfrac{1}{2}; e^{iu}). \tag{6.28}$$

Indeed, this is clearly an element of $\mathscr{S}^*_{2m}$, while for integer ν $S_{2m}(\nu + \frac{1}{2}; e^{iu}) = t^\nu S_{2m}(\frac{1}{2}; e^{iu})$ and therefore by (6.28)

$$S^*_{2m}(\nu; e^{iu}) = t^\nu. \tag{6.29}$$

We may let Figure 1 do double duty and regard it as showing the graph of

$$\Gamma^*_{2m}: z = S^*_{2m}(x; e^{iu}), \qquad -\infty < x < \infty. \tag{6.30}$$

For instance, if $m = 1$ we find that Γ^*_2 is composed of a train of parabolic arcs, the arc AB being one of them. Its parabola is characterized by having its axis along the axis of reals, and being orthogonal to the lines L, L', at the points B and A, respectively. If $u = \pi$, hence $t = -1$, then

$$S^*_{2m}(x; -1) = \mathscr{E}_{2m}(x)$$

is the even-degree Euler spline of Theorem 5 of the next Lecture 4.

LECTURE 4

Cardinal Spline Interpolation

We deal now with the simplest and most important case of the central problem of these lectures. After a brief analysis of its nature (§ 1), the main results are stated in Theorems 1, 2, 3, below. The proof of uniqueness of solutions in Theorem 1 is based (§ 4) on the eigensplines (5.6) of Lecture 3. The proof of existence of solutions (§ 5) uses the Euler–Frobenius polynomials $\Pi_n(t)$ generated by (1.6) of Lecture 3. Theorem 1 is applied to periodic splines and to the Bernoulli monospline in § 6. The last § 7 states extensions of our results to bivariate cardinal splines.

1. The problem and a preliminary answer. The problem of cardinal spline interpolation (C.S.I.) is the following. A sequence of numbers, real or complex,

$$y = (y_\nu), \qquad \nu = 0, \pm 1, \pm 2, \cdots, \tag{1.1}$$

being prescribed, we are to find $S(x) \in \mathcal{S}_n$ such that

$$S(\nu) = y_\nu \quad \text{for all} \quad \nu. \tag{1.2}$$

A second problem arises if we require that $S(x) \in \mathcal{S}_n^*$.

If $n = 1$ and we ask that $S(x) \in \mathcal{S}_1$, then the problem is trivial, because it is clearly uniquely solved by the linear spline $S_1(x)$ obtained by linear interpolation between every pair of consecutive data. A good deal of light is thrown on the problem by the following almost trivial lemma.

LEMMA 1. 1. *The problem* (1.2), *with the requirement that* $S(x) \in \mathcal{S}_n$, *where* $n > 1$, *has infinitely many solutions forming a linear manifold in* $\mathcal{S}_n$ *of dimension*

$$d = n - 1. \tag{1.3}$$

2. *The problem* (1.2), *with* $S(x) \in \mathcal{S}_n^*$, *has infinitely many solutions forming a linear manifold in* $\mathcal{S}_n^*$ *of dimension*

$$d^* = n. \tag{1.4}$$

We give proofs only for simple and fundamental results. This is one of them.

Proof. 1. We claim that every $S(x) \in \mathcal{S}_n$ may be uniquely represented in the form

$$\begin{aligned} S(x) = P(x) &+ a_1(x-1)_+^n + a_2(x-2)_+^n + \cdots \\ &+ a_0(-x)_+^n + a_{-1}(-x-1)_+^n + \cdots, \end{aligned} \tag{1.5}$$

where $P(x) \in \pi_n$ and the a_ν are constants. In the first place it is clear that (1.5) always represents an element of $\mathcal{S}_n$. Conversely, if $S(x)$ is given, then $P(x) = S(x)$ in $[0, 1]$

and then the a_ν are the values of the jumps of the step function $S^{(n)}(x)$. Let us now determine (1.5) so as to satisfy the relations (1.2). Select $P(x) \in \pi_n$ arbitrarily such as to satisfy the relations

$$(1.6) \qquad P(0) = y_0, \qquad P(1) = y_1.$$

$P(x)$ having been selected, we readily see that the coefficients $a_1, a_2, \cdots$ are successively and uniquely determined by the conditions (1.2) for $\nu = 2, 3, \cdots$, respectively. Likewise $a_0, a_{-1}, \cdots$ are successively and uniquely determined by (1.2) for $\nu = -1, -2, \cdots$, respectively. Since $P(x)$, satisfying (1.6), depends linearly on $(n + 1) - 2 = n - 1$ parameters, the result (1.3) follows.

2. Every $S(x) \in S_n^*$ may be uniquely written as

$$S(x) = Q(x) + b_1(x - \tfrac{1}{2})_+^n + b_2(x - \tfrac{3}{2})_+^n + \cdots$$
$$+ b_0(-x - \tfrac{1}{2})_+^n + b_{-1}(-x - \tfrac{3}{2})_+^n + \cdots.$$

If

$$(1.7) \qquad Q(0) = y_0,$$

$Q(x) \in \pi_n$, then the b_ν are uniquely determined, as before, from the relations (1.2) $(\nu \neq 0)$. Now $Q(x)$, subject to (1.7), depends on $(n + 1) - 1 = n$ parameters and (1.4) is established.

2. The main results. Lemma 1 suggests that our problem requires some kind of boundary conditions, i.e., growth conditions. Let $\gamma \geqq 0$ and let us define the following classes of sequences and cardinal splines:

$$(2.1) \qquad \mathcal{Y}_\gamma = \{(y_\nu); y_\nu = O(|\nu|^\gamma) \text{ as } \nu \to \pm\infty\},$$

$$(2.2) \qquad \mathcal{S}_{n,\gamma} = \{S(x); S(x) \in \mathcal{S}_n, S(x) = O(|x|^\gamma) \text{ as } x \to \pm\infty\}.$$

$$(2.3) \qquad \mathcal{S}_{n,\gamma}^* = \{S(x); S(x) \in \mathcal{S}_n^*, S(x) = O(|x|^\gamma) \text{ as } x \to \pm\infty\},$$

THEOREM 1. *Let*

$$(2.4) \qquad (y_\nu) \in \mathcal{Y}_\gamma.$$

1. *If n is odd, $n = 2m - 1$, then the interpolation problem* (1.2) *has a unique solution $S(x)$ such that*

$$(2.5) \qquad S(x) \in \mathcal{S}_{n,\gamma}.$$

2. *If n is even, $n = 2m$, then the problem* (1.2) *has a unique solution $S(x)$ such that*

$$(2.6) \qquad S(x) \in \mathcal{S}_{n,\gamma}^*.$$

These results show that the interpolation relations (1.2) induce a $1 - 1$ correspondence between the space $\mathcal{Y}_\gamma$ and $\mathcal{S}_{n,\gamma}$ if n is odd, and between $\mathcal{Y}_\gamma$ and $\mathcal{S}_{n,\gamma}^*$ if n is even. Of particular interest is the case $\gamma = 0$, of *bounded sequences* and *bounded splines*, first established by Subbotin in [85]. For Theorem 1 as stated, see [2].

It can be shown that if n is odd, then the $1-1$ correspondence

$$l_\infty \leftrightarrow \mathscr{S}_n \cap L_\infty(\mathbb{R}) \quad \text{induced by (1.2)},$$

can be described more precisely as follows: *If* $1 \leqq p \leqq \infty$ *and* $(y_v) \in l_p$, *then* (1.2) *has a unique solution* $S(x) \in \mathscr{S}_n \cap L_p(\mathbb{R})$. If n is even, then there is a similar result concerning $\mathscr{S}_n^*$.

We turn now to the *fundamental functions* of our interpolation process. The special sequence

$$\delta_v = \begin{cases} 1 & \text{if} \quad v = 0, \\ 0 & \text{if} \quad v \neq 0, \end{cases} \tag{2.7}$$

being bounded, we may apply to the C.I.P. $S(v) = \delta_v$ our Theorem 1 with $\gamma = 0$. We conclude that there exists a uniquely defined function $L_n(x)$ having the following properties:

$$L_n(x) \in \begin{cases} \mathscr{S}_n & \text{if} \quad n \quad \text{is odd}, \\ \mathscr{S}_n^* & \text{if} \quad n \quad \text{is even}, \end{cases} \tag{2.8}$$

$$L_n(v) = \delta_v \quad \text{for all} \quad v, \tag{2.9}$$

$$L_n(x) \quad \text{is bounded on} \quad \mathbb{R}. \tag{2.10}$$

The following two theorems hold [2].

THEOREM 2. *For appropriate positive constants* A_n *and* α_n *we have*

$$|L_n(x)| < A_n e^{-\alpha_n|x|} \quad \textit{for all real} \quad x. \tag{2.11}$$

THEOREM 3. *The unique solution* $S(x)$ *in Theorem* 1, *satisfying* (2.5) *if* n *is odd and* (2.6) *if* n *is even, is represented by the expansion*

$$S(x) = \sum_{-\infty}^{\infty} y_v L_n(x - v), \qquad -\infty < x < \infty. \tag{2.12}$$

This expansion, which can be called the cardinal Lagrange interpolation formula of degree n, *converges absolutely and uniformly in every finite interval.*

3. A basis for the space of null-splines: the eigensplines. In this section and the next we deal with the question of unicity in Theorem 1. Since (1.2) is a linear problem, we consider its null-space defined by

$$\mathring{\mathscr{S}}_n = \{S(x); S(x) \in \mathscr{S}_n,\ S(v) = 0 \text{ for all integers } v\} \tag{3.1}$$

and we refer to its elements as *null-splines*. As a special case of Lemma 1 (§ 1) we conclude that $\mathring{\mathscr{S}}_n$ is a linear subspace of $\mathscr{S}_n$ of dimension

$$d = n - 1. \tag{3.2}$$

We need now the eigensplines (5.6) of Lecture 3 and state the following lemma.

LEMMA 2. *A basis for the space* $\mathring{\mathscr{S}}_n$, *defined by* (3.1), *is provided by the eigensplines*

$$S_v(x) = \Phi_n(x; \lambda_v), \qquad v = 1, \cdots, n-1. \tag{3.3}$$

Proof. Observe that the eigensplines (3.3) are elements of $\mathring{\mathscr{S}}_n$ by (5.7) of Lecture 3. Their number $n-1$ being equal to the dimension (3.2) of the space, it suffices to show that

(3.4) *the eigensplines* $S_\nu(x)$ *are linearly independent.*

Proof of (3.4). Let us show that

$$\sum_1^{n-1} c_\nu S_\nu(x) = 0, \quad x \in \mathbb{R}, \quad \text{implies that} \quad c_1 = c_2 = \cdots = c_{n-1} = 0.$$

Since

$$S_\nu(x) = \sum_{-\infty}^{\infty} (\lambda_\nu)^j Q_{n+1}(x-j), \tag{3.5}$$

we obtain

$$\sum_1^{n-1} c_\nu S_\nu(x) = \sum_{j=-\infty}^{\infty} \left\{ \sum_{\nu=1}^{n-1} c_\nu (\lambda_\nu)^j \right\} Q_{n+1}(x-j).$$

The assumed vanishing of this sum implies, by Theorem 1 of Lecture 2, that

$$\sum_{\nu=1}^{n-1} c_\nu (\lambda_\nu)^j = 0 \quad \text{for all} \quad j.$$

Using these relations only for $j = 0, 1, \cdots, n-2$, and the fact that the λ_ν are pairwise distinct, we obtain $c_\nu = 0$, $\nu = 1, \cdots, n-1$.

The class of null-splines of $\mathscr{S}_n^*$ is

$$\mathring{\mathscr{S}}_n^* = \{S(x);\, S(x) \in \mathscr{S}_n^*,\, S(\nu) = 0 \text{ for all integers } \nu\} \tag{3.6}$$

having the dimension

$$d^* = n \tag{3.7}$$

by Lemma 1. As an analogue of Lemma 2 we have a further lemma.

LEMMA 3. *A basis of the space* $\mathring{\mathscr{S}}_n^*$, *defined by* (3.6), *is provided by the eigensplines*

$$S_\nu^*(x) = \Phi_n(x + \tfrac{1}{2}; \mu_\nu), \qquad \nu = 1, \cdots, n, \tag{3.8}$$

of Corollary 4 *of Lecture* 3.

4. Proof of unicity in Theorem 1. We assume (1.2), (2.4) and (2.5) to hold, and suppose that $S(x)$ and $\tilde{S}(x)$ are two splines satisfying the conditions. It follows that their difference

$$s(x) = S(x) - \tilde{S}(x) \in \mathscr{S}_{n,\gamma} \cap \mathring{\mathscr{S}}_n. \tag{4.1}$$

By Lemma 2 we conclude that

$$s(x) = \sum_1^{n-1} c_\nu S_\nu(x). \tag{4.2}$$

We must distinguish two cases:

1. $n = 2m - 1$ *is odd*. From the simplicity of the zeros λ_ν of $\Pi_n(x)$ and the relations (1.10), (1.11) of Lecture 3, we conclude that

$$\lambda_{2m-2} < \cdots < \lambda_m < -1 < \lambda_{m-1} < \cdots < \lambda_1 < 0 \tag{4.3}$$

and the behavior of the $S_\nu(x)$ at $\pm\infty$, together with $s(x) = \sum c_\nu S_\nu(x) \in \mathcal{S}_{n,\gamma}$, show that we must have $c_\nu = 0$ for $\nu = 1, \cdots, n-1$. Therefore $s(x) = 0$ for all x and the unicity is established.

2. $n = 2m$ *is even*. In this case, the inequalities (4.3) are replaced by

$$\lambda_{2m-1} < \cdots < \lambda_{m+1} < \lambda_m = -1 < \lambda_{m-1} < \cdots < \lambda_1 < 0, \tag{4.4}$$

hence $\lambda_m = -1$ is an eigenvalue, and from the previous argument we can only conclude that

$$s(x) = S(x) - \tilde{S}(x) = c_m S_m(x), \tag{4.5}$$

which is a *bounded* function: There is no unicity.

If we now look for spline interpolants in $\mathcal{S}_n^*$, we find the situation to be reversed, the inequalities between the eigenvalues μ_ν being as follows:

$$\text{if } n = 2m-1: \mu_{2m-1} < \cdots < \mu_{m+1} < \mu_m = -1 < \mu_{m-1} < \cdots < \mu_1 < 0; \tag{4.6}$$

$$\text{if } n = 2m: \mu_{2m} < \cdots < \mu_{m+1} < -1 < \mu_m < \cdots < \mu_1 < 0. \tag{4.7}$$

If $s(x) = S(x) - \tilde{S}(x)$ is the difference of two solutions of (1.2) satisfying (2.6), then $s(x) \in \mathring{\mathcal{S}}_n^*$ and Lemma 3 shows that $s(x)$ is a linear combination of the eigenspline $S_\nu^*(x)$, $\nu = 1, \cdots, n$. In the case (4.7) we obtain unicity, i.e., $s(x) = 0$ for all x. However, in the case (4.6) we can only conclude that $s(x) = c_m S_m^*(x)$ and unicity fails to hold.

We have not only established the uniqueness part of Theorem 1, but have also discovered the reason for its breakdown if n has the wrong parity. The two exponential splines

$$\Phi_{2m}(x; -1), \qquad \Phi_{2m-1}(x + \tfrac{1}{2}; -1), \tag{4.8}$$

which were the reason for the breakdown, are perhaps the two most remarkable cardinal spline functions.

5. Proof of existence in Theorem 1. This will be done by constructing the Lagrange expansion (2.12) of Theorem 3, thereby exhibiting the solution. We first construct the fundamental function $L_n(x)$. For this purpose we prefer the central B-spline $M_{n+1}(x)$, related to $Q_{n+1}(x)$ by

$$M_{n+1}(x) = Q_{n+1}\left(x + \frac{n+1}{2}\right). \tag{5.1}$$

We begin with the case that n is odd, $n = 2m - 1$. The Euler–Frobenius polynomial (1.7) of Lecture 3 may, by (5.1), also be written as

$$\Pi_{2m-1}(t) = (2m-1)!\, t^{m-1} \sum_{-(m-1)}^{m-1} M_{2m}(j) t^j. \tag{5.2}$$

Its $2m-2$ zeros λ_ν satisfy the inequalities (4.3), and it follows that this polynomial is $\neq 0$ in the ring $|\lambda_{m-1}| < |t| < |\lambda_m|$ of the complex t-plane, which ring contains the circumference $|t| = 1$. We may therefore expand its reciprocal in a Laurent series to obtain

$$1 \Big/ \sum_{-(m-1)}^{m-1} M_{2m}(\nu)t^\nu = \sum_{-\infty}^{\infty} \omega_j t^j, \qquad |t| = 1, \tag{5.3}$$

where

$$\omega_j = \omega_{-j} \quad \text{for all } j, \qquad \omega_j = O(|\lambda_{m-1}|^j) \quad \text{as} \quad j \to +\infty. \tag{5.4}$$

We claim that the function

$$L_{2m-1}(x) = \sum_{-\infty}^{\infty} \omega_j M_{2m}(x-j) \tag{5.5}$$

has all the required properties (2.8), (2.9), and (2.10). Since (2.8) is implied by (5.5), while (2.10) is implied by (5.4), there remains to show that

$$L_{2m-1}(\nu) = \sum_{j=-\infty}^{\infty} \omega_j M_{2m}(\nu - j) = \delta_\nu. \tag{5.6}$$

This follows from (5.3) if we observe that the product of the two sums is to be identically $=1$. For the easy proofs of the inequality (2.11) and the convergence of the series, we refer to [2, §§ 4, 4]. This completes an existence proof in Theorem 1 for n odd.

Let us now assume that n is even, $n = 2m$. Expressing the midpoint Euler–Frobenius polynomial (4.8) of Lecture 3 by central B-splines we find that

$$\rho_{2m}(t) = 2^{2m}(2m)!\, t^m \sum_{j=-m}^{m} M_{2m+1}(j)t^j. \tag{5.7}$$

From the inequalities (4.7) satisfied by its $2m$ zeros μ_ν, we conclude that (5.7) does not vanish in the ring $|\mu_m| < |t| < |\mu_{m+1}|$ containing $|t| = 1$, and this establishes an expansion

$$1 \Big/ \sum_{-m}^{m} M_{2m+1}(\nu)t^\nu = \sum_{-\infty}^{\infty} \omega_j^* t^j, \qquad |t| = 1, \tag{5.8}$$

where

$$\omega_j^* = \omega_{-j}^* \quad \text{for all } j, \qquad \omega_j^* = O(|\mu_m|^j) \quad \text{as} \quad j \to +\infty. \tag{5.9}$$

Arguments analogous to those used in the previous case show that

$$L_{2m}(x) = \sum_{-\infty}^{\infty} \omega_j^* M_{2m+1}(x-j) \tag{5.10}$$

is the fundamental function having the properties required by Theorems 2 and 3. This completes an existence proof in Theorem 1 for even n, and thereby Theorems 1, 2, 3, are established.

The function $L_n(x)$ assumes simple forms for the cases $n = 2$ and $n = 3$.

1. $n = 2$. From (4.13) of Lecture 3 we find (5.7) to be $\rho_2(t) = t^2 + 6t + 1$. In terms of its zero $\mu = -3 + 2\sqrt{2} = -.17158$, we find (5.10) to become

$$L_2(x) = \sqrt{2} \sum_{-\infty}^{\infty} \mu^{|j|} M_3(x - j). \tag{5.11}$$

2. $n = 3$. From (1.12) of Lecture 3, we find (5.2) to be $\Pi_3(x) = t^2 + 4t + 1$. In terms of its zero $\lambda = -2 + \sqrt{3} = -.26795$, we find that (5.5) becomes

$$L_3(x) = \sqrt{3} \sum_{-\infty}^{\infty} \lambda^{|j|} M_4(x - j). \tag{5.12}$$

These are the only cases whose solution depends on solving a quadratic equation. However, also $L_5(x)$ depends on quadratics (see [75, p. 1036]). For very accurate values of the zeros of $\Pi_{2m-1}(x)$ for $m = 2, 3, \cdots, 7$, see [78].

6. A few applications.

A. *The spline interpolation of periodic sequences.* Let $y = (y_\nu)$ be a periodic sequence of period k, $k \geqq 2$; hence,

$$y_\nu = y_{\nu+k} \quad \text{for all} \quad \nu. \tag{6.1}$$

Since (y_ν) is a bounded sequence we conclude from Theorem 1 that there is a unique bounded interpolating $S(x)$ in $\mathscr{S}_n$ if n is odd, in $\mathscr{S}_n^*$ if n is even. Let us show that $S(x)$ is periodic of period k. Indeed, by (6.1), $S(\nu + k) = y_{\nu+k} = y_\nu$; hence, $S(\nu + k) = y_\nu$ for all ν. Therefore $S(x + k)$ is also a bounded solution and their unicity shows that $S(x + k) = S(x)$ for all real x.

THEOREM 4. *A periodic sequence (y_ν) of period k admits a unique bounded cardinal spline interpolant $S(x)$ of degree n. Here $S(x) \in \mathscr{S}_n$, or $S(x) \in \mathscr{S}_n^*$, depending on whether n is odd or even. Moreover, $S(x)$ is periodic of period k.*

Observe that this theorem characterizes periodic spline interpolants within the class of cardinal spline interpolants. For periodic data the Lagrange expansion (2.12) assumes a finite form: writing

$$l_n(x) = \sum_{j=-\infty}^{\infty} L_n(x - jk), \tag{6.2}$$

then (2.12) reduces to

$$S(x) = \sum_{0}^{k-1} y_\nu l_n(x - \nu), \tag{6.3}$$

where $l_n(x)$ is the fundamental function (see [67, Part II]). Concerning the norm of the interpolation operator in this case, see [58].

B. *The Euler splines.* The simplest nontrivial periodic sequence is $y_\nu = (-1)^\nu$ for all ν. As a special case of Theorem 4 we obtain the following theorem.

THEOREM 5. *For each natural number n there is a unique bounded function* $\mathscr{E}_n(x)$ *such that*

$$\mathscr{E}_n(\nu) = (-1)^\nu \quad \textit{for all} \quad \nu, \tag{6.4}$$

and such that

$$\mathscr{E}_n(x) \in \begin{cases} \mathscr{S}_n & \textit{if} \quad n \quad \textit{is odd}, \\ \mathscr{S}_n^* & \textit{if} \quad n \quad \textit{is even}. \end{cases} \tag{6.5}$$

These functions are well known (see [52, p. 34]). I call them *Euler splines* because their polynomial components are essentially the classical Euler polynomial appearing in (1.13) of Lecture 3. In the Russian literature they are often called Favard functions, because they are the extremizing functions in Favard's determination of the best constants in Jackson's theorem for periodic functions. We might also call them Kolmogorov functions, because they are extremizing functions in Kolmogorov's general solution of 1939 of a problem proposed and solved by Landau in 1913 for $n = 2$ (see [43], [14], [77]). Some of these problems will be taken up in Lecture 9. Notice that the two functions (4.8) that appeared in the course of the proof of Theorem 1, are Euler splines, up to a constant factor and a shift of origin by 1/2.

C. *The Bernoulli monosplines.* In Theorem 1 (§ 2) we replace n by $n - 1$ and apply it to interpolate the sequence $y_\nu = \nu^n$, i.e., we interpolate the function x^n at the integers. By Theorem 3 (§ 2) the interpolant is the cardinal spline

$$S(x) = \sum_{-\infty}^{\infty} \nu^n L_{n-1}(x - \nu). \tag{6.6}$$

If we define $R(x)$ by

$$R(x) = x^n - S(x), \tag{6.7}$$

then

$$R(\nu) = 0 \quad \text{for all} \quad \nu. \tag{6.8}$$

Moreover, (6.7) shows that

$$R(x) = O(|x|^\gamma) \quad \text{as} \quad x \to \pm\infty, \qquad \gamma \geqq 0, \tag{6.9}$$

where $\gamma = n$ will do.

DEFINITION 1. A function of the form (6.7), where $S(x) \in \mathscr{S}_{n-1}$ or $S(x) \in \mathscr{S}_{n-1}^*$, is called a *cardinal monospline of degree n.*

Let $B_n(x)$ denote the Bernoulli polynomial in [0, 1], and $\bar{B}_n(x)$ its periodic extension of period 1. It is known that $\bar{B}_n(x)$ is a cardinal monospline of degree n. Moreover, $\bar{B}_n(x) - B_n$ and $\bar{B}_n(x + \frac{1}{2})$ are known to vanish at the integers, the first if n is even, the second if n is odd. In terms of the function (6.7) we must therefore have the identities

$$\bar{B}_n(x) - B_n = R(x) \quad \text{if} \quad n \quad \text{is even}, \tag{6.10}$$

$$\bar{B}_n(x + \tfrac{1}{2}) = R(x) \quad \text{if} \quad n \quad \text{is odd}. \tag{6.11}$$

However, we may also state the following.

THEOREM 6. *The functions appearing on the left sides of* (6.10), (6.11), *are uniquely characterized within the class of cardinal monosplines of degree n by the conditions* (6.8) *and* (6.9).

The Bernoulli functions can be *defined* by these conditions and their further properties developed by starting from this definition.

7. The bivariate cardinal splines. How do cardinal splines and the C.I.P. generalize to the plane? We begin with the classes $\mathscr{S}_n$ and $\mathscr{S}_n^*$ of § 1 of Lecture 1, and define

$$\mathscr{S}_{n,n} = \{S(x, y)\} \tag{7.1}$$

as the class of functions $S(x, y)$ having the following two properties: 1. *In each square* $R_{ij} = [i - 1, i) \times [j - 1, j]$,

$$S(x, y) = \sum_{\alpha=0}^{n} \sum_{\beta=0}^{n} a_{\alpha,\beta}^{(i,j)} x^\alpha y^\beta, \tag{7.2}$$

depending on $(n + 1)^2$ *parameters.*

2. *The* n^2 *partial derivatives*

$$\frac{\partial^{\alpha+\beta}}{\partial x^\alpha\, \partial y^\beta} S(x, y), \quad 0 \leqq \alpha \leqq n - 1, 0 \leqq \beta \leqq n - 1, \tag{7.3}$$

are continuous in the entire plane $\mathbb{R}^2$.

We call $S(x, y)$ a *bivariate* cardinal spline of degree n (or bilinear, biquadratic a.s.f). Observe that the sections $S(x, y_0)$ and $S(x_0, y)$ are elements of $\mathscr{S}_n$ in the remaining variable. We also define the class

$$\mathscr{S}_{n,n}^* = \{S(x, y); S(x + \tfrac{1}{2}, y + \tfrac{1}{2}) \in \mathscr{S}_{n,n}\}. \tag{7.4}$$

With these definitions, Theorem 1 of Lecture 2 has the following analogue.

THEOREM 7. *Every element of* $\mathscr{S}_{n,n}$ *admits a unique representation of the form*

$$S(x, n) = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} c_{ij} Q_{n+1}(x - i) Q_{n+1}(y - j) \tag{7.5}$$

and conversely.

Although the right side of (7.5) is so obviously a formal tensor product, Theorem 7 requires a proof (unpublished) which is modeled after § 2 of Lecture 2, but considerably more intricate.

Let (y_{ij}) be a prescribed double-sequence, and let us seek $S(x, y)$ satisfying

$$S(i, j) = y_{ij} \quad \text{for all} \quad (i, j). \tag{7.6}$$

We assume that y_{ij} are of *power-growth.* This means that for some $K > 0$ and $\gamma \geqq 0$,

$$|y_{ij}| < K(|i| + |j| + 1)^\gamma \quad \text{for all} \quad (i, j). \tag{7.7}$$

THEOREM 8. 1. *If n is odd, then there is a unique $S(x, y) \in \mathscr{S}_{n,n}$, satisfying* (7.6), *and*

$$|S(x, y)| < K_1(|x| + |y| + 1)^\gamma \quad \text{for all} \quad (x, y) \in \mathbb{R}^2. \tag{7.8}$$

2. *If n is even, then there is a unique $S(x, y) \in \mathscr{S}^*_{n,n}$ such that* (7.6) *and* (7.8) *hold.*

A proof follows easily from the univariate result (Theorem 1). To show uniqueness, let us assume (7.8) and that $S(i, j) = 0$ for all (i, j), and *let us show that $S(x, y) = 0$ for all (x, y).*

Proof. By our Theorem 1, $S(v, y) = 0$ for all y, and all integers v. But then $S(x; y_0)$ is a cardinal spline in x vanishing for all integers, and therefore $S(x_0, y_0) = 0$ for all (x_0, y_0).

The existence of solutions of the interpolation problem (7.6) is assured by the bivariate Lagrange expansion.

$$S(x, y) = \sum_i \sum_j y_{ij} L_n(x - i) L_n(y - j), \tag{7.9}$$

where $L_n(x)$ is the fundamental function of § 1. Our bivariate case is a simple example of the so very popular "finite element" method. See also the second part of our last Lecture 10.

LECTURE 5

Cardinal Hermite Interpolation

We do not promote Hermite into the highest ranks of the hierarchy, but only wish to add his name to the description of the problem. As one would expect, this means that we are interpolating Hermite data, i.e., also a certain number of *consecutive* derivatives. To preserve translation invariance, we take the same number r of data at each integer. In this case, essentially more difficult, much remains yet to be done, especially concerning the corresponding B-splines. Our presentation is based on the papers [3] and [5], with the contribution from [5] somewhat improved (§ 7). Results of Gantmacher and Krein [91] on eigenvalues of oscillation matrices are used at a decisive point (§ 3). Our developments have been very recently largely generalized and deepened by Karlin and Pinkus in [41].

1. The problem and a preliminary answer. Let r be a natural number and assume that we are prescribed r sequences of data

$$y = (y_\nu), \quad y' = (y'_\nu), \cdots, y^{(r-1)} = (y_\nu^{(r-1)}), \qquad -\infty < \nu < \infty. \tag{1.1}$$

The problem is to find an appropriate spline function $S(x)$ such that

$$S(\nu) = y_\nu, \qquad S'(\nu) = y'_\nu, \cdots, S^{(r-1)}(\nu) = y_\nu^{(r-1)} \qquad \text{for all integers } \nu. \tag{1.2}$$

Let us choose m, $m \geqq r$, and consider for the moment the class $\mathscr{S}_{2m-1}$ of cardinal splines of degree $2m-1$ and class C^{2m-2}. Within this class we could solve the C.I.P. (1.2) of Lecture 4. As we have to satisfy now so many more interpolation conditions, it is clear that we now need a class of functions depending on a correspondingly larger number of parameters. These we provide by relaxing the continuity conditions on its elements. This justifies the following definition.

DEFINITION 1. We denote by

$$\mathscr{S}_{2m-1,r} = \{S(x)\}, \qquad 1 \leqq r \leqq m, \tag{1.3}$$

the class of functions $S(x)$ such that

$$S(x) \in C^{2m-1-r}(\mathbb{R}), \tag{1.4}$$

$$S(x) \in \pi_{2m-1} \qquad \text{in each interval} \quad j, j+1. \tag{1.5}$$

Thus $\mathscr{S}_{2m-1,1} = \mathscr{S}_{2m-1}$ is the old class of Lecture 4. One way of expressing (1.4) is to say that *the integers are knots of $S(x)$ of multiplicity r*. We may now formulate the problem of cardinal Hermite interpolation as follows: *To find*

$$S(x) \in \mathscr{S}_{2m-1,r} \tag{1.6}$$

such as to satisfy the relations (1.2).

The knots are at the integers throughout, and we discuss only splines of the odd degree $2m-1$, often writing $n = 2m-1$. The analogue of Lemma 1 of Lecture 4 is the following lemma.

LEMMA 1. *The interpolation problem* (1.2), (1.6) *has always solutions and the totality of its solutions form a linear manifold in* $\mathscr{S}_{2m-1,r}$ *of dimension*

$$d = 2m - 2r. \tag{1.7}$$

Proof. The representation (1.5) of Lecture 4 is to be now replaced by

$$\begin{aligned} S(x) = P(x) &+ \sum_{s=0}^{r-1} c_1^{(s)}(x-1)_+^{2m-s-1} + \sum_{s=0}^{r-1} c_2^{(s)}(x-2)_+^{2m-s-1} + \cdots \\ &+ \sum_{s=0}^{r-1} c_0^{(s)}(-x)_+^{2m-s-1} + \sum_{s=0}^{r-1} c_{-1}^{(s)}(-x-1)_+^{2m-s-1} + \cdots, \end{aligned} \tag{1.8}$$

to account for the relaxed continuity requirements (1.4). Here $P(x) \in \pi_{2m-1}$. If we select a fixed polynomial $P(x)$ such as to satisfy the conditions

$$P^{(s)}(0) = y_0^{(s)}, \qquad P^{(s)}(1) = y_1^{(s)}, \qquad s = 0, \cdots, r-1, \tag{1.9}$$

then, as in the case of Lecture 4, we see that the coefficients $c_1^{(s)}, c_2^{(s)}, \cdots$ and also $c_0^{(s)}, c_{-1}^{(s)}, \cdots$, are successively and uniquely determined from the remaining interpolation conditions (1.2). Since $P(x)$ satisfying (1.9) still depends on $2m - 2r$ parameters, the lemma is established.

Observe the need for the inequality $r \leqq m$. If $r > m$, we could not even always satisfy (1.9). *If $r = m$, then the interpolation problem becomes trivial*, because it falls apart into an infinite succession of 2-point Hermite interpolation problems each having a unique solution. The resulting interpolant $S(x)$ is a spline of degree $2m-1$ and class C^{m-1}. Some authors call them *Hermite splines.* If $r = m = 1$ we have the familiar case of linear spline interpolation.

2. The main results. Let $\gamma \geqq 0$. We recall the class of sequences

$$\mathscr{Y}_\gamma = \{(y_\nu); y_\nu = O(|\nu|^\gamma) \text{ as } \nu \to \pm\infty\}. \tag{2.1}$$

Let $S(x) \in \mathscr{S}_{2m-1,r}$. An easy application of Markov's theorem on bounds for the derivatives of a bounded polynomial (see [59, Chap. 1, § 1.2], or [48]) will show that if $S(x) = O(|x|^\gamma)$ as $x \to \pm\infty$, then also $S^{(s)}(x) = O(|x|^\gamma)$ as $x \to \pm\infty$, and this for $r = 1, 2, \cdots, 2m-1$. Accordingly we consider the class

$$\mathscr{S}_{2m-1,r,\gamma} = \{S(x); S(x) \in \mathscr{S}_{2m-1,r}, S(x) = O(|x|^\gamma) \text{ as } x \to \pm\infty\}. \tag{2.2}$$

Our main result is the following theorem.

THEOREM 1 (Lipow–Schoenberg). *If*

$$(y_\nu^{(s)}) \in \mathscr{Y}_\gamma, \qquad s = 0, 1, \cdots, r-1, \tag{2.3}$$

then the interpolation problem

$$S^{(s)}(y_\nu) = y_\nu^{(s)}, s = 0, \cdots, r-1, \quad y = 0, \pm 1, \cdots, \tag{2.4}$$

has a unique solution $S(x)$ such that

$$S(x) \in \mathscr{S}_{2m-1,r,\gamma}. \tag{2.5}$$

We see that the relations (2.4) induce a $1-1$ correspondence between the product space $\mathscr{Y}_\gamma \times \mathscr{Y}_\gamma \times \cdots \times \mathscr{Y}_\gamma$ (of r factors) and $\mathscr{S}_{2m-1,r,\gamma}$. The case $\gamma = 0$ corresponds to *bounded* sequences and splines. This correspondence for $\gamma = 0$ can be refined to a correspondence between sequences in l_p and splines in $L_p(\mathbb{R})$, for $0 \leqq p \leqq \infty$ [3, Theorem 2].

Let s be given, $s = 0, \cdots, r-1$. Corresponding to s we define the r sequences of *unit-data*

$$y_\nu^{(\rho)} = \begin{cases} 1 & \text{if} \quad (\nu, \rho) = (0, s), \\ 0 & \text{if} \quad (\nu, \rho) \neq (0, s). \end{cases} \tag{2.6}$$

These r sequences being *bounded*, by Theorem 1, with $\gamma = 0$, we can interpolate them to obtain a solution $S(x) = L_{n,r,s}(x)$ (where $n = 2m - 1$) having the following properties:

$$L_{n,r,s}^{(\rho)}(\nu) = \delta_\nu \cdot \delta_{\rho - s}, \quad \rho = 0, \cdots, r-1, \quad \text{all } \nu, \tag{2.7}$$

and

$$L_{n,r,s}(x) \in \mathscr{S}_{n,r,0}. \tag{2.8}$$

These are the fundamental functions of our interpolation process. That they decay exponentially is asserted by the following.

THEOREM 2. *There are positive constants A and α, depending on m and r, such that*

$$|L_{2m-1,r,s}^{(\rho)}(x)| < A\, e^{-\alpha|x|} \quad \textit{for all real} \quad x. \tag{2.9}$$

To simplify notations we write

$$L_{2m-1,r,s}(x) = L_s(x), \qquad s = 0, \cdots, r-1. \tag{2.10}$$

From the relations (2.7) it is clear that the expression

$$S(x) = \sum_{-\infty}^{\infty} \{y_\nu L_0(x-\nu) + y_\nu' L_1(x-\nu) + \cdots + y_\nu^{(r-1)} L_{r-1}(x-\nu)\} \tag{2.11}$$

furnishes a formal solution of the interpolation problem (2.4). That it actually solves the problem, we state as the next theorem.

THEOREM 3. *The spline function $S(x)$ which is the unique solution of the problem* (2.4), *as described in Theorem* 1, *is explicitly given by the Lagrange–Hermite expansion* (2.11), *which converges absolutely and locally uniformly in x.*

3. The eigensplines of $\mathscr{S}_{2m-1,r}$. The theory of the cardinal Hermite interpolation problem is not as well developed as in the case $r = 1$ of Lecture 4. In the case $r = 1$ we had the B-splines $Q_{n+1}(x-j)$, and the exponential spline $\Phi_{2m-1}(x;t)$ (§ 5 of Lecture 3) easily led to the eigensplines of $\mathscr{S}_{2m-1} = \mathscr{S}_{2m-1,1}$ in § 3 of Lecture 4. In the present case, when $r > 1$, the B-splines are not immediately available. Of course, we might think of the B-splines with multiple knots as described in [23, Part I], but these were not found useful in the present context. As the eigensplines of $\mathscr{S}_{2m-1,r}$ are indispensable, they will be constructed directly, by appealing to a fundamental result of Gantmacher and Krein from total positivity. As soon as the eigensplines are available, the fundamental functions (2.10) can be derived in terms of these.

DEFINITION 2. The *null-space of* $\mathscr{S}_{2m-1,r}$ is defined by

$$\mathring{\mathscr{S}}_{2m-1,r} = \{S(x); S(x) \in \mathscr{S}_{2m-1,r}, \quad S^{(s)}(\nu) = 0, \text{ for } s = 0, \cdots, r-1, \text{ and all integers } \nu\}. \tag{3.1}$$

By Lemma 1 we know that

$$\mathring{\mathscr{S}}_{2m-1,r} \textit{ is a linear space of dimension } d = 2m - 2r. \tag{3.2}$$

DEFINITION 3. Let $S(x) \in \mathring{\mathscr{S}}_{2m-1,r}$. $S(x)$ is called an *eigenspline* of $\mathscr{S}_{2m-1,r}$, provided that it satisfies the functional equation

$$S(x+1) = \lambda S(x), \tag{3.3}$$

for an appropriate constant $\lambda \neq 0$. We call λ the corresponding *eigenvalue.*

Eigensplines have properties induced by the additive group-structure of the integers. The simplest is given in the following lemma.

LEMMA 2. *If $S(x)$ is an eigenspline with eigenvalue λ, then*

$$s(x) = S(-x) \tag{3.4}$$

is also an eigenspline for the eigenvalue λ^{-1}.

Proof. Clearly $s(x) = S(-x) \in \mathring{\mathscr{S}}_{2m-1,r}$. From (3.3), $\lambda s(x+1) = \lambda S(-x-1) = S(-x) = s(x)$ and therefore $s(x+1) = \lambda^{-1}s(x)$.

Let $S(x)$ be an eigenspline, hence satisfying (3.3). Differentiation of (3.3), in view of (1.4), yields the relations

$$S^{(s)}(1) = \lambda S^{(s)}(0), \qquad s = 0, \cdots, 2m - r - 1. \tag{3.5}$$

Let $P(x)$ be the element of π_{2m-1} that represents $S(x)$ in the interval $[0, 1]$. $P(x)$ also satisfies (3.5). However, of these relations we only retain the last $2m - 2r$ relations

$$P^{(s)}(1) = \lambda P^{(s)}(0), \quad s = r, r+1, \cdots, 2m - r - 1, \tag{3.6}$$

while replacing the first r relations (3.5) by the stronger information expressed by

$$P^{(s)}(0) = P^{(s)}(1) = 0, \qquad s = 0, \cdots, r-1, \tag{3.7}$$

and due to the fact that $S(x)$ is a null-spline, hence $S(x) \in \mathring{\mathscr{S}}_{2m-1,r}$. In (3.6), (3.7), we have a total of $(2m - 2r) + 2r = 2m$ homogeneous linear relations which will allow us to determine $P(x)$ up to a multiplicative factor, provided that λ assumes certain appropriate values (the eigenvalues).

By (3.7) we may write

$$P(x) = a_0x^n + \binom{n}{1}a_1x^{n-1} + \cdots + \binom{n}{n-r}a_{n-r}x^r, \quad n = 2m - 1, \tag{3.8}$$

with coefficients yet to be determined. Writing the relations (3.6) in the reverse order of *decreasing* s, followed by the relations $P^{(s)}(1) = 0$, $s = 0, \cdots, r-1$, also written in reverse order, we obtain a system of $2m - r$ linear homogeneous equations in the same number $n - r + 1 = 2m - r$ of unknowns $a_0, a_1, \cdots, a_{n-r}$. The matrix of this system is found to be

$$(3.9)\qquad \Delta_{r,d}(\lambda) = \left\|\begin{matrix} 1 & \binom{r}{1} & \binom{r}{2} & \cdots & \binom{r}{r-1} & 1-\lambda & 0 & \cdots & & 0 \\ 1 & \binom{r+1}{1} & \binom{r+1}{2} & \cdots & \binom{r+1}{r-1} & \binom{r+1}{r} & 1-\lambda & \cdots & & 0 \\ \vdots & & & & & & & & & \vdots \\ 1 & \binom{n-r}{1} & \cdots & & & \binom{n-r}{r} & \cdots & \binom{n-r}{n-r-1} & & 1-\lambda \\ 1 & \binom{n-r+1}{1} & \cdots & & & & & & & \binom{n-r+1}{n-r} \\ \vdots & & & & & & & & & \vdots \\ 1 & \binom{n}{1} & \cdots & & & & & & & \binom{n}{n-r} \end{matrix}\right\|$$

In terms of this matrix we define the polynomial

$$(3.10)\qquad \Pi_{n,r}(\lambda) = (-1)^{m(r-1)}|\Delta_{r,d}(\lambda)|, \qquad n = 2m-1.$$

Observe that the element $1 - \lambda$ appears in the matrix (3.9) a total of $(n - r + 1) - r = 2m - 2r = d$ times. Therefore (3.10) is a polynomial of degree d, in λ, and d is also the dimension of the space $\mathring{\mathscr{S}}_{2m-1,r}$, by (3.2). We shall see that the zeros of (3.10) are precisely the eigenvalues we are looking for. Concerning (3.10) the following theorem holds.

THEOREM 4. *The polynomial*

$$(3.11)\qquad \Pi_{n,r}(x) = \sum_{\nu=0}^{2m-2r} c_{\nu-(m-r)}x^{\nu},$$

of degree $d = 2m - 2r$, *has integer coefficients* c_j *such that*

$$(3.12)\qquad c_0 > 0, \quad c_j = c_{-j}, \quad j = 1, \cdots, m-r, \quad c_{m-r} = c_{-(m-r)} = \pm 1.$$

The polynomial is therefore reciprocal and monic, except for the sign. Moreover, all its zeros are real, simple, and have the sign of $(-1)^r$. *If we label these zeros in the order of increasing values, then*

$$(3.13)\qquad 0 < |\lambda_1| < |\lambda_2| < \cdots < |\lambda_{m-r}| < 1 < |\lambda_{m-r+1}| < \cdots < |\lambda_{2m-2r}|,$$

and

$$(3.14)\qquad \lambda_1\lambda_{2m-2r} = \lambda_2\lambda_{2m-2r-1} = \cdots = 1.$$

Notice, in particular, that due to the reciprocity of the polynomial and simplicity of its zeros, *the value* $(-1)^r$ *is not a zero*, as indicated in (3.13). For $r = 1$, $\Pi_{n,1}(x) = \Pi_n(x)$ is identical with the even degree Euler Frobenius polynomial of Lecture 3.

For this reason we may call $\Pi_{n,r}(x)$ the *Euler Frobenius polynomial for the multiplicity r.*

Explicit examples are the following:

$m = 3, r = 2$:

$$(3.15) \qquad \Pi_{5,2}(x) = -\begin{vmatrix} 1 & 2 & 1-x & 0 \\ 1 & 3 & 3 & 1-x \\ 1 & 4 & 6 & 4 \\ 1 & 5 & 10 & 10 \end{vmatrix} = -x^2 + 6x - 1.$$

$m = 4, r = 2$:

$$(3.16) \qquad \Pi_{7,2}(x) = \begin{Vmatrix} 1 & 2 & 1-x & 0 & 0 & 0 \\ 1 & 3 & 3 & 1-x & 0 & 0 \\ 1 & 4 & 6 & 4 & 1-x & 0 \\ 1 & 5 & 10 & 10 & 5 & 1-x \\ 1 & 6 & 15 & 20 & 15 & 6 \\ 1 & 7 & 21 & 35 & 35 & 21 \end{Vmatrix} = x^4 - 72x^3 + 262x^2 - 72x + 1.$$

For complete proof see [3; §§3, 4], including Greville's proof of the indentity (3.19) below. Here we merely sketch the main lines of the argument which are as follows: In terms of the two $d \times d$ matrices

$$(3.17) \qquad P_{r,d} = \left\Vert \binom{i}{j} \right\Vert, \quad i = r, \cdots, r+d-1, \quad j = 0, \cdots, d-1,$$

$$(3.18) \qquad J_d = \Vert e_{ij} \Vert, \qquad e_{ij} = \begin{cases} 1 & \text{if} \quad i \leqq j, \\ 0 & \text{if} \quad i > j, \end{cases}$$

the following identity is established:

$$(3.19) \qquad |\Delta_{r,d}(\lambda)| = (-1)^{rd} |A_d - \lambda I_d|,$$

where the $d \times d$ matrix A_d is defined by

$$(3.20) \qquad (-1)^r A_d = (J_d)^r P_{r,d}.$$

Using (3.19), we may write (3.10) in the form

$$(3.21) \qquad (-1)^{m(r-1)} \Pi_{n,r}(\lambda) = |(-1)^r A_d - (-1)^r \lambda I_d|.$$

The matrices (3.17) and (3.18) are known to be *totally positive.* From general properties of such matrices it follows from (3.20) that also $(-1)^r A_d$ *is totally positive.* It is shown, moreover, that a certain integral power of this matrix is strictly totally positive. Square matrices M which are totally positive and have a *strictly* totally positive power M^k, are called *oscillatory matrices* by Gantmacher and Krein [91, p. 87]. By a fundamental result of these authors, an oscillatory matrix has only simple and positive eigenvalues [91, Satz 6 on p. 100]. Our matrix $(-1)^r A_d$ being oscillatory, the properties of the zeros of $\Pi_{n,r}(x)$ stated in Theorem 4, follow from the identity (3.21).

It should now be clear how the eigensplines of $\mathring{\mathscr{S}}_{2m-1,r}$ are to be constructed. For each of the $d(=2m-2r)$ eigenvalues λ_ν, we determine $S_\nu(x)$ so as to be identical with the corresponding polynomial (3.8) in $[0,1]$, the extension to all real x being done by the functional equation (3.3), hence

$$S_\nu(x+1) = \lambda_\nu S_\nu(x), \qquad \nu = 1, \cdots, d = 2m-2r. \tag{3.22}$$

From the inequalities (3.13) it follows that the $S_\nu(x)$ are linearly independent in $\mathbb{R}$. As they are all elements of $\mathring{\mathscr{S}}_{2m-1,r}$, by construction, and their number agrees with the dimension of this space by (3.2), we have established the following lemma.

LEMMA 3. *Every element $S(x)$ of $\mathring{\mathscr{S}}_{2m-1,r}$ admits a unique representation*

$$S(x) = \sum_1^d c_j S_j(x), \qquad d = 2m-2r, \tag{3.23}$$

for appropriate constants c_j.

4. A proof of uniqueness in Theorem 1. This follows from by now familiar arguments. The difference $S(x)$ of two solutions of the interpolation problem (2.3), (2.4), (2.5) is such that

$$S(x) \in \mathring{\mathscr{S}}_{2m-1,r} \cap \mathscr{S}_{2m-1,r,\gamma}. \tag{4.1}$$

On the other hand, $S(x)$ is of the form (3.23) by Lemma 3. Since the eigensplines $S_\nu(x)$ have, by (3.13) and (3.22), at infinity oscillatory exponential behavior of different exponential types, we conclude from (4.1) that all coefficients c_j in (3.23) must vanish.

5. The fundamental functions and a proof of existence in Theorem 1. Notice that we have not mentioned any B-splines. We recall that the B-splines allowed a natural construction of the eigensplines and of the fundamental functions (f.f. for short) for the case $r = 1$ of Lecture 4. In fact we shall discuss the B-spline approach for the present problem when $r > 1$ in our next section. However, at the moment the B-splines are as yet unavailable and we construct the f.f. directly as follows.

We are looking for r functions

$$L_s(x) = L_{2m-1,r,s}(x), \qquad s = 0, \cdots, r-1, \tag{5.1}$$

with the following properties:

$$L_s(x) \in \mathscr{S}_{2m-1,r}, \tag{5.2}$$

$$L_s^{(s)}(0) = 1, \quad L_s^{(\rho)}(0) = 0 \quad \text{if} \quad \rho \neq s, \quad \rho = 0, \cdots, r-1, \tag{5.3}$$

$$L_s(\nu) = L_s'(\nu) = \cdots = L_s^{(r-1)}(\nu) = 0 \quad \text{for all} \quad \nu \neq 0, \tag{5.4}$$

and finally also the essential requirement that all

$$L_s(x) \quad \textit{are bounded for all real} \quad x. \tag{5.5}$$

For a detailed account of their construction, see [3, § 7]. Here we give an outline. Let us analyze the properties of $L_s(x)$. By (5.4) we see that its restriction to $[1, +\infty)$ has, at the positive integers, the properties of elements of $\mathring{\mathscr{S}}_{2m-1,r}$ as defined by (3.1). It follows from Lemma 1 that we can extend this restriction to all of $\mathbb{R}$, call this extension $\tilde{L}_s(x)$, so that $\tilde{L}_s(x) \in \mathring{\mathscr{S}}_{2m-1,r}$. Now Lemma 3 becomes applicable and shows that $\tilde{L}_s(x)$ may be expressed as a linear combination of the eigensplines $S_1(x), \cdots, S_d(x)$. However, since $\tilde{L}_s(x) = L_s(x)$, if $x \geqq 1$, we see from (5.5) that $\tilde{L}_s(x)$ must be bounded for $x \geqq 1$. It follows from (3.13) and (3.23) that only those eigenspline may be used that $\to 0$ as $x \to +\infty$. It follows that the representation (3.23) must reduce to

$$L_s(x) = \sum_{1}^{m-r} c_j S_j(x) \quad \text{for} \quad x \geqq 1. \tag{5.6}$$

On the other hand, the already established unicity in Theorem 1 shows easily that $L_s(x)$, *if it exists*, must be an *even* or *odd* function, depending on whether s is an *even* or *odd* number; hence,

$$L_s(-x) = (-1)^s L_s(x), \qquad s = 0, \cdots, r-1. \tag{5.7}$$

It therefore suffices to construct $L_s(x)$ for $x \geqq 0$, consistent with (5.7) and the requirements (5.2) and (5.3), since (5.4) and (5.5) are already taken care of by (5.6).

Let

$$P(x) = L_s(x) \quad \text{in} \quad 0 \leqq x \leqq 1, \tag{5.8}$$

be the polynomial that is to represent L_s in $[0, 1]$. Analyzing the requirements (5.3) we find that $P(x)$ must have the following structure depending on whether r and s have the same parity or not:

1. *If r and s are of the same parity, then*

$$\begin{aligned} P(x) = \frac{1}{s!}x^s + a_1 x^r + a_2 x^{r+2} + \cdots + a_{m-r}x^{2m-r-2} + a_{m-r+1}x^{2m-r} \\ + a_{m-r+2}x^{2m-r+1} + \cdots + a_m x^{2m-1}. \end{aligned} \tag{5.9}$$

2. *If r and s are of opposite parities, then*

$$\begin{aligned} P(x) = \frac{1}{s!}x^s + a_1 x^{r+1} + a_2 x^{r+3} + \cdots + a_{m-r}x^{2m-r-1} + a_{m-r+1}x^{2m-r} \\ + a_{m-r+2}x^{2m-r+1} + \cdots + a_m x^{2m-1}. \end{aligned} \tag{5.10}$$

These expressions already take care of the requirements (5.3). To enforce also (5.2) there remains to join $P(x)$, $0 \leqq x \leqq 1$, to $L_s(x)$, $x \geqq 1$, correctly at $x = 1$. By (5.6), this amounts to the relations

$$P^{(\rho)}(1) = \sum_{j=1}^{m-r} c_j S_j^{(\rho)}(1), \qquad \rho = 0, \cdots, 2m-r-1. \tag{5.11}$$

These are $2m - r$ nonhomogeneous linear equations that are to determine the $m + (m - r) = 2m - r$ unknowns a_i and c_j. See [4, § 7] for the simple argument showing that (5.11) is a nonsingular system. It is based on the fact that the trivial element $S(x) \equiv 0$ is the only element of $\mathring{\mathcal{S}}_{2m-1,r}$ that is bounded on $\mathbb{R}$. Observe that the $L_s(x)$ just obtained have, beyond (5.5), the much stronger property of satisfying the inequality (2.9) of Theorem 2.

The existence part of Theorem 1 is now easily settled by showing that the expansion (2.11) actually furnishes a solution that satisfies all required conditions.

6. The B-splines. B-splines have so far not been mentioned in the genuinely Hermite case ($r > 1$) of our problem. An attempt will now be made in this direction. The direction of attack is no doubt the right one, even if its execution is as yet far from perfect, and some open problems will be formulated later.

We consider the r spaces of *partial null-splines*

$$\mathcal{S}^{(s)}_{2m-1,r} = \{S(x); S(x) \in \mathcal{S}_{2m-1,r}, S^{(\rho)}(\nu) = 0 \text{ for all } \nu, \text{ if } \rho \neq s\}, \qquad s = 0, \cdots, r-1. \tag{6.1}$$

We are already familiar with some elements of these spaces, since the f.f. (5.1) satisfy

$$L_s(x) \in \mathcal{S}^{(s)}_{2m-1,r}, \qquad s = 0, \cdots, r-1. \tag{6.2}$$

We wish to find convenient bases for the spaces (6.1). In this quest we are guided by the experience in § 2 of Lecture 2, and are looking for *elements of* (6.1) *having shortest support*. This aim is only imperfectly attained, as we base our discussion on an arithmetic assumption to be mentioned later.

DEFINITION 4. We define the r elements of $\mathcal{S}_{2m-1,r}$

$$N_0(x), N_1(x), \cdots, N_{r-1}(s), \tag{6.3}$$

by the relations

$$N_s(x) = \sum_{-(m-r)}^{m-r} c_\nu L_s(x - \nu), \qquad s = 0, \cdots, r-1. \tag{6.4}$$

Here the c_ν are the coefficients of the polynomial (3.11), and the $L_s(x)$ are the f.f. (5.1). We call $N_s(x)$ the *B-spline of the space* $\mathcal{S}^{(s)}_{2m-1,r}$.

See [5, § 1] for the easy verification that, if $r = 1$, then $N_0(x) = (2m - 1)!M_{2m}(x)$, and this is indeed the B-spline of $\mathcal{S}_{2m-1,1}$ (Lecture 2).

LEMMA 4. *The B-splines $N_s(x)$ have their supports in the interval*

$$I = (-(m - r + 1), m - r + 1); \tag{6.5}$$

hence,

$$N_s(x) = 0 \quad \textit{if} \quad x \leqq -(m - r + 1) \quad \textit{or if} \quad x \geqq m - r + 1. \tag{6.6}$$

Proof. We know from (5.6) that

$$L_s(x) = \sum_{j=1}^{m-r} a_{j,s} S_j(x) \quad \text{for} \quad x \geqq 1, \tag{6.7}$$

where $a_{j,s}$ are constants and the $S_j(x)$ are the "decreasing" eigensplines. If

$$x \geqq m - r + 1, \tag{6.8}$$

then (6.7) may be used for all terms on the right side of (6.4). Therefore: If (6.8) holds, then

$$\begin{aligned} N_s(x) &= \sum_{-(m-r)}^{m-r} c_\nu L_s(x - \nu) = \sum_\nu c_\nu \sum_{j=1}^{m-r} a_{j,s} S_j(x - \nu) \\ &= \sum_{j=1}^{m-r} a_{j,s} \sum_\nu c_\nu S_j(x - \nu), \end{aligned}$$

and the vanishing of this double-sum will follow as soon as we show that:

$$\sum_{\nu=-(m-r)}^{m-r} c_\nu S_j(x - \nu) = 0 \quad \textit{for all real} \quad x. \tag{6.9}$$

However, by (3.22), $S_j(x - \nu) = \lambda_j^{-\nu} S_j(x)$, and substituting into the left side of (6.9) we obtain

$$\sum_{-(m-r)}^{m-r} c_\nu S_j(x - \nu) = \sum_\nu c_\nu \lambda_j^{-\nu} S_j(x) = S_j(x) \sum_{-(m-r)}^{m-r} c_\nu \lambda_j^{-\nu}.$$

Here the last sum vanishes, because λ_j is a zero of the reciprocal polynomial (3.11). This establishes the second half of the statement (6.6). The proof will be complete as soon as we establish the symmetry relations

$$N_s(-x) = (-1)^s N_s(x), \qquad s = 0, \cdots, r - 1. \tag{6.10}$$

These follow from (5.7) and (3.12):

$$\begin{aligned} N_s(-x) &= \sum_\nu c_\nu L_s(-x - \nu) = (-1)^s \sum_\nu c_\nu L_s(x + \nu) \\ &= (-1)^s \sum_\nu c_{-\nu} L_s(x - \nu) = (-1)^s \sum_\nu c_\nu L_s(x - \nu) = (-1)^s N_s(x). \end{aligned}$$

From their definition (6.4), and in view of (5.3), (5.4), we see that *our new B-splines enjoy the following interpolatory properties*:

$$N_s^{(s)}(\nu) = c_\nu, \qquad N_s^{(\rho)}(\nu) = 0 \quad \textit{if} \quad \rho = 0, \cdots, r - 1, \quad \rho \neq s, \tag{6.11}$$
$$\nu = -(m - r), \cdots, m - r.$$

In fact these hold for all integers ν if we define

$$c_\nu = 0 \quad \text{if} \quad \nu < -(m - r) \quad \text{or if} \quad \nu > m - r. \tag{6.12}$$

7. Properties of B-splines and their construction. We recall that the $L_s(x)$ were constructed in § 5, and that the B-splines were defined in terms of the $L_s(x)$ by (6.4). Our aim is to use them as bases in the respective spaces $\mathscr{S}^{(s)}_{2m-1,r}$, in the same way that $M_{2m}(x)$ was shown in Lecture 2 to furnish a basis for $\mathscr{S}_{2m-1,1}$. Success depends on whether $N_s(x)$ is the element of $\mathscr{S}^{(s)}_{2m-1,r}$ having least support. In trying to decide this point, let us actually construct $N_s(x)$, where s has a fixed value $= 0, \cdots, r-1$.

In view of Lemma 4, it is clear that $N_s(x)$ must be of the form

$$N(x) = \sum_{\nu=-(m-r+1)}^{m-r+1} \sum_{h=1}^{r} b_{\nu,h}(x-\nu)_+^{2m-h} \qquad (b_{\nu,h} \text{ are constants}) \tag{7.1}$$

under further appropriate conditions. By Lemma 4 we require that $N(x) = 0$ if $x > m - r + 1$. This is equivalent to the identity

$$\sum_\nu \sum_h b_{\nu,h}(x-\nu)^{2m-h} = 0 \quad \text{for all real} \quad x, \tag{7.2}$$

where we have removed the subscript "+". Secondly, from $N(x) \in \mathscr{S}^{(s)}_{2m-1,r}$ we obtain the conditions

$$N^{(\rho)}(\nu) = 0 \quad \text{if} \quad \nu = -(m-r), \cdots, m-r, \quad \text{for all} \quad \rho \neq s. \tag{7.3}$$

In (7.1) we have $(2m - 2r + 3)r$ parameters. The conditions (7.2) furnish $2m$ homogeneous equations, and (7.3) give $(2m - 2r + 1)(r - 1)$ additional ones. This gives *a total of* $(2m - 2r + 3)r - 1$ *homogeneous equations for the* $b_{\nu,h}$.

Let R_s *denote the rank of the matrix of the system resulting from* (7.2) *and* (7.3). If

$$R_s = (2m - 2r + 3)r - 1, \tag{7.4}$$

then $N_s(x)$ is uniquely defined up to a constant factor. If

$$R_s < (2m - 2r + 3)r - 1, \tag{7.5}$$

then we have at least two linearly independent solutions.

For computational applications we may use the symmetry relation (6.10) to reduce the linear system to half its size. Thus, if

$$m = 3, \qquad r = 2, \tag{7.6}$$

we find that the alternative (7.4) holds, and the B-splines $N_0(x)$, $N_1(x)$, with support in $(-2, 2)$ are explicitly given by

$$N_0(x) = \begin{cases} 8(1-x)_+^5 - 50(1-x)_+^4 + 4(2-x)_+^5 - 5(2-x)_+^4 & \text{if} \quad x \geqq 0, \\ N_0(-x) & \text{if} \quad x < 0, \end{cases} \tag{7.7}$$

$$N_1(x) = \begin{cases} 10(1-x)_+^5 - 26(1-x)_+^4 + (2-x)_+^5 - (2-x)_+^4 & \text{if} \quad x \geqq 0, \\ -N_1(-x) & \text{if} \quad x < 0. \end{cases} \tag{7.8}$$

Their graphs are sketched in Fig. 1. These graphs illustrate nicely the relations (6.11): The *ordinates* of the first function for $x = -1, 0, 1$, are $-1, 6, -1$, while these are the values of the *slopes* of the second at these points. Notice that $-1, 6, -1$ are the coefficients of the polynomial (3.15).

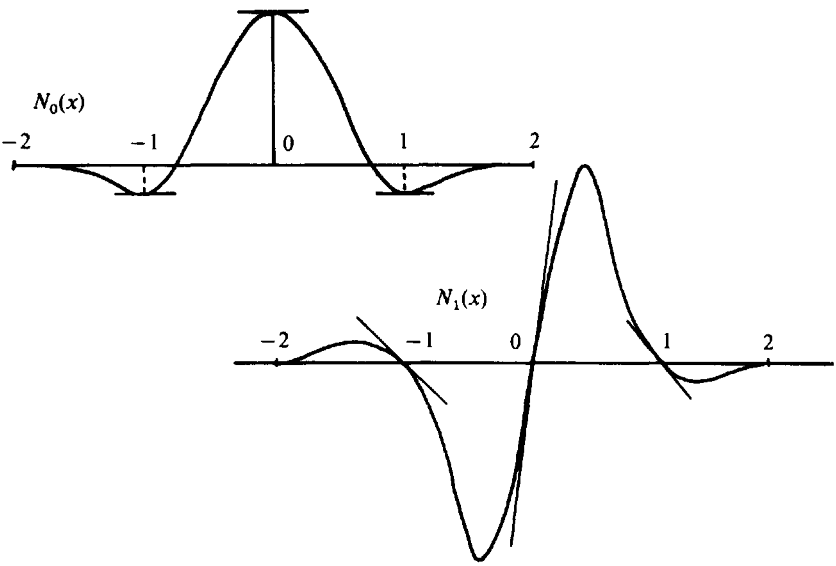

FIG. 1.

The author conjectures that the alternative (7.4) always holds, but he is unable to prove this. To make up for his lack of skill in this matter, he now introduces the following ad hoc assumption.

ASSUMPTION 1. *The integers m and r are such that the Euler–Frobenius polynomial*

$$\Pi_{2m-1,r}(x) = \sum_{\nu=0}^{2m-2r} c_{\nu-(m-r)} x^{\nu} \tag{7.9}$$

is irreducible in the rational field.

This assumption holds for the two polynomials (3.15), (3.16). Also for $\Pi_{7,3}(x)$ ($m = 4$, $r = 3$). Assumption 1 was recently shown, by Sharma and Straus [81], to hold if r is odd and $2m - r$ is a prime number. We conjecture that Assumption 1 always holds. The relevance of Assumption 1 to our problem is shown by the following theorem.

THEOREM 5. *If the integers m and r are such that Assumption* 1 *holds, then, for every* $s = 0, \cdots, r - 1$, *every*

$$S(x) \in \mathscr{S}^{(s)}_{2m-1,r} \tag{7.10}$$

admits a unique representation of the form

$$S(x) = \sum_{-\infty}^{\infty} c_j N_s(x - j). \tag{7.11}$$

Proof. We assume that the polynomial (7.9) is irreducible over the rationals, and claim that:

(7.12) *the alternative* (7.4) *holds.*

This we establish by contradiction by assuming that (7.5) holds. We work now exclusively within the field Q of rationals. It follows from (7.5) that the linear system (7.2), (7.3), has two linearly independent rational solutions. This means that we can find two linearly independent functions $N(x)$ and $N_*(x)$, which are elements of $\mathscr{S}^{(s)}_{2m-1,r}$, and satisfying (7.2) and (7.3). If it so happens that

$$N^{(s)}(-(m-r)) = 0, \tag{7.13}$$

then we do nothing further, and continue our arguments with $N(x)$ only. If (7.13) does not hold, then surely a nontrivial linear combination of $N(x)$ and $N_*(x)$ can be produced, call it again $N(x)$, such that (7.13) does hold. We now consider the family

$$\varphi(x;t) = \sum_{-\infty}^{\infty} t^j N(x-j), \qquad t \neq 0, \tag{7.14}$$

of elements $\mathscr{S}^{(s)}_{2m-1,r}$. It satisfies the functional equation

$$\varphi(x+1;t) = t\varphi(x;t) \tag{7.15}$$

and therefore also

$$\varphi^{(s)}(x+1;t) = t\varphi^{(s)}(x;t). \tag{7.16}$$

If $t = t_0$ *is such that* $t_0 \neq 0$ and

$$\varphi^{(s)}(0;t_0) = 0, \tag{7.17}$$

then also $\varphi^{(s)}(v;t_0) = 0$ for all v, by (7.16), so that

$$\varphi(x;t_0) \in \mathring{\mathscr{S}}_{2m-1,r}. \tag{7.18}$$

By (7.14) the equation (7.17) becomes

$$0 = \sum_{-\infty}^{\infty} t_0^j N^{(s)}(-j) = \sum_j N^{(s)}(m-r-j)t_0^{r-m+j}$$

or

$$\sum_{j=0}^{2m-2r} N^{(s)}(m-r-j)t_0^j = 0. \tag{7.19}$$

However, by our choice of $N(x)$ we know that (7.13) holds and therefore (7.19) reduces to

$$\sum_{j=0}^{2m-2r-1} N^{(s)}(m-r-j)t_0^j = 0. \tag{7.20}$$

Moreover, observe that: 1°. The coefficients of the left-side polynomial are rational, as we worked in Q. 2°. The left side is a nontrivial polynomial, i.e., it has *some non-vanishing coefficients*. For if they were it would follow that $N(x) \in \mathring{\mathscr{S}}_{2m-1,r}$. However, since $N(x)$ is surely bounded, we would conclude that $N(x) \equiv 0$, which contradicts our assumption.

If t_0 is a nonvanishing root of (7.20), then $\varphi(x;t_0)$ is an eigenspline of $\mathscr{S}_{2m-1,r}$. However, all eigenspline solutions of $S(x+1) = tS(x)$ have been determined and found to correspond to the zeros $\lambda_1, \cdots, \lambda_{2m-2r}$ of $\Pi_{2m-1,r}(x)$. We conclude that the polynomial on the left side of (7.20) has a zero t_0 in common with $\Pi_{2m-1,r}(x)$, in contradiction with our Assumption 1.

We are yet to show that (7.17), i.e., the equation (7.19) has a nonvanishing root t_0. If this were not the case, then all $N^{(s)}(\nu)$ would vanish except one, for $\nu = \alpha$ say, and we may as well assume that $N^{(s)}(\alpha) = 1$. However, this is impossible, because it implies that $N(x)$ satisfies all the conditions imposed on the f.f. $L_s(x - \alpha)$. From the unicity of the latter we obtain that $N(x) = L_s(x - \alpha)$, for all x, which again is impossible because $N(x)$ vanishes for large x.

The remainder of the proof of Theorem 5 may be omitted. Indeed, it now follows easily that $I = (-(m - r + 1), m - r + 1)$ is the least support of nontrivial elements of $\mathscr{S}^{(s)}_{2m-1,r}$, and then a proof of the possibility and uniqueness of the representation (7.11) can be carried out along the lines of the proof of Theorem 1 of Lecture 2, § 2.

Remark 1. Theorem 5 shows that if Assumption 1 is satisfied, we get functions $N_s(x)$ which really deserve the name of B-splines. We have already exhibited the B-splines (7.7), (7.8) for the case when $m = 3$ and $r = 2$. Their role in the *finite* quintic Hermite interpolation problem will be the subject of § 5 of Lecture 7.

In terms of the $N_s(x)$ we obtain the f.f. $L_s(x)$ of Theorems 2 and 3 as follows: If we expand in a Laurent series the reciprocal of $\Pi_{2m-1,r}(x)$ by

$$1 \Big/ \sum_{-(m-r)}^{m-r} c_\nu x^\nu = \sum_{-\infty}^{\infty} \omega_j x^j, \qquad |x| = 1, \tag{7.21}$$

then we can invert the convolution (6.4) to yield

$$L_s(x) = \sum_{-\infty}^{\infty} \omega_j N_s(x - j), \qquad s = 0, \cdots, r - 1. \tag{7.22}$$

For the quintic case $m = 3$, $r = 2$, we found that $\Pi_{5,2}(x) = -x^2 + 6x - 1$ and therefore

$$1/(-x + 6 - x^{-1}) = \frac{1}{4\sqrt{2}} \sum_{-\infty}^{\infty} \lambda^{|j|} x^j, \qquad |x| = 1, \tag{7.23}$$

where $\lambda = 3 - 2\sqrt{2} = 0.17158$. It follows that

$$L_s(x) = \frac{1}{4\sqrt{2}} \sum_{-\infty}^{\infty} \lambda^{|j|} N_s(x - j), \qquad s = 0,1, \tag{7.24}$$

where $N_0(x)$ and $N_1(x)$ are the B-splines (7.7) and (7.8).

Remark 2. The developments of this lecture will leave the reader, as they leave the author, with the feeling that the last word on this subject has not been said. Assumption 1 concerns too deep an arithmetic problem in comparison with the linear algebra nature of the interpolation problem. A general proof of Theorem 5, freed of the Assumption 1, could be given, if a certain determinant could be shown to be different from zero. A yet more natural approach would be to start describing the B-splines $N_s(x)$ directly, as a first step, and derive afterwards the remainder of the theory, including the Euler–Frobenius polynomials, the function $L_s(x)$ and Theorem 1.

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LECTURE 6

Other Spaces and Semi-Cardinal Interpolation

In this lecture we deal with three subjects. In § 1 we investigate the C.I.P. by spline functions belonging to classes that are much narrower than the classes of splines of power growth of Lecture 4: We assume them to have a square integrable mth derivative. In § 2 we deal with semi-cardinal interpolation, i.e., interpolation at the nonnegative integers only, for data of power growth. Finally, in § 3 we study again semi-cardinal interpolation, but for the same spaces as in § 1. This last problem depends on a special discrete analogue of the Wiener–Hopf problem.

1. Cardinal interpolation in L_2^m and l_2^m. We return to the C.I.P. of Lecture 4. Following a suggestion from the optimal property of so-called *natural* spline interpolation (see, for example, [34, § 7], or [59, § 4.4.2] for the case of cubic splines) we investigate the possibility of finding a solution of the C.I.P.

$$f(\nu) = y_\nu \quad \text{for all integer} \quad \nu, \tag{1.1}$$

with the property of minimizing the integral

$$\int_{-\infty}^{\infty} |f^{(m)}(x)|^2 \, dx \tag{1.2}$$

among all other possible solutions of (1.1). Here m is a preassigned natural number and we must assume that $f(x)$ belongs to the class

$$L_2^m = \{f(x); f, \cdots, f^{(m-1)} \text{ are abs. cont.}, f^{(m)}(x) \in L_2(\mathbb{R})\}. \tag{1.3}$$

This class is evidently identical with the class of m-fold repeated integrals of functions in $L_2(\mathbb{R})$.

Let us find out what conditions on the sequence (y_ν) are required for the existence of such solutions $f(x)$ of (1.1). Let $f(x) \in L_2^m$ and denote the integral (1.2) by $(\|f^{(m)}\|_2)^2$, as usual. From the relation (1.3) of Lecture 2 we obtain

$$\Delta^m f(j) = \int_j^{j+m} f^{(m)}(x) Q_m(x-j)\, dx, \tag{1.4}$$

whence

$$|\Delta^m f(j)|^2 \leqq \int_j^{j+m} |f^{(m)}(x)|^2 \, dx \cdot \int_j^{j+m} (Q_m(x-j))^2 \, dx = c \int_j^{j+m} |f^{(m)}(x)|^2 \, dx,$$

where $c = \int Q_m^2(x)\,dx$. Summing the extreme terms over all integers j we obtain

$$\sum_{-\infty}^{\infty} |\Delta^m f(j)|^2 \leqq mc\|f^{(m)}\|_2^2.$$

This relation shows that if (1.1) is to have any solutions whatever in L_2^m, the sequence (y_ν) must satisfy the necessary condition that

$$\sum_{-\infty}^{\infty} |\Delta^m y_\nu|^2 < \infty. \tag{1.5}$$

This suggests defining the class of sequences

$$l_2^m = \{(y_\nu);\ \text{the series (1.5) converges}\}. \tag{1.6}$$

The following theorem will not only show that the necessary condition (1.5) is also sufficient for the existence of interpolants $f(x) \in L_2^m$, but will also identify the unique interpolant $f(x)$ that minimizes the integral (1.2).

THEOREM 1. *Let*

$$(y_\nu) \in l_2^m. \tag{1.7}$$

There is a unique spline function $S(x)$ *such that*

$$S(x) \in \mathscr{S}_{2m-1} \cap L_2^m \tag{1.8}$$

and satisfying

$$S(\nu) = y_\nu \quad \textit{for all} \quad \nu. \tag{1.9}$$

2. *This solution* $S(x)$ *has the property that*

$$\|f^{(m)}\|_2 > \|S^{(m)}\|_2 \tag{1.10}$$

for any $f(x) \in L_2^m$ *that satisfies* (1.1), *unless* $f(x) = S(x)$ *for all real* x.

Proof of 1. The uniqueness of the solution $S(x)$ of (1.7), (1.8), is easily obtained by means of the eigensplines of $\mathscr{S}_{2m-1}$ (§§ 3, 4 of Lecture 4). From § 3 of Lecture 4 we know that the eigensplines

$$S_1(x), S_2(x), \cdots, S_{2m-2}(x) \tag{1.11}$$

of $\mathscr{S}_{2m-1}$ form a basis for the space $\mathring{\mathscr{S}}_{2m-1}$ of null-splines $S(x)$ satisfying the homogeneous system $S(\nu) = 0$ for all integers ν. This implies that the *difference* $s(x)$ of any *two* solutions of (1.8), (1.9), is on the one hand representable as

$$s(x) = \sum_{1}^{2m-2} a_\nu S_\nu(x), \tag{1.12}$$

while on the other, clearly

$$s^{(m)}(x) \in L_2(\mathbb{R}). \tag{1.13}$$

It should be equally clear that both relations (1.12) and (1.13) together imply that $s(x) = 0$ for all real x. This settles the uniqueness.

On the question of existence we would be tempted to argue as follows. Surely (1.7), i.e., (1.5), implies that $y_\nu = O(|\nu|^\gamma)$ as $\nu \to \pm\infty$, for some exponent $\gamma \geqq 0$. By Theorems 1 and 3 of Lecture 4 we conclude that $S(x) = \sum y_\nu L_{2m-1}(x-\nu)$ is a solution of (1.9). However, it would now be awkward to show that the spline $S(x)$ so obtained belongs to L_2^m. For this reason we proceed quite differently.

Observe that (1.9) implies the system of relations

$$\Delta^m S(j) = \Delta^m y_j \quad \text{for all} \quad j, \tag{1.14}$$

where, by (1.4),

$$\Delta^m S(j) = \int_{-\infty}^{\infty} S^{(m)}(x) Q_m(x-j)\,dx. \tag{1.15}$$

On the other hand $S^{(m)}(x)$, being in $\mathscr{S}_{m-1}$, admits a representation

$$S^{(m)}(x) = \sum_{-\infty}^{\infty} c_k Q_m(x-k). \tag{1.16}$$

We also know from Theorem 3 of Lecture 2 (§ 3) that

$$S^{(m)}(x) \in L_2(\mathbb{R}) \tag{1.17}$$

if and only if

$$(c_k) \in l_2. \tag{1.18}$$

Finally, observe that if we substitute (1.16) into (1.15) and use the relation

$$\int_{-\infty}^{\infty} Q_m(x-j) Q_m(x-k)\,dx = M_{2m}(j-k), \tag{1.19}$$

then (1.14) becomes

$$\sum_k M_{2m}(j-k) c_k = \Delta^m y_j \quad \text{for all} \quad j. \tag{1.20}$$

This is a convolution transformation $(c_k) \to (\Delta^m y_j)$ which is a bounded linear transformation of l_2 into l_2 (see [1, § 5]). From (1.7) we conclude that (1.20) has a unique solution $(c_k) \in l_2$. Moreover, (1.20) has an inverse giving this solution explicitly. From the expansion (5.3) of Lecture 4 we find the inverse of (1.20) to be

$$c_k = \sum_j \omega_{k-j} \Delta^m y_j. \tag{1.21}$$

Notice again the role of the zeros (4.3) of Lecture 4, of the polynomial $\Pi_{2m-1}(x)$ and especially of the fact that none of these is on $|t| = 1$.

Having found the (c_k) satisfying (1.18) and (1.20), we obtain $S^{(m)}(x)$ by (1.16), and call $S_*(x)$ any m-fold integral of $S^{(m)}(x)$. This $S_*(x)$ satisfies (1.14); hence, $\Delta^m(S_*(j) - y_j) = 0$ for all j. Therefore, there exists a $P(x) \in \pi_{m-1}$ such that $S_*(j) + P(j) = y_j$ for all j. But then we have the spline function

$$S(x) = S_*(x) + P(x) \in \mathscr{S}_{2m-1} \cap L_2^m$$

satisfying all the conditions of the first part of Theorem 1.

Proof of 2. Let $S(x)$ be the interpolant (1.8), let $f(x)$ be any interpolant in L_2^m, and *let us establish the inequality* (1.10). We start from the identity

$$\int_{-\infty}^{\infty} |f^{(m)} - S^{(m)}|^2\,dx = \int_{-\infty}^{\infty} |f^{(m)}|^2\,dx + \int_{-\infty}^{\infty} S^{(m)}\bar{S}^{(m)}\,dx - 2\,\mathrm{Re}\int_{-\infty}^{\infty} S^{(m)}\bar{f}^{(m)}\,dx$$

which we may also write as

$$\int_{-\infty}^{\infty} |f^{(m)} - S^{(m)}|^2\,dx = \int_{-\infty}^{\infty} |f^{(m)}|^2\,dx - \int_{-\infty}^{\infty} |S^{(m)}|^2\,dx + 2\,\mathrm{Re}\int_{-\infty}^{\infty} S^{(m)}(\bar{S}^{(m)} - \bar{f}^{(m)})\,dx, \tag{1.22}$$

and wish to show that the last integral vanishes. Writing $R(x) = S(x) - f(x)$, (1.16) implies

$$\int_{-\infty}^{\infty} \bar{R}^{(m)}(x)S^{(m)}(x)\,dx = \int_{-\infty}^{\infty} \bar{R}^{(m)}(x)\left\{\sum_j c_j Q_m(x-j)\right\}dx, \tag{1.23}$$

while

$$\int_{-\infty}^{\infty} \bar{R}^{(m)}(x)\left\{\sum_{-n}^{n} c_j Q_m(x-j)\right\}dx = \sum_{-n}^{n} c_j \int_{-\infty}^{\infty} \bar{R}^{(m)} Q_m(x-j)\,dx. \tag{1.24}$$

We claim that *the left side of* (1.24) *converges to the right side of* (1.23) *as* $n \to \infty$. Indeed, the integrand on the left of (1.24) is dominated by the function

$$|\bar{R}^{(m)}(x)| \cdot \sum_{-\infty}^{\infty} |c_j| Q_m(x-j), \tag{1.25}$$

independent of n, and this function is in $L_1(\mathbb{R})$ for the following reason: From $\sum |c_j|^2 < \infty$ and Theorem 3 of Lecture 2, we conclude that the second factor of (1.25) is in $L_2(\mathbb{R})$. Since $\bar{R}^{(m)} \in L_2$, we see that the function (1.25) is summable by Schwarz's inequality. This establishes our italicized statement by the bounded convergence theorem. On the other hand, observe that

$$\int \bar{R}^{(m)}(x) Q_m(x-j)\,dx = \overline{\Delta^m R(j)} = 0 \quad \text{for all} \quad j,$$

because $R(x)$ vanishes at all integers. It follows that the left side of (1.24) vanishes, for all n, and therefore also it limits (1.23). Thus (1.22) reduces to

$$\int_{-\infty}^{\infty} |f^{(m)}|^2\,dx = \int_{-\infty}^{\infty} |S^{(m)}|^2\,dx + \int_{-\infty}^{\infty} |f^{(m)} - S^{(m)}|^2\,dx, \tag{1.26}$$

which evidently implies the desired conclusion.

Remark 1. The first part of Theorem 1 extends from L_2 and l_2, to L_p and l_p, respectively, for any p such that $1 \leqq p < \infty$.

Remark 2. A wide generalization of the problem of cardinal interpolation by functions in L_2^m was given by M. Golomb and the author in [32]. The problem considered is as follows. Suppose that $f(x)$ is defined on a closed set $A \subset \mathbb{R}$.

Under what conditions can $f(x)$ be extended to the entire $\mathbb{R}$ so as to result in a function $F(x) \in L_2^m$? And if this can be done, what are the extensions minimizing $\int |F^{(m)}|^2\, dx$ over $\mathbb{R}$? This paper is as yet unpublished, but an account of the main results can be found in [70].

2. Semi-cardinal interpolation of data of power growth. The results of Lecture 4 have analogues for the problem of finding $S(x)$ such that

$$S(\nu) = y_\nu, \qquad \nu = 0, 1, \cdots. \tag{2.1}$$

We call this the problem of *semi-cardinal interpolation*. Throughout this section $n = 2m - 1$ denotes a positive odd integer.

DEFINITION 1. We denote by

$$\mathscr{S}_n^+ = \{S(x)\}, \qquad n = 2m - 1, \tag{2.2}$$

the class of functions $S(x)$ such that

1. $S(x) \in C^{2m-2}(\mathbb{R})$,
2. $S(x) \in \pi_{2m-1}$ in $[\nu, \nu + 1]$, $\nu = 0, 1, \cdots$,
3. $S(x) \in \pi_{m-1}$ in $(-\infty, 0]$.

Such functions may be called *natural semi-cardinal splines* of degree $2m - 1$. Observe that $\pi_{m-1} \subset \mathscr{S}_n^+$. It will at times be convenient to consider the elements of $\mathscr{S}_n^+$ as defined on $\mathbb{R}^+ = [0, \infty)$ only, and we denote the class of these restrictions by $\mathscr{S}_n^+(\mathbb{R}^+)$. They may evidently be defined independently as the class of functions $S(x)$ satisfying condition 2 above, while replacing conditions 1 and 3 by:

1′. $S(x) \in C^{2m-2}(\mathbb{R}^+)$, and

3′. $S^{(m)}(0) = S^{(m+1)}(0) = \cdots = S^{(2m-2)}(0) = 0$,

respectively. Our main result is as follows.

THEOREM 2. *If the sequence*

$$y^+ = (y_\nu), \qquad \nu = 0, 1, \cdots, \tag{2.3}$$

is prescribed, such that

$$y_\nu = O(\nu^\gamma) \quad as \quad \nu \to \infty \quad for\ some \quad \gamma \geqq 0, \tag{2.4}$$

then the interpolation problem (2.1) *has a unique solution $S(x)$ such that*

$$S(x) \in \mathscr{S}_{2m-1}^+ \quad and \quad S(x) = O(x^\gamma) \quad as \quad x \to \infty. \tag{2.5}$$

For each nonnegative j, the sequence $\delta_{\nu,j}$, $\nu = 0, 1, \cdots$, satisfies (2.4) with $\gamma = 0$. By Theorem 2 we conclude the existence of uniquely defined *bounded* functions $L_j(x)$, $j = 0, 1, \cdots$, such that

$$L_j(x) \in \mathscr{S}_{2m-1}^+ \quad \text{for all} \quad j, \tag{2.6}$$

and

$$L_j(\nu) = \delta_{j,\nu}, \qquad j, \nu = 0, 1, \cdots. \tag{2.7}$$

These are the *fundamental functions* (f.f.) of our interpolation process.

THEOREM 3. *For appropriate constants A_m and α_m, independent of j, the inequality*

$$|L_j(x)| < A_m e^{-\alpha_m|x-j|} \quad \textit{holds for} \quad x \in \mathbb{R}^+, \qquad j = 0, 1, \cdots. \tag{2.8}$$

THEOREM 4. *In Theorem 2 the unique solution of* (2.1), *satisfying* (2.6), *may be represented by the Lagrange expansion*

$$S(x) = \sum_0^\infty y_\nu L_\nu(x), \tag{2.9}$$

which converges locally uniformly and absolutely.

For an arbitrary function $f(x)$ we may write

$$f(x) = \sum_0^\infty f(\nu) L_\nu(x) + R_f(x), \tag{2.10}$$

and call this the *natural semi-cardinal spline interpolation formula of degree* $2m - 1$.

All these results are trivially true if $m = 1$ and we interpolate by linear splines.

Let now $m \geqq 2$; hence, $n = 2m - 1 \geqq 3$. As in the cardinal case, also here we begin by observing the following: *Within the class* $\mathcal{S}_n^+$ *the problem* (2.1) *has infinitely many solutions* $S(x)$ *forming a linear manifold in* $\mathcal{S}_n^+$ *of dimensions*

$$d^+ = m - 1. \tag{2.11}$$

This is almost self-evident if we realize that every element of $\mathcal{S}_n^+$ has a unique representation

$$S(x) = P(x) + \sum_0^\infty c_\nu (x - \nu)_+^{2m-1}, \quad \text{where} \quad P(x) \in \pi_{m-1}. \tag{2.12}$$

If we choose $P(x)$ arbitrarily such that $P(0) = y_0$, then the c_ν are successively and uniquely determined from (2.1) for $\nu = 1, 2, \cdots$. Since $P(x)$ still depends on $m - 1$ parameters, the statement (2.11) is established.

It follows, in particular, that *the null-space*

$$\mathring{\mathcal{S}}_n^+ = \{S(x); S(x) \in \mathcal{S}_n^+, S(\nu) = 0, \nu = 0, 1, \cdots\} \tag{2.13}$$

is a linear subspace of $\mathcal{S}_n^+$ of dimension $d^+ = m - 1$.

Proof of uniqueness in Theorem 2. The difference $S(x)$ of two solutions of (2.1), such that (2.6) holds, evidently has the properties

$$S(x) \in \mathring{\mathcal{S}}_n^+, \quad S(x) = O(x^\gamma) \quad \text{as} \quad x \to \infty. \tag{2.14}$$

Let us show that $S(x) = 0$ *for all real* x. For this purpose we first modify $S(x)$ for $x < 0$ by defining

$$\tilde{S}(x) = S(x) + \sum_{\nu=0}^\infty c_\nu(-x - \nu)_+^{2m-1}, \qquad x \in \mathbb{R}. \tag{2.15}$$

Here we determine successively the c_ν such that $\tilde{S}(-1) = 0, \tilde{S}(-2) = 0, \cdots$. This done it is clear that $\tilde{S}(x) \in \mathring{\mathcal{S}}_{2m-1}$. At this point we may apply the result of

§ 3 of Lecture 4, to conclude that $\tilde{S}(x)$ is a linear combination of the $2m - 2$ eigensplines of $\mathcal{S}_{2m-1}$. However, the second relation (2.14) shows that in this representation only the "decreasing" eigensplines may occur, so that $\tilde{S}(x) = \sum_1^{m-1} a_j S_j(x)$ for all x, *and therefore by* (2.15), *that*

$$S(x) = \sum_1^{m-1} a_j S_j(x) \quad \textit{if} \quad x \geqq 0. \tag{2.16}$$

This implies that

$$\lim_{x \to +\infty} S^{(r)}(x) = 0 \quad \text{if} \quad r = 0, 1, \cdots, 2m - 2. \tag{2.17}$$

On the other hand, the first relation (2.14) shows that

$$S^{(r)}(0) = 0 \quad \text{if} \quad r = m, m + 1, \cdots, 2m - 2. \tag{2.18}$$

If we consider the integral

$$J = \int_0^\infty (S^{(m)}(x))^2 \, dx \tag{2.19}$$

and apply successive integrations by parts, we obtain from (2.17) and (2.18), that

$$J = \int_0^\infty S^{(m)} S^{(m)} \, dx = -\int_0^\infty S^{(m+1)} S^{(m-1)} \, dx = \cdots = \pm \int_0^\infty S^{(2m-1)} S' \, dx;$$

hence,

$$J = \pm \sum_{i=0}^{\infty} \int_i^{i+1} S^{(2m-1)}(x) S'(x) \, dx. \tag{2.20}$$

Here $S^{(2m-1)}(x)$ is a step-function, while $\int_i^{i+1} S'(x)\,dx = S(i + 1) - S(i) = 0$ for all i. It follows that $J = 0$ and this easily shows that $S(x) = 0$ for $x \geqq 0$.

To prepare a proof of existence we first establish the following lemma.

LEMMA 1. *For the "decreasing" eigensplines*

$$S_1(x), S_2(x), \cdots, S_{m-1}(x) \tag{2.21}$$

of $\mathcal{S}_{2m-1}$, *the determinant*

$$\Delta_m = \det \|S_{j+1}^{(m+i)}(0)\| \neq 0, \quad \textit{where} \quad i, j = 0, 1, \cdots, m - 2. \tag{2.22}$$

Proof. Let us assume that $\Delta_m = 0$ and reach a contradiction. $\Delta_m = 0$ implies the existence of a *nontrivial*

$$S(x) = \sum_1^{m-1} c_\nu S_\nu(x), \tag{2.23}$$

satisfying

$$S^{(m)}(0) = S^{(m+1)}(0) = \cdots = S^{(2m-2)}(0) = 0. \tag{2.24}$$

This shows that the restriction of $S(x)$ to $[0, \infty)$ satisfies $S(x) \in \mathcal{S}_n^+(\mathbb{R}^+)$, while (2.23) shows that $S(x)$ tends to zero exponentially, $x \to \infty$, together with all its derivatives up to $S^{(2m-2)}(x)$. But then we can again show, as for (2.19), that the integral of $(S^{(m)}(x))^2$ over $\mathbb{R}^+$, vanishes, hence $S(x) = 0$ for all x. This conclusion contradicts the assumption that (2.23) was nontrivial.

Proof of existence in Theorem 2. This is done by a construction of the f.f. $L_\nu(x)$ satisfying (2.6), (2.7), and here the eigensplines again come into their own. Let

$$\mathscr{L}_n(x), \quad n = 2m - 1, \tag{2.25}$$

denote the f.f. of $\mathscr{S}_{2m-1}$ having the properties (2.9) and (2.11) of Lecture 4. Let us construct the f.f. $L_\nu(x)$ such that

$$L_\nu(x) \in \mathscr{S}_n^+ \cap L_1(\mathbb{R}^+), \qquad L_\nu(j) = \delta_{\nu,j}, \qquad j = 0, 1, \cdots. \tag{2.26}$$

Observe that the restriction of $\mathscr{L}_n(x - \nu)$ to $\mathbb{R}^+$ already satisfies the *second* relations (2.26), but not necessarily the *first*. For this reason we set tentatively

$$L_\nu(x) = \mathscr{L}_n(x - \nu) + \sum_1^{m-1} c_j S_j(x) \quad \text{for} \quad x \geqq 0, \tag{2.27}$$

and try to determine the c_j so that we also have

$$L_\nu^{(\rho)}(0) = 0 \quad \text{for} \quad \rho = m, m + 1, \cdots, 2m - 2.$$

Using (2.27) we get for the c_j a linear system of equations which is nonsingular by Lemma 1.

We may omit the easy proofs of Theorems 3 and 4 (see [2, § 5]) and take it for granted that the expansion (2.9) exhibits the desired solution.

3. Semi-cardinal interpolation in L_2^m and l_2^m. In this section we deal with the semi-cardinal analogues of the problems and results of § 1. Also here we consider for simplicity only the case when $p = 2$, although most of the results extend to L_p^m and l_p^m, $1 \leqq p \leqq \infty$. The analogues of the spaces (1.3) and (1.6) are

$$L_{2,+}^m = \{f(x); f, \cdots, f^{(m-1)} \text{ are abs. cont.}, f^{(m)}(x) \in L_2(\mathbb{R}^+)\}, \tag{3.1}$$

$$l_{2,+}^m = \left\{(y_\nu); \sum_0^\infty |\Delta^m y_\nu|^2 < \infty\right\}, \tag{3.2}$$

and are provided with the semi-norms $\|f^{(m)}\|_{2,+} = (\int_0^\infty |f^{(m)}|\,dx)^{1/2}$ and $\|\Delta^m y^+\|_{2,+} = (\sum_0^\infty |\Delta^m y_\nu|^2)^{1/2}$, respectively.

Finally, the analogue of Theorem 1 is as follows.

THEOREM 5. 1. *Let*

$$(y_\nu) \in l_{2,+}^m. \tag{3.3}$$

There is a unique spline function $S(x)$ such that

$$S(x) \in \mathscr{S}_{2m-1}^+ \cap L_{2,+}^m, \tag{3.4}$$

and satisfying

$$S(\nu) = y_\nu, \qquad \nu = 0, 1, \cdots, \tag{3.5}$$

where $\mathscr{S}_{2m-1}^+$ is defined by (2.2).

2. *This solution* $S(x)$ *has the property that*

$$\|f^{(m)}\|_{2,+} > \|S^{(m)}\|_{2,+} \tag{3.6}$$

for any $f(x) \in L_{2,+}^m$ *that satisfies* $f(\nu) = y_\nu$, $\nu = 0, 1, \cdots$, *unless* $f(x) = S(x)$ *if* $x \in \mathbb{R}^+$.

Proof. We follow closely the approach used in § 1 in proving Theorem 1. Suppose that $S(x)$ satisfies (3.4) and (3.5). From (3.5) we obtain

$$\Delta^m S(j) = \Delta^m y_j, \qquad j = 0, 1, \cdots, \tag{3.7}$$

which now replaces (1.14). Since $S(x) \in \pi_{m-1}$ in $(-\infty, 0)$, by (3.4), the representation (1.16) is now replaced by

$$S^{(m)}(x) = \sum_{k=0}^{\infty} c_k Q_m(x - k), \tag{3.8}$$

where $S^{(m)}(x) \in L_{2,+}$ implies and is implied by

$$(c_k) \in l_{2,+}. \tag{3.9}$$

Finally, the relation (1.19) shows that (3.7) is equivalent to the system

$$\sum_{k=0}^{\infty} M_{2m}(j - k)c_k = \Delta^m y_j, \qquad j = 0, 1, \cdots. \tag{3.10}$$

Notice that the convolution relations (1.20) of § 1 are now replaced by the system (3.10) which we are to solve for the sequence $c^+ = (c_k)$. This is an example of a so-called discrete Wiener–Hopf problem. Their general theory was independently discovered by Krein [44] and by Calderon, Spitzer and Widom [20]. We follow closely Krein's exposition [44, pp. 227–231]. Specializing Krein's general discussion we find that (3.10), transforming (c_k) into $(\Delta^m y_j)$, is a bounded linear transformation of $l_{2,+}$ into itself; in fact, the inequality

$$\|\Delta^m y\|_{2,+} \leqq \|c^+\|_{2,+} \tag{3.11}$$

holds. Moreover, (3.10) has an inverse that can be constructed. The theory requires that several conditions be satisfied, and these will be shown to hold in the case of (3.10). These conditions are as follows.

1. *The function*

$$a(z) = \sum_{-(m-1)}^{m-1} M_{2m}(j)z^j \tag{3.12}$$

is to be $\neq 0$ *on the circumference* $|z| = 1$.

If we recall the identity (1.7) of Lecture 3, and $M_{2m}(x) = Q_{2m}(x + m)$ we find that

$$a(z) = \frac{1}{(2m - 1)!} z^{-(m-1)} \cdot \Pi_{2m-1}(z), \tag{3.13}$$

and this is indeed $\neq 0$, on $|z| = 1$, in view of the location of the zeros of $\Pi_{2m-1}(z)$ as described by the inequalities (4.3) of Lecture 4.

2. *If z describes the unit circle $|z| = 1$ just once, the winding number* ind $a(z)$ *of* $a(z)$ *in the complex plane and with respect to the origin, should vanish.*

Since $\Pi_{2m-1}(z)$ has precisely $m - 1$ zeros in $|z| < 1$, the argument of Π_{2m-1} is increased by $2\pi(m - 1)$ as z describes $|z| = 1$ in the positive sense. This increase is exactly compensated by the similar decrease of the argument of the first factor on the right side of (3.13).

3. *The reciprocal of $a(z)$ can be represented on $|z| = 1$ in the form of a product*

$$1/a(z) = Cg_+(z)g_-(z), \qquad |z| = 1, \tag{3.14}$$

where C is a positive constant, $g_+(z)$ is regular in the closed disc $|z| \leqq 1$, while $g_-(z)$ is regular in $|z| \geqq 1$, including the point at ∞.

The pairwise reciprocity of the zeros λ_ν shows that $\Pi_{2m-1}(z) = \prod_1^{m-1}(z - \lambda_\nu) \prod_1^{m-1}(z - \lambda_\nu^{-1})$. Using this, (3.13) shows that (3.14) holds, where

$$C = (2m - 1)! \prod_1^{m-1} |\lambda_\nu|, \tag{3.15}$$

$$g_+(z) = \prod_1^{m-1} (1 - \lambda_\nu z)^{-1} = \sum_0^\infty \gamma_j z^j \quad \text{is regular in} \quad |z| \leqq 1, \tag{3.16}$$

$$g_-(z) = \prod (1 - \lambda_\nu z^{-1})^{-1} = \sum_0^\infty \gamma_j z^{-j} \quad \text{is regular in} \quad |z| \geqq 1. \tag{3.17}$$

The general theory shows that the matrix

$$\Lambda = \|M_{2m}(j - k)\|, \qquad j, k = 0, 1, \cdots, \tag{3.18}$$

has an inverse that may be expressed in terms of the above quantities by

$$\Lambda^{-1} = \|\omega_{j,k}\| = C \cdot \left\| \begin{matrix} \gamma_0 & 0 & 0 & \cdots \\ \gamma_1 & \gamma_0 & 0 & \cdots \\ \gamma_2 & \gamma_1 & \gamma_0 & \cdots \\ \vdots & \vdots & & \end{matrix} \right\| \left\| \begin{matrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots \\ 0 & \gamma_0 & \gamma_1 & \\ 0 & 0 & \gamma_0 & \\ \vdots & & & \end{matrix} \right\|. \tag{3.19}$$

It follows from the assumption (3.3) that the system (3.10) has a unique solution $(c_k) \in l_{2,+}$, and that this solution is given by

$$c_k = \sum_{j=0}^\infty \omega_{k,j} \Delta^m y_j, \qquad k = 0, 1, \cdots. \tag{3.20}$$

A proof of the first part of Theorem 5 is completed as follows. With the c_k as given by (3.20), we define $S^{(m)}(x)$ by (3.8), and we are assured that the relations (3.7) hold. Let $S_*(x)$ be any m-fold integral of $S^{(m)}(x)$. We may write (3.7) as $\Delta^m(S_*(j) - y_j) = 0, j = 0, 1, \cdots$. Therefore a $P(x) \in \pi_{m-1}$ exists, such that $S_*(j) + P(j) = y_j$ for $j = 0, 1, \cdots$. It follows that

$$S(x) = S_*(x) + P(x) \in \mathscr{S}_{2m-1}^+ \cap L_2^m(\mathbb{R}^+)$$

satisfies all required conditions.

Proof of the second part of Theorem 5. The proof of the similar statement of Theorem 1, as expressed by the relations (1.22) to (1.26), carries over to the present case, provided that we replace the interval of integration from $\mathbb{R}$ to $\mathbb{R}^+$, and use the expansion (3.8) rather than (1.16).

In §§ 1 and 3 we have not even mentioned the Lagrange expansions in terms of fundamental functions. However, the following should be clear: If

$$y \in l_2^m \quad \text{or if} \quad y^+ \in l_{2,+}^m, \tag{3.21}$$

then the data y_ν are in each case also of power growth (see [6, § 6]). Therefore the Lagrange expansions (2.12) of Lecture 4, and (2.9) of the present Lecture 6, hold also for the two cases (3.21), respectively.

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LECTURE 7

Finite Spline Interpolation Problems

When sometime in 1945 the author showed S. Bochner the cardinal spline interpolation formula (2.12) of Lecture 4, he said: "Fine, but what do you do with a finite table?" This question has since been thoroughly answered, as shown in the present lecture. True, it does not concern cardinal interpolation, but there are two-way relationships. In the first place, cardinal and semi-cardinal interpolation formulae are the limiting forms of finite formulae as the number of data tends to infinity both ways, or only in one direction. Secondly, we shall show that the eigensplines of Lecture 4 allow us to express the finite formulae in explicit form (§§ 4, 5). Finally, the cardinal and semi-cardinal formulae may be used away from the ends, or near the ends, of extensive tables, respectively, with good accuracy.

1. Lagrange interpolation with general boundary conditions. We are given the finite table of data (y_ν), $\nu = 0, 1, \cdots, n$, which we are to interpolate in the interval $[0, n]$. Let m be a natural number and let

$$\mathcal{S}_{2m-1}[0, n] = \{S(x)\} \tag{1.1}$$

denote the class of splines $S(x)$ of degree $2m - 1$, defined in $[0, n]$, and having the $n - 1$ simple knots $1, 2, \cdots, n - 1$; hence, $S(x) \in C^{2m-2}[0, n]$. We wish to find $S(x)$ such that

$$S(\nu) = y_\nu, \qquad \nu = 0, \cdots, n. \tag{1.2}$$

Except for the trivial case $m = 1$ of linear spline interpolation, this problem is indeterminate. For $S(x)$ depends linearly on $2m + n - 1$ parameters as seen from the representation $S(x) = P(x) + \sum_1^{n-1} c_\nu(x - \nu)_+^{2m-1}$, where $P(x) \in \pi_{2m-1}$, while $2m + n - 1 > n + 1$, unless $m = 1$. The problem may perhaps become sensible only if we prescribe $2m + n - 1 - (n + 1) = 2m - 2$ additional data.

These $2m - 2$ additional data or conditions may be chosen judiciously, or arbitrarily, in an enormous number of different ways. For instance, it follows from a result in [66, Theorem 2] that we may proceed as follows: *Select at will the $2m - 2$ points (x_i, z_i), $i = 1, \cdots, 2m - 2$, such that $0 < x_1 < x_2 < \cdots < x_{2m-2} < n$, while none of the x_i are integers, and require the spline $S(x)$ to satisfy* (1.2) *and the additional conditions $S(x_i) = z_i$, $i = 1, \cdots, 2m - 2$. This spline exists and is uniquely determined.*

Usually, however, the additional data are values assigned to some of the derivatives of $S(x)$ at the endpoints 0 and n, i.e., in the form of *boundary conditions.* Let us describe a general way of choosing these boundary conditions that contains as special cases all of the choices that have been found of interest: *We partition the numbers* $1, 2, \cdots, 2m-2$ *into the* $m-1$ *disjoint pairs*

$$(1, 2m-2), (2, 2m-3), (3, 2m-4), \cdots, (m-1, m). \tag{1.3}$$

Note that in each pair the sum of elements is $2m-1$. *Let* I *be a set of* $m-1$ *elements obtained by choosing one and only one element from each of the pairs* (1.3). We call such sets I *admissible*, and exhibit their elements in increasing order.

Examples of admissible sets are

$$I_1 = \{1, 2, \cdots, m-1\}, \tag{1.4}$$

$$I_2 = \{1, 3, 5, \cdots, 2m-3\}, \tag{1.5}$$

$$I_3 = \{m, m+1, \cdots, 2m-2\}. \tag{1.6}$$

These are important examples from a total of 2^{m-1} possible choices of I.

Let I and J be any two admissible sets (of indices) and let us consider the interpolation problem

$$\begin{cases} S(\nu) = y_\nu, & \nu = 0, 1, \cdots, n, \\ S^{(i)}(0) = y_0^{(i)} \quad \text{if} \quad i \in I; \quad S^{(j)}(n) = y_n^{(j)} \quad \text{if} \quad j \in J. \end{cases} \tag{1.7}$$

Here $y_0^{(i)}$ and $y_n^{(j)}$ are $2m-2$ additional prescribed data.

DEFINITION 1. We say that the problem (1.7) is *m-poised*, provided that the assumptions

$$P(x) \in \pi_{m-1}, \tag{1.8}$$

$$\begin{cases} P(\nu) = 0, & \nu = 0, \cdots, n, \\ P^{(i)}(0) = 0 \quad \text{if} \quad i \in I, \quad P^{(j)}(n) = 0 \quad \text{if} \quad j \in J, \end{cases} \tag{1.9}$$

imply that $P(x) \equiv 0$.

With these definitions we can state the following.

THEOREM 1. *If the interpolation problem* (1.7) *is m-poised, then it has a unique solution*

$$S(x) \in \mathscr{S}_{2m-1}[0, n]. \tag{1.10}$$

Proof. We already know that we have a linear system of $2m + n - 1$ equations in as many unknowns. It suffices therefore to show that if $S(x)$ satisfies (1.10) and the homogeneous system

$$S(\nu) = 0, \qquad \nu = 0, \cdots, n, \tag{1.11}$$

$$S^{(i)}(0) = 0, \quad i \in I, \quad S^{(j)}(n) = 0, \quad j \in J, \tag{1.12}$$

then

$$S(x) = 0 \quad \text{for all} \quad x \quad \text{in} \quad [0, n]. \tag{1.13}$$

To show this, we consider the integral

$$H = \int_0^n (S^{(m)}(x))^2\,dx, \tag{1.14}$$

and integrate it repeatedly by parts, thus

$$\begin{aligned} H &= \int_0^n S^{(m)}\,dS^{(m-1)} = -\int_0^n S^{(m+1)}S^{(m-1)}\,dx = -\int_0^n S^{(m+1)}\,dS^{(m-2)} \\ &= \cdots = \pm\int_0^n S^{(\beta)}\,dS^{(\alpha)}, \end{aligned} \tag{1.15}$$

where the integers α and β satisfy the conditions

$$1 \leqq \alpha \leqq m-1, \qquad m \leqq \beta \leqq 2m-2, \qquad \alpha + \beta = 2m-1. \tag{1.16}$$

Notice that we have written (1.15) as if all "finite parts" drop out at each endpoint and at each step of the process. As a matter of fact, they do: It is seen by our definition of admissible set (and our I and J are admissible) that for each pair of number (α, β), satisfying (1.16), we have

$$S^{(\beta)}(0)S^{(\alpha)}(0) = S^{(\beta)}(n)S^{(\alpha)}(n) = 0, \tag{1.17}$$

because of the boundary conditions (1.12). Indeed, in each product of (1.17), at least one of the two factors $=0$. But then we can continue the integrations by parts in (1.15) until we reach

$$H = \pm\int_0^n S^{(2m-1)}\,dS = \pm\sum_{i=0}^{n-1}\int_i^{i+1} S^{(2m-1)}(x)S'(x)\,dx. \tag{1.18}$$

Here $S^{(2m-1)}(x)$ is a step function and each term of the last sum vanishes, because its integral equals the constant $S^{(2m-1)}(x)$ multiplied by $S(i+1) - S(i)$, which vanishes by (1.11). Thus $H = 0$ and therefore $S^{(n)}(x) = 0$ for all x, whence $S(x) \in \pi_{m-1}$. This conclusion and the relations (1.11) and (1.12) establish (1.13), in view of Definition 1 of an m-poised system.

Concerning the early developments on finite spline interpolation, see [57], [35], [19], [67]. The account just given is based on [34, § 13] and [72]. Very simple necessary and sufficient conditions for the system (1.7) to be m-poised are available. They were introduced by Pólya for the case when $n = 1$ in [55], and generalized to the present situation in [68], [69]. Let

$$I_s = I \cap \{1, 2, \cdots, s\}, \quad J_s = J \cap \{1, 2, \cdots, s\}, \tag{1.19}$$
$$0 \leqq s \leqq m-1,\ I_0 = J_0 = \varnothing,$$

and let $|I_s|$ and $|J_s|$ denote the number of elements of these sets. Clearly,

$$M_s = n + 1 + |I_s| + |J_s| \tag{1.20}$$

equals the number of relations in (1.7) involving derivatives of S of orders $\leqq s$. The criterion is as follows.

LEMMA 1. *The system* (1.7) *is m-poised if and only if*

$$M_s \geqq s + 1 \quad \text{for} \quad s = 0, 1, \cdots, m - 1. \tag{1.21}$$

The inequalities (1.21) are called the *Pólya conditions*. For a proof see [68], [69, Theorem 5]. Clearly, the conditions for "m-poised" involve only the sections I_{m-1}, J_{m-1}. From Lemma 1 we see that the inequality

$$n + 1 + |I_{m-1}| + |J_{m-1}| \geqq m \tag{1.22}$$

is a *necessary* condition. However, it is *not* sufficient, as shown by the example when $m = 4$, $n = 1$, and $I = J = \{3, 5, 6\}$, which are admissible sets. Here (1.22) is satisfied since $1 + 1 + 1 + 1 \geqq 4$, while the corresponding system (1.7) is not 4-poised. Indeed, $P(x) = x(x - 1) \in \pi_3$ and satisfies (1.9) without vanishing identically. In contrast, the inequality

$$n + 1 \geqq m \tag{1.23}$$

is *sufficient*, because $M_s \geqq n + 1$ for all relevant s, hence $M_s \geqq n + 1 \geqq m \geqq s + 1$. However, (1.23) is *not* necessary as shown by obvious examples.

2. Interpolation by splines having the optimal property. We retain all notations of § 1. Let us write

$$I' = I_{m-1} = I \cap \{1, 2, \cdots, m - 1\}, \quad J' = J_{m-1} = J \cap \{1, 2, \cdots, m - 1\}, \tag{2.1}$$

$$I'' = I \cap \{m, m + 1, \cdots, 2m - 2\}, \quad J'' = J \cap \{m, \cdots, 2m - 2\}. \tag{2.2}$$

Of particular interest is the following special case of the system (1.7):

$$\begin{cases} S_0(\nu) = y_\nu, & \nu = 0, \cdots, n, \\ S_0^{(i)}(0) = y_0^{(i)} \quad \text{if} \quad i \in I', \quad S_0^{(j)}(n) = y_n^{(j)} \quad \text{if} \quad j \in J', & \\ S_0^{(i)}(0) = 0 \quad \text{if} \quad i \in I'', \quad S_0^{(j)}(n) = 0 \quad \text{if} \quad j \in J''. & \end{cases} \tag{2.3}$$

In words: We have specialized (1.7) by requiring that the endpoint derivatives of order $\geqq m$ should vanish. We assume that:

(2.4) *the problem* (2.3) *is m-poised.*

Since (2.3) is a special case of (1.7), we may apply to (2.3) Theorem 1 and conclude that (2.3) has a unique solution $S_0(x)$ such that

$$S_0(x) \in \mathscr{S}_{2m-1}[0, n]. \tag{2.5}$$

This solution has an optimal property described by the following theorem.

THEOREM 2. *If $f(x)$ is a function in $L_2^m[0, n]$ satisfying the system*

$$\begin{cases} f(\nu) = y_\nu, \\ f^{(i)}(0) = y_0^{(i)} \quad \text{if} \quad i \in I', \quad f^{(j)}(n) = y_n^{(j)} \quad \text{if} \quad j \in J', \end{cases} \tag{2.6}$$

then

$$\int_0^n (f^{(m)}(x))^2\,dx > \int_0^n (S_0^{(m)}(x))^2\,dx, \tag{2.7}$$

unless $f(x) = S_0(x)$ *in* $[0, n]$.

Proof. Setting $R(x) = f(x) - S_0(x)$, we find that

$$\begin{cases} R(v) = 0, \\ R^{(i)}(0) = 0 \quad \text{if} \quad i \in I', \quad R^{(j)}(n) = 0 \quad \text{if} \quad j \in J'. \end{cases} \tag{2.8}$$

Using the identity analogous to the identity (1.22) of Lecture 6 (this time for real-valued functions!), we wish to show that

$$H = \int_0^n S_0^{(m)}(x)R^{(m)}(x)\,dx = 0. \tag{2.9}$$

This follows in the way that the vanishing of the integral (1.14) was established. Now we write

$$H = \int_0^n S_0^{(m)}\,dR^{(m-1)} = \cdots = \pm\int_0^n S_0^{(\beta)}(x)\,dR^{(\alpha)},$$

again for integers α, β satisfying (1.16). Again "finite parts" drop out because $S_0^{(\beta)}(0)R^{(\alpha)}(0) = S_0^{(\beta)}(n)R^{(\alpha)}(n) = 0$, as in (1.17), and for the same reasons. We finally reach

$$H = \pm\sum_{i=0}^{n-1}\int_i^{i+1} S_0^{(2m-1)}(x)R'(x)\,dx,$$

where all terms vanish. This establishes the identity

$$\int_0^n (f^{(m)})^2\,dx = \int_0^n (S_0^{(m)})^2\,dx + \int_0^n (f^{(m)} - S_0^{(m)})^2\,dx,$$

which proves (2.7). Indeed, equality in (2.7) shows that $R(x) = f(x) - S_0(x) \in \pi_{m-1}$, hence $f(x) = S_0(x)$ for all x, because the system (2.8) is m-poised.

We may summarize our discussion as follows: Let the problem (1.7) be m-poised. Given a function $f(x)$ defined in $[0, n]$, let $S(x) \in \mathscr{S}_{2m-1}[0, n]$ be such that

$$\begin{aligned} &S(v) = f(v), && v = 0, \cdots, n, \\ &S^{(i)}(0) = f^{(i)}(0) \quad \text{if} \quad i \in I, \quad S^{(j)}(n) = f^{(j)}(n) \quad \text{if} \quad j \in J. \end{aligned} \tag{2.10}$$

Let $S_0(x) \in \mathscr{S}_{2m-1}[0, n]$ be such that

$$\begin{aligned} &S_0(v) = f(v), \\ &S_0^{(i)}(0) = f^{(i)}(0) \quad \text{if} \quad i \in I', \quad S_0^{(j)}(n) = f^{(j)}(n) \quad \text{if} \quad j \in J', \\ &S_0^{(i)}(0) = 0 \quad \text{if} \quad i \in I'', \quad S_0^{(j)}(n) = 0 \quad \text{if} \quad j \in J''. \end{aligned} \tag{2.11}$$

Here I', I'', J', J'' are defined by (2.1), (2.2), hence $I = I' \cup I''$, $J = J' \cup J''$. We may call $S(x)$ the *complete* spline interpolant of $f(x)$, and $S_0(x)$ the *natural* spline interpolant. By Theorem 1 we may express $S(x)$ as the right-hand side (without remainder term) of the formula

$$(2.12)\qquad f(x) = \sum_0^n f(\nu)L_\nu(x) + \sum_{i\in I} f^{(i)}(0)L_{i,0}(x) + \sum_{j\in J} f^{(j)}(n)L_{j,n}(x) + R(x).$$

Here L_ν, $L_{i,0}$, $L_{j,n}$ are the fundamental functions. It is equally clear by (2.11) that the natural spline interpolant $S_0(x)$ is the right side of

$$(2.13)\qquad f(x) = \sum_0^n f(\nu)L_\nu(x) + \sum_{i\in I'} f'(0)L_{i,0}(x) + \sum_{j\in J'} f^{(j)}(n)L_{j,n}(x) + R_0(x).$$

The interpolation formulae (2.12), (2.13) may also be called *complete* and *natural*, respectively. The uniqueness of the spline interpolants also implies the following corollary.

COROLLARY 1. 1. *The complete interpolation formula* (2.12) *is exact, i.e.*, $R(x) \equiv 0$, *if*

$$(2.14)\qquad f(x) \in \mathscr{S}_{2m-1}[0, n]$$

and in particular if $f(x) \in \pi_{2m-1}$.

2. *The natural interpolation formula* (2.13) *is exact for those elements* $S(x)$ *of* $\mathscr{S}_{2m-1}[0, n]$ *that satisfy the boundary conditions*

$$(2.15)\qquad S^{(i)}(0) = 0 \quad \textit{if} \quad i \in I'', \qquad S^{(j)}(n) = 0 \quad \textit{if} \quad i \in J'',$$

and in particular if $S(x) \in \pi_{m-1}$, *when* (2.15) *are automatically fulfilled. In fact, if* $k = \min\{i; i \in I'' \cup J''\}$, *then* (2.13) *is exact if* $f(x) \in \pi_{k-1}$.

The fact that the nodes of interpolation were the consecutive integers $0, \cdots, n$, is of course immaterial, and they may be replaced by $x_0 < x_1 < \cdots < x_n$. They may even coalesce, when we get a so-called quasi-Hermite problem [68]. The "inside" derivatives $f^{(r)}(x_i)$ need not even have to be consecutive and we have a *Hermite–Birkhoff problem* (see [9], [69]).

3. Special boundary conditions of interest. Among the $(2^{m-1})^2$ pairs (I, J) of admissible sets that can be selected from the pairs (1.3), we single out for special mention the following three:

(i) *The most remarkable choice is when*

$$(3.1)\qquad I = I' = J = J' = \{1, 2, \cdots, m-1\}, \quad \textit{hence} \quad I'' = J'' = \varnothing.$$

For this choice, and for this choice only, the two formulae (2.12) and (2.13) become identical. It follows that the resulting formula

$$(3.2)\qquad f(x) = \sum_0^n f(\nu)L_\nu(x) + \sum_{i=1}^{m-1} f^{(i)}(0)L_{i,0}(x) + \sum_{j=1}^{m-1} f^{(j)}(n)L_{j,n}(x) + R(x)$$

enjoys the properties of each of the formulae (2.12) and (2.13). Since $M_s = n + 1 + 2s > s + 1$ for $s = 0, \cdots, m-1$, the Pólya conditions (1.21) of Lemma 1 are always satisfied and the problem is m-poised.

THEOREM 3. *The interpolation formula* (3.2) *is exact if* (2.14) *holds, and the interpolating spline*

$$S(x) = \sum_{0}^{n} f(v)L_v(x) + \sum_{1}^{m-1} f^{(i)}(0)L_{i,0}(x) + \sum_{1}^{m-1} f^{(j)}(n)L_{j,n}(x) \tag{3.3}$$

minimizes the integral $\int_0^n (f^{(m)}(x))^2\,dx$ *among all interpolants* $f(x) \in L_2^m$, *by Theorem* 2.

The formula (3.2) is the most accurate of all these interpolation methods and may be called *the* formula of complete spline interpolation.

(ii) *A second choice*, which does not deserve the popularity which it enjoys, is when

$$I = I'' = J = J'' = \{m, m+1, \cdots, 2m-2\}, \quad \text{hence,} \quad I' = J' = \varnothing. \tag{3.4}$$

In this case (2.12) and (2.13) become

$$f(x) = \sum_{0}^{n} f(v)L_v(x) + \sum_{m}^{2m-2} f^{(i)}(0)L_{i,0}(x) + \sum_{m}^{2m-2} f^{(j)}(0)L_{j,n}(x) + R(x) \tag{3.5}$$

and

$$f(x) = \sum_{0}^{n} f(v)L_v(x) + R(x), \tag{3.6}$$

respectively. By Lemma 1, $M_{m-1} = n + 1 \geq m$ is *necessary*, and by (1.23) also sufficient for the I.P. to be m-poised. Therefore the choice (3.4) can be made if and only if $m \leq n + 1$. The $L_v(x)$ in (3.5) and (3.6) are, of course, the same. However, only (3.5) is exact if (2.14) holds, while (3.6) is exact only for those elements of $\mathscr{S}_{2m-1}[0, n]$ that satisfy the boundary conditions

$$S^{(m)} = S^{(m+1)} = \cdots = S^{(2m-2)} = 0 \quad \text{for} \quad x = 0 \quad \text{and for} \quad x = n. \tag{3.7}$$

These are sometimes called *natural splines*.

The formula (3.6), called the formula of natural spline interpolation, does have two advantages: 1. *It requires no endpoint derivatives*. 2. *The spline* $S(x)$ *represented by the sum in* (3.6) *minimizes the integral* $\int (F^{(m)})^2\,dx$ *over* $[0, n]$, *among all possible functions* $F(x)$ *that interpolate* $f(x)$ *at* $x = 0, 1, \cdots, n$. Surely a striking property. Formula (3.6) is identical with Sard's *best special interpolation formula* [61, p. 91].

The use of (3.6) in numerical applications can be avoided as follows. One can use (3.3) instead, with values of the required endpoint derivatives that are estimated in terms of the $f(v)$ by finite difference methods (the "Gregory-idea"; see [87]).

(iii) *Euler–Maclaurin data*. Our third and last choice is

$$I = J = \{1, 3, 5, 7, \cdots, 2m-5, 2m-3\}. \tag{3.8}$$

That these are *admissible* sets is clear: The number of elements is $m - 1$; all its elements are odd, and therefore no two among them can belong to the same pair (1.3). For this choice, the formula (2.12) becomes

$$(3.9)\qquad \begin{aligned} f(x) = \sum_{0}^{n} f(\nu)L_\nu(x) &+ \sum_{r=1}^{m-1} f^{(2r-1)}(0)L_{2r-1,0}(x) \\ &+ \sum_{1}^{m-1} f^{(2r-1)}(n)L_{2r-1,n}(x) + R(x). \end{aligned}$$

As for any of the formulae (2.12), we know (3.9) to be exact if

$$(3.10)\qquad f(x) \in \mathscr{S}_{2m-1}[0, n].$$

The data appearing in (3.9) look familiar. To refresh the readers' memory we write the Euler–Maclaurin quadrature (or summation) formula

$$(3.11)\qquad \begin{aligned} \int_0^n f(x)\,dx = T_n &+ \sum_{r=1}^{m-1} \frac{B_{2r}}{(2r)!}(f^{(2r-1)}(0) - f^{(2r-1)}(n)) \\ &+ \frac{1}{(2m)!}\int_0^n (\bar{B}_{2m}(x) - B_{2m})f^{(2m)}(x)\,dx, \end{aligned}$$

where

$$(3.12)\qquad T_n = \tfrac{1}{2}f(0) + f(1) + \cdots + \tfrac{1}{2}f(n).$$

THEOREM 4. *If we integrate both sides of* (3.9) *between the limits* 0 *and* n, *we obtain the quadrature formula* (3.11), (3.12). *This means that*

$$(3.13)\qquad \int_0^n L_\nu(x) = \begin{cases} \frac{1}{2} & \text{if } \nu = 0, n, \\ 1 & \text{if } 0 < \nu < n, \end{cases}$$

and

$$(3.14)\qquad \int_0^n L_{2r-1,0}(x)\,dx = -\int_0^n L_{2r-1,n}(x)\,dx = \frac{B_{2r}}{(2r)!}, \qquad r = 1, \cdots, m-1.$$

Proof. A proof will follow in a roundabout way, as it must, because we know so little about the fundamental functions of (3.9). It goes like this: If we integrate (3.9) we get a quadrature formula of the form

$$(3.15)\qquad \int_0^n f(x)\,dx = \sum_0^n a_\nu f(\nu) + \sum_1^{m-1} b_r f^{(2r-1)}(0) + \sum_1^{m-1} c_r f^{(2r-1)}(n) + Rf,$$

where the values of the coefficients a_ν, b_r, c_r, are as yet not known. However, since (3.9) is exact for the class of functions (3.10), it follows that the quadrature formula (3.15), derived from it, is also exact for the class (3.10). On the other hand it is known (see [72, p. 298]) that (3.11) *is exact for the class* (3.10), *and is characterized among the quadrature formulae of the general form* (3.15) *by this property.* It follows that the formulae (3.11) and (3.15) must be identical. This establishes the relations (3.13) and (3.14).

I like to refer to (3.9) as "the interpolatory background of the Euler–Maclaurin quadrature formula." As this is too long, it may be abbreviated to *the "Euler–*

Maclaurin" interpolation formula. The quotation marks are to remind us that its fundamental functions are not known explicitly in terms of m and n, nor do I expect that they ever will be. That this can be done for each fixed m, and a general n, is shown in [75] for the case when $m = 3$. Actually the I.F. (3.9) is not that important for numerical applications, for it stands to reason that the complete I.F. (3.2) should, in general, give a smaller error. I base this statement on the following heuristic grounds: Both (3.9) and (3.2) use the same number $m - 1$ of derivatives at each endpoint. However the derivatives in (3.9) are of higher order, hence more remote, and therefore harder to compute and also less likely to convey useful information. The same "argument" should also "show" that the quadrature formula derived from (3.2) by integration, should produce a smaller error than the Euler–Maclaurin formula (3.11). However, the beauty of (3.11), (3.12) is unsurpassed.

4. The complete cubic spline interpolation formula. We are here concerned with the actual construction of some of the interpolation methods discussed in §§ 1–3, and begin with the complete formula (3.3) for the simplest nontrivial case which is $m = 2$. We are to find

$$S(x) \in \mathscr{S}_3[0, n] \tag{4.1}$$

such that

$$S(\nu) = f(\nu), \qquad \nu = 0, \cdots, n, \tag{4.2}$$

$$S'(0) = f'(0), \qquad S'(n) = f'(n). \tag{4.3}$$

The computationally simplest approach seems to be the following. It is based on the fact that

$$S(x) = \sum_{-1}^{n+1} C_j M_4(x - j), \qquad 0 \leqq x \leqq n, \tag{4.4}$$

represents uniquely every function satisfying (4.1), where $M_4(x)$ is the cubic B-spline of § 1 of Lecture 2. From (1.4), (1.5), of Lecture 2 we find

$$\begin{aligned} &M_4(-1) = 1/6, \quad M_4(0) = 4/6, \quad M_4(1) = 1/6, \quad M_4(\nu) = 0 \quad \text{if} \quad |\nu| > 1, \\ &M_4'(-1) = 1/2, \quad M_4'(0) = 0, \quad M_4'(1) = -1/2, \quad M_4'(\nu) = 0 \quad \text{if} \quad |\nu| > 1. \end{aligned} \tag{4.5}$$

Substituting (4.4) into (4.2), (4.3), and using (4.5) we obtain the system of $n + 3$ equations in as many unknowns C_j

$$\begin{aligned} -C_{-1} \qquad\quad + C_1 \qquad\qquad\qquad\qquad &= 2f'(0), \\ C_{-1} + 4C_0 + C_1 \qquad\qquad\qquad\qquad &= 6f(0), \\ C_0 + 4C_1 + C_2 \qquad\qquad\quad &= 6f(1), \\ \ddots \qquad\qquad\qquad &\;\;\vdots \\ C_{n-1} + 4C_n + C_{n+1} &= 6f(n), \\ -C_{n-1} \qquad\quad + C_{n+1} &= 2f'(n). \end{aligned} \tag{4.6}$$

Adding together the first two equations and also subtracting the last from the one before it, this amounts to solving the system

$$
(4.7)\qquad
\begin{aligned}
2C_0 + C_1 \qquad\qquad\qquad\qquad\qquad\qquad &= 3f(0) + f'(0),\\
C_0 + 4C_1 + C_2 \qquad\qquad\qquad\qquad\quad &= 6f(1),\\
\ddots \qquad\qquad\qquad &\ \ \vdots\\
C_{n-2} + 4C_{n-1} + C_n &= 6f(n-1),\\
C_{n-1} + 2C_n &= 3f(n) - f'(n),
\end{aligned}
$$

afterwards using

$$
(4.8)\qquad C_{-1} = C_1 - 2f'(0), \qquad C_{n+1} = C_{n-1} + 2f'(n),
$$

to complete the determination of all coefficients of (4.4). *It is important to use the interpolant* $S(x)$ *only in its standard form* (4.4). An auxilliary table of $M_4(x)$ in its support $(-2, 2)$ allows us to do that easily, since the sum (4.4) has for each x at most four nonvanishing consecutive terms.

There is also a formal solution which amounts to inverting the system (4.6). We determine the f. functions of the interpolation formula

$$
(4.9)\qquad f(x) = \sum_0^n f(\nu)L_\nu(x) + f'(0)L_{-1}(x) + f'(n)L_{n+1}(x) + R(x)
$$

in their form (4.4) which we write as

$$
(4.10)\qquad L_\nu(x) = \sum_{-1}^{n+1} c_{j,\nu}M_4(x-j), \qquad \nu = -1, 0, \cdots, n+1.
$$

The method by which the explicit value of $c_{j,\nu}$ can be found is briefly as follows.

We only discuss the determination of $L_\nu(x)$, for $0 < \nu < n$, as the other cases are treated similarly. We start with the cardinal cubic fundamental function (5.12) of Lecture 4, which is

$$
(4.11)\qquad \mathscr{L}_3(x) = \sqrt{3}\sum_{-\infty}^{\infty} \lambda^{|j|}M_4(x-j), \qquad \lambda = -2 + \sqrt{3}.
$$

The restriction of $\mathscr{L}_3(x-\nu)$ to $[0, n]$ already satisfies all conditions required of $L_\nu(x)$, except the two boundary conditions

$$
(4.12)\qquad L'_\nu(0) = 0, \qquad L'_\nu(n) = 0.
$$

In order to satisfy these, we tentatively set

$$
(4.13)\qquad L_\nu(x) = \mathscr{L}_3(x-\nu) + c_1S_1(x) + c_2S_2(x),
$$

where

$$
(4.14)\qquad S_1(x) = \sum_{-\infty}^{\infty} \lambda^j M_4(x-j), \qquad S_2(x) = \sum_{-\infty}^{\infty} \lambda^{-j}M_4(x-j)
$$

are the two eigensplines of $\mathscr{S}_3$ (§ 3 of Lecture 4). The two relations (4.12) allow us to determine the values of the coefficients c_1, c_2 of (4.13).

Let us describe the results. That the coefficients $c_{j,\nu}$ of (4.10) are rational numbers was clear a priori. Actually they can be explicitly described as follows. We form two sequences of integers (a_k) and (b_k) for all integers k, both satisfying the recurrence relation

$$x_{k+2} + 4x_{k+1} + x_k = 0 \quad \text{for all} \quad k,$$

and determined by their initial values

$$a_0 = 1, \qquad a_1 = -2, \quad \text{and} \quad b_0 = 0, \qquad b_1 = 1.$$

A short table of values is as follows:

k	-1	0	1	2	3	4	5	6
a_k	-2	1	-2	7	-26	97	-362	1351
b_k	-1	0	1	-4	15	-56	209	-780

We may state the following theorem.

THEOREM 5. *The coefficients* $c_{j,\nu}$ *of the fundamental functions* (4.10) *of the complete cubic spline interpolation formula* (4.9) *are as follows*:

$$c_{j,0} = -a_{n-|j|}/b_n, \tag{4.15}$$

$$c_{j,\nu} = \begin{cases} -2a_j a_{n-\nu}/b_n & \text{if } j \leqq \nu, \quad 0 < \nu < n. \\ -2a_\nu a_{n-j}/b_n & \text{if } \nu \leqq j, \quad 0 < \nu < n. \end{cases} \tag{4.16}$$

The symmetry relations

$$c_{j,\nu} = c_{n-j,n-\nu} \quad \textit{hold if} \quad 0 \leqq \nu \leqq n, \tag{4.17}$$

and for $\nu = n$ *we find*

$$c_{j,n} = c_{n-j,0} \tag{4.18}$$

already described by (4.15). *Finally,*

$$c_{j,-1} = -\tfrac{1}{3}a_{n-j}/b_n \tag{4.19}$$

and

$$c_{j,n+1} = -c_{n-j,-1} \tag{4.20}$$

already described by (4.19).

Writing

$$\Gamma_n = \|c_{j,\nu}\|, \qquad j, \nu = -1, 0, \cdots, n+1, \tag{4.21}$$

we easily find for $n = 2$ that

$$\Gamma_2 = \tfrac{1}{4}\cdot\left\|\begin{matrix} -26/3 & -2 & 8 & -2 & 2/3 \\ 7/3 & 7 & -4 & 1 & -1/3 \\ -2/3 & -2 & 8 & -2 & 2/3 \\ 1/3 & 1 & -4 & 7 & -7/3 \\ -2/3 & -2 & 8 & -2 & 26/3 \end{matrix}\right\|.$$

Observe the symmetry and anti-symmetry expressed by (4.17) and (4.20), respectively. The matrix (4.21) is the inverse of the matrix of the system (4.6) if we divide its first and last equation by 2, the others by 6. How to invert 3-diagonal, or Jacobi matrices, was shown by Gantmacher and Krein in [91, p. 95] and rediscovered for the special case needed here by Kershaw [42].

5. Finite Hermite spline interpolation problems. The contents of §§ 1–3 generalize to the case of finite *Hermite* spline interpolation. Without going into the details, we only mention the main points of this generalization. Let $\mathcal{S}_{2m-1,r}$ be the class of cardinal splines of Lecture 5, and let $\mathcal{S}_{2m-1,r}[0, n]$ be the class of their *restrictions* to the interval $[0, n]$. The finite problem is as follows: *To find $S(x)$ in $\mathcal{S}_{2m-1,r}[0, n]$ so as to satisfy the relations*

(5.1) $$S(v) = f(v), \quad S'(v) = f'(v), \cdots, S^{(r-1)}(v) = f^{(r-1)}(v) \quad \textit{for} \quad v = 0, 1, \cdots, n.$$

A count of parameters and equations shows that we need $2m - 2r$ additional boundary conditions. The idea of admissible sets I based on the set of pairs (1.3) goes over nicely: We remove from (1.3) the first $r - 1$ pairs, and write the remaining $m - r$ pairs

(5.2) $$(r, 2m - r - 1), (r + 1, 2m - r - 2), \cdots, (m - 1, m).$$

We choose just one element from each of these pairs, to obtain an *admissible* set $I = \{i\}$ of indices. If J is another admissible set, we adjoint to (5.1) the boundary conditions

(5.3) $$S^{(i)}(0) = f^{(i)}(0) \quad (i \in I), \qquad S^{(j)}(n) = f^{(j)}(n) \quad (j \in J).$$

The definition of "m-poised" is unchanged, except that the equations (1.9) are to be replaced by the *homogeneous* relations derived from (5.1), (5.3). With these definitions we can state the following theorem.

THEOREM 6. *If the interpolation problem* (5.1), (5.3), *is m-poised, then it has a unique solution*

(5.4) $$S(x) \in \mathcal{S}_{2m-1,r}[0, n].$$

Proof. The proof of Theorem 1 ($r = 1$) generalizes easily. We have $2m + (n - 1)r$ equations in as many unknowns. If $S(x)$ is a spline satisfying the *homogeneous* equations (5.1), (5.3), we are again to show that the integral (1.14) vanishes. But this time we can continue the integrations by parts in (1.15) only until $\beta = 2m - r - 1$ and $\alpha = r$, when $S^{(\beta)}$ and $S^{(\alpha)}$ are still continuous. However, splitting the integral thus

$$H = \pm \sum_{i=0}^{n-1} \int_i^{i+1} S^{(2m-r-1)}\,dS^{(r)},$$

we find that in each integral we can continue the integrations by parts, with all finite parts vanishing. The final conclusion is that $H = 0$, hence $S(x) \equiv 0$.

The contents of § 2 concerning I', I'', J', J'', again defined by (2.1), (2.2) generalize easily and the analogue of Theorem 2 holds. Rather than pursuing the general case, *let us discuss in some detail the special case when*

$$m = 3, \qquad r = 2, \tag{5.5}$$

for which the B-splines $N_0(x)$, $N_1(x)$, defined explicitly by (7.7), (7.8) of Lecture 5, are available. In this case (5.2) *reduces to the single pair* (2, 3) and only two admissible sets $I = \{2\}$, or $\{3\}$, are possible. We choose $I = J = \{2\}$ to obtain the *complete* interpolation problem

$$S(\nu) = f(\nu), \qquad S'(\nu) = f'(\nu), \qquad \nu = 0, \cdots, n, \tag{5.6}$$

$$S''(0) = f''(0), \qquad S''(n) = f''(n). \tag{5.7}$$

Here we have two spaces of "partial null-splines" defined by (6.1) of Lecture 5:

$$\mathcal{S}_{5,2}^{(0)} = \{S(x); S'(\nu) = 0 \text{ for all } \nu\}, \qquad \mathcal{S}_{5,2}^{(1)} = \{S(x); S(\nu) = 0 \text{ for all } \nu\}. \tag{5.8}$$

Bases for these are provided by $N_0(x - \nu)$ and $N_1(x - \nu)$, respectively, by Theorem 5 (§ 7) of Lecture 5. We consider also the classes of *restrictions* of the elements of (5.8) and denote them by

$$\mathcal{S}_{5,2}^{(0)}[0, n], \qquad \mathcal{S}_{5,2}^{(1)}[0, n]. \tag{5.9}$$

We add (a proof is easy) that the formulae

$$s_0(x) = \sum_{1}^{n+1} c_j N_0(x - j), \qquad s_1(x) = \sum_{-1}^{n+1} \gamma_j N_1(x - j), \tag{5.10}$$

represent uniquely the most general elements of the classes (5.9), respectively. The following theorem holds.

THEOREM 7. *If we determine $s_0(x)$ and $s_1(x)$ such as to satisfy*

$$s_0(\nu) = f(\nu), \quad \nu = 0, \cdots, n, \qquad s_0''(0) = f''(0), \quad s_0''(n) = f''(n), \tag{5.11}$$

and

$$s_1'(\nu) = f'(\nu), \quad \nu = 0, \cdots, n, \qquad s_1''(0) = 0, \qquad s_1''(n) = 0, \tag{5.12}$$

then

$$S(x) = s_0(x) + s_1(x) \tag{5.13}$$

is the unique solution of the interpolation problem (5.6), (5.7).

Actually, this describes an *eminently applicable algorithm* rather than a theorem. Using the table of values

(5.14)

x	$N_0(x)$	$N_0'(x)$	$N_0''(x)$	$N_1(x)$	$N_1'(x)$	$N_1''(x)$
-1	-1	0	20	0	-1	-8
0	6	0	-40	0	6	0
1	-1	0	20	0	-1	8

we find that the determination of the coefficients c_j, in (5.10), from (5.11), depends on a linear system having a 3-diagonal matrix with elements $-1, 6, -1$. Similarly the matrix for $s_1(x)$, from (5.12), is also 3-diagonal with elements $-1, 6, -1$. For each of these partial splines $s_0(x)$, $s_1(x)$, the situation is much like the one for cubic spline interpolation in § 4. *The splitting of the linear problem into two linear problems of roughly half the size is due to the peculiar properties of the two B-splines $N_0(x)$ and $N_1(x)$.*

As a numerical example we choose in (5.6), (5.7), $n = 2$, $f(x) = (x - 1)^6$, and find $c_{-1} = 61/24, c_0 = 5/8, c_1 = 5/24, c_2 = 5/8, c_3 = 61/24$ and $\gamma_{-1} = 0, \gamma_0 = -1, \gamma_1 = 0, \gamma_2 = 1, \gamma_3 = 0$.

LECTURE 8

Semi-Cardinal Interpolation and Quadratures with General Boundary Conditions

In the present lecture we return to the S.–C.I.F. of Lecture 6 with the more general boundary conditions that were introduced in Lecture 7. We also consider quadrature formulae for the first time in these lectures. In § 6 we consider again the case when no differential correction terms are present, in order to describe results concerning some conjectures of Meyers and Sard on Sard's best quadrature formulae. The results of §§ 1, 2, have not appeared elsewhere and complete details are given here.

1. Semi-cardinal interpolation with general boundary conditions. Let $m \geqq 2$, and let $I = \{i_1, i_2, \cdots, i_{m-1}\}$, $1 \leqq i_1 < \cdots < i_{m-1} \leqq 2m - 2$, be an *admissible* set of indices chosen from the pairs (1.3) of Lecture 7. Using I, we modify the S.–C.I.F. of Lecture 6 to be

$$S(\nu) = y_\nu, \qquad \nu = 0, 1, \cdots, \tag{1.1}$$

$$S^{(i)}(0) = y_0^{(i)} \quad \text{if} \quad i \in I. \tag{1.2}$$

Furthermore, *let*

$$\mathscr{S}_{2m-1}(\mathbb{R}^+) = \{S(x)\} \tag{1.3}$$

denote the class of the restrictions to $\mathbb{R}^+ = [0, \infty)$ *of the elements of* $\mathscr{S}_{2m-1}$. The main results are as follows.

THEOREM 1. *If*

$$y_\nu = O(\nu^\gamma) \quad as \quad \nu \to \infty, \quad \textit{for some constant} \quad \gamma \geqq 0, \tag{1.4}$$

then there exists a unique element $S(x)$ of (1.3) *such that*

$$S(x) = O(x^\gamma) \quad as \quad x \to \infty, \tag{1.5}$$

and satisfying the relations (1.1) *and* (1.2).

Applying Theorem 1 with $\gamma = 0$, hence for *bounded* sequences and functions, we obtain the fundamental functions

$$L_\nu(x), \qquad \nu = 0, 1, \cdots, \qquad \Lambda_i(x), \qquad i \in I, \tag{1.6}$$

of our process, as the bounded elements of $\mathscr{S}_{2m-1}(\mathbb{R}^+)$ such that

$$L_\nu(j) = \delta_{\nu,j}, \qquad L_\nu^{(i)}(0) = 0, \qquad i \in I, \tag{1.7}$$

$$\Lambda_i(\nu) = 0, \qquad \Lambda_i^{(i')}(0) = \delta_{i,i'}, \qquad i \in I, \quad i' \in I. \tag{1.8}$$

THEOREM 2. *The f.f.* (1.6) *satisfy the inequalities*

$$(1.9) \qquad |L_\nu(x)| < A\, e^{-\alpha|x-\nu|}, \qquad |\Lambda_i(x)| < A\, e^{-\alpha x} \quad \textit{for} \quad x \geqq 0,$$

for appropriate positive constants A and α depending only on m.

THEOREM 3. *The unique spline $S(x)$ of Theorem 1 is given by*

$$(1.10) \qquad S(x) = \sum_0^\infty y_\nu L_\nu(x) + \sum_{i \in I} y_0^{(i)} \Lambda_i(x),$$

where the series converge absolutely and locally uniformly.

We may also describe the interpolation process by the S.–C.I.F.

$$(1.11) \qquad f(x) = \sum_0^\infty f(\nu) L_\nu(x) + \sum_{i \in I} f^{(i)}(0) \Lambda_i(x) + R_f(x).$$

The unicity in Theorem 1 shows that (1.11) is exact, i.e., $R_f(x) = 0$ for all x, if $f(x)$ is an element of (1.3) that is of power growth.

Outstandingly simple are the three admissible sets (1.4), (1.5), (1.6), of Lecture 7. These sets I_1, I_2, I_3, give rise, respectively, to the following S.–C.I.F:

$$(1.12) \qquad f(x) = \sum_0^\infty f(\nu) L_\nu(x) + \sum_{i=1}^{m-1} f^{(i)}(0) \Lambda_i(x) + R_1(x),$$

called the *complete* I.F.,

$$(1.13) \qquad f(x) = \sum_0^\infty f(\nu) L_\nu(x) + \sum_{i=1}^{m-1} f^{(2i-1)}(0) \Lambda_{2i-1}(x) + R_2(x),$$

called the "*Euler–Maclaurin*" I.F., and

$$(1.14) \qquad f(x) = \sum_0^\infty f(\nu) L_\nu(x) + \sum_{i=m}^{2m-2} f^{(i)}(0) \Lambda_i(x) + R_3(x),$$

called the *natural* I.F. with forcing terms. Clearly, also the f.f. $L_\nu(x)$ and Λ_i should have subscripts 1, 2, 3, which we suppressed. The I.F. (1.12) is the most important, being the most accurate. We shall see below that integration of (1.13) over $\mathbb{R}^+$ leads to the Euler–Maclaurin Q.F., whence the name. If $S(x) = \sum_0^\infty f(\nu) L_\nu(x)$ represents the first sum (1.14), then $S^{(m)}(0) = S^{(m+1)}(0) = \cdots = S^{(2m-2)}(0) = 0$, and $S(x)$ is identical with the natural interpolant of § 2 of Lecture 6. Adding to $S(x)$ the second sum of (1.14) (the "forcing" terms) makes the I.F. exact for all (1.3) that are of power growth.

For the statement that (1.12) is the *most accurate*, and (1.14) the *least accurate* of the three I.F., we advance here the following heuristic argument: The I.F. (1.14) uses the remotest derivatives, hence harder to compute and less likely to convey useful information on $f(x)$. Observe that if $m = 2$ then (1.12) and (1.13) are identical.

2. Proof of Theorems 1, 2, 3. We follow closely the discussion of § 2 of Lecture 6 which concerned the set I_3 of (1.14). We establish first *uniqueness*. The difference $s(x)$ of two solutions of (1.1), (1.2), (1.5), satisfies

$$s(\nu) = 0, \qquad \nu = 0, 1, \cdots, \tag{2.1}$$

$$s^{(i)}(0) = 0, \qquad i \in I, \tag{2.2}$$

$$s(x) = O(x^\gamma) \quad \text{as} \quad x \to \infty. \tag{2.3}$$

As in § 2 of Lecture 6, (2.3) easily implies that

$$s(x) = \sum_1^{m-1} c_j S_j(x), \qquad x \geqq 0, \tag{2.4}$$

where $S_j(x)$ are the "decreasing" eigensplines of $\mathcal{S}_{2m-1}$. But then $s(x) \in L_1(\mathbb{R}^+)$, and so are all its derivatives. From the nature of the set I we conclude by integration by parts, as in § 1 of Lecture 7, that

$$\int_0^\infty (s^{(m)}(x))^2\, dx = 0, \tag{2.5}$$

and therefore $s(x) \in \pi_{m-1}$. This and (2.1) show that $s(x) = 0$ if $x \geqq 0$.

Existence in Theorem 1 is shown by constructing the f.f. (1.6). Denoting by $\mathcal{L}_{2m-1}(x)$ the f.f. of $\mathcal{S}_{2m-1}$ (§ 2 of Lecture 4), we try to construct $L_\nu(x)$ by writing ($\nu \geqq 0$)

$$L_\nu(x) = \mathcal{L}_{2m-1}(x - \nu) + \sum_1^{m-1} c_j S_j(x), \tag{2.6}$$

with appropriate constants c_j. The first relations (1.7) are thereby satisfied. From the second relations (1.7) we obtain the system

$$\mathcal{L}_{2m-1}^{(i)}(-\nu) + \sum_{j=1}^{m-1} c_j S_j^{(i)}(0) = 0, \qquad i \in I. \tag{2.7}$$

Let us show that this system is nonsingular; hence,

$$\Delta = \det \|S_j^{(i)}(0)\| \neq 0, \qquad i \in I, j = 1, \cdots, m-1. \tag{2.8}$$

Clearly so, for otherwise we could find a *nontrivial* $s(x) = \sum_1^{m-1} a_j S_j(x)$ such that $s^{(i)}(0) = 0$, $i \in I$. But now the argument that proved the relation (2.5) applies here as well and shows that $s(x) = 0$, $x \geqq 0$, in contradiction to our assumption that $s(x)$ is nontrivial.

To construct $\Lambda_{i_0}(x)$, $i_0 \in I$, we set $\Lambda_{i_0}(x) = \sum_1^{m-1} c_j S_j(x)$ and determine the c_j from the second relations (1.8) as above. Finally, proofs of Theorems 2 and 3 are very much like those in [2, §§ 4, 5] and may be omitted. The existence proof in Theorem 1 is completed by exhibiting the solution $S(x)$ as given by (1.10).

3. Semi-cardinal monosplines. We apply the general interpolation formula (1.11) to the function

$$f(x) = x^{2m}/(2m)!, \tag{3.1}$$

which is surely of power growth, obtaining the spline interpolant $S(x)$ and the remainder $K(x)$, defined by

$$K(x) = x^{2m}/(2m)! - S(x), \qquad x \geqq 0. \tag{3.2}$$

Because of its form we call $K(x)$ a *monospline*. Its properties are described by the following theorem.

THEOREM 4. *The monospline $K(x)$, defined by* (3.2) *and* $S(x) \in \mathscr{S}_{2m-1}(\mathbb{R}^+)$, *is uniquely characterized by the three requirements*:

$$K(v) = 0, \qquad v = 0, 1, \cdots, \tag{3.3}$$

$$K^{(i)}(0) = 0, \qquad i \in I, \tag{3.4}$$

$$K(x) \quad \textit{is of power growth as} \quad x \to \infty. \tag{3.5}$$

In fact, $K(x)$ is bounded on $\mathbb{R}^+$ and more precisely

$$K(x) = \frac{1}{(2m)!}(\bar{B}_{2m}(x) - B_{2m}) + o(1) \quad \textit{as} \quad x \to \infty, \tag{3.6}$$

where $\bar{B}_{2m}(x)$ is the extension of period 1 *of the Bernoulli polynomial $B_n(x)$, $0 \leqq x \leqq 1$, and $B_{2m} = B_{2m}(0)$ is the Bernoulli number.*

Let us use the abbreviation

$$b_{2m}(x) = (\bar{B}_{2m}(x) - B_{2m})/(2m)!. \tag{3.7}$$

Observe that for the special case when $I = I_2 = \{1, 3, 5, \cdots, 2m - 3\}$ is the admissible set I of Theorem 4, then the monospline (3.2), which we now denote by $K_2(x)$, has the properties

$$K_2(v) = 0, \qquad v = 0, 1, \cdots, \tag{3.8}$$

$$K_2^{(2r-1)}(0) = 0, \qquad r = 1, \cdots, m - 1, \tag{3.9}$$

$$K_2(x) = b_{2m}(x) + o(1) \quad \text{as} \quad x \to \infty. \tag{3.10}$$

Let us show that

$$K_2(x) = b_{2m}(x) \quad \textit{if} \quad x \geqq 0. \tag{3.11}$$

This will follow from Theorem 4 (which is a corollary of Theorem 1) for $I = I_2$, as soon as we show that the monospline $b_{2m}(x)$ satisfies the conditions (3.8), (3.9), and (3.10). The conditions (3.8), (3.10), are evidently satisfied, while

$$b_2^{(2r-1)}(0) = B_{2m-2r+1}(0)/(2m - 2r + 1)! = 0$$

because $B_k(0) = B_k$ and $B_3 = B_5 = B_7 = \cdots = 0$. This proves (3.11).

Proof of Theorem 4. The only matter requiring a proof is the relation (3.6). Let us form the difference

$$s(x) = K(x) - b_{2m}(x), \qquad x \geqq 0. \tag{3.12}$$

This is an element of $\mathscr{S}_{2m-1}(\mathbb{R}^+)$ of power growth and $s(v) = 0$ if $v \geqq 0$. Therefore $s(x) = \sum_1^{m-1} c_j S_j(x)$, in terms of the "decreasing" eigensplines. Now (3.12) shows that $K(x) = b_{2m}(x) + s(x)$, where $s(x) = O(|\lambda_{m-1}|^x)$, showing that the $o(1)$ term in (3.6) decays exponentially.

4. Quadrature formulae. It is well known that monosplines and approximate Q.F. are equivalent subjects in the sense that there is a $1-1$ correspondence between them obtained by integrations by parts. See [71, § 3], [39]. In § 5 below we establish the following theorem.

THEOREM 5. *Let us assume that* $f(x) \in C^{2m}(\mathbb{R}^+)$, *and that* $f^{(2m)}(x)$, $f(x)$, *are in* $L_1(\mathbb{R}^+)$ *and* $\to 0$ *as* $x \to \infty$. *If we integrate the I.F.* (1.11) *we obtain a quadrature formula*

$$\int_0^\infty f(x)\,dx = \sum_0^\infty H_\nu f(\nu) + \sum_{i \in I} A_i f^{(i)}(0) + Rf, \tag{4.1}$$

where

$$H_\nu = \int_0^\infty L_\nu(x)\,dx, \qquad A_i = \int_0^\infty \Lambda_i(x)\,dx \tag{4.2}$$

and

$$Rf = \int_0^\infty K(x) f^{(2m)}(x)\,dx, \tag{4.3}$$

where $K(x)$ *is the monospline* (3.2) *of Theorem* 4.

Moreover, the Q.F. (4.1) *is uniquely characterized among all quadrature formulae of its type* (*i.e., involving the* $f(\nu)$ *and* $f^{(i)}(0)$ *for* $i \in I$) *subject to the condition*

$$H_\nu = O(1) \quad as \quad \nu \to \infty, \tag{4.4}$$

by the condition of being exact, i.e., $Rf = 0$, *if*

$$f(x) \in \mathcal{S}_{2m-1}(\mathbb{R}^+) \cap L_1(\mathbb{R}^+). \tag{4.5}$$

From the three interpolation formulae (1.12), (1.13) and (1.14) we obtain by integration the Q.F.

$$\int_0^\infty f(x)\,dx = \sum_0^\infty H_\nu^{(1)} f(\nu) + \sum_{i=1}^{m-1} A_i^{(1)} f^{(i)}(0) + \int_0^\infty K_1(x) f^{(2m)}(x)\,dx, \tag{4.6}$$

$$\begin{aligned} \int_0^\infty f(x)\,dx &= \tfrac{1}{2} f(0) + f(1) + f(2) + \cdots + \sum_{r=1}^{m-1} \frac{B_{2r}}{(2r)!} f^{(2r-1)}(0) \\ &\quad + \int_0^\infty K_2(x) f^{(2m)}(x)\,dx, \end{aligned} \tag{4.7}$$

$$\int_0^\infty f(x)\,dx = \sum_0^\infty H_\nu^{(3)} f(\nu) + \sum_{i=m}^{2m-2} A_i^{(3)} f^{(i)}(0) + \int_0^\infty K_3(x) f^{(2m)}(x)\,dx, \tag{4.8}$$

respectively. The first and third Q.F. may be called *complete* and *natural*, respectively. The second is the Euler–Maclaurin Q.F. and its kernel $K_2(x) = b_{2m}(x)$, defined by (3.7). The characterization Theorem 4 applies, of course, to $K_1(x)$ and $K_3(x)$.

In § 6 below we discuss the Q.F. (4.8), especially in its form

$$\int_0^\infty f(x)\,dx = \sum_0^\infty H_\nu^{(3)} f(\nu) + \int_0^\infty K_{3,m}(x) f^{(m)}(x)\,dx, \tag{4.9}$$

which is exact if

$$f(x) \in \mathcal{S}_n^+(\mathbb{R}^+) \cap L_1(\mathbb{R}^+) \tag{4.10}$$

(see Definition 1 of § 2 of Lecture 6).

Let us comment on a comparison between (4.6) and (4.7). By Theorem 4 we know that $K_1(t)$ and $K_2(t)$ are characterized among monosplines by the common conditions (3.3), (3.5), and

$$K_1^{(i)}(0) = 0, \qquad i = 1, 2, \cdots, m-1, \tag{4.11}$$

$$K_2^{(2i-1)}(0) = 0, \qquad i = 1, \cdots, m-1, \tag{4.12}$$

respectively. These show that $K_1(x)$ has an m-fold zero at $x = 0$ in the most important range (near zero), and is therefore likely to lead to a smaller error term.

S. D. Silliman [82], [83], expresses the complete Q.F. (4.6) in the following interesting hybrid form.

THEOREM 6 (S. D. Silliman). *Let $S(x)$ denote the spline in $\mathcal{S}_{2m-1}(\mathbb{R}^+)$ that interpolates $f(x)$ by the complete I.F.* (1.12). *Then the complete Q.F.* (4.6) *may be written as*

$$\begin{aligned}\int_0^\infty f(x)\,dx &= \tfrac{1}{2} f(0) + f(1) + \cdots + \sum_{2i-1 \leqq m-1} \frac{B_{2i}}{(2i)!} f^{(2i-1)}(0)\\ &\quad + \sum_{2i-1 \geqq m} \frac{B_{2i}}{(2i)!} S^{(2i-1)}(0) + \int_0^\infty K_1(x) f^{(2m)}(x)\,dx.\end{aligned} \tag{4.13}$$

Thus for $m = 3$, this becomes

$$\begin{aligned}\int_0^\infty f(x)\,dx &= \tfrac{1}{2} f(0) + f(1) + \cdots + \tfrac{1}{12} f'(0) - \tfrac{1}{720} S'''(0)\\ &\quad + \int_0^\infty K_1(x) f^{(6)}(x)\,dx.\end{aligned} \tag{4.14}$$

It does seem strange that the numerical results obtained by applying the Euler–Maclaurin Q.F. should, in general, become more accurate if we replace in it the *exact values* of the "upper half" of the odd-order derivatives by their *approximations* obtained from the complete interpolating spline $S(x)$. This reminds one of the French saying: "Il faut reculer pour mieux sauter." Let $\sum_1$ and $\sum_2$ stand for the two sums in (4.13). To use (4.13) in numerical applications we need the numerical values of the coefficients in the expansion

$$(4.15)\qquad \sum_1 + \sum_2 = \sum_{v=0}^{\infty} a_v^{(m)} f(v) + \sum_{i=1}^{m-1} b_i^{(m)} f^{(i)}(0).$$

Because $a_v^{(m)} \to 0$ as $v \to \infty$, exponentially and rather rapidly if m is not too large, the listing of these values should be easy. These coefficients should be tabulated for $m = 3$ and 4, say.

Silliman reports in a letter the following numerical example. Applying (4.7), for $m = 3$, to the function $f(x) = e^{-x}$, he found its integral over $\mathbb{R}^+$ to be $= 1.00003226$, while the modified form (4.13) (hence (4.6) for $m = 3$) gave the slightly better value 1.00002094. In both cases the step $h = 1$ was used.

Proof of Theorem 6. Let $S(x)$ be the spline solution of the I.P.

$$S(v) = f(v),\quad v = 0, 1, \cdots,\qquad S^{(j)}(0) = f^{(j)}(0),\quad j = 1, \cdots, m-1.$$

This $S(x)$ being the complete spline interpolant, the complete Q.F. (4.6) gives for the integral $\int_0^\infty f(x)\,dx$ the approximation $\int_0^\infty S(x)\,dx$. On the other hand, the Euler–Maclaurin Q.F. (4.7), being exact for such splines, shows that

$$\begin{aligned}\int_0^\infty S(x)\,dx &= \tfrac{1}{2}S(0) + S(1) + \cdots + \sum_{v=1}^{m-1} B_{2v} S^{(2v-1)}(0)/(2v)! \\ &= \tfrac{1}{2}f(0) + f(1) + \cdots + \sum_{2v-1 \leqq m-1} B_{2v} f^{(2v-1)}(0)/(2v)! \\ &\quad + \sum_{2v-1 \geqq m} B_{2v} S^{(2v-1)}(0)/(2v)!.\end{aligned}$$

5. Proof of Theorem 5. The main difficulty is to show that the remainder Rf of (4.1), has the form (4.3). Since (4.1) was obtained by integrating (1.11), the Q.F. (4.1) inherits from (1.11) the property of being exact for the class (4.5). We shall now take for granted the second paragraph of Theorem 5 describing a characterization of the Q.F. (4.1). This can be shown by the method of generating functions used in [78]. Now we proceed as follows. We derive from the functional

$$(5.1)\qquad Rf = \int_0^\infty K(x) f^{(2m)}(x)\,dx = \int_0^\infty K(x)\,df^{(2m-1)}(x)$$

a Q.F. which will be shown to have the same structure as (4.1) and that satisfies the properties that characterize (4.1). This will identify the two quadrature formulae.

We integrate (5.1) by parts repeatedly. Ignoring for the moment the finite parts at the origin, we see from $K(x) \in C^{2m-2}(\mathbb{R}^+)$ that we can continue the integrations until we reach the integral $-\int K^{(2m-1)}(x)\,df(x)$. At this point we break up the last integral, and integrating by parts the individual terms we obtain

$$\begin{aligned}&-\sum_{v=0}^{\infty} \int_v^{v+1} K^{(2m-1)}(x)\,df(x) \\ (5.2)\qquad &= -\sum_0^\infty \left\{K^{(2m-1)}(v+1-0)f(v+1) - K^{(2m-1)}(v+0)f(v) - \int_v^{v+1} f(x)\,dx\right\}\end{aligned}$$

because $K^{(2m-1)}(x) = x -$ (step function). We claim that the final result is a Q.F. of the same form as (4.1). This we see as follows. The ith intermediate integral

$$\pm \int_0^\infty K^{(i)}(x)\, df^{(2m-1-i)}(x), \qquad 0 \leqq i \leqq 2m-2,$$

leads to the finite part $\mp K^{(i)}(0) f^{(2m-1-i)}(0)$, and this can be $\neq 0$ if and only if $i \notin I$, because of (3.4), hence if and only if $2m-1-i \in I$, from the definition of an admissible set. This shows that the final result is a Q.F. of the form

$$\int_0^\infty f(x)\,dx = \sum_0^\infty \tilde{H}_\nu f(\nu) + \sum_{i\in I} \tilde{A}_i f^{(i)}(0) + \int_0^\infty K(x) f^{(2m)}(x)\,dx. \tag{5.3}$$

Moreover, (5.2) shows that $\tilde{H}_\nu = K^{(2m-1)}(\nu+0) - K^{(2m-1)}(\nu-0)$ if $\nu > 0$, while from (3.11) we obtain that

$$K(x) = b_{2m}(x) + \sum_1^{m-1} c_j S_j(x). \tag{5.4}$$

The first term on the right has period 1, and therefore its contribution to $\tilde{H}_\nu$ does not depend on ν. The contribution of the second term (the sum) $\to 0$ exponentially as $\nu \to \infty$. We therefore conclude that

$$\tilde{H}_\nu = O(1) \quad \text{as} \quad \nu \to \infty. \tag{5.5}$$

The exactness of (5.3) for the class (4.5) follows from (3.3). Indeed, if $f(x) = (n-x)_+^{2m-1}/(2m-1)!$, then (5.1) shows that

$$Rf = -\int_0^\infty K(x)\, d(n-x)_+^0 = K(n) = 0.$$

Therefore (5.3) is exact for all elements of $\mathscr{S}_{2m-1}(\mathbb{R}^+)$ having finite support. From this we conclude the exactness of (5.3) for the entire class (4.5) by an argument used in [78, § 5]. We have established the identity of the two formulae (4.1) and (5.3).

6. The conjectures of Meyers and Sard. In the present last section we consider the Q.F. (4.8), denoting its kernel by $K(x)$ rather than $K_3(x)$. This kernel is characterized among monosplines of degree $2m$ by the properties

$$K(\nu) = 0, \qquad \nu = 0, 1, \cdots, \tag{6.1}$$

$$K^{(i)}(0) = 0 \quad \text{if} \quad i = m, m+1, \cdots, 2m-2, \tag{6.2}$$

$$K(x) = \frac{1}{(2m)!}(\bar{B}_{2m}(x) - B_{2m}) + o(1) \quad \text{as} \quad x \to \infty. \tag{6.3}$$

If we integrate by parts m times the remainder of (4.8), the terms of the second sum will cancel against the finite parts and we obtain

$$\int_0^\infty f(x)\,dx = \sum_0^\infty H_\nu f(\nu) + (-1)^m \int_0^\infty K^{(m)}(x) f^{(m)}(x)\,dx. \tag{6.4}$$

This is also the Q.F. that we obtain if we integrate the S.–C.I.F. (2.10) of Lecture 6. It is characterized by having bounded coefficients and being exact for elements of the class $\mathscr{S}^+_{2m-1}(\mathbb{R}^+) \cap L_1(\mathbb{R}^+)$.

At this point we recall the I.F. (3.6) of Lecture 7. As we shall vary n, we write it as

$$f(x) = \sum_{\nu=0}^{n} f(\nu) L_{\nu,n}(x) + R(x). \tag{6.5}$$

Its f.f. $L_{\nu,n}(x)$ are uniquely defined as elements of $\mathscr{S}_{2m-1}[0, n]$ such that $L_{\nu,n}(j)$ $\delta_{\nu,j}$, $\nu, j = 0, \cdots, n$, $L^{(\rho)}_{\nu,n}(0) = L^{(\rho)}_{\nu,n}(n) = 0$, $\rho = m, \cdots, 2m - 2$. If we integrate (6.5) we get the Q.F.

$$\int_0^n f(x)\, dx = \sum_{\nu=0}^{n} H_{\nu,n} f(\nu) + R_n f, \tag{6.6}$$

where

$$H_{\nu,n} = \int_0^n L_{\nu,n}(x)\, dx. \tag{6.7}$$

The Q.F. (6.6) is the *best Q.F. in the sense of Sard*, among all formulae of the form (6.6) that are exact if $f(x) \in \pi_{m-1}$. The term "best" refers to the fact that its remainder is of the form

$$R_n f = \int_0^n K_n(x) f^{(m)}(x)\, dx, \tag{6.8}$$

where

$$J_n = \int_0^n (K_n(x))^2\, dx \tag{6.9}$$

has the least value among all such formulae (see [61, Chap. 2], [34, § 11]).

Meyers and Sard [61, pp. 60–61] have studied the dependence of the Q.F. (6.6) on the number n, and formulated the following conjectures.

CONJECTURE 1 (Meyers–Sard). *The limits*

$$\lim_{n \to \infty} H_{\nu,n}, \qquad \nu = 0, 1, \cdots, \tag{6.10}$$

exist.

CONJECTURE 2 (Meyers–Sard). *The relation*

$$\lim_{n \to \infty} H_{[n/2]+k,n=1} \tag{6.11}$$

holds for each fixed integer k. Here $[n/2]$ has its usual arithmetic meaning.

CONJECTURE 3 (Meyers–Sard). J_n *being defined by* (6.9), *the limit*

$$\lim_{n \to \infty} (J_n - J_{n-1}) \tag{6.12}$$

exists.

Meyers and Sard established Conjecture 1 for $m = 2$ (the case of cubic splines) only (for a reference see [61, p. 61]). The key to these conjectures is the observation that the finite I.F. (6.5) "converges" to the semi-cardinal I.F. (2.10) of Lecture 6, and that the finite Q.F. (6.6) also "converges" to the semi-cardinal Q.F. (6.4). The precise statements are as follows.

THEOREM 7. *The relations*

$$\lim_{n\to\infty} L_{v,n}^{(s)}(x) = L_v^{(s)}(x), \qquad s = 0, 1, \cdots, 2m-2, \tag{6.13}$$

hold uniformly in x, in the finite but variable interval $0 \leqq x \leqq n$.

THEOREM 8. *The relation*

$$\lim_{n\to\infty} H_{v,n} = H_v = \int_0^\infty L_v(x)\,dx \tag{6.14}$$

holds.

THEORER 9. *The relation* (6.11) *holds.*

THEOREM 10. *The relation*

$$\lim_{v\to\infty} H_v = 1 \tag{6.15}$$

holds.

THEOREM 11. *If we write* $J_n = J_n^{(m)}$, *then*

$$\lim_{n\to\infty} (J_n^{(m)} - J_{n-1}^{(m)}) = (-1)^{m-1} B_{2m}/(2m)!, \tag{6.16}$$

where B_{2m} *is the Bernoulli number.*

THEOREM 12. *The kernel* $K^{(m)}(x)$ *of* (6.4) *has the property that*

$$K^{(m)}(x) = \frac{1}{m!}\bar{B}_m(x) + o(1) \quad \text{as} \quad x \to \infty. \tag{6.17}$$

The last theorem is the easiest to establish: As the error term in (6.3) is a linear combination of the "decreasing" eigensplines, we can differentiate (6.3) m times to obtain (6.17). For the proofs of these results which rely heavily on the properties of the eigensplines of $\mathscr{S}_{2m-1}$, we refer to [6, Part III].

LECTURE 9

Extremum and Limit Properties

In § 2 of Lecture 7 we found that the interpolating splines of the odd degree $2m - 1$ minimize the L_2-norm of the mth derivative of interpolating functions. even more frequently splines are the solutions of extremum problems if the Chebyshev-norm is involved. The first such case was discovered by Landau [46]. This is the topic of § 1. The results of § 4 of Lecture 1 were of a similar nature. In §§ 2, 3, we deal with splines and monosplines that are extremizing functions of problems in finite or infinite intervals. Our second topic is the limit properties of §§ 4 and 5. Here there is need for unifying results and much remains to be done.

1. The Landau problem and the Euler splines. We are here concerned with the Euler splines $\mathscr{E}_n(x)$ of § 6B of Lecture 4. We recall that $\mathscr{E}_n(x)$ is a spline of degree n and period 2, and is uniquely defined as the bounded cardinal spline of degree n such that

$$\mathscr{E}_n(v) = (-1)^v \quad \text{for all} \quad v, \quad \mathscr{E}_n(x) \in \begin{cases} \mathscr{S}_n & \text{if } n \text{ is odd}, \\ \mathscr{S}_n^* & \text{if } n \text{ is even}. \end{cases} \tag{1.1}$$

Using the supremum-norm on $\mathbb{R}$ we easily find that

$$\|\mathscr{E}_n^{(v)}\| = \begin{cases} |\mathscr{E}^{(v)}(0)| & \text{if } v \text{ is even}, \quad v \leqq n, \\ |\mathscr{E}^{(v)}(\frac{1}{2})| & \text{if } v \text{ is odd}, \quad v \leqq n. \end{cases} \tag{1.2}$$

Let us write

$$\|\mathscr{E}_n^{(v)}\| = \gamma_{n,v}, \quad v = 0, \cdots, n, \quad \gamma_{n,0} = 1. \tag{1.3}$$

The $\gamma_{n,v}$ are rational numbers that can be expressed in terms of Bernoulli and Euler numbers. The general result is the following theorem.

THEOREM 1 (Kolmogorov). *If $f(x) \in C^{n-1}(\mathbb{R})$, with $f^{(n-1)}(x)$ satisfying a Lipschitz condition on $\mathbb{R}$, is such that*

$$\|f\| \leqq 1, \qquad \|f^{(n)}\| \leqq \gamma_{n,n}, \tag{1.4}$$

then

$$\|f^{(v)}\| \leqq \gamma_{n,v}, \qquad v = 1, \cdots, n-1. \tag{1.5}$$

The theorem for the case when $n = 2$ was established by Landau [46], for $n = 3$ and $n = 4$ by G. E. Šilov [84], and for general n by Kolmogorov [43]. A second proof is due to Bang [14], and recently a third proof was given by the author [8].

For the corresponding problem for the half-line $\mathbb{R}^+ = [0, \infty)$, see [94]. A discussion of the elementary cases when $n = 2$, or $n = 3$, for both $\mathbb{R}$ and $\mathbb{R}^+$, is found in [77].

We wish to present here the approach used in [8], but for simplicity we limit ourselves to establish Šilov's case when $n = 4$, and this only for $\nu = 1$. We easily find that

$$\mathscr{E}_4(x) = 1 - \tfrac{24}{5}x^2 + \tfrac{16}{5}x^4 \quad \text{in} \quad -\tfrac{1}{2} \leqq x < \tfrac{1}{2}, \tag{1.6}$$

and using (1.2) we find the relevant constants to be

$$\gamma_{4,0} = 1, \quad \gamma_{4,1} = 16/5, \quad \gamma_{4,2} = 48/5, \quad \gamma_{4,3} = 192/5, \quad \gamma_{4,4} = 384/5. \tag{1.7}$$

For $n = 4$, Theorem 1 therefore reduces to the following.

THEOREM 2 (G. E. Šilov). *If $f(x) \in C^3(\mathbb{R})$, with f''' satisfying a Lipschitz condition, is such that*

$$\|f\| \leqq 1, \qquad \|f^{(4)}\| \leqq 384/5, \tag{1.8}$$

then

$$\|f'\| \leqq 16/5, \qquad \|f''\| \leqq 48/5, \qquad \|f'''\| \leqq 192/5. \tag{1.9}$$

Here we propose to establish only the first inequality

$$\|f'\| \leqq 16/5, \tag{1.10}$$

the proofs of the others requiring different but similar tools and reasonings.

Proof of (1.10). We need the "approximate" differentiation formula

$$\begin{aligned} f'(\tfrac{1}{2}) &= \mu f(1) + \mu\lambda f(2) + \mu\lambda^2 f(3) + \cdots \\ &\quad - \mu f(0) - \mu\lambda f(-1) - \mu\lambda^2 f(-2) + \cdots + \int_{-\infty}^{\infty} K(x) f^{(4)}(x)\,dx, \end{aligned} \tag{1.11}$$

where

$$\mu = 12(-5 + \sqrt{30})/5 = 1.14534, \qquad \lambda = -11 + 2\sqrt{30} = -.045548. \tag{1.12}$$

This relation is valid for all $f(x)$ of Theorem 2. The kernel $K(x)$ is a cardinal cubic spline, i.e., having its knots at the integers, except that at $x = \frac{1}{2}$ its second derivative has a discontinuity such that

$$K''(\tfrac{1}{2} + 0) - K''(\tfrac{1}{2} - 0) = 1. \tag{1.13}$$

Moreover, $K(x)$ decays exponentially as $x \to \pm\infty$, hence $K(x) \in L_1(\mathbb{R})$, and

$$K(x) = -K(1 - x) \quad \text{if} \quad x \in \mathbb{R}, \tag{1.14}$$

so that $K(x)$ is odd about $x = \frac{1}{2}$. Finally,

$$K(\nu + \tfrac{1}{2}) = 0 \quad \text{for all integers} \quad \nu, \tag{1.15}$$

and

$$(-1)^{\nu+1}K(x) > 0 \quad \text{if} \quad \nu - \tfrac{1}{2} < x < \nu + \tfrac{1}{2}, \quad \text{for all} \quad \nu. \tag{1.16}$$

The existence of the spline $K(x)$ with these properties being granted for the moment, the differentiation formula (1.11) is obtained by integrations by parts of its remainder term. The coefficient of $f(\nu)$ in (1.11) is $= K'''(\nu - 0) - K'''(\nu + 0)$, and the single term $f'(\frac{1}{2})$ is due to the discontinuity (1.13). Postponing the construction of $K(x)$, *let us see how* (1.11) *implies* (1.10) in a few lines. If we substitute into (1.11) the function

$$f_0(x) = -\mathscr{E}_4(x),$$

we find that $K(x)f_0^{(4)}(x)$ is positive for all x, except that it vanishes for all $x = \nu + \frac{1}{2}$ by (1.15). Since $f_0^{(4)}(x) = \pm 384/5$ by (1.7), we obtain

$$\text{(1.17)} \qquad \frac{16}{5} = f_0'(\tfrac{1}{2}) = 2\mu \sum_0^\infty |\lambda|^\nu + \frac{384}{5}\int_{-\infty}^{\infty} |K(x)|\, dx.$$

If $f(x)$ is *any* function satisfying (1.8), and assuming $f'(\frac{1}{2}) \geqq 0$ or else we replace f by $-f$, we obtain from (1.11) and (1.8) the inequality

$$0 \leqq f'(\tfrac{1}{2}) \leqq 2\mu \sum_0^\infty |\lambda|^\nu + \frac{384}{5}\int_{-\infty}^{\infty} |K(x)|\, dx,$$

which is $= 16/5$ by (1.17). Therefore, $|f'(\frac{1}{2})| \leqq 16/5$ and (1.10) follows by a shift of origin.

The relation (1.17) has such a tight hold on the problem that the following *converse* follows easily: *If* $f(x)$ *satisfies* (1.8) *and* $f'(\frac{1}{2}) = 16/5$, *then* $f(x) = -\mathscr{E}_4(x)$. We shall return to this question at the end of this section.

There remains to construct $K(x)$. We do it by starting with the cubic exponential spline

$$\text{(1.18)} \qquad \Phi_3(x; t) = \sum_{-\infty}^{\infty} t^j Q_4(x - j),$$

of § 4 of Lecture 2, and want to determine t such that the function vanishes for all $x = \nu + \frac{1}{2}$. Since (1.18) satisfies $\Phi_3(x + 1; t) = t\Phi_3(x; t)$, it suffices to enforce its vanishing for $x = \frac{1}{2}$ only. From the values

x	1/2	3/2	5/2	7/2
$48Q(x)$	1	23	23	1
$8Q_4'(x)$	1	5	-5	-1
$2Q_4''(x)$	1	-1	-1	1
$Q_4'''(x)$	1	-3	3	-1

we obtain the equation

$$\text{(1.19)} \qquad \Phi_3(\tfrac{1}{2}; t) = \sum_{-3}^{0} t^j Q_4(\tfrac{1}{2} - j) = \tfrac{1}{48}(t^{-3} + 23t^{-2} + 23t^{-1} + 1) = 0.$$

The roots of this are the zeros of the polynomial $\rho_3(t) = t^3 + 23t^2 + 23t + 1 = (t + 1)(t^2 + 22t + 1)$, which are the eigenvalues of $\mathscr{S}_3^*$ (see § 4 of Lecture 3).

However, among the three functions (1.18) so obtained, only

$$\varphi(x) = \sum_{-\infty}^{\infty} \lambda^j Q_4(x-j), \quad \text{where} \quad \lambda = -11 + 2\sqrt{30} = -.045548, \tag{1.20}$$

is seen to belong to $L_1(\mathbb{R}^+)$. We now define our kernel by

$$K(x) = \begin{cases} \varphi(x)(2\varphi''(\frac{1}{2}))^{-1} & \text{if} \quad x \geqq \frac{1}{2}, \\ -K(1-x) & \text{if} \quad x < \frac{1}{2}, \end{cases} \tag{1.21}$$

since this choice will ensure (1.13) and (1.14), and automatically take care of (1.15). From the table (1.19) we find that

$$\begin{aligned} \varphi''(\tfrac{1}{2}) &= (1 - \lambda^{-1} - \lambda^{-2} + \lambda^3)/2 = -24(115 + 21\sqrt{30}), \\ \varphi'''(1+0) - \varphi'''(1-0) &= \lambda(1-\lambda^{-1})^4 = -576(11 + 2\sqrt{30}), \end{aligned} \tag{1.22}$$

and dividing this by 2 times the value (1.22) we obtain that

$$K'''(1-0) - K'''(1+0) = \tfrac{12}{5}(-5 + \sqrt{30}) = \mu.$$

This is therefore the coefficient of $f(1)$ in (1.11). From $\varphi(x+1) = \lambda\varphi(x)$ it follows that the coefficient of $f(v)$, $v > 1$, is $\mu\lambda^{v-1}$. Lastly, we easily find from (1.19), (1.20) that $\varphi'(\frac{1}{2}) = (-\lambda^{-3} - 8\lambda^{-2} + 8\lambda^{-1} + 1)/8$ is negative, and therefore by (1.21), (1.22), that $K'(\frac{1}{2}) > 0$. This establishes (1.16).

Similar cubic spline kernels $K(x)$, with (1.13) replaced by other suitable discontinuities at $x = 0$ or $x = \frac{1}{2}$, allow us to establish the remaining inequalities (1.9). A discussion of the nature of the extremizing functions is a new contribution of our approach. The following theorems hold.

THEOREM 3. *If $f(x)$ satisfies the conditions* (1.8), *and if we have the equality sign in one of the inequalities* (1.9), *the corresponding supremum being actually attained, i.e., it is a maximum, then $f(x) = \mathscr{E}_4(x-c)$, for a suitable real c.*

THEOREM 4. *There exist functions $f(x)$ such that*

$$\|f\| = 1, \quad \|f'\| = 16/5, \quad \|f''\| = 48/5, \quad \|f'''\| = 192/5, \quad \|f^{(4)}\| = 384/5, \tag{1.23}$$

while none of these five extrema is actually attained for a finite x.

THEOREM 5. *Let $f(x)$ satisfy* (1.8) *and therefore also* (1.9). *If the equality sign holds in one of the inequalities* (1.9), *then the equations* (1.23) *hold.*

Functions $f(x)$ as described by Theorem 4 may appropriately be called extremizing functions *in the weak sense.* The remaining kind of extremizing functions may be called *in the strong sense.* Theorem 3 shows that the only extremizing functions in the strong sense are the Euler splines $\mathscr{E}_4(x-c)$. See [77, § 9]. For an entirely different extremum property of the Euler polynomial $E_n(x)$ with a function-theoretic application, see [95].

2. Monosplines and perfect splines of least L_∞-norms. Approximation theory began when Chebyshev proposed and solved the following problem: To find the polynomial $P_{n-1}(x) \in \pi_{n-1}$ such that

$$\|2^{n-1}x^n - P_{n-1}(x)\|_\infty = \max_{-1 \leqq x \leqq 1} |2^{n-1}x^n - P_{n-1}(x)| \tag{2.1}$$

should be least. The solution is given by $2^{n-1}x^n - P_{n-1}(x) = T_n(x)$, where $T_n(x)$ is the Chebyshev polynomial. Since $T_n(\cos\theta) = \cos n\theta$, we see that $|T_n(x)|$ reaches its maximum value 1 in exactly $n + 1$ points of $[-1, 1]$ of which -1 is the first and 1 the last, and this is what one would expect, because $2^{n-1}x - P_{n-1}(x)$ depends on n parameters.

Worthwhile generalizations of this problem are obtained on passing from polynomials to spline functions. This can be done in a variety of ways of which two will now be discussed.

A. Monosplines. Let

$$\xi_0 = -1 < \xi_1 < \xi_2 < \cdots < \xi_k < 1 = \xi_{k+1}, \tag{2.2}$$

and let us denote by $\mathscr{S}_{n-1,k} = \{S(x)\}$ the class of spline functions of degree $n - 1$, defined in $[-1, 1]$, and having the k points (2.2) as simple knots; hence, $S(x) \in C^{n-2}$ in $[-1, 1]$. As in § 3 of Lecture 8 we call $F(x) = 2^{n-1}x^n - S(x)$ a monospline, and the problem is to find a specific $F(x)$ such that

$$\|F\|_\infty = \inf_{S\in\mathscr{S}_{n-1,k}} \|2^{n-1}x^n - S(x)\|. \tag{2.3}$$

Observe that the knots $\xi_1, \cdots, \xi_k$ are variable; hence, $F(x)$ is seen to depend on $n + 2k$ parameters. For the solution $F(x)$ satisfying (2.3) we would expect $n + 2k + 1$ values of x at which the extreme values $\pm\|F\|_\infty$ are assumed with alternation of signs. In fact, Johnson [37] established the following.

Theorem 6 (R. S. Johnson). *There is a unique monospline $F(x)$ of least L_∞-norm having the following two properties*:

(i) $|F(x)| = \|F\|_\infty$ *holds for exactly $n + 2k + 1$ increasing values of x in* $[-1, 1]$, *the first being -1 and the last $+1$.*

(ii) *$F(x)$ alternates in sign as x assumes these $n + 2k + 1$ values in increasing order. Moreover,* $F(-1) = (-1)^n\|F\|_\infty$.

The unicity implies that the properties (i) *and* (ii) *characterize the monospline of least* $\|F\|_\infty$.

Let $T_{n,k}(x)$ denote the solution of the problem and let us call it a *Chebyshev–monospline* or *T-monospline*. Clearly $T_{n,0}(x) = T_n(x)$. For the deviation

$$\sigma_{n,k} = \|T_{n,k}\|_\infty \tag{2.4}$$

only rough estimates are available [37, § 4] and more precise information would be desirable. It could also be of interest to replace in (2.3) the monomial $2^{n-1}x^n$ by a preassigned function $f(x)$. This is the important problem of best Chebyshev approximation of $f(x)$ by splines of degree $n - 1$ with a given number k of *variable* knots ξ_i. A characterization theorem, like Theorem 6, does not hold for the general case of a continuous $f(x)$. Most likely such a theorem will hold for $f(x)$ satisfying a suitable higher order convexity condition like $f^{(n)}(x) > 0$ in $[-1, 1]$. Concerning higher order convexity, see [93]. See [73, § 1] for an explicit construction of $T_{n,k}(x)$, for $n = 1, 2, 3, 4$, and all values of k: Some simple surgery applied to the graph of $T_n(x)$ is shown to lead to the graph of $T_{n,k}(x)$, which is identified as such by the characterization Theorem 6. This method does not work if $n \geqq 5$. For some open problems suggested by this construction, see [73, § 2].

B. PERFECT SPLINES. We have already met such splines in § 4 of Lecture 1. Here we consider yet another generalization of Chebyshev's original problem (2.1). Again we start with the set of knots (2.2), but in place of the monospline we consider a spline $S(x)$ of degree n in the interval $[-1, 1]$ such that

$$S^{(n)}(x) = (-1)^i 2^{n-1} n! \quad \text{if} \quad \xi_i < x < \xi_{i+1}, \qquad i = 0, \cdots, k. \tag{2.5}$$

We call such functions *perfect splines* and denote their class by the symbol $\mathscr{P}_{n,k}$, the knots (2.2) being free to vary. Observe that the most general element of $\mathscr{P}_{n,k}$ is of the form

$$S(x) = 2^{n-1}x^n + \sum_{i=1}^{k} (-1)^i 2^n (x - \xi_i)_+^n + \sum_{0}^{n-1} a_\nu x^\nu \quad \text{in} \quad -1 \leqq x \leqq 1. \tag{2.6}$$

Again we raise Chebyshev's question and ask for $t_{n,k}(x) \in \mathscr{P}_{n,k}$ such that

$$\rho_{n,k} = \|t_{n,k}\|_\infty = \inf_{S \in \mathscr{P}_{n,k}} \|S\|_\infty. \tag{2.7}$$

Since (2.6) depends on the $k + n$ parameters ξ_i and a_ν, we expect the solution $t_{n,k}(x)$ to have $k + n + 1$ points of equi-oscillation. This expectation is verified by the following analogue of Theorem 6.

THEOREM 7 (Tihomirov, Schoenberg–Cavaretta). *There is a unique $t_{n,k}(x) \in \mathscr{P}_{n,k}$ of least norm, and it is characterized by the presence of $n + k + 1$ points of equi-oscillation. This means that if $\rho_{n,k} = \|t_{n,k}\|_\infty$, then there are exactly $n + k + 1$ points*

$$-1 = x_1 < x_2 < \cdots < x_{n+k} < x_{n+k+1} = 1 \tag{2.8}$$

such that

$$t_{n,k}(x_i) = (-1)^{n+i-1}\rho_{n,k}, \qquad i = 1, \cdots, n + k + 1, \tag{2.9}$$

and $|t_{n,k}(x)| < \rho_{n,k}$ if $x \neq x_i$ for all i. $t_{n,k}(x)$ is an even or odd function, depending on whether $n + k$ is even or odd.

We call these functions *Chebyshev–Euler splines*, because they are the analogues of the Euler splines $\mathscr{E}_n(x)$ for the interval $[-1, 1]$. Of course, again $t_{n,0}(x) = T_n(x)$ (see [98], [94, § 2]). Like Theorem 6, also Theorem 7 can be applied to determine explicitly (by suitable surgery performed on the graph of $T_n(x)$) the functions $t_{n,k}(x)$, for $n = 1, 2, 3$, and all values of k (see [94, § 2, Fig. 1]).

In [94] the Landau problem of § 1 was solved for the half-line R^+. Its solution $\mathscr{E}_n^+(x)$, called the *one-sided Euler spline*, was obtained as the limit of the suitably placed and normalized functions $t_{n,k}(x)$, as $k \to \infty$.

C. CARDINAL MONOSPLINES. Let m and r be integers, $n \geqq 2$ and

$$-1 \leqq r \leqq n - 2, \tag{2.10}$$

and let

$$\mathscr{M}_n^r = \{F(x)\} \tag{2.11}$$

denote the class of cardinal monosplines

$$F(x) = x^n - S(x), \quad \text{where} \quad S(x) \in \mathscr{S}_{n-1} \cap C^r(\mathbb{R}). \tag{2.12}$$

We propose the following problem.

PROBLEM 1. *To determine $F(x) \in \mathcal{M}_n^r$ having the least norm $\|F\|_\infty$ on $\mathbb{R}$.*

We first discuss the cases where r assumes one of the extreme values in (2.10). If $r = -1$, then there are no continuity requirements at the integers. The solution of Problem 1 is then evidently the periodic extension of period 1 of the monic Chebyshev polynomial for $[0, 1]$. For the other extreme case we have the following theorem.

THEOREM 8. *If $m \geqq 2$ and $r = n - 2$, then Problem 1 has a unique solution $F(x)$ given by*

$$F(x) = \bar{B}_n(x) - \frac{1}{2^n} B_n, \qquad x \in \mathbb{R}. \tag{2.13}$$

We subtracted the constant $2^{-n}B_n$ in (2.13) in order to equalize the moduli of the maximal and minimal values of $F(x)$.

The solution of Problem 1 if $0 \leqq r \leqq n - 3$, is more complicated and requires new classes of polynomials that enjoy a blend of the properties of the Bernoulli and the Chebyshev polynomials. They are called Bernoulli–Chebyshev polynomials or BT-polynomials, and are characterized by the following properties.

THEOREM 9. *Let p and q be integers such that $1 \leqq p \leqq q$.*

1. *There is a unique polynomial $B(x) = B_{2q+1,2p-1}(x)$ satisfying the three conditions:*

$$B(x) = x^{2q+1} + \textit{lower degree terms}, \tag{2.14}$$

$$B^{(\nu)}(0) = B^{(\nu)}(1), \qquad \nu = 0, 1, \cdots, 2p-1, \tag{2.15}$$

(2.16) *$B(x)$ has least L_∞-norm in $[0, 1]$ among all polynomials satisfying* (2.14), (2.15).

2. *There is a unique polynomial $B(x) = B_{2q,2p-2}(x)$ satisfying the three conditions:*

$$B(x) = x^{2q} + \textit{lower degree terms}, \tag{2.17}$$

$$B^{(\nu)}(0) = B^{(\nu)}(1), \qquad \nu = 0, \cdots, 2p-2, \tag{2.18}$$

(2.19) *$B(x)$ has least L_∞-norm in $[0, 1]$ among all polynomials satisfying* (2.17) *and* (2.18).

The properties (2.15), (2.18) are Bernoullian, while (2.16), (2.19) are Chebyshevian. Theorem 9 defines polynomials and splines are not mentioned. However, its proof depends not only on Chebyshevian properties of the Bernoulli polynomials, but also on oscillation properties of cardinal spline functions.

THEOREM 10. *Let $1 \leqq p \leqq q$. 1. The element $\mathcal{M}_{2q+1}^{2p-1}$ of least L_∞-norm on $\mathbb{R}$ is unique and is identical with the periodic extension $\bar{B}_{2q+1,2p-1}(x)$. 2. The element of $\mathcal{M}_{2q}^{2p-2}$ of least L_∞-norm on $\mathbb{R}$ is the periodic extension $\bar{B}_{2q,2p-2}(x)$.*

In Theorem 9 the difference between the degree of $B(x)$ and its order of "continuity" is always an even number. Problem 1 for the classes $\mathcal{M}_n^r$, with $n - r$ being an odd number, is solved by the following theorem.

THEOREM 11. 1. *The element of $\mathcal{M}_{2q+1}^{2p-1}$ that is of least norm is also the element of least norm within the wider class $\mathcal{M}_{2q+1}^{2p-2}$. 2. The element of $\mathcal{M}_{2q}^{2p-2}$ that is of least norm is also the element of least norm within the wider class $\mathcal{M}_{2q}^{2p-3}$.*

The unicity of these solutions is an open question (for further details and proofs see [73]).

D. CARDINAL PERFECT SPLINES. The perfect spline analogue of Problem 1 was investigated by A. Cavaretta in [21]. Let again n and r be integers, $-1 \leqq r \leqq n-1$, and let $\mathscr{P}_n^r = \{S(x)\}$ be the class of cardinal splines $S(x)$ of degree n, such that: 1. $S(x) \in C^r(\mathbb{R})$. 2. $S^{(n)}(x) = (-1)^\nu n!$ if $\nu < x < \nu + 1$, for all integers ν. The problem is to find in $\mathscr{P}_n^r$ the element $S(x)$ of least $\|S\|_\infty$ on $\mathbb{R}$. If $r = n-1$, then the solution is the Euler-spline $n!\gamma_{nn}^{-1}\mathscr{E}_n(x + \frac{1}{2}(n+1))$, where we use the notation (1.3). If $r = -1$ we again obtain essentially the Chebyshev polynomial suitably reflected and shifted. For a discussion of the polynomials $E_{n,r}(x)$, which appear if $0 \leqq r \leqq n-2$, and which may be called Euler–Chebyshev polynomials, we refer to [21].

In the present § 2 we have introduced a number of hybrid functions, such as $T_{n,k}(x)$ (Chebyshev monosplines), $t_{n,k}(x)$ (Chebyshev–Euler splines), $B_{n,r}(x)$ (Bernoulli–Chebyshev polynomials), $E_{n,r}(x)$ (Euler–Chebyshev polynomials), all described by extremum norm properties. However, we acknowledge that the original Chebyshev, Bernoulli, and Euler polynomials are *God given* (in French: *Dieudonné*, in Hebrew: *Netanyahu*, to mention only names of well-known mathematicians). Their coefficients are nice rationals and they enjoy many properties. Our hybrid functions are hard to determine, except for low degrees. What other properties they may have the future will tell.

3. Behavior of periodic spline interpolants as their degree tends to infinity. In § 6 of Lecture 3 we studied the behavior of the exponential Euler spline $S_n(x; t)$ as $n \to \infty$, and found that it converges to t^x if t is not a negative real. In § 3 and § 4 we discuss the same problem in different settings. In the present section we depart from our subject of cardinal splines, and consider instead the problem of interpolation by periodic splines of the odd degree $n = 2m - 1$ having the period 2π. Suppose that we have in $(0, 2\pi]$ exactly k data

$$(3.1) \qquad (x_i, y_i), \quad i = 1, \cdots, k, \qquad 0 < x_1 \leqq x_2 \leqq \cdots \leqq x_k \leqq 2\pi, \quad k \geqq 2.$$

The abscissae x_i may coalesce to any degree, except that we assume that $x_i < x_{i+m}$ for all relevant i, and that at multiple nodes x_i the requisite number of consecutive derivatives $y_i', y_i'', \cdots$, are also given. In any case we have a total of k data, $k \geqq 2$.

Let $\mathscr{S}_{n,2\pi} = \{S(x)\}$ denote the class of periodic splines of degree $n = 2m - 1$ having the x_i as knots. It is well known that there is a unique element of $\mathscr{S}_{n,2\pi}$ satisfying the interpolatory conditions

$$(3.2) \qquad S_n(x_i) = y_i, \qquad i = 1, \cdots, k,$$

or their modifications in the case of Hermite data.

PROBLEM 2. *How does $S_n(x)$ behave as $n = 2m - 1 \to \infty$?*

The correct answer to this question was conjectured by the author in [67, § 8]. It was recently established independently by v. Golitschek [30] and by the author [96]. To describe it we must distinguish two cases.

Case 1. $k = 2q + 1$ *is odd.* In this case it is known that the interpolation problem (3.2) admits a unique trigonometric solution

$$(3.3) \qquad T(x) = \frac{1}{2}a_0 + \sum_1^q (a_\nu \cos \nu x + b_\nu \sin \nu x).$$

THEOREM 12. *The relation*

$$\lim_{n\to\infty} S_n^{(s)}(x) = T^{(s)}(x), \qquad s = 0, 1, 2, \cdots, \tag{3.4}$$

holds uniformly in x.

Case 2. $k = 2q$ *is even.* In this case the author could find no answer in the literature as to how to choose one among the infinitely many interpolants of the form (3.3), except for the case of equidistant data. However, the spline interpolants $S_n(x)$ do converge to one among these interpolants, to be now described.

DEFINITION 1. Among the 1-parameter family of functions of the form (3.3) that interpolate the data (3.1), let again $T(x)$ denote the interpolant (3.3) characterized by the condition that

$$a_q^2 + b_q^2 = \text{minimum}. \tag{3.5}$$

In words: The amplitude of the terms of highest frequency is least. Let us call this $T(x)$ the *proximal interpolant* of the data (3.1).

THEOREM 13. *If* $k = 2q$, *then the relations*

$$\lim_{n\to\infty} S_n^{(s)}(x) = T^{(s)}(x), \qquad s = 0, 1, \cdots, \tag{3.6}$$

hold uniformly in x, *where* $T(x)$ *is the proximal interpolant.*

If $k = 2q$, then the proximal interpolant of Definition 1 is identified as the interpolant of the form

$$T(x) = \frac{1}{2}a_0 + \sum_1^{q-1} (a_\nu \cos \nu x + b_\nu \sin \nu x) + A \sin\left(qx - \frac{1}{2}\sum_1^{2q} x_i\right) \tag{3.7}$$

(see [96, Theorem 1]). For the case of equidistant data, $x_{i+1} - x_i = 2\pi/2q$, $i = 1, \cdots, 2q - 1$, this is seen to agree with the form of the interpolant used in the procedures of practical harmonic analysis. For the case of an *even* number of *equidistant* data, Theorem 13 was already established by Quade and Collatz in [57].

4. The interpolatory properties of a certain class of entire functions. Let

$$y = (y_\nu), \qquad \nu = 0, \pm 1, \cdots, \tag{4.1}$$

be a sequence of numbers, real or complex. As usual we write $y \in l_2$, provided that $\sum |y_\nu|^2$ converges. We are interested in the space l_2^k of sequences obtained from sequences in l_2 by a succession of k summations. A direct definition of this space is

$$l_2^k = \{(y_\nu); (\Delta^k y_\nu) \in l_2\}. \tag{4.2}$$

From the inequality

$$|\Delta^{k+p} y_\nu|^2 = \left|\sum_j (-1)^j \binom{p}{j} \Delta^k y_{\nu+j}\right|^2 \leqq \sum_{j=0}^p \binom{p}{j}^2 \cdot \sum_{j=0}^p |\Delta^k y_{\nu+j}|^2$$

we obtain

$$|\Delta^{k+p} y_\nu|^2 \leqq \binom{2p}{p} \sum_{j=0}^p |\Delta^k y_{\nu+j}|^2. \tag{4.3}$$

Assuming that $(y_\nu) \in l_2^k$ and summing (4.3) over all ν, we find that

$$\|\Delta^{k+p}y\|_2^2 \leqq (p+1)\binom{2p}{p}\|\Delta^k y\|_2^2.$$

This shows that if $k' > k$, then $y \in l_2^k$ implies that $y \in l_2^{k'}$, and establishes the inclusions

$$l_2 \subset l_2^1 \subset \cdots \subset l_2^k \subset l_2^{k+1} \subset \cdots. \tag{4.4}$$

Sometimes we shall use their union

$$l_2^* = \bigcup_{k=0}^{\infty} l_2^k. \tag{4.5}$$

Observe that $y \in l_2^*$ implies that (y_ν) is of power growth, i.e., $y_\nu = O(|\nu|^\gamma)$ as $\nu \to \pm\infty$, for a suitable $\gamma \geqq 0$. The converse is not true, as shown by the sequence $y_\nu = (-1)^\nu$ which is bounded but is not in l_2^*.

We are also concerned with functions $F(x)$, of the complex variable x, defined by

$$F(x) = \int_{-\pi}^{\pi} g(u)\, e^{-iux}\, du, \quad \text{where} \quad g(u) \in L_2(-\pi, \pi). \tag{4.6}$$

Clearly, *$F(x)$ is an entire function of exponential type $\leqq \pi$*. Moreover, Plancherel's theorem shows that *the restriction of $F(x)$ to the real axis $\mathbb{R}$ is in $L_2(\mathbb{R})$*. The classical theorem of Paley and Wiener states that the two italicized statements characterize the class of functions $F(x)$ defined by (4.6). For a particularly beautiful proof of this, see [56]. *We denote this class of functions by the symbol PW_π*. We also need the space PW_π^k of functions $F(x)$ that are obtained from the elements of PW_π by a succession of k integrations. A direct definition of this class is

$$PW_\pi^k = \{F(x); F(x) \text{ is entire and } F^{(k)}(x) \in PW_\pi\}. \tag{4.7}$$

If $F(x)$ belongs to this class, then $F^{(k)}(x)$ is of the form (4.6) and p differentiations show that $F^{(k+p)}(x) = \int_{-\pi}^{\pi} (-iu)^p g(u)\, e^{-iux}\, du$. The integrand being evidently in $L_2(-\pi, \pi)$, we conclude that if $k' > k$, then $F(x) \in PW_\pi^k$ implies that $F(x) \in PW_\pi^{k'}$. Therefore,

$$PW_\pi \subset PW_\pi^1 \subset \cdots \subset PW_\pi^k \subset PW_\pi^{k+1} \subset \cdots. \tag{4.8}$$

We need also the union

$$PW_\pi^* = \bigcup_{k=0}^{\infty} PW_\pi^k. \tag{4.9}$$

A polynomial $P(x)$, no matter of what degree k, belongs to this class, because $P^{(k+1)}(x) = 0 \in PW_\pi$, hence $P(x) \in PW_\pi^{k+1}$.

A relation between the class of sequence l_2^k and the corresponding class of functions PW_π^k is stated by the following theorem.

THEOREM 1. *If*

$$(y_\nu) \in l_2^k, \tag{4.10}$$

then there is a unique

$$F(x) \in PW_\pi^k \tag{4.11}$$

such that

$$F(v) = y_v \quad \textit{for all} \quad v. \tag{4.12}$$

Conversely, if (4.11) *holds, and we define* (y_v) *by* (4.12), *then* (4.10) *holds.*

For an elegant proof due to Richard Askey, see [7, § 2]. Theorem 1 shows that the interpolatory relations (4.12) induce a $1-1$ correspondence between the classes l_2^k and PW_π^k, and therefore also between l_2^* and PW_π^*. It follows, for instance, that the function

$$C(x) = \sin \pi x/(\pi x) \tag{4.13}$$

is the only element of PW_π^* which interpolates the unit-sequence (δ_v). Also that if $F(x) = PW_\pi^*$ and $F(v) = 0$ for all integers v, then $F(x) \equiv 0$.

5. The convergence of cardinal splines that interpolate sequences in l_2^*. Let $y = (y_v) \in l_2^*$, and therefore for some k,

$$(y_v) \in l_2^k. \tag{5.1}$$

We recall Theorem 1 of § 1 of Lecture 6, the first half of which is as follows.

THEOREM 2 (§ 1 of Lecture 6). *If*

$$(y_v) \in l_2^m, \tag{5.2}$$

then there is a unique $S_m(x)$ *such that*

$$S_m(x) \in \mathscr{S}_{2m-1} \cap L_2^m \tag{5.3}$$

and

$$S_m(v) = y_v \quad \textit{for all} \quad v. \tag{5.4}$$

We return to (5.1) and let $m \geqq k$. The inclusion $l_2^k \subset l_2^m$ and Theorem 2 show that the sequence (y_v) can be interpolated by a cardinal spline $S_m(x)$, satisfying (5.3), and this for every $m \geqq k$. This raises the following problem.

PROBLEM 3. *The sequence* (y_v) *being fixed, how do the interpolants* $S_m(x)$ *behave as* $m \to \infty$?

We have so arranged matters that the answer is close at hand. The assumption (5.1) and Theorem 1, of § 4, imply the existence of a unique

$$F(x) \in PW_\pi^k \tag{5.5}$$

such that

$$F(v) = y_v \quad \text{for all} \quad v. \tag{5.6}$$

We may now state the following theorem.

THEOREM 3. *The relation*

$$\lim_{m\to\infty} S_m^{(s)}(x) = F^{(s)}(x), \qquad s = 0, 1, \cdots, \tag{5.7}$$

holds uniformly in x on $\mathbb{R}$.

The special sequence (δ_ν), $\delta_0 = 1$, $\delta_\nu = 0$ if $\nu \neq 0$, surely satisfies (5.1) with $k = 0$. The corresponding spline interpolant $S_m(x)$ is identical with the fundamental function $L_{2m-1}(x)$ of Theorems 2 and 3 of § 2 of Lecture 4. On the other hand, the interpolating function $F(x)$ is the function $C(x)$ of (4.13). By Theorem 3 we conclude that

$$\lim_{m\to\infty} L_{2m-1}(x) = \sin \pi x/(\pi x) \quad \text{uniformly for} \quad x \in \mathbb{R}. \tag{5.8}$$

The sequence (y_ν), satisfying (5.1), is also necessarily of power growth. It follows from Theorem 3 of Lecture 4 that the $S_m(x)$ of Theorem 2 may be represented by the Lagrange expansion

$$S_m(x) = \sum_{-\infty}^{\infty} y_\nu L_{2m-1}(x - \nu). \tag{5.9}$$

This situation suggests the following *summability method for the cardinal series*.

DEFINITION 2. Given a sequence $(y_\nu) \in l_2^*$, we define the *S-sum* (read: *spline-sum*) of the cardinal series to be the interpolating $F(x) \in PW_\pi^*$, and write

$$(\mathrm{S}) \sum_{-\infty}^{\infty} y_\nu \frac{\sin \pi(x - \nu)}{\pi(x - \nu)} = F(x). \tag{5.10}$$

The term "spline-sum" is suggested because, by Theorem 3,

$$\lim_{m\to\infty} S_m(x) = F(x) \quad \text{uniformly in} \quad x \in \mathbb{R}. \tag{5.11}$$

Observe that the summability method here suggested is not of the standard variety which operates with linear means. Let us conclude with an example. For the sequence $y_\nu = |\nu|$ for all ν, (5.1) holds with $k = 2$. It is easily shown [7, § 3] that the interpolating element of PW_π^2 is

$$F(x) = \frac{1}{\pi}\int_0^\pi \left\{\frac{\sin (ux/2)}{\sin (u/2)}\right\}^2 du.$$

This, then, is the limit of the splines $S_m(x)$ interpolating the sequence $(|\nu|)$, and (5.10) suggests writing our result as

$$(\mathrm{S}) \sum_{-\infty}^{\infty} |\nu| \frac{\sin \pi(x - \nu)}{\pi(x - \nu)} = F(x).$$

An open problem is as follows. We have met with three distinct results on the convergence of the cardinal spline interpolants as their degree tends to infinity: 1.

The convergence of the exponential Euler spline $S_n(x;t)$ to t^x (§ 6 of Lecture 3). 2. The convergence of periodic spline interpolants to the (proximal) trigonometric interpolant (§ 3). 3. The present discussion for $(y_\nu) \in l_2^*$. The question arises as to the existence of a comprehensive theory that would cover these cases, or at least the second and third case.

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LECTURE 10

Applications: 1. Approximations of Fourier Transforms. 2. The Smoothing of Histograms

In this last lecture we apply equidistant spline interpolation to the two topics of the title. 1. The approximation of Fourier transforms is one of the subjects of S. D. Silliman's dissertation [82]. By using the exponential Euler spline of § 6 of Lecture 3, we can easily obtain his remarkable Theorem 2 below, with a slight improvement in the constant of the error term. 2. In their paper [18] Boneva, Kendall and Stefanov have reinvented cardinal cubic spline interpolation for the purpose of obtaining approximations to the frequency (density) function of a histogram. Using results from Lecture 7 we can proceed more realistically and apply finite cubic or quintic spline interpolation to histograms of finite support.

1. Approximations of the Fourier transform. The basic idea is simple and primitive, in fact the same as that used by Newton to approximate quadratures: Operate on the interpolant. The application of a cardinal *linear* spline interpolant can be found in the Handbook of Krylov and Skoblya [45, pp. 22–23]. The first attack on the problem by cubic spline interpolation is due to B. Einarsson [25]. More recently, splines of higher odd degree were used by S. D. Silliman in [82], [83], and our account is mainly based on his work. To find an approximation for the Fourier integral

$$F(u) = \int_{-\infty}^{\infty} e^{ixu} f(x)\,dx, \tag{1.1}$$

we interpolate $f(x)$ at the integers (for simplicity we choose the step $h = 1$) by a cardinal spline $S(x) \in \mathcal{S}_{2m-1}$ and derive the transform of $S(x)$ which is to serve as an approximation to $F(u)$. Under appropriate conditions we have

$$S(x) = \sum_{-\infty}^{\infty} f(v) L_{2m-1}(x - v), \tag{1.2}$$

where $L_{2m-1}(x)$ is the fundamental function of $\mathcal{S}_{2m-1}$ (§ 5 of Lecture 4). We evidently need the Fourier transform of $L_{2m-1}(x)$ and we have all the necessary tools to obtain it (see [63, § 4.2]).

It was shown in § 5 of Lecture 4 that

$$L_{2m-1}(x) = \sum_{-\infty}^{\infty} \omega_j M_{2m}(x - j), \tag{1.3}$$

where ω_j are the Fourier coefficients of the expansion

$$1 \Big/ \sum_{-(m-1)}^{m-1} M_{2m}(v)\, e^{ivu} = \sum_{-\infty}^{\infty} \omega_v\, e^{ivu}. \tag{1.4}$$

From § 1 of Lecture 2 we recall that

$$\int_{-\infty}^{\infty} M_{2m}(x)\, e^{-ixu}\, dx = \psi_{2m}(u), \quad \text{where} \quad \psi_{2m}(u) = \left(\frac{\sin u/2}{u/2}\right)^{2m}. \tag{1.5}$$

Inverting this transform we obtain

$$M_{2m}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi_{2m}(u)\, e^{ixu}\, du, \tag{1.6}$$

and letting $x = v$ be an integer we obtain, following a general idea of Poisson,

$$\begin{aligned} M(v) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(u)\, e^{ivu}\, du \\ &= \frac{1}{2\pi}\sum_j \int_{2\pi j}^{2\pi(j+1)} \psi(u)\, e^{ivu}\, du = \frac{1}{2\pi}\sum_j \int_0^{2\pi} \psi(u + 2\pi j)\, e^{ivu}\, du. \end{aligned}$$

If we define

$$\phi_{2m}(u) = \sum_{-\infty}^{\infty} \psi_{2m}(u + 2\pi j), \tag{1.7}$$

which is visibly a periodic function, we obtain

$$M_{2m}(v) = \frac{1}{2\pi}\int_0^{2\pi} \phi_{2m}(u)\, e^{ivu}\, du \tag{1.8}$$

and therefore,

$$\phi_{2m}(u) = \sum_{-(m-1)}^{m-1} M_{2m}(v)\, e^{-ivu}. \tag{1.9}$$

Using the new function ϕ_{2m} we may rewrite (1.4) as

$$\frac{1}{\phi_{2m}(u)} = \sum_{-\infty}^{\infty} \omega_v\, e^{-ivu}, \qquad \omega_v = \omega_{-v}. \tag{1.10}$$

On multiplying this by $\psi_{2m}(u)\, e^{ixu}/2\pi$ and integrating with respect to u, we obtain

$$\begin{aligned} \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\psi_{2m}(u)}{\phi_{2m}(u)}\, e^{ixu}\, du &= \sum_v \omega_v \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi_{2m}(u)\, e^{i(x-v)}\, du \\ &= \sum_v \omega_v M_{2m}(x - v) \end{aligned}$$

by (1.6). A comparison with (1.3) shows that

$$L_{2m-1}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\psi_{2m}(u)}{\phi_{2m}(u)} e^{ixu}\, du, \tag{1.11}$$

and therefore,

$$\frac{\psi_{2m}(u)}{\phi_{2m}(u)} = \int_{-\infty}^{\infty} L_{2m-1}(x)\, e^{-ixu}\, dx. \tag{1.12}$$

Applying our result to (1.2) we find

$$\int_{-\infty}^{\infty} S(x)\, e^{ixu}\, dx = \frac{\psi_{2m}(u)}{\phi_{2m}(u)} \sum_{-\infty}^{\infty} f(v)\, e^{ivu} \tag{1.13}$$

to be the formal Fourier transform of our cardinal spline $S(x)$. If we apply our result to $f(hx)$, where h is a fixed positive step, we find

$$\int_{-\infty}^{\infty} S_h(x)\, e^{ixu}\, dx = \frac{\psi_{2m}(uh)}{\phi_{2m}(uh)} \cdot h \sum_{-\infty}^{\infty} f(vh)\, e^{ivhu}. \tag{1.14}$$

For $m = 1$, the case of linear splines, we find this relation in Krylov–Skoblya [45, pp. 22–23]. The analogous problem of approximating the Fourier coefficients of a *periodic* function by means of *periodic* spline interpolation, was already discussed by Quade and Collatz in 1938 in [57]. They obtained a "reduction factor" analogous to our ψ_{2m}/ϕ_{2m} and, of course, related to it.

An interesting interpretation due to Silliman [82, Theorem 1] of the quadrature formula

$$\int_{-\infty}^{\infty} f(x)\, e^{ixu}\, dx = \frac{\psi_{2m}(u)}{\phi_{2m}(u)} \sum_{-\infty}^{\infty} f(v)\, e^{ivu} + R_f(u) \tag{1.15}$$

is as follows.

THEOREM 1 (S. D. Silliman). *Among the Q. F. of the form*

$$\int_{-\infty}^{\infty} f(x)\, e^{ixu} dx = \sum_{-\infty}^{\infty} H_v(u) f(v) + R_f(u) \tag{1.16}$$

having bounded coefficients $H_v(u)$ for each fixed u, hence,

$$|H_v(u)| < A(u) \quad \textit{for all} \quad v \quad \textit{and all} \quad u, \tag{1.17}$$

only the Q. F. (1.15) *satisfies the condition that*

$$R_f(u) \equiv 0 \quad \textit{if} \quad f(x) \in \mathcal{S}_{2m-1} \cap L_1(\mathbb{R}). \tag{1.18}$$

In this sense, therefore, is (1.15) characterized by the exactness condition (1.18). From the implication (see [6, Lemma 5, § 9])

$$f(x) \in \mathcal{S}_{2m-1} \cap L_1(\mathbb{R}) \quad \text{implies that} \quad \sum_{-\infty}^{\infty} |f(v)| < \infty, \tag{1.19}$$

and the condition (1.17), it follows that the operator $R_f(u)$ is well defined for all such splines f. Silliman establishes Theorem 1 by the method of generating functions.

2. Bounds for the error term. Under certain conditions on $f(x)$ Silliman derives interesting and useful bounds for the error obtained if we use the approximation (1.14). His analysis is based on a result of M. Golomb [31, Lemma 6.3 on p. 41, for $s = 0$]. We derive them here by a different approach using the exponential Euler splines and their properties developed in § 6 of Lecture 3. Now we no longer work formally.

We consider the exponential function

$$e^{iux}, \quad -\infty < x < \infty, \quad \text{where} \quad u \quad \text{is a real constant,} \tag{2.1}$$

and interpolate it at all integer values of x by the corresponding exponential Euler spline of degree $2m - 1$

$$S(x) = S_{2m-1}(x; e^{iu}). \tag{2.2}$$

Here the base of the exponential is $t = e^{iu}$ and from our discussion of Lecture 3 we must assume that

$$e^{iu} \neq 1. \tag{2.3}$$

Because the function (2.1) is *bounded*, we may also think of $S(x)$ as the unique *bounded* cardinal spline of degree $2m - 1$ interpolating the exponential (2.1) at the integers. Let

$$K(x) = e^{iux} - S_{2m-1}(x; e^{iu}), \tag{2.4}$$

it being clear that

$$K(\nu) = 0 \quad \text{for all integers} \quad \nu. \tag{2.5}$$

Let $f(x)$ be a function which is $2m$ times differentiable on $\mathbb{R}$ and such that

$$f^{(\nu)}(x) \in L_1(\mathbb{R}), \qquad \nu = 0, 1, \cdots, 2m. \tag{2.6}$$

We also assume that

$$\lim_{x \to \pm\infty} f^{(\nu)}(x) = 0, \qquad \nu = 0, 1, \cdots, 2m - 1, \tag{2.7}$$

our assumptions being very likely redundant. In any case they surely imply that

$$\sum_{\nu=-\infty}^{\infty} |f(\nu h)| < \infty \quad \text{if} \quad h > 0. \tag{2.8}$$

We consider the functional

$$Rf = \int_{-\infty}^{\infty} K(x) f^{(2m)}(x)\, dx = \int_{-\infty}^{\infty} K(x)\, df^{(2m-1)}(x), \tag{2.9}$$

and keep integrating it by parts up to the point where the higher derivative of $K(x)$ becomes possibly discontinuous. This point is reached when we obtain

$$Rf = (-1)^{2m-1}\int_{-\infty}^{\infty} K^{(2m-1)}(x)\,df(x)$$
$$= -\int_{-\infty}^{\infty} \{(iu)^{2m-1} e^{iux} - S^{(2m-1)}(x)\}\,df(x).$$

Writing the last integral as a sum of two integrals, and integrating by parts once more the first of these, and also solving for the transform of $f(x)$, we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} e^{iux} f(x)\,dx &= (iu)^{-2m}\sum_{-\infty}^{\infty}(S^{(2m-1)}(v+0) - S^{(2m-1)}(v-0))f(v)\\ &\quad + (iu)^{-2m}\int_{-\infty}^{\infty} K(x) f^{(2m)}(x)\,dx.\end{aligned} \tag{2.10}$$

LEMMA 1. *We have the relations*

$$(iu)^{-2m}(S^{(2m-1)}(v+0) - S^{(2m-1)}(v-0)) = \frac{\psi_{2m}(u)}{\phi_{2m}(u)} e^{ivu}, \quad \text{for all} \quad v, \tag{2.11}$$

showing that the coefficients of (2.10) *are identical with those of* (1.15).

We prove this in two different ways.

First proof. Let $f(x)$ be an element of $\mathscr{S}_{2m-1}$ of *finite support*; hence, $f(x) = 0$ if $|x|$ is sufficiently large. The second expression (2.9) of Rf shows that

$$Rf = \int_{-\infty}^{\infty} K\,df^{(2m-1)} = \sum_{v} K(v)(f^{(2m-1)}(v+0) - f^{(2m-1)}(v-0)),$$

which is really only a finite sum by our assumption on the spline $f(x)$, and this sum vanishes in view of (2.5). Observe next that the coefficients of the Q. F. (2.10) are surely *bounded*, in fact from properties of the Euler spline we have

$$S^{(2m-1)}(v+0) - S^{(2m-1)}(v-0) = (S^{(2m-1)}(+0) - S^{(2m-1)}(-0))\,e^{ivu}. \tag{2.12}$$

The boundedness of the coefficients of (2.10), together with its exactness for splines of finite support allow us easily (see [78, § 5]) to conclude that the Q. F. (2.10) is exact for all splines of the class $\mathscr{S}_{2m-1} \cap L_1(\mathbb{R})$. Applying Silliman's characterization Theorem 1 we conclude that the relations (2.11) must hold.

Second proof. As we have not established Theorem 1, we give a direct derivation of (2.11). We write $n = 2m - 1$. At the end of Lecture 2 it was shown that the exponential spline $\Phi_n(x;t) = \sum_j t^j Q_{n+1}(x-j)$ has the properties

$$\Phi_n^{(n)}(x;t) = \begin{cases}(1-t^{-1})^n & \text{in} \quad 0 < x < 1,\\ (1-t^{-1})^n t^{-1} & \text{in} \quad -1 < x < 0,\end{cases}$$

the expression in $(-1, 0)$ following from the first by the functional equation (2.5) of Lecture 3. But then $\Phi_n^{(n)}(+0) - \Phi_n^{(n)}(-0) = (1 - t^{-1})^{2m}$. From the definition

(5.4) of Lecture 3 we conclude that $S^{(n)}(+0) - S^{(n)}(-0) = (1 - t^{-1})^{2m}/\Phi_n(0; t)$, while

$$\Phi_n(0; t) = \frac{1}{n!} a_n(t)(1 - t^{-1})^n = \frac{1}{n!} \frac{\Pi_n(t)}{(t-1)^n}(1 - t^{-1})^n$$

$$= \frac{1}{n!}\Pi_n(t)t^{-n} = t^{-n} \sum_{0}^{2m-2} Q_{2m}(j+1)t^j = e^{-imu} \sum_{-(m-1)}^{m-1} M_{2m}(j)\, e^{iju}.$$

Therefore,

$$(iu)^{-2m}(S^{(n)}(+0) - S^{(n)}(-0)) = \frac{(1 - e^{-iu})^{2m}\, e^{imu}}{\phi_{2m}(u)}(iu)^{-2m} = \psi_{2m}(u)/\phi_{2m}(u),$$

where we have used the relations (1.5) and (1.9). The relations (2.11) now follow from (2.12).

We think of u as fixed and pass from the step 1 to the step h, i.e., we apply the relation (2.10) to the new function $f(vh)$. Then (2.10) becomes (after replacing x by x/h, and u by uh)

$$\int_{-\infty}^{\infty} e^{iux} f(x)\, dx = \frac{\psi_{2m}(hu)}{\phi_{2m}(hu)} h \sum_{-\infty}^{\infty} f(vh)\, e^{ivhu} + R_h(u), \tag{2.13}$$

where

$$R_h(u) = (iuh)^{-2m}h^{2m+1} \int_{-\infty}^{\infty} (e^{iuhx} - S(x; e^{iuh}))f^{(2m)}(xh)\, dx.$$

Finally, replacing x by x/h in the last integral we obtain the final form

$$R_h(u) = (iu)^{-2m} \int_{-\infty}^{\infty} K_h(x)\, f^{(2m)}(xh)\, dx. \tag{2.14}$$

where

$$K_h(x) = e^{iux} - S\left(\frac{x}{h}; e^{iuh}\right). \tag{2.15}$$

Here we interpolate the same exponential $t^x = e^{iux}$, but as far as its interpolation is concerned we have replaced $t = e^{iu}$ by $t_h = e^{iuh}$. The result is that the Euler spline in (2.15) interpolates e^{iux} at the points $x = 0, \pm h, \pm 2h, \cdots$. Naturally the remainder $K_h(x)$ should therefore be small if h is small. How small it becomes will be shown by Theorem 6 of § 6 of Lecture 3. Its statement is as follows:

If

$$-\pi \leqq u \leqq \pi, \tag{2.16}$$

then

$$|e^{iux} - S_{2m-1}(x; e^{iu})| \leqq A_m \left(\frac{|u|}{\pi}\right)^{2m} \quad \textit{for all real} \quad x, \tag{2.17}$$

where

$$A_m = 2\left(1 + 2\sum_{\nu=1}^{\infty} \frac{1}{(2\nu+1)^{2m}}\right) < 3 \quad \textit{for} \quad m = 1, 2, \cdots. \tag{2.18}$$

This we apply as follows. Keeping the old u fixed, we select $h > 0$ such that

$$-\frac{\pi}{h} \leqq u \leqq \frac{\pi}{h}. \tag{2.19}$$

We may therefore apply (2.17) for *u replaced by uh*, at the same time replacing x by x/h, to obtain

$$|e^{iux} - S_{2m-1}\left(\frac{x}{h}; e^{iuh}\right)| \leqq A_m \left(\frac{|u|h}{\pi}\right)^{2m} \quad \text{for all} \quad x. \tag{2.20}$$

Using (2.14) and (2.15) we obtain the estimate

$$|R_h(u)| \leqq A_m \left(\frac{h}{\pi}\right)^{2m} \cdot \|f^{(2m)}\|_{L_1(\mathbb{R})}. \tag{2.21}$$

We have just established the following theorem.

THEOREM 2 (S. D. Silliman). *If u and h satisfy* (2.19), *then* (2.13) *holds, with the estimate* (2.21).

From the oscillatory character of the kernel $K_h(x)$ it is clear that the actual error will be much smaller than the bound (2.21) indicates. Silliman obtains (2.21) with A_m replaced by the slightly larger number 4. This theorem shows that the approximation (2.13) is of the order $O(h^{2m})$.

We have discussed the Fourier transform because it also takes care of the approximation of the inverse Laplace transform, if applied to the complex inversion formula for the latter. Silliman considers also the cosine and sine transforms. Approximations for these are also obtained by integrations by parts of (2.9), except that the integrations are now from 0 to $+\infty$ only, and the result contains additional terms in the derivatives of $f(x)$ at the origin. For the coefficients of these derivatives Silliman obtains remarkable explicit expressions (see [82], [83] also for numerical applications).

3. Smoothing of univariate histograms. Let

$$H = (h_j), \qquad -\infty < j < \infty, \tag{3.1}$$

be a so-called *histogram*, where h_j denotes the observed frequency in the class-interval $(j-1, j)$. Observations that fall on the boundary point of two adjacent class-intervals count as $\frac{1}{2}$ in each of them. The following crucial definition is due to Boneva, Kendall and Stefanov [18].

DEFINITION 1. We say that the integrable function $f(x)$, defined on $\mathbb{R}$, satisfies the *area-matching property* for the histogram H, provided that

$$\int_{j-1}^{j} f(x)\,dx = h_j \quad \text{for all} \quad j. \tag{3.2}$$

Let the symbol $AM(H)$ denote the class of functions satisfying this condition.

Boneva, Kendall and Stepanov obtain a cardinal quadratic spline $S_2(x)$ in $AM(H)$ by a procedure equivalent to the following. We associate with H the cumulative sequence

$$F = F_H = (F_j), \qquad -\infty < j < \infty, \tag{3.3}$$

defined by

$$F_i = \sum_{j \leq i} h_j, \qquad -\infty < i < \infty. \tag{3.4}$$

Since H comes from a finite sample of some statistic, we must have $h_j \geqq 0$, for all j, and $\sum h_j < \infty$. Therefore (F_j) is a bounded nondecreasing sequence. From Theorem 1 of Lecture 4, for $n = 3$, we are assured of the existence of a unique cubic spline $S(x) \in \mathscr{S}_3$ which is *bounded* and satisfies

$$S(j) = F_j \quad \text{for all} \quad j. \tag{3.5}$$

If we write

$$\sigma(x) = S'(x), \tag{3.6}$$

then clearly $\sigma(x) \in \mathscr{S}_2$. Moreover, from (3.5) and (3.6) we obtain

$$\int_{j-1}^{j} \sigma(x)\,dx = \int_{j-1}^{j} S'(x)\,dx = S(j) - S(j-1) = F_j - F_{j-1} = h_j, \tag{3.7}$$

in view of (3.4). Therefore $\sigma(x) \in AM(H)$. Boneva, Kendall and Stefanov call $\sigma(x)$ the histospline of H and regard it as an approximation to the unknown frequency function of the underlying statistic. This is the B.K.S. procedure described in terms of the results of Lecture 4.

The author believes that the results of Lecture 7 are more relevant, especially if we assume that we deal with a finite histogram

$$H = (h_j), \qquad j = 1, 2, \cdots, m. \tag{3.8}$$

We also assume that h_1 and h_m are small compared with $\max h_j$; hence, H is assumed to have "thin tails". The corresponding $F = (F_j)$, of (3.4), is described by

$$F_0 = 0, \qquad F_i = \sum_{j=1}^{i} h_j, \qquad i = 1, \cdots, m. \tag{3.8'}$$

In this situation it seems appropriate to apply for the interpolation of F the complete interpolation formula (3.2) of Lecture 7, for low degree splines such as cubic and quintic. In this way we obtain the following theorems.

THEOREM 3. *There is a unique*

$$\sigma_2(x) \in \mathscr{S}_2[0, m] \cap AM(H) \tag{3.9}$$

such that

$$\sigma_2(0) = 0, \qquad \sigma_2(m) = 0. \tag{3.10}$$

THEOREM 4. *There is a unique*

$$\sigma_4(x) \in \mathscr{S}_4[0, m] \cap AM(H) \tag{3.11}$$

such that

$$\sigma_4(0) = \sigma_4'(0) = 0, \qquad \sigma_4(m) = \sigma_4'(m) = 0. \tag{3.12}$$

The proofs are immediate. Let $S_3(x)$ denote the cubic spline defined in $[0, m]$ and interpolating the sequence (3.8′) with the boundary conditions $S_3'(0) = S_3'(m) = 0$. Writing $\sigma_2(x) = S_3'(x)$ we find, as in (3.7) that $\sigma_2(x)$ satisfies the conditions (3.9) and (3.10). Likewise, if $S_5(x)$ is the quintic spline interpolating (3.8′) with the boundary conditions $S_5'(0) = S_5''(0) = S_5'(m) = S_5''(m) = 0$, then $\sigma_4(x) = S_5'(x)$ is a quartic spline such that (3.11) and (3.12) hold.

From § 2 of Lecture 7 we know that the interpolants $S_3(x)$ and $S_5(x)$ have certain optimal properties which are inherited by their derivatives as follows.

THEOREM 5. 1. *The quadratic histospline $\sigma_2(x)$, of Theorem* 3, *minimizes the integral*

$$\int_0^m (\sigma'(x))^2\,dx \tag{3.13}$$

among all functions $\sigma(x)$ that belong to $AM(H)$ and satisfy the boundary conditions (3.10).

2. *The quartic histospline $\sigma_4(x)$, of Theorem* 4, *minimizes the integral*

$$\int_0^m (\sigma''(x))^2\,dx \tag{3.14}$$

among all functions $\sigma(x)$ that belong to $AM(H)$ and satisfy the boundary conditions (3.12).

We illustrate Theorem 4 by the following example.

The age distribution of Bulgarian mothers of 1963. From [18, p. 21] we take the table below giving the age distribution of 50226 Bulgarian mothers during the year 1963:

j	1	2	3	4	5	6	7
Age group	15–20^-	20^+–25^-	25^+–30^-	30^+–35^-	35^+–40^-	40^+–45^-	45^+–50^-
h_j	7442	19261	14385	6547	2123	451	17

The corresponding histogram is shown in Fig. 1 where we have changed scale and origin by setting age $= 15 + 5x$, $0 \leqq x \leqq 7$. The observed frequencies $h_1, \cdots, h_7$ are the areas, and therefore also the heights, of the seven rectangles of Fig. 1, of which the last does not show at our scale of ordinates due to its small height. The curve shown in Fig. 1 is the graph of the quartic histospline of Theorem 4. The graph shows clearly its area-matching property. The curve owes its nicely balanced shape perhaps to the optimal property of minimizing the integral (3.14). The area matching property is close in spirit to P.-J. Laurent's Example 4 (fonctions spline par moyenne locale) in [47, p. 225].

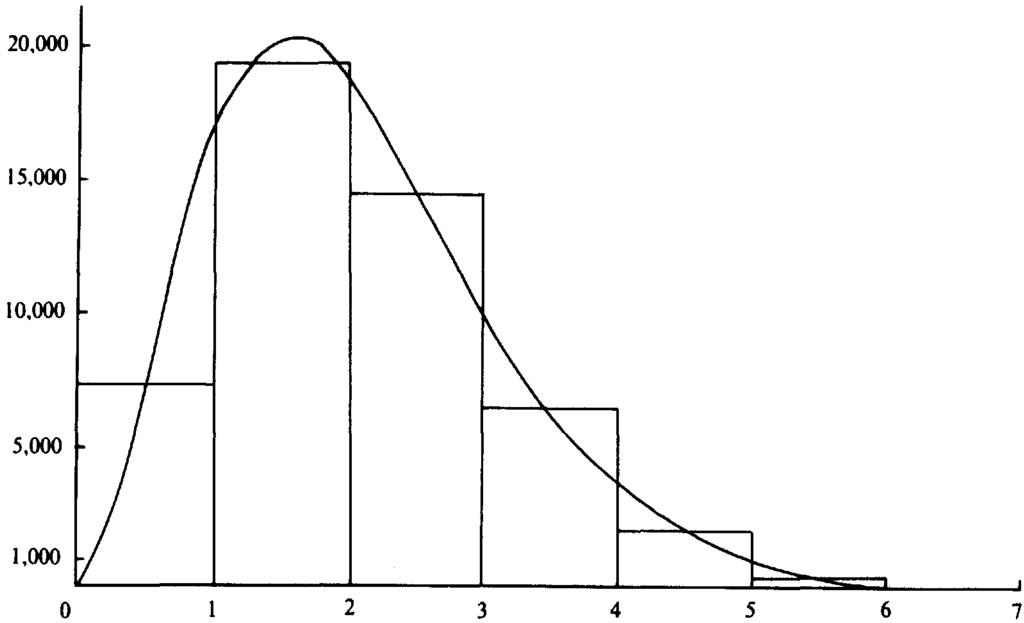

FIG. 1.

4. Bivariate histograms. Paralleling the developments of § 3 we pass to the approximation of bivariate frequency functions by means of biquadratic and biquartic spline functions. We could follow the univariate approach of Boneva, Kendall and Stefanov and apply the "bicardinal" results of § 7 of Lecture 4 to a bivariate histogram whose cells are unit squares covering the entire (x, y)-plane. However, for brevity we pass immediately to the case of a finite histogram in a rectangle. The problem and our terminology are as follows.

Let

$$R = [0, m] \times [0, n] \tag{4.1}$$

be a rectangle in the (x, y)-plane of dimensions m and n, where m, n are natural numbers. We think of R as dissected into mn unit squares

$$R_{ij} = [i - 1, i] \times [j - 1, j]. \tag{4.2}$$

A bivariate histogram

$$H = (h_{ij}), \qquad i = 1, \cdots, m; j = 1, \cdots, n, \tag{4.3}$$

is a matrix of observed frequencies, where h_{ij} is the number of observed "points" falling within the square R_{ij}.

DEFINITION 2. An integrable function $f(x, y)$, defined in R, is said to possess *the volume-matching property*, with respect to H, provided that

$$\iint_{R_{ij}} f(x, y)\, dx\, dy = h_{ij} \quad \text{for all pairs} \quad (i, j). \tag{4.4}$$

We denote by the symbol $VM(H)$ the class of functions that match the mn volumes of H.

DEFINITION 3. Let k be a natural number. We denote by

$$\mathscr{S}_{k,k}(R) = \{S(x, y)\} \tag{4.5}$$

the class of functions $S(x, y)$ having the properties (7.2) and (7.3) of § 7 of Lecture 4 (with n replaced by k) within the mn squares R_{ij} of R.

These functions are called *bivariate splines of degree* (k, k) in R.

If $k = 1, 2, 3, \cdots$ we speak of bilinear, biquadratic, bicubic $\cdots$ splines. The main results are the following two theorems.

THEOREM 6. *There is a unique* $\sigma_2(x, y)$ *such that*

$$\sigma_2(x, y) \in \mathscr{S}_{2,2}(R) \cap VM(H) \tag{4.6}$$

and satisfying the boundary conditions

$$\sigma_2(x, y) = 0 \quad \textit{along the entire boundary of } R. \tag{4.7}$$

THEOREM 7. *There is a unique* $\sigma_4(x, y)$ *such that*

$$\sigma_4(x, y) \in \mathscr{S}_{4,4}(R) \cap VM(H) \tag{4.8}$$

and satisfying the boundary conditions

$$\sigma_4(x, y) = \frac{\partial}{\partial x}\sigma_4(x, y) = \frac{\partial}{\partial y}\sigma_4(x, y) = 0 \quad \textit{along the boundary of } R. \tag{4.9}$$

The purpose of the boundary conditions (4.7) and (4.9) should be clear: Theorems 6 and 7 always hold. However, the results of their application should be satisfactory only if applied to histograms having frequencies h_{ij} *near the boundary of* R, that are small compared to the value of max h_{ij}. For proofs of these theorems and the construction of σ_2 and σ_4, see [97]. In a beautifully written Appendix to [97] C. de Boor explains the extent to which Theorem 6 and 7 can be derived from Theorems 3 and 4 by using the concept and properties of a *tensor product*. Even the algorithms for the construction of the bivariate histosplines $\sigma_2(x, y)$ and $\sigma_4(x, y)$ become simpler in the tensor product approach.

We conclude this last lecture by quoting the last paragraph of the Introduction to [97]: "In the present paper probability considerations and criteria are conspicuous by their absence. A mathematical analyst can provide the statistician with new tools of approximation. The statisticians must decide on their usefulness and their reliability at different levels of probability."

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A last reference on the combinatorial significance of $\Pi_n(t)$:

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Added in proof, April 1973. The following additional references became known since these lectures went to the printers.

[100] C. de Boor and G. J. Fix, *Spline approximation by quasiinterpolants*, J. Approximation Theory, 7 (1973), to appear.

The authors construct an important class of spline approximants having many applications. For the special case of cardinal interpolation it furnishes our Theorem 2 of Lecture 2.

[101] I. J. Schoenberg, *Notes on spline functions III. On the convergence of interpolating cardinal splines as their degree tends to infinity*, MRC T.S.R. #1326, April, 1973.

[102] F. B. Richards and I. J. Schoenberg, *Notes on spline functions IV. A cardinal spline analogue of the theorem of the brothers Markov*, MRC T.S.R. #1330, April, 1973. Submitted to the Israel J. Math.

The papers [101] and [102] are contributions to the problems of the last paragraph of Lecture 9. The first gives *sufficient* conditions for the convergence of the interpolating cardinal splines, and the second gives *necessary* conditions for the convergence to take place. There is still a wide gap between the two sets of conditions.

[103] I. J. Schoenberg, *Notes on spline functions V. Orthogonal or Legendre splines*, MRC T.S.R. #1360, to appear.

A main point of this paper is a simple derivation of a generalization of Theorem 2 of Lecture 2, § 3. It furnishes the cardinal splines that interpolate polynomials, of whatever degree, at the integers.

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