Je suggère un

"Practical usability of second order Hermite interpolation"

(plus général)

Hermite Polynomials snakes order 2

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Contents

I Translation of Schoenberg's 1973 paper

include actual bibliography

The following is simply a reminder of some of the results found by I.J Schoenberg in his paper *Cardinal Interpolation and Spline Functions. III Cardinal Hermite interpolation.* Let's reintroduce notations of the article and make somehow more explicits what the objects they encode are.

Let r and m be positive integers that satisfy $r \leq m$. The set of cardinal splines of order $2m$ with knot multiplicity r is denoted by $S_{2m,r}$. Note that using De Boor's notations for splines set we have the following

↳ c'est pas plutôt "with regularity r at the junctions" ? Il me semble que Hermite ordre 3 serait dans $S_{6,2}$ alors même qu'il n'y aurait que des noeuds simples.

je ne suis pas sûre de comprendre cette notation

$$S_{2m,r} = \$_{2m,\mathbb{Z}_r} = \Pi_{<2m,\mathbb{Z},2m-r} \tag{1}$$

where $\mathbb{Z}_3$ denotes the sequence of knots $(\dots,-1,\dots,-1,0,\dots,0,1,\dots,1,\dots)$ that is integers with multiplicity r. It is clear from these notations that $S_{2m,r} \subset \mathcal{C}^{2m-r-1}$.

c'est qui lui?

Theorem 1. *Let S be either of the vector spaces $\mathcal{L}_{p,r}, F_{\gamma,r}$ with $\gamma \geq 0$, $p \in \mathbb{N}^*$. Provided a solution to C.H.I.P $\left(y_\nu,\dots,y_\nu^{(r-1)}, S_{2m,r} \cap S\right)$ exists, it is uniquely given by*

↳ Aucune condition sur y_ν ? Il me semble qu'il manque l'info que y est une fonction, $y^{(n)}$ sa dérivée n-ième, et $y_\nu^{(n)}$ la valeur de sa n-ième dérivée en un entier ν.

$$\forall x \in \mathbb{R} \qquad S(x) = \sum_{\nu=-\infty}^{\infty} y_\nu L_0(x-\nu) + \dots + y_\nu^{(r-1)} L_{r-1}(x-\nu) \tag{2}$$

in practice (pas que pour les active contours!)

In order to specify a usable model ~~for active contours~~ it remains to determine explicit expressions for the basis functions $L_0,\dots,L_{r-1}$, ~~In the article they~~ which are determined by solving a set of $2m-r$ linear equations. This system is obtained by considering separately the function L_s on $[1,\infty)$ and $[0,1]$. ~~Note that~~ S specifying the function on ~~both~~ two these intervals completely determine L_s as the latter is even (if s is even) or odd (if s

is odd).

On $[1, \infty)$, L_s can be decomposed into

$$L_s = \sum_{j=1}^{m-r} c_j S_j$$

where $(c_j)_{j=1}^{m-r}$ are $(m-r)$ unknown coefficients to be determined and S_j are the eigensplines for the first $m-r$ eigenvalues λ_j, solutions to $|\Delta_{r,d}(\lambda)| = 0$.

→ le polynôme est pas plutôt de degré 2m-1 ?

On $[0, 1]$, L_s is given by a polynomial P of order $2m$ that takes a specific form according to the parity of s and r (~~we refer to~~ see equations (7.13) and (7.14)) ~~in the article~~. This polynomial introduces m unknown coefficients $(a_j)_{j=1}^{m}$. To determine a total of $m + m - r = 2m - r$ unknown coefficients we make use of the $2m - r$ equality conditions at 1 $P^{(\rho)}(1) = L_s^{(\rho)}(1)$. We end up of a system of $2m - r$ equations for $2m - r$ unknowns that can be solved exactly provided the matrix of the system is nonsingular. Schoenberg proves with a very nice argument that the matrix of the system is always non singular.

↳ Si tu dis ça il faudrait donner en une phrase l'intuition du nice argument ! :)

I.1 General case $m = r$

~~In the case~~ When $m = r$, splines L_s $(s = 0, \ldots, r-1)$ vanish outside $[-1, 1]$ and therefore give rise to a local interpolation scheme as given below for the case of periodic sequences. Let ~~denote~~ us rewrite $\phi_s = L_{s-1}$ those elements of $S_{2r,r}$ ~~with~~ supported in $[-1, 1]$ and that together satisfy the Hermite interpolation conditions.

Corollary 1. *Given r M-periodic sequences* $(r[k], \ldots, r^{(r-1)}[k])_{k\in\mathbb{Z}}$ *in* $\mathbb{R}^d$*, there exists a unique spline curve of order 2r whose value and derivatives agree with the sequence of coefficients at integers locations. This spline curve and its derivatives are everywhere bounded and take the following form, for* $t \in \mathbb{R}$:

C'est plutôt confusant que la sequence et l'ordre max de dérivée ont le même nom.
Suggestion : renommer la sequence, c par exemple.

$$r(t) = \sum_{k\in\mathbb{Z}} \sum_{l=1}^{r} r^{(l-1)}[k]\phi_l(t-k) \tag{3}$$

$$= \sum_{k=0}^{M-1} \sum_{l=1}^{r} r^{(l-1)}[k]\phi_{l,per}(t-k), \tag{4}$$

where $\phi_{l,per} = \sum_{k\in\mathbb{Z}} \phi_l(. - Mk)$

~~In case~~ When the sequences are not periodic but have a finite number of M non-zero entries ~~say M~~, we can similarly restrict the infinite sum to M elements without periodizing the basis functions.

↱ pourquoi ces étranges brackets ?

Corollary 2. *Given r sequences* $(r[k], \ldots, r^{(r-1)}[k])_{k\in\mathbb{Z}}$ *in* $\mathbb{R}^d$ *that vanish outside* $[\![0, M]\!]$*, there exists a unique spline curve of order 2r whose value and derivatives agree with the sequence of coefficients at integers locations. This spline curve and its derivatives have compact support and take the following form, for* $t \in \mathbb{R}$:

$$r(t) = \sum_{k\in\mathbb{Z}} \sum_{l=1}^{r} r^{(l-1)}[k]\phi_l(t-k) \tag{5}$$

$$= \sum_{k=0}^{M-1} \sum_{l=1}^{r} r^{(l-1)}[k]\phi_l(t-k). \tag{6}$$

I.2 Case $m = r = 3$

In the case $m = r = 3$, L_0, L_1, L_2 are 0 on $[1, \infty)$ and on $[0, 1]$ are given by

Pourquoi ceux-ci ne sont pas des paramètres également ? (comme les a_1, a_2, a_3)

$$L_0(x) = 1 + a_1 x^3 + a_2 x^4 + a_3 x^5 \quad (7)$$
$$L_1(x) = x + a_1 x^3 + a_2 x^4 + a_3 x^5 \quad (8)$$
$$L_2(x) = \frac{1}{2} x^2 + a_1 x^3 + a_2 x^4 + a_3 x^5 \quad (9)$$

C'est confusant qu'ils aient le même nom alors qu'ils sont différent → pourquoi ne pas les nommer e.g. a_i^3 avec i le numéro du générateur ?

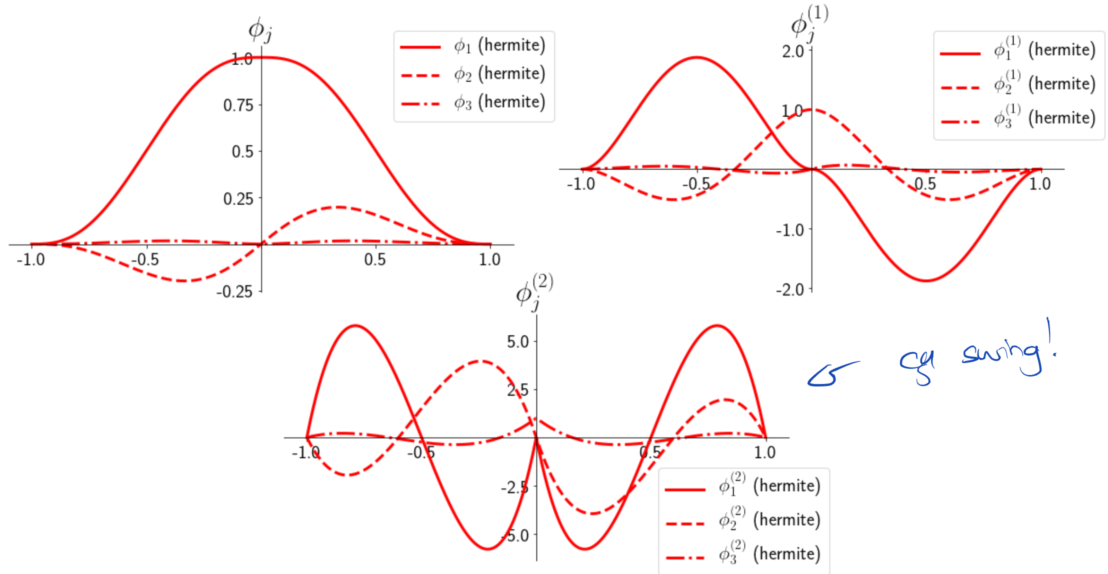
Bonne question :)

where the coefficients for each generator are unrelated. Note that L_s have finite support *because* $m = r$. If that was not the case the term $\sum_{j=1}^{m-r} c_j S_j$ may not be 0 and therefore L_s would be non zero on $[1, \infty)$! Can it happen though that $m > r$ and $(c_j)_{j=1}^{m-r}$ are 0? To determine the coefficients above we need to solve independently for each generator the 3 equations $P^{(\rho)}(1) = 0$. This leads to the following systems

$$\begin{cases} a_1 + a_2 + a_3 = -1 \\ 3a_1 + 4a_2 + 5a_3 = 0 \\ 3a_1 + 6a_2 + 10a_3 = 0 \end{cases} \begin{cases} a_1 + a_2 + a_3 = -1 \\ 3a_1 + 4a_2 + 5a_3 = -1 \\ 3a_1 + 6a_2 + 10a_3 = 0 \end{cases} \begin{cases} a_1 + a_2 + a_3 = -\frac{1}{2} \\ 3a_1 + 4a_2 + 5a_3 = -1 \\ 3a_1 + 6a_2 + 10a_3 = -\frac{1}{2} \end{cases}$$

II The resulting snake scheme for $m = r = 3$

II.1 Generating functions



ça swing!

Figure 1: Generators for C.H.I.P with $m = r = 3$

Solving the linear systems written in the previous section yields explicit formulas for the Schoenberg basis generators L_0, L_1, L_2, that we rename ϕ_1, ϕ_2, ϕ_3 in accordance with modern notations (see ~~V. Uhlmann *Hermite Snakes with Controls of Tangents*~~). The formulas are the following.

put direct ref [X]

$$\phi_1(x) = \begin{cases} 1 - 10x^3 + 15x^4 - 6x^5 & \text{if } 0 \le x \le 1 \\ 1 + 10x^3 + 15x^4 + 6x^5 & \text{if } -1 \le x < 0 \end{cases} \tag{10}$$

$$\phi_2(x) = \begin{cases} x - 6x^3 + 8x^4 - 3x^5 & \text{if } 0 \le x \le 1 \\ x - 6x^3 - 8x^4 - 3x^5 & \text{if } -1 \le x < 0 \end{cases} \tag{11}$$

$$\phi_3(x) = \begin{cases} 0.5x^2 - 1.5x^3 + 1.5x^4 - 0.5x^5 & \text{if } 0 \le x \le 1 \\ 0.5x^2 + 1.5x^3 + 1.5x^4 + 0.5x^5 & \text{if } -1 \le x < 0 \end{cases} \tag{12}$$

~~In figure 1 are displayed the values of~~ These functions as well as their two first derivatives are displayed in figure 1. As mentioned in the previous section, the generators L_s are elements of $S_{2m,r} = S_{6,3}$ which is a subset of $\mathcal{C}^{2m-r-1} = C^2$. It is apparent in the figure that these functions have continuous derivatives up to order 2 ~~but~~ that higher order derivatives do not exist in the neighborhoods of $-1, 0$ and 1.

pourquoi n'est-ce pas le même C?

II.2 Closed planar curves or "contours"

Consider a positive integer M and an M**-periodic parametrized closed curve** $r : \mathbb{R} \to \mathbb{R}^2$ for which we have local derivatives up to order 2 at M location ~~sites~~ s regularly spaced. That is, we know $(r[k], r'[k], r''[k])_{k=0}^{M-1}$.

Corollary 3. *Given M periodic sequences $(r[k], r'[k], r''[k])_{k\in\mathbb{Z}}$ in $\mathbb{R}^2$, there exists a unique spline curve of order 6 whose value and derivatives agree with the sequence of coefficients at integers locations. This spline curve and its derivatives are everywhere bounded, and take the following form, for $t \in \mathbb{R}$,*

Je trouve confusant de noter $(r[k], r'[k], r''[k])$ in $\mathbb{R}^2$: ce que tu veux dire c'est que chaque élément de ce triplet est dans $\mathbb{R}^2$. Pour ça je suggérais d'abord d'utiliser des notations bold pour les r (pour montrer qu'il s'agit de vecteurs), les définir, et changer "in $\mathbb{R}^2$" pour "whose each element is a sequence in $\mathbb{R}^2$"

$$r(t) = \sum_{k\in\mathbb{Z}} r[k]\phi_1(t-k) + r'[k]\phi_2(t-k) + r''[k]\phi_3(t-k) \tag{13}$$

upright (\mathrm{ })

$$= \sum_{k=0}^{M-1} r[k]\phi_{1,per}(t-k) + r'[k]\phi_{2,per}(t-k) + r''[k]\phi_{3,per}(t-k)\,. \tag{14}$$

Proof. As the sequence of coefficients $(r[k], r'[k], r''[k])_{k\in\mathbb{Z}}$ are in $Y_{\gamma,r} = Y_{0,3}$ (i.e., they are bounded), ~~application of~~ Schoenberg's Theorem 1 (ref) yields existence and unicity of an interpolating function in $S_{6,3} \cap F_{0,3}$. ~~Application of~~ Theorem 4 (ref) then leads to the explicit formulation given above. $\square$

Remark 1. *It is convenient to normalize the continuous parameter to the $[0,1]$ interval ~~as is usual in the~~ for implementations purpose. For that, let the renormalized curve $s(t) = r(Mt)$ for $t \in [0,1]$. Note that this completely describes the curve as it is enough to describe the curve r on $[0, M]$. Differentiating this equality twice yields $r[k] = s[\frac{k}{M}], r'[k] = \frac{1}{M}s'[\frac{k}{M}], r''[k] = \frac{1}{M^2}s''[\frac{k}{M}]$. Therefore, ~~equation~~ (14) is rewritten, for $t \in [0,1]$, as*

$$s(t) = \sum_{k=0}^{M-1} s[\frac{k}{M}]\phi_{1,per}(Mt-k) + \frac{1}{M}s'[\frac{k}{M}]\phi_{2,per}(Mt-k) + \frac{1}{M^2}s''[\frac{k}{M}]\phi_{3,per}(Mt-k)\,. \tag{15}$$

In the rest of this document, we will reuse the notation r for the normalized curve and won't make use of the notation s anymore. ~~Equation (15) is~~ The **mathematical representation of a planar curve** r is given by (15) and ~~we call it "snake" or~~ can be used in an "active contour" framework. By playing with the coefficients we can capture a wide variety of contours ~~that arise from closed~~ of objects in 2D images, like a cell membrane in a bioimage.

"active contour" c'est l'algorithme complet (un modèle de courbe + son énergie et une méthode d'optimisation). Ce dont tu parles ici c'est le "deformable curve model".

II.3 Open planar curves

Consider again a positive integer M and a **parametrized open curve** $r : \mathbb{R} \to \mathbb{R}^2$ for which we have local derivatives up to order 2 at M location ~~sites~~ s regularly spaced. That is, we know $(r[k], r'[k], r''[k])_{k=0}^{M-1}$. By "open", we mean a curve that is not periodic.

I think it is pretty obvious, by "open" you mean a curve that is not closed :)

Corollary 4. *Given biinfinite sequences of coefficients $(\ldots, 0, r[0], \ldots, r[M-1], 0, \ldots)$, $(\ldots, 0, r'[0], \ldots, r'[M-1], 0, \ldots)$, $(\ldots, 0, r''[0], \ldots, r''[M-1], 0, \ldots)$ in $\mathbb{R}^2$, there exists a unique spline curve of order 6 whose value*

↳ voir ma remarque au corollaire 3

and derivatives agree with the sequence of coefficients at integers locations. This spline curve and its derivatives have compact support and take the, form, for $t \in \mathbb{R}$:

following

$$r(t) = \sum_{k\in\mathbb{Z}} r[k]\phi_1(t-k) + r'[k]\phi_2(t-k) + r''[k]\phi_3(t-k)$$
$$= \sum_{k=0}^{M-1} r[k]\phi_1(t-k) + r'[k]\phi_2(t-k) + r''[k]\phi_3(t-k) .$$

[X] (cited ref)

Proof. This result is again a simple application of Theorems 1 and 4 given in ~~Schoenberg's paper of 1981~~. □

Remark 2. *In this setting, we are only interested in the curve lying between our coefficients; that is, the interpolated points with continuous parameter in the interval $[0, M-1]$ The normalization factor is therefore $M-1$ and the renormalized open curve $r(t) = r((M-1)t)$ takes the form*

$$r(t) = \sum_{k=0}^{M-1} r[\frac{k}{M-1}]\phi_1((M-1)t-k) + r'[\frac{k}{M-1}]\frac{\phi_2((M-1)t-k)}{M-1} + r''[\frac{k}{M-1}]\frac{\phi_3((M-1)t-k)}{(M-1)^2} \quad (16)$$

II.4 Closed sphere-like surfaces

model

In my research project, we are interested in developing a ~~mathematical methods~~ for representing ~~a certain~~ ~~types~~ of surfaces with explicit control of local properties including first-order derivatives and curvature. As a consequence, extension of the schemes given in ~~equations~~ (15) and (16) to tensor-product surfaces (that is, surfaces parametrized by 2 continuous parameters in a way that each continuous parameter appear in separate functions) ~~may be~~ is relevant ~~for the questions we have~~.

Consider positives integers M_1 and M_2 and a **sphere-like parametrized** surface $\sigma : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ with U a subset (closed in our case) of the plane. By "sphere-like" we mean an object that can be described with closed curves on latitudes (u varies while v is fixed) and open curves on longitudes (v varies while u is fixed). Suppose we have local properties of the surface at $M_1 \times (M_2+1)$ locations (counted with multiplicity as some locations may coalesce) on a regular grid.

je connaissais pas cette notation. c'est standard?

Corollary 5. *Given the biinfinite sequences of coefficients $(\partial^{i,j}\sigma(k,l))_{k,l\in\mathbb{Z}^2,(i,j)\in\{0,1,2\}^2}$ in $\mathbb{R}^3$, that are M_1 periodic in the first coordinate and vanish when the second coordinate is outside $[\![0, M_2]\!]$, there exists a unique interpolating tensor-product spline curve of order 6 whose value and partial derivatives agree with the sequence of coefficients at the integers grid locations. This tensor-product spline curve and its derivatives are everywhere bounded and take the, form, for $(u,v) \in [0, M_1] \times [0, M_2]$ (or equiv. $\mathbb{R} \times [0, M_2]$),*

following

$$\sigma(u,v) = \sum_{k=0}^{M_1-1}\sum_{l=0}^{M_2}\sum_{i,j=0}^{2} \partial^{i,j}\sigma(k,l)\phi_{i+1,per}(u-k)\phi_{j+1}(v-l) . \quad (17)$$

Proof. This is a simple application of corollaries 1 and 2 given before. □

Remark 3.
- *Not all surfaces admit a tensor-product representation, which limits the range of surfaces one can reach with this kind of interpolation scheme. Tensor-product spline has not yet been defined and we just mention here that we call a tensor-product spline a map $f : U_1 \times U_2 \subseteq \mathbb{R}^2 \to \mathbb{R}$ that can be written as $f_1(u) \times f_2(v)$ with f_1 and f_2 each splines on U_1 and U_2 respectively.*
- *The continuity of the basis functions is automatically transferred to the each coordinate of the spline curve, resulting in surfaces with parametrizations that are twice continuously differentiable. Is this preventing us from representing surfaces with singular points as we would like to have? This question is crucial to our objective and will be addressed later in more details.*

- *Normalizing each continuous parameter to the interval* $[0,1]$ *yields the following representation of the surface (where we note again* σ *the surface with normalized parameters)*

$$\sigma(u,v) = \sum_{k=0}^{M_1-1}\sum_{l=0}^{M_2}\sum_{i,j=0}^{2} \frac{1}{M_1^i M_2^j}\partial^{i,j}\sigma(\frac{k}{M_1},\frac{l}{M_2})\phi_{i+1,per}(M_1u-k)\phi_{j+1}(M_2v-l) \tag{18}$$

III Properties of the interpolation scheme

III.1 The Riesz-Schauder basis property

III.1.1 Definition

Definition 1. *Let* $\mathcal{H}$ *be a Hilbert space over the field* $\mathbb{K}=\mathbb{R}$ *or* $\mathbb{C}$ *of real or complex numbers. A basis* $\{\phi_k\}_{k\in\mathbb{Z}}$ *of* $\mathcal{H}$ *is said to be a Riesz-Schauder basis if there exist positive constants* $0<m\leq M$ *such that*

$$\forall c \in l_2(\mathbb{Z})(\subset \mathbb{K}^{\mathbb{Z}}) \qquad m\|c\|_{l_2}^2 \leq \|\sum_{k\in\mathbb{Z}} c_k\phi_k\|_{\mathcal{H}}^2 \leq M\|c\|_{l_2}^2 \tag{19}$$

il s'est un peu perdu en bas!

Let n be a positive integer, f,g functions $\mathbb{R}^n\to\mathbb{C}$. Define λ the Lebesgue measure on $\mathbb{R}^n$ and the map

$$\langle f,g\rangle = \int_{\mathbb{R}^n} f\bar{g}d\lambda$$

This map is an inner product on the vector space over $\mathbb{C}$ of all measurable functions for which $\langle f,f\rangle$ is finite (to be precise we consider quotient space so that its elements are equivalence classes for the relation that two functions are equivalent if they agree λ-a.e). The latter, endowed with this inner product, is therefore a pre-Hilbert space. It is a known result that this space is complete and therefore a Hilbert space. add a reference! It is usually denoted by $L_2(\mathbb{R}^n)$. For an element f of that space, we denote $\hat{f}$ the extension (by density) to functions in $L_2(\mathbb{R}^n)$ of the Fourier transform as usually defined in mathematics textbooks over functions in $L_1(\mathbb{R}^n)$.

III.1.2 Characterization in the Fourier domain

Theorem 2. *~~In case~~ If the Hilbert space* $\mathcal{H}$ *is* $L_2(\mathbb{R}^n)$ *and* $\phi\in\mathcal{H}$*, the following statements for some* $0<m\leq M$ *are equivalent*

1. $\forall c\in l_2(\mathbb{Z}),\ m\|c\|_{l_2}^2 \leq \|\sum_{k\in\mathbb{Z}} c_k\phi(.-k)\|_{L_2}^2 \leq M\|c\|_{l_2}^2$;

2. $m \leq \sum_{k\in\mathbb{Z}} |\hat{\phi}(w+2k\pi)|^2 \leq M$.

J'aime mieux la notation $c=\{c[k]\}_{k\in\mathbb{Z}}\in l_2(\mathbb{Z})$, mais c'est peut être un peu lourd

use \color instead

~~with obvious notations. In case~~ If any of the two statements holds, the subspace $V=\{\sum_{k\in\mathbb{Z}} c_k\phi(.-k)|c\in l_2(\mathbb{Z})\}$ *of* $L_2(\mathbb{R}^n)$ *is closed and thus a Hilbert space itself, and* $\{\phi(.-k)\}$ *is a Riesz-Schauder basis.*

use \cite{} and put direct ref

Proof. This theorem is exactly Theorem 2 as given in ~~A. Aldroubi, M. Unser, "Sampling procedures in function spaces and asymptotic equivalence with Shannon's sampling theory," Numer. Funct. Anal. Optim., vol. 15, nos. 1-2, pp. 1-21, May 1994~~ and is proved there for the case $n=1$ ~~but~~, Extension to domain of generic dimension is straightforward. □

Regarding the needs of having ~~this~~ a Riesz-Schauder basis property equivalent for vector-valued basis functions, ~~let's give an~~ let us consider the immediate corollary ~~for that case. For that let's~~ define the map $\langle\cdot,\cdot\rangle$ over functions $\mathbb{R}^n\to\mathbb{C}^m$ by

following

$$\langle f,g\rangle = \sum_{r=1}^{m}\int_{\mathbb{R}^n} f_r\bar{g}_r d\lambda$$

remove indent

with obvious notations. This map defines an inner product and the space $L_2(\mathbb{R}^n,\mathbb{C}^m)$ of functions having finite norm as induced by this inner product is a Hilbert space.

Theorem 3. ~~*In case*~~ If *the Hilbert space $\mathcal{H}$ is $L_2(\mathbb{R}^n, \mathbb{C}^m)$ and $\phi \in \mathcal{H}$, the following statements for some $0 < m \le M$ are equivalent*

1. $\forall c \in l_2(\mathbb{Z})$, $m\|c\|_{l_2}^2 \le \|\sum_{k\in\mathbb{Z}} c_k\phi(.-k)\|^2_{L_2(\mathbb{R}^n,\mathbb{C}^m)} \le M\|c\|_{l_2}^2$;
2. $m \le \sum_{k\in\mathbb{Z}}\sum_{r=1}^{m} |\hat{\phi}_r(w+2k\pi)|^2 = \sum_{k\in\mathbb{Z}} \|\hat{\phi}(w+2k\pi)\|_2^2 \le M$.

~~*with obvious notations.*~~ ~~*In case*~~ If *any of the two statements holds, the subspace $V = \{\sum_{k\in\mathbb{Z}} c_k\phi(.-k) | c \in l_2(\mathbb{Z})\}$ of $L_2(\mathbb{R}^n, \mathbb{C}^m)$ is closed and thus a Hilbert space itself, and $\{\phi(.-k)\}$ is a Riesz-Schauder basis.*

III.1.3 Characterization from Gram matrix

In ~~cases where there are several~~ the multi- generators ~~, like the~~ case such as Hermite polynomial interpolation, one can characterize the ~~property of~~ Riesz-Schauder basis property from the Gram matrix of the set of generators $\{\phi_i(.-k)\}$. For that, let $\{\phi^i\}_{i=1}^q$ be functions in a Hilbert space $\mathcal{H}$ and ~~let~~ O be a unitary operator on $\mathcal{H}$, i.e., $OO^* = O^*O = I$ (~~in our case of~~ for Hermite interpolation, $O\phi = \phi(.-1)$).

Consider the subspace $V = \{\sum_{k\in\mathbb{Z}} C(k)O^k\Phi = \sum_{k\in\mathbb{Z}}\sum_{i=1}^{q} c^i(k)O^k\phi^i | c^1, \ldots, c^q \in l_2(\mathbb{Z})\}$. As seen before, V is a well-defined closed subspace of $\mathcal{H}$ if there exists constants $0 < m \le M$ such that

$$\forall C \in l_2^q(\mathbb{Z}) \quad m\|C\|_{l_2^q}^2 \le \|\sum_{k\in\mathbb{Z}} C(k)O^k\Phi\|^2_{\mathcal{H}} \le M\|C\|_{l_2^r}^2 . \tag{20}$$

Note that the central term of the inequality above can be rewritten as

$$\begin{aligned}\langle \sum_{k\in\mathbb{Z}} C(k)O^k\Phi, \sum_{l\in\mathbb{Z}} C(l)O^l\Phi\rangle &= \sum_{(k,l)\in\mathbb{Z}^2} C(k)\langle O^k\Phi, O^l\Phi\rangle C(l)^* \\ &= \sum_{(k,l)\in\mathbb{Z}^2} C(k)A(l-k)C(l)^* ,\end{aligned}$$

where A is a $q\times q$ matrix with elements $A_{i,j}(k) = \langle\phi^i, O^k\phi^j\rangle$. Now, the above can be viewed as

$$\sum_{(k,l)\in\mathbb{Z}^2} C(k)A(l-k)C(l)^* = \sum_{i,j=1}^{q}\sum_{l\in\mathbb{Z}} (c^i * A_{i,j})(l){c^j}^*(l) ,$$

Using Parseval's theorem for discrete time Fourier transform, that is, $\hat{c}(w) = \sum_{k\in\mathbb{Z}} c(k)e^{-jwk}$ or, in f-notation, $\hat{c}(f) = \sum_{k\in\mathbb{Z}} c(k)e^{-j2\pi fk}$. thus ~~We~~ we have that

$$\begin{aligned}\sum_{i,j=1}^{q}\sum_{l\in\mathbb{Z}} (c^i * A_{i,j})(l){c^j}^*(l) &= \sum_{i,j=1}^{q} \frac{1}{2\pi}\int_0^{2\pi} \hat{c^i}(w)\hat{A}_{i,j}(w)\hat{c^j}^*(w)\frac{dw}{2\pi} \\ &= \frac{1}{2\pi}\int_0^{2\pi} \hat{C}(w)\hat{A}(w)\hat{C}^*(w)\frac{dw}{2\pi} \\ &= \int_0^1 \hat{C}(f)\hat{A}(f)\hat{C}^*(f)df .\end{aligned}$$

The matrix $\hat{A}(w)$ with elements $A_{i,j}(w) = \sum_{k\in\mathbb{Z}}\langle\phi^i, O^k\phi^j\rangle e^{-jwk}$ is called the **Gram matrix**. Now, Theorem 2.1 from ~~1996 paper by Adroubi *Oblique Projections in Atomic Spaces*~~ cite 33 ~~makes~~ establishes the link between ~~m, M~~ the constants m and M characterizing the Riesz-Schauder basis property and the minimum and maximum eigenvalue of the Gram matrix as follows.

Theorem 4. *The space V is a well-defined closed subspace of $\mathcal{H}$ with Riesz-Schauder basis $\{O^k\phi^i\}_{i=1,\dots,q,k\in\mathbb{Z}}$ if and only if the $q\times q$ Gram matrix $\hat{A}(w)$ is hermitian positive for almost all w, and there exist two constants $0<m\leq M$ such that*

$$m \leq \underset{w\in[-\pi,\pi]}{\text{ess inf}}\ \lambda_{min}(\hat{A}(w)) \leq \underset{w\in[-\pi,\pi]}{\text{ess sup}}\ \lambda_{max}(\hat{A}(w)) \leq M. \tag{21}$$

Proof. This is ~~exactly~~ Theorem 2.1 as stated in ~~the article mentioned~~ [X] and the proof is given there. □

III.2 Riesz-Schauder basis for Hermite interpolation order r

Let r be a positive integer and $\phi_1,\dots,\phi_r$ the elements of $S_{2r,r}\cap\mathcal{L}_{1,r}$ with support in $[-1,1]$ that satisfy Hermite interpolation conditions as introduced in Subsection I.1. Consider a positive integer d representing the dimension of the space in which the coefficients of the resulting scheme live, i.e.,

où apparaît le shift ? je mettrais plutôt $\phi_1(.-k)\dots\phi_r(.-k)$

$$V=\{\sum_{k\in\mathbb{Z}}C(k)\Phi(.-k)|C\in l_2^{d\times r}(\mathbb{Z})\}, \tag{22}$$

with $\Phi(.-k)=\begin{pmatrix}\phi_1 & \cdots & \phi_r\end{pmatrix}^T$, $C(k)=\begin{pmatrix}c_1(k) & \cdots & c_r(k)\end{pmatrix}$ and $c_l\in l_2^d(\mathbb{Z})$. Note that V is a subspace of $L_2(\mathbb{R},\mathbb{R}^d)$. To state properly the Riesz-basis property, we need to exhibit functions that are elements of $L_2(\mathbb{R},\mathbb{R}^d)$. For that, consider the functions ϕ_je_i for $j=1,\dots,r$, $i=1,\dots,d$, with $(e_i)_{i=1}^d$ the canonical basis of $\mathbb{R}^d$. Then, V can be rewritten as

$$V=\{\sum_{k\in\mathbb{Z}}\sum_{j=1}^r\sum_{i=1}^d c_{j,i}(k)\phi_j(.-k)e_i|c\in l_2(\mathbb{Z})\}, \tag{23}$$

which amounts to taking $\Phi(.-k)=\begin{pmatrix}\phi_1e_1 & \cdots & \phi_1e_d & \cdots & \phi_re_1 & \cdots & \phi_re_d\end{pmatrix}^T$, $C(k)=\begin{pmatrix}c_1(k) & \cdots & c_{rd}(k)\end{pmatrix}$ Following the ~~in~~ notations of (22). Theorem 5 shows that the Riesz-Schauder basis property holds at any order r for Hermite polynomial interpolation.

Theorem 5. *Let r be a positive integer, $\phi_1,\dots,\phi_r$ the elements of $S_{2r,r}\cap\mathcal{L}_{1,r}$ with support in $[-1,1]$ that satisfy Hermite interpolation conditions, and d the dimension of the coefficients of the induced scheme. Then $\{\phi_j(.-k)e_i\}_{j=1,\dots,r,i=1,\dots,d,k\in\mathbb{Z}}$ is a Riesz-Schauder basis of*

$$V=\{\sum_{k=-\infty}^{\infty}\sum_{j=1}^r\sum_{i=1}^d c_{k,i}^j\phi_j(.-k)e_i|c\in l_2(\mathbb{Z})\} \tag{24}$$

Remark 4. *Having a Riesz-Schauder basis makes V a closed subspace of $L_2(\mathbb{R},\mathbb{R}^d)$. The latter being a Hilbert space, V is a Hilbert space itself.*

Proof. In order to prove that our set of generators is a Riesz-Schauder basis we are going to use characterization from the Gram matrix $\hat{A}(w)$.

1. **Expression of the Gram matrix**

   The generators are elements of $L_2(\mathbb{R},\mathbb{R}^d)$ with inner product $\langle\phi,\psi\rangle=\sum_{l=1}^d\langle\phi_l,\psi_l\rangle_{L_2}$. As a consequence, $\langle\phi_{j_1}e_{i_1},\phi_{j_2}(.-k)e_{i_2}\rangle=0$ whenever $i_1\neq i_2$, and the Gram matrix entries are non-zero only at indices that are multiple integers of d. It easier to express the Gram matrix as the block matrix

   Je suis pas fan de cette notation mais j'ai pas mieux à proposer là comme ça

$$\hat{A}(w)=\begin{pmatrix}\hat{B}_{1,1}(w) & \hat{B}_{1,2}(w) & \cdots & \hat{B}_{1,r}(w)\\ \vdots & \vdots & \ddots & \vdots\\ \hat{B}_{r,1}(w) & \hat{B}_{r,2}(w) & \cdots & \hat{B}_{r,r}(w)\end{pmatrix} \tag{25}$$

   with the blocks being diagonal matrices given by

$$\hat{B}_{i,j}(w) = \sum_{k\in\mathbb{Z}} \langle \phi_i, \phi_j(. - k)\rangle e^{-jwk} I_d \tag{26}$$

One can show by recurrence (**to be done**) that the determinant of $rd \times rd$ matrices as in (25) ~~have a determinant which~~ corresponds to ~~is~~ the d^{th} exponent of the determinant of the submatrix (27) of size $r \times r$

$$\hat{G}(w) = \sum_{k\in\mathbb{Z}} \begin{pmatrix} \langle \phi_1, \phi_1(. - k)\rangle & \cdots & \langle \phi_1, \phi_r(. - k)\rangle \\ \vdots & \ddots & \vdots \\ \langle \phi_r, \phi_1(. - k)\rangle & \cdots & \langle \phi_r, \phi_r(. - k)\rangle \end{pmatrix} e^{-jwk} \tag{27}$$

This ~~is~~ also applies to $\hat{A}(w) - \lambda I$, whose determinant is the characteristic polynomial of $\hat{A}$. Therefore, $\chi_{\hat{A}(w)}(\lambda) = \chi_{\hat{G}(w)}(\lambda)^d$, which means that $\hat{A}(w)$ and $\hat{G}(w)$ share the same eigenvalues. The study of eigenvalues of $\hat{A}(w)$ can thus be replaced by that of $\hat{G}(w)$.

2. **Gram matrix is self-adjoint**

   To prove ~~this~~ that the Gram matrix is self-adjoint, it is enough to prove that $\overline{\hat{B}_{j,i}(w)} = \hat{B}_{i,j}(w)$ ~~which is proven below~~:

$$\begin{aligned} \overline{\hat{B}_{j,i}(w)} &= \sum_{k\in\mathbb{Z}} \overline{\langle \phi_j, \phi_i(. - k)\rangle e^{-jwk}} I_d \\ &= \sum_{k\in\mathbb{Z}} \langle \phi_i(. - k), \phi_j\rangle e^{jwk} I_d \\ &= \sum_{k\in\mathbb{Z}} \langle \phi_i(. + k), \phi_j\rangle e^{-jwk} I_d \\ &= \sum_{k\in\mathbb{Z}} \langle \phi_i, \phi_j(. - k)\rangle e^{-jwk} I_d \\ &= \hat{B}_{i,j}(w) \,. \end{aligned}$$

3. **Gram matrix is positive definite a.e.**

   Suppose, on the contrary, that $\hat{A}$ is not positive definite a.e. Given that $\hat{A} : w \to \hat{A}(w)$ is 2π-periodic, this means that there exists $E \subseteq [0, 2\pi]$ whose Lebesgue measure $\lambda(E) > 0$ and such that

$$\forall w \in E, \exists \hat{X}(w) \in \mathbb{C}^{rd} \setminus \{0\}, \quad \hat{X}(w)\hat{A}(w)\hat{X}(w)^* \leq 0$$

Let $\hat{C} : \mathbb{R} \to \mathbb{C}^{rd}$ the 2π-periodic function such that $\hat{C}_{|[0,2\pi]} = \hat{X}(w)\mathbb{1}_E(w)$. $\hat{C}$ is the Fourier transform of the discrete function $C : \mathbb{Z} \to \mathbb{C}^{rd}$, with $C(k) = \frac{1}{2\pi}\int_0^{2\pi} \hat{C}(w)e^{-jwk}dw$ and

$$\int_0^{2\pi} \hat{C}(w)\hat{A}(w)\hat{C}(w)^* \leq 0 \,.$$

However, from Parseval's theorem and properties of the Fourier transform, we have

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} \hat{C}(w)\hat{A}(w)\hat{C}^*(w)dw &= \sum_{k\in\mathbb{Z}} C(k)A(l-k)C(l)^* \\ &= \langle \sum_{k\in\mathbb{Z}} C(k)\Phi(. - k), \sum_{l\in\mathbb{Z}} C(l)\Phi(. - l)\rangle \\ &= \| \sum_{k\in\mathbb{Z}} C(k)\Phi(. - k)\|^2_{L_2(\mathbb{R},\mathbb{C}^d)} \,. \end{aligned}$$

Consequently, the norm above is 0 (~~reminder: it is~~ we recall that we consider the norm on $L_2(\mathbb{R}, \mathbb{C}^d)$, whose elements are equivalent classes). As our function is continuous, this leads to

$$\sum_{k\in\mathbb{Z}} C(k)\Phi(.-k) = 0 . \tag{28}$$

We now make use of the Hermite interpolation conditions of $\Phi = \begin{pmatrix}\phi_1 e_1 & \cdots & \phi_1 e_d & \cdots & \phi_r e_1 & \cdots & \phi_r e_d\end{pmatrix}^T$ at integer locations. Let $k_0 \in \mathbb{Z}$. Evaluating (28) and its derivatives up to $(r-1)$ at k_0 leads to $C(k_0) = 0$. As k_0 ~~being~~ is arbitrary chosen, we have $C \equiv 0$, which in turn leads to $\hat{X} \equiv 0$. This contradicts our initial choice of E and $\hat{X}$. Consequently, $\hat{A}$ is positive definite almost everywhere.

J'imagine que ceci a disparu dans la dernière version ?

4. **Continuity and compacity**

→ Je ne connais pas cette notation - est-ce standard d'avoir un exposant ℤ comme ça ?

For the needs of what follows, let's prove that the map $T : (\mathbb{C}^{rd})^{\mathbb{Z}} \to L_2(\mathbb{R},\mathbb{C})^d$ such that $T(C) = \sum_{k\in\mathbb{Z}} C(k)\Phi(.-k)$ is continuous and that $\{C \in (\mathbb{C}^{rd})^{\mathbb{Z}} | \|C\|^2_{l_2^{rd}} = 1\}$ is compact. For that, notice that Φ has compact support and thus only a finite number of inner products $\langle \Phi, \Phi(.-k)\rangle$ are non zero. ~~Let~~ Consider k_Φ such that $\langle \Phi, \Phi(.-k)\rangle = 0$ for $|k| > k_\Phi$. Consequently,

$$\begin{aligned}
\|\sum_{k\in\mathbb{Z}} C(k)\Phi(.-k)\|^2_{L_2(\mathbb{R},\mathbb{C}^d)} &= \sum_{(k,l)\in\mathbb{Z}^2} C(k)C(l)^*\langle \Phi, \Phi(.-(l-k))\rangle \\
&\le \sup_{|k|\le k_\Phi} \langle \Phi, \Phi(.-k)\rangle \sum_{|k-l|\le k_\Phi} |C(k)C(l)^*| \\
&\le \sup_{|k|\le k_\Phi} \langle \Phi, \Phi(.-k)\rangle \sum_{l=-k_\Phi}^{k_\Phi} \sum_{k\in\mathbb{Z}} |C(k)||C(l+k)|^* \\
&\le (2k_\Phi + 1) \sup_{|k|\le k_\Phi} \langle \Phi, \Phi(.-k)\rangle \|C\|^2_{l_2^{rd}},
\end{aligned}$$

using Cauchy-Schwarz inequality for the inner product on $(\mathbb{C}^{rd})^{\mathbb{Z}}$ given by $\langle C_1, C_2\rangle = \sum_{k\in\mathbb{Z}} C_1(k)C_2(k)^*$.

Now, consider ~~now~~ the map $\psi : (\mathbb{C}^{rd})^{\mathbb{Z}} \to \bar{\mathbb{R}}$ such that $\psi(C) = \|C\|^2_{l_2^{rd}}$. Given that $|\psi(C)| \to \infty$ as $\|C\| \to \infty$, ψ is a proper function. Therefore the preimage of every compact set of $\bar{\mathbb{R}}$ is a compact set hence $\{C \in (\mathbb{C}^{rd})^{\mathbb{Z}} | \|C\|^2_{l_2^{rd}} = 1\} = \psi^{-1}(\{1\})$ is compact.

Ah je vois de remarque que la n'a pas la dernière version :-)

5. **Eigenvalues of $\hat{A}$ are essentially bounded**

Suppose that the essential supremum of $\lambda_{max} : w \to \max \mathrm{sp}(\hat{A}(w))$ is infinite. Using 2π-periodicity of $\hat{A}$, it means that

→ upright (\mathrm) - valide pour tous les "max" ci-dessous

$$\forall p \in \mathbb{N}^*, \exists E_p \subseteq [0, 2\pi], \forall w \in E_p, \quad \lambda_{max}(w) > p .$$

Now, for each $w \in E_p$, let $\hat{V}_{max}(w)$ be a normalized eigenvector, i.e, $\hat{A}(w)\hat{V}_{max}(w) = \lambda_{max}(w)\hat{V}_{max}(w)$ and $\|\hat{V}_{max}(w)\|^2_{\mathbb{C}^{rd}} = 2\pi$. Let $\hat{C}_p$ be the 2π-periodic function such that $\hat{C}_{p|_{[0,2\pi]}}(w) = \lambda(E_p)^{-0.5}\mathbb{1}_{E_p}(w)\hat{V}_{max}(w)$ and let C_p be the discrete function that maps $k \in \mathbb{Z}$ to $C_p(k) = \frac{1}{2\pi}\int_0^{2\pi} \hat{C}_p(w)e^{-jwk}dw$. From Parseval's theorem, $\|C_p\|^2_{l_2^{rd}} = 1$ ~~and with~~ following the same calculations as in ~~the previous point~~ 4., we have

$$\frac{1}{2\pi}\int_0^{2\pi} \hat{C}_p(w)\hat{A}(w)\hat{C}_p(w)^* > p,$$

while

$$
\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi}\hat{C_p}(w)\hat{A}(w)\hat{C_p}^*(w)dw &= \sum_{k\in\mathbb{Z}} C_p(k)A(l-k)C_p(l)^* \\
&= \langle \sum_{k\in\mathbb{Z}} C_p(k)\Phi(.-k), \sum_{l\in\mathbb{Z}} C_p(l)\Phi(.-l)\rangle \\
&= \| \sum_{k\in\mathbb{Z}} C_p(k)\Phi(.-k)\|^2_{L_2(\mathbb{R},\mathbb{C}^d)} .
\end{aligned}
$$

Therefore,

$$
\forall p \in \mathbb{N}^*, \exists C_p \in (\mathbb{C}^{rd})^{\mathbb{Z}}, \quad \|C_p\|^2_{l_2^{rd}} = 1, \| \sum_{k\in\mathbb{Z}} C_p(k)\Phi(.-k)\|^2_{L_2(\mathbb{R},\mathbb{C}^d)} > p . \tag{29}
$$

This is absurd ~~for the reason that~~ since $T : C \in (\mathbb{C}^{rd})^{\mathbb{Z}} \to \sum_{k\in\mathbb{Z}} C(k)\Phi(.-l) \in L_2(\mathbb{R},\mathbb{C})^d$ is continuous, and therefore its image of the compact set $K = \{C \in (\mathbb{C}^{rd})^{\mathbb{Z}} | \|C\|^2_{l_2^{rd}} = 1\}$ is compact.

Similarly, suppose that the essential infinimum of $\lambda_{min} : w \to \min \mathrm{sp}(\hat{A}(w))$ is 0. Using 2π-periodicity of $\hat{A}$ it means that

même remarque que pour max

$$
\forall p \in \mathbb{N}^*, \exists E_p \subseteq [0, 2\pi], \forall w \in E_p, \quad \lambda_{min}(w) < \frac{1}{p} .
$$

~~Same~~ As previously, we then prove that

$$
\forall p \in \mathbb{N}^*, \exists C_p \in (\mathbb{C}^{rd})^{\mathbb{Z}}, \quad \|C_p\|^2_{l_2^{rd}} = 1, \| \sum_{k\in\mathbb{Z}} C_p(k)\Phi(.-k)\|^2_{L_2(\mathbb{R},\mathbb{C}^d)} < \frac{1}{p} . \tag{30}
$$

This is also absurd ~~for the reason that~~ as the image of the compact set K by T is compact and therefore has a minimum that is reached by some $C_m \in K$. Given the result above, this minimum is 0, which means that $\sum_{k\in\mathbb{Z}} C_m(k)\Phi(.-k) = 0$. As shown previousy, this is true if and only if $C_m \equiv 0$, which contradicts $C \in K$.

□

III.3 Application to the case $r = 3$

The scheme described for $r = 3$ and $d = 2$, i.e, 2D vector coefficients, can be rewritten as scalar combinations of integer shifts of functions in $L_2(\mathbb{R}, \mathbb{R}^2)$ as follows

$$
r(t) = \sum_{k=-\infty}^{\infty} \sum_{j=1}^{3} \sum_{i=1}^{2} c_{j,i}(k)\phi_j(t-k)e_i , \tag{31}
$$

with $c_{j,i}(k) = r^{(j)}[k]_i$, and $e_1 = (1, 0)$ and $e_2 = (0, 1)$ the canonical basis of $\mathbb{R}^2$. It is thus an element of

$$
\begin{aligned}
V &= \{ \sum_{k=-\infty}^{\infty} c_1(k)\phi_1(t-k) + c_2(j)\phi_2(t-k) + c_3(k)\phi_3(t-k) | c_1, c_2, c_3 \in l_2(\mathbb{Z})^2 \} \\
&= \{ \sum_{k=-\infty}^{\infty} \sum_{j=1}^{3} \sum_{i=1}^{2} c_{j,i}(k)\phi_j(t-k)e_i | c \in l_2(\mathbb{Z}) \} ,
\end{aligned}
$$

which is a subspace of $L_2(\mathbb{R}, \mathbb{R}^2)$. A direct application of Theorem 5 proves that $\{\phi_j(t-k)e_i\}_{i=1,2,j=1,2,3,k\in\mathbb{Z}}$ is a Riesz-Schauder basis. Therefore V, is a Hilbert space ~~itself~~.

The Gram matrix of $\{\phi_1 e_1(.-k), \phi_1 e_2(.-k), \phi_2 e_1(.-k), \phi_2 e_2(.-k), \phi_3 e_1(.-k), \phi_3 e_2(.-k)\}_{k\in\mathbb{Z}}$ is ~~is~~ a 6×6 matrix. It is explicitly given by

$$\sum_{k\in\mathbb{Z}} \begin{pmatrix} \langle\phi_1,\phi_1(.-k)\rangle & 0 & \langle\phi_1,\phi_2(.-k)\rangle & 0 & \langle\phi_1,\phi_3(.-k)\rangle & 0 \\ 0 & \langle\phi_1,\phi_1(.-k)\rangle & 0 & \langle\phi_1,\phi_2(.-k)\rangle & 0 & \langle\phi_1,\phi_3(.-k)\rangle \\ \langle\phi_2,\phi_1(.-k)\rangle & 0 & \langle\phi_2,\phi_2(.-k)\rangle & 0 & \langle\phi_2,\phi_3(.-k)\rangle & 0 \\ 0 & \langle\phi_2,\phi_1(.-k)\rangle & 0 & \langle\phi_2,\phi_2(.-k)\rangle & 0 & \langle\phi_2,\phi_3(.-k)\rangle \\ \langle\phi_3,\phi_1(.-k)\rangle & 0 & \langle\phi_3,\phi_2(.-k)\rangle & 0 & \langle\phi_3,\phi_3(.-k)\rangle & 0 \\ 0 & \langle\phi_3,\phi_1(.-k)\rangle & 0 & \langle\phi_3,\phi_2(.-k)\rangle & 0 & \langle\phi_3,\phi_3(.-k)\rangle \end{pmatrix} e^{-jwk} \tag{32}$$

where the inner product is that on $L_2(\mathbb{R},\mathbb{C})$ (and actually even $L_2(\mathbb{R},\mathbb{R})$, ~~as~~ since the ϕ_j are real-valued).

As explained before, looking into the essential infinimum and supremum eigenvalues of (32) yields ~~the best achievabel~~ Riesz-Schauder basis bounds m and M. As mentionned in the proof, the matrix above has a characteristic polynomial that is the square of the characteristic polynomial of the submatrix

$$\hat{G}(w) = \sum_{k\in\mathbb{Z}} \begin{pmatrix} \langle\phi_1,\phi_1(.-k)\rangle & \langle\phi_1,\phi_2(.-k)\rangle & \langle\phi_1,\phi_3(.-k)\rangle \\ \langle\phi_2,\phi_1(.-k)\rangle & \langle\phi_2,\phi_2(.-k)\rangle & \langle\phi_2,\phi_3(.-k)\rangle \\ \langle\phi_3,\phi_1(.-k)\rangle & \langle\phi_3,\phi_2(.-k)\rangle & \langle\phi_3,\phi_3(.-k)\rangle \end{pmatrix} e^{-jwk}. \tag{33}$$

Therefore, it is equivalent to look into essential infinimum and supremum of the spectrum of $\hat{G}$ if one is interested in knowing the best achievable parameters m and M.