Hermite Polynomials snakes order 2

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I Translation of Schoenberg's 1973 paper for the case $r = 3, m = 3$

The following is simply a reminder of some of the results found by I.J Schoenberg in his paper *Cardinal Interpolation and Spline Functions. III Cardinal Hermite interpolation.* Let's reintroduce notations of the article and make somehow more explicits what the objects they encode are.

Let r and m be positive integers that satisfy $r \leq m$. The set of cardinal splines of order $2m$ with knot multiplicity r is denoted by $S_{2m,r}$. Note that using De Boor's notations for splines set we have the following

$$S_{2m,r} = \$_{2m,\mathbb{Z}_3} = \Pi_{<2m,\mathbb{Z},2m-r} \tag{1}$$

where $\mathbb{Z}_3$ denotes the sequence of knots $(\dots,-1,-1,-1,0,0,0,1,1,1,\dots)$. It is clear from these notations that $S_{2m,r} \subset \mathcal{C}^{2m-r-1}$.

Theorem 1. *Let S be either of the vector spaces $\mathcal{L}_{p,r}, F_{\gamma,r}$ with $\gamma \geq 0$, $p \in \mathbb{N}^*$. Provided a solution to C.H.I.P $\left(y_\nu,\dots,y_\nu^{(r-1)}, S_{2m,r} \cap S\right)$ exists, it is uniquely given by*

$$\forall x \in \mathbb{R} \qquad S(x) = \sum_{\nu=-\infty}^{\infty} y_\nu L_0(x-\nu) + \dots + y_\nu^{(r-1)} L_{r-1}(x-\nu) \tag{2}$$

In order to specify a usable model for active contours it remains to determine explicit expressions for the basis functions $L_0,\dots,L_{r-1}$. In the article they are determined by solving a set of $2m-r$ linear equations. This system is obtained by considering separately the function L_s on $[1,\infty)$ and $[0,1]$. Note that specifying the function on both these intervals completely determine L_s as the latter is even (if s is even) or odd (if s is odd).

On $[1,\infty)$, L_s can be decomposed into

$$L_s = \sum_{j=1}^{m-r} c_j S_j$$

where $(c_j)_{j=1}^{m-r}$ are $(m-r)$ unknown coefficients to be determined and S_j are the eigensplines for the first $m-r$ "eigenvalues" λ_j, solutions to $|\Delta_{r,d}(\lambda)| = 0$.

On $[0,1]$, L_s is given by a polynomial P of order $2m$ that takes a specific form according to the parities of s and r (we refer to equations (7.13) and (7.14)) in the article. This polynomial introduces m unknown coefficients $(a_j)_{j=1}^m$. To determine a total of $m+m-r = 2m-r$ unknown coefficients we make use of the $2m-r$ equality conditions at 1 $P^{(\rho)}(1) = L_s^{(\rho)}(1)$. We end up of a system of $2m-r$ equations for $2m-r$ unknowns that can be solved exactly provided the matrix of the system is non singular. Schoenberg proves with a very nice argument that the matrix of the system is always non singular.

In the case $m = r = 3$, L_0, L_1, L_2 are 0 on $[1, \infty)$ and on $[0, 1]$ are given by

$$L_0(x) = 1 + a_1 x^3 + a_2 x^4 + a_3 x^5 \tag{3}$$
$$L_1(x) = x + a_1 x^3 + a_2 x^4 + a_3 x^5 \tag{4}$$
$$L_2(x) = \frac{1}{2} x^2 + a_1 x^3 + a_2 x^4 + a_3 x^5 \tag{5}$$

where the coefficients for each generator are unrelated. Note that L_s have finite support *because* $m = r$. If that was not the case the term $\sum_{j=1}^{m-r} c_j S_j$ may not be 0 and therefore L_s would be non zero on $[1, \infty)$! Can it happen though that $m > r$ and $(c_j)_{j=1}^{m-r}$ are 0? To determine the coefficients above we need to solve independently for each generator the 3 equations $P^{(\rho)}(1) = 0$. This leads to the following systems

$$\begin{cases} a_1 + a_2 + a_3 = -1 \\ 3a_1 + 4a_2 + 5a_3 = 0 \\ 3a_1 + 6a_2 + 10a_3 = 0 \end{cases} \begin{cases} a_1 + a_2 + a_3 = -1 \\ 3a_1 + 4a_2 + 5a_3 = -1 \\ 3a_1 + 6a_2 + 10a_3 = 0 \end{cases} \begin{cases} a_1 + a_2 + a_3 = -\frac{1}{2} \\ 3a_1 + 4a_2 + 5a_3 = -1 \\ 3a_1 + 6a_2 + 10a_3 = -\frac{1}{2} \end{cases}$$

II The resulting snake scheme

II.1 Generating functions

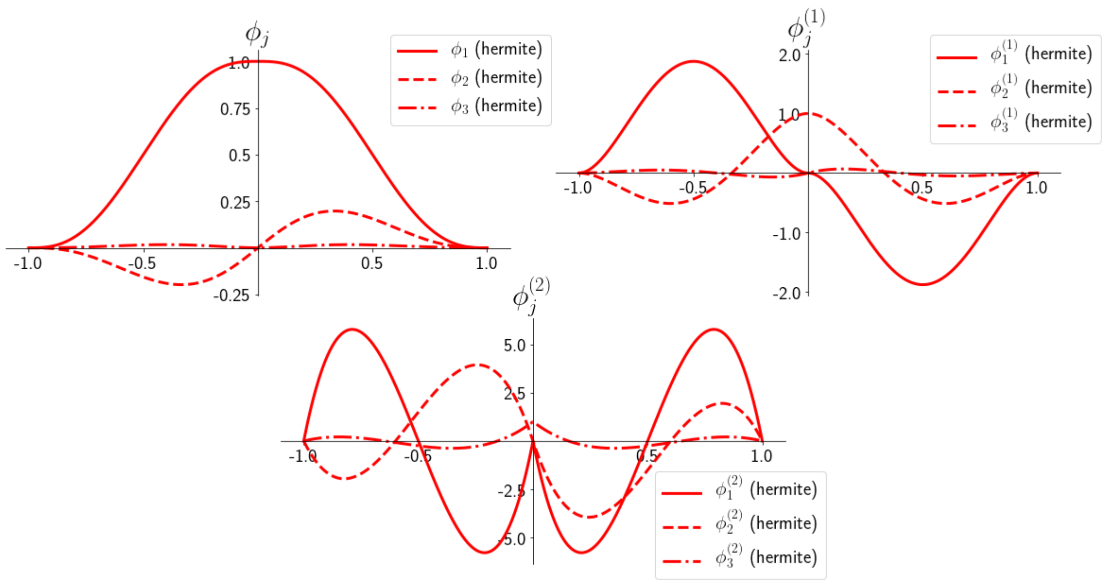

Figure 1: Generators for C.H.I.P with $m = r = 3$

Solving the linear systems written in the previous section yields explicit formulas for the Schoenberg basis generators L_0, L_1, L_2, that we rename ϕ_1, ϕ_2, ϕ_3 in accordance with modern notations (see V. Uhlmann *Hermite Snakes with Controls of Tangents*). The formulas are the following.

$$\phi_1(x) = \begin{cases} 1 - 10x^3 + 15x^4 - 6x^5 & \text{if } 0 \le x \le 1 \\ 1 + 10x^3 + 15x^4 + 6x^5 & \text{if } -1 \le x < 0 \end{cases} \tag{6}$$

$$\phi_2(x) = \begin{cases} x - 6x^3 + 8x^4 - 3x^5 & \text{if } 0 \le x \le 1 \\ x - 6x^3 - 8x^4 - 3x^5 & \text{if } -1 \le x < 0 \end{cases} \tag{7}$$

$$\phi_3(x) = \begin{cases} 0.5x^2 - 1.5x^3 + 1.5x^4 - 0.5x^5 & \text{if } 0 \le x \le 1 \\ 0.5x^2 + 1.5x^3 + 1.5x^4 + 0.5x^5 & \text{if } -1 \le x < 0 \end{cases} \tag{8}$$

In figure 1 are displayed the values of these functions as well as their two first derivatives. As mentioned in the previous section, the generators L_s are elements of $S_{2m,r} = S_{6,3}$ which is a subset of $\mathcal{C}^{2m-r-1} = C^2$. It is apparent in the figure that these functions have continuous derivatives up to order 2 but that higher order derivatives do not exist in neighborhoods of $-1, 0$ and 1.

II.2 Closed planar curves or "contours"

Consider a positive integer M and an M**-periodic parametrized closed curve** $r : \mathbb{R} \to \mathbb{R}^2$ for which we have local derivatives up to order 2 at M location sites regularly spaced that is we know $(r[k], r'[k], r''[k])_{k=0}^{M-1}$.

Corollary 1. *Given M periodic sequences $(r[k], r'[k], r''[k])_{k\in\mathbb{Z}}$, there exists a unique spline of order 6 whose value and derivatives agree with the sequence of coefficients at integers locations. This spline and its derivatives are everywhere bounded and take the form for $t \in \mathbb{R}$*

$$r(t) = \sum_{k\in\mathbb{Z}} r[k]\phi_1(t-k) + r'[k]\phi_2(t-k) + r''[k]\phi_3(t-k) \tag{9}$$

$$= \sum_{k=0}^{M-1} r[k]\phi_{1,per}(t-k) + r'[k]\phi_{2,per}(t-k) + r''[k]\phi_{3,per}(t-k) \tag{10}$$

Proof. As the sequence of coefficients $(r[k], r'[k], r''[k])_{k\in\mathbb{Z}}$ are in $Y_{\gamma,r} = Y_{0,3}$ (i.e they are bounded), application of Schoenberg's theorem 1 yields existence and unicity of a interpolating function in $S_{6,3} \cap F_{0,3}$. Application of theorem 4 then leads to the explicit formulation given above. □

Remark 1. *It is convenient to normalize the continuous parameter to the $[0, 1]$ interval as is usual in the implementations. For that let the renormalized curve $s(t) = r(Mt)$ for $t \in [0, 1]$. Note that this completely describes the curve as it is enough to describe the curve r on $[0, M]$. Differentiating this equality twice yields $r[k] = s[\frac{k}{M}], r'[k] = \frac{1}{M}s'[\frac{k}{M}], r''[k] = \frac{1}{M^2}s''[\frac{k}{M}]$. Therefore equation (10) is rewritten for $t \in [0, 1]$*

$$s(t) = \sum_{k=0}^{M-1} s[\frac{k}{M}]\phi_{1,per}(Mt-k) + \frac{1}{M}s'[\frac{k}{M}]\phi_{2,per}(Mt-k) + \frac{1}{M^2}s''[\frac{k}{M}]\phi_{3,per}(Mt-k) \tag{11}$$

In the rest of this document we will reuse the notation r for the normalized curve and won't make use of the notation s anymore. Equation (11) is the **mathematical representation of a planar curve** and we call it "snake" or "active contour". By playing with the coefficients we can capture a wide variety of contours that arise from closed objects in 2D images like cells membrane in a bioimage.

II.3 Open planar curves

Consider again a positive integer M and a **parametrized open curve** $r : \mathbb{R} \to \mathbb{R}^2$ for which we have local derivatives up to order 2 at M location sites regularly spaced that is we know $(r[k], r'[k], r''[k])_{k=0}^{M-1}$. By "open" we mean a curve that is not periodic.

Corollary 2. *Given biinfinite sequences of coefficients $(\ldots, 0, r[0], \ldots, r[M-1], 0, \ldots)$, $(\ldots, 0, r'[0], \ldots, r'[M-1], 0, \ldots)$, $(\ldots, 0, r''[0], \ldots, r''[M-1], 0, \ldots)$ there exists a unique spline of order 6 whose value and derivatives agree with the sequence of coefficients at integers locations. This spline and its derivatives have compact support and take the form for $t \in \mathbb{R}$*

$$
\begin{aligned}
r(t) &= \sum_{k\in\mathbb{Z}} r[k]\phi_1(t-k) + r'[k]\phi_2(t-k) + r''[k]\phi_3(t-k) \\
&= \sum_{k=0}^{M-1} r[k]\phi_1(t-k) + r'[k]\phi_2(t-k) + r''[k]\phi_3(t-k)
\end{aligned}
$$

Proof. This result is again a simple application of theorem 1 and 4 given in Schoenberg's paper of 1981. □

Remark 2. *In this setting we are only interested in the curve lying between our coefficients that is the interpolated points with continuous parameter in the interval* $[0, M-1]$ *The normalization factor is therefore* $M-1$ *and the renormalized open curve* $s(t) = r((M-1)t)$ *takes the form*

$$
s(t) = \sum_{k=0}^{M-1} s[\frac{k}{M-1}]\phi_1((M-1)t-k) + \frac{1}{M-1}s'[\frac{k}{M-1}]\phi_2((M-1)t-k) + \frac{1}{(M-1)^2}s''[\frac{k}{M-1}]\phi_3((M-1)t-k) \tag{12}
$$

which we will also renote r.

II.4 Closed sphere-like surfaces

In my research project we are interested in developing a mathematical methods for representing a certain type of surfaces with explicit control of local properties including first-order derivatives and curvature. As a consequence extension of the schemes given in equations (11) and (12) to tensor-product surfaces (that is surfaces parametrized by 2 continuous parameters in a way that each continuous parameter appear in separate functions) may be relevant for the questions we have.

Consider positives integers M_1 and M_2 and a **sphere-like parametrized** surface $\sigma : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ with U a subset (closed in our case) of the plane. By "sphere-like" we mean an object that can be described with closed curves on latitudes (u varies while v is fixed) and open curves on longitudes (v varies while u is fixed). Suppose we have local properties of the surface at $M_1 \times (M_2+1)$ locations (counted with multiplicity as some locations may coalesce) on a regular grid.

Given the 9 sequences of coefficients $(\partial^{i,j}\sigma(k,l))_{k,l\in\mathbb{Z}^2,(i,j)\in\{0,1,2\}^2}$ that are M_1 periodic in the first coordinate and 0 when the second coordinate is outside $[0, M_2]$, there exists a unique interpolating tensor-product spline of order 6 that is bounded and takes the form for $(u,v) \in [0,1]^2$

$$
\sigma(u,v) = \sum_{k=0}^{M_1-1}\sum_{l=0}^{M_2}\sum_{i,j=0}^{2} \frac{1}{M_1^i M_2^j}\partial^{i,j}\sigma(\frac{k}{M_1},\frac{l}{M_2})\phi_{i+1,per}(M_1u-k)\phi_{j+1}(M_2v-l) \tag{13}
$$