Summary paper 33: Wavelets of multiplicity r

Yoann Pradat

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1 Introduction

This paper by T. Goodman applies the general theory that derive necessary and sufficient conditions for translates of some functions $\phi_1, \dots, \phi_r$, $\psi_1, \dots, \psi_r$ to form a Riesz basis for V_1 in order to construct spline wavelets with multiple knots.

Notations

- $l^2(\mathbb{Z})^r = \{(s_1, \dots, s_r) | s_j \in l^2(\mathbb{Z}), j = 0, \dots, r-1\}$

Definition 1. *A* multiresolution approximation *of multiplicity r is a sequence of closed subspaces $(V_m)_{m \in \mathbb{Z}}$ of L^2 such that*

1. $V_m \subset V_{m+1}$
2. $\bigcup_{m \in \mathbb{Z}} V_m$ *is dense in* L^2 *and* $\bigcap_{m \in \mathbb{Z}} V_m = \{0\}$
3. $f \in V_m \implies D_2 f \in V_{m+1}$ *with* $D_a f(x) = f(ax)$
4. $f \in V_m \implies T_{2^{-m}n} f \in V_m$ *for all* $n \in \mathbb{Z}$ *with* $T_\tau f(x) = f(x - \tau)$
5. $\exists$ *isomorphism* $\mathcal{I} : V_0 \to l^2(\mathbb{Z})^r$ *which commutes with the action of* $\mathbb{Z}$ *i.e* $\mathcal{I} T_k = t_k \mathcal{I}$ *with* t_k *the translation by* k *on sequences of* $l^2(\mathbb{Z})^r$.

In his famous (among many) article[1], Mallat has given a general construction of wavelets $\phi \in V_0$ and $\psi \in W_0$ ($V_1 = V_0 \bigoplus W_0$) such that $\{T_n \phi\}_n, \{T_n \psi\}_n$ are orthonormal bases of V_0 and W_0 respectively. Furthermore

$$\phi_{m,n} = \sqrt{2^m} \phi(2^m x - n) \quad (m,n) \in \mathbb{Z}^2$$
$$\psi_{m,n} = \sqrt{2^m} \psi(2^m x - n) \quad (m,n) \in \mathbb{Z}^2$$

are orthonormal basis for V_m and W_m respectively ($V_{m+1} = V_m \bigoplus W_m$). Goodman refers to another of his works *Wavelets in wandering spaces* where he extended Mallat's results to multiresolution approximation of multiplicity r.

Cardinal B-splines generate a large class of simple multiresolution approximations. This was studied extensively by Chu et Wang in *A cardinal spline approach to wavelets* and *A general framework of compactly supported splines and wavelets*. The present papers has following contributions

1. New general duality principle
2. Cardinal spline wavelets with multiple knots. They generate nonorthonormal Riesz spaces for the V_m.
3. Considers special case $n = r - 1$, $n = 2r - 1$ with n the degree of spline functions

2 Wavelets for multiresolution approximations of multiplicity r

Note that $\{f \in L^1, \hat{f} \in L^1\} \subsetneq L^1 \cap L^2$. Indeed if $\hat{f} \in L^1$, $\|f\|_\infty \leq \frac{\|\hat{f}\|_{L^1}}{(2\pi)^d}$ and thus $f \in L^2$. However $\mathbb{1}_{[-1,1]^d}$ is in $L^1 \cap L^2$ but not in the first set. Also if $f \in L^1$ is such that $\hat{f} \in L^2$ then $f \in L^2$.

Fourier transform of $f \in L^2(\mathbb{R})$ is said to be regular if $\hat{f}$ is continuous and $\hat{f}(w) = \mathcal{O}(|w|^{-1})$ as $|w| \to \infty$. This makes $\hat{f}$ a function in L^2. In fact, $\mathcal{F}$ is a continuous automorphism of L^2 and therefore the simple fact of f being in L^2 makes its Fourier transform an element of L^2. Is the author's condition stronger?

Let $\phi = (\phi_j)_{j=1}^r \in L^2$ such that $\sum_{n\in\mathbb{Z}} s(n) T_n \phi_j \in L^2$ for any $(s(n)) \in l^2$ and similarly let $\psi = (\psi_j)_{j=1}^r \in L^2$ such that $\sum_{n\in\mathbb{Z}} s(n) T_n \psi_j \in L^2$ for any $(s(n)) \in l^2$. ϕ_j and ψ_j are **assumed to be regular**.

<u>Notations</u>

- $V_0 = \operatorname{span}\overline{\{T_n\phi_j, n \in \mathbb{Z}, 1 \leq j \leq r\}}$
- $V_1 = \{D_2 f, f \in V_0\}$, **assume** $V_0 \subset V_1$
- **Assume** $\psi_j \in V_1$, $j = 1, \ldots, r$.
- $V_1 = V_0 \bigoplus W_0$
- $\tilde{L}^2_{r\times r}(0, 2\pi)$ $r \times r$ matrices with entries 2π-periodic and square-integrable.
- $\tilde{\mathcal{C}}^2_{r\times r}(0, 2\pi)$ $r \times r$ matrices with entries 2π-periodic and continuous.

As ϕ_j, ψ_j are elements of V_1 while $\{T_n D_2 \phi_j\}$ spans V_1, there exist matrices P_n and Q_n such that

$$\phi(x) = 2\sum_{n\in\mathbb{Z}} P_n \phi(2x - n)$$
$$\psi(x) = 2\sum_{n\in\mathbb{Z}} Q_n \phi(2x - n)$$

Goodman then introduces $r \times r$ Gram matrices for ϕ and ψ as

$$\Phi_{jk}(u) = \sum_{n\in\mathbb{Z}} \hat{\phi}_j(u + 2n\pi)\overline{\hat{\phi}_k(u + 2n\pi)}$$
$$\Psi_{jk}(u) = \sum_{n\in\mathbb{Z}} \hat{\psi}_j(u + 2n\pi)\overline{\hat{\psi}_k(u + 2n\pi)}$$

and the "dual-Gram" matrix

$$\Omega_{jk}(u) = \sum_{n\in\mathbb{Z}} \hat{\phi}_j(u + 2n\pi)\overline{\hat{\psi}_k(u + 2n\pi)}$$

Φ, Ψ are Hermitian matrices and so is $M = \begin{pmatrix} \Phi & \Omega \\ \Omega^* & \Psi \end{pmatrix}$.

Proposition 2.1. *The set $\{T_n\phi_j\}$ is a Riesz basis iif Φ is invertible, $\{T_n\psi_j\}$ is a Riesz-basis iif Ψ is invertible and $\{T_n\phi_j, T_n\psi_j\}$ is a Riesz basis iif M is invertible.*

The author claims that Φ, Ψ, Ω are invertible iif the eigenvalues are bounded away from zero. First this is not precise. It is invertibility everywhere or a.e? Same question for the boundedness away from 0? It is clear that latter implies the former, i.e eigenvalues bounded away from zero everywhere (resp a.e) implies invertibility everywhere (a.e). The converse implication is hard though. Invertibility everywhere (resp a.e) leads to eigenvalues strictly positive everywhere (resp a.e) from which we cannot deduce boundedness away from 0 everywhere (resp a.e) in general. In case the entries of the matrices are continuous functions of u, $\lambda_j(u)$ are also continuous and also 2π-periodic. Then strict positivity everywhere leads to boundedness away from 0 everywhere (continuous function on compact reaches its bounds) but the same does not apply to a.e case.

The article Goodman refers to is actually just saying in the case where ϕ_j, ψ_j are regular, Φ, Ψ are invertible everywhere iif $\exists 0 < A \leq B$ such that $A \leq \lambda_j \leq B$ everywhere (equiv to a.e as eigenvalues are continuous here) which is clear.

Regarding lemma 2.1, we have

$$\begin{aligned}\Phi(2u) &= \sum_{n\in\mathbb{Z}} \hat{\phi}(2u+2n\pi)\hat{\phi}(2u+2n\pi)^* \\ &= \sum_{n\in\mathbb{Z}} P(u+n\pi)\hat{\phi}(u+n\pi)\hat{\phi}(u+n\pi)^* P(u+n\pi)^* \\ &= P(u)\Phi(u)P(u)^* + P(u+\pi)\Phi(u+\pi)P(u+\pi)^*\end{aligned}$$

A similar expression holds for $\Psi(2u)$ with P replaced by Q and for $\Omega(2u)$ with P on the right only replaced by Q.

Theorem 2.1. *The following are equivalent*

1. *$\{T_n\phi_j, T_n\psi_j\}$ forms a Riesz basis for V_1*
2. *$\begin{bmatrix} P(u) & P(u+\pi) \\ Q(u) & Q(u+\pi)\end{bmatrix}$ and $\Phi(u)$ are invertible.*

If this holds, for any sequences of matrices $r \times r$ $(A_n), (B_n), (C_n)$ with FT $A(u), B(u), C(u)$ in $\tilde{L}^2_{r\times r}(0, 2\pi)$, we have

$$\sum_{n\in\mathbb{Z}} C_n\phi(2x-n) = \sum_{n\in\mathbb{Z}} A_n\phi(x-n) + B_n\psi(x-n) \tag{1}$$

$$\iff \frac{1}{2}\begin{bmatrix} C(u) & C(u+2\pi)\end{bmatrix} = \begin{bmatrix} A(2u) & B(2u)\end{bmatrix}\begin{bmatrix} P(u) & P(u+\pi) \\ Q(u) & Q(u+\pi)\end{bmatrix} \tag{2}$$

If the first equivalence holds, $\Phi(u)$ and $\Psi(u)$ are positive definite and have inverses in $\tilde{\mathcal{C}}_{r\times r}(0, 2\pi)$. Functions ψ_j belongs to W_0 if and only if

$$P(u)\Phi(u)Q(u)^* + P(u+\pi)\Phi(u+\pi)Q(u+\pi)^* = 0 \tag{3}$$

We **now assume** that $\{T_n\phi_j, T_n\psi_j\}$ forms a Riesz basis for V_1. The author choose sequences (G_n), (H_n) so that for any integer l

$$\phi(2x-l) = \sum_{n\in\mathbb{Z}} G_{2n-l}\phi(x-n) + \sum_{n\in\mathbb{Z}} H_{2n-l}\psi(x-n) \tag{4}$$

I understand now where $2n$ subscript comes from. Even rank elements of $(G_n), (H_n)$ (G^0, H^0), are used for $\phi(2x)$, odd rank (G^1, H^1) are used for $\phi(2x-1)$ i.e

$$\phi(2x) = \sum_{n\in\mathbb{Z}} G_{2n}\phi_n(x) + H_{2n}\psi_n(x) \tag{5}$$

$$\phi(2x-1) = \sum_{n\in\mathbb{Z}} G_{2n-1}\phi_n(x) + H_{2n-1}\psi_n(x) \tag{6}$$

Note that

$$\begin{aligned}
G(u) &= \sum_{n\in\mathbb{Z}} G_n e^{-iun} \\
G(u) + G(u+\pi) &= 2\sum_{n\in\mathbb{Z}} G_{2n} e^{-2iun} \\
&= 2G^0(2u) \\
G(u) - G(u+\pi) &= 2\sum_{n\in\mathbb{Z}} G_{2n-1} e^{-2iun+iu} \\
&= 2G^1(2u)e^{iu}
\end{aligned}$$

and the same holds for H. Taking the Fourier transform of equations (5), (6) leads to

$$\frac{1}{2}\hat{\phi}(u) = G^0(2u)P(u)\hat{\phi}(u) + H^0(2u)Q(u)\hat{\phi}(u) \tag{7}$$

$$\frac{1}{2}\hat{\phi}(u)e^{-iu} = G^1(2u)P(u)\hat{\phi}(u) + H^1(2u)Q(u)\hat{\phi}(u) \tag{8}$$

Theorem 2.2. *If $\{T_n\phi_j\}$, $\{T_n\psi_j\}$ are Riesz basis of V_0 and W_0 respectively, then G and H are given by*

$$\begin{aligned}
G(u) &= \Phi(u)P(u)^*\Phi(2u)^{-1} \\
H(u) &= \Phi(u)Q(u)^*\Psi(2u)^{-1}
\end{aligned}$$

Proof. Goodman claims that equations of equivalence of theorem 2.1 holds but **why**? Because $V_1 = V_0 \bigoplus W_0$. The rest of the proof is simply assembling previous equations. □

Assume equivalence of theorem 2.1 holds. By the way **is it true that**

$$\{T_n\phi_j, T_n\psi_j\} \text{ Riesz basis for } V_1 \iff \{T_n\phi_j\}, \{T_n\psi_j\} \text{ Riesz basis for } V_0, W_0 \text{ respectively?}$$

Let $\tilde{\phi}, \tilde{\psi}$ in $V_1 \times \cdots \times V_1$ defined in the Fourier domain by

$$\hat{\tilde{\phi}} = \Phi^{-1}\hat{\phi}, \quad \hat{\tilde{\psi}} = \Psi^{-1}\hat{\psi}$$

Then $\langle \phi_j, T_n\tilde{\phi}_k\rangle = \delta_{j,k}\delta_{n,0}$ that is ϕ and $\tilde{\phi}$ are dual basis. Similarly $\tilde{\psi}$ is dual to ψ.

Theorem 2.3. *If $\{T_n\phi_j\}$, $\{T_n\psi_j\}$ are Riesz basis of V_0 and W_0 respectively, dual functions are such that*

$$\begin{aligned}
\tilde{\phi}(x) &= 2\sum_{n\in\mathbb{Z}} G_n^*\tilde{\phi}(2x-n) \\
\tilde{\psi}(x) &= 2\sum_{n\in\mathbb{Z}} H_n^*\tilde{\psi}(2x-n)
\end{aligned}$$

and

$$\tilde{\phi}(2x-l) = \sum_{n\in\mathbb{Z}} P_{2n-l}^*\tilde{\phi}(x-n) + \sum_{n\in\mathbb{Z}} Q_{2n-l}^*\tilde{\psi}(x-n) \tag{9}$$

3 Spline wavelets of multiplicity r

Notations

- $\zeta_{n,r}(S)$ space of spline functions of degree n on $\mathbb{R}$ with knots multiplicity r on set S. Note that $\zeta_{n,r}(\mathbb{Z}) = \$_{n+1,\mathbb{Z}_r}$ in De Boor's notation.
- $V_0 = \zeta_{n,r}(\mathbb{Z}) \cap L_2$, $V_1 = \zeta_{n,r}(\frac{1}{2}\mathbb{Z}) \cap L_2$.
- W_0 orthogonal complement to V_0 in V_1 i.e $V_1 = V_0 \bigoplus W_0$.
- $(t_i)_{i\in\mathbb{Z}} = \mathbb{Z}_r$ with $t_i = j$ for $jr \le i \le (j+1)r - 1$.

- N_i $i = 0, \ldots, r-1$, B-spline in $\zeta_{n,r}(\mathbb{Z})$ with support in $[t_i, t_{i+n+1}]$ and knots at $t_i, \ldots, t_{i+n+1}$ normalized so that $\sum_i N_i^n = 1$. This is exactly De Boor's B-splines i.e

$$N_i^n(t) = (t_{i+n+1} - t_i)[t_i, \ldots, t_{i+n+1}](\cdot - t)_+^n \text{ or eq. } N_i^n(t) = B_{i,n+1,\mathbb{Z}_r}$$

From the work of De Boor (see [2] chapter XI), we know that $\{N_i^n\}_{i\in\mathbb{Z}}$ is a Riesz-basis for the infinite norm as for any bounded sequence $(c_i)_{i\in\mathbb{Z}}$ we have

$$D_n^{-1}\|c\|_\infty \leq \|\sum_{i\in\mathbb{Z}} c_i N_i^n\|_\infty \leq \|c\|_\infty$$

Does this also hold for l^2, L^2 norms? Be aware that in infinite dimension $\|\|_\infty$ and $\|\|_{l^2}$ are not equivalent.

- $U = \{f \in \zeta_{2n+1,r}(\frac{1}{2}\mathbb{Z}), f^{(i)}_{|\mathbb{Z}} = 0, i = 0, \ldots, r-1\}$. **Is this a subset of L_2?** Following Schoenberg Hermite interpolation theorems, it is the case provided results extend to half-integer knots. More specifically, let $(y_\nu)_{\nu\in\frac{1}{2}\mathbb{Z}}, \ldots, (y_\nu^{(r-1)})_{\nu\in\frac{1}{2}\mathbb{Z}}$ sequences on the half integers. Let $(z_\nu^{(j)} = y^{(j)}_{\frac{\nu}{2}})_{\nu\in\mathbb{Z}}$ sequences on the integers. Provided $z^{(j)} \in l_p$ for $j = 0, \ldots, r-1$, $\exists! L \in \zeta_{2m-1,r}(\mathbb{Z}) \cap \mathcal{L}_{p,r}$, $m \geq r$ such that L interpolates $z^{(j)}$ and it is given by

$$\forall x, \quad L(x) = \sum_{\nu\in\mathbb{Z}} \sum_{j=0}^{r-1} z_\nu^{(j)} L_j(x - \nu)$$

Note now that $D_2 L \in \zeta_{2m-1,r}(\frac{1}{2}\mathbb{Z})$ and so do $D_2 L_j$, $j = 0, \ldots, r-1$. Consequently,

$$\begin{aligned} D_2 L(x) &= \sum_{\nu\in\mathbb{Z}} \sum_{j=0}^{r-1} z_\nu^{(j)} L_j(2x - \nu) \\ &= \sum_{\nu\in\frac{1}{2}\mathbb{Z}} \sum_{j=0}^{r-1} z_{2\nu}^{(j)} L_j(2x - 2\nu) \\ &= \sum_{\nu\in\frac{1}{2}\mathbb{Z}} \sum_{j=0}^{r-1} y_\nu^{(j)} D_2 L_j(x - \nu) \end{aligned}$$

The r null sequences $(0)_{i\in\frac{1}{2}\mathbb{Z}}$ being obviously in l^2, there exists a unique $L \in \zeta_{2m-1,r}(\frac{1}{2}\mathbb{Z}) \cap \mathcal{L}_{2,r}$ with $m \geq r$ that has vanishing $(r-1)$ first derivatives on $\frac{1}{2}\mathbb{Z}$. U **is a subset of** L_2 (and L_p for any $p \geq 1$) provided $n + 1 \geq r$ which is part of the assumptions.

Theorem 3.1. *For each $i \in \mathbb{Z}$, $\exists! \psi_i \in W_0$ with support on $[t_i, \ldots, t_{i+2n+2-r}]$ and integer knots $t_i, \ldots, t_{i+2n+2-r}$. ψ_i does not have smaller support nor a smaller set of knots.*

For that Goodman first constructs functions Ψ_i in U.

Lemma 3.1. *For each $f \in W_0$ with $\operatorname{supp} f \subset [a, b]$, $\exists! g \in U$ with $\operatorname{supp} g \subset [a, b]$ such that $g^{(n+1)} = f$. Conversely, if $g \in U$ has support in $[a, b]$ then $g^{(n+1)}$ is in W_0.*

Proof. Goodman says the proof is easy using integration by parts but **I don't see that**. ☐

Theorem 3.2. *For each $i \in \mathbb{Z}$, $\exists! \Psi_i \in U$ with support on $[t_i, \ldots, t_{i+2n+2-r}]$ and integer knots $t_i, \ldots, t_{i+2n+2-r}$. Ψ_i does not have smaller support nor a smaller set of knots.*

Normalization is chosen so that $\Psi_{i+r} = \Psi_i(\cdot - 1), i \in \mathbb{Z}$ and $\psi_i = \Psi_i^{(n+1)}$ gives theorem 3.1.

4 Linear combinations of wavelets

Theorem 4.1. *The sequence* $(\Psi_i)_{i\in\mathbb{Z}}$ *is locally linearly independent on any interval.*

Corollary 4.1. *Any function* $f \in U$ *can be written uniquely in the form*

$$f = \sum_{i\in\mathbb{Z}} c_i \Psi_i$$

for some constants c_i*. Moreover if support of* Ψ_i *overlaps* $(j, j+1)$ *then*

$$|c_i| \leq K \|f_{|[j,j+1]}\|_\infty$$

with K *independent of* i, j *and* f*.*

Proof. Goodman says that it is easily seen that $\zeta = \zeta_{2n+1,r}(\frac{1}{2}\mathbb{Z})_{|[j,j+M]}$ has dimension $2n+2+r(2M-1)$. I personally don't see this easily but following Curry-Schoenberg (see [2], p97) theorem one can notice that

$$\zeta_{2n+1,r}(\frac{1}{2}\mathbb{Z})_{|[j,j+M]} = \Pi_{<2n+2,\frac{1}{2}\mathbb{Z},\boldsymbol{\nu}}{}_{|[j,j+M]}$$

with $\nu_j = 2n+2-r$ for all $j \in \mathbb{Z}$. Now we have

$$\Pi_{<2n+2,\frac{1}{2}\mathbb{Z},\boldsymbol{\nu}}{}_{|[j,j+M]} = \Pi_{<2n+2,\frac{1}{2}\mathbb{Z}\cap[j,j+M],\boldsymbol{\nu}}{}_{|[j,j+M]}$$

The latter is the space of piecewe polynomials of order $2n+2$ on the finite knot sequence $\frac{1}{2}\mathbb{Z} \cap [j, j+M]$ of $2M+1$ elements each with multiplicity r. From (see [2], p84), a basis of $2n+2+r(2M-1)$ elements can be built for this space which is thus of dimension $2n+2+r(2M-1)$.

The dimension of $\zeta_0 = U_{|[j,j+M]} = \{f_{|[j,j+M]} \in \zeta_{|[j,j+M]} | f^{(j)}_{|\mathbb{Z}\cap[j,j+M]} = 0, j = 0, \ldots, r-1\}$ is $r(M+1)$ less than the dimension of ζ that is $2n+2+r(M-2)$. Functions Ψ_i have support overlapping $[j, j+M]$ for $(j+2)r - 2n - 2 \leq i \leq (j+M)r - 1$ that is $2n+2+r(M-2)$ of them. Functions $\Psi_{i|[j,j+M]}$ for such i are all in ζ_0 and are linearly independent which makes $\{\Psi_{i|[j,j+M]}\}_{(j+2)r-2n-2\leq i\leq(j+M)r-1}$ a basis for ζ_0.

Then **comes the magic**. Goodman deduces from the fact that $\{\Psi_{i|[j,j+1]}\}_{(j+2)r-2n-2\leq i\leq(j+1)r-1}$ is a basis for ζ_0 and from the fact that norms on finite dimension spaces are equivalent to get that $\exists K > 0$ such that

$$\forall f \in U, \quad \max_{(j+2)r-2n-2\leq i\leq(j+1)r-1} |c_i| \leq K \|f_{|[j,j+1]}\|_\infty$$

□

Theorem 4.2. *The sequence* $(\psi_i)_{i=-\infty}^{\infty}$ *is locally linearly independent on any interval* $(j, j+M)$ *for any* j *and* M *such that* $r(M+1) \geq n+1$*.*

Corollary 4.2. *For integers* j*,* M *with* $M \geq 1$*,*

$$W_{0|[j,j+M]} = \left\{ f \in \zeta_{n,r}(\frac{1}{2}\mathbb{Z}) | \int_{-\infty}^{\infty} fg = 0, g \in \zeta_{n,r}(\mathbb{Z}), \operatorname{supp} g \subset [j, j+M] \right\} \tag{10}$$

Corollary 4.3. *Any function in* W_0 *can be uniquely written* $\sum_{i\in\mathbb{Z}} c_i\psi_i$ *for some constants* c_i*.*

Corollary 4.4. *The functions* $\{\psi_i\}_{i=-\infty}^{\infty}$ *form a Riesz basis for* W_0*.*

Proof. Let M such that $r(M+1) \geq n+1$. $\psi_{i|[j,j+M]}$ for $l = (j+2)r - 2n - 2 \leq i \leq L = (j+M)r - 1$ form a basis for $W_{0|[j,j+M]}$. Goodman **claims then there exists constants A, B independent of j such that for any $f = \sum_{i\in\mathbb{Z}} c_i\psi_i$ in W_0 we have**

$$A \int_j^{j+M} f^2 \leq \sum_{i=l}^{L} c_i^2 \leq B \int_j^{j+M} f^2$$

Where does that come from??? □

5 Construction of the wavelets

Ψ_i with support in $[0,T]=[t_i,t_{i+2n+2-r}]$ is an element of $\zeta_{2n+1,r}(\frac{1}{2}\mathbb{Z})$ and can be written

$$\Psi_i(x)=\sum_{j=0}^{(T-1)r} c_j N_{i+j}^{2n+1}(2x)$$

$(T-1)r+1$ coefficients c_j are determined by the conditions $(T-1)r$ conditions $\Psi_i^{(j)}(k)=0$ for $j=0,\dots,r-1$, $k=1,\dots T-1$ and by a normalisation condition.

Consider the case $r=n+1$. Note then that

$$t_i=0, \quad t_{i+2n+2-r}=t_{i+r}=1, \quad \text{for } i=0,\dots,r-1$$

i.e $T=1$. Then

$$\Psi_i(x)=N_i^{2n+1}(2x) \quad i=0,\dots,n \tag{11}$$

In De Boor's notation we have $N_j^{2n+1}(t)=B_{j,2n+2,\mathbb{Z}_{n+1}}(t)=(t_{j+2n+2}-t_j)[t_j,\dots,t_{j+2n+2}](\cdot-t)_+^{2n+1}$. Be aware that the recurrence relationship of De Boor's B-splines does not transpose directly to the N_j^n as varying n implies varying the sequences of knots whereas in De Boor's relation the sequence of knots is assumed to be fixed.

Remember that ψ is related to Ψ by $\psi_i=\Psi_i^{(n+1)}$ and also $\Psi_{i+r}=\Psi_i(\cdot-1)$.

Case $n=0, r=1$
Then

$$\Psi_0(x)=N_0^1(2x)=\begin{cases}2x & 0\le x<\frac{1}{2}\\ 2-2x & \frac{1}{2}\le x<1\end{cases}$$

$$\psi_0(x)=\begin{cases}2 & 0\le x<\frac{1}{2}\\ -2 & \frac{1}{2}\le x<1\end{cases}$$

However applying Goodman's relation (5.4) leads to $\psi_0(x)=N_0^0(2x)-N_0^0(2x-1)=\begin{cases}1 & 0\le x<\frac{1}{2}\\ -1 & \frac{1}{2}\le x<1\end{cases}$.

A factor 2 is missing.

Case $n=1, r=2$
In that case we have $\Psi_j(x)=N_j^3(2x)=B_{j,4,\mathbb{Z}_2}(2x)$, $j=0,1$. Let's compute the latter. Note $I_j=[j,j+1]$

$$B_{2j,1}=0 \qquad B_{2j+1,1}(t)=\begin{cases}1 & I_j\end{cases}$$

$$B_{2j,2}=\begin{cases}j+1-t & I_j\end{cases} \qquad B_{2j+1,2}=\begin{cases}t-j & I_j\end{cases}$$

$$B_{2j,3}=\begin{cases}2(j+1-t)(t-j) & I_j\end{cases} \qquad B_{2j+1,3}=\begin{cases}(t-j)^2 & I_j\\ (j+2-t)^2 & I_{j+1}\end{cases}$$

$$B_{2j,4}=\begin{cases}\frac{1}{2}(5(j+1-t)+1)(t-j)^2 & I_j\\ \frac{1}{2}(j+2-t)^3 & I_{j+1}\end{cases} \qquad B_{2j+1,4}=\begin{cases}\frac{1}{2}(t-j)^3 & I_j\\ \frac{1}{2}(5(t-j-1)+1)(j+2-t)^2 & I_{j+1}\end{cases}$$

Consequently,

$$\Psi_0(x) = \begin{cases} 4(3-5x)x^2 & 0 \le x < \frac{1}{2} \\ 4(1-x)^3 & \frac{1}{2} \le x < 1 \end{cases}$$

$$\Psi_1(x) = \begin{cases} 4x^3 & 0 \le x < \frac{1}{2} \\ 4(5x-2)(1-x)^2 & \frac{1}{2} \le x < 1 \end{cases}$$

$$\psi_0(x) = \begin{cases} 24(1-5x) & 0 \le x < \frac{1}{2} \\ 24(1-x) & \frac{1}{2} \le x < 1 \end{cases}$$

$$\psi_1(x) = \begin{cases} 24x & 0 \le x < \frac{1}{2} \\ 24(5x-4) & \frac{1}{2} \le x < 1 \end{cases}$$

Again relation (5.4) leads to $a_{0,0} = 2, a_{0,1} = -3, a_{1,0} = 0, a_{1,1} = -1$ and

$$\psi_0(x) = 2N_0^1(2x) - 3N_1^1(2x) - N_0^1(2x-1) = \begin{cases} 2-10x & 0 \le x < \frac{1}{2} \\ 2-2x & \frac{1}{2} \le x < 1 \end{cases}$$

$$\psi_1(x) = -N_1^1(2x) + 3N_0^1(2x-1) - 2N_1^1(2x-1) = \begin{cases} 2x & 0 \le x < \frac{1}{2} \\ -10x+8 & \frac{1}{2} \le x < 1 \end{cases}$$

A factor 12 or -12 is missing.

References

[1] S.Mallat. Multiresolution approximations and wavelet orthonormal bases of $L^2(\mathbb{R})$. pages 69–87, 1989.

[2] Carl de Boor. *A practical guide to splines.* Springer, revised edition, 2001.