Hermite Spline wavelets

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Contents

I A cardinal spline approach to wavelets

Notations

- $N_m(t) = m[0, \dots, m](\cdot - t)_+^{m-1} = N_1 * \dots * N_1(t)$
- $\phi \in L^2$, $\phi_{kj}(x) := \phi(2^k x - j)$
- $\psi_{kj}(x) := \psi(2^k x - j)$
- $V_k = \text{Clos}_{L^2}\text{span}\{\phi_{kj}\}_{j\in\mathbb{Z}}$

It is well-kwnon that the m^{th} order B-spline N_m with integer knots generate a mulitresolution analysis with m^{th} order of approximation (references?). This paper achieves the following

1. $\psi(x) = L_{2m}^{(m)}(2x-1)$ generates the orthogonal wavelet spaces W_k.
2. Gives exact formulation of $N_m(2\cdot -1)$ in terms of N_m and $L_{2m}^{(m)}$.

An *unconditional* basis is (Mallat, Multiresolution Approximations and Wavelet orthonormal bases of $L^2(\mathbb{R})$) a sequence of functions $(e_\lambda)_{\lambda\in\Lambda}$ such that for any sequence of numbers $\alpha := (\alpha_\lambda)_{\lambda\in\Lambda}$

$$A\|\alpha\|_{l^2} \leq \left\| \sum_{\lambda\in\Lambda} \alpha_\lambda e_\lambda \right\|_{L^2} \leq B\|\alpha\|_{l^2}$$

ψ is a basic wavelet relative to ϕ if $W_0 = \text{Clos}_{L^2}\text{span}\{\psi(\cdot - j)\}_j$ is the orthogonal complement to V_0 in V_1. **Let us assume** that $V_1 = V_0 \bigoplus W_0$. Let's prove that the following holds then

Proposition 1. *1.* $V_{k+1} = V_k \bigoplus W_k$

2. $W_k \perp W_j$ *if* $j \neq k$

3. $L^2 = \bigoplus_k W_k$

Proof. As $V_1 = V_0 \bigoplus W_0$, there exists sequences $(a_i), (\tilde{a}_i), (b_i), (\tilde{b}_i)$ such that

$$\phi(2x) = \sum_i a_i\phi(x-i) + b_i\psi(x-i)$$
$$\phi(2x-1) = \sum_i \tilde{a}_i\phi(x-i) + \tilde{b}_i\psi(x-i)$$

Defining (α_i) and (β_i) so that $\begin{cases}\alpha_{2i} &= a_i \\ \alpha_{2i-1} &= \tilde{a}_i\end{cases}$ we get that for any integer l

$$\phi(2x-l) = \sum_i \alpha_{2i-l}\phi(x-i) + \beta_{2i-l}\psi(x-i) \tag{1}$$

Changing x in $2x$ leads to

$$\phi(4x-l) = \sum_i \alpha_{2i-l}\phi(2x-i) + \beta_{2i-l}\psi(2x-i) \tag{2}$$

that is $V_2 \subseteq V_1 + W_1$.

As $V_0 \subset V_1$, there exists (γ_i) such that

$$\phi(x) = \sum_i \gamma_i \phi(2x-i)$$

so that

$$\phi(2x-l) = \sum_i \gamma_{i-2l}\phi(4x-i)$$

i.e $V_1 \subset V_2$. Similarly $W_1 \subset V_2$ hence $V_1 + W_1 \subseteq V_2$.

As for the orthogonality notice that $\int \phi(x-k)\overline{\psi(x-l)}dx = 2\int \phi(2x-k)\overline{\psi(2x-l)}dx$ so that $V_0 \perp W_0 \implies V_1 \perp W_1$. Eventually $V_1 \bigoplus W_1 = V_2$. The reasoning extends to any integer k. $\square$

Standard method to determine $\tilde{\psi}$ from ϕ is: orthormalize $\{\phi_{0j}\}$ into $\{\tilde{\phi}_{0j}\}$, find two-scale relation of $\tilde{\phi}_0$ in terms of $\{\tilde{\phi}_{1j}\}$, alternate the signs cleverly in the sequence to form $\tilde{\psi}_0$. See Mallat's paper for linear and cubic spline i.e $m = 2$ and $m = 4$.

In the paper authors do not impose orthogonality of $\{\psi_{0j}\}$ and instead focus on representing ϕ_{1j} in terms of $\{\phi_{0j}\}$ and $\{\psi_{0j}\}$ with fast decaying sequences.

I.1 Main results

Note that the m^{th} order B-spline N_m is such that

$$\begin{aligned} N_m(t) &= \sum_{k=0}^{m} (-1)^{m-k}\binom{m}{k}(k-t)_+^{m-1}\frac{1}{(m-1)!} && (3)\\ &= \sum_{k=0}^{m} (-1)^{k}\binom{m}{k}(t-k)_+^{m-1}\frac{1}{(m-1)!} && (4)\\ & && (5)\end{aligned}$$

These two writing are equivalent for the reason that $(t-k)_+^{m-1} = (t-k)^{m-1} + (-1)^m(k-t)_+^{m-1}$. Notice that $D^m\frac{(t-k)_+^{m-1}}{(m-1)!} = \delta_k$ in the sense of distribution hence

$$D^m N_m = \sum_{k=0}^{m}(-1)^k\binom{m}{k}\delta_k$$

$\phi = N_m$ generates a multiresolution analysis. The fundamental splines L_{2m} is an element of $\mathcal{S}_{2m}$ hence the decomposition

$$L_{2m}(t) = \sum_i \alpha_i N_{2m}(t+m-j) \tag{6}$$

where the $+m$ in the B-spline is equivalent to considering a decomposition in the central B-splines M_{2m}. Using the z-transforms $A(z) = \sum_i \alpha_i z^i$, $B(z) = \sum_{i=-(m-1)}^{m-1} N_{2m}(i+m)z^i$, equation (6) becomes

$$A(z)B(z) = 1 \tag{7}$$

As detailed by Schoenberg in constructing L_{2m}, we have $B(z) = \frac{\Pi_{2m-1}(z)}{(2m-1)!z^{m-1}}$ with Π_{2m} the Euler-Froebenius polynomial of order $2m-1$. The basice wavelet ψ_m related to ϕ_m is

$$\psi_m(t) = L_{2m}^{(m)}(2t-1) \tag{8}$$

Supposedly, we have the 2-scale relation

$$N_m(t) = \sum_{j=0}^{m} 2^{-(m-1)} \binom{m}{j} N_m(2t-j) \tag{9}$$

proving that $V_0 \subset V_1$. As $L_{2m} \in \mathcal{S}_{2m}$, $L_{2m}^{(m)} \in \mathcal{S}_m$ so that $W_0 \subset V_1$. Property $W_k \subset V_{k+1}$ is a consequence of lemma 1. It remains to show that $V_1 = V_0 + W_0$ i.e find (a_n), (b_n) such that

$$\phi(2t-l) = \sum_n a_{2n-l}\phi(t-l) + b_{2n-l}\psi(t-l)$$

(a_n) and (b_n) are the coefficients of the Laurent series $G(z)$, $H(z)$.

II Goodman's papers

Notations

- $\zeta_{2r-1,r}(\mathbb{Z}) = \$_{2r,\mathbb{Z}_r}$ the set of cardinal splines of order $2r$ with knots of multiplicity r
- $V_0 = \zeta_{2r-1,r}(\mathbb{Z}) \cap L_2$
- $V_1 = \zeta_{2r-1,r}(\frac{1}{2}\mathbb{Z}) \cap L_2$

In the introductory part of its 1994 article *Interpolatory Hermite Spline wavelets*, Goodman discussed the B-splines introduced by Schoenberg and Sharma for the problem of Hermite interpolation. Schoenberg and Sharma proposed as generators for the solution of Hermite interpolation problem the functions $L_s \in \zeta_{2r-1,r-1}(\mathbb{Z})$, $s = 0,\dots,r-1$, with support in $[-1,1]$ satisfying for all integers i

$$L_s^{(j)}(i) = \delta_{sj}\delta_i$$

These functions are splines of order $2r$ and are symmetric (s even) or antisymmetric (s odd) depending on the parity of s.

Goodman considers in his paper $\{B_s = L_s(.-1)\}_{0\leq s\leq r-1}$, a causal version of these B-splines with support in $[0,2]$. He also introduces N_s the B-spline in $\zeta_{2r-1,r}(\mathbb{Z})$ also supported in $[0,2]$ with knots at 0, 1 and 2 of multiplicity $r-s, r, s+1$ for $s = 0,\dots,r-1$ ($N_s \propto [0_{r-s}, 1_r, 2_{s+1}](.-t)_+^{(2r-1)}$). He claims that $B_0,\dots,B_{r-1}$ and $N_0,\dots,N_{r-1}$ are equivalent bases while the latter (with all integer shifts) is a Riesz basis for V_0. This would have the consequence that $\{B_s(.-i)\}_{0\leq s\leq r-1, i\in\mathbb{Z}}$ is also a Riesz basis for V_0, a result that is hard to prove otherwise in the general case. Having a Riesz basis for V_0 makes it a closed subspace of the Hilbert space L_2 and therefore a Hilbert space itself.

Goodman makes reference to another of his articles published in 1993 when stating that claim, the article *Wavelets of multiplicity r*.