JOURNAL OF APPROXIMATION THEORY 18, 205–212 (1976)

Exponential Hermite–Euler Splines

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Communicated by Richard S. Varga

Received April 10, 1975

Recently I. J. Schoenberg studied the cardinal splines that interpolate the function λ^x at the integers, where λ is a complex number. This paper deals with cardinal splines which together with their successive derivatives interpolate λ^x and its successive derivatives at the integers.

INTRODUCTION

Let n, r be positive integers such that $n \geqslant 2r - 1$. The class $\mathscr{S}_{n,r}$ of cardinal splines of degree n with integer knots of multiplicity r consists of the functions $S(x)$ such that $S(x)$ is a polynomial of degree n in each of the intervals $[\nu, \nu + 1]$ ($\nu = 0, \pm 1, \pm 2,...$) and $S(x) \in C^{n-r}(-\infty, \infty)$.

In an interesting paper [4] Schoenberg studied the cardinal splines $S_n(x; \lambda)$, called the exponential Euler splines, that interpolate the function λ^x at the integers, where λ is a complex number (see also [7]). These exponential Euler splines $S_n(x; \lambda)$ are extremely useful (see [5, 7]). It turns out that $S_n(x; \lambda)$ are "periodic extensions" of the exponential Euler polynomials $A_n(x; \lambda)$ introduced by Euler [1]. These polynomials are generated by the relation

$$\frac{\lambda - 1}{\lambda - e^z} e^{xz} = \sum_{n=0}^{\infty} \frac{A_n(x; \lambda)}{n!} z^n. \tag{1}$$

The essential properties of $A_n(x; \lambda)$ are given in [4].

This paper deals with cardinal splines $S_{n,r}(x; \lambda) \in \mathscr{S}_{n,r}$ which together with their successive derivatives interpolate the function λ^x and its successive derivatives at the integers, i.e.,

$$S_{n,r}^{(\rho)}(\nu) = \lambda^\nu (\log \lambda)^\rho \qquad (\rho = 0, 1,..., r - 1) \ \forall \text{ integers}. \tag{2}$$

In Section 1 we introduce the polynomials $A_{n,r,s}(x; \lambda)$ from which the splines $S_{n,r,s}(x; \lambda)$ are constructed in Section 2. The representations of $S_{n,r,s}(x; \lambda)$ in terms of B-splines are given in Section 3. In Section 4 we study the behavior of $S_{n,r}(x; \lambda)$ as n tends to infinity, and in the last section we give a complete proof of the convergence theorem for the case $r = 2$.

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1. The Polynomial $A_{n,r,s}(x;\lambda)$

Let $s = 0, 1, \ldots, r-1$ be a fixed integer and set

$$A_{n,r,s}(x;\lambda) = \begin{vmatrix} \frac{A_n(0;\lambda)}{n!} & \frac{A_{n-1}(0;\lambda)}{(n-1)!} \cdots & \frac{A_{n-s+1}(0;\lambda)}{(n-s+1)!} & \frac{A_n(x;\lambda)}{n!} & \frac{A_{n-s-1}(0;\lambda)}{(n-s-1)!} \cdots & \frac{A_{n-r+1}(0;\lambda)}{(n-r+1)!} \\ \frac{A_{n-1}(0;\lambda)}{(n-1)!} & \frac{A_{n-2}(0;\lambda)}{(n-2)!} \cdots & \frac{A_{n-s}(0;\lambda)}{(n-s)!} & \frac{A_{n-1}(x;\lambda)}{(n-1)!} & \frac{A_{n-s-2}(0;\lambda)}{(n-s-2)!} \cdots & \frac{A_{n-r}(0;\lambda)}{(n-r)!} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{A_{n-r+1}(0;\lambda)}{(n-r+1)!} & \frac{A_{n-r}(0;\lambda)}{(n-r)!} \cdots & \frac{A_{n-r-s+2}(0;\lambda)}{(n-r-s+2)!} & \frac{A_{n-r+1}(x;\lambda)}{(n-r+1)!} & \frac{A_{n-r-s}(0;\lambda)}{(n-r-s)!} \cdots & \frac{A_{n-2r+2}(0;\lambda)}{(n-2r+2)!} \end{vmatrix} \quad (1.1)$$

where $A_n(x;\lambda)$ are the exponential Euler polynomials. From the relation $A_n'(x;\lambda)/n! = A_{n-1}(x;\lambda)/(n-1)!$ it is easy to see that

$$A^{(s)}_{n,r,s}(0;\lambda) = H_r(A_n(0;\lambda)/n!), \quad (1.2)$$

where $H_r(a_n)$ denotes the Hankel determinant of order r given by

$$H_r(a_n) = \begin{vmatrix} a_n & a_{n-1} & \cdots & a_{n-r+1} \\ a_{n-1} & a_{n-2} & \cdots & a_{n-r} \\ \vdots & \vdots & & \vdots \\ a_{n-r+1} & a_{n-r} & \cdots & a_{n-2r+2} \end{vmatrix}.$$

Using the relation

$$H_r(\Pi_n(\lambda)/n!) = (-1)^{[r/2]}\, C(n,r)\, \Pi_{n,r}(\lambda), \quad (1.3)$$

where

$$C(n,r) = \frac{1!\,2!\cdots(r-1)!}{n!(n-1)!\cdots(n-r+1)!}$$

(see [3]), it follows that

$$A^{(s)}_{n,r,s}(0;\lambda) = (-1)^{[r/2]+(r-1)(n-r+1)}\, \frac{C(n,r)\,\Pi_{n,r}(\lambda)}{(\lambda-1)^{n-r+1}}, \quad (1.4)$$

where $\Pi_n(\lambda) = (\lambda - 1)^n A_n(0; \lambda)$. Further, from the properties $A_n^{(\rho)}(1; \lambda) = \lambda A_n^{(\rho)}(0; \lambda)$ ($\rho = 0, 1, ..., n - 1$), it is easy to check that $A_{n,r,s}(x; \lambda)$ satisfy the relations

$$A_{n,r,s}^{(\rho)}(1; \lambda) = \lambda A_{n,r,s}^{(\rho)}(0; \lambda) \qquad (\rho = 0, 1, ..., n - r), \tag{1.5}$$

$$\left.\begin{aligned} &A_{n,r,s}^{(\rho)}(1; \lambda) = A_{n,r,s}^{(\rho)}(0; \lambda) = 0 \qquad (\rho = 0, 1, ..., r - 1, \rho \neq s), \\ &A_{n,r,s}^{(s)}(0; \lambda)/H_r(A_n(0; \lambda)/n!) = 1, \end{aligned}\right\} \tag{1.6}$$

provided $\lambda \neq 1$ and λ is not a zero of $\Pi_{n,r}(\lambda)$, an assumption which we shall impose throughout this paper.

2. The Exponential Hermite–Euler Splines $S_{n,r}(x; \lambda)$

Let us define a function $S_{n,r,s}(x; \lambda)$ ($s = 0, 1, ..., r - 1$) such that

$$\left.\begin{aligned} S_{n,r,s}(x; \lambda) &= A_{n,r,s}(x; \lambda)/H_r(A_n(0; \lambda)/n!) \qquad (0 \leqslant x \leqslant 1) \\ S_{n,r,s}(x + 1; \lambda) &= \lambda S_{n,r,s}(x; \lambda) \qquad \forall \text{ real } x. \end{aligned}\right\} \tag{2.1}$$

It follows from (1.5) and (1.6) that $S_{n,r,s}(x) \in C^{n-r}(-\infty, \infty)$ and

$$\left.\begin{aligned} &S_{n,r,s}^{(\rho)}(\nu, \lambda) = 0 \qquad (\rho = 0, 1, ..., r - 1, \rho \neq s), \\ &S_{n,r,s}^{(s)}(\nu, \lambda) = \lambda^\nu \qquad (\nu = 0, \pm 1, \pm 2, ...), \end{aligned}\right\} \tag{2.2}$$

so that it is cardinal spline belonging to the class

$$\mathscr{S}_{n,r}^{(s)} = \{S(x) \in \mathscr{S}_{n,r} : S^{(\rho)}(\nu) = 0 \ \forall \text{ integers}, \rho = 0, 1, ..., r - 1, \rho \neq s\}.$$

When $r = 1$ (in which case $s = 0$), $S_{n,1,0}(x; \lambda) = S_n(x; \lambda)$ are the exponential Euler splines considered by Schoenberg [4].

Now, set

$$S_{n,r}(x; \lambda) = \sum_{s=0}^{r-1} (\log \lambda)^s S_{n,r,s}(x; \lambda) \qquad (x \in R). \tag{2.3}$$

The following theorem is an easy consequence of (2.2).

THEOREM 2.1. *The spline functions $S_{n,r}(x; \lambda)$ belong to $\mathscr{S}_{n,r}$ and satisfy the interpolatory conditions*

$$S_{n,r}^{(\rho)}(\nu; \lambda) = (\log \lambda)^\rho \lambda^\nu \qquad (\rho = 0, 1, ..., r - 1) \qquad \textit{for all } \nu = 0, \pm 1, \pm 2, ..., \tag{2.4}$$

3. Representation of $S_{2m-1,r,s}(x;\lambda)$ in Terms of B-Splines

The B-splines for cardinal Hermite interpolation, denoted by $N_s(x)$ $(s = 0, 1,..., r - 1)$, were introduced by Schoenberg and Sharma [6]. These B-splines belong to the spaces $\mathscr{S}_{2m-1,r}^{(s)}$, have support in $(-(m - r + 1), (m - r + 1))$, and satisfy the interpolatory properties

$$\begin{aligned} N_s^{(s)}(\nu) &= C_\nu \qquad (\nu = -(m-r),..., (m-r)), \\ &= 0 \qquad \text{otherwise}, \end{aligned} \tag{3.1}$$

where C_ν are the coefficients of the monic reciprocal polynomial $\Pi_{2m-1,r}(\lambda) = \sum_{\nu=0}^{2m-2r} C_{\nu-(m-r)}\lambda^\nu$.

It was shown in [2] that the 'polynomial component of the spline $s!\,\lambda^{(m-r)}\sum_{-\infty}^{\infty}\lambda^\nu N_s(x-\nu)$ in $[0, 1]$ is given explicitly by the determinant

$$\left|\begin{array}{cccccccc} x^s & 1 & \binom{s}{1} & \cdots (1-\lambda) & 0 \cdots 0 & 0 & \cdots & 0 \\ x^r & 1 & \binom{r}{1} & \cdots & \binom{r}{r-1}(1-\lambda) & 0 & \cdots & 0 \\ x^{r+1} & 1 & \binom{r+1}{1} & & & (1-\lambda)\ 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & & & \ddots & \vdots \\ & & & & & & & 0 \\ x^{2m-r-1} & 1 & \binom{2m-r-1}{1} & \cdots & \cdots & \cdots & \cdots & (1-\lambda) \\ \vdots & \vdots & \vdots & & & & & \vdots \\ x^{2m-2} & 1 & \binom{2m-2}{1} & \cdots & \cdots & \cdots & \cdots & \binom{2m-2}{2m-r-1} \\ x^{2m-1} & 1 & \binom{2m-1}{1} & \cdots & \cdots & \cdots & \cdots & \binom{2m-1}{2m-r-1} \end{array}\right|. \tag{3.2}$$

By an argument similar to that in [2], using the properties (1.5) and (1.6), it can be shown that the polynomial

$$\frac{s!\,\Pi_{2m-1,r}(\lambda)\,A_{2m-1,r,s}(x;\lambda)}{H_r(A_{2m-1}(0;\lambda)/(2m-1!))} \qquad (x \in [0, 1])$$

is also given by (3.2), provided λ is not a zero of $\Pi_{2m-1,r}(\lambda)$. Hence

$$\frac{A_{2m-1,r,s}(x;\lambda)}{H_r(A_{2m-1}(0;\lambda)/(2m-1)!)} = \frac{1}{\Pi_{2m-1,r}(\lambda)}\sum_{-\infty}^{\infty}\lambda^{(m-r)+\nu}N_s(x-\nu) \quad (x \in [0, 1]). \tag{3.3}$$

From (2.1) and (3.3) we easily deduce the following

THEOREM 3.1. *The exponential Hermite–Euler spline $S_{2m-1,r,s}(x;\lambda)$ is expressible in terms of the B-spline $N_s(x)$ by*

$$S_{2m-1,r,s}(x;\lambda) = \frac{1}{\Pi_{2m-1,r}(\lambda)} \sum_{\infty}^{\infty} \lambda^{\nu} N_s(x + (m-r) - \nu). \tag{3.4}$$

4. CONVERGENCE OF EXPONENTIAL HERMITE–EULER SPLINES

When $r = 1$, Schoenberg [4] proved that $\lim_{n\to\infty} S_n(x;\lambda) \to \lambda^x$ uniformly for x belonging to a finite interval, if λ is a nonnegative complex number. In general we have the following result.

THEOREM 4.1. *If λ is a complex number which is not of sign $(-1)^r$, then*

$$\lim_{n\to\infty} S_{n,r}^{(\rho)}(x;\lambda) = (\log \lambda)^{\rho}\, \lambda^x \qquad (\rho = 0, 1, 2,..., r-1). \tag{4.1}$$

uniformly for x belonging to a finite interval.

The results of the above theorem follow from the corresponding results for the functions $S_{n,r,s}(x;\lambda)$. In order to state the latter results we write $\lambda = |\lambda|\, e^{i\alpha}$ and $\lambda_k = \log|\lambda| + i(\alpha + 2\pi k)$ $(k = 0, \pm 1, \pm 2,...)$. In [4] it was shown that the exponential Euler polynomial $A_n(x;\lambda)$ has the following expansion.

$$A_n(x;\lambda)/n! = (\lambda - 1)\,\lambda^{-1}\,\lambda^{x'} \sum_{-\infty}^{\infty} e^{2\pi i k x}/\lambda_k^{n+1}. \tag{4.2}$$

If we define a numerical sequence $\{\mu_k\}$ $(k = 0, 1, 2,...)$ by

$$\mu_0 = \lambda_0\,,\quad \mu_1 = \lambda_{-1}\,,\quad \mu_2 = \lambda_1\,,\quad \mu_3 = \lambda_{-2}\,,\quad \mu_4 = \lambda_2\,,..., \tag{4.3}$$

and the corresponding sequence of functions $\{u_k(x)\}$ $(k = 0, 1, 2,...)$ by

$$u_0(x) = 1,\quad u_1(x) = e^{-2\pi i x},\quad u_2(x) = e^{2\pi i x},\quad u_3(x) = e^{-2\pi 2 i x},..., \tag{4.4}$$

then (4.2) can be written as

$$A_n(x;\lambda)/n! = (\lambda - 1)\,\lambda^{-1}\lambda^{x} \sum_{0}^{\infty} u_k(x)/\mu_k^{n+1}. \tag{4.5}$$

Next, we introduce the notation $V(a_0, a_1,..., a_{r-1})$ to stand for the Vandermonte determinant

$$V(a_0, a_1,..., a_{r-1}) = \begin{vmatrix} 1 & a_0 & a_0{}^2 & \cdots & a_0^{r-1} \\ 1 & a_1 & a_1{}^2 & \cdots & a_1^{r-1} \\ \vdots & \vdots & & & \vdots \\ 1 & a_{r-1} & a_{r-1}^2 & \cdots & a_{r-1}^{r-1} \end{vmatrix} \tag{4.6}$$

and let $V_s(a_0, a_1, \ldots, a_{r-1}; u(x))$ $(s = 0, 1, \ldots, r-1)$ be the determinants obtained from (4.6) by replacing the sth column by the column vector $(u_0(x), u_1(x), u_2(x), \ldots, u_{r-1}(x))^T$. For each $s = 0, 1, \ldots, r-1$, define

$$\phi_s(x; \lambda) = \lambda^x \frac{V_s(\mu_0, \mu_1, \ldots, \mu_{r-1}; u(x))}{V(\mu_0, \mu_1, \ldots, \mu_{r-1})}. \tag{4.7}$$

The behavior of the exponential Hermite–Euler splines $S_{n,r,s}(x; \lambda)$ as $n \to \infty$ is described by the following

THEOREM 4.2. *Let $\lambda = |\lambda| e^{i\alpha}$. The following relation holds uniformly for x belonging to a finite interval*:

$$\lim_{n\to\infty} S_{n,r,s}^{(\rho)}(x; \lambda) = \phi_s^{(\rho)}(x; \lambda) \cdot \quad (\rho = 0, 1, \ldots, r-1) \tag{4.8}$$

for $-\pi < \alpha < \pi$ when r is odd, and for $0 < \alpha < 2\pi$ when r is even.

The proofs of Theorem 4.1 and 4.2 involve tedious determinantal manipulations. We shall give a complete proof only for the case $r = 2$.

5. CONVERGENCE FOR THE CASE $r = 2$

When $r = 2$, the results of Theorem 4.2 can be expressed in a simple form in terms of the functions

$$\alpha(x) = \lambda^x e^{-\pi i x} (\sin \pi x)/\pi, \tag{5.1}$$

$$\beta(x) = \lambda^x - (\log \lambda)\, \alpha(x). \tag{5.2}$$

More precisely we have

THEOREM 5.1. *Let $\lambda = |\lambda| e^{i\alpha}$. If $0 < \alpha < 2\pi$, the following relations hold uniformly for all x belonging to finite interval*:

$$\lim_{n\to\infty} S_{n,2,0}^{(\rho)}(x; \lambda) = \beta^{(\rho)}(x) \qquad (\rho = 0, 1), \tag{5.3}$$

and

$$\lim_{n\to\infty} S_{n,2,1}^{(\rho)}(x; \lambda) = \alpha^{(\rho)}(x) \qquad (\rho = 0, 1). \tag{5.4}$$

Clearly, the results of Theorem 4.1 for the case $r = 2$ follow from (5.3) and (5.4). A proof of Theorem 5.1 depends on the following lemma.

LEMMA 5.2. *Let* $\lambda = |\lambda| e^{i\alpha}$ *and* $\lambda_k = \log|\lambda| + i(\alpha + 2\pi k)$. *The following relations hold uniformly for all x in* [0, 1]:

$$\lim_{n\to\infty} \lambda_0^{n+1} A_n(x;\lambda)/n! = (\lambda - 1)\,\lambda^{-1}\lambda^x \qquad (-\pi < \alpha \leqslant \pi), \tag{5.5}$$

$$\begin{aligned}&\lim_{n\to\infty} \lambda_1^{n+1}\{A_{n-1}(x;\lambda)/n-1)! - \lambda_0 A_n(x;\lambda)/n!\} \\ &\qquad = (\lambda + 1)\,\lambda^{-1}\,\lambda^x e^{2\pi i x}(2\pi i) \qquad (-\pi < \alpha < 0),\end{aligned} \tag{5.6}$$

$$\begin{aligned}&\lim_{n\to\infty} \lambda_{-1}^{n+1}\{A_{n-1}(x;\lambda)/(n-1)! - \lambda_0 A_n(x;\lambda)/n!\} \\ &\qquad = (\lambda - 1)\,\lambda^{-1}\lambda^x e^{-2\pi i x}(-2\pi i) \qquad (0 < \alpha \leqslant \pi).\end{aligned} \tag{5.7}$$

Proof. Using the expansion (4.2) we have

$$\lambda_0^{n+1} A_n(x;\lambda)/n! = (\lambda - 1)\,\lambda^{-1}\lambda^x \sum_{-\infty}^{\infty} e^{2\pi i k x}(\lambda_0/\lambda_k)^{n+1}. \tag{5.8}$$

Since $|\lambda_0| < |\lambda_k|\ \forall k \neq 0$, (5.5) follows from (5.8). Also from (4.2) we have

$$\begin{aligned}&\lambda_1^{n+1}\{A_{n-1}(x;\lambda)/(n-1)! - \lambda_0 A_n(x;\lambda)/n!\} \\ &\qquad = (\lambda - 1)\,\lambda^{-1}\lambda^x \sum_{k\neq 0} (\lambda_k - \lambda_0)(\lambda_1/\lambda_k)^{n+1}\, e^{2\pi i k x}.\end{aligned} \tag{5.9}$$

If $-\pi < \alpha < 0$, $|\lambda_1| < |\lambda_k|\ \forall k \neq 0, 1$, and (5.6) follows from (5.9). The limit (5.7) is proved in the same way. ∎

Proof of Theorem 5.1. We shall prove only the relation

$$\lim_{n\to\infty} S_{n,2,0}(x;\lambda) = \beta(x). \tag{5.10}$$

The rest are proved in the same way.

We can write

$$\lambda_0^{n+1}\lambda_{-1}^{n+1} A_{n,2,0}(x;\lambda) = \begin{vmatrix} \lambda_0^{n+1} A_n(x;\lambda)/n! & \lambda_0^{n+1} A_{n-1}(0;\lambda)/(n-1)! \\ \lambda_{-1}^{n+1}\{A_{n-1}(x;\lambda)/(n-1)! - \lambda_0 A_n(x;\lambda)/n!\} & \lambda_{-1}^{n+1}\{A_{n-2}(0;\lambda)/(n-2)! - \lambda_0 A_{n-1}(0;\lambda)/(n-1)!\} \end{vmatrix}.$$

If $0 < \alpha \leqslant \pi$, it follows from (5.5) and (5.7) that

$$\lambda_0^{n+1}\lambda_{-1}^{n+1} A_{n,2,0}(x;\lambda) \to (\lambda - 1)^2\,\lambda^{-2}\lambda^x(\lambda_{-1} - \lambda_0)(\lambda_{-1} - \lambda_0 e^{-2\pi i x}). \tag{5.11}$$

Hence from (2.1) and (5.11) we have

$$\lim_{n\to\infty} S_{n,2,0}(x;\lambda) = \lambda^x(\lambda_{-1} - \lambda_0 e^{-2\pi i x})/(\lambda_{-1} - \lambda_0)$$
$$= \lambda^x\{1 - (\log\lambda)(1 - e^{-2\pi i x})/2\pi i\} \qquad (0 < \alpha \leqslant \pi). \quad (5.12)$$

Similarly,

$$\begin{aligned}\lim_{n\to\infty} S_{n,2,0}(x;\lambda) &= \lambda^x\{1 + (\log\lambda)(1 - e^{2\pi i x})/2\pi i\}\\ &= \lambda^x e^{2\pi i x}\{1 - (\log\lambda + 2\pi i)(1 - e^{-2\pi i x})/2\pi i\}\\ &\qquad (-\pi < \alpha < 0).\end{aligned} \quad (5.13)$$

Combining (5.12) and (5.13) we obtain

$$\lim_{n\to\infty} S_{n,2,0}(x;\lambda) = \lambda^x\{1 - (\log\lambda)(1 - e^{-2\pi i x})/2\pi i\} \quad (5.14)$$

when $\lambda = |\lambda| e^{i\alpha}$ for $0 < \alpha < 2\pi$, from which (5.10) follows. ∎

Acknowledgment

Part of this paper is taken from the author's doctoral thesis written under the supervision of Professor A. Sharma at the University of Alberta, Edmonton, Alberta, Canada. The author wishes to thank Professor Sharma for his kind help and encouragement.

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