

Parametrisation of the sphere

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1 Cardinal Hermite exponential splines

1.1 The parametric model

Conti et al's paper Ellipse-preserving interpolation and subdivision scheme introduces two basis functions from the space $\mathcal{E}_4 = \langle 1, x, e^{-iw_1x}, e^{iw_1x} \rangle$ where $w = \frac{2\pi}{M}$ to reproduce closed curves with M control points. The corresponding parametric representation is

$$r(t) = \sum_{k \in \mathbb{Z}} r(k) \phi_{1,w}(t - k) + r'(k) \phi_{2,w}(t - k) \quad (1)$$

with r and r' assumed to be M -periodic.

The basis functions are **cycloidal splines** (Exponential splines? Exponential B-splines?) given by

$$\phi_{1,w}(x) = \begin{cases} g_{1,w}(x) & \text{for } x \geq 0 \\ g_{1,w}(-x) & \text{for } x < 0 \end{cases} \quad \phi_{2,w}(x) = \begin{cases} g_{2,w}(x) & \text{for } x \geq 0 \\ -g_{2,w}(-x) & \text{for } x < 0 \end{cases} \quad (2)$$

The resulting parametric model has the following properties

1. Unique and stable representation ($\{\phi_{\mathbf{w}}(\cdot - \mathbf{k}) = (\phi_{1,w}(\cdot - k), \phi_{2,w}(\cdot - k))\}_k$ Riesz basis)
2. Affine invariance (partition unity condition on ϕ_1)
3. Perfectly reproduce sinusoids of period M
4. Exact interpolation of points and first derivative
5. Support of ϕ_1, ϕ_2 is $[-1, 1]$
6. Hermite interpolation property of order 1
7. C^1 -continuous

1.2 The unit sphere with scaling factors w_1, w_2

The usual continuous representation of the sphere is given by

$$\sigma(u, v) = (\cos(2\pi u) \sin(\pi v), \sin(2\pi u) \sin(\pi v), \cos(\pi v)) \quad (u, v) \in [0, 1]^2 \quad (3)$$

Suppose we have M_1 control points on latitudes, M_2 control points on meridians. The control points are then $c[k, l]_{k \in [0, \dots, M_1-1], l \in [0, \dots, M_2-1]}$. Let $w_1 = \frac{2\pi}{M_1}, w_2 = \frac{\pi}{M_2}$.

From the paper we have (also holds for sin functions)

$$\begin{aligned} \forall u \in [0, M_1] \quad \cos(w_1 u) &= \sum_{k \in \mathbb{Z}} \cos(w_1 k) \phi_{1,w_1}(u - k) - w_1 \sin(w_1 k) \phi_{2,w_1}(u - k) \\ \forall v \in [0, 2M_2] \quad \cos(w_2 v) &= \sum_{l \in \mathbb{Z}} \cos(w_2 l) \phi_{1,w_2}(v - l) - w_2 \sin(w_2 l) \phi_{2,w_2}(v - l) \end{aligned}$$

Normalizing the the continuous parameters leads to

$$\begin{aligned}\forall u \in [0, 1] \quad \cos(2\pi u) &= \sum_{k \in \mathbb{Z}} \cos(w_1 k) \phi_{1,w_1}(M_1 u - k) - w_1 \sin(w_1 k) \phi_{2,w_1}(M_1 u - k) \\ \forall v \in [0, 2] \quad \cos(\pi v) &= \sum_{l \in \mathbb{Z}} \cos(w_2 l) \phi_{1,w_2}(M_2 v - l) - w_2 \sin(w_2 l) \phi_{2,w_2}(M_2 v - l)\end{aligned}$$

Be aware that in the first representations above $\{\cos(w_1 k), -w_1 \sin(w_1 k)\}$ is (M_1, M_1) -periodic i.e we need point and first derivative values at M_1 control points for a full representation. However in the second representation $\{\cos(w_2 l), -w_2 \sin(w_2 l)\}$ are $(2M_2, 2M_2)$ -periodic i.e we need point and first derivative values at $2M_2$ control points for a full representation.

1.3 Representation of the sphere

For all $(u, v) \in [0, 1]^2$

$$\begin{aligned}\sigma(u, v) &= \sum_{(k,l) \in \mathbb{Z}^2} c_1[k, l] \phi_{1,w_1}(M_1 u - k) \phi_{1,w_2}(M_2 v - l) \\ &+ \sum_{(k,l) \in \mathbb{Z}^2} c_2[k, l] \phi_{1,w_1}(M_1 u - k) \phi_{2,w_2}(M_2 v - l) \\ &+ \sum_{(k,l) \in \mathbb{Z}^2} c_3[k, l] \phi_{2,w_1}(M_1 u - k) \phi_{1,w_2}(M_2 v - l) \\ &+ \sum_{(k,l) \in \mathbb{Z}^2} c_4[k, l] \phi_{2,w_1}(M_1 u - k) \phi_{2,w_2}(M_2 v - l)\end{aligned}$$

Or equivalently for all $(u, v) \in [0, 1]^2$

$$\begin{aligned}\sigma(u, v) &= \sum_{k=0}^{M_1} \sum_{l=0}^{M_2} c_1[k, l] \phi_{1,w_1}(M_1 u - k) \phi_{1,w_2}(M_2 v - l) \\ &+ \sum_{k=0}^{M_1} \sum_{l=0}^{M_2} c_2[k, l] \phi_{1,w_1}(M_1 u - k) \phi_{2,w_2}(M_2 v - l) \\ &+ \sum_{k=0}^{M_1} \sum_{l=0}^{M_2} c_3[k, l] \phi_{2,w_1}(M_1 u - k) \phi_{1,w_2}(M_2 v - l) \\ &+ \sum_{k=0}^{M_1} \sum_{l=0}^{M_2} c_4[k, l] \phi_{2,w_1}(M_1 u - k) \phi_{2,w_2}(M_2 v - l)\end{aligned}$$

Or equivalently for all $(u, v) \in [0, 1]^2$

$$\begin{aligned}\sigma(u, v) &= \sum_{k=0}^{M_1-1} \sum_{l=0}^{2M_2-1} c_1[k, l] \phi_{1,w_1,per}(M_1 u - k) \phi_{1,w_2,per}(M_2 v - l) \\ &+ \sum_{k=0}^{M_1-1} \sum_{l=0}^{2M_2-1} c_2[k, l] \phi_{1,w_1,per}(M_1 u - k) \phi_{2,w_2,per}(M_2 v - l) \\ &+ \sum_{k=0}^{M_1-1} \sum_{l=0}^{2M_2-1} c_3[k, l] \phi_{2,w_1,per}(M_1 u - k) \phi_{1,w_2,per}(M_2 v - l) \\ &+ \sum_{k=0}^{M_1-1} \sum_{l=0}^{2M_2-1} c_4[k, l] \phi_{2,w_1,per}(M_1 u - k) \phi_{2,w_2,per}(M_2 v - l)\end{aligned}$$

Or equivalently for all $(u, v) \in [0, 1]^2$

$$\begin{aligned}\sigma(u, v) = & \sum_{k=0}^{M_1-1} \sum_{l=0}^{M_2} c_1[k, l] \phi_{1,w_1,per}(M_1 u - k) \phi_{1,w_2}(M_2 v - l) \\ & + \sum_{k=0}^{M_1-1} \sum_{l=0}^{M_2} c_2[k, l] \phi_{1,w_1,per}(M_1 u - k) \phi_{2,w_2}(M_2 v - l) \\ & + \sum_{k=0}^{M_1-1} \sum_{l=0}^{M_2} c_3[k, l] \phi_{2,w_1,per}(M_1 u - k) \phi_{1,w_2}(M_2 v - l) \\ & + \sum_{k=0}^{M_1-1} \sum_{l=0}^{M_2} c_4[k, l] \phi_{2,w_1,per}(M_1 u - k) \phi_{2,w_2}(M_2 v - l)\end{aligned}$$

$$\begin{aligned}c_1[k, l] &= \begin{bmatrix} \cos(w_1 k) \sin(w_2 l) \\ \sin(w_1 k) \sin(w_2 l) \\ \cos(w_2 l) \end{bmatrix} = \sigma\left(\frac{k}{M_1}, \frac{l}{M_2}\right) & c_2[k, l] &= \begin{bmatrix} w_2 \cos(w_1 k) \cos(w_2 l) \\ w_2 \sin(w_1 k) \cos(w_2 l) \\ -w_2 \sin(w_2 l) \end{bmatrix} = \frac{1}{M_2} \frac{\partial \sigma}{\partial v}\left(\frac{k}{M_1}, \frac{l}{M_2}\right) \\ c_3[k, l] &= \begin{bmatrix} -w_1 \sin(w_1 k) \sin(w_2 l) \\ w_1 \cos(w_1 k) \sin(w_2 l) \\ 0 \end{bmatrix} = \frac{1}{M_1} \frac{\partial \sigma}{\partial u}\left(\frac{k}{M_1}, \frac{l}{M_2}\right) & c_4[k, l] &= \begin{bmatrix} -w_1 w_2 \sin(w_1 k) \cos(w_2 l) \\ w_1 w_2 \cos(w_1 k) \cos(w_2 l) \\ 0 \end{bmatrix} = \frac{1}{M_1 M_2} \frac{\partial^2 \sigma}{\partial u \partial v}\left(\frac{k}{M_1}, \frac{l}{M_2}\right)\end{aligned}$$

$$\begin{aligned}\phi_{1,w_1,per}(\cdot) &= \sum_{k \in \mathbb{Z}} \phi_{1,w_1}(\cdot - M_1 k) \\ \phi_{2,w_1,per}(\cdot) &= \sum_{k \in \mathbb{Z}} \phi_{2,w_1}(\cdot - M_1 k)\end{aligned}$$

$$\begin{aligned}\phi_{1,w_2,per}(\cdot) &= \sum_{k \in \mathbb{Z}} \phi_{1,w_2}(\cdot - 2M_2 k) \\ \phi_{2,w_2,per}(\cdot) &= \sum_{k \in \mathbb{Z}} \phi_{2,w_2}(\cdot - 2M_2 k)\end{aligned}$$

2 Exponential B-splines in 3D

2.1 The parametric model

Delgado et al's paper [Spline-based deforming ellipsoids for 3D bioimage segmentation](#) derive an exponential B-splines-based model that allow to reproduce ellipsoids. The model can well approximate blobs and perfectly spheres and ellipsoids. The corresponding parametric representation is

$$\sigma(u, v) = \sum_{(i,j) \in \mathbb{Z}^2} c[i, j] \phi_1\left(\frac{u}{T_1} - i\right) \phi_2\left(\frac{v}{T_2} - j\right) \quad (4)$$

with $T_1, T_2 > 0$ sampling steps for each parametric dimension and $\{c[i, j]\}_{(i,j) \in \mathbb{Z}^2}$ are the 3D control points.

The basis functions, reproducing unit period sinusoids with M coefficients, are exponential B-splines given by

$$\varphi_M(\cdot) = \sum_{k=0}^3 (-1)^k h_M[k] \varsigma_M(\cdot + \frac{3}{2} - k) \quad (5)$$

where $\varsigma_M(\cdot) = \frac{1}{4} \text{sgn}(\cdot) \frac{\sin^2(\frac{\pi}{M} \cdot)}{\sin^2(\frac{\pi}{M})}$ and $h_M = [1, 1 + 2 \cos(\frac{2\pi}{M}), 1 + 2 \cos(\frac{2\pi}{M}), 1]$.

Suppose we have M_1 control points on latitudes, M_2 control points on meridians. The resulting parametric model has the following properties

1. Unique and stable representation (sufficient is $\{\phi_1(\cdot - k)\}_k, \{\phi_2(\cdot - k)\}_k$ Riesz basis)
2. Affine invariance (partition unity condition on ϕ_1, ϕ_2)
3. Well-defined Gaussian curvature. ϕ_1, ϕ_2 are twice differentiable with bounded second derivative
4. Perfectly reproduce ellipsoids
5. Support of $\phi_1 = \varphi_{M_1}, \phi_2 = \varphi_{2M_2}$ is $[-\frac{3}{2}, \frac{3}{2}]$

2.2 Conditions for representing the unit sphere

The parametric representation of a closed surface with sphere-like topology, M_1 control points on latitudes and M_2 control points on meridians is

$$\forall (u, v) \in [0, 1]^2 \quad \sigma(u, v) = \sum_{k=0}^{M_1-1} \sum_{l=-1}^{M_2+1} c[k, l] \phi_1(M_1 u - k) \phi_2(M_2 v - l) \quad (6)$$

Unlike before, continuity of points and tangents at poles is not guaranteed. Conditions are

$$\forall k = 0, \dots, M_1 - 1 \quad \mathbf{c}_N = c[k, 1] \phi_2(-1) + c[k, 0] \phi_2(0) + c[k, -1] \phi_2(1) \quad (7)$$

$$\mathbf{c}_S = c[k, M_2 + 1] \phi_2(-1) + c[k, M_2] \phi_2(0) + c[k, M_2 - 1] \phi_2(1) \quad (8)$$

$$\mathbf{T}_{1,N} \cos(2\pi u) + \mathbf{T}_{2,N} \sin(2\pi u) = M_2 \sum_{k=0}^{M_1-1} \sum_{l=-1}^{M_2+1} c[k, l] \phi_1(M_1 u - k) \phi_2'(-l) \quad (9)$$

$$\mathbf{T}_{1,S} \cos(2\pi u) + \mathbf{T}_{2,S} \sin(2\pi u) = M_2 \sum_{k=0}^{M_1-1} \sum_{l=-1}^{M_2+1} c[k, l] \phi_1(M_1 u - k) \phi_2'(M_2 - l) \quad (10)$$

$$(11)$$

Incorporating such conditions in the model, a parametric splines-based surface with a sphere-like topology, C^1 continuity and ellipsoid-reproducing capabilities (all positions and orientations) is given by

$$\forall (u, v) \in [0, 1]^2 \quad \sigma(u, v) = \sum_{k=0}^{M_1-1} \sum_{l=-1}^{M_2+1} c[k, l] \phi_{1,per}(M_1 u - k) \phi_2(M_2 v - l) \quad (12)$$

where $\{c[i, j]\}_{i \in [0, \dots, M_1-1], j \in [1, \dots, M_2-1]}$, $\mathbf{T}_{1,N}, \mathbf{T}_{2,N}, \mathbf{T}_{1,S}, \mathbf{T}_{2,S}, \mathbf{c}_N, \mathbf{c}_S$ are free parameters that is $M_1(M_2 - 1) + 6$ control points.

$c[k, -1], c[k, 0], c[k, M_2], c[k, M_2 + 1]$ are constrained by the values of the free parameters.

2.3 Representation of the sphere

The unit sphere is thus represented by

$$\forall (u, v) \in [0, 1]^2 \quad \sigma(u, v) = \sum_{k=0}^{M_1-1} \sum_{l=-1}^{M_2+1} c[k, l] \phi_{1,per}(M_1 u - k) \phi_2(M_2 v - l) \quad (13)$$

With coefficients are given by

$$c[k, l] = \begin{bmatrix} c_{M_1}[k] s_{2M_2}[l] \\ s_{M_1}[k] s_{2M_2}[l] \\ c_{2M_2}[l] \end{bmatrix} = \begin{bmatrix} \frac{2(1 - \cos(\frac{2\pi}{M_1}))}{\cos(\frac{\pi}{M_1}) - \cos(\frac{3\pi}{M_1})} \frac{2(1 - \cos(\frac{\pi}{M_2}))}{\cos(\frac{\pi}{2M_2}) - \cos(\frac{3\pi}{2M_2})} \cos(\frac{2\pi k}{M_1}) \sin(\frac{\pi l}{M_2}) \\ \frac{2(1 - \cos(\frac{2\pi}{M_1}))}{\cos(\frac{\pi}{M_1}) - \cos(\frac{3\pi}{M_1})} \frac{2(1 - \cos(\frac{\pi}{M_2}))}{\cos(\frac{\pi}{2M_2}) - \cos(\frac{3\pi}{2M_2})} \sin(\frac{2\pi k}{M_1}) \sin(\frac{\pi l}{M_2}) \\ \frac{2(1 - \cos(\frac{\pi}{M_2}))}{\cos(\frac{\pi}{2M_2}) - \cos(\frac{3\pi}{2M_2})} \cos(\frac{\pi l}{M_2}) \end{bmatrix} \quad (14)$$

and

$$c_M[k] = \frac{2(1 - \cos(\frac{2\pi}{M}))}{\cos(\frac{\pi}{M}) - \cos(\frac{3\pi}{M})} \cos(\frac{2\pi k}{M})$$

$$s_M[k] = \frac{2(1 - \cos(\frac{2\pi}{M}))}{\cos(\frac{\pi}{M}) - \cos(\frac{3\pi}{M})} \sin(\frac{2\pi k}{M})$$

These coefficients satisfy the constraints with

$$\begin{array}{lll} \mathbf{c}_N = [0 \ 0 \ 1]^T & \mathbf{c}_N = [0 \ 0 \ -1]^T & \mathbf{T}_{1,N} = [\pi \ 0 \ 0]^T \\ \mathbf{T}_{2,N} = [0 \ \pi \ 0]^T & \mathbf{T}_{1,S} = [-\pi \ 0 \ 0]^T & \mathbf{T}_{2,S} = [0 \ -\pi \ 0]^T \end{array}$$

3 Compactly-supported smooth interpolators for shape modeling

3.1 The parametric model

Schmitter et al's paper Compactly-supported smooth interpolators for shape modeling with varying resolution propose a continuous representation of curves and surfaces with the help of generators that have the advantages of both continuous and discrete schemes. The generator is expressed as a linear combination of half integer shifts of exponential B-spline of vector $\alpha \in \mathbb{C}^n$ i.e

$$\phi_{\lambda,\alpha}(t) = \sum_{k \in \mathbb{Z}} \lambda[k] \beta_{\alpha}(t - \frac{k}{2}) \quad (15)$$

β_{α} has support $[-\frac{n}{2}, \frac{n}{2}]$. In what follows we choose to have $\lambda[k] = 0$ for $k \notin \llbracket -n+2, n-2 \rrbracket$ and $\lambda[-k] = \lambda[k]$. There are therefore $(n-1)$ unknowns $\lambda[0], \dots, \lambda[n-2]$. We also impose that elements in α are 0 or come in complex conjugate pairs and that no pair of purely imaginary elements of α is separated by integer multiple of $2j\pi$ (for Riesz basis property).

This function is interpolatory if and only if $\phi_{\lambda,\alpha}(0) = 1$ and $\phi_{\lambda,\alpha}(1) = \dots = \phi_{\lambda,\alpha}(n-2) = 0$. This defines a system of $n-1$ equations with $n-1$ unknowns. The system has a solution if the matrix defined by $k, l = 0, \dots, n-2$

$$[A_{\alpha}]_{k+1,l+1} = \begin{cases} \beta_{\alpha}(k) & \text{if } l = 0 \\ \beta_{\alpha}(k - \frac{l}{2}) + \beta_{\alpha}(k + \frac{l}{2}) & \text{else} \end{cases} \quad (16)$$

is invertible. In that case $\lambda = A_{\alpha}^{-1}(1, 0, \dots, 0)$ and we define $\phi_{\alpha} = \phi_{\lambda,\alpha}$. Tensor-product surfaces are represented with the help of two generators in the form

$$\sigma(u, v) = \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} \sigma[k, l] \phi_{\alpha_1}(u - k) \phi_{\alpha_2}(v - l) \quad (17)$$

The resulting interpolation scheme has the following properties

1. Unique and stable representation ($\alpha_m - \alpha_n \notin 2j\pi\mathbb{Z}$ Riesz basis)
2. Affine invariance ($0 \in \alpha_1, 0 \in \alpha_2$)
3. Perfectly reproduce ellipsoids (conditions on α)
4. ϕ_{α} is interpolatory
5. ϕ_{α} is smooth i.e at least \mathcal{C}^1
6. Can reproduce the nullspace \mathcal{N}_{α}
7. Can reproduce shapes at various resolutions
8. ϕ_{α} is compactly supported on $[-n+1, n-1]$

3.2 Conditions for representing the unit sphere

Let M_1 be the number of control points in u and M_2 the number of control points in v . For ϕ_{α_1} to be able to reproduce $\cos(\frac{2\pi \cdot}{M_1}), \sin(\frac{2\pi \cdot}{M_1})$ we need to have $(\frac{-2i\pi}{M_1}, \frac{2i\pi}{M_1}) \in \alpha_1$. Adding affine invariance condition, $\phi_{(0, \frac{-2i\pi}{M_1}, \frac{2i\pi}{M_1})}$ can reproduce constants and M_1 -periodic sinusoids with M_1 control points as follows

$$\cos(\frac{2\pi \cdot}{M_1}) = \sum_{k \in \mathbb{Z}} \cos(\frac{2\pi k}{M_1}) \phi_{\alpha_1}(\cdot - k) \quad (18)$$

Similarly $\phi_{(0, \frac{-i\pi}{M_2}, \frac{i\pi}{M_2})}$ can reproduce constants and $2M_2$ -periodic sinusoids with $2M_2$ control points as follows

$$\cos\left(\frac{\pi \cdot}{M_2}\right) = \sum_{k \in \mathbb{Z}} \cos\left(\frac{\pi k}{M_2}\right) \phi_{\alpha_2}(\cdot - k) \quad (19)$$

Generators $\phi_{\alpha_1}, \phi_{\alpha_2}$ both have support of size 4 ($n = 3$) so that they are given by

$$\begin{aligned} \phi_{\alpha_1}(t) &= \lambda_1[0]\beta_{\alpha_1}(t) + \lambda_1[1](\beta_{\alpha_1}(t - 1/2) + \beta_{\alpha_1}(t + 1/2)) \\ \phi_{\alpha_2}(t) &= \lambda_2[0]\beta_{\alpha_2}(t) + \lambda_2[1](\beta_{\alpha_2}(t - 1/2) + \beta_{\alpha_2}(t + 1/2)) \end{aligned}$$

In order to find $\lambda_1[0], \lambda_1[1]$ one has to solve $\phi_{\alpha_1}(0) = 1, \phi_{\alpha_1}(1) = 0$.

Aparte on tempered distributions

Green function of operator $L_\alpha : \mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}'(\mathbb{R})$ is an element ρ_α of $\mathcal{S}(\mathbb{R})$ that satisfies $L\{\rho_\alpha\} = \delta$ where δ is the Dirac tempered distribution. There is a unique such function (to be proved?) that also satisfies $\forall t < 0 \rho_\alpha(t) < 0$. The tempered distribution $T_{\rho_\alpha} : \phi \mapsto \int_0^\infty e^{\alpha t} \phi(t) dt$ is such that the associated element of $\mathcal{S}(\mathbb{R})$ (bijection $\mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}'(\mathbb{R})$? probably not true), ρ_α satisfies $L_\alpha\{\rho_\alpha\} = \delta$.

Consequently β_α^+ is an element of Schwartz space $\mathcal{S}(\mathbb{R})$. **Is that right?** With abuse of notation we write $\beta_\alpha^+(t) = e^{\alpha t} \chi_{[0,1]}(t)$. Using distribution then we would have

$$\beta_\alpha^{+'}(t) = \delta(t) + \alpha e^{\alpha t} \chi_{[0,1]}(t)$$

The equality is to be taken in the distribution sense.

3.3 Representation of the sphere

The unit sphere is thus represented by

$$\forall (u, v) \in [0, 1]^2 \quad \sigma(u, v) = \sum_{(k, l) \in \mathbb{Z}^2} c[k, l] \phi_{\alpha_1}(M_1 u - k) \phi_{\alpha_2}(M_2 v - l) \quad (20)$$

Or equivalently

$$\forall (u, v) \in [0, 1]^2 \quad \sigma(u, v) = \sum_{k=0}^{M_1-1} \sum_{l \in \mathbb{Z}} c[k, l] \phi_{\alpha_1, per}(M_1 u - k) \phi_{\alpha_2}(M_2 v - l) \quad (21)$$

Or equivalently

$$\boxed{\forall (u, v) \in [0, 1]^2 \quad \sigma(u, v) = \sum_{k=0}^{M_1-1} \sum_{l=-1}^{M_2+1} c[k, l] \phi_{\alpha_1, per}(M_1 u - k) \phi_{\alpha_2}(M_2 v - l)} \quad (22)$$

Denoting $w_1 = \frac{2\pi}{M_1}, w_2 = \frac{\pi}{M_2}$, the coefficients are given by

$$\boxed{c[k, l] = \begin{bmatrix} \cos(w_1 k) \sin(w_2 l) \\ \sin(w_1 k) \sin(w_2 l) \\ \cos(w_2 l) \end{bmatrix} = \sigma\left(\frac{k}{M_1}, \frac{l}{M_2}\right)} \quad (23)$$

4 Smooth shapes with spherical topology

4.1 The parametric model

In 2017 Schmitter et al's paper Smooth shapes with spherical topology derive a parametric model very similar to that presented in Spline-based deforming ellipsoids for 3D bioimage segmentation. In user interactive applications one usually wants a curve/shape reproducing model to have some or all following properties: 1.intuitive manipulation, 2.stable deformation, 3.shape deformation as optimization process requiring fast evaluation of surface and volume integrals, 4.smooth representation. It is usually impossible to find a model optimal w.r.t to all these requirements. In practice a compromise is made with existing models based on polygon meshes, subdivision or NURBS.

Parametric shapes are built as linear combinations of integers shifts of a generator function φ that is to say

$$r(t) = \sum_{k \in \mathbb{Z}} c[k] \varphi(t - k) \quad (24)$$

φ is piecewise exponential. It is the smoothed version of third order exponential B-spline that is $\varphi = \beta * \psi$ with ψ an appropriate smoothing kernel. The model can be extended to tensor-product surfaces $\sigma(u, v)$ as previously done in previous representations.

$$\sigma(u, v) = \sum_{(k, l) \in \mathbb{Z}^2} c[k, l] \varphi_1(u - k) \varphi_2(v - l)$$

Authors define

$$\begin{aligned} \phi_1(t) &= \varphi_{M_1, per}(t) = \sum_{n \in \mathbb{Z}} \varphi_{M_1}(t - M_1 n) & \phi_2(t) &= \varphi_{2M_2}(t) \\ \forall k \in \mathbb{Z} \quad \phi_{1,k}(t) &= \phi_1(M_1 t - k) & \phi_{2,k}(t) &= \phi_2(M_2 t - k) \end{aligned}$$

The resulting interpolation scheme has the following properties

1. Unique and stable representation ($\{\varphi_M(\cdot - k)\}_k$ Riesz basis)
2. Affine invariance (partition unity condition on φ_M)
3. Well-defined Gaussian curvature. φ_M is twice differentiable with bounded second derivative
4. Perfectly reproduce ellipsoids for $M \geq 3$
5. φ_M is interpolatory
6. Support of φ_M is in $[-2, 2]$

4.2 Conditions for representing the unit sphere

The parametric representation of a closed surface with sphere-like topology, M_1 control points on latitudes and M_2 control points on meridians is

$$\forall (u, v) \in [0, 1]^2 \quad \sigma(u, v) = \sum_{k=0}^{M_1-1} \sum_{l=-1}^{M_2+1} c[k, l] \phi_{1,k}(u) \phi_{2,k}(v) \quad (25)$$

As for the model from article 7, continuity of points and tangents at poles is not guaranteed. The exact same conditions are used leading to

$$\forall k = 0, \dots, M_1 - 1 \quad \mathbf{c}_N = c[k, 0] \quad (26)$$

$$\mathbf{c}_S = c[k, M_2] \quad (27)$$

$$\mathbf{T}_{1,N} \cos(2\pi u) + \mathbf{T}_{2,N} \sin(2\pi u) = M_2 \sum_{k=0}^{M_1-1} \sum_{l=-1}^{M_2+1} c[k, l] \varphi_{M_1}(M_1 u - k) \varphi'_{2M_2}(-l) \quad (28)$$

$$\mathbf{T}_{1,S} \cos(2\pi u) + \mathbf{T}_{2,S} \sin(2\pi u) = M_2 \sum_{k=0}^{M_1-1} \sum_{l=-1}^{M_2+1} c[k, l] \varphi_{M_1}(M_1 u - k) \varphi'_{2M_2}(M_2 - l) \quad (29)$$

$$(30)$$

By incorporating conditions in the model to ensure continuity of the surface and of the tangent plane at poles we obtain constraints on $c[k, -1], c[k, 0], c[k, M_2], c[k, M_2 + 1]$. Other values $c[k, l]$ are free parameters as well as $\mathbf{c}_N, \mathbf{c}_S, \mathbf{T}_{1,N}, \mathbf{T}_{2,N}, \mathbf{T}_{1,S}, \mathbf{T}_{2,S}$ describing the poles.

φ_{M_1} can reproduce $\cos(\frac{2\pi \cdot}{M_1})$ with M_1 control points

$$\begin{aligned} \cos\left(\frac{2\pi u}{M_1}\right) &= \sum_{k \in \mathbb{Z}} \cos\left(\frac{2\pi k}{M_1}\right) \varphi_{M_1}(u - k) \\ \cos(2\pi u) &= \sum_{k=0}^{M_1-1} \cos\left(\frac{2\pi k}{M_1}\right) \phi_{1,k}(u) \end{aligned}$$

In a similar fashion φ_{2M_2} can reproduce $\cos(\frac{\pi \cdot}{M_2})$ with $2M_2$ control points i.e

$$\begin{aligned} \cos\left(\frac{\pi v}{M_2}\right) &= \sum_{k \in \mathbb{Z}} \cos\left(\frac{\pi k}{M_2}\right) \varphi_{2M_2}(v - k) \\ \cos(\pi v) &= \sum_{k \in \mathbb{Z}} \cos\left(\frac{\pi k}{M_2}\right) \phi_{2,k}(v) \end{aligned}$$

4.3 Representation of the sphere

Given the usual representation of the unit sphere, it can be represented in our scheme as follows

$$\sigma(u, v) = \sum_{k=0}^{M_1-1} \sum_{l \in \mathbb{Z}} c[k, l] \phi_{1,k}(u) \phi_{2,k}(v) \quad (31)$$

or using the fact φ_{2M_2} has support of size 4

$$\boxed{\forall (u, v) \in [0, 1]^2 \quad \sigma(u, v) = \sum_{k=0}^{M_1-1} \sum_{l=-1}^{M_2+1} c[k, l] \phi_{1,k}(u) \phi_{2,k}(v)} \quad (32)$$

Denoting $w_1 = \frac{2\pi}{M_1}$, $w_2 = \frac{\pi}{M_2}$, the coefficients are given by

$$\boxed{c[k, l] = \begin{bmatrix} \cos(w_1 k) \sin(w_2 l) \\ \sin(w_1 k) \sin(w_2 l) \\ \cos(w_2 l) \end{bmatrix} = \sigma\left(\frac{k}{M_1}, \frac{l}{M_2}\right)} \quad (33)$$