Hermite Polynomials snakes order 2

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I Translation of Schoenberg's 1973 paper for the case $r = 3, m = 3$

The following is simply a reminder of some of the results found by I.J Schoenberg in his paper *Cardinal Interpolation and Spline Functions. III Cardinal Hermite interpolation.* Let's reintroduce notations of the article and make somehow more explicits what the objects they encode are.

Let r and m be positive integers that satisfy $r \leq m$. The set of cardinal splines of order $2m$ with knot multiplicity r is denoted by $S_{2m,r}$. Note that using De Boor's notations for splines set we have the following

$$S_{2m,r} = \$_{2m,\mathbb{Z}_3} = \Pi_{<2m,\mathbb{Z},2m-r} \tag{1}$$

where $\mathbb{Z}_3$ denotes the sequence of knots $(\dots, -1, -1, -1, 0, 0, 0, 1, 1, 1, \dots)$. It is clear from these notations that $S_{2m,r} \subset \mathcal{C}^{2m-r-1}$.

Theorem 1. *Let S be either of the vector spaces $\mathcal{L}_{p,r}, F_{\gamma,r}$ with $\gamma \geq 0$, $p \in \mathbb{N}^*$. Provided a solution to C.H.I.P $\left(y_\nu, \dots, y_\nu^{(r-1)}, S_{2m,r} \cap S\right)$ exists, it is uniquely given by*

$$\forall x \in \mathbb{R} \qquad S(x) = \sum_{\nu=-\infty}^{\infty} y_\nu L_0(x-\nu) + \dots + y_\nu^{(r-1)} L_{r-1}(x-\nu) \tag{2}$$

In order to specify a usable model for active contours it remains to determine explicit expressions for the basis functions $L_0, \dots, L_{r-1}$. In the article they are determined by solving a set of $2m-r$ linear equations. This system is obtained by considering separately the function L_s on $[1, \infty)$ and $[0, 1]$. Note that specifying the function on both these intervals completely determine L_s as the latter is even (if s is even) or odd (if s is odd).

On $[1, \infty)$, L_s can be decomposed into

$$L_s = \sum_{j=1}^{m-r} c_j S_j$$

where $(c_j)_{j=1}^{m-r}$ are $(m-r)$ unknown coefficients to be determined and S_j are the eigensplines for the first $m-r$ "eigenvalues" λ_j, solutions to $|\Delta_{r,d}(\lambda)| = 0$.

On $[0, 1]$, L_s is given by a polynomial P of order $2m$ that takes a specific form according to the parities of s and r (we refer to equations (7.13) and (7.14)) in the article. This polynomial introduces m unknown coefficients $(a_j)_{j=1}^m$. To determine a total of $m + m - r = 2m - r$ unknown coefficients we make use of the $2m - r$ equality conditions at 1 $P^{(\rho)}(1) = L_s^{(\rho)}(1)$. We end up of a system of $2m - r$ equations for $2m - r$ unknowns that can be solved exactly provided the matrix of the system is non singular. Schoenberg proves with a very nice argument that the matrix of the system is always non singular.

In the case $m = r = 3$, L_0, L_1, L_2 are 0 on $[1, \infty)$ and on $[0, 1]$ are given by

$$L_0(x) = 1 + a_1 x^3 + a_2 x^4 + a_3 x^5 \tag{3}$$
$$L_1(x) = x + a_1 x^3 + a_2 x^4 + a_3 x^5 \tag{4}$$
$$L_2(x) = \frac{1}{2} x^2 + a_1 x^3 + a_2 x^4 + a_3 x^5 \tag{5}$$

where the coefficients for each generator are unrelated. Note that L_s have finite support *because* $m = r$. If that was not the case the term $\sum_{j=1}^{m-r} c_j S_j$ may not be 0 and therefore L_s would be non zero on $[1, \infty)$! Can it happen though that $m > r$ and $(c_j)_{j=1}^{m-r}$ are 0? To determine the coefficients above we need to solve independently for each generator the 3 equations $P^{(\rho)}(1) = 0$. This leads to the following systems

$$\begin{cases} a_1 + a_2 + a_3 = -1 \\ 3a_1 + 4a_2 + 5a_3 = 0 \\ 3a_1 + 6a_2 + 10a_3 = 0 \end{cases} \quad \begin{cases} a_1 + a_2 + a_3 = -1 \\ 3a_1 + 4a_2 + 5a_3 = -1 \\ 3a_1 + 6a_2 + 10a_3 = 0 \end{cases} \quad \begin{cases} a_1 + a_2 + a_3 = -\frac{1}{2} \\ 3a_1 + 4a_2 + 5a_3 = -1 \\ 3a_1 + 6a_2 + 10a_3 = -\frac{1}{2} \end{cases}$$

II The resulting snake scheme

II.1 Generating functions

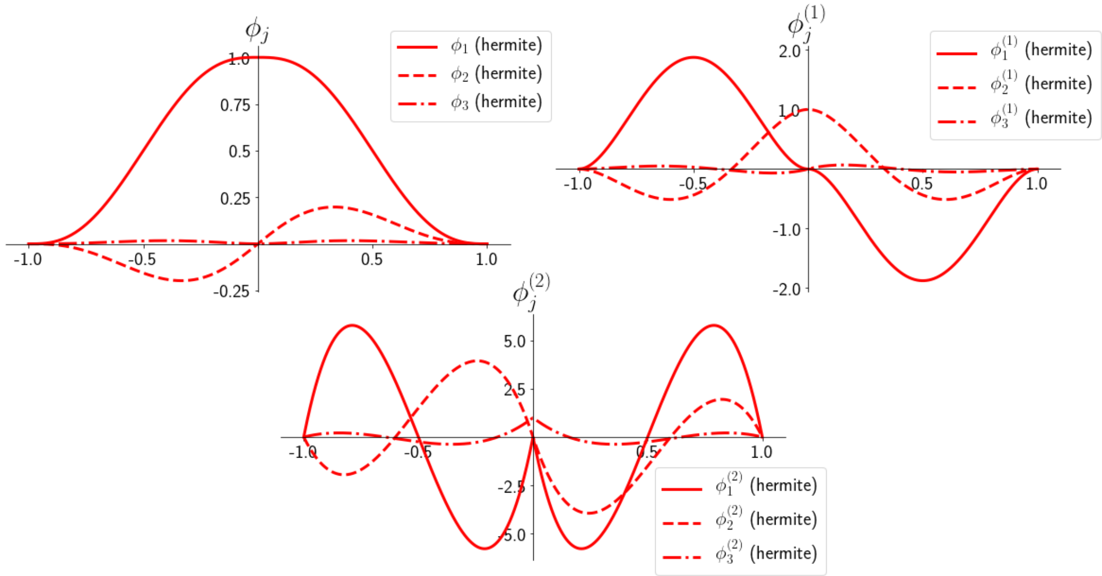

Figure 1: Generators for C.H.I.P with $m = r = 3$

Solving the linear systems written in the previous section yields explicit formulas for the Schoenberg basis generators L_0, L_1, L_2, that we rename ϕ_1, ϕ_2, ϕ_3 in accordance with modern notations (see V. Uhlmann *Hermite Snakes with Controls of Tangents*). The formulas are the following.

$$\phi_1(x) = \begin{cases} 1 - 10x^3 + 15x^4 - 6x^5 & \text{if } 0 \le x \le 1 \\ 1 + 10x^3 + 15x^4 + 6x^5 & \text{if } -1 \le x < 0 \end{cases} \tag{6}$$

$$\phi_2(x) = \begin{cases} x - 6x^3 + 8x^4 - 3x^5 & \text{if } 0 \le x \le 1 \\ x - 6x^3 - 8x^4 - 3x^5 & \text{if } -1 \le x < 0 \end{cases} \tag{7}$$

$$\phi_3(x) = \begin{cases} 0.5x^2 - 1.5x^3 + 1.5x^4 - 0.5x^5 & \text{if } 0 \le x \le 1 \\ 0.5x^2 + 1.5x^3 + 1.5x^4 + 0.5x^5 & \text{if } -1 \le x < 0 \end{cases} \tag{8}$$

In figure 1 are displayed the values of these functions as well as their two first derivatives. As mentioned in the previous section, the generators L_s are elements of $S_{2m,r} = S_{6,3}$ which is a subset of $\mathcal{C}^{2m-r-1} = C^2$. It is apparent in the figure that these functions have continuous derivatives up to order 2 but that higher order derivatives do not exist in neighborhoods of $-1, 0$ and 1.

II.2 Closed planar curves or "contours"

Consider a positive integer M and an M-**periodic parametrized closed curve** $r : \mathbb{R} \to \mathbb{R}^2$ for which we have local derivatives up to order 2 at M location sites regularly spaced that is we know $(r[k], r'[k], r''[k])_{k=0}^{M-1}$.

Corollary 1. *Given M periodic sequences $(r[k], r'[k], r''[k])_{k\in\mathbb{Z}}$, there exists a unique spline curve of order 6 whose value and derivatives agree with the sequence of coefficients at integers locations. This spline curve and its derivatives are everywhere bounded and take the form for $t \in \mathbb{R}$*

$$r(t) = \sum_{k\in\mathbb{Z}} r[k]\phi_1(t-k) + r'[k]\phi_2(t-k) + r''[k]\phi_3(t-k) \tag{9}$$

$$= \sum_{k=0}^{M-1} r[k]\phi_{1,per}(t-k) + r'[k]\phi_{2,per}(t-k) + r''[k]\phi_{3,per}(t-k) \tag{10}$$

Proof. As the sequence of coefficients $(r[k], r'[k], r''[k])_{k\in\mathbb{Z}}$ are in $Y_{\gamma,r} = Y_{0,3}$ (i.e they are bounded), application of Schoenberg's theorem 1 yields existence and unicity of a interpolating function in $S_{6,3} \cap F_{0,3}$. Application of theorem 4 then leads to the explicit formulation given above. □

Remark 1. *It is convenient to normalize the continuous parameter to the $[0, 1]$ interval as is usual in the implementations. For that let the renormalized curve $s(t) = r(Mt)$ for $t \in [0, 1]$. Note that this completely describes the curve as it is enough to describe the curve r on $[0, M]$. Differentiating this equality twice yields $r[k] = s[\frac{k}{M}], r'[k] = \frac{1}{M}s'[\frac{k}{M}], r''[k] = \frac{1}{M^2}s''[\frac{k}{M}]$. Therefore equation (10) is rewritten for $t \in [0, 1]$*

$$s(t) = \sum_{k=0}^{M-1} s[\frac{k}{M}]\phi_{1,per}(Mt-k) + \frac{1}{M}s'[\frac{k}{M}]\phi_{2,per}(Mt-k) + \frac{1}{M^2}s''[\frac{k}{M}]\phi_{3,per}(Mt-k) \tag{11}$$

In the rest of this document we will reuse the notation r for the normalized curve and won't make use of the notation s anymore. Equation (11) is the **mathematical representation of a planar curve** and we call it "snake" or "active contour". By playing with the coefficients we can capture a wide variety of contours that arise from closed objects in 2D images like cells membrane in a bioimage.

II.3 Open planar curves

Consider again a positive integer M and a **parametrized open curve** $r : \mathbb{R} \to \mathbb{R}^2$ for which we have local derivatives up to order 2 at M location sites regularly spaced that is we know $(r[k], r'[k], r''[k])_{k=0}^{M-1}$. By "open" we mean a curve that is not periodic.

Corollary 2. *Given biinfinite sequences of coefficients $(\ldots, 0, r[0], \ldots, r[M-1], 0, \ldots)$, $(\ldots, 0, r'[0], \ldots, r'[M-1], 0, \ldots)$, $(\ldots, 0, r''[0], \ldots, r''[M-1], 0, \ldots)$there exists a unique spline curve of order 6 whose value and* *derivatives agree with the sequence of coefficients at integers locations. This spline curve and its derivatives have compact support and take the form for $t \in \mathbb{R}$*

$$\begin{aligned} r(t) &= \sum_{k\in\mathbb{Z}} r[k]\phi_1(t-k) + r'[k]\phi_2(t-k) + r''[k]\phi_3(t-k) \\ &= \sum_{k=0}^{M-1} r[k]\phi_1(t-k) + r'[k]\phi_2(t-k) + r''[k]\phi_3(t-k) \end{aligned}$$

Proof. This result is again a simple application of theorem 1 and 4 given in Schoenberg's paper of 1981. □

Remark 2. *In this setting we are only interested in the curve lying between our coefficients that is the interpolated points with continuous parameter in the interval $[0, M-1]$ The normalization factor is therefore $M-1$ and the renormalized open curve $s(t) = r((M-1)t)$ takes the form*

$$s(t) = \sum_{k=0}^{M-1} s[\frac{k}{M-1}]\phi_1((M-1)t-k) + \frac{1}{M-1}s'[\frac{k}{M-1}]\phi_2((M-1)t-k) + \frac{1}{(M-1)^2}s''[\frac{k}{M-1}]\phi_3((M-1)t-k) \tag{12}$$

which we will also renote r.

II.4 Closed sphere-like surfaces

In my research project we are interested in developing a mathematical methods for representing a certain type of surfaces with explicit control of local properties including first-order derivatives and curvature. As a consequence extension of the schemes given in equations (11) and (12) to tensor-product surfaces (that is surfaces parametrized by 2 continuous parameters in a way that each continuous parameter appear in separate functions) may be relevant for the questions we have.

Consider positives integers M_1 and M_2 and a **sphere-like parametrized** surface $\sigma : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ with U a subset (closed in our case) of the plane. By "sphere-like" we mean an object that can be described with closed curves on latitudes (u varies while v is fixed) and open curves on longitudes (v varies while u is fixed). Suppose we have local properties of the surface at $M_1 \times (M_2+1)$ locations (counted with multiplicity as some locations may coalesce) on a regular grid.

Corollary 3. *Given the biinfinite sequences of coefficients $(\partial^{i,j}\sigma(k,l))_{k,l\in\mathbb{Z}^2,(i,j)\in\{0,1,2\}^2}$ that are M_1 periodic in the first coordinate and vanish when the second coordinate is outside $[0, M_2]$, there exists a unique interpolating tensor-product spline curve of order 6 whose value and partial derivatives agree with the sequence of coefficients at the integers grid locations. This tensor-product spline curve and its derivatives are everywhere bounded and take the form for $(u,v) \in [0, M_1] \times [0, M_2]$ (or equiv. $\mathbb{R} \times [0, M_2]$)*

$$\sigma(u,v) = \sum_{k=0}^{M_1-1}\sum_{l=0}^{M_2}\sum_{i,j=0}^{2} \partial^{i,j}\sigma(k,l)\phi_{i+1,per}(u-k)\phi_{j+1}(v-l) \tag{13}$$

Proof. This is a simple application of corollaries 1 and 2 given before. □

Remark 3.

- *Not all surfaces admit a tensor-product representation which limits the range of surfaces one can reach with this kind of interpolation scheme. Tensor-product spline has not yet been defined and therefore just mention here mention what we call a tensor-product spline a map $f : U_1 \times U_2 \subseteq \mathbb{R}^2 \to \mathbb{R}$ that can be written as $f_1(u) \times f_2(v)$ with f_1 and f_2 each splines on U_1 and U_2 respectively.*
- *The continuity of the basis functions is automatically transferred to the each coordinate of the spline curve, resulting in surfaces with parametrizations that are twice continuously differentiable. Is this preventing us from representing surfaces with singular points as we would like to have? This question is crucial to our objective and will be addressed later in more details.*

- *Normalizing each continuous parameter to the interval* $[0,1]$ *yields the following representation of the surface (where we note again* σ *the surface with normalized parameters)*

$$\sigma(u,v) = \sum_{k=0}^{M_1-1} \sum_{l=0}^{M_2} \sum_{i,j=0}^{2} \frac{1}{M_1^i M_2^j} \partial^{i,j} \sigma(\frac{k}{M_1}, \frac{l}{M_2}) \phi_{i+1,per}(M_1 u - k) \phi_{j+1}(M_2 v - l) \tag{14}$$

III Properties of the interpolation scheme