B-Splines for Cardinal Hermite Interpolation

S. L. Lee
Department of Mathematics
University of Alberta
Edmonton, Alberta, Canada

Submitted by Hans Schneider

ABSTRACT

We give a proof of a conjecture of I. J. Schoenberg on B-splines for Cardinal Hermite interpolation without the assumption that the characteristic polynomial $\Pi_{n,r}(\lambda)$ *is irreducible over the rational field.*

1. INTRODUCTION

In the study of Cardinal Hermite interpolation with values and first $r-1$ $(r \geqslant 1)$ derivatives prescribed at the integers, Lipow and Schoenberg [4] introduced the class $\mathcal{S}_{2m-1,r}$ $(m \geqslant r)$ of Cardinal spline functions $S(x) \in C^{2m-r-1}(-\infty,\infty)$ which are piecewise polynomials of degree $2m-1$ in each of the intervals $[\nu,\nu+1]$ $\forall$ integers ν. Subsequently Schoenberg and Sharma [6] introduced the B-spline $N_s(x)$ $(s=0,1,\dots,r-1)$ belonging to the space

$$\mathcal{S}^{(s)}_{2m-1,r} = \left\{ S(x) \in \mathcal{S}_{2m-1,r} : S^{(\rho)}(\nu)=0 \ (\rho=0,1,\dots,r-1,\ \rho \neq s) \ \forall \text{ integers } \nu \right\} \tag{1.1}$$

such that $N_s(x)$ vanishes outside the interval $\big(-(m-r+1),(m-r+1)\big)$ and

$$N_s^{(s)}(\nu)=c_\nu \qquad [\nu=-(m-r),-(m-r)+1,\dots,(m-r)] \tag{1.2}$$

where c_ν are the coefficients of the monic reciprocal polynomial

$$\Pi_{2m-1,r}(\lambda)=\sum_{\nu=0}^{2m-2r} c_{\nu-(m-r)}\lambda^{\nu}. \tag{1.3}$$

The main result of Schoenberg and Sharma asserts that every $S(x) \in \mathcal{S}_{2m-1,r}^{(s)}$ admits a unique representation of the form

$$S(x) = \sum_{\nu=-\infty}^{\infty} a_\nu N_s(x-\nu). \tag{1.4}$$

However, their proof is based upon the following ad hoc assumption:

ASSUMPTION 1. $\Pi_{2m-1,r}(\lambda)$ *is irreducible over the rational field.*

That this holds if r is odd and $2m-r$ is a prime number has been shown recently by Sharma and Strauss [7]. However Schoenberg [5] remarked that "this assumption concerns too deep an arithmetic problem in comparison with the linear algebra nature of the interpolation problem". He also remarked [6] that it would be interesting to establish the result without this assumption. The aim of this note is to give a proof of the representation (1.4) without Assumption 1.

2. PRELIMINARIES AND THE MAIN THEOREM

Let m, r be positive integers with $m \geqslant r$. The Cardinal Hermite interpolation problem (C.H.I.P.) is posed as follows:

Given r bi-infinite sequences of numbers

$$y^{(\rho)} = (y_\nu^{(\rho)}) \qquad (\rho = 0, 1, \ldots, r-1), \tag{2.1}$$

find $S(x) \in \mathcal{S}_{2m-1,r}$ *such that*

$$S^{(\rho)}(\nu) = y_\nu^{(\rho)} \qquad (\rho = 0, 1, \ldots, r-1) \tag{2.2}$$

$\forall$ *integers* ν.Lipow and Schoenberg [4] proved that if the data (2.1) satisfy the condition

$$y_\nu^{(\rho)} = 0(|\nu|^\gamma) \qquad (\rho = 0, 1, \ldots, r-1) \tag{2.3}$$

for some $\gamma > 0$, then the C.H.I.P. has a unique solution $S(x) \in \mathcal{S}_{2m-1,r}$ such that $S(x) = O(|x|^\gamma)$. It was also shown that the null space

$$\mathring{\mathcal{S}}_{2m-1,r} = \{ S(x) \in \mathcal{S}_{2m-1,r} : S^{(\rho)}(\nu) = 0 \ (\rho = 0, 1, \ldots, r-1) \ \forall \text{ integers } \nu \} \tag{2.4}$$

is a linear space of dimension $d=2m-2r$ spanned by the eigensplines $S_j(x)$ $(j=1,2,\dots,d)$ which satisfy the functional relation

$$S_j(x+1)=\lambda_j S_j(x) \qquad \forall x\in\mathbf{R}, \tag{2.5}$$

where λ_j $(j=1,2,\dots,d)$ are the zeros (real and simple) of $\Pi_{2m-1,r}(\lambda)$, which is given explicitly by

$$\Pi_{2m-1,r}(\lambda)=P\begin{pmatrix} r,\ r+1,\dots,2m-1 & \\ 0,\quad 1,\dots,2m-r-1 & :\lambda \end{pmatrix}, \tag{2.6}$$

where we use the notation $P\begin{pmatrix} i_0,i_1,\dots,i_\nu & \\ j_0,j_1,\dots,j_\nu & :\lambda\end{pmatrix}$ $(\nu=0,1,2,\dots)$ to denote the determinant obtained from the matrix $\|\binom{i}{j}-\lambda\delta_{ij}\|$ $(i,j=0,1,2,\dots)$ by deleting all the rows and columns except those labeled $\{i_0,i_1,\dots,i_\nu\}$ and $\{j_0,j_1,\dots,j_\nu\}$ respectively, and $P\begin{pmatrix} i_0,i_1,\dots,i_\nu \\ j_0,j_1,\dots,j_\nu\end{pmatrix}$ the corresponding determinant obtained from the matrix $P=\|\binom{i}{j}\|$ $(i,j=0,1,2,\dots)$. Lipow and Schoenberg [4] proved that the zeros of the polynomial $\Pi_{n,r}(\lambda)$ are real, simple and of sign $(-1)^r$, and it was shown in [3] (see also [2]) that the zeros of $\Pi_{n,r}(\lambda)$ and $\Pi_{n-1,r}(\lambda)$ strictly interlace.

We shall prove the following theorem.

THEOREM 1. *Every* $S(x)\in \mathbb{S}^{(s)}_{2m-1,r}$ $(s=0,1,\dots,r-1)$ *admits a unique representation of the form* $S(x)=\sum_{\nu=-\infty}^{\infty} a_\nu N_s(x-\nu)$.

The proof of Schoenberg and Sharma [6] for this theorem depends on the following lemma under the condition that Assumption 1 holds.

LEMMA 1. *For every* $s=0,1,\dots,r-1$, *the* $2m-2r+2$ *polynomials*

$$N_s(x),N_s(x+1),\dots,N_s(x+2m-2r+1) \qquad [x\in(-(m-r+1),-(m-r))] \tag{2.7}$$

are linearly independent.

In order to prove the theorem it is enough to prove Lemma 1 without Assumption 1. The remaining proof is the same as in [6].

3. PROOF OF LEMMA 1

We shall show that the $2m-2r+2$ polynomials given by (2.7) are linearly independent. Suppose

$$a_0N_s(x)+a_1N_s(x+1)+\cdots+a_{2m-2r+1}N_s(x+2m-2r+1)=0$$

$$\forall x\in(-(m-r+1),-(m-r)). \tag{3.1}$$

Then

$$a_0N_s^{(\rho)}(-(m-r))+a_1N_s^{(\rho)}(-(m-r)+1)+\cdots+a_{2m-2r}N_s^{(\rho)}((m-r))=0$$

$$(\rho=s,r,r+1,\ldots,2m-r-1), \tag{3.2}$$

since $N_s^{(\rho)}\ ((m-r)+1)=0$ $(\rho=0,1,\ldots,2m-r-1)$. Now (3.2) gives a homogeneous system of $2m-2r+1$ equations in $2m-2r+1$ unknowns a_0,-$a_1,\ldots,a_{2m-2r}$, whose determinant is given by

$$\Delta=\begin{vmatrix} N_s^{(s)}(-(m-r)) & N_s^{(s)}(-(m-r)+1) & \cdots & N_s^{(s)}((m-r)) \\ N_s^{(r)}(-(m-r)) & N_s^{(r)}(-(m-r)+1) & \cdots & N_s^{(r)}((m-r)) \\ N_s^{(r+1)}(-(m-r)) & N_s^{(r+1)}(-(m-r)+1) & \cdots & N_s^{(r+1)}((m-r)) \\ \vdots & \vdots & & \vdots \\ N_s^{(2m-r-1)}(-(m-r)) & N_s^{(2m-r-1)}(-(m-r)+1) & \cdots & N_s^{(2m-r-1)}((m-r)) \end{vmatrix}. \tag{3.3}$$

Next we show that this determinant is non-zero. For this purpose we multiply Δ by the following determinant:

$$V=\begin{vmatrix} 1 & \lambda_1^{(m-r)} & \lambda_2^{(m-r)} & \cdots & \lambda_{2m-2r}^{(m-r)} \\ 1 & \lambda_1^{(m-r)-1} & \lambda_2^{(m-r)-1} & \cdots & \lambda_{2m-2r}^{(m-r)-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \lambda_1^{-(m-r)} & \lambda_2^{-(m-r)} & \cdots & \lambda_{2m-2r}^{-(m-r)} \end{vmatrix}$$

to obtain

$$\Delta\cdot V=\begin{vmatrix} \sum_{-(m-r)}^{m-r} N_s^{(s)}(-\nu) & \sum_{-(m-r)}^{m-r} \lambda_1^\nu N_s^{(s)}(-\nu) & \cdots & \sum_{-(m-r)}^{m-r} \lambda_{2m-2r}^\nu N_s^{(s)}(-\nu) \\ \sum_{-(m-r)}^{m-r} N_s^{(r)}(-\nu) & \sum_{-(m-r)}^{m-r} \lambda_1^\nu N_s^{(r)}(-\nu) & \cdots & \sum_{-(m-r)}^{m-r} \lambda_{2m-2r}^\nu N_s^{(r)}(-\nu) \\ \vdots & \vdots & & \vdots \\ \sum_{-(m-r)}^{m-r} N_s^{(2m-r-1)}(-\nu) & \sum_{-(m-r)}^{m-r} \lambda_1^\nu N_s^{(2m-r-1)}(-\nu) & \cdots & \sum_{-(m-r)}^{m-r} \lambda_{2m-2r}^\nu N_s^{(2m-2r-1)}(-\nu) \end{vmatrix}, \tag{3.4}$$

where λ_j $(j=1,2,\ldots,2m-2r)$ are the zeros of $\Pi_{2m-1,r}(\lambda)$. Observe that the elements on the first row of (3.4) are all zero except the first element, which is $\Pi_{2m-1,r}(1)$. Hence we have

$$\Delta\cdot V=\Pi_{2m-1,r}(1)\begin{vmatrix} B_1^{(r)}(0) & B_2^{(r)}(0) & \cdots & B_{2m-2r}^{(r)}(0) \\ B_1^{(r+1)}(0) & B_2^{(r+1)}(0) & \cdots & B_{2m-2r}^{(r+1)}(0) \\ \vdots & \vdots & & \vdots \\ B_1^{(2m-r-1)}(0) & B_2^{(2m-r-1)}(0) & \cdots & B_{2m-2r}^{(2m-r-1)}(0) \end{vmatrix} \tag{3.5}$$

where

$$B_j(x)=\sum_{\nu=-\infty}^{\infty} \lambda_j^\nu N_s(x-\nu) \qquad (j=1,2,\ldots,2m-2r). \tag{3.6}$$

In Sec. 4 it is shown that the splines $B_j(x)$ $(j=1,2,\ldots,2m-2r)$ are equal to the eigensplines $S_j(x)$ defined by the functional equation (2.5), and that the columns of the determinant on the right of (3.5) are linearly independent vectors in $\mathbf{R}^{2m-2r}$. Since $\Pi_{2m-1,r}(1)\neq 0$, we conclude that $\Delta\cdot V\neq 0$ and therefore $\Delta\neq 0$. It follows from (3.2) that $a_j=0$ $(j=1,2,\ldots,2m-2r)$ and then from (3.1) that $a_{2m-2r+1}$ is also zero. Thus the polynomials given by (2.7) are linearly independent. ■

4. THE SPLINES $B_j(x)$

In proving Lemma 1 we made use of the following

LEMMA 2. *Let $B_j(x)$ $(j=1,2,\ldots,2m-2r)$ be defined by* (3.6). *Then the* *vectors*

$$\left(B_j^{(r)}(0), B_j^{(r+1)}(0), \ldots, B_j^{(2m-r-1)}(0)\right) \in \mathbf{R}^{2m-2r} \tag{4.1}$$

are linearly independent.

In order to prove Lemma 2 we shall first of all show that $B_j(x)$ ($j = 1, 2, \ldots, 2m-2r$) are equal to the eigensplines $S_j(x) \in \mathring{S}_{2m-1,r}$ defined by the functional relation (2.5).

Clearly it follows from (3.6) that $B_j(x)$ satisfies the functional relation

$$B_j(x+1) = \lambda_j B_j(x) \qquad (x \in \mathbf{R}), \tag{4.2}$$

and we have only to show that it is not identically zero. For this purpose let us set, for $s = 0, 1, \ldots, r-1$,

$$\phi_s(x;\lambda) \equiv \phi_{2m-1,r,s}(x;\lambda) = \lambda^{(m-r)} \sum_{\nu=-\infty}^{\infty} \lambda^\nu N_s(x-\nu)$$

$$= \lambda^{(m-r)} \sum_{\nu=-(m-r)}^{(m-r)+1} \lambda^\nu N_s(x-\nu) \qquad (x \in [0,1]). \tag{4.3}$$

Observe that $\phi_s(x;\lambda)$ is a polynomial in $x \in [0,1]$ of degree $2m-1$ when λ is fixed, and

$$\phi_s(x;\lambda_j) = \lambda_j^{(m-r)} B_j(x) \qquad (x \in [0,1]), \tag{4.4}$$

where λ_j ($j = 1, 2, \ldots, 2m-2r$) are the zeros of $\Pi_{2m-1,r}(\lambda)$. Furthermore,

$$\phi_s^{(s)}(0;\lambda) = \Pi_{2m-1,r}(\lambda) \tag{4.5}$$

where the differentiation is with respect to x.

If we set

$$\Pi_{n,r,s}(\lambda) = P\begin{pmatrix} s, r+1, \ldots, n & \\ 0, \quad 1, \ldots, n-r & \end{pmatrix} : \lambda \Bigg) \qquad (s = 0, 1, \ldots, r-1), \tag{4.6}$$

we have

Lemma 3. *For every $s = 0, 1, \ldots, r-1$,*

$$\phi_s^{(r)}(0;\lambda) = \frac{r!}{s!} \Pi_{2m-1,r,s}(\lambda) \qquad (\lambda \in \mathbf{R}), \tag{4.7}$$

where $\phi_s(x;\lambda)$ is given by (4.3) and the derivative is with respect to x.

Proof. If λ is not a zero of $\Pi_{2m-1,r}(\lambda)$, it follows from (4.5) that $\phi_s(x;\lambda)$, as a polynomial in x, is not identically zero. It is easy to see from (4.3) that

$$\phi_s^{(\rho)}(1;\lambda)=\phi_s^{(\rho)}(0;\lambda)=0 \qquad (\rho=0,1,\dots,r-1,\ \rho\neq s)$$

$$\phi_s^{(\rho)}(1;\lambda)=\lambda\phi_s^{(\rho)}(0;\lambda) \qquad (\rho=s,r,r+1,\dots,2m-r-1), \tag{4.8}$$

so that we can write

$$\phi_s(x;\lambda)=a_0x^{2m-1}+a_1x^{2m-2}+\cdots+a_{2m-r-1}x^r+\left(\Pi_{2m-1,r}(\lambda)\right)\frac{x^s}{s!}. \tag{4.9}$$

Writing (4.9) first followed by (4.8) in increasing ρ, we obtain a homogeneous system of $2m-r+1$ equations. Eliminating the unknowns, we see that if λ is not a zero of $\Pi_{2m-1,r}(\lambda)$, then

$$s!\phi_s(x;\lambda)=$$

$$\begin{vmatrix}
x^s & 1 & \binom{s}{1} & \cdots & 1-\lambda & 0 & \cdots & \cdots & \cdots & \cdots & 0\\
x^r & 1 & \binom{r}{1} & \cdots & \cdots & \cdots & \binom{r}{r-1} & 1-\lambda & 0 & \cdots & 0\\
x^{r+1} & 1 & \binom{r+1}{1} & \cdots & \cdots & \cdots & \binom{r+1}{r} & 1-\lambda & \cdots & & 0\\
\vdots & \vdots & \vdots & & & & & \ddots & \ddots & & \vdots\\
x^{2m-r-1} & 1 & \binom{2m-r-1}{1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1-\lambda\\
\vdots & \vdots & \vdots & & & & & & & & \\
x^{2m-2} & 1 & \binom{2m-2}{1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \binom{2m-2}{2m-r-1}\\
x^{2m-1} & 1 & \binom{2m-1}{1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \binom{2m-1}{2m-r-1}
\end{vmatrix}$$

By continuity we conclude that (4.10) holds for all real λ. It is clear that (4.10) implies (4.7). ■

Next we establish a useful identity for the polynomials $\Pi_{n,r}(\lambda)$ and $\Pi_{n,r,s}(\lambda)$.

Lemma 4. *Let n,r be positive integers such that $n\geqslant 2r+1$. If $0\leqslant s\leqslant r-1$, then*

$$\Pi_{n,r}(\lambda)\Pi_{n-1,r,s}(\lambda)-\Pi_{n-1,r}(\lambda)\Pi_{n,r,s}(\lambda)=-\Pi_{n,r+1}(\lambda)\Pi_{n-1,r-1,s}(\lambda). \tag{4.11}$$

Proof. Consider the following $(n-r+2)\times(n-r+1)$ matrix

$$\left|\begin{matrix}
1 & \binom{s}{1} & \cdots & 1-\lambda & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
1 & \binom{r}{1} & \cdots & \cdots & \cdots & \binom{r}{r-1} & 1-\lambda & 0 & \cdots & 0 \\
1 & \binom{r+1}{1} & \cdots & \cdots & \cdots & \cdots & \binom{r+1}{r} & (1-\lambda) & \cdots & 0 \\
\vdots & \vdots & & & & & & \ddots & \ddots & \vdots \\
1 & \binom{n-r}{1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \binom{n-r}{n-r-1} & (1-\lambda) \\
1 & \binom{n-r+1}{1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \binom{n-r+1}{n-r-1} & \binom{n-r+1}{n-r} \\
\vdots & \vdots & & & & & & & \vdots & \vdots \\
1 & \binom{n}{1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \binom{n}{n-r-1} & \binom{n}{n-r}
\end{matrix}\right| . \tag{4.12}$$

Let $\mathbf{f}^{(\nu)}$ denote the νth column of (4.12) ($\nu=1,2,\ldots,n-r$) and $\mathbf{d}$ denote the last column. Further, let $\mathbf{a}=(1,0,\ldots,0)^T$, $\mathbf{b}=(0,1,0,\ldots,0)^T$ and $\mathbf{c}=(0,0,\ldots,0,1)^T$ be column vectors in $\mathbf{R}^{n-r+2}$. If we denote by $D(\mathbf{a},\mathbf{b},\mathbf{f})\equiv D(\mathbf{a},\mathbf{b},\mathbf{f}^{(1)},\mathbf{f}^{(2)},\ldots,\mathbf{f}^{(n-r)})$ the determinant whose columns are the vectors $\mathbf{a},\mathbf{b},\mathbf{f}^{(1)},\ldots,\mathbf{f}^{(n-r)}$ in this order, then (4.11) follows easily from the following identity (see [1], p. 7):

$$\begin{vmatrix} D(\mathbf{a},\mathbf{c},\mathbf{f}) & D(\mathbf{b},\mathbf{c},\mathbf{f}) \\ D(\mathbf{a},\mathbf{d},\mathbf{f}) & D(\mathbf{b},\mathbf{d},\mathbf{f}) \end{vmatrix} = D(\mathbf{a},\mathbf{b},\mathbf{f})D(\mathbf{c},\mathbf{d},\mathbf{f}).$$

■

LEMMA 5. *Let $n\geqslant 2r+1$. If $(r-s)$ is even, then*

$$\Pi_{n,r,s}(\lambda)>0 \qquad \text{for } (-1)^{r+1}\lambda\geqslant 0. \tag{4.13}$$

where $\Pi_{n,r,s}(\lambda)$ is given by (4.6).

Proof. If we expand the determinantal representation of $\Pi_{n,r,s}(\lambda)$ in powers of λ, and take into account that $r-s$ is even, we obtain

$$\begin{aligned}\Pi_{n,r,s}(\lambda) = {} & a_0\left[(-1)^{r+1}\lambda\right]^{n-2r+1}+a_1\left[(-1)^{r+1}\lambda\right]^{n-2r}\\ & +a_2\left[(-1)^{r+1}\lambda\right]^{n-2r-1}+\cdots+a_{n-2r+1},\end{aligned} \tag{4.14}$$

where

$$a_0=P\begin{pmatrix} n-r+1, & n-r+2, & \ldots, & \ldots, & \ldots, & \ldots, & n \\ 0, & 1, & \ldots, & s-1, & s+1, & \ldots, & r \end{pmatrix},$$

$$a_{n-2r+1}=P\begin{pmatrix} s, & r+1, & r+2, & \ldots, & n \\ 0, & 1, & \ldots, & \ldots, & n-r \end{pmatrix} \quad \text{and}$$

$$\begin{aligned} a_k = {} & \sum P\begin{pmatrix} s,\nu_2,\ldots,\nu_k,n-r+1,n-r+2,\ldots,n \\ 0,1,\ldots,s-1,s,s+1,\ldots,r,\nu_2,\ldots,\nu_k \end{pmatrix} \\ & +\sum P\begin{pmatrix} \gamma_1,\gamma_2,\ldots,\gamma_k,n-r+1,n-r+2,\ldots,n \\ 0,1,\ldots,s-1,s+1,\ldots,r,\gamma_1,\gamma_2,\ldots,\gamma_k \end{pmatrix} \qquad (k=1,2,\ldots,n-2r). \end{aligned} \tag{4.15}$$

The first summation on the right hand side of the last equation of (4.15) is over all possible choices of $\{\nu_2,\nu_3,\ldots,\nu_k\}$ from $\{r+1,r+2,\ldots,n-r\}$, while the second is over all possible choices of $\{\gamma_1,\gamma_2,\ldots,\gamma_k\}$ from $\{r+1,r+2,\ldots,n-r\}$.

By Lemma 6.2 in [2], $a_k>0$ for all $k=0,1,\ldots,n-2r+1$. Hence (4.13) follows from (4.14). ■

REMARK. *It was in fact given in* [2] *that if* $r-s$ *is even,* $\Pi_{n,r,s}(\lambda)$ *has real simple zeros of sign* $(-1)^r$.

LEMMA 6. *Let* n,r *be positive integers such that* $n\geqslant 2r+1$. *If* $0\leqslant s\leqslant r-1$ *and* $r-s$ *is odd, the zeros of* $\Pi_{n,r,s}(\lambda)$ *are real simple zeros; one of which is* $(-1)^{r-1}$, *and the remaining ones are all of sign* $(-1)^r$.

Proof. Without loss of generality we assume that r is even. By Theorem 7.3 in [2] the zeros of $\Pi_{n,r}(\lambda)$ and $\Pi_{n-1,r}(\lambda)$ are positive, simple and interlacing. Let us denote the zeros of $\Pi_{n,r}(\lambda)$ by $\{\lambda_j^{(n)}\}$ $(j=1,2,\ldots,l)$, where $l=n-2r+1$. Then

$$0<\lambda_1^{(n)}<\lambda_1^{(n-1)}<\lambda_2^{(n)}<\cdots<\lambda_{l-1}^{(n-1)}<\lambda_l^{(n)}. \tag{4.16}$$

From (4.11) we have

$$\Pi_{n-1,r}(\lambda_j^{(n)})\Pi_{n,r,s}(\lambda_j^{(n)})=\Pi_{n,r+1}(\lambda_j^{(n)})\Pi_{n-1,r-1,s}(\lambda_j^{(n)}). \tag{4.17}$$

Since $r-1-s$ is even, it follows from Lemma 5 that the right hand side of (4.17) is positive for $j=1,2,\ldots,l$. Hence

$$\operatorname{sgn}\Pi_{n-1,r}(\lambda_j^{(n)})=\operatorname{sgn}\Pi_{n,r,s}(\lambda_j^{(n)}), \tag{4.18}$$

and it follows from (4.16) that $\Pi_{n,r,s}(\lambda)$ has exactly one zero in each of the intervals $(\lambda_j^{(n)},\lambda_{j+1}^{(n)})$ $(j=1,2,\ldots,l-1)$. Since $\Pi_{n,r,s}(\lambda)$ is a reciprocal polynomial (see [2]), the remaining zero must be -1. ■

LEMMA 7. *The splines* $B_j(x)$ $(j=1,2,\ldots,2m-2r)$ *defined in* (3.6) *are not identically zero.*

Proof. In view of (4.4) and (4.7) we have only to show that

$$\Pi_{2m-1,r,s}(\lambda_j)\neq 0 \qquad (j=1,2,\ldots,2m-2r), \tag{4.19}$$

where λ_j are the zeros of $\Pi_{2m-1,r}(\lambda)$. From the relation (4.11) we have

$$\Pi_{2m-2,r}(\lambda_j)\Pi_{2m-1,r,s}(\lambda_j)=\Pi_{2m-1,r+1}(\lambda_j)\Pi_{2m-2,r-1,s}(\lambda_j). \tag{4.20}$$

Since λ_j $(j=1,2,\ldots,d)$ are of sign $(-1)^r$, $\Pi_{2m-1,r+1}(\lambda_j)\neq 0$. Also $\Pi_{2m-2,r-1,s}(\lambda_j)\neq 0$ in view of Lemmas 5 and 6, since $\lambda_j\neq(-1)^r$. Hence $\Pi_{2m-1,r,s}(\lambda_j)\neq 0$ $(j=1,2,\ldots,d)$. ∎

COROLLARY 1. *The splines $B_j(x)$ $(j=1,2,\ldots,2m-2r)$ belong to $\mathring{S}_{2m-1,r}$ and are equal to the eigensplines $S_j(x)$ defined by the relation* (2.5) *up to a constant factor.*

Proof of Lemma 2. Define a linear transformation $\Phi: \mathring{S}_{2m-1,r}\rightarrow \mathbf{R}^{2m-2r}$ by

$$\Phi(S(x))=\left(S^{(r)}(0),S^{(r+1)}(0),\ldots,S^{(2m-r-1)}(0)\right)\quad \forall S(x)\in \mathring{S}_{2m-1,r}. \tag{4.21}$$

Clearly Φ is linear, and we shall show that it is non-singular.

For this purpose, let $(a_1,a_2,\ldots,a_{2m-2r})\in\mathbf{R}^{2m-2r}$ and define a polynomial in $[0,1]$ by

$$\begin{aligned}P(x)=&\frac{b_0x^{2m-1}}{(2m-1)!}+\frac{b_1x^{2m-2}}{(2m-2)!}+\cdots+\frac{b_{r-1}x^{2m-r}}{(2m-r)!}\\&+\frac{a_{2m-2r}x^{2m-r-1}}{(2m-r-1)!}+\cdots+\frac{a_1x^r}{r!}.\end{aligned} \tag{4.22}$$

If we set

$$P^{(\rho)}(1)=0\qquad(\rho=0,1,\ldots,r-1), \tag{4.23}$$

we obtain a non-homogeneous system of r equations in r unknowns $b_0,b_1,\ldots,b_{r-1}$ which can be uniquely solved in terms of $a_1,a_2,\ldots,a_{2m-2r}$. Thus each vector $(a_1,a_2,\ldots,a_{2m-2r})\in\mathbf{R}^{2m-2r}$ determines a unique polynomial $P(x)$ satisfying

$$P^{(\rho)}(1)=P^{(\rho)}(0)=0\qquad(\rho=0,1,\ldots,r-1). \tag{4.24}$$

By the same argument as in [4], $P(x)$ determines a unique spline $S(x)\in\mathring{S}_{2m-1,r}$ such that

$$S(x)=P(x)\qquad(x\in[0,1]). \tag{4.25}$$

It follows from (4.22) and (4.25) that $\Phi(S(x))=(a_1,a_2,\ldots,a_{2m-2r})$. Hence Φ is non-singular.

Since the eigensplines in $\mathring{S}_{2m-1,r}$ are linearly independent, it follows from Corollary 1 that $\Phi(B_j(x)) = (B_j^{(r)}(0), B_j^{(r+1)}(0), \ldots, B_j^{(2m-r-1)}(0))$ $(j = 1, 2, \ldots, 2m-2r)$ are linearly independent. ■

REMARK. *The fact that the splines* $B_j(x)$ $(j = 1, 2, \ldots, 2m-2r)$ *defined by* (3.6) *are not identically zero is essential for us to conclude that they are equal to the eigensplines except for a constant factor. Professor I. J. Schoenberg has kindly pointed out that* $B_j(x) \equiv 0$ *does not follow from Corollary* 1 *of* [6]. *In fact the uniqueness assertion of Corollary* 1 *in* [6] *is true only in a certain restricted sense.*

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