Estimation of the twist vector

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1 Estimation technique of the twist vector

Suppose we would like to represent a surface over the patch $[0,1]^2$ from data at each of the 4 corners. Depending on the available data, different interpolation techniques (with different properties) can be used. In case of bicubic polynomial interpolation ("bicubic patch") one can represent a surface from knowledge of coordinates of the surface value and it's first-order derivatives (that is 16 vectors in total) as $\sigma : [0,1]^2 \to \mathbb{R}^3$ given by

$$\sigma(u,v) = \begin{bmatrix} f_1(u) \\ f_1(1-u) \\ f_2(u) \\ -f_2(1-u) \end{bmatrix}^T \begin{bmatrix} \sigma(0,0) & \sigma(0,1) & \sigma_2(0,0) & \sigma_2(0,1) \\ \sigma(1,0) & \sigma(1,1) & \sigma_2(1,0) & \sigma_2(1,1) \\ \sigma_1(0,0) & \sigma_1(0,1) & \sigma_{12}(0,0) & \sigma_{12}(0,1) \\ \sigma_1(1,0) & \sigma_1(1,1) & \sigma_{12}(1,0) & \sigma_{12}(1,1) \end{bmatrix} \begin{bmatrix} f_1(v) \\ f_1(1-v) \\ f_2(v) \\ -f_2(1-v) \end{bmatrix} \tag{1}$$

With f_1 and f_2 being the cubic Hermite polynomials over $[0,1]$

$$f_1(u) = 1 - 3u^2 + 2u^3$$
$$f_2(u) = u - 2u^2 + u^3$$

The parameters σ_{12} are the cross-derivatives according to each direction and as such do not lend itself to a consistent interpretation. As a consequence, techniques have been developed to estimate the "optimal" twist vector with different optimality criterions.

1.1 Selesnick's method

In his paper of 1980 Selesnick proposes to estimate separately the normal and tangential component of the twist vector **from the rest of the parameters and the Gaussian curvature K.**

Let's denote $n = \sigma_1 \wedge \sigma_2$ the normal vector and $\hat{n} = \frac{n}{|n|}$ it's normalized version. As for the surface with denote we subscript 1 and 2 the derivative with respect to u and v in that order. Then the normal component of the twist vector can be obtained from

$$\hat{n}.\sigma_{12} = \pm\sqrt{\hat{n}.\sigma_{11}\hat{n}.\sigma_{22} - K\left(\sigma_1^2\sigma_2^2 - \sigma_1.\sigma_2\right)} \tag{2}$$

How to get the values for σ_{11}, σ_{22}? Supposing $\sigma, \sigma_1, \sigma_2$ are known at the corners, one can readily compute the values of σ_{11} and σ_{22}. This is a consequence of having interpolators that are *twice differentiable* and *satisfy Hermite interpolation properties* that is to say

$$f_1(\nu) = \delta_\nu \quad f_1^{(1)}(\nu) = 0 \quad f_2(\nu) = 0 \quad f_2^{(1)}(\nu) = \delta_\nu \tag{3}$$

as the terms corresponding to the twist vectors will appear with factor $f_2^{(2)}(u)f_2(v)$, which is equal to 0 when u and v are 0 or 1. Knowing all quantities on the right-hand side of the equation above, we can estimate the normal component of the twist vector. As for the sign, one can choose say positive sign for the first corner and deduce the signs for all other corners under the assumption that the variation of the interpolated surface between the data points should less than the variation implied by the points themselves.

To get a complete description of the twist vector it is enough to compute projection of the vector on two additional vectors that are each orthogonal to the normal and between themselves. One is naturally led towards computing tangential components of the twist vector that is $\sigma_1.\sigma_{12}$ and $\sigma_2.\sigma_{12}$. Introducing the parametrisation by arclength s for say u at fixed v, it is straightforward to show that

$$\sigma_1.\sigma_{12} = \left[\frac{ds}{du}\right]\frac{\partial}{\partial v}\left[\frac{ds}{du}\right] \tag{4}$$

with $\frac{ds}{du} = |\sigma_1|$.
In a similar fashion the other tangential component is obtained by

$$\sigma_2.\sigma_{12} = \left[\frac{ds}{dv}\right] \frac{\partial}{\partial u} \left[\frac{ds}{dv}\right] \tag{5}$$

with $\frac{ds}{du} = |\sigma_2|$.

1.2 Minimum quadratic oscillation

A piecewise Coon's surface is described on a square $[a, b] \times [c, d]$ by $M_1 \times M_2$ patches of the form $I_{k,l} = [u_k, u_{k+1}] \times [v_l, v_{l+1}]$.

In case of a *bilinear interpolating patch*, the surface on $I_{k,l}$ is given by

$$\sigma(u, v) = \begin{bmatrix} f_1(u) \\ f_1(1-u) \end{bmatrix}^T \begin{bmatrix} \sigma(0,0) & \sigma(0,1) \\ \sigma(1,0) & \sigma(1,1) \end{bmatrix} \begin{bmatrix} f_1(v) \\ f_1(1-v) \end{bmatrix} \tag{6}$$

wiht $f_1(u) = u$.

In case of *bicubic Coon's patch*, the surface on $I_{k,l}$ is given by

$$\sigma(u, v) = \begin{bmatrix} f_1(u) \\ f_1(1-u) \\ f_2(u) \\ -f_2(1-u) \end{bmatrix}^T \begin{bmatrix} \sigma(0,0) & \sigma(0,1) & \sigma_2(0,0) & \sigma_2(0,1) \\ \sigma(1,0) & \sigma(1,1) & \sigma_2(1,0) & \sigma_2(1,1) \\ \sigma_1(0,0) & \sigma_1(0,1) & \sigma_{12}(0,0) & \sigma_{12}(0,1) \\ \sigma_1(1,0) & \sigma_1(1,1) & \sigma_{12}(1,0) & \sigma_{12}(1,1) \end{bmatrix} \begin{bmatrix} f_1(v) \\ f_1(1-v) \\ f_2(v) \\ -f_2(1-v) \end{bmatrix} \tag{7}$$

with renoting $u \leftarrow \frac{u-u_k}{\Delta u_k}$, $v \leftarrow \frac{v-v_l}{\Delta v_l}$, $\sigma(u, v) \leftarrow \sigma(\frac{u-u_k}{\Delta u_k}, \frac{v-v_l}{\Delta v_l})$ and f_1 and f_2 being the cubic Hermite polynomials over $[0, 1]$

$$f_1(u) = 1 - 3u^2 + 2u^3$$
$$f_2(u) = u - 2u^2 + u^3$$

Remark Renoting $\sigma(u, v) \leftarrow \sigma(\frac{u-u_k}{\Delta u_k}, \frac{v-v_l}{\Delta v_l})$ means we note again σ the function on $[0, 1]^2$ that is equal to σ on $I_{k,l}$ mapped to $[0, 1]^2$ that is $\sigma(u, v) = \sigma(\Delta u_k u + u_k, \Delta v_l v + v_l)$. As a consequence, $\sigma_1(0, 0)$ in new notation corresponds to $\Delta u_k \sigma_1(u_k, v_l)$ in old notation. In coherence with previous usage we would write $\sigma_1(u, v) \leftarrow \Delta u_k \sigma_1(\frac{u-u_k}{\Delta u_k}, \frac{v-v_l}{\Delta v_l})$.

In their paper of 2017, X. Guo et X. Han derives a method to determine twist vectors that are optimal in the sense that they minimize the quadratic oscillation in average that is to say they minimize the distance between the bicubic and bilinear interpolant

$$\int_a^b \int_c^d \|Q(x, y) - L(x, y)\|^2 dx dy \tag{8}$$

2 Coon's formulation for polynomial and exponential cubic Hermite splines

2.1 Bicubic Coon's is cubic Hermite spline interpolation

Despite having a different name, bicubic Coon's patches are exactly the same as a cubic Hermite spline interpolation for tensor-product surface. Indeed, for the latter case the surface is given by

$$
\begin{aligned}
\sigma(u,v) = &\sum_{k=0}^{M_1}\sum_{l=0}^{M_2} c_1[k,l]\phi_1(M_1u-k)\phi_1(M_2v-l) \\
&+\sum_{k=0}^{M_1}\sum_{l=0}^{M_2} c_2[k,l]\phi_1(M_1u-k)\phi_2(M_2v-l) \\
&+\sum_{k=0}^{M_1}\sum_{l=0}^{M_2} c_3[k,l]\phi_2(M_1u-k)\phi_1(M_2v-l) \\
&+\sum_{k=0}^{M_1}\sum_{l=0}^{M_2} c_4[k,l]\phi_2(M_1u-k)\phi_2(M_2v-l)
\end{aligned}
$$

With

$$
\phi_1(x) = \begin{cases} f_1(x) & \text{for } 0 \le x \le 1 \\ f_1(-x) & \text{for } -1 \le x < 0 \end{cases} \qquad \phi_{2,w}(x) = \begin{cases} f_2(x) & \text{for } 0 \le x \le 1 \\ -f_2(-x) & \text{for } -1 \le x < 0 \end{cases}
$$

and f_1, f_2 are as previously the cubic Hermite polynomials.

Restricting our attention to a single patch $I_{k,l} = [\frac{k}{M_1}, \frac{k+1}{M_1}] \times [\frac{l}{M_2}, \frac{l+1}{M_2}]$, and *because* ϕ_1, ϕ_2 have support $[-1,1]$, the expression above boils down to

$$
\boxed{\sigma(u,v) = \begin{bmatrix} f_1(u) \\ f_1(1-u) \\ f_2(u) \\ -f_2(1-u) \end{bmatrix}^T \begin{bmatrix} \sigma(0,0) & \sigma(0,1) & \sigma_2(0,0) & \sigma_2(0,1) \\ \sigma(1,0) & \sigma(1,1) & \sigma_2(1,0) & \sigma_2(1,1) \\ \sigma_1(0,0) & \sigma_1(0,1) & \sigma_{12}(0,0) & \sigma_{12}(0,1) \\ \sigma_1(1,0) & \sigma_1(1,1) & \sigma_{12}(1,0) & \sigma_{12}(1,1) \end{bmatrix} \begin{bmatrix} f_1(v) \\ f_1(1-v) \\ f_2(v) \\ -f_2(1-v) \end{bmatrix}} \tag{9}
$$

with renoting $u \leftarrow M_1u-k, v \leftarrow M_2v-l$ and $\sigma(u,v) \leftarrow \sigma(M_1u-k, M_1v-l)$.

2.2 Exponential cubic Hermite splines

Conti et al's paper Ellipse-preserving interpolation and subdivision scheme introduces two basis functions from the space $\mathcal{E}_4 =< 1, x, e^{-iw_1x}, e^{iw_1x} >$ where $w = \frac{2\pi}{M}$ to reproduce closed curves with M control points. The basis functions are **cycloidal splines** (Exponential splines? Exponential B-splines?) given by

$$
\phi_{1,w}(x) = \begin{cases} g_{1,w}(x) & \text{for } x \ge 0 \\ g_{1,w}(-x) & \text{for } x < 0 \end{cases} \qquad \phi_{2,w}(x) = \begin{cases} g_{2,w}(x) & \text{for } x \ge 0 \\ -g_{2,w}(-x) & \text{for } x < 0 \end{cases} \tag{10}
$$

The surface with $M_1 \times (M_2+1)$ (M_1 because of periodicity over u) control points is given by
For all $(u,v) \in [0,1]^2$

$$
\begin{aligned}
\sigma(u,v) = &\sum_{k=0}^{M_1}\sum_{l=0}^{M_2} c_1[k,l]\phi_{1,w_1}(M_1u-k)\phi_{1,w_2}(M_2v-l) \\
&+\sum_{k=0}^{M_1}\sum_{l=0}^{M_2} c_2[k,l]\phi_{1,w_1}(M_1u-k)\phi_{2,w_2}(M_2v-l) \\
&+\sum_{k=0}^{M_1}\sum_{l=0}^{M_2} c_3[k,l]\phi_{2,w_1}(M_1u-k)\phi_{1,w_2}(M_2v-l) \\
&+\sum_{k=0}^{M_1}\sum_{l=0}^{M_2} c_4[k,l]\phi_{2,w_1}(M_1u-k)\phi_{2,w_2}(M_2v-l)
\end{aligned}
$$

$$c_1[k,l] = \begin{bmatrix} \cos(w_1 k)\sin(w_2 l) \\ \sin(w_1 k)\sin(w_2 l) \\ \cos(w_2 l) \end{bmatrix} = \sigma(\frac{k}{M_1}, \frac{l}{M_2}) \qquad c_2[k,l] = \begin{bmatrix} w_2\cos(w_1 k)\cos(w_2 l) \\ w_2\sin(w_1 k)\cos(w_2 l) \\ -w_2\sin(w_2 l) \end{bmatrix} = \frac{1}{M_2}\frac{\partial\sigma}{\partial v}(\frac{k}{M_1}, \frac{l}{M_2})$$

$$c_3[k,l] = \begin{bmatrix} -w_1\sin(w_1 k)\sin(w_2 l) \\ w_1\cos(w_1 k)\sin(w_2 l) \\ 0 \end{bmatrix} = \frac{1}{M_1}\frac{\partial\sigma}{\partial u}(\frac{k}{M_1}, \frac{l}{M_2}) \quad c_4[k,l] = \begin{bmatrix} -w_1 w_2\sin(w_1 k)\cos(w_2 l) \\ w_1 w_2\cos(w_1 k)\cos(w_2 l) \\ 0 \end{bmatrix} = \frac{1}{M_1 M_2}\frac{\partial^2\sigma}{\partial u\partial v}(\frac{k}{M_1}, \frac{l}{M_2})$$

The parameters $c_4[k,l]$ are the twist vectors at each location of the control points.

Again restricting our attention to a single patch $I_{k,l} = [\frac{k}{M_1}, \frac{k+1}{M_1}] \times [\frac{l}{M_2}, \frac{l+1}{M_2}]$, and *because* $\phi_{1,w_i}, \phi_{2,w_i}$ have support $[-1,1]$, the expression above boils down to

$$\boxed{\sigma(u,v) = \begin{bmatrix} g_{1,w_1}(u) \\ g_{1,w_1}(1-u) \\ g_{2,w_1}(u) \\ -g_{2,w_1}(1-u) \end{bmatrix}^T \begin{bmatrix} \sigma(0,0) & \sigma(0,1) & \sigma_2(0,0) & \sigma_2(0,1) \\ \sigma(1,0) & \sigma(1,1) & \sigma_2(1,0) & \sigma_2(1,1) \\ \sigma_1(0,0) & \sigma_1(0,1) & \sigma_{12}(0,0) & \sigma_{12}(0,1) \\ \sigma_1(1,0) & \sigma_1(1,1) & \sigma_{12}(1,0) & \sigma_{12}(1,1) \end{bmatrix} \begin{bmatrix} g_{1,w_2}(v) \\ g_{1,w_2}(1-v) \\ g_{2,w_2}(v) \\ -g_{2,w_2}(1-v) \end{bmatrix}} \tag{11}$$

with renoting $u \leftarrow M_1 u - k, v \leftarrow M_2 v - l$ and $\sigma(u,v) \leftarrow \sigma(M_1 u - k, M_1 v - l)$.

The ressemblance with Hermite cubic interpolation (or bicubic Coon's patch equivalently) is striking. The only difference is the use of different basis functions on each continuous direction, the difference disappearing in case $M_1 = 2M_2$. Observe also that g is not polynomial but an exponential polynomial. However the values taken by g and its derivative over $[0,1]$ are very close to that taken by f and its derivative over $[0,1]$.

3 Twist estimation for the sphere

In this work we try to get a mathematical representation of a surface from information at a few points. This information usually consists in the surface value and potentially local tangents and/or local curvature. The mathematical object representing the surface is a linear combination of shifts of a given basis function. The coefficients of that linear combination are the parameters we are solving for when fitting the surface to the input data. Some of these coefficients directly relate to interpretable local quantities while others do not have a consistent interpretation and therefore need to be estimated somehow. In what follows we are going to present different techniques for estimating the unknown coefficients and assess the quality of the estimations on a couple of surfaces. This will help us understand the issues that arise. As all coefficients relate to the value and derivatives of the surface, it is equivalent to estimate these derivatives and compare them to their theoretical counterpart. As specificities may arise at poles (for example the cross-product of partial derivatives is 0 at the poles of a sphere) we are going to consider quantities at a pole point and at a non-pole point.

In the rest of this section numerical values are that for a sphere of radius $R = 1$ at locations $u = 0, v = 0$ (pole) and $u = 0, v = 0.2$ (non-pole). These choices are arbitrary, the idea is simply to illustrate the differences between the different interpolation schemes.

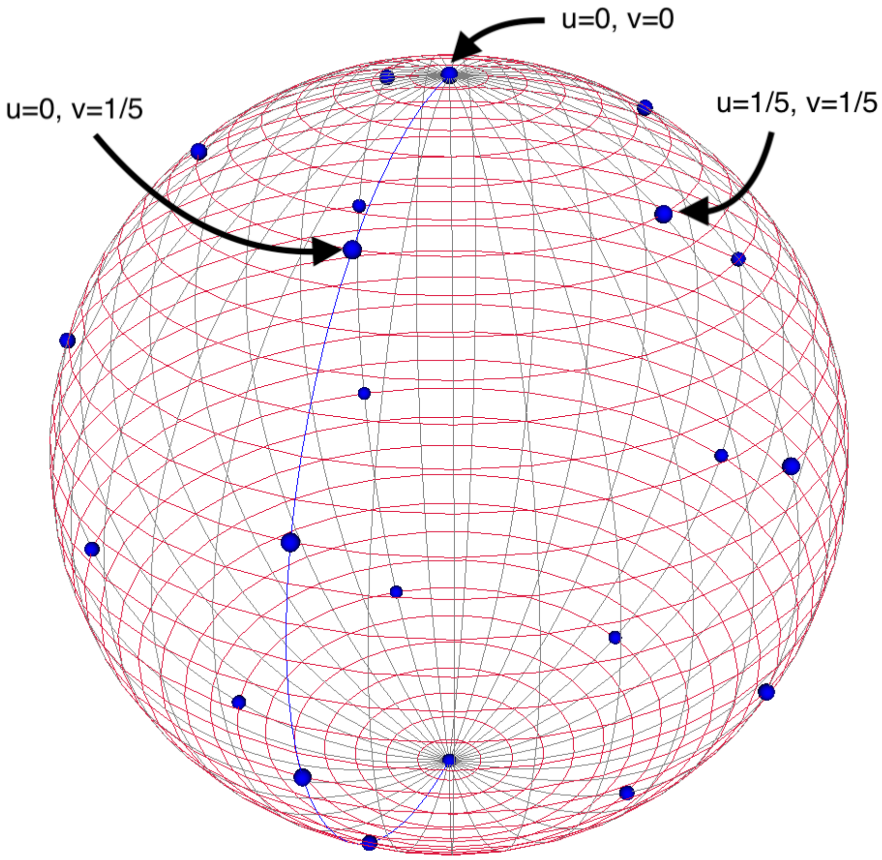

Figure 1: Sphere using Hermite exponentials

The usual parametric representation of the unit sphere is

$$\sigma(u, v) = \begin{pmatrix} \cos(2\pi u)\sin(\pi v) \\ \sin(2\pi u)\sin(\pi v) \\ \cos(\pi v) \end{pmatrix} \tag{12}$$

At $u = 0, v = 0$,

$$\sigma = (0,0,1) \qquad K = 1 \qquad n = (0,0,0) \qquad \hat{n} = (0,0,1)$$
$$\sigma_1 = (0,0,0) \qquad \sigma_2 = (0.628,0,0) \qquad \sigma_{11} = (0,0,0) \qquad \sigma_{22} = (0,0,-0.395)$$
$$\sigma_{12}^{can} = (0,0.790,0) \qquad \sigma_{12}^{nor} = (0,790,0)$$

At $u = 0, v = 0.2$,

$$\sigma = (0.588,0,0.809) \qquad K = 1 \qquad n = (-0.273,0,-0.375) \qquad \hat{n} = (-0.588,0,-0.809)$$
$$\sigma_1 = (0,0.739,0) \qquad \sigma_2 = (0.508,0,-0.369) \qquad \sigma_{11} = (-0.928,0,0) \qquad \sigma_{22} = (-0.232,0,-0.319)$$
$$\sigma_{12}^{can} = (0,0.639,0) \qquad \sigma_{12}^{nor} = (0,0.639,0)$$

Superscript “can” denotes the canonical base (e_1, e_2, e_3) whereas “nor” denote the base made of $(\hat{n}, \hat{\sigma_1}, \hat{\sigma_2})$. Note that the latter is indeed a base if none of its components is 0. You may argue that at the pole $\sigma_1(0,0)$ is null vector but we will take as definition for the normalized versions of the normal and tangents the following

$$\hat{n}(u_0, v_0) = \lim_{u\to u_0, v\to v_0} \frac{n}{|n|}(u,v) \qquad \hat{\sigma}_i(u_0, v_0) = \lim_{u\to u_0, v\to v_0} \frac{\sigma_i}{|\sigma_i|}(u,v)$$

Using such definitions one can easily check that for the sphere $\hat{n}(0,0) = (0,0,1)$ and $\hat{\sigma}_1(0,0) = (0,1,0)$.

3.1 Hermite polynomial order 1

The scheme is the extension to surfaces of what V. Uhlmann presented in her article “Hermite snakes with control of tangents”. Given that a sphere has closed latitudes and open longitudes, the representation for sphere-like surfaces with M_1 control points in latitudes and $M_2 + 1$ in longitudes is

$$\begin{aligned}\sigma(u,v) = \sum_{k=0}^{M_1-1}\sum_{l=0}^{M_2} &\sigma\left(\frac{k}{M_1}, \frac{l}{M_2}\right)\phi_{1per}(M_1u-k)\phi_1(M_2v-l)\\ &+\frac{1}{M_2}\sigma_2\left(\frac{k}{M_1}, \frac{l}{M_2}\right)\phi_{1,per}(M_1u-k)\phi_2(M_2v-l)\\ &+\frac{1}{M_1}\sigma_1\left(\frac{k}{M_1}, \frac{l}{M_2}\right)\phi_{2,per}(M_1u-k)\phi_1(M_2v-l)\\ &+\frac{1}{M_1M_2}\sigma_{12}\left(\frac{k}{M_1}, \frac{l}{M_2}\right)\phi_{2,per}(M_1u-k)\phi_2(M_2v-l)\end{aligned}$$

In the figure on the right a fixed view of the 3D sphere generated by such a scheme for $M_1 = 5, M_2 = 5$. The scheme is theoretically not able to reproduce exactly a sphere as cosinus and sinus functions cannot be represented by finite polynomials. However as you can see the approximation is very good. In the case of $M_1 = 5, M_2 = 5$ the average norm of the error at each control point site (30 control points with multiplicity) is 2.24×10^{-3}.

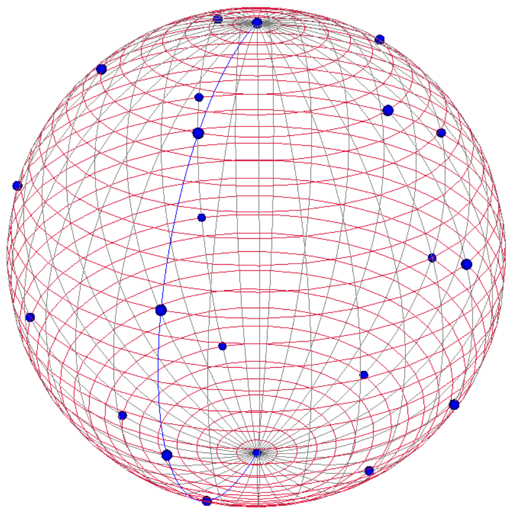

Figure 2: Sphere using Hermite polynomials

3.1.1 Twist estimation from $\sigma, \sigma_1, \sigma_2$

Consider the setting where samples of the surface and the first derivatives are known on a regular grid. In order to represent a surface that interpolates this data with a (first-order) Hermite polynomial scheme it remains to specify the value of the coefficient (a 3D vector) that goes in front of the term $\phi_{2,per}(M_1. - k)\phi_2(M_2. - l)$. As mentionned above this term is the scaled first order cross-derivative (or twist vector). Therefore if we can find a way to estimate this twist vector from the rest of the data our scheme would be complete.

Naive technique

A *naive* way of doing this is to interpolate for example σ_1 at the different control points on a given longitude and compute the derivative of the interpolant at the control points locations.

Interpolate σ_1 on each longitude

Interpolating the $M_2 + 1 = 6$ available values of each coordinate of σ_1 for each longitude allows to estimate σ_{12} at all control points. The average norm error is 9.47×10^{-3}. Example values

At $u = 0, v = 0$,

$$\text{estimated } \sigma_{12}^{can} = (0, 0.810, 0) \qquad \text{truth } \sigma_{12}^{can} = (0, 0.790, 0)$$

At $u = 0, v = \frac{1}{5}$,

$$\text{estimated } \sigma_{12}^{can} = (0, 0.632, 0) \qquad \text{truth } \sigma_{12}^{can} = (0, 0.639, 0)$$

The estimation is very close to the real value. This is because we have a couple of points but most of all because the coordinates of the twist vector are very smooth function in this case! Below is a plot showing the interpolation result and it's derivative for the y coordinate of the $\sigma_1(u, .)$ (u=0).

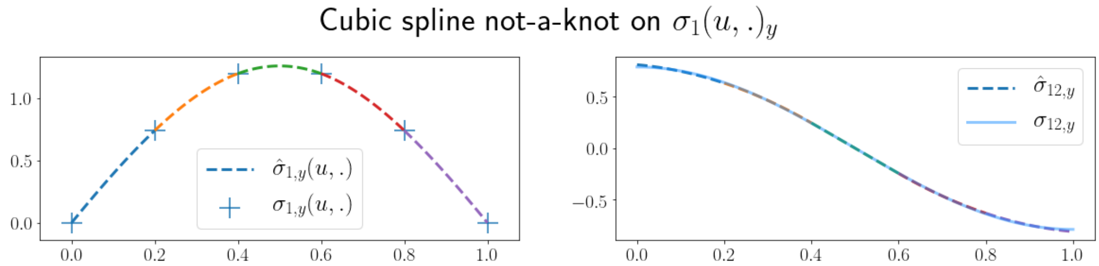

Figure 3: Cubic spline not-a-knot interpolation of $\left(\sigma_{1,y}(0, \frac{l}{M_2})\right)_{l=0}^{M_2}$

As detailed by De Boor in "A practical guide to splines" the inteprolation power of this scheme is $\mathcal{O}(|\tau|^4)$ (chapter IV p.45) for the function but he doesn't say for the derivative. If of any interest we can work out approximation power to the derivative. Estimation works fine for the sphere but it could well be that we have a surface where first derivative σ_1 is 0 at all control points while the cross-derivative is not. However in that case the twist estimated by this technique will obviously be 0. That illustrates one of the many cases where it will fail.

Interpolate σ_2 on each latitude

Interpolating the $M_1 = 5$ available values of each coordinate of σ_2 for each latitude allows to estimate $\sigma_{21} = \sigma_{12}$ at all control points. The average norm error is 0.113. Example values

At $u = 0, v = \frac{1}{5}$,

$$\text{estimated } \sigma_{12}^{can} = (0.085, 0.855, 0) \qquad \text{truth } \sigma_{12}^{can} = (0, 0.639, 0)$$

At $u = \frac{1}{5}, v = \frac{1}{5}$,

$$\text{estimated } \sigma_{12}^{can} = (-0.623, 0.131, 0) \qquad \text{truth } \sigma_{12}^{can} = (-0.608, 0.197, 0)$$

The error is 10 times higher in that case! First we are using one point less but it also may be that values of σ_2 on a latitude have higher amplitude than σ_1 on longitude. Below is a representation of the interpolation at latitude $v = 0.2$.

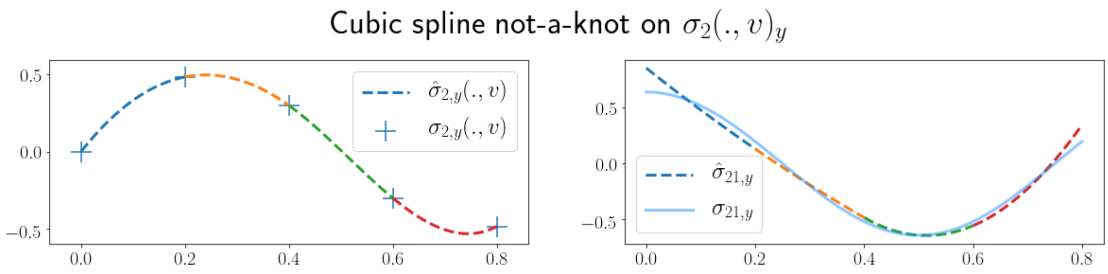

Figure 4: Cubic spline not-a-knot interpolation of $\left(\sigma_{2,y}(\frac{k}{M_1}, 0.2)\right)_{l=0}^{M_1-1}$

Minimum quadratic oscillation

In order to treat the general case of non uniform spacing, let's adopt the following notations:

1. The index i denotes the u direction and ranges from 0 to $M_1 - 1$. Similarly index j denotes the v direction and ranges from 0 to M_2.
2. Subscript $i = -1$ or $i = M_1$ are to be considered with M_1 periodicity. For instance $r_{-1,j}$ stands for $r_{M_1-1,j}$ while $r_{M_1,j}$ stands for $r_{0,j}$.
3. For $i = 0, \ldots, M_1 - 1$, h_i denotes the spacing $u_{i+1} - u_i$ in the u direction with the previous convention on indices outside $[0, M_1 - 1]$.
4. For $j = 0, \ldots, M_2 - 1$, l_j denotes the spacing $v_{j+1} - v_i$ in the v direction with no convention.
5. For $i = 0, \ldots, M_1 - 1, j = 0, \ldots, M_2 - 1$, $I_{i,j}$ denotes the square $[\frac{i}{M_1}, \frac{i+1}{M_1}] \times [\frac{j}{M_2}, \frac{j+1}{M_2}]$.
6. $f_0^h(u) = 1 - 3u^2 + 2u^3, f_1^h(u) = 3u^2 - 2u^3$ and $g_0^h(u) = u(u-1)^2, g_1^h(u) = u^2(u-1)$ denote the Hermite polynomials at order 3.
7. $f_0(u), f_0(v), f_1(u), f_1(v), g_0(u), g_0(v), g_1(u), g_1(v)$ are as defined in the article, where the dependence on h_i and l_j has been dropped for lighter notations but must no be forgotten.
8. $$G_0 = -\int_0^1 \int_0^1 g_0^h(u) g_0^h(v) H(u,v) dudv \qquad G_1 = -\int_0^1 \int_0^1 g_1^h(u) g_0^h(v) H(u,v) dudv$$
$$G_2 = -\int_0^1 \int_0^1 g_0^h(u) g_1^h(v) H(u,v) dudv \qquad G_3 = -\int_0^1 \int_0^1 g_1^h(u) g_1^h(v) H(u,v) dudv$$

On $I_{i,j}$ the cubic polynomial Hermite surface can be rewritten as

$$Q_{i,j}(u,v) = \begin{bmatrix} f_0(u) \\ f_1(u) \\ g_0(u) \\ g_1(u) \end{bmatrix}^T \begin{bmatrix} p_{i,j} & p_{i,j+1} & g_{i,j} & g_{i,j+1} \\ p_{i+1,j} & p_{i+1,j+1} & g_{i+1,j} & g_{i+1,j+1} \\ f_{i,j} & f_{i,j+1} & r_{i,j} & r_{i,j+1} \\ f_{i+1,j} & f_{i+1,j+1} & r_{i+1,j} & r_{i+1,j+1} \end{bmatrix} \begin{bmatrix} f_0(v) \\ f_1(v) \\ g_0(v) \\ g_1(v) \end{bmatrix} \tag{13}$$

where

$$p_{i,j} = c_1[i,j] \qquad l_j g_{i,j} = c_2[i,j]$$
$$h_i f_{i,j} = c_3[i,j] \qquad h_i l_j r_{i,j} = c_4[i,j]$$

The linear Hermite polynomial surface can be rewritten in a similar way

$$L_{i,j}(u,v)=\begin{bmatrix}f_0(u)\\ f_1(u)\\ g_0(u)\\ g_1(u)\end{bmatrix}^T\begin{bmatrix}p_{i,j} & p_{i,j+1} & \bar{g}_{i,j} & \bar{g}_{i,j+1}\\ p_{i+1,j} & p_{i+1,j+1} & \bar{g}_{i+1,j} & \bar{g}_{i+1,j+1}\\ \bar{f}_{i,j} & \bar{f}_{i,j+1} & \bar{r}_{i,j} & \bar{r}_{i,j+1}\\ \bar{f}_{i+1,j} & \bar{f}_{i+1,j+1} & \bar{r}_{i+1,j} & \bar{r}_{i+1,j+1}\end{bmatrix}\begin{bmatrix}f_0(v)\\ f_1(v)\\ g_0(v)\\ g_1(v)\end{bmatrix} \tag{14}$$

The minimization problem is

$$\min_{r_{i,j}}\sum_{i=1}^{M_1}\sum_{j=1}^{M_2}E_{i,j}$$

with

$$E_{i,j}=h_i l_j\int_0^1\int_0^1\|Q_{i,j}(u,v)-L_{i,j}(u,v)\|^2dudv$$

The term inside the norm can be rewritten as

$$Q_{i,j}(u,v)-L_{i,j}(u,v)=\mathrm{Tr}(H(u,v)B_{i,j})+\sum_{s,t\in\{0,1\}}r_{i+s,j+t}g_s(u),g_t(v)$$

Remark 1. *$B_{i,j}$ defined as in the article except that $\tilde{r}_{i,j}=-\bar{r}_{i,j}$ (it is an error in the article).*

The partial derivative of $E_{i,j}$ w.r.t to $r_{i+k,j+l}$ for $k,l\in\{0,1\}$ is such that

$$\frac{1}{2}\frac{\partial E_{i,j}}{\partial r_{i+k,j+l}}=h_i^2l_j^2\,\mathrm{Tr}\left(\int_0^1\int_0^1 g_k^h(u)g_l^h(v)H(u,v)dudvB_{i,j}\right)+h_i^3l_j^3\int_0^1\int_0^1\sum_{s,t\in\{0,1\}}r_{i+s,j+l}g_s^h(u)g_t^h(v)g_k^h(u)g_l^h(v)dudv$$

Remark 2. *Again this is slightly different from what is written in the article. The coefficient in front of the integral changes as k,l change but it should not. It has no impact in case of uniform spacing but it would otherwise.*

Now the system defining the optimal solution differs slightly because of the closure along latitudes. Indeed, the twists $r_{i,0},r_{i,M_2}$ at north and south poles are shared between two adjacent patches for all i while all other twists $r_{i,j}$ are shared between four adjacent patches. Note also that $r_{0,j}=r_{M_1,j}$ for all j that is the number of free vectors to optimize is $M_1\times(M_2+1)$. This observation leads to the following variation of the system of equations given in the article

$$\frac{\partial E_{i-1,0}}{\partial r_{i,0}}+\frac{\partial E_{i,0}}{\partial r_{i,0}}=0 \qquad \text{for } i=0,\dots,M_1-1 \tag{15}$$

$$\frac{\partial E_{i-1,M_2-1}}{\partial r_{i,M_2}}+\frac{\partial E_{i,M_2-1}}{\partial r_{i,M_2}}=0 \qquad \text{for } i=0,\dots,M_1-1 \tag{16}$$

$$\frac{\partial E_{i-1,j-1}}{\partial r_{i,j}}+\frac{\partial E_{i,j-1}}{\partial r_{i,j}}+\frac{\partial E_{i-1,j}}{\partial r_{i,j}}+\frac{\partial E_{i,j}}{\partial r_{i,j}}=0 \qquad \text{for } i=0,\dots M_1-1, j=1,\dots,M_2-1 \tag{17}$$

with the M_1 periodic notation $E_{-1,j}=E_{M_1-1,j}$ for all j. This makes $M_1\times(M_2+1)$ linear equations for the $M_1\times(M_2+1)$ variables $R=[r_{0,0},\dots,r_{M_1-1,0},\dots,r_{0,M_2},\dots,r_{M_1-1,M_2}]^T$.

Note that in the expression of partial derivative of $E_{i,j}$, elements of the unknown vector R comes in factor of at most 9 different values (down from 16 by symmetry) that are $\int_0^1\int_0^1 g_s^h(u)g_k^h(u)g_t^h(v)g_l^h(v)dudv$ for $s,t,k,l\in\{0,1\}$. Actually, it so happens that $\int_0^1 g_0^h(u)g_0^h(u)=\int_0^1 g_0^h(u)g_0^h(u)=\frac{1}{105}$ and $\int_0^1 g_0^h(u)g_1^h(u)=\frac{-1}{140}$ so that only 3 different factors can appear in front of the unknown quantities $r_{i,j}$. Noting that 176400 can be divided by $105^2, 105\times 140$ and 140^2, it will be convenient to factorize by this number.

Equation (15) can be rewritten for $i=0,\dots,M_1-1$

$$\begin{aligned}&l_0^3\left(-12h_{i-1}^3r_{i-1,0}+16(h_{i-1}^3+h_i^3)r_{i,0}-12h_i^3r_{i+1,0}\right)+l_0^3\left(9h_{i-1}^3r_{i-1,1}-12(h_{i-1}^3+h_i^3)r_{i,1}+9h_i^3r_{i+1,1}\right)\\&=h_{i-1}^2l_0^2\,\mathrm{Tr}\left(G_1B_{i-1,0}\right)+h_i^2l_0^2\,\mathrm{Tr}\left(G_0B_{i,0}\right)\end{aligned} \tag{18}$$

while equation (16) can be rewritten for $i = 0, \dots, M_1 - 1$

$$
\begin{aligned}
&l_{M_2-1}^3 \left(-12h_{i-1}^3 r_{i-1,M_2-1} + 16(h_{i-1}^3 + h_i^3) r_{i,M_2-1} - 12h_i^3 r_{i+1,M_2-1}\right) \\
&\quad + l_{M_2-1}^3 \left(9h_{i-1}^3 r_{i-1,M_2} - 12(h_{i-1}^3 + h_i^3) r_{i,M_2} + 9h_i^3 r_{i+1,M_2}\right) = h_{i-1}^2 l_{M_2-1}^2 \operatorname{Tr}\left(G_1 B_{i-1,M_2-1}\right) \\
&\qquad + h_i^2 l_{M_2-1}^2 \operatorname{Tr}\left(G_0 B_{i,M_2-1}\right)
\end{aligned}
\tag{19}
$$

This calls for a slight modification of the M matrix given in the article. Indeed, in the article the twist vector at corners of the surface contribute to 1 unique patch whereas they contribute to 2 patches for the case of closed latitudes. The modification is simply a “periodization” of the first and last rows of the matrix $L = W * D$

$$
K = \begin{bmatrix}
4l_0^3 & -3l_0^3 & & & & \\
-3l_0^3 & 4(l_0^3 + l_1^3) & -3l_1^3 & & & \\
& \ddots & \ddots & \ddots & & \\
& & & -3l_{M_2-2}^3 & 4(l_{M_2-2}^3 + l_{M_2-1}^3) & -3l_{M_2-1}^3 \\
& & & & -3l_{M_2-1}^3 & 4l_{M_2-1}^3
\end{bmatrix}
\tag{20}
$$

$$
L = \begin{bmatrix}
4(h_{M_1-1}^3 + h_0^3) & -3h_0^3 & & & & -3h_{M_1-1}^3 \\
-3h_0^3 & 4(h_0^3 + h_1^3) & -3h_1^3 & & & \\
& \ddots & \ddots & \ddots & & \\
& & & -3h_{M_1-3}^3 & 4(h_{M_1-3}^3 + h_{M_1-2}^3) & -3h_{M_1-2}^3 \\
-3h_{M_1-1}^3 & & & & -3h_{M_1-2}^3 & 4(h_{M_1-2}^3 + h_{M_1-1}^3)
\end{bmatrix}
\tag{21}
$$

The sequence of optimal twist in the sense of minimal quadratic oscillation is solution to

$$
(K \bigotimes L) R = 176400 C \tag{22}
$$

with $C = [c_{0,0}, \dots, c_{M_1-1,0}, \dots, c_{0,M_2}, \dots, c_{M_1-1,M_2}]^T$ and for $i = 0, \dots, M_1 - 1$

$$
\begin{aligned}
c_{i,0} &= h_{i-1}^2 l_0^2 \operatorname{Tr}\left(G_1 B_{i-1,0}\right) + h_i^2 l_0^2 \operatorname{Tr}\left(G_0 B_{i,0}\right) \\
c_{i,M_2} &= h_{i-1}^2 l_{M_2-1}^2 \operatorname{Tr}\left(G_3 B_{i-1,M_2-1}\right) + h_i^2 l_{M_2-1}^2 \operatorname{Tr}\left(G_2 B_{i,M_2-1}\right)
\end{aligned}
$$

and for $i = 0, \dots, M_1 - 1$, $j = 1, \dots, M_2 - 1$

$$
c_{i,j} = h_{i-1}^2 l_{j-1}^2 \operatorname{Tr}\left(G_3 B_{i-1,j-1}\right) + h_i^2 l_{j-1}^2 \operatorname{Tr}\left(G_2 B_{i,j-1}\right) + h_{i-1}^2 l_j^2 \operatorname{Tr}\left(G_1 B_{i-1,j}\right) + h_i^2 l_j^2 \operatorname{Tr}\left(G_0 B_{i,j}\right)
$$

As is true for our case, uniform spacing greatly reduces the complexity of the formula with $h_i = \frac{1}{M_1}, l_j = \frac{1}{M_2}$ for all i and j. In order to compute the matrices G_0 to G_3, consider the quantities

$$
\begin{aligned}
&\int f_0^h f_0^h = \frac{13}{35} && \int f_0^h f_1^h = \frac{9}{70} && \int f_1^h f_1^h = \frac{13}{35} \\
&\int g_0^h g_0^h = \frac{1}{105} && \int g_0^h g_1^h = \frac{-1}{140} && \int g_1^h g_1^h = \frac{1}{105} \\
&\int f_1^h g_0^h = -\int f_0^h g_1^h = \frac{13}{420} && \int f_0^h g_0^h = -\int f_1^h g_1^h = \frac{11}{120}
\end{aligned}
$$

3.1.2 Twist estimation from $\sigma, \sigma_1, \sigma_2, K$

Suppose now that as additional data we have the values of the local curvature (Gaussian curvature) at the control points locations. Is there a way to extract the twist vector from that additional information? At a point p on a regular surface in $\mathbb{R}^3$, the Gaussian curvature is the determinant of the shape operator that is $K(p) = \det\, S(p)$. It can also be expressed as the ratio of the second fundamental form to the first fundamental form. **What are the minimum requirements on the surface to define these two forms**? In case the surface is the trace of the differentiable (definition of differentiable?) map σ and at the point p we have $\sigma_1^2\sigma_2^2 - (\sigma_1.\sigma_2)^2 \neq 0$ (i.e > 0) the Gaussian curvature is expressed by

$$K = \frac{\hat{n}.\sigma_{11}\hat{n}.\sigma_{22} - \hat{n}.\sigma_{12}{}^2}{\sigma_1^2\sigma_2^2 - (\sigma_1.\sigma_2)^2} \tag{23}$$

At the point $u = 0$, $v = 0$ of the sphere with parametrisation (12), σ_1 is 0 (also at the south pole) and thus the above expression cannot be used at this point. At points where determinant of first fundamental form is not 0, this formula can be reversed to extract normal component of the twist vector as explained in section 1.1 (Selesnick's method).

In order to inverse formula (23) to extract $\hat{n}.\sigma_{12}$ one needs to compute somehow second-order derivatives σ_{11}, σ_{22} at control points. Consider the surface on a single patch of the form $I_{k,l} = [u_k, u_{k+1}] \times [v_l, v_{l+1}]$ the surface is given by

$$\sigma(u,v) = \begin{bmatrix} f_1(u) \\ f_1(1-u) \\ f_2(u) \\ -f_2(1-u) \end{bmatrix}^T \begin{bmatrix} \sigma(0,0) & \sigma(0,1) & \sigma_2(0,0) & \sigma_2(0,1) \\ \sigma(1,0) & \sigma(1,1) & \sigma_2(1,0) & \sigma_2(1,1) \\ \sigma_1(0,0) & \sigma_1(0,1) & \sigma_{12}(0,0) & \sigma_{12}(0,1) \\ \sigma_1(1,0) & \sigma_1(1,1) & \sigma_{12}(1,0) & \sigma_{12}(1,1) \end{bmatrix} \begin{bmatrix} f_1(v) \\ f_1(1-v) \\ f_2(v) \\ -f_2(1-v) \end{bmatrix} \tag{24}$$

with renoting $u \leftarrow \frac{u-u_k}{\Delta u_k}$, $v \leftarrow \frac{v-v_l}{\Delta v_l}$, $\sigma(u,v) \leftarrow \sigma(\frac{u-u_k}{\Delta u_k}, \frac{v-v_l}{\Delta v_l})$ and f_1 and f_2 being the cubic Hermite polynomials over $[0,1]$

$$\begin{aligned} f_1(u) &= 1 - 3u^2 + 2u^3 \\ f_2(u) &= u - 2u^2 + u^3 \end{aligned}$$

Let's focus on the control point at $u = 0, v = 0.2$. Four adjacent patches share this control point as a corner namely $I_{0,0} = [0, 0.2] \times [0, 0.2]$ (corner (0,1)), $I_{0,1} = [0, 0.2] \times [0.2, 0.4]$ (corner (0,0)), $I_{4,0} = [0.8, 0] \times [0, 0.2]$ (corner (1,1)), $I_{4,1} = [0.8, 0] \times [0.2, 0.4]$ (corner (1, 0)). For each patch, the second derivative at corner (u, v) $(u = 0, 1, v = 0, 1)$ is calculated by

$$\sigma_{11}(u,v) = \begin{bmatrix} f_1^{(2)}(u) \\ f_1^{(2)}(1-u) \\ f_2^{(2)}(u) \\ -f_2^{(2)}(1-u) \end{bmatrix}^T \begin{bmatrix} \sigma(0,0)f_1(v) + \sigma(0,1)f_1(1-v) \\ \sigma(1,0)f_1(v) + \sigma(1,1)f_1(1-v) \\ \sigma_1(0,0)f_1(v) + \sigma_1(0,1)f_1(1-v) \\ \sigma_1(1,0)f_1(v) + \sigma_1(1,1)f_1(1-v) \end{bmatrix} \tag{25}$$

where we used the fact that f_2 vanishes at 0 and 1. A similar expression holds for calculating σ_{22}. The problem with this calculation is when second derivative f_1, f_2 do not have the same value at ends 0 and 1 as is the case here. f_1 and f_2 coincide with ϕ_1 and ϕ_2 on $[0,1]$ respectively.

Let's look at σ_{11} calculation at $u = 0, v = 0.2$ at all four adjacent patches. σ in the formulas below is the parametric representation of the whole sphere on $[0,1]^2$.

For $I_{0,0}$,

$$\sigma_{11}(u,v) = \begin{bmatrix} -6 \\ 6 \\ -4 \\ -2 \end{bmatrix}^T \begin{bmatrix} \sigma(0,0.2) \\ \sigma(0.2,0.2) \\ \frac{1}{M_1}\sigma_1(0,0.2) \\ \frac{1}{M_1}\sigma_1(0.2,0.2) \end{bmatrix} = (-1.032, -0.057, 0)$$

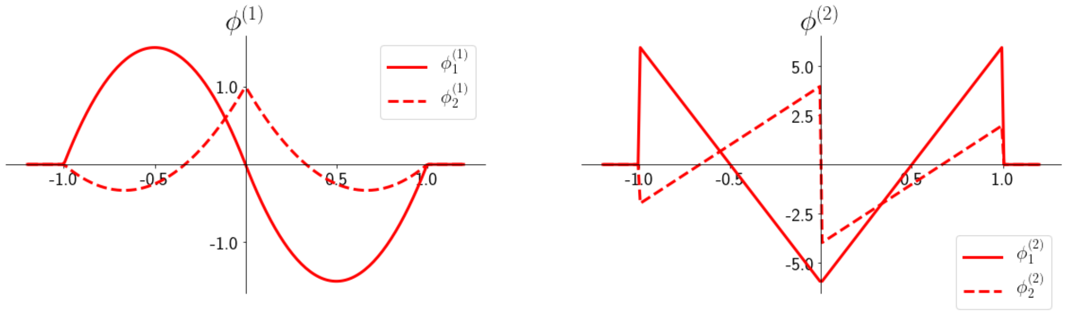

Figure 5: First and second derivatives of ϕ_1 and ϕ_2 on their domain

For $I_{0,1}$,

$$\sigma_{11}(u,v) = \begin{bmatrix} -6 \\ 6 \\ -4 \\ -2 \end{bmatrix}^T \begin{bmatrix} \sigma(0,0.2) \\ \sigma(0.2,0.2) \\ \frac{1}{M_1}\sigma_1(0,0.2) \\ \frac{1}{M_1}\sigma_1(0.2,0.2) \end{bmatrix} = (-1.032, -0.057, 0)$$

For $I_{4,0}$,

$$\sigma_{11}(u,v) = \begin{bmatrix} 6 \\ -6 \\ 2 \\ 4 \end{bmatrix}^T \begin{bmatrix} \sigma(0.8,0.2) \\ \sigma(0,0.2) \\ \frac{1}{M_1}\sigma_1(0.8,0.2) \\ \frac{1}{M_1}\sigma_1(0,0.2) \end{bmatrix} = (-1.032, +0.057, 0)$$

For $I_{4,1}$,

$$\sigma_{11}(u,v) = \begin{bmatrix} 6 \\ -6 \\ 2 \\ 4 \end{bmatrix}^T \begin{bmatrix} \sigma(0.8,0.2) \\ \sigma(0,0.2) \\ \frac{1}{M_1}\sigma_1(0.8,0.2) \\ \frac{1}{M_1}\sigma_1(0,0.2) \end{bmatrix} = (-1.032, +0.057, 0)$$

One notices the inconsistent estimation of the y coordinate of the σ_{11} at $u = 0, v = 0.2$. As a reminder the real value of this vector is $(-0.928, 0, 0)$. As for the x coordinate the estimated value is inexact. **Why is it inexact?**.

Once σ_{11}, σ_{22} are known at control points, it remain to determine the sign in order to compute the normal component of the twist. It is chosen to be positive at $u = 0, v = 0$ and then alternating from one corner to the next. In order to estimate the tangential components of the twist, one needs to estimate derivative with 1 of $|\sigma_1|$ and derivative with 2 of $|\sigma_2|$ at control points. These are obtained from the slopes at interpolation sites of cubic spline interpolation with not-a-knot condition of $|\sigma_1[0,v]|, \ldots, |\sigma_1[M_1-1,v]|$ and $|\sigma_2[u,0]|, \ldots, |\sigma_2[u,M_2]|$ respectively. For the sake of completeness here are the four estimations of σ_{22} and σ_{12} at $u = 0, v = 0.2$.

For $I_{0,0}$,

$$\sigma_{22}(u,v) = (-0.237, 0, -0.331) \qquad \sigma_{12}(u,v) = (-0.105, 0.632, -0.144)$$

For $I_{0,1}$,

$$\sigma_{22}(u,v) = (-0.242, 0, -0.328) \qquad \sigma_{12}(u,v) = (0.105, 0.632, 0.144)$$

For $I_{4,0}$,

$$\sigma_{22}(u,v) = (-0.242, 0, -0.328) \qquad \sigma_{12}(u,v) = (0.105, 0.632, 0.144)$$

For $I_{4,1}$,

$$\sigma_{22}(u,v) = (-0.237, 0, -0.331) \qquad \sigma_{12}(u,v) = (-0.105, 0.632, -0.144)$$

Again notices inconsistent estimates of second derivatives and hence of the twist vector.

3.1.3 Twist estimation from $\sigma, \hat{n}, K$

3.2 Hermite exponential order 1