Final Report

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Abstract

Traveling Salesman Problem (TSP) is a classic mathematical problem which asks about one question: Given a list of cities and the traveling cost between each pair of cities, what is the cheapest possible route that visits each city exactly once and returns to the origin city? (From wikipedia) This is a NP-hard problem, unless P=NP or there is no reasonable results can be provided. So in this report, I am going to use the Semidefinite Programming (SDP) in convex optimization to get an approximation result, more specifically, the lower bound of the original TSP.

Introduction

During a long time, the most common way to solve a NP-hard problem was using Polyhedral Combinatorics. By having a partial description of the convex hull of all feasible solutions of an integer program, one can easily solve the linear programming problems. This is kind like to the method of enumerating.

For this TSP, We consider an abstract combinatorial optimization problem (COP) given as follows. Let E being the edge set of the underlying TSP graph G and let F be a (finite) family of subsets of *E*. An edge set f $\in$ F is the feasible solutions of the TSP (That all the cities are contained once). Each $e \in E$ has a given integer cost C_e. We define the cost $C(F)$ of $f \in$ F to be

$$\mathrm{c(F)} = \sum_{e \in \mathrm{E}} C_e$$

The problem now consists in finding a feasible solution *F* of minimum cost:

$$Z^* = min\{c(f) : \ f \in F\}$$

By assigning to each f $\in$ F a characteristic vector $x_f \in \{0,1\}^n$ with $(x_f)e = 1$ if and only if e $\in$ F, we can write (COP) as a linear program as follows. Let P := conv$\{x_f$: f $\in$ F$\}$ denote the convex hull of the incidence vectors of feasible solutions. Then it is clear that

$$Z^* = min\{c^T x_f : f \in F\} = min\{c^T x : x \in P\}$$

Then we make some changes on the $min\{c^T x\}$, we use the $x^T Qx$ to replace it. The Q is a matrix with each line's cost on the dialog. So, the problem is reformulated like this:

$$\begin{array}{ll} \text{minimize} & x^T Qx \\ \text{subject to} & x \in \{0,1\}^n \end{array}$$

more constrains like the number of lines across each point should be 2 must be concluded to make this a complete graph.

Lagrange duality analysis

In this section , we discuss a Lagrange relaxation for obtaining a nontrivial lower bound on f^*.

$$\begin{array}{ll} \text{minimize} & x^T Qx \\ \text{subject to} & x \in \{0,1\}^n \end{array}$$

Formulation – (1)

The constraint $x \in \{0,1\}^n$ is not convex, so we need to find an equivalent constraint to replace it and get an approximation results.

We note a simple fact that integer point x satisfies either $x_i \leqslant 0$ or $x_i \geqslant 1$ for all i. Equivalently, this condition can be written as $x_i(x_i - 1) \geqslant 0$ for all i. Using this, we relax the integer constraint $x \in Zn$ into a set of nonconvex quadratic constraints: $x_i(x_i - 1) \geqslant 0$ for all i. The following nonconvex problem is then a relaxation of the original problem:

$$\begin{array}{ll} \text{minimize} & x^T Qx \\ \text{subject to} & x_i(x_i - 1) \geq 0, \mathrm{i} = 1, \ldots.., \mathrm{n} \end{array}$$

Formulation – (2)

Note that the optimal value of Formulation-(2) is a lower bound of the Formulation-(1). The reason is that the feasible set of the relaxation includes the feasible set of the Formulation-(1). It follows that the Formulation-(1) is infeasible if the relaxation is infeasible, and that the optimal value of the relaxation is less than or equal to the optimal value of the Formulation-(1).

The Lagrangian of the Formulation-(2) is given by

$$\mathrm{L}(\mathrm{x},\gamma) = x^T Qx - \sum_{i=1}^{n} \gamma_i x_i (x_i - 1) = x^T \big(Q - diag(\gamma)\big) x + \gamma^T x$$

Where $\gamma \in R^n$ is the vector of dual variable, and $\gamma_i \geq 0$.

$$\mathrm{g}(\gamma) = \inf\big(L(x,\gamma)\big) \quad \rightarrow \quad \nabla_x L(x,\gamma) = 0$$

$$x^* = -\frac{1}{2}\gamma(Q - diag(\gamma))^{-1}$$

$$\mathrm{g}(\gamma) = L(x^*,\gamma) = -\tfrac{1}{4}\gamma^2 (Q - diag(\gamma))^{-1}$$

So, the problem goes to:

$$\begin{array}{ll} \text{maximize} & g(\gamma) \\ \text{subject to} & Q - \text{diag}(\gamma) \geq 0 \\ & \gamma \geq 0 \end{array}$$

Formulation – (3)

By applying the epigraph form, we change this to a SDP:

$$\begin{array}{ll} \text{maxmize} & -\theta \\ \text{subject to} & \begin{bmatrix} Q - \text{diag}(\gamma) & \gamma \\ \gamma & \theta \end{bmatrix} \geq 0 \end{array}$$

Formulation – (4)

In the variables $\gamma \in R^n$ and $\theta \in R$. We note that this SDP is derived from a non-convex problem, but now is a convex problem, and thus can be solved in polynomial time.

Semidefinite relaxation

In this section, I will form a SDP relaxation of the original problem.
By introduce a new variable $X = xx^T$, we can reformulate (1) as:

$$\begin{array}{ll} \text{minimize} & TR(QX) \\ \text{subject to} & \text{diag}(X) \geq x, \\ & X = xx^T \\ & sum_i(x) = 2, \quad i = 1, \dots n \end{array}$$

Formulation – (5)

The first two constrains is just a rewriting of $x_i(x_i - 1) \geq 0$ for i=1,…,n, which was a constrain in (Formulation – 2). The last constraint is new here to limit the number of lines that cross each points, make sure the graph is a complete one.

Then, we relax the nonconvex constraint $X = xx^T$ into a convex constraint $X \geq xx^T$ then the formulation changes to:

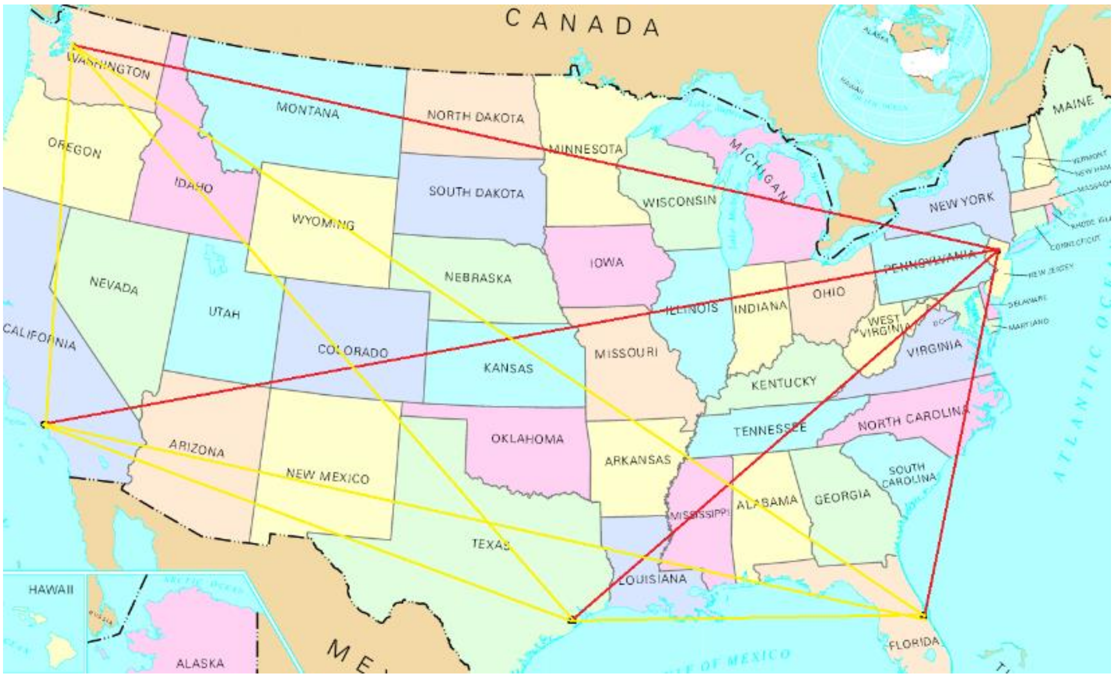
$$\begin{array}{ll} \text{minimize} & TR(QX) \\ \text{subject to} & \text{diag}(X) \geq x, \\ & \begin{bmatrix} X & x \\ x^T & 1 \end{bmatrix} \geq 0 \\ & sum_i(x) = 2, \quad i = 1, \dots n \end{array}$$

Formulation – (6)

Example

In the real practice, I choose 5 city as to form the TSP map, and write down the real airline price among these cities:

	New Jersey	Seattle	Orlando	Los Angeles	Houston
New Jersey	0	216	129	216	249
Seattle	216	0	316	138	137
Orlando	129	316	0	211	140
Los Angeles	216	138	211	0	129
Houston	249	137	140	129	0



As you can see, there are total 10 lines among these cities, and the salesman must go over all the cities with a minimal cost.

The results are as follows.

	1
1	1.0000
2	1.0000
3	8.5320e-08
4	1.8713e-09
5	5.6375e-10
6	1.0000
7	7.8567e-08
8	6.9496e-09
9	1.0000
10	1.0000

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Elapsed time: 0.49438 seconds
Optimal value is 752.00000 >>
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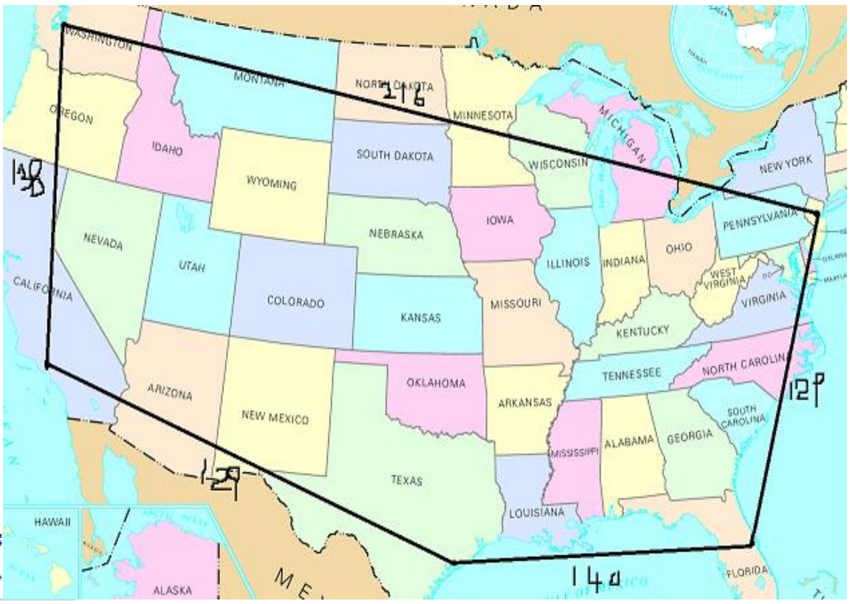


some analysis about this case:
There are only 5 city in this example (the test data is pretty small), and the result just show the cheapest way of travel. when I increase the number of cities, the results will not show the actual travel route, some of the weights would be like 0.5. But still the optimal value is the lower bound of the actual cost. (At the Appendix section there is a example of 10 city)

Another SDP

In this section, the formulation is more about applying the graph method, mainly use the adjacency matrix to form the constrains and objective function. Note that the power of the adjacency matrix can show the connect situation of the corresponding graph.
In this section we show that the optimal value of the following semidefinite program provides a lower bound on the length TSP_{opt} of an optimal tour:

$$\begin{array}{lll} \text{minimize} & \frac{1}{2} trace(QX^{(1)}) & \\ \text{subject to} & X^{(K)} \geq 0 & k = 1, \ldots. d \\ & \sum_{k=1}^{d} X^{(k)} + I = J & i = 1, \ldots.. d \\ & X^{(k)} \in S^n & k = 1, \ldots. d \end{array}$$

Where d=[$\frac{1}{2}n$]

Constrain 1 make sure there is no sub circle inside this graph. Constrain 2 makes sure every point will be reached. Constrain 3 make the X a symmetric matrix. (Actually I try to combine these two models' constrains, but the performance differs a little.)

Note that this problem involves nonnegative matrix variables $X^{(1)}, \ldots, X^{(k)}$ of order n.
In this reference paper, these matrix variables are called association scheme, following are some brief introduction about these variables:
Given a connected graph G = (V,E) with diameter d, we define $|V| \times |V|$ matrices $X^{(k)}$ (k = 1,…., d) as follows:

$$X_{ij}^{(k)} = \begin{cases} 1 & if\ dist(i,j) = k \\ 0 & else \end{cases} \qquad (i, j \in V)$$

and $X^{(1)}$ would be the adjacency matrix of the graph.(C_1 is the basic adjacency matrix and $X^{(1)} = X^T C_1 X$ is the transaction of C_1)

$$C1 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 1 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & & & 0 & 1 \\ 1 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$$

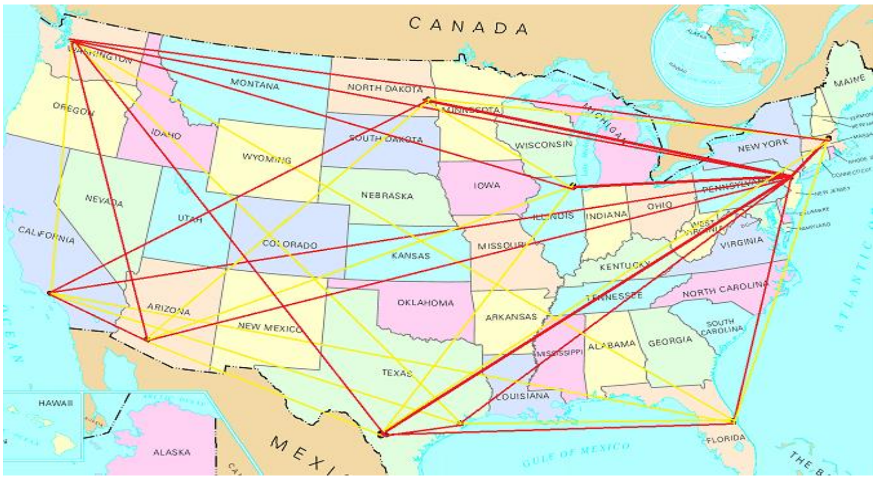
Appendix 1

More test on the 10-city graph

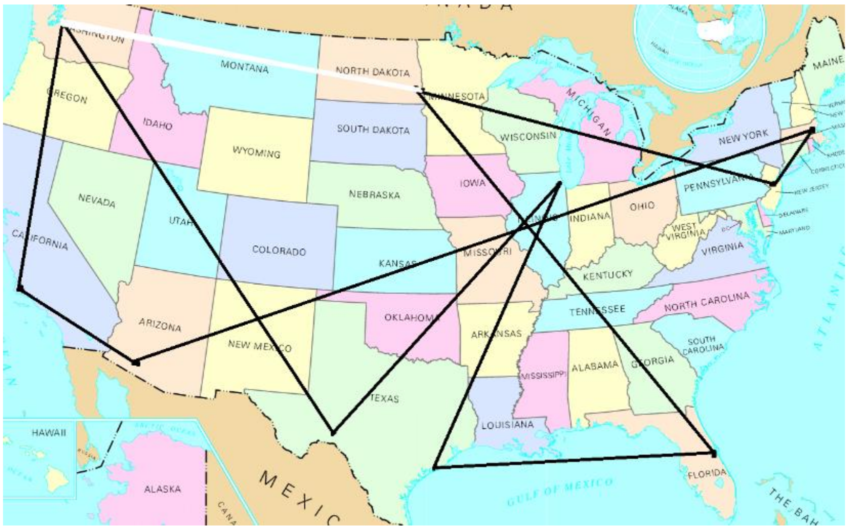
Here is the cost among these cities:

	newJersey	Seattle	Orlando	Los Angeles	Houston	San Antonia	Fargo	Chicago	Phoenix	Boston
newJersey	0	216	129	216	249	162	251	116	174	64
Seattle	216	0	316	138	137	136	317	188	288	216
Orlando	129	316	0	211	140	214	267	156	205	234
Los Angeles	216	138	211	0	129	179	213	178	58	167
Houston	249	137	140	129	0	168	256	132	155	189
San Antonia	162	136	214	179	168	0	363	179	177	201
Fargo	251	317	267	213	256	363	0	160	205	319
Chicago	116	188	156	178	132	179	160	0	118	145
Phoenix	174	288	205	58	155	177	205	118	0	144
Boston	64	216	234	167	189	201	319	145	144	0

Here is the map among these cities (total 45 lines):



And the optimal value is 1273 (no actual route, following is an approximation route based on the results.).

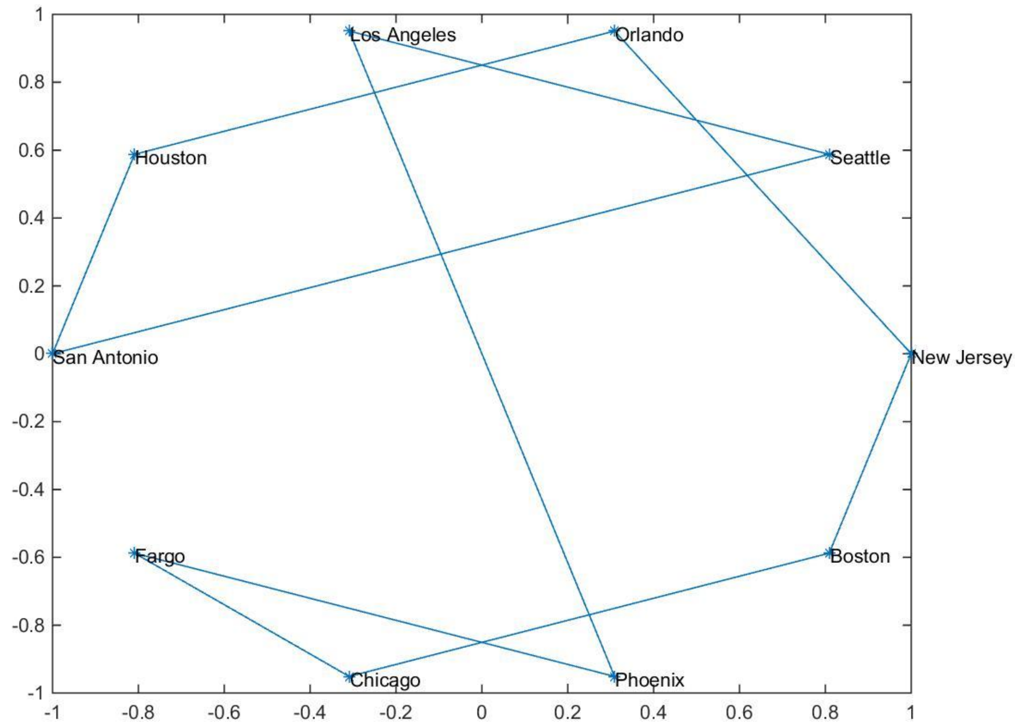


Appendix 2

A LP model is provided to make a comparison with the SDP:

$$\begin{aligned}
&\textit{minimize} && \sum_{i=0}^{n}\sum_{j=0}^{n} c_{ij} * x_{ij} \\
&\text{subject to} && 0 \le x_{ij} \le 1 \qquad i,j = 0, \dots n \\
& && \sum_{i=0, i\neq j}^{n} x_{ij} = 1 \qquad i = 0, \dots, n \\
& && \sum_{j=0, i\neq j}^{n} x_{ij} = 1 \qquad j = 0, \dots, n \\
& && u_i \in Z \\
& && u_i - u_j + n x_{ij} \le n - 1 \qquad 1 \le i \neq j \le n
\end{aligned}$$

This formulations use the variable u_i to prevent the situation of sub circles showing up.
And the results are like following:



Appendix 3

These two models are both the relaxation of the original problem.
Comparison between these two models:

	SDP	LP
Optimal value	1378	1273

The result shows that the SDP lower bound is stronger than the LP model.
Basically, half of the routes are the same.

Reference

1. https://en.wikipedia.org/wiki/Travelling_salesman_problem
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   Written by Jaehyun Park and Stephen Boyd, February 5, 2016
4. On semidefinite programming relaxations of the traveling salesman problem
   Written by Etienne de Klerk, Dmitrii V. Pasechnik, Renata Sotirov, February 11, 2009