CSMATH2 Problem Set #2

Section: **S12**
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Problem 3: Permanent

Let $A \in M_{n,n}(\mathbb{R})$. The permanent of $A = [a_{ij}]$ is defined to be

$$\text{per(A)} = \sum_{j=1} a_{ij} M_{ij},$$

where M_{ij} is the permanent of the submatrix formed by removing the row and column index by a_{ij}. For example, in the 2 x 2 case,

$$\text{per}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad + bc.$$

Find the permanent of the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Solution:

To find the permanent of the 4x4 matrix A, we must find the permanents of every submatrix of A along the first row.

$$per(A) = 1 \cdot per(M_{11}) + 2 \cdot per(M_{12}) + 1 \cdot per(M_{13}) + 0 \cdot per(M_{14})$$

$$per(M_{11}) = per\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= 1 \cdot per\begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix} + 1 \cdot per\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} + 0 \cdot per\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

$$= 1(2) + 1(3) + 0$$

$$= 5$$

$$per(M_{12}) \;=\; per\begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \; 0 \cdot per\begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix} + 1 \cdot per\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} + 0 \cdot per\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

$$= \; 0 + 1(0) + 0$$

$$= \; 0$$

$$per(M_{13}) \;=\; per\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \; 0 \cdot per\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} + 1 \cdot per\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} + 0 \cdot per\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \; 0 + 1(0) + 0$$

$$= \; 0$$

$$per(A) \;=\; 1(5) + 2(0) + 1(0) + 0$$

$$per(A) \;=\; 5$$

Problem 4: Trace

Let $A \in M_{3,3}(\mathbb{R})$. The trace of A is the scalar

$$tr(A) \;=\; \sum_{j=1}^{n} a_{ij}.$$

That is, the trace of a square matrix is the sum of its diagonal entries.

Consider the vector space $M_{3,3}(\mathbb{R})$ with its usual operations. Show that

$$\mathbf{Z} = \{A \; \in M_{3,3}(\mathbb{R}) \mid tr(A) = 0\}$$

is a subspace of $M_{3,3}(\mathbb{R})$. What is the value of dim(**Z**)?

Solution:

Given the vector space, we need to show that this matrix

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

in which $a + e + i = 0$. Algebraically, we can isolate one of the three variables in order to cancel out the resulting sum of the other two free variables. In our example, we will isolate i and substitute it into the matrix.

$$\begin{aligned} a + e + i &= 0 \\ i &= -(a + e) \end{aligned}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & -(a + e) \end{bmatrix}$$

Performing the subspace test, we will check if **Z** is a nonempty set.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathbf{Z}$$

We will now check if it is closed under vector addition.

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & h_1 & -(a_1 + e_1) \end{bmatrix} + \begin{bmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \\ g_2 & h_2 & -(a_2 + e_2) \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 & c_1 + c_2 \\ d_1 + d_2 & e_1 + e_2 & f_1 + f_2 \\ g_1 + g_2 & h_1 + h_2 & -[(a_1 + a_2) + (e_1 + e_2)] \end{bmatrix} \in \mathbf{Z}$$

Finally, we will check if it is closed under scalar vector multiplication.

$$k\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & -(a + e) \end{bmatrix} = \begin{bmatrix} ka & kb & kc \\ kd & ke & kf \\ kg & kh & -(ka + ke) \end{bmatrix} \in \mathbf{Z}$$

Therefore, **Z** is a subspace of $M_{3,3}(\mathbb{R})$.

Since dim $M_{m,n}(\mathbb{R}) = mn$, dim $M_{3,3}(\mathbb{R}) = 3 \times 3 = 9$. As the definition of **Z** constricts A with the definition $tr(A) = 0$, this helped reduce dim(**Z**) by 1, as how we've previously substituted a free variable from the definition. Therefore, dim(**Z**) $= 8$.

Problem 5: Vector Spaces of $\mathcal{F}_2$

We define the binary field $\mathcal{F}_2$ to be the set with only two elements 0 and 1 satisfying the addition and multiplication table given below.

$+$	0	1
0	0	1
1	1	0

$\times$	0	1
0	0	0
1	0	1

Let $\mathcal{F}_2^n$ be the set fo all column vectors whose entries come form the binaary field $\mathcal{F}_2$ it turns out that $\mathcal{F}_2^n$ is a vector spave space over $\mathcal{F}_2$ under the usual operations. Unlike the Euclidean space $\mathbb{R}^n$ over $\mathbb{R}$, the vector space $\mathcal{F}_2^n$ has a finite number of vectors.

a. How many vectors are in $\mathcal{F}_2^3$?

b. How many subspaces does $\mathcal{F}_2^3$ have? Hint: consider counting the subspaces by dimension.

Solution:

a. The vector space $\mathcal{F}_2^3$ has 3-dimensional vectors over $\mathcal{F}_2$. Each each component has a possible values of 0 or 1, so each of the three entries has 2 possible values. So the total number of vectors is 2^3 = 8.

$$\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}$$

b. To count the subspaces we count them by dimension and there are four dimensions: 0, 1, 2, and 3

- Dimension 0
  - Only contains the zero or trivial subspace

$$\begin{bmatrix}0\\0\\0\end{bmatrix}$$

  = 1 subspace

- Dimension 1
  - Consists of 2 vectors and spanned by 1 non-zero vector.

$$\left\{\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}\right\}, \left\{\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}, \left\{\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}, \left\{\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\}, \left\{\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}, \left\{\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right\}, \left\{\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$$

  = 7 subspaces

- Dimension 2
  - Contains 4 vectors with one vector being the zero vector.

$$\left\{\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\}, \left\{\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}, \left\{\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right\},$$

$$\left\{\begin{bmatrix}0\\0\\0\end{bmatrix},\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}1\\1\\1\end{bmatrix}\right\},\ \left\{\begin{bmatrix}0\\0\\0\end{bmatrix},\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix},\begin{bmatrix}1\\1\\1\end{bmatrix}\right\},\left\{\begin{bmatrix}0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix},\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}1\\1\\1\end{bmatrix}\right\},\ \left\{\begin{bmatrix}0\\0\\0\end{bmatrix},\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$$

= 7 subspaces

- Dimension 3
  - The entire space of $\mathcal{F}_2^3$, only 1 subspace.

$$\left\{\begin{bmatrix}0\\0\\0\end{bmatrix},\ \begin{bmatrix}0\\0\\1\end{bmatrix},\ \begin{bmatrix}0\\1\\0\end{bmatrix},\ \begin{bmatrix}0\\1\\1\end{bmatrix},\ \begin{bmatrix}1\\0\\0\end{bmatrix},\ \begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$$

= 1 subspace

- TOTAL SUBSPACES = 1 + 7 + 7 + 1 = 16