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Random Attractors of Boussinesq Equations with Multiplicative Noise

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Abstract We study the random dynamical system (RDS) generated by the Benald flow problem with multiplicative noise and prove the existence of a compact random attractor for such RDS.

Keywords random dynamical systems, random attractor, Boussinesq equation, white noise, Wiener process

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1 Introduction

The Boussinesq equation is a mathematics model of thermohydraulics, which consists of equations of fluid and temperature in the Boussinesq approximation. The deterministic case has been studied systematically by many authors (for instance, see [1–3 etc.]), the attractor problem in the deterministic case has been investigated by other papers (see [4–7]). However, in many practical circumstances, small irregularity has to be taken account. Thus it is necessary to add to the equation a random force, which is in general a space-time white noise, as considered recently by many authors for other equations (see [8–18]).

Let us consider the stochastic two-dimensional Boussinesq equation perturbed by a multiplicative white noise of Stratonovich form:

$$dv + [(v\cdot\nabla)v - \nu\triangle v + \nabla p]dt = e_2(T - T_1)dt + bv \circ dW(t), \tag{1.1}$$

$$dT + [(v\cdot\nabla)T - \kappa\triangle T)]dt = 0, \tag{1.2}$$

$$\mathrm{div} v = 0. \tag{1.3}$$

The domain occupied by the fluid is $D = (0,1)\times(0,1)$, e_1, e_2 are the canonical basis of $\mathbb{R}^2$. The unknown $v = (v_1, v_2), T, p$ stand for the velocity vector, temperature and pressure respectively. T_1 is the temperature at the top, $x_2 = 1$, while $T_0 = T_1 + 1$ is the temperature at

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the boundary below, $x_2 = 0$. The constant numbers $\nu > 0, \kappa > 0, b > 0$ are related to the usual Grashof, Prandtl and Rayleigh numbers.

The white noise described by a process $W(t)$ results from the fact small regularity has to be taken account in some circumstances. Here, we assume that $W(t)$ is a two-sided Wiener process on the probability space $(\Omega, \mathscr{F}, P)$, where $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \ \omega(0) = 0\}$.$\mathscr{F}$ is the Borel sigma-algebra induced by the compact-open topology of Ω, P is a Wiener measure.

We supplement (1.1)–(1.3) with the following boundary conditions:

$$v = 0 \quad \text{at} \quad x_2 = 0, x_2 = 1, \tag{1.4}$$

$$T = T_0 \quad \text{at} \quad x_2 = 0 \ \text{ and } \ T = T_1 = T_0 - 1 \ \text{ at } \ x_2 = 1, \tag{1.5}$$

$$\psi|_{x_1=0} = \psi|_{x_1=1} \quad \text{for} \quad \psi = v, T, p, \frac{\partial v}{\partial x_1}, \frac{\partial T}{\partial x_1}. \tag{1.6}$$

The equation (1.1)–(1.6) can be solved pathwise by writing in an abstract operator form and by using the Galerkin approximation and a priori estimates. For the unperturbed case ($b = 0$),we refer to Temma [1] and Guo [2–3]. Of course, the unique solution of equations (1.1)–(1.6) will generate a random dynamical system(RDS), this allows us to consider the random attractor of the RDS.

To prove the existence of attractors, one does not apply directly the method of the abstract operator form, even in the unperturbed case (see Temma [1]). Temma [1] introduces the maximum principle to overcome this difficulty and obtains the desired result in the unperturbed case. However, the maximum principle is valid only for the time $t \in [0, +\infty)$, (more precisely, the temperature $T(t)$ is not necessarily bounded on $t \in (-\infty, 0)$), thus the maximum principle is not useful for proving the existence of random attractor, which is a "pullback" attractor. We must go the other way to obtain the random attractor for the white noise case.

This paper is arranged as follows. Section 2 presents some preliminaries on the theory of random dynamical system (RDS) and introduces the notations of random attractor. In Section 3, we solve the stochastic Boussinesq equation and get the corresponding RDS. In Section 4, we prove the existence of random attractors for this RDS.

2 Preliminaries on RDS

Let $(\Omega, \mathscr{F}, P)$ be a probability space and $\{\theta_t : \Omega \to \Omega, t \in \mathbb{R}\}$ a family of measure preserving transformations such that $(t, \omega) \mapsto \theta_t \omega$ is measurable, $\theta_0 = \text{id}$, and $\theta_{t+s} = \theta_t \theta_s$ for all $s, t \in \mathbb{R}$. The flow θ_t together with the corresponding probability space $(\Omega, \mathscr{F}, P, \theta_t)$ is called a measurable dynamical system.

A continuous random dynamical system (RDS) on a Polish space (X, d) with Borel sigma-algebra $\mathbb{B}$ over θ on $(\Omega, \mathscr{F}, P)$ is a measurable map

$$\varphi : \ \mathbb{R}^+ \times \Omega \times X \to X, \quad (t, \omega, x) \mapsto \varphi(t, \omega)x$$

such that P-a.s.

(i) $\varphi(0, \omega) =$id on X;

(ii) $\varphi(t + s, \omega) = \varphi(t, \theta_s \omega)\varphi(s, \omega)$ for all $s, t \in \mathbb{R}^+$; (cocycle property)

(iii) $\varphi(t,\omega): X \to X$ is continuous.

A random compact set $\{K(\omega)\}_{\omega\in\Omega}$ is a family of compact sets indexed by ω such that for every $x \in X$ the map $\omega \mapsto \mathrm{d}(x, K(\omega))$ is measurable with respect to $\mathscr{F}$.

Let $\mathscr{A}(\omega)$ be a random set and $B \subset X$. We say $\mathscr{A}(\omega)$ attracts B if

$$\lim_{t\to\infty} \mathrm{dist}(\varphi(t,\theta_{-t}\omega)B, \mathscr{A}(\omega)) = 0,\ P\text{-a.s.},$$

where $\mathrm{dist}(\cdot,\cdot)$ denotes the Hausdorff semi-distance in X. We say $\mathscr{A}(\omega)$ absorbs B if there exists $t_B(\omega)$ such that for all $t \geq t_B(\omega)$,

$$\varphi(t,\theta_{-t}\omega)B \subset \mathscr{A}(\omega),\ P\text{-a.s.}.$$

A random set $\mathscr{A}(\omega)$ is said to be a random attractor for the RDS φ if P-a.s.

(i) $\mathscr{A}(\omega)$ is a random compact set;

(ii) $\mathscr{A}(\omega)$ is invariant, that is, $\varphi(t,\omega)\mathscr{A}(\omega) = \mathscr{A}(\theta_t\omega)$, for all $t \geq 0$.

(iii) $\mathscr{A}(\omega)$ attracts all deterministic bounded sets $B \in X$.

Theorem 1 If there exists a random compact set absorbing every bounded set $B \subset X$, then the RDS φ possesses a random attractor $\mathscr{A}(\omega)$,

$$\mathscr{A}(\omega) = \overline{\bigcup_{B\subset X} \Lambda_B(\omega)},$$

where $\Lambda_B(\omega) := \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} \varphi(t,\theta_{-t}\omega)B}$ is the omega-limit set of B.

For the details of the theory of RDS, we refer to [8–13]. For the theory of random attractors for other equations, we refer to [14–18].

3 Solve the Equations and Generate RDS

Let

$$\eta = T - T_0 + x_2, \tag{3.1}$$

and change p to $p - (x_2 + x_2^2/2)$. The equations (1.1)–(1.3) can be rewritten as

$$\mathrm{d}v + [(v\cdot\nabla)v - \nu\triangle v + \nabla p]\mathrm{d}t = e_2\eta \mathrm{d}t + bv \circ \mathrm{d}W, \tag{3.2}$$

$$\mathrm{d}\eta + [(v\cdot\nabla)\eta - \kappa\triangle\eta]\mathrm{d}t = v_2\mathrm{d}t, \tag{3.3}$$

$$\mathrm{div} v = 0. \tag{3.4}$$

Let the process

$$\alpha(t) := \mathrm{e}^{-bW(t)}. \tag{3.5}$$

Then $d\alpha = -b\alpha \circ dW$, if we let

$$\xi = \alpha v, \tag{3.6}$$

we get the new equations (no stochastic differential appears here)

$$\frac{d\xi}{dt} + \alpha^{-1}(\xi\cdot\nabla)\xi - \nu\triangle\xi + \alpha\nabla p = \alpha e_2\eta, \tag{3.7}$$

$$\frac{d\eta}{dt} + \alpha^{-1}(\xi\cdot\nabla) - \kappa\triangle\eta = \alpha^{-1}\xi_2, \tag{3.8}$$

$$\mathrm{div}\xi = 0. \tag{3.9}$$

To solve the equations, we consider the Hilbert space $H = H_1 \times H_2$ with the scalar product $(\cdot, \cdot)$ and norms $|\cdot|$, where $H_2 = L^2(D)$ and

$$H_1 = \{\xi \in L^2(D)^2 : \ \mathrm{div}\xi = 0, \xi_i|_{x_i=0} = \xi_i|_{x_i=1}, i = 1, 2\}.$$

We also consider the subspace $V = V_1 \times V_2$, where V_2 is the space of functions in $H^1(D)$ vanishing at $x_2 = 0$ and $x_2 = 1$ and periodic in the direction of x_1, V_2 is a Hilbert space for the scalar product and the norm:

$$((\eta_1, \eta_2)) = \int_D \mathrm{grad}\eta_1 \mathrm{grad}\eta_2 dx, \quad \|\eta\| = ((\eta, \eta))^{\frac{1}{2}}$$

and $V_1 = \{\xi \in V_2^2 : \ \mathrm{div}\xi = 0\}$.

The bilinear form

$$a(u_1, u_2) = \nu((\xi_1, \xi_2)) + \kappa((\eta_1, \eta_2)), \ \forall\, u_i = \{\xi_i, \eta_i\} \in V, \ i = 1, 2,$$

determines a linear isomorphism A from $D(A)$ into H and from V into the dual space V', defined by

$$(Au_1, u_2) = a(u_1, u_2), \quad \forall\, u_i = \{\xi_i, \eta_i\} \in V, \ i = 1, 2, \tag{3.10}$$

with $D(A) = D(A_1) \times D(A_2)$, where

$$D(A_1) = \left\{\xi \in V_1 \cap H^2(D)^2 : \ \frac{\partial \xi}{\partial x_1}|_{x_1=0} = \frac{\partial \xi}{\partial x_1}|_{x_1=1}\right\},$$
$$D(A_2) = \left\{\eta \in V_2 \cap H^2(D) : \ \frac{\partial \eta}{\partial x_1}|_{x_1=0} = \frac{\partial \eta}{\partial x_1}|_{x_1=1}\right\}.$$

It is well-known that A is a self-adjoint and positive operator and A^{-1} is a compact self-adjoint operator in H.

We also consider the trilinear forms b on V defined by

$$b(u_1, u_2, u_3) = ((\xi_1 \cdot \nabla)\xi_2, \xi_3) + (\xi_1 \cdot \nabla)\eta_2, \eta_3) \ \ \forall\, u_i = \{\xi_i, \eta_i\} \in V, \ i = 1, 2, 3.$$

The trilinear form b is continuous on V or even on $H^1(D)^2 \times H^1(D)$. We associate with the form b the bilinear continuous operator B which maps $V \times V$ into V' and $D(A) \times D(A)$ into H, defined by

$$(B(u_1, u_2), u_3) = b(u_1, u_2, u_3), \quad \forall\, u_1, u_2, u_3 \in V. \tag{3.11}$$

Finally, we define a family of linear continuous operators $R(t)$ in H

$$R(t) : u = \{\xi, \eta\} \to Ru = \{\alpha(t)e_2\eta, \ \alpha^{-1}(t)\xi_2\}. \tag{3.12}$$

Now we can set the equations (3.7)–(3.9) in the operator form. If $u = \{\xi, \eta\}$ is the solution of (3.7)–(3.9) and $y = \{f, g\}$ is a test function in V, we multiply (3.7) by f, (3.8) by g, integrate over D and add the resulting equation. The pressure term disappears and after simplification we find

$$\frac{d}{dt}(u, y) + \alpha^{-1}b(u, u, y) + a(u, y) + (R(t)u, y) = 0, \quad \forall\, y \in V$$

which can be reinterpreted as

$$\frac{du}{dt} + Au + \alpha^{-1}B(u,u) + R(t)u = 0. \tag{3.13}$$

The above equation (which depends on a random parameter) can be solved pathwise. Using the Galerkin approximation and a priori estimates, similarly to the proof Theorem 3.3.1 of Temma [1], one can show that P-a.s.

(i) For all $t_0 \in \mathbb{R}$ and $u_0 \in H$ there exists a unique solution $u \in C([t_0,\infty);H) \cap L^2_{\text{loc}}(t_0,\infty;V)$ of equation (3.13) with $u(t_0) = u_0$.

(ii) If $u_0 \in V$, the solution is $v \in C([t_0,\infty);V) \cap L^2_{\text{loc}}(t_0,\infty;D(A))$.

(iii) For every $\epsilon > 0$, $u \in C([t_0+\epsilon,\infty);V) \cap L^2_{\text{loc}}(t_0+\epsilon,\infty;D(A))$ for $u_0 \in H$.

(iv) Denoting such solution by $u(t,\omega;t_0,u_0)$ the mapping $u_0 \mapsto u(t,\omega;t_0,u_0)$ is continuous for all $t \geq t_0$.

Define a stochastic flow $\varphi(t,\omega)$ by

$$\varphi(t,\omega)z_0 = \{v,\eta\} := \{\alpha(t,\omega)^{-1}Q_{H_1}u(t,\omega,0,z_0), Q_{H_2}u(t,\omega,0,z_0)\} \tag{3.14}$$

for all $z_0 = \{v_0,\eta_0\} \in H$, where $Q_{H_i} : H = H_1 \times H_2 \to H_i$, $i = 1,2$, are projection operators.

It is easy, by above (i)–(iv),to prove that φ is a continuous RDS on H over θ on $(\Omega,\mathscr{F},P)$, where θ is a measurable dynamical system determined by the two-sided Wiener process $W(t,\omega) := \omega(t)$,

$$\theta_s\omega(t) = \omega(t+s) - \omega(s), \quad \forall\ s,t \in \mathbb{R}. \tag{3.15}$$

· We also have the following facts:

(1) $z = \{v,\eta\} = \varphi(t,\omega)z_0$ is just the unique solution of equations (3.2)–(3.4) with $z(0) = z_0$.

(2) The solution $z = \{v,\eta\} = z(t,\omega;t_0,z_0)$ of equations (3.2)–(3.3) with $z(t_0) = z_0 = \{v_0,\eta_0\}$, can be formulated as

$$v = \alpha(t,\omega)^{-1}Q_{H_1}u(t,\omega;t_0,u_0) = \alpha(t,\omega)^{-1}\xi(t,\omega;t_0,\alpha(t_0,\omega)v_0)$$

and $\eta = Q_{H_2}u(t,w;t_0,u_0) = \eta(t,w;t_0,\eta_0)$, where $u_0 = \{\alpha(t_0,\omega)v_0,\eta_0\}$.

(3) $\varphi(s,\theta_{-s}\omega)z_0 = z(0,\omega,-s,z_0)$, for $z_0 \in H$, and can be interpreted as the position at time 0 of the trajectory which was at z_0 at time $-s$.

4 Random Attractors

In this section, we'll show that there exists a compact random attractor for RDS φ as introduced in Section 3 (3.14). By Theorem 1, we have to get a compact random absorbing set.

Lemma 4.1 *Let $u = \{\xi,\eta\}$ be a solution of* (3.7)–(3.9). *Then*

$$|\eta(t)| \leq (|\eta(t_0)| + c_1)\mathrm{e}^{2\kappa(t_0-t)} + c_1,\ \forall\ t_0 \leq t \tag{4.1}$$

where $c_1 = 2T_0|D|^{\frac{1}{2}}$ is a determinative constant.

Proof The results are proved more naturally on the equation (1.2) for the T. Multiplying (1.2) by T and integrating over D we obtain

$$\frac{1}{2}\frac{d}{dt}|T|^2 + \kappa\|T\|^2 = 0.$$

Note that the Poincare inequality in V, V_1, V_2 implies

$$|u| \le \|u\|, \ |\xi| \le \|\xi\|, \ |\eta| \le \|\eta\|, \ \forall \ u \in V, \ \xi \in V_1, \ \eta \in V_2. \tag{4.2}$$

(For the fact that the Poincare number $c_0 = 1$ here comes from an elementary and easy computation, see Temma [1]), we get

$$\frac{d}{dt}|T|^2 + 2\kappa |T|^2 \le 0$$

which implies by Gronwall lemma that

$$|T(t)| \le |T(t_0)| \mathrm{e}^{2\kappa(t_0 - t)}, \quad \forall \, t \ge t_0.$$

A rephrasing of this result in terms of $\eta := T - T_0 + x_2$ implies the desired result.

Lemma 4.2 *Assume $u = \{\xi, \eta\}$ is a solution of* (3.7)–(3.9). *Then P-a.s.*

$$|\xi(t)|^2 \le \mathrm{e}^{-\nu t}\left[|\xi(t_0)|^2 \mathrm{e}^{\nu t_0} + \frac{1}{\nu}\int_{t_0}^{t} |\eta(s)|^2 \alpha^2(s) \mathrm{e}^{\nu s} ds\right], \quad \forall \, t \ge t_0. \tag{4.3}$$

Proof Taking the scalar product of (3.7) with ξ and using the Poincare inequality (4.2) and Young inequality, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|\xi|^2 + \nu \|\xi\|^2 \ &= \ \alpha(e_2 \eta, \xi) = \alpha(\eta, \xi_2) \le \alpha |\eta||\xi_2| \\ &\le \ \alpha |\eta||\xi| \le \alpha |\eta| \|\xi\| \le \frac{\nu}{2}\|\xi\|^2 + \frac{1}{2\nu}|\alpha \eta|^2, \end{aligned}$$

which implies that

$$\frac{d}{dt}|\xi|^2 + \nu \|\xi\|^2 \le \frac{1}{\nu}\alpha^2 |\eta|^2. \tag{4.4}$$

Using the Poincare inequality (4.2) we have

$$\frac{d}{dt}|\xi|^2 + \nu \|\xi\|^2 \le \frac{1}{\nu}\alpha^2 |\eta|^2,$$

which, by Gronwall lemma, implies the desired inequality (4.3).

Proposition 4.3 *There exist a number C and a random variable $\gamma(\omega)$ satisfying the following property: For every $\rho > 0$, there exists $t(\omega, \rho) \le -1$, such that for any $z_0 = \{v_0, \eta_0\} \in H$ with $|z_0| < \rho$, and for any $t_0 \le t(\omega, \rho)$, the following estimates hold P-a.s.*

$$|\eta(t, \omega; t_0, \eta_0)| \le C, \quad \forall \, t \in [-1, 0]; \tag{4.5}$$

$$|\xi(t, \omega; t_0, \alpha(t_0, \omega) v_0)| \le \gamma(\omega) \quad \forall \, t \in [-1, 0]. \tag{4.6}$$

In particular, $B(0, C + \gamma(\omega))$ is a random absorbing set for RDS φ in H.

Proof By Lemma 4.1, for all $-1 \le t \le 0$ and $t_0 \le 1$, we have

$$|\eta(t)| \le (|\eta(t_0)| + c_1)\mathrm{e}^{2\kappa(t_0 - t)} \le (\rho + c_1)\mathrm{e}^{2\kappa(t_0 + 1)} + c_1.$$

Choose $t_1(\rho) \le -1$ such that $(\rho + c_1)\mathrm{e}^{2\kappa(t_1 + 1)} \le 1$. Then

$$|\eta(t, \omega; t_0, \eta_0)| \le 1 + c_1 = C, \quad \forall \, -1 \le t \le 0$$

provided $t_0 \le t_1(\rho)$, which proves (4.5).

To prove (4.6), we see from (4.3) that P-a.s.

$$|\xi(t,\omega;t_0,\alpha(t_0,\omega)v_0|^2 \leq \mathrm{e}^{\nu}[|v_0|^2\alpha^2(t_0)\mathrm{e}^{\nu t_0} + \frac{1}{\nu}\int_{t_0}^{t}|\eta(s)|^2\alpha^2(s)\mathrm{e}^{\nu s}ds \tag{4.7}$$

holds for all $-1 \leq t \leq 0$ and $t_0 \leq -1$. By a standard argument (see [5]), we have

$$\lim_{s\to-\infty}\frac{W(s)}{s} = 0, \quad P\text{-a.s.} \tag{4.8}$$

It easily follows that

$$\alpha^2(t_0)e^{\nu t_0} \to 0 \quad \text{as} \ \ t_0 \to -\infty$$

which implies that there exists $t_2(\omega,\rho) \leq -1$ such that if $t_0 \leq t_2$,

$$|v_0|^2\alpha^2(t_0,\omega)\mathrm{e}^{\nu t_0} \leq \rho^2\alpha^2(t_0,\omega)\mathrm{e}^{\nu t_0} \leq 1. \tag{4.9}$$

To estimate the integrals on the right hand of (4.7) is more difficult (since $|\eta|$ is not necessarily bounded on $(-\infty,-1]$). For this reason, we let

$$I = \int_{t_0}^{t}|\eta(s)|^2\alpha^2(s)\mathrm{e}^{\nu s}ds = \left(\int_{\frac{t_0}{2}}^{t} + \int_{t_0}^{\frac{t_0}{2}}\right)|\eta(s)|^2\alpha^2(s)\mathrm{e}^{\nu s}ds = I_1 + I_2.$$

To estimate I_1, we see from (4.1) that if $\frac{t_0}{2} \leq s \leq t$ then

$$|\eta(s)| \leq (\rho + c_1)\mathrm{e}^{2\kappa(t_0 - s)} + c_1 \leq (\rho + c_1)\mathrm{e}^{2\kappa\cdot\frac{t_0}{2}} + c_1 \leq 1 + c_1 = C$$

provided $t_0 \leq t_3(\rho)$ for some chosen $t_3(\rho) \leq -1$. The above inequality implies now that, for $t_0 \leq t_3(\rho)$, $t \in [-1,0]$, P-a.s.

$$I_1 = \int_{\frac{t_0}{2}}^{t}|\eta(s)|^2\alpha^2(s)\mathrm{e}^{\nu s}ds \leq C^2\int_{-\infty}^{0}\alpha^2(s,\omega)\mathrm{e}^{\nu s}ds \leq \gamma_1(\omega), \tag{4.10}$$

where $\gamma_1(\omega)$ is a random variable being independent of ρ, because the ergodic property (4.8) implies easily that $s \mapsto \alpha^2(s)\mathrm{e}^{\nu s}$ is pathwise integrable over $(-\infty,0]$.

To estimate I_2, we choose $t_4(\rho) \leq -1$ such that

$$(\rho + 2c_1)^2\mathrm{e}^{\frac{\nu}{2}\cdot t_4} \leq 1.$$

Since, by (4.1), $|\eta(s)| \leq \rho + 2c_1$ for all $s \in [t_0, \frac{t_0}{2}]$, it follows that if $\frac{t_0}{2} \leq t_4(\rho)$ then

$$\begin{aligned} I_2 = \int_{t_0}^{\frac{t_0}{2}}|\eta(s)|^2\alpha^2(s)\mathrm{e}^{\nu s}ds &\leq \int_{t_0}^{\frac{t_0}{2}}[(\rho + 2c_1)^2\mathrm{e}^{\frac{\nu}{2}s}]\alpha^2(s)\mathrm{e}^{\frac{\nu}{2}s}ds \\ \leq \int_{t_0}^{\frac{t_0}{2}}\alpha^2(s)\mathrm{e}^{\frac{\nu}{2}s}ds &\leq \int_{-\infty}^{0}\alpha^2(s)\mathrm{e}^{\frac{\nu}{2}s}ds = \gamma_2(\omega). \end{aligned}$$

The above inequality, together with (4.7), (4.9) and (4.10), implies (4.7) by taking $t(\omega,\rho) = \min\{t_1, t_2, t_3, 2t_4\}$ and

$$\gamma^2(\omega) = \mathrm{e}^{\nu}\left[1 + \frac{1}{\nu}(\gamma_1(\omega) + \gamma_2(\omega))\right].$$

Finally, since $\varphi(-t_0,\theta_{t_0}\omega)z_0 = z(0,\omega;,t_0,z_0) = \{\xi(0,\omega;t_0,\alpha(t_0,\omega)v_0),\eta(0,\omega;t_0,\eta_0)\}$, it follows from (4.5) and (4.6) that $|\varphi(-t_0,\theta_{t_0}\omega)z_0| \leq C+\gamma_(\omega)$ if $t_0 \leq t(\omega,\rho)$, which means $B(0,C+\gamma_1(\omega))$ is a random absorbing set in H.

Lemma 4.4 *Under the same assumptions and notations of Proposition* 4.3, *there exist two random variables* $C_1(\omega)$ *and* $C_2(\omega)$ *such that for* $t_0 \le t(\omega, \rho)$,

$$\int_{-1}^{0} \|\eta(s,\omega;t_0,\eta_0)\|^2 \le C_1(\omega); \tag{4.11}$$

$$\int_{-1}^{0} \|\xi(s,\omega;t_0,\alpha(t_0,\omega)v_0)\|^2 \le C_2(\omega). \tag{4.12}$$

Proof An energy equation for η can be easily obtained by multiplying (3.8) by η and integrating over D. After simplification we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\eta|^2 + \kappa\|\eta\|^2 &= \alpha^{(-1)}(\xi_2,\eta) \le \alpha^{-1}|\xi||\eta| \\ &\le \|\eta\| \cdot |\alpha^{-1}\xi| \le \frac{\kappa}{2}\|\eta\|^2 + \frac{1}{2\kappa}\alpha^{-2}|\xi|^2,\end{aligned}$$

which implies that

$$\frac{d}{dt}|\eta|^2 + \kappa\|\eta\|^2 \le \frac{1}{\kappa}\alpha^{-2}|\xi|^2. \tag{4.13}$$

By integration of (4.13) between -1 and 0, we have

$$|\eta(0)|^2 - |\eta(-1)|^2 + \kappa\int_{-1}^{0}\|\eta(s)\|^2 ds \le \frac{1}{\kappa}\int_{-1}^{0}\alpha^{-2}(s)|\xi(s)|^2 ds.$$

Thus

$$\begin{aligned}\int_{-1}^{0}\|\eta(s)\|^2 ds &\le \frac{1}{\kappa}|\eta(-1)|^2 + \frac{1}{\kappa^2}\int_{-1}^{0}\alpha^{-2}(s)|\xi(s)|^2 ds \\ &\le \frac{1}{\kappa}C^2 + \frac{1}{\kappa^2}\gamma^2(\omega)\int_{-1}^{0}\alpha^{-2}(s)ds := C_1(\omega),\end{aligned}$$

which proves (4.11). Similarly, by integration of (4.4) between -1 and 0 we get

$$|\xi(0)|^2 - |\xi(-1)|^2 + \nu\int_{-1}^{0}\|\xi(s)\|^2 ds \le \frac{1}{\nu}\int_{-1}^{0}\alpha^2(s)|\eta(s)|^2 ds.$$

Therefore,

$$\int_{-1}^{0}\|\xi(s)\|^2 ds \le \frac{1}{\nu}\gamma^2(\omega) + \frac{C^2}{\nu^2}\int_{-1}^{0}\alpha^2(s)ds := C_2(\omega).$$

Proposition 4.5 *There exist two random radii* $C_3(\omega)$, $C_4(\omega)$ *satisfying the following properties: For all* $\rho > 0$ *there exists* $t(\omega,\rho) \le -1$ *such that the following hold* P*-a.s., for all* $t_0 \le t(\omega,\rho)$ *and* $z_0 = \{v_0, \eta_0\} \in H$ *with* $|z_0| < \rho$:

$$\|\eta(0,\omega;t_0,\eta_0)\| \le C_3(\omega); \tag{4.14}$$

$$\|\xi(0,\omega;t_0,\alpha(t_0,\omega)v_0\| \le C_4(\omega). \tag{4.15}$$

In particular, the ball $B(0, C_3(\omega) + C_4(\omega))$ *is a random absorbing set in* V.

Proof Taking the scalar product of the equation (3.7) with $-\triangle\xi$ in $L^2(D)^2$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\xi\|^2 + \nu|\triangle\xi|^2 &= -\alpha(\eta,\triangle\xi_2) + \alpha^{-1}b_1(\xi,\xi,\triangle\xi) \\ &\le \alpha|\eta||\triangle\xi| + \alpha^{-1}c_3|\xi|^{\frac{1}{2}}\|\xi\|\|\triangle\xi\|^{\frac{3}{2}} \\ &\le \frac{\nu}{2}|\triangle\xi|^2 + c_4|\alpha\eta|^2 + c_5\alpha^{-4}|\xi|^2\|\xi\|^4,\end{aligned}$$

which implies that

$$\frac{d}{dt}\|\xi\|^2 \le (c_6\alpha^{-4}|\xi|^2\|\xi\|^2)\cdot\|\xi\|^2 + c_7|\alpha\eta|^2.$$

Letting

$$f(\tau) = c_6\alpha^{-4}|\xi(\tau)|^2\|\xi(\tau)\|^2$$

and applying the Gronwall lemma over an arbitrary interval $[s,0]\subset[-1,0]$ we obtain

$$\|\xi(0)\|^2 \le \mathrm{e}^{\int_s^0 f(\tau)d\tau}\Big[\|\xi(s)\|^2 + c_7\int_s^0 \alpha^2(\tau)|\eta(\tau)|^2 d\tau\Big].$$

Integrating w.r.t. s over the interval $[-1,0]$ yields

$$\|\xi(0)\|^2 \le \mathrm{e}^{\int_{-1}^0 f(\tau)d\tau}\Big[\int_{-1}^0 \|\xi(s)\|^2 ds + c_7\int_{-1}^0 \alpha^2(\tau)|\eta(\tau)|^2 d\tau\Big].$$

Since

$$\int_{-1}^0 f(\tau)d\tau \le c_6 \sup_{-1\le t\le 0}|\alpha(t)|^{-4}\cdot \sup_{-1\le t\le 0}|\xi(t)|^2\cdot\int_{-1}^0\|\xi(\tau)\|^2 d\tau \le c_6c_8(\omega)\gamma^2(\omega)C_2(\omega)$$

and

$$\int_{-1}^0 \alpha^2(\tau)|\eta(\tau)|^2 d\tau \le \sup_{-1\le t\le 0}\alpha^2(t)\cdot\sup_{-1\le t\le 0}|\eta(t)|^2 \le c_8(\omega)C,$$

then (4.15) follows by taking

$$C_4(\omega) = \mathrm{e}^{c_6c_8(\omega)\gamma^2(\omega)}[\gamma^2(\omega) + c_7c_8(\omega)C].$$

Thus we obtain the final conclusion.

Theorem 4.6 *The random dynamical system associated with the stochastic Boussinesq equations* (3.1)–(3.6) *has a compact random attractor.*

References

[1] Temam, R.: Infinite-dimensional dynamical systems in Mechanics and Physics, Springer-Verlag, New York, 1988

[2] Guo, B. L.: Nonlinear Galerkin methods for solving two dimensional Boussinesq equations. *Chin. Ann. of Math.*, **16B**, 379–390 (1995)

[3] Guo, B. L.: Spectral methods for solving two dimension Newton–Boussinesq equations. *Acta Math. Appl. Sinica*, **5**, 208–218 (1989)

[4] Guo, B. L., Wang, B. X.: Exponential attractors for the generalized Ginzburg–Landau equation. *Acta Mathematica Sinica, Chinese Series*, **16**(3), 515–526 (2000)

[5] Lu, S. J., Lu, Q. S.: Exponential attractors of the 3D derivative Ginzburg–Landau equation. *Acta Mathematica Sinica, English Series*, **24**(5), 809–828 (2008)

[6] Shen, C. X., Guo, B. L.: Ill-posedness for the nonlinear Davey–Stewartson equation. *Acta Mathematica Sinica, English Series*, **24**(1), 117–127 (2008)

[7] Zhang, W. N.: Dimension of Maximal attractors for the m-dimensional Cahn–Hilliard system. *Acta Mathematica Sinica, English Series*, **21**(6), 1487–1494 (2005)

[8] Amold, L.: "Random dynamical system", Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998

[9] Caraballo, T., Langa, J. A.: Stability and random attractors for a reaction-diffusion equation with multiplicative noise. *Discrete Contin. Dyn. Syst.* **6**, 875–892 (2000)

[10] Caraballo, T., Langa, J. A., Robinson, J. C.: Upper semicontinuity of attractors for random perturbations of dynamical systems. *Commun. Partial Diff. Equ.*, **23**, 1557–1581 (1998)

[11] Crauel, H., Debussche, A., Flandoli, F.: Random attractors. *J. Dyn Diff. Equ.*, **9**, 307–341 (1995)

[12] Crauel, H., Flandoli, F.: Attractors for random dynamical systems. *Probab. Theory Relat. Fields*, **100**, 365–393 (1994)

[13] Debussche, A.: Hausdorff dimension of a random invariant set. *J. Math. Pures Appl.*, **77**, 967–988 (1998)
[14] Fan, X. M.: Random attractor for a damped sine-Gordon equation with white noise. *Pacific J. Math.*, **216**, 63–76 (2004)
[15] Prato, G., Da Debussche, A., Temam, R.: Stochastic Burger's equations. *Nonlin. Diff. Eq. Appl.*, **1**, 389–402 (1994)
[16] Robinson, J. C.: Stability of random attractors under perturbation and approximation. *J. Diff. Equ.*, **186**, 652–669 (2002)
[17] Wang, G. L., Guo, B. L., Li, Y. R.: The asymptotic behavior of the stochastic Ginzburg–Landau equation with additive noise. *Appl. Math. Computation*, **198**, 849–857 (2008)
[18] Yang, D. S.: The asymptotic behavior of the stochastic Ginzburg–Landau equation with multiplicative noise. *J. Math. Physics*, **45**, 4064–4076 (2004)