

# **Aprendizaje Reforzado**

David Peña Peralta

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# Table of contents

Preface	3
1 Introduction	11
2 Summary	12
References	13

# Preface

This is a Quarto book.

To learn more about Quarto books visit <https://quarto.org/docs/books>.

```
import numpy as np
```

```
x = 10
```

Ejercicio 4

Considerando  $J_N$  como sigue

$$J_N(x) = \beta^N x^{1-\gamma} K_N,$$

con  $K_N = 1$  bajo la hipótesis de que

$$c_k(x, a) = \beta^k a^{1-\gamma}$$

calculamos  $J_{N-1}$ .

$$\begin{aligned} J_{N-1}(x) &= \max_{a \in A(x)} \{c_{N-1}(x, a) + J_N((1+i)(x-a))\} \\ &= \max_{a \in A(x)} \{\beta^{N-1} a^{1-\gamma} + \beta^N ((1+i)(x-a))^{1-\gamma}\} \end{aligned}$$

Definimos el argumento como una función  $q$ .

$$\begin{aligned} q(x, a) &= \beta^{N-1} a^{1-\gamma} + \beta^N ((1+i)(x-a))^{1-\gamma} \\ &= C_1 a^{1-\gamma} + C_2 (x-a)^{1-\gamma}, \end{aligned}$$

donde  $C_1 = \beta^{N-1}$  y  $C_2 = \beta^N (1+i)^{1-\gamma} K_N$ . Como  $q$  es continua en  $(x, a)$ . Podemos calcular el máximo mediante el gradiente.

$$\partial_a q = C_1 (1 - \gamma) a^{-\gamma} - C_2 (1 - \gamma) (x - a)^{-\gamma}.$$

Igualando,  $\partial_a q = 0$ .

$$\begin{aligned} C_1 a^{-\gamma} &= C_2 (x - a)^{-\gamma} \\ \frac{C_1}{C_2} &= \left( \frac{x - a}{a} \right)^{-\gamma} \\ \left( \frac{C_1}{C_2} \right)^{-\frac{1}{\gamma}} &= \frac{x}{a} - 1 \\ \left( \frac{C_1}{C_2} \right)^{-\frac{1}{\gamma}} + 1 &= \frac{x}{a} \\ a &= \frac{x}{\left( \frac{C_1}{C_2} \right)^{-\frac{1}{\gamma}} + 1} \end{aligned}$$

Finalmente

$$a = h(x) = \frac{x}{(\beta(1+i)^{1-\gamma})^{\frac{1}{\gamma}} + 1}$$

Definiendo  $\eta = (\beta(1+i)^{1-\gamma})^{\frac{1}{\gamma}} + 1$ ,  $\eta - 1 = (\beta(1+i)^{1-\gamma})^{\frac{1}{\gamma}}$

entonces

$$h(x) = \frac{x}{\eta},$$

$$\begin{aligned}
J_{N-1}(x) &= \beta^{N-1} \left( \frac{x}{\eta} \right)^{1-\gamma} + \beta^N \left( (1+i) \left( x - \frac{x}{\eta} \right) \right)^{1-\gamma} \\
&= \beta^{N-1} x^{1-\gamma} \left( \eta^{\gamma-1} + \beta (1+i)^{1-\gamma} \left( \frac{\eta-1}{\eta} \right)^{1-\gamma} \right) \\
&= \beta^{N-1} x^{1-\gamma} \eta^{\gamma-1} \left( 1 + \beta (1+i)^{1-\gamma} (\eta-1)^{1-\gamma} \right) \\
&= \beta^{N-1} x^{1-\gamma} \eta^{\gamma-1} \left( 1 + \beta (1+i)^{1-\gamma} (\eta-1)^{1-\gamma} \right) \\
&= \beta^{N-1} x^{1-\gamma} \eta^{\gamma-1} \left( 1 + \beta (1+i)^{1-\gamma} \left( (\beta(1+i)^{1-\gamma})^{\frac{1}{\gamma}} \right)^{1-\gamma} \right) \\
&= \beta^{N-1} x^{1-\gamma} \eta^{\gamma-1} \left( 1 + \beta (1+i)^{1-\gamma} (\beta(1+i)^{1-\gamma})^{\frac{1}{\gamma}-1} \right) \\
&= \beta^{N-1} x^{1-\gamma} \eta^{\gamma-1} \left( 1 + \beta^{\frac{1}{\gamma}} (1+i)^{(1-\gamma)(\frac{1}{\gamma}-1)+1-\gamma} \right) \\
&= \beta^{N-1} x^{1-\gamma} \eta^{\gamma-1} \left( 1 + \beta^{\frac{1}{\gamma}} (1+i)^{(\frac{1}{\gamma}-1)} \right) \\
&= \beta^{N-1} x^{1-\gamma} \eta^{\gamma},
\end{aligned}$$

Entonces

$$K_{N-1} = \eta^\gamma, h_{k-1}(x) = \frac{x}{(K_{N-1})^{1/\gamma}}$$

Ahora calculamos  $J_{N-2}$

$$\begin{aligned}
J_{N-2}(x) &= \max_{a \in A(x)} \left\{ \beta^{N-2} a^{1-\gamma} + \beta^{N-1} [(1+i)(x-a)]^{1-\gamma} \eta^\gamma \right\} \\
&= \max_{a \in A(x)} \{ q(x, a) \},
\end{aligned}$$

donde

$$q(x, a) = C_1 a^{1-\gamma} + C_2 (x-a)^{1-\gamma},$$

con  $C_1 = \beta^{N-2}$  y  $C_2 = \beta^{N-1} (1+i)^{1-\gamma} K_{N-1}$ . Obteniendo, por recursividad

$$\begin{aligned} h_{N-2} &= \frac{x}{\left(\frac{C_1}{C_2}\right)^{-\frac{1}{\gamma}} + 1} \\ &= \frac{x}{\left(\frac{1}{\beta(1+i)^{1-\gamma} K_{N-1}}\right)^{-\frac{1}{\gamma}} + 1} \\ &= \frac{x}{\left(\beta(1+i)^{1-\gamma} K_{N-1}\right)^{\frac{1}{\gamma}} + 1} \end{aligned}$$

Entonces, sea

$$\eta' = \left(\beta(1+i)^{1-\gamma} K_{N-1}\right)^{\frac{1}{\gamma}} + 1.$$

Repitiendo, el caso anterior, tenemos que

$$\begin{aligned} J_{N-2}(x) &= \beta^{N-2} x^{1-\gamma} \eta_i^{\gamma-1} \left(1 + K_{N-1} \beta(1+i)^{1-\gamma} \left((\beta(1+i)^{1-\gamma} K_{N-1})^{\frac{1}{\gamma}}\right)^{1-\gamma}\right) \\ &= \beta^{N-2} x^{1-\gamma} \eta_i^{\gamma-1} \left(1 + K_{N-1} \beta(1+i)^{1-\gamma} \left((\beta(1+i)^{1-\gamma} K_{N-1})^{\frac{1}{\gamma}}\right)^{1-\gamma}\right) \\ &= \beta^{N-2} x^{1-\gamma} \eta_i^{\gamma-1} \left(1 + K_{N-1} \beta(1+i)^{1-\gamma} (\beta(1+i)^{1-\gamma} K_{N-1})^{\frac{1}{\gamma}-1}\right) \\ &= \beta^{N-2} x^{1-\gamma} \eta_i^{\gamma-1} \left(1 + K_{N-1} \beta(1+i)^{1-\gamma} (1+i)^{(1-\gamma)(\frac{1}{\gamma}-1)} K_{N-1}^{\frac{1}{\gamma}-1}\right) \\ &= \beta^{N-2} x^{1-\gamma} \eta_i^{\gamma-1} \left(1 + K_{N-1} \beta^{1/\gamma} (1+i)^{\frac{1}{\gamma}-1} K_{N-1}^{\frac{1}{\gamma}-1}\right) \\ &= \beta^{N-2} x^{1-\gamma} \eta_i^{\gamma-1} \left(1 + \beta^{1/\gamma} (1+i)^{\frac{1}{\gamma}-1} K_{N-1}^{\frac{1}{\gamma}}\right) \\ &= \beta^{N-2} x^{1-\gamma} \eta_i^{\gamma}, \end{aligned}$$

entonces

$$K_{N-2} = \eta'^{\gamma},$$

y

$$h_{N-2} = \frac{x}{K_{N-2}^{1/\gamma}}$$

Por lo tanto, tenemos que

$$K_n = \left(\beta(1+i)^{1-\gamma} K_{n+1}\right)^{\frac{1}{\gamma}} + 1, n = 0, 1, 2, \dots, N,$$

con  $K_N = 1$ . Obteniendo así

$$J_n(x) = \beta^n x^{1-\gamma} K_n$$

$$h_n(x) = \frac{x}{K_n^{1/\gamma}}$$

Ejercicio 5

Para el caso infinito. Estamos considerando

$$c(x, a) = a^{1-\gamma}$$

Entonces

$$\nu(x) = \max_{a \in A(x)} \{a^{1-\gamma} + \beta \nu((1+i)(x-a))\},$$

considerando  $\nu(x) = cx^{1-\gamma}$ . Entonces

$$\nu(x) = \max_{a \in A(x)} \{a^{1-\gamma} + \beta c[(1+i)(x-a)]^{1-\gamma}\},$$

definimos

$$q(x, a) = a^{1-\gamma} + \beta c[(1+i)(x-a)]^{1-\gamma},$$

entonces

$$\partial_a q = (1-\gamma) a^{-\gamma} + \beta c(1-\gamma)(1+i)^{1-\gamma}(-1)(x-a)^{-\gamma}.$$

Si  $\partial_a q = 0$ . Entonces

$$a^{-\gamma} = \beta c(1+i)^{1-\gamma}(x-a)^{-\gamma}$$

$$(\beta c(1+i)^{1-\gamma})^{-1} = \left(\frac{x-a}{a}\right)^{-\gamma}$$

$$\beta^{-1} c^{-1} (1+i)^{\gamma-1} = \left(\frac{x}{a} - 1\right)^{-\gamma}$$

$$\left[\beta^{-1} c^{-1} (1+i)^{\gamma-1}\right]^{-\frac{1}{\gamma}} + 1 = \frac{x}{a}$$

Por lo tanto

$$a = \frac{x}{\left[\beta^{-1} c^{-1} (1+i)^{\gamma-1}\right]^{-\frac{1}{\gamma}} + 1}$$

$$= \frac{x}{\left[\beta c(1+i)^{1-\gamma}\right]^{\frac{1}{\gamma}} + 1}$$

Ahora remplazamos en  $q$

$$\begin{aligned}
\nu(x) &= \left( \frac{x}{[\beta c(1+i)^{1-\gamma}]^{\frac{1}{\gamma}} + 1} \right)^{1-\gamma} + \beta c \left[ (1+i) \left( x - \frac{x}{[\beta c(1+i)^{1-\gamma}]^{\frac{1}{\gamma}} + 1} \right) \right]^{1-\gamma} \\
&= x^{1-\gamma} \left( \frac{1}{[\beta c(1+i)^{1-\gamma}]^{\frac{1}{\gamma}} + 1} \right) + x^{1-\gamma} (1+i)^{1-\gamma} \beta c \left( 1 - \frac{1}{[\beta c(1+i)^{1-\gamma}]^{\frac{1}{\gamma}} + 1} \right)^{1-\gamma} \\
&= x^{1-\gamma} \left[ \left( \frac{1}{[\beta c(1+i)^{1-\gamma}]^{\frac{1}{\gamma}} + 1} \right) + (1+i)^{1-\gamma} \beta c \left( 1 - \frac{1}{[\beta c(1+i)^{1-\gamma}]^{\frac{1}{\gamma}} + 1} \right)^{1-\gamma} \right].
\end{aligned}$$

Entonces

$$\begin{aligned}
c &= \left( \frac{1}{[\beta c(1+i)^{1-\gamma}]^{\frac{1}{\gamma}} + 1} \right)^{1-\gamma} + (1+i)^{1-\gamma} \beta c \left( 1 - \frac{1}{[\beta c(1+i)^{1-\gamma}]^{\frac{1}{\gamma}} + 1} \right)^{1-\gamma} \\
&= \left( \frac{1}{[\beta c(1+i)^{1-\gamma}]^{\frac{1}{\gamma}} + 1} \right)^{1-\gamma} + (1+i)^{1-\gamma} \beta c \left( \frac{[\beta c(1+i)^{1-\gamma}]^{\frac{1}{\gamma}}}{[\beta c(1+i)^{1-\gamma}]^{\frac{1}{\gamma}} + 1} \right)^{1-\gamma} \\
&= \left( \frac{1}{[\beta c(1+i)^{1-\gamma}]^{\frac{1}{\gamma}} + 1} \right)^{1-\gamma} \left( 1 + (1+i)^{1-\gamma} \beta c \left( [\beta c(1+i)^{1-\gamma}]^{\frac{1}{\gamma}} \right)^{1-\gamma} \right) \\
&= \left( \frac{1}{[\beta c(1+i)^{1-\gamma}]^{\frac{1}{\gamma}} + 1} \right)^{1-\gamma} \left( 1 + (1+i)^{1-\gamma} \beta c [\beta c(1+i)^{1-\gamma}]^{\frac{1}{\gamma}-1} \right) \\
&= \left( \frac{1}{[\beta c(1+i)^{1-\gamma}]^{\frac{1}{\gamma}} + 1} \right)^{1-\gamma} \left( 1 + [\beta c(1+i)^{1-\gamma}]^{\frac{1}{\gamma}} \right) \\
&= \left( [\beta c(1+i)^{1-\gamma}]^{\frac{1}{\gamma}} + 1 \right)^{\gamma-1} \left( 1 + [\beta c(1+i)^{1-\gamma}]^{\frac{1}{\gamma}} \right) \\
c &= \left( [\beta c(1+i)^{1-\gamma}]^{\frac{1}{\gamma}} + 1 \right)^{\gamma}
\end{aligned}$$



Ahora, nos queda despejar  $c$ .

$$\begin{aligned} c^{\frac{1}{\gamma}} &= \beta^{\frac{1}{\gamma}} c^{\frac{1}{\gamma}} (1+i)^{\frac{1}{\gamma}-1} + 1 \\ 1 &= \beta^{\frac{1}{\gamma}} (1+i)^{\frac{1}{\gamma}-1} + c^{-\frac{1}{\gamma}} \\ c^{-\frac{1}{\gamma}} &= 1 - \beta^{\frac{1}{\gamma}} (1+i)^{\frac{1}{\gamma}-1} \\ c &= \left(1 - \beta^{\frac{1}{\gamma}} (1+i)^{\frac{1}{\gamma}-1}\right)^{-\gamma} \end{aligned}$$

Ejercicio 6

$$\begin{aligned} \nu(x) &= \max_{a \in A(x)} \{c(x, a) + \nu(f(x, a))\} \\ &= \max_{a \in A(x)} \{a^2 + x^2 + E[\nu(x + a + \xi)]\} \end{aligned}$$

Para  $\nu(x) = ax^2 + b$

$$\begin{aligned} \nu(x) &= \max_{a \in A(x)} \{c(x, a) + \beta E[\nu(f(x, a))]\} \\ &= \max_{a \in A(x)} \{A^2 + x^2 + \beta(E[a(f^2(x, a))] + b)\} \\ &= \max_{a \in A(x)} \{A^2 + x^2 + \beta(aE[f^2(x, a)] + b)\} \end{aligned}$$

Notemos que

$$\begin{aligned} E[f^2(x, a)] &= E[(x + A + \xi)^2] \\ &= E[x^2 + A^2 + \xi^2 + 2xA + 2x\xi + 2\xi A] \\ &= x^2 + A^2 + E[\xi^2] + 2xA + 2xE[\xi] + 2AE[\xi] \\ &= x^2 + A^2 + d + 2xA \end{aligned}$$

Entonces

$$\begin{aligned} ax^2 + b &= \max_{a \in A(x)} \{A^2 + x^2 + \beta[a(x^2 + A^2 + d + 2xA) + b]\} \\ &= \max_{a \in A(x)} \{A^2 + x^2 + \beta a(x^2 + A^2 + d + 2xA) + \beta b\} \\ &= \max_{a \in A(x)} \{A^2 + x^2 + \beta ax^2 + \beta aA^2 + \beta ad + 2\beta axA + \beta b\} \\ &= \max_{a \in A(x)} \{A^2(\beta a + 1) + 2\beta axA + x^2 + \beta ax^2 + \beta ad + \beta b\} \end{aligned}$$

Definimos

$$w(x, A) = A^2 (\beta a + 1) + 2\beta axA + x^2 + \beta ax^2 + \beta ad + \beta b,$$

entonces

$$\partial_A w = 2A (\beta a + 1) + 2\beta ax.$$

Si  $\partial_A w = 0$ , entonces

$$A = -\frac{\beta ax}{\beta a + 1}$$

Entonces

$$\begin{aligned} \nu(x) &= (\beta ax)^2 - 2\frac{(\beta ax)^2}{\beta a + 1} + x^2 + \beta ax^2 + \beta ad + \beta b \\ &= x^2 \left( [\beta a]^2 - 2\frac{(\beta a)^2}{\beta a + 1} + 1 + \beta a \right) + \beta ad + \beta b \end{aligned}$$

Entonces

$$\begin{aligned} a &= [\beta a]^2 - 2\frac{(\beta a)^2}{\beta a + 1} + 1 + \beta a \\ b &= \beta ad + \beta b, \end{aligned}$$

de forma rapida

$$b = \frac{\beta ad}{1 - \beta},$$

entonces queda pendiente calcular  $a$

$$\begin{aligned} a &= [\beta a]^2 - 2\frac{(\beta a)^2}{\beta a + 1} + 1 + \beta a. \\ 0 &= (\beta a)^2 \left( 1 - \frac{2}{\beta a + 1} \right) + 1 + (\beta - 1) a \\ &= (\beta a)^2 (\beta a + 1 - 2) + \beta a + 1 + (a\beta - a) (\beta a + 1) \\ &= (\beta a)^2 (\beta a - 1) + \beta a + 1 + [(a\beta)^2 + a\beta - \beta a^2 - a] \\ &= (\beta a)^3 + 2a\beta + 1 - \beta a^2 - a \\ &= \beta^3 a^3 - \beta a^2 + (2\beta - 1) a + 1 \end{aligned}$$

# 1 Introduction

This is a book created from markdown and executable code.

See Knuth (1984) for additional discussion of literate programming.

## 2 Summary

In summary, this book has no content whatsoever.

## References

Knuth, Donald E. 1984. “Literate Programming.” *Comput. J.* 27 (2): 97–111. <https://doi.org/10.1093/comjnl/27.2.97>.