ReinforcedLearningClass

Norah Jones

2024-09-09

Table of contents

Preface

This is a Quarto book.

To learn more about Quarto books visit https://quarto.org/docs/books.

1 Tarea 1 (Fecha de Entrega 20 Septiembre 2024 12:00:00)

Exercise 1.1. Read (Sec 1.1, pp 1-2 Sutton and Barto 2018) and answer the following. Explain why Reinforcement Learning differs for supervised and unsupervised.

El aprendizaje supervisado requiere de ejemplos de soluciones. Mientras que el reforzado requiere una función de valor.

Exercise 1.2. See the first Brunton's youtube about Reinforced Learning. Then acordingly to its presentation explain what is the meaning of the following expression.

$$V_\pi(s) = E\left(\sum_t \gamma^t r_t \mid s_0 = s\right)$$

La expresión presentada en el video Reinforcement Learning.

$$V_\pi(s) = E\left[\sum_t \gamma^t r_t \middle| s_0 = s\right]$$

hace referencia a la función de valor del problema de optimización representada por la recompensa esperada dado la politica π y el estado inicial s. Aquí γ es el factor de descuento y r_t es la recompensa por etapa t.

Exercise 1.3. Form (see Sutton and Barto 2018, sec. 1.7) obtain a time line pear year from 1950 to 2012.

```r
library(devtools)
```

```
Warning: package 'devtools' was built under R version 4.3.3
```

```
Loading required package: usethis
```

```
Warning: package 'usethis' was built under R version 4.3.3
```

```r
library(milestones)

library(tidyverse)
```

```
Warning: package 'ggplot2' was built under R version 4.3.3
```

```
-- Attaching core tidyverse packages ------------------------ tidyverse 2.0.0 --
v dplyr     1.1.4     v readr     2.1.5
v forcats   1.0.0     v stringr   1.5.1
v ggplot2   3.5.1     v tibble    3.2.1
v lubridate 1.9.3     v tidyr     1.3.1
v purrr     1.0.2
```

```
-- Conflicts ------------------------------------------ tidyverse_conflicts() --
x dplyr::filter() masks stats::filter()
x dplyr::lag()    masks stats::lag()
i Use the conflicted package (<http://conflicted.r-lib.org/>) to force all conflicts to becom
```

```r
library(gt)
```

```
Warning: package 'gt' was built under R version 4.3.3
```

```r
library(bibtex)
```

```
Warning: package 'bibtex' was built under R version 4.3.3
```

```r
## Activate the Core Packages
biblio <- bibtex::read.bib("references.bib")

## Initialize defaults
column <- lolli_styles()

data <- read_csv(col_names=TRUE, show_col_types=FALSE, file='rl_time_line.csv')


data <- data |> arrange(date)
```

```r
## Build a table
gt(data) |>
  #cols_hide(columns = event) |>
  tab_style(cell_text(v_align = "top"),
            locations = cells_body(columns = date)) |>
  tab_source_note(source_note = "Source: Sutton and Barto (2018)")
```

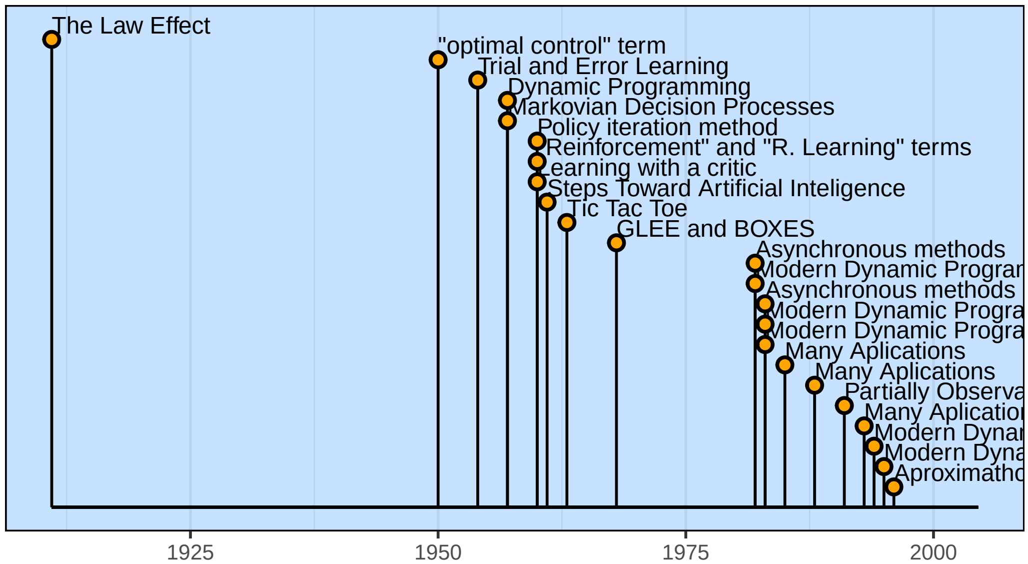
date	event	reference
1911	The Law Effect	Thorndike, 1911
1950	”optimal control” term	NA
1954	Trial and Error Learning	Minsky,Farley, Clark 1954
1957	Dynamic Programming	Bellman 1957a
1957	Markovian Decision Processes	Bellman 1957b
1960	Policy iteration method	Ron Howard (1960)
1960	”Reinforcement” and ”R. Learning” terms	NA
1960	Learning with a critic	Hoff, 1960
1961	Steps Toward Artificial Inteligence	Minsky, 1961
1963	Tic Tac Toe	Donald Michie, 1961
1968	GLEE and BOXES	Michie and Chambers 1968
1982	Asynchronous methods	Bertsekas, 1982
1982	Modern Dynamic Programming	White 1982
1983	Asynchronous methods	Bertsekas, 1983
1983	Modern Dynamic Programming	Ross, 1983
1983	Modern Dynamic Programming	White 1983
1985	Many Aplications	White, 1995
1988	Many Aplications	White, 1998
1991	Partially Observable MDPs	Lovejoy, 1991
1993	Many Aplications	White 1993
1994	Modern Dynamic Programming	Puterman, 1994
1995	Modern Dynamic Programming	Bertsekas, 1995
1996	Aproximathon methods	Rust 1996

Source: Sutton and Barto (2018)

```r
column$color <- "orange"
column$size  <- 15
column$source_info <- "Source: Sutton and Barto (2018)"

## Milestones timeline
```

```r
milestones(datatable = data, styles = column)
```



Source: Sutton and Barto (2018)

Exercise 1.4. Consider the following comsuption-saving problem with dynamics

$$x_{k+1} = (1+r)(x_k - a_k), k = 0, 1, \dots, N-1$$

and utility function

$$\beta^N (x_N)^{1-\gamma} + \sum_{k=0}^{N-1} \beta^k (a_k)^{1-\gamma}.$$

Show that the value functions of the DP alghorithm the form

$$J_k(x) = A_k k \beta^k x^{1-\gamma},$$

where $A_N = 1$ and for $k = N-1, \dots, 0$,

$$A_k = \left[1 + ((1+r)\beta A_{k+1})^{1/\gamma}\right]^{\gamma}.$$

Show also that the optimal policies are $h_k(x) = A^{-1/\gamma} x$, for $k = N-1, \dots, 0$.

Considerando J_N como sigue

$$J_N(x) = \beta^N x^{1-\gamma} K_N,$$

con $K_N = 1$ bajo la hipótesis de que

$$c_k(x,a) = \beta^k a^{1-\gamma}$$

calculamos J_{N-1}.

$$\begin{aligned} J_{N-1}(x) &= \max_{a \in A(x)} \{c_{N-1}(x,a) + J_N((1+i)(x-a))\} \\ &= \max_{a \in A(x)} \left\{\beta^{N-1} a^{1-\gamma} + \beta^N \left((1+i)(x-a)\right)^{1-\gamma}\right\} \end{aligned}$$

Definimos el argumento como una función q.

$$\begin{aligned} q(x,a) &= \beta^{N-1} a^{1-\gamma} + \beta^N \left((1+i)(x-a)\right)^{1-\gamma} \\ &= C_1 a^{1-\gamma} + C_2 \left(x-a\right)^{1-\gamma}, \end{aligned}$$

donde $C_1 = \beta^{N-1}$ y $C_2 = \beta^N (1+i)^{1-\gamma} K_N$. Como q es continua en (x,a). Podemos calcular el máximo mediante el gradiente.

$$\partial_a q = C_1 (1-\gamma) a^{-\gamma} - C_2 (1-\gamma) (x-a)^{-\gamma}.$$

Igualando, $\partial_a q = 0$.

$$\begin{aligned} C_1 a^{-\gamma} &= C_2 \left(x-a\right)^{-\gamma} \\ \frac{C_1}{C_2} &= \left(\frac{x-a}{a}\right)^{-\gamma} \\ \left(\frac{C_1}{C_2}\right)^{-\frac{1}{\gamma}} &= \frac{x}{a} - 1 \\ \left(\frac{C_1}{C_2}\right)^{-\frac{1}{\gamma}} + 1 &= \frac{x}{a} \\ a &= \frac{x}{\left(\frac{C_1}{C_2}\right)^{-\frac{1}{\gamma}} + 1} \end{aligned}$$

Finalmente

$$a = h(x) = \frac{x}{\left(\beta(1+i)^{1-\gamma}\right)^{\frac{1}{\gamma}} + 1}$$

Definiendo $\eta = \left(\beta(1+i)^{1-\gamma}\right)^{\frac{1}{\gamma}} + 1$, $\eta - 1 = \left(\beta(1+i)^{1-\gamma}\right)^{\frac{1}{\gamma}}$

entonces

$$h(x) = \frac{x}{\eta},$$

$$\begin{aligned}
J_{N-1}(x) &= \beta^{N-1}\left(\frac{x}{\eta}\right)^{1-\gamma} + \beta^{N}\left((1+i)\left(x - \frac{x}{\eta}\right)\right)^{1-\gamma} \\
&= \beta^{N-1}x^{1-\gamma}\left(\eta^{\gamma-1} + \beta\left(1+i\right)^{1-\gamma}\left(\frac{\eta-1}{\eta}\right)^{1-\gamma}\right) \\
&= \beta^{N-1}x^{1-\gamma}\eta^{\gamma-1}\left(1 + \beta\left(1+i\right)^{1-\gamma}\left(\eta-1\right)^{1-\gamma}\right) \\
&= \beta^{N-1}x^{1-\gamma}\eta^{\gamma-1}\left(1 + \beta\left(1+i\right)^{1-\gamma}\left(\eta-1\right)^{1-\gamma}\right) \\
&= \beta^{N-1}x^{1-\gamma}\eta^{\gamma-1}\left(1 + \beta\left(1+i\right)^{1-\gamma}\left(\left(\beta(1+i)^{1-\gamma}\right)^{\frac{1}{\gamma}}\right)^{1-\gamma}\right) \\
&= \beta^{N-1}x^{1-\gamma}\eta^{\gamma-1}\left(1 + \beta\left(1+i\right)^{1-\gamma}\left(\beta(1+i)^{1-\gamma}\right)^{\frac{1}{\gamma}-1}\right) \\
&= \beta^{N-1}x^{1-\gamma}\eta^{\gamma-1}\left(1 + \beta^{\frac{1}{\gamma}}\left(1+i\right)^{(1-\gamma)\left(\frac{1}{\gamma}-1\right)+1-\gamma}\right) \\
&= \beta^{N-1}x^{1-\gamma}\eta^{\gamma-1}\left(1 + \beta^{\frac{1}{\gamma}}\left(1+i\right)^{\left(\frac{1}{\gamma}-1\right)}\right) \\
&= \beta^{N-1}x^{1-\gamma}\eta^{\gamma},
\end{aligned}$$

Entonces

$$K_{N-1} = \eta^{\gamma}, h_{k-1}\left(x\right) = \frac{x}{\left(K_{N-1}\right)^{1/\gamma}}$$

Ahora calculamos J_{N-2}

$$\begin{aligned}
J_{N-2}\left(x\right) &= \max_{a\in A(x)}\left\{\beta^{N-2}a^{1-\gamma} + \beta^{N-1}\left[\left(1+i\right)\left(x-a\right)\right]^{1-\gamma}\eta^{\gamma}\right\} \\
&= \max_{a\in A(x)}\left\{q\left(x,a\right)\right\},
\end{aligned}$$

donde

$$q\left(x,a\right) = C_1 a^{1-\gamma} + C_2\left(x-a\right)^{1-\gamma},$$

con $C_1 = \beta^{N-2}$ y $C_2 = \beta^{N-1}\left(1+i\right)^{1-\gamma} K_{N-1}$. Obteniendo, por recursividad

$$\begin{aligned} h_{N-2} &= \frac{x}{\left(\frac{C_1}{C_2}\right)^{-\frac{1}{\gamma}}+1} \\ &= \frac{x}{\left(\frac{1}{\beta\left(1+i\right)^{1-\gamma} K_{N-1}}\right)^{-\frac{1}{\gamma}}+1} \\ &= \frac{x}{\left(\beta\left(1+i\right)^{1-\gamma} K_{N-1}\right)^{\frac{1}{\gamma}}+1} \end{aligned}$$

Entonces, sea

$$\eta' = \left(\beta\left(1+i\right)^{1-\gamma} K_{N-1}\right)^{\frac{1}{\gamma}}+1.$$

Repitiendo, el caso anterior, tenemos que

$$\begin{aligned} J_{N-2}\left(x\right) &= \beta^{N-2} x^{1-\gamma} \eta'^{\gamma-1}\left(1+K_{N-1}\beta\left(1+i\right)^{1-\gamma}\left(\left(\beta(1+i)^{1-\gamma}K_{N-1}\right)^{\frac{1}{\gamma}}\right)^{1-\gamma}\right) \\ &= \beta^{N-2} x^{1-\gamma} \eta'^{\gamma-1}\left(1+K_{N-1}\beta\left(1+i\right)^{1-\gamma}\left(\left(\beta(1+i)^{1-\gamma}K_{N-1}\right)^{\frac{1}{\gamma}}\right)^{1-\gamma}\right) \\ &= \beta^{N-2} x^{1-\gamma} \eta'^{\gamma-1}\left(1+K_{N-1}\beta\left(1+i\right)^{1-\gamma}\left(\beta(1+i)^{1-\gamma}K_{N-1}\right)^{\frac{1}{\gamma}-1}\right) \\ &= \beta^{N-2} x^{1-\gamma} \eta'^{\gamma-1}\left(1+K_{N-1}\beta\left(1+i\right)^{1-\gamma}\left(1+i\right)^{(1-\gamma)\left(\frac{1}{\gamma}-1\right)}K_{N-1}^{\frac{1}{\gamma}-1}\right) \\ &= \beta^{N-2} x^{1-\gamma} \eta'^{\gamma-1}\left(1+K_{N-1}\beta^{1/\gamma}\left(1+i\right)^{\frac{1}{\gamma}-1}K_{N-1}^{\frac{1}{\gamma}-1}\right) \\ &= \beta^{N-2} x^{1-\gamma} \eta'^{\gamma-1}\left(1+\beta^{1/\gamma}\left(1+i\right)^{\frac{1}{\gamma}-1}K_{N-1}^{\frac{1}{\gamma}}\right) \\ &= \beta^{N-2} x^{1-\gamma} \eta'^{\gamma}, \end{aligned}$$

entonces

$$K_{N-2} = \eta'^{\gamma},$$

y

$$h_{N-2} = \frac{x}{K_{N-2}^{1/\gamma}}$$

Por lo tanto, tenemos que

$$K_n = \left(\beta\left(1+i\right)^{1-\gamma} K_{n+1}\right)^{\frac{1}{\gamma}}+1, n = 0, 1, 2, \ldots, N,$$

con $K_N = 1$.

Obteniendo así

$$\begin{aligned} J_n(x) &= \beta^n x^{1-\gamma} K_n \\ h_n(x) &= \frac{x}{K_n^{1/\gamma}} \end{aligned}$$

Exercise 1.5. Consider now the infinite-horizon version of the above compsumption problem.

1. Write down the associated Bellman equation.
2. Argue why a solution to the Bellman equation should be the form

$$v(x) = cx^{1-\gamma},$$

where c is constant. Find the constant c and the stationary optimal policy.

Para el caso infinito. Estamos considerando

$$c(x,a) = a^{1-\gamma}$$

Entonces

$$\nu(x) = \max_{a\in A(x)} \left\{ a^{1-\gamma} + \beta\nu\left((1+i)(x-a)\right)\right\},$$

considerando $\nu(x) = cx^{1-\gamma}$. Entonces

$$\nu(x) = \max_{a\in A(x)} \left\{ a^{1-\gamma} + \beta c\left[(1+i)(x-a)\right]^{1-\gamma}\right\},$$

definimos

$$q(x,a) = a^{1-\gamma} + \beta c\left[(1+i)(x-a)\right]^{1-\gamma},$$

entonces

$$\partial_a q = (1-\gamma)a^{-\gamma} + \beta c(1-\gamma)(1+i)^{1-\gamma}(-1)(x-a)^{-\gamma}.$$

Si $\partial_a q = 0$. Entonces

$$\begin{aligned} a^{-\gamma} &= \beta c(1+i)^{1-\gamma}(x-a)^{-\gamma} \\ \left(\beta c(1+i)^{1-\gamma}\right)^{-1} &= \left(\frac{x-a}{a}\right)^{-\gamma} \\ \beta^{-1}c^{-1}(1+i)^{\gamma-1} &= \left(\frac{x}{a}-1\right)^{-\gamma} \\ \left[\beta^{-1}c^{-1}(1+i)^{\gamma-1}\right]^{-\frac{1}{\gamma}} + 1 &= \frac{x}{a} \end{aligned}$$

Por lo tanto

$$
\begin{aligned}
a &= \frac{x}{\left[\beta^{-1}c^{-1}\left(1+i\right)^{\gamma-1}\right]^{-\frac{1}{\gamma}}+1} \\
&= \frac{x}{\left[\beta c\left(1+i\right)^{1-\gamma}\right]^{\frac{1}{\gamma}}+1}
\end{aligned}
$$

Ahora remplazamos en q

$$
\begin{aligned}
\nu\left(x\right) &= \left(\frac{x}{\left[\beta c\left(1+i\right)^{1-\gamma}\right]^{\frac{1}{\gamma}}+1}\right)^{1-\gamma} + \beta c\left[\left(1+i\right)\left(x-\frac{x}{\left[\beta c\left(1+i\right)^{1-\gamma}\right]^{\frac{1}{\gamma}}+1}\right)\right]^{1-\gamma} \\
&= x^{1-\gamma}\left(\frac{1}{\left[\beta c\left(1+i\right)^{1-\gamma}\right]^{\frac{1}{\gamma}}+1}\right) + x^{1-\gamma}\left(1+i\right)^{1-\gamma}\beta c\left(1-\frac{1}{\left[\beta c\left(1+i\right)^{1-\gamma}\right]^{\frac{1}{\gamma}}+1}\right)^{1-\gamma} \\
&= x^{1-\gamma}\left[\left(\frac{1}{\left[\beta c\left(1+i\right)^{1-\gamma}\right]^{\frac{1}{\gamma}}+1}\right) + \left(1+i\right)^{1-\gamma}\beta c\left(1-\frac{1}{\left[\beta c\left(1+i\right)^{1-\gamma}\right]^{\frac{1}{\gamma}}+1}\right)^{1-\gamma}\right].
\end{aligned}
$$

Entonces

$$
\begin{aligned}
c &= \left(\frac{1}{\left[\beta c\left(1+i\right)^{1-\gamma}\right]^{\frac{1}{\gamma}}+1}\right)^{1-\gamma} + \left(1+i\right)^{1-\gamma}\beta c\left(1-\frac{1}{\left[\beta c\left(1+i\right)^{1-\gamma}\right]^{\frac{1}{\gamma}}+1}\right)^{1-\gamma} \\
&= \left(\frac{1}{\left[\beta c\left(1+i\right)^{1-\gamma}\right]^{\frac{1}{\gamma}}+1}\right)^{1-\gamma} + \left(1+i\right)^{1-\gamma}\beta c\left(\frac{\left[\beta c\left(1+i\right)^{1-\gamma}\right]^{\frac{1}{\gamma}}}{\left[\beta c\left(1+i\right)^{1-\gamma}\right]^{\frac{1}{\gamma}}+1}\right)^{1-\gamma} \\
&= \left(\frac{1}{\left[\beta c\left(1+i\right)^{1-\gamma}\right]^{\frac{1}{\gamma}}+1}\right)^{1-\gamma}\left(1+\left(1+i\right)^{1-\gamma}\beta c\left(\left[\beta c\left(1+i\right)^{1-\gamma}\right]^{\frac{1}{\gamma}}\right)^{1-\gamma}\right) \\
&= \left(\frac{1}{\left[\beta c\left(1+i\right)^{1-\gamma}\right]^{\frac{1}{\gamma}}+1}\right)^{1-\gamma}\left(1+\left(1+i\right)^{1-\gamma}\beta c\left[\beta c\left(1+i\right)^{1-\gamma}\right]^{\frac{1}{\gamma}-1}\right) \\
&= \left(\frac{1}{\left[\beta c\left(1+i\right)^{1-\gamma}\right]^{\frac{1}{\gamma}}+1}\right)^{1-\gamma}\left(1+\left[\beta c\left(1+i\right)^{1-\gamma}\right]^{\frac{1}{\gamma}}\right) \\
&= \left(\left[\beta c\left(1+i\right)^{1-\gamma}\right]^{\frac{1}{\gamma}}+1\right)^{\gamma-1}\left(1+\left[\beta c\left(1+i\right)^{1-\gamma}\right]^{\frac{1}{\gamma}}\right) \\
c &= \left(\left[\beta c\left(1+i\right)^{1-\gamma}\right]^{\frac{1}{\gamma}}+1\right)^{\gamma}
\end{aligned}
$$

Ahora, nos queda despejar c.

$$
\begin{aligned}
c^{\frac{1}{\gamma}} &= \beta^{\frac{1}{\gamma}}c^{\frac{1}{\gamma}}\left(1+i\right)^{\frac{1}{\gamma}-1}+1 \\
1 &= \beta^{\frac{1}{\gamma}}\left(1+i\right)^{\frac{1}{\gamma}-1}+c^{-\frac{1}{\gamma}} \\
c^{-\frac{1}{\gamma}} &= 1-\beta^{\frac{1}{\gamma}}\left(1+i\right)^{\frac{1}{\gamma}-1} \\
c &= \left(1-\beta^{\frac{1}{\gamma}}\left(1+i\right)^{\frac{1}{\gamma}-1}\right)^{-\gamma}
\end{aligned}
$$

Exercise 1.6. Let $\{\xi_k\}$ be a dynamics of iid random variables such that $E\left[\xi\right]=0$ and $E\left[\xi^2\right]=d$. Consider the dynamics

$$x_{k+1}=x_k+a_k+\xi_k, k=0,1,2,\ldots,$$

and the discounted cost

$$E\left[\sum\beta^k\left(a_k^2+x_k^2\right)\right]$$

1. Write down the associated Bellman equation.
2. Conjecture that the solution to the Bellman equation takes the form $v(x) = ax^2 + b$, where a and b are constant.
3. Determine the constants a and b.
4. Conjecture that the solution to the Bellman equation takes the form $v(x) = ax^2 + b$, where a and b are constant. Determine the constants a and b.

$$\begin{aligned}
\nu(x) &= \max_{a\in A(x)} \{c(x,a) + E[\nu(f(x,a))]\} \\
&= \max_{a\in A(x)} \left\{a^2 + x^2 + E[\nu(x+a+\xi)]\right\}
\end{aligned}$$

Para $\nu(x) = ax^2 + b$

$$\begin{aligned}
\nu(x) &= \max_{a\in A(x)} \{c(x,a) + \beta E[\nu(f(x,a))]\} \\
&= \max_{a\in A(x)} \left\{A^2 + x^2 + \beta\left(E\left[a\left(f^2(x,a)\right)\right] + b\right)\right\} \\
&= \max_{a\in A(x)} \left\{A^2 + x^2 + \beta\left(aE\left[f^2(x,a)\right] + b\right)\right\}
\end{aligned}$$

Notemos que

$$\begin{aligned}
E\left[f^2(x,A)\right] &= E\left[(x+A+\xi)^2\right] \\
&= E\left[x^2 + A^2 + \xi^2 + 2xA + 2x\xi + 2\xi A\right] \\
&= x^2 + A^2 + E\left[\xi^2\right] + 2xA + 2xE[\xi] + 2AE[\xi] \\
&= x^2 + A^2 + d + 2xA
\end{aligned}$$

Entonces

$$\begin{aligned}
ax^2 + b &= \max_{a\in A(x)} \left\{A^2 + x^2 + \beta\left[a\left(x^2 + A^2 + d + 2xA\right) + b\right]\right\} \\
&= \max_{a\in A(x)} \left\{A^2 + x^2 + \beta a\left(x^2 + A^2 + d + 2xA\right) + \beta b\right\} \\
&= \max_{a\in A(x)} \left\{A^2 + x^2 + \beta ax^2 + \beta aA^2 + \beta ad + 2\beta axA + \beta b\right\} \\
&= \max_{a\in A(x)} \left\{A^2(\beta a + 1) + 2\beta axA + x^2 + \beta ax^2 + \beta ad + \beta b\right\}
\end{aligned}$$

Definimos

$$w(x,A) = A^2(\beta a + 1) + 2\beta axA + x^2 + \beta ax^2 + \beta ad + \beta b,$$

entonces

$$\partial_A w = 2A\left(\beta a + 1\right) + 2\beta a x.$$

Si $\partial_A w = 0$, entonces

$$A = -\frac{\beta a x}{\beta a + 1}$$

Entonces

$$\begin{aligned}
\nu\left(x\right) &= \left(\beta a x\right)^2 - 2\frac{\left(\beta a x\right)^2}{\beta a + 1} + x^2 + \beta a x^2 + \beta a d + \beta b \\
&= x^2\left(\left[\beta a\right]^2 - 2\frac{\left(\beta a\right)^2}{\beta a + 1} + 1 + \beta a\right) + \beta a d + \beta b
\end{aligned}$$

Entonces

$$\begin{aligned}
a &= \left[\beta a\right]^2 - 2\frac{\left(\beta a\right)^2}{\beta a + 1} + 1 + \beta a \\
b &= \beta a d + \beta b,
\end{aligned}$$

de forma rapida

$$b = \frac{\beta a d}{1 - \beta},$$

entonces queda pendiente calcular a

$$\begin{aligned}
a &= \left[\beta a\right]^2 - 2\frac{\left(\beta a\right)^2}{\beta a + 1} + 1 + \beta a. \\
0 &= \left(\beta a\right)^2\left(1 - \frac{2}{\beta a + 1}\right) + 1 + \left(\beta - 1\right)a \\
&= \left(\beta a\right)^2\left(\beta a + 1 - 2\right) + \beta a + 1 + \left(a\beta - a\right)\left(\beta a + 1\right) \\
&= \left(\beta a\right)^2\left(\beta a - 1\right) + \beta a + 1 + \left[\left(a\beta\right)^2 + a\beta - \beta a^2 - a\right] \\
&= \left(\beta a\right)^3 + 2a\beta + 1 - \beta a^2 - a \\
&= \beta^3 a^3 - \beta a^2 + \left(2\beta - 1\right)a + 1
\end{aligned}$$

2 Tarea 2

References