mma_hws

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Lista de Tareas.

Este libro es una recopilación de las tareas presentadas a lo largo del curso de Métodos de Matemáticas Aplicadas. Enfocada principalmente sobre Cálculo Estocástico.

1 Codigos Usados en las Tareas. (Se encuentran en la carpeta PythonScripts)

1.1 Movimiento Browniano

```
import numpy as np

def u(t: float, n_points: int):
    dt = t / (n_points - 1)
    dw = np.sqrt(dt) * np.random.standard_normal(n_points - 1)
    w = np.zeros(n_points)
    w[1:] = dw.cumsum()
    time = np.linspace(0, t, n_points)
    return time, w
```

1.2 Intregral de Ito

```
import numpy as np
def f(x: float, t: float):
    y = x ** 2
    return y


def fB(partition: np.array, x: float, t: float):
    y = 0
    for i in range(len(partition) - 1):
        if partition[i] <= t < partition[i + 1]:
            y = f(x, t)
    return y

def ito_n(n_points: int, t: float):
    time, w = bw.u(t, n_points)
```

```
    integral = np.zeros(n_points)
    for i in range(n_points - 1):
        integral[i] = fB(time, w[i], time[i]) * (w[i + 1] - w[i])
    ito = integral.sum()
    return w, ito
```

1.3 Integral de Riemann

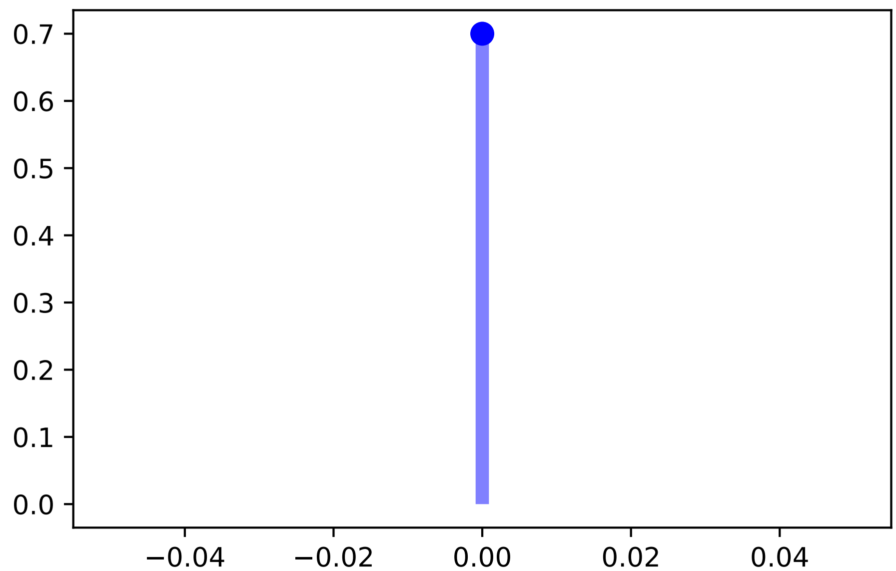
```
import numpy as np

def v(t: float):
    y = np.exp( - t)
    return y

def riemann_integral(a, b, n_points):
    r = np.zeros(n_points)
    time = np.linspace(a,b,n_points)
    for i in range(n_points - 1):
        r[i] = v(time[i]) * (time[i + 1] - time[i])
    riemann = r.sum()
    return riemann
```

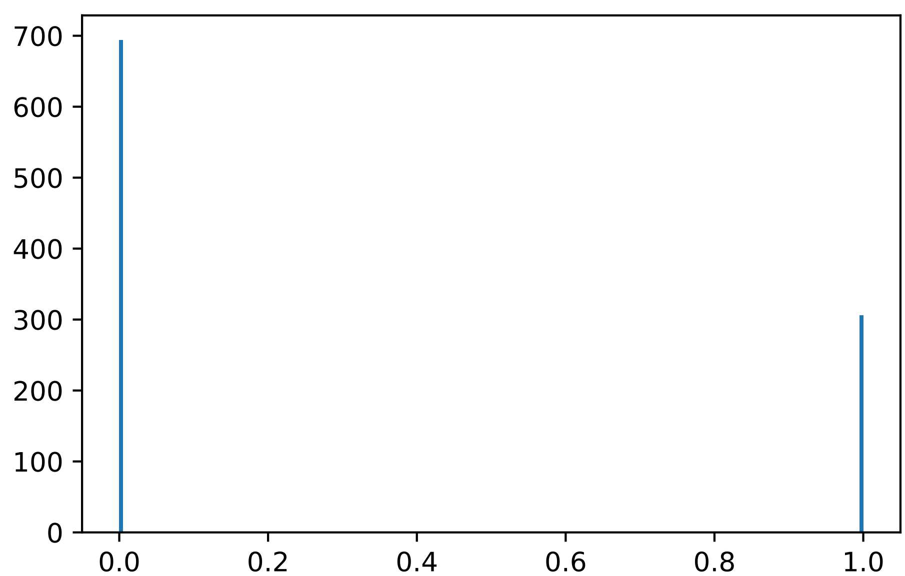
2 Tarea 1

Exercise 2.1. Ejecute y explica en pocas palabras la salida del código ex_001.py

```
from scipy.stats import multivariate_normal
from mpl_toolkits.mplot3d import axes3d
from scipy.stats import norm
import numpy as np
from scipy.stats import bernoulli
import matplotlib.pyplot as plt
fig_01, ax_01 = plt.subplots(1, 1)
fig_02, ax_02 = plt.subplots(1, 1)
p = 0.3
mean, var, skew, kurt = bernoulli.stats(p, moments='mvsk')
print(mean, var, skew,kurt)

x = np.arange(bernoulli.ppf(0.01, p), bernoulli.ppf(0.99, p))
ax_01.plot(x, bernoulli.pmf(x, p), 'bo', ms=8, label='bernoulli pmf')
ax_01.vlines(x, 0, bernoulli.pmf(x, p), colors='b', lw=5, alpha=0.5)
r = bernoulli.rvs(p, size=1000)
ax_02.hist(r, bins=200)
plt.show()
```

```
0.3 0.21 0.8728715609439694 -1.2380952380952381
```





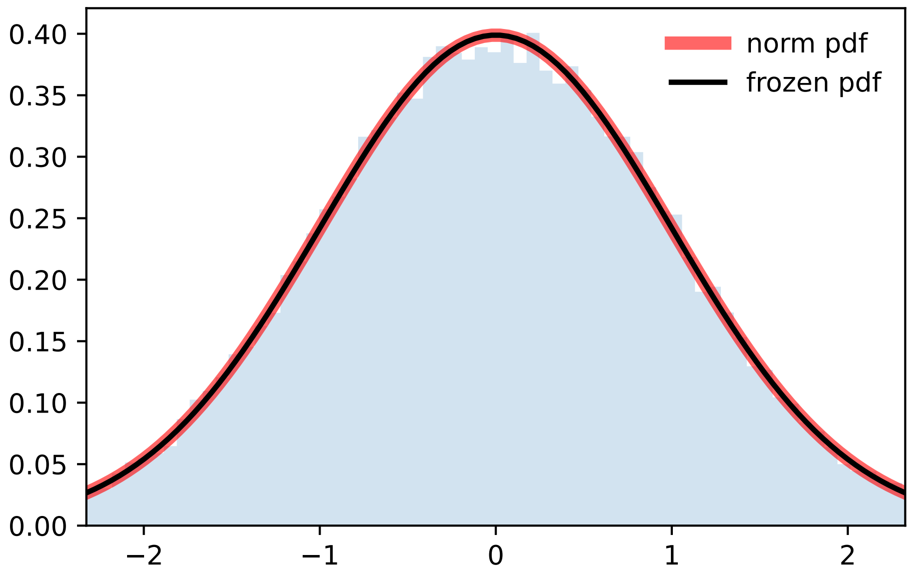
El código posee 3 salidas: * Un vector $[0.3, 0.21, 0.87, -1.23]$ * Dos gráficas.

El vector hace referencia a los momentos de la distribución bernoulli con parámetro $p = 0.3$. * mean hace referencia a la media. * var hace referencia a la varianza. * skew hace referencia al sesgo. * kurt hace referencia a la kurtosis.

Finalmente, las dos gráficas: * La primera hace referencia a la función de probabilidad. Notemos que $\mathcal{P}[X = 0] = 0.7$, lo que muestra la gráfica. Notemos que la gráfica va de -0.04 a 0.04, por lo tanto no se iba a mostrar el caso $X = 1$.
* La segunda hace referencia a una simulación: Se generaron una muestra de tamaño N variables aleatorias con distribución bernoulli. Como la distribución bernoulli tiene media Np, pretende mostrar que en efecto, habrá de forma aproximada Np valores igual a 1 y $N(1-p)$ valores igual a 0.

Exercise 2.2. Ejecute y explica en pocas palabras la salida del código ex_002.py

```
fig, ax = plt.subplots(1, 1)
mean, var, skew, kurt = norm.stats(moments='mvsk')

x = np.linspace(norm.ppf(0.01), norm.ppf(0.99), 100)
ax.plot(
    x,
    norm.pdf(x),
    'r-',
    lw=5,
    alpha=0.6,
    label='norm pdf'
)
rv = norm()
ax.plot(x, rv.pdf(x), 'k-', lw=2, label='frozen pdf')
vals = norm.ppf([0.001, 0.5, 0.999])

np.allclose([0.001, 0.5, 0.999], norm.cdf(vals))

r = norm.rvs(size=50000)

ax.hist(r, density=True, bins='auto', histtype='stepfilled', alpha=0.2)
ax.set_xlim([x[0], x[-1]])
ax.legend(loc='best', frameon=False)
plt.show()
```



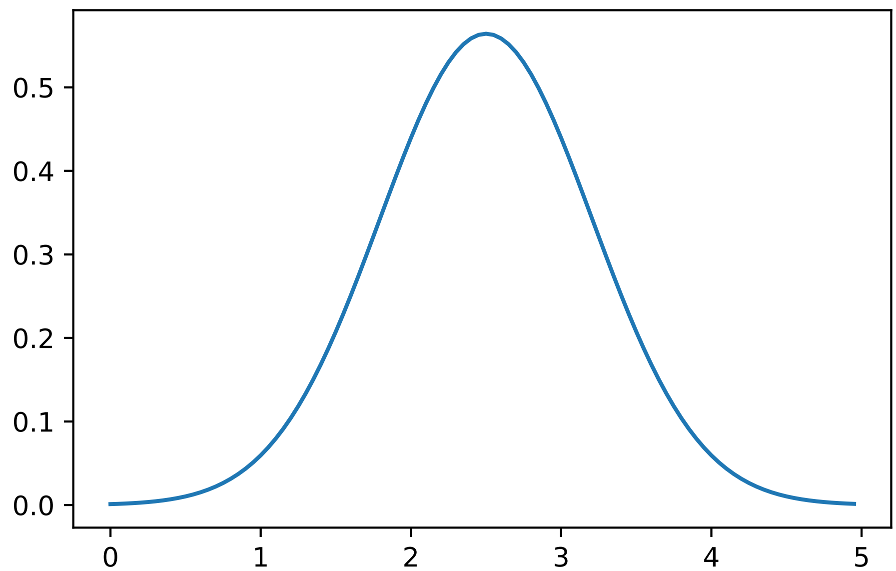
El código posee una gráfica. Que hace referencia a una simulación de variables aleatorias normales. Notemos que * El elemento en azul, hace referencia a un histograma que refleja las frecuencias de los valores generados. * Mientras que la linea roja, muestra la función de densidad de una variable aleatoria estándar.

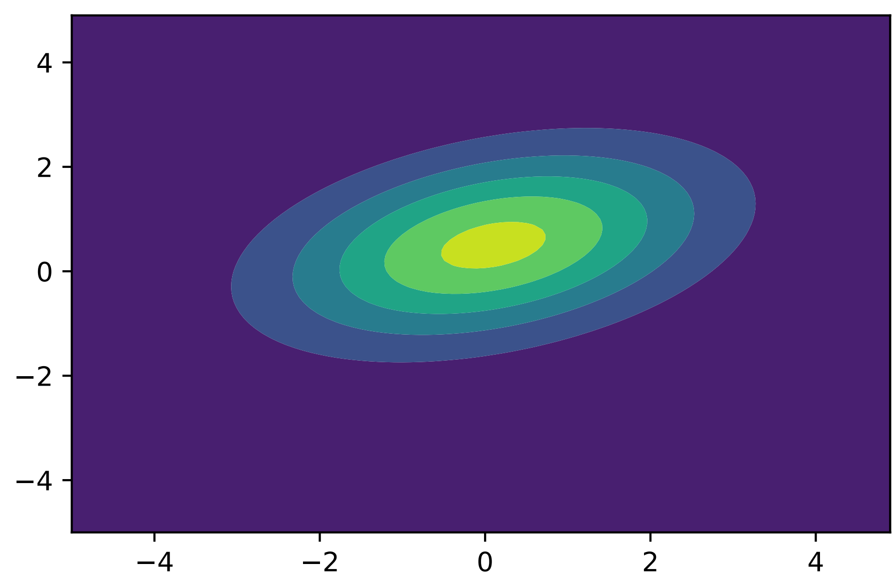
Exercise 2.3. Ejecute y explica en pocas palabras la salida del código ex_003.py

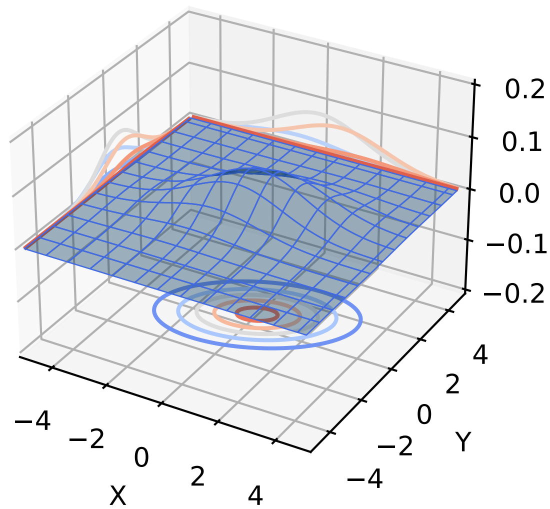
Para cambiar el vector de medias μ y la matriz Σ hay que prestar atención en la linea donde aparece la función `multivariate_normal()` que de forma simple posee dos parámetros: * El vector de medias $\mu = [0.5, -0.2]$ * La matriz de covarianza $\Sigma = [[2.0, 0.3], [0.3, 0.5]]$

```
x = np.linspace(0, 5, 100, endpoint=False)
y = multivariate_normal.pdf(x, mean=2.5, cov=0.5)

fig1 = plt.figure()
ax = fig1.add_subplot(111)
ax.plot(x, y)
# plt.show()

x, y = np.mgrid[-5:5:.1, -5:5:.1]
pos = np.dstack((x, y))
rv = multivariate_normal([0.1, 0.5], [[3.0, 0.3], [0.75, 1.5]])
fig2 = plt.figure()
ax2 = fig2.add_subplot(111)
```

```python
ax2.contourf(x, y, rv.pdf(pos))
# plt.show()

ax = plt.figure().add_subplot(projection='3d')
ax.plot_surface(
    x,
    y,
    rv.pdf(pos),
    edgecolor='royalblue',
    lw=0.5,
    rstride=8,
    cstride=8,
    alpha=0.4
)
ax.contour(x, y, rv.pdf(pos), zdir='z', offset=-.2, cmap='coolwarm')
ax.contour(x, y, rv.pdf(pos), zdir='x', offset=-5, cmap='coolwarm')
ax.contour(x, y, rv.pdf(pos), zdir='y', offset=5, cmap='coolwarm')

ax.set(
    xlim=(-5, 5),
    ylim=(-5, 5),
    zlim=(-0.2, 0.2),
    xlabel='X',
    ylabel='Y',
    zlabel='Z'
)
plt.show()
```


0.5
0.4
0.3
0.2
0.1
0.0
0
1
2
3
4
5


4
2
0
−2
−4
−4
−2
0
2
4

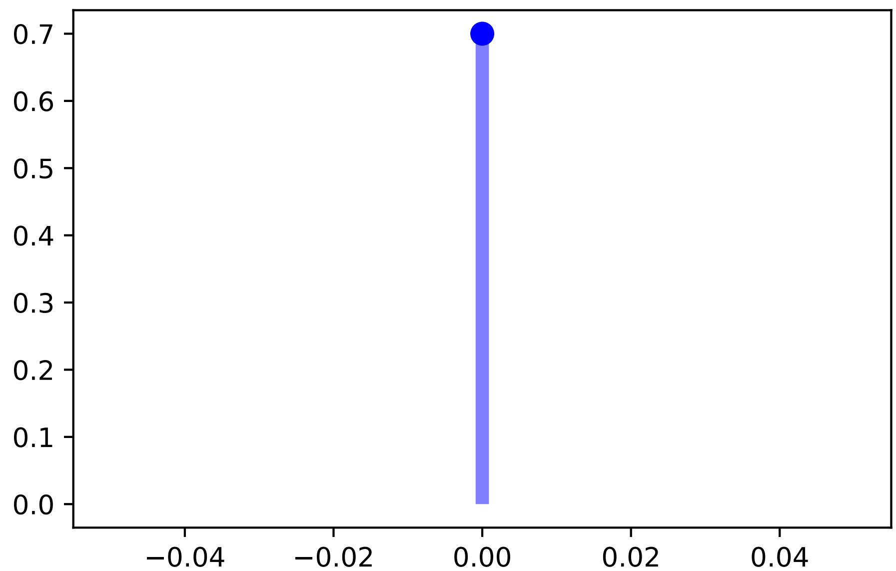


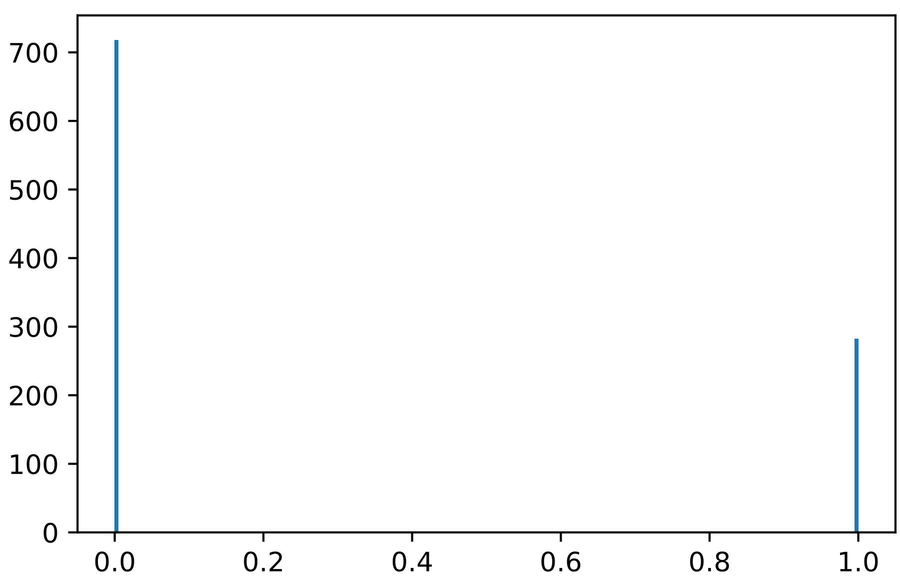
Exercise 2.4. Generando Normales

```
import numpy as np
from scipy.stats import bernoulli
import matplotlib.pyplot as plt

fig_01, ax_01 = plt.subplots(1, 1)
fig_02, ax_02 = plt.subplots(1, 1)

p = 0.3

x = np.arange(bernoulli.ppf(0.01, p), bernoulli.ppf(0.99, p))
ax_01.plot(x, bernoulli.pmf(x, p), 'bo', ms = 8, label = 'bernoulli pmf')
ax_01.vlines(x, 0, bernoulli.pmf(x, p), colors = 'b', lw = 5, alpha = 0.5)

r = bernoulli.rvs(p, size = 1000)
ax_02.hist(r, bins = 200)
```

```
(array([718.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,
          0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,
          0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,
          0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,
          0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,
          0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,
```

```
        0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,
        0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,
        0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,
        0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,
        0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,
        0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,
        0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,
        0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,
        0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,
        0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,
        0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,
        0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,   0.,
        0., 282.]),
array([0.   , 0.005, 0.01 , 0.015, 0.02 , 0.025, 0.03 , 0.035, 0.04 ,
       0.045, 0.05 , 0.055, 0.06 , 0.065, 0.07 , 0.075, 0.08 , 0.085,
       0.09 , 0.095, 0.1  , 0.105, 0.11 , 0.115, 0.12 , 0.125, 0.13 ,
       0.135, 0.14 , 0.145, 0.15 , 0.155, 0.16 , 0.165, 0.17 , 0.175,
       0.18 , 0.185, 0.19 , 0.195, 0.2  , 0.205, 0.21 , 0.215, 0.22 ,
       0.225, 0.23 , 0.235, 0.24 , 0.245, 0.25 , 0.255, 0.26 , 0.265,
       0.27 , 0.275, 0.28 , 0.285, 0.29 , 0.295, 0.3  , 0.305, 0.31 ,
       0.315, 0.32 , 0.325, 0.33 , 0.335, 0.34 , 0.345, 0.35 , 0.355,
       0.36 , 0.365, 0.37 , 0.375, 0.38 , 0.385, 0.39 , 0.395, 0.4  ,
       0.405, 0.41 , 0.415, 0.42 , 0.425, 0.43 , 0.435, 0.44 , 0.445,
       0.45 , 0.455, 0.46 , 0.465, 0.47 , 0.475, 0.48 , 0.485, 0.49 ,
       0.495, 0.5  , 0.505, 0.51 , 0.515, 0.52 , 0.525, 0.53 , 0.535,
       0.54 , 0.545, 0.55 , 0.555, 0.56 , 0.565, 0.57 , 0.575, 0.58 ,
       0.585, 0.59 , 0.595, 0.6  , 0.605, 0.61 , 0.615, 0.62 , 0.625,
       0.63 , 0.635, 0.64 , 0.645, 0.65 , 0.655, 0.66 , 0.665, 0.67 ,
       0.675, 0.68 , 0.685, 0.69 , 0.695, 0.7  , 0.705, 0.71 , 0.715,
       0.72 , 0.725, 0.73 , 0.735, 0.74 , 0.745, 0.75 , 0.755, 0.76 ,
       0.765, 0.77 , 0.775, 0.78 , 0.785, 0.79 , 0.795, 0.8  , 0.805,
       0.81 , 0.815, 0.82 , 0.825, 0.83 , 0.835, 0.84 , 0.845, 0.85 ,
       0.855, 0.86 , 0.865, 0.87 , 0.875, 0.88 , 0.885, 0.89 , 0.895,
       0.9  , 0.905, 0.91 , 0.915, 0.92 , 0.925, 0.93 , 0.935, 0.94 ,
       0.945, 0.95 , 0.955, 0.96 , 0.965, 0.97 , 0.975, 0.98 , 0.985,
       0.99 , 0.995, 1.   ]),
<BarContainer object of 200 artists>)
```



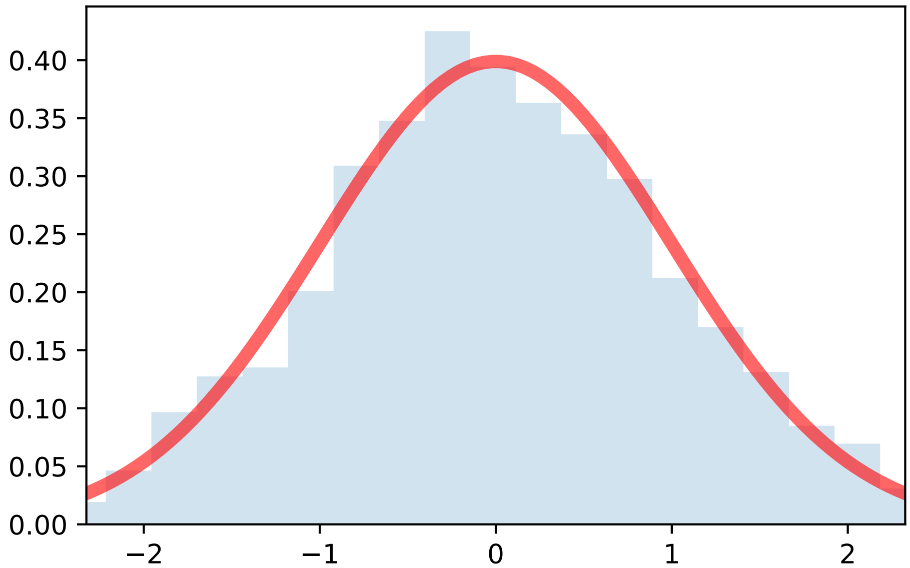



```
from scipy.stats import norm

fig, ax = plt.subplots(1,1)

x = np.linspace(norm.ppf(0.01),norm.ppf(0.99), 100)
```

```
ax.plot(x, norm.pdf(x),'r-', lw = 5, alpha = 0.6)

r = norm.rvs(size = 1000)

ax.hist(r, density = True, bins = 'auto', histtype = 'stepfilled', alpha = 0.2)
ax.set_xlim(x[0], x[-1])
ax.legend(loc='best', frameon = False)
```

```
No artists with labels found to put in legend.  Note that artists whose label start with an u
```

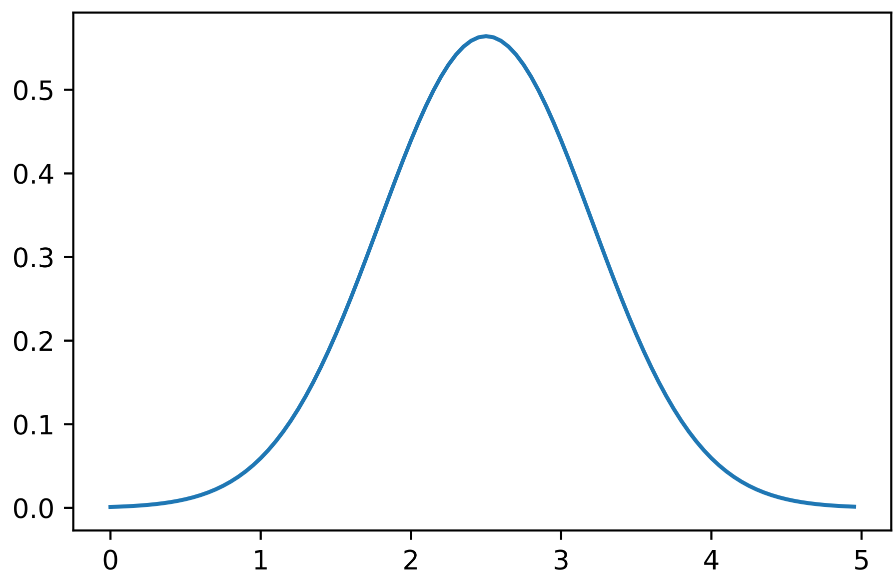
```
<matplotlib.legend.Legend at 0x256704e5960>
```

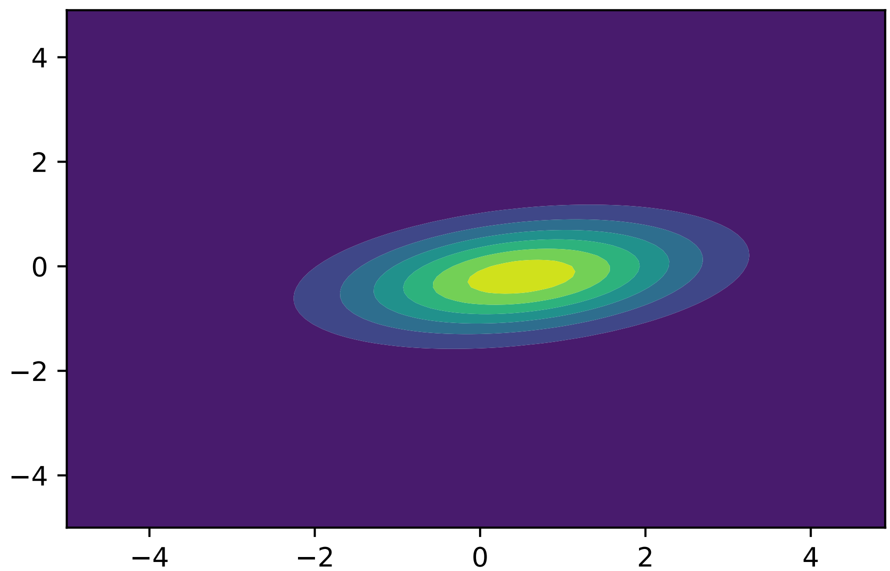


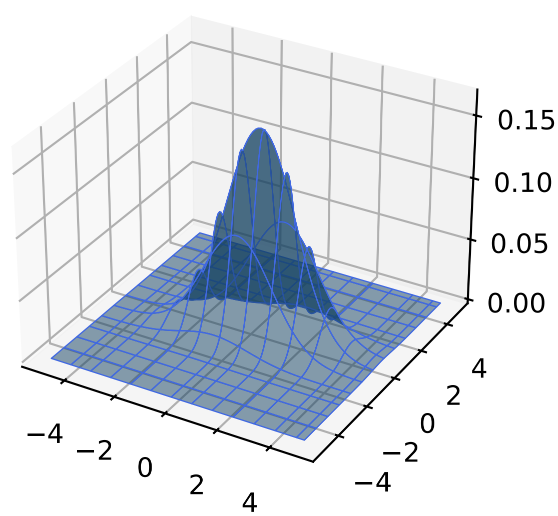

```
from mpl_toolkits.mplot3d import axes3d
from scipy.stats import multivariate_normal

x = np.linspace(0,5,100,endpoint = False)
y = multivariate_normal.pdf(x , mean = 2.5, cov = 0.5)

fig1 = plt.figure()
ax = fig1.add_subplot(111)
ax.plot(x,y)
```

```
x, y = np.mgrid[-5:5:.1, -5:5:.1]

pos = np.dstack((x,y))
rv = multivariate_normal([0.5, -0.2], [[2.0,0.3], [0.3,0.5]])
fig2 = plt.figure()
ax2 = fig2.add_subplot(111)
ax2.contourf(x,y, rv.pdf(pos))

ax = plt.figure().add_subplot(projection = '3d')
ax.plot_surface(x,
                y,
                rv.pdf(pos),
                edgecolor = 'royalblue',
                lw = 0.5,
                rstride = 8,
                cstride = 8,
                alpha = 0.5)
```

```
<mpl_toolkits.mplot3d.art3d.Poly3DCollection at 0x25671a3e320>
```




4
2
0
−2
−4
−4
−2
0
2
4


0.15
0.10
0.05
0.00
−4
−2
0
2
4
4
2
0
−2
−4

3 Tarea 2

Exercise 3.1. Sea X_i v.a.i.i.d tales que

$$\mathcal{P}\left[X_i = h\right] = \mathcal{P}\left[X_i = -h\right] = \frac{1}{2}, \forall i,$$

entonces definimos $Y_{n,h}$.

Queremos calcular la función característica de $Y_{n,\delta}$.

$$E\left[i\lambda Y_{n,\delta}\left(t\right)\right],$$

Aprovechando que para cada X_i son v.a.i.i.d. Entonces, tenemos lo siguiente

$$\begin{aligned} E\left[i\lambda Y_{n,\delta}\left(t\right)\right] &= \left(\cos\left(\lambda h\right)\right)^{t/\delta} \\ &= u^t, \end{aligned}$$

donde

$$\begin{aligned} u &= \left[\cos\left(\lambda h\right)\right]^{1/\delta} \\ \ln\left(u\right) &= \frac{1}{\delta}\ln\left[\cos\left(\lambda h\right)\right] \end{aligned}$$

Entonces, aproximaremos $\cos\left(\lambda h\right)$ con su expansión de Taylor.

$$\cos\left(\lambda h\right) \approx 1 - \frac{\left(\lambda h\right)^2}{2!} + \frac{\left(\lambda h\right)^4}{4!},$$

entonces

$$\begin{aligned} \ln\left(\cos\left(\lambda h\right)\right) &\approx \ln\left[1 - \frac{\left(\lambda h\right)^2}{2} + \frac{\left(\lambda h\right)^4}{4!}\right] \\ &\approx -\frac{\left(\lambda h\right)^2}{2!} + \frac{\left(\lambda h\right)^4}{4!} \end{aligned}$$

Entonces

$$
\begin{aligned}
u^t &\approx \exp\left[\frac{t}{\delta}\left(-\frac{(\lambda h)^2}{2!}+\frac{(\lambda h)^4}{4!}\right)\right],\\
&\approx \exp\left[-\frac{t}{\delta}\left(\frac{\lambda^2 h^2}{2}-\frac{\lambda^4 h^4}{24}\right)\right],
\end{aligned}
$$

Calculando el limite

$$
\lim_{\delta\to 0} E\left[\exp\left(i\lambda Y_{n,\delta}\left(t\right)\right)\right]=\lim_{\delta\to 0}\exp\left[-t\left(\left[\frac{h^2}{\delta}\right]\left(\frac{\lambda^2}{2}-\frac{\lambda^4 h^2}{24}\right)\right)\right],
$$

si $h^2/\delta\to\infty$ Segun la sucesión $\delta_n\to 0$ tenemos limites diferentes, por lo tanto, este no existe. Ahora, usando la normalización, retomando las operaciones anteriores,

$$
\begin{aligned}
E\left[\exp\left(i\lambda Y_{n,\delta}\left(t\right)+\frac{th^2\lambda^2}{2}\right)\right] &= E\left[\exp\left(i\lambda\sum_{i=0}^{n}X_i+\frac{th^2\lambda^2}{2\delta}\right)\right]\\
&= E\left[\exp\left(i\lambda\sum_{i=0}^{n}X_i\right)\right]\exp\left(\frac{th^2\lambda^2}{2\delta}\right)\\
&= \left(\left[\cos\left(\lambda h\right)\right]^{1/\delta}\exp\left(\frac{h^2\lambda^2}{2\delta}\right)\right)^t,
\end{aligned}
$$

entonces,

$$
\begin{aligned}
v &= \left[\cos\left(\lambda h\right)\right]^{1/\delta}\exp\left(\frac{h^2\lambda^2}{2}\right)\\
\ln v &= \ln\left[\left[\cos\left(\lambda h\right)\right]^{1/\delta}\exp\left(\frac{h^2\lambda^2}{2}\right)\right]\\
&= \frac{1}{\delta}\ln\left[\cos\left(\lambda h\right)\right]+\frac{h^2\lambda^2}{2\delta}\\
&= \frac{1}{\delta}\left(\ln\left[\cos\left(\lambda h\right)\right]+\frac{h^2\lambda^2}{2}\right)\\
&\approx \frac{1}{\delta}\left(\ln\left[1\underbrace{-\frac{(\lambda h)^2}{2}+\frac{(\lambda h)^4}{4!}}_{x_0}\right]+\frac{h^2\lambda^2}{2}\right)
\end{aligned}
$$

recordando que

$$
\ln\left(1+x\right)\approx x-\frac{x^2}{2},
$$

entonces

$$\ln v \approx \frac{1}{\delta}\left(\left[-\frac{(\lambda h)^2}{2}+\frac{(\lambda h)^4}{4!}-\frac{\left(-\frac{(\lambda h)^2}{2}+\frac{(\lambda h)^4}{4!}\right)^2}{2}\right]+\frac{h^2\lambda^2}{2}\right),$$

bajo la simplificación de que $o\left(h^k\right) \equiv 0, k \geq 4$, entonces

$$\begin{aligned}
\ln v &\approx \frac{1}{\delta}\left(\frac{(\lambda h)^4}{24}-\frac{(\lambda h)^4}{8}\right)\\
&\approx \frac{1}{\delta}\left(\frac{(\lambda h)^4}{24}-\frac{3\,(\lambda h)^4}{24}\right)\\
v &\approx \exp\left(-\frac{(\lambda h)^4}{12\delta}\right)
\end{aligned}$$

por lo tanto, si $h^4/\delta \to 0$

$$\lim_{\delta\to 0} E\left[\exp\left(i\lambda Y_{n,\delta}(t)+\frac{th^2\lambda^2}{2}\right)\right] = \lim_{\delta\to 0}\exp\left(-\frac{(\lambda h)^4}{12\delta}\right) = 1$$

4 Tarea 3

Exercise 4.1. Si $X \sim N(\mu, \sigma)$ entonces $\left(\dfrac{X-\mu}{\sigma}\right) \sim N(0,1)$.

Calculemos la función característica de la variable $\dfrac{X-\mu}{\sigma}$,

$$\begin{aligned}
\varphi_{\frac{X-\mu}{\sigma}}(t) &= E\left[e^{it\left(\frac{X-\mu}{\sigma}\right)}\right] \\
&= E\left[e^{\left(\frac{itX}{\sigma}-\frac{it\mu}{\sigma}\right)}\right] \\
&= e^{-\frac{it\mu}{\sigma}} E\left[e^{\left(\frac{itX}{\sigma}\right)}\right] \\
&= e^{-\frac{it\mu}{\sigma}} \int_{-\infty}^{\infty} e^{\frac{itx}{\sigma}} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&= e^{-\frac{it\mu}{\sigma}} \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{itx}{\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&= e^{-\frac{it\mu}{\sigma}} \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{itx}{\sigma}-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&= e^{-\frac{it\mu}{\sigma}} \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\frac{(x-\mu)^2-2itx\sigma}{\sigma^2}} dx \qquad (4.1)
\end{aligned}$$

Observemos que,

$$\begin{aligned}
\frac{(x-\mu)^2-2itx\sigma}{\sigma^2} &= \frac{x^2-2x\mu+\mu^2-2itx\sigma}{\sigma^2} \\
&= \frac{x^2}{\sigma^2} - \frac{2x\mu}{\sigma^2} + \frac{\mu^2}{\sigma^2} - \frac{2itx\sigma}{\sigma^2} \\
&= \frac{x^2}{\sigma^2} - \frac{2x}{\sigma}\left(\frac{\mu+it\sigma}{\sigma^2}\right) + \frac{\mu^2}{\sigma^2} \\
&= \left(\frac{x}{\sigma} - \left(\frac{\mu+it\sigma}{\sigma}\right)\right)^2 - \left(\frac{\mu+it\sigma}{\sigma}\right)^2 + \frac{\mu^2}{\sigma^2} \\
&= \left(\frac{x}{\sigma} - \left(\frac{\mu+it\sigma}{\sigma}\right)\right)^2 - \frac{2it\sigma\mu}{\sigma^2} - \frac{(it\sigma)^2}{\sigma^2} \\
&= \left(\frac{x}{\sigma} - \left(\frac{\mu+it\sigma}{\sigma}\right)\right)^2 - \frac{2it\mu}{\sigma} + t^2. \qquad (4.2)
\end{aligned}$$

Sustituyendo (4.2) en (4.1), resulta

$$
\begin{aligned}
\varphi_{\frac{X-\mu}{\sigma}}(t) &= e^{-\frac{it\mu}{\sigma}} \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[\left(\frac{x}{\sigma}-\left(\frac{\mu+it\sigma}{\sigma}\right)\right)^2-\frac{2it\mu}{\sigma}+t^2\right]} dx \\
&= e^{-\frac{it\mu}{\sigma}} e^{\frac{it\mu}{\sigma}-\frac{t^2}{2}} \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x}{\sigma}-\left(\frac{\mu+it\sigma}{\sigma}\right)\right)^2} dx \\
&= e^{-\frac{t^2}{2}} \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x}{\sigma}-\left(\frac{\mu+it\sigma}{\sigma}\right)\right)^2} dx \qquad (4.3)
\end{aligned}
$$

Sea $u = \frac{x}{\sigma} - \left(\frac{\mu+it\sigma}{\sigma}\right) \Longrightarrow du = \frac{1}{\sigma}dx$, sustituyendo esto en (4.3), resulta

$$
\varphi_{\frac{X-\mu}{\sigma}}(t) = e^{-\frac{t^2}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} du \qquad (4.4)
$$

de aquí se sigue que $u \sim N(0,1)$, entonces

$$
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} dx = 1.
$$

sustituyendo esto ultimo en (4.4), se tiene,

$$
\varphi_{\frac{X-\mu}{\sigma}}(t) = e^{-\frac{t^2}{2}},
$$

que es la función característica de una Normal estándar, como las funciones características coinciden se concluye que $\frac{X-\mu}{\sigma} \sim N(0,1)$.

Exercise 4.2. Si $Y \sim N(0,1)$ entonces $\sigma Y + \mu \sim N(\mu, \sigma)$. Calculemos la función característica de la variable $\sigma Y + \mu$,

$$
\begin{aligned}
\varphi_{\sigma Y+\mu}(t) &= E\left[e^{it(\sigma Y+\mu)}\right] \\
&= E\left[e^{it\sigma Y+it\mu}\right] \\
&= e^{it\mu} E\left[e^{it\sigma Y}\right] \\
&= e^{it\mu} \int_{-\infty}^{\infty} e^{it\sigma y} \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}} dy \\
&= e^{it\mu} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(y^2-2yit\sigma)} dy. \qquad (4.5)
\end{aligned}
$$

Observemos que,

$$\begin{aligned} y^2 - 2yit\sigma &= (y - it\sigma)^2 - (it\sigma)^2 \\ &= (y - it\sigma)^2 + t^2\sigma^2. \end{aligned} \tag{4.6}$$

Sustituyendo, (4.6) en (4.5) resulta

$$\begin{aligned} \varphi_{\sigma Y+\mu}(t) &= e^{it\mu}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}((y-it\sigma)^2+t^2\sigma^2)}dy \\ &= e^{it\mu}e^{-\frac{1}{2}t^2\sigma^2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(y-it\sigma)^2}dy \end{aligned} \tag{4.7}$$

Tomando $u = y - it\sigma \Longrightarrow du = dy$, se tiene que

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(y-it\sigma)^2}dy = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{u^2}{2}}du,$$

entonces $U \sim N(0,1)$, por lo tanto,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(y-it\sigma)^2}dy = 1$$

sustituyendo esto ultimo en (4.7), resulta,

$$\varphi_{\sigma Y+\mu}(t) = e^{it\mu}e^{-\frac{1}{2}t^2\sigma^2} = e^{it\mu-\frac{t^2\sigma^2}{2}}.$$

Sea Z una variable aleatoria tal que $Z \sim N(\mu,\sigma)$ sabemos que,

$$\varphi_Z(t) = e^{it\mu-\frac{t^2\sigma^2}{2}}.$$

De estas dos ultimas igualdades se sigue que,

$$\varphi_Z(t) = \varphi_{\sigma Y+\mu}(t).$$

Dado que tienen iguales funciones características se concluye que,

$$\sigma Y + \mu \sim N(\mu,\sigma)$$

Exercise 4.3. Si $X \sim N(\mu_1,\sigma_1^2)$, $Y \sim N(\mu_2,\sigma_2^2)$ además X y Y son independientes entonces $X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$. Por definición, se tiene que,

$$
\begin{aligned}
\varphi_{X+Y}(t) &= E[e^{it(X+Y)}] \\
&= E[e^{itX}e^{itY}] \text{ por ser independientes, del ejercicio 4} \\
&= E[e^{itX}]E[e^{itY}] \\
&= \varphi_X(t)\varphi_Y(t). \qquad (4.8)
\end{aligned}
$$

Por otro lado, sea Z una variables aleatoria tal que, $Z \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$, sabemos que la función característica de Z, esta dada por,

$$
\begin{aligned}
\varphi_Z(t) = &\quad e^{it(\mu_1+\mu_2)-\frac{t^2}{2}(\sigma_1^2+\sigma_2^2)} \\
= &\quad e^{it\mu_1-\frac{t^2\sigma_1^2}{2}+it\mu_2-\frac{t^2\sigma_2^2}{2}} \\
= &\quad e^{it\mu_1-\frac{t^2\sigma_1^2}{2}}e^{it\mu_2-\frac{t^2\sigma_2^2}{2}} \\
= &\quad \varphi_X(t)\varphi_Y(t),
\end{aligned}
$$

entonces, de esta ultima igualdad y de (**??**) se sigue que,

$$\varphi_Z(t) = \varphi_{X+Y}(t).$$

Como las funciones características coinciden se sigue que, $X+Y \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$.

Exercise 4.4. Si X, Y son variables normales independientes. Entonces $E[XY] = E[X]E[Y]$.

Recordemos que

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{XY}(x,y)\,\mathrm{d}x\mathrm{d}y$$

Como X, Y son independientes

$$f_{XY}(x,y) = f_X(x)\,f_Y(y)$$

Entonces

$$
\begin{aligned}
E[XY] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{XY}(x,y)\,\mathrm{d}x\mathrm{d}y \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_X(x)\,f_y(y)\,\mathrm{d}x\mathrm{d}y \\
&= \left[\int_{-\infty}^{\infty} xf_X(x)\,\mathrm{d}x\right]\left[\int_{-\infty}^{\infty} yf_y(y)\,\mathrm{d}y\right] \\
&= E[X]\,E[Y]
\end{aligned}
$$

Exercise 4.5. Por Demostrar

$$\mathcal{P}\left[|X-\mu| \geq \epsilon\right] \leq \frac{\operatorname{Var}[X]}{\epsilon^2}$$

Por la desigualdad de Chebysev, para X una variable aleatoria.

$$\mathcal{P}\left[X \geq \epsilon\right] \leq \frac{E[X]}{\epsilon}$$

Entonces, sea $Y = |X-\mu|, \mu = E[X]$

$$\begin{aligned}\mathcal{P}\left[|X-\mu| \geq \epsilon\right] &= \mathcal{P}\left[|X-\mu|^2 \geq \epsilon^2\right] \\ &\leq \frac{E\left[(X-\mu)^2\right]}{\epsilon^2} = \frac{\operatorname{Var}[X]}{\epsilon^2}\end{aligned}$$

::: {#exr-1}

Por demostrar

Sean $X_1, X_2, \ldots, X_n$ variables aleatorias independienes con esperanza finita $\mu = E\left[X_j\right]$ y varianza infinita. $\sigma^2 = \operatorname{Var}\left(X_j\right)$. Sean $S_n = X_1 + X_2 + \ldots + X_n$. Entonces para cada $\epsilon > 0$.

:::

$$\mathcal{P}\left[\left|\frac{S_n}{n} - \mu\right| \geq \epsilon\right] \to 0$$

Notemos que

$$\begin{aligned}\operatorname{Var}\left[\frac{S_n}{n} - \mu\right] &= \frac{1}{n^2}\operatorname{Var}\left(S_n\right) = \frac{1}{n^2}\sum_{i=1}^{n}\operatorname{Var}\left(X_i\right) \\ &= \frac{\sigma^2}{n}\end{aligned}$$

Entonces, por el Teorema de Chebysev

$$\mathcal{P}\left[\left|\frac{S_n}{n} - \mu\right| \geq \epsilon\right] \leq \frac{\sigma^2}{n\epsilon},$$

notemos que para $n \to \infty$

$$\frac{\sigma^2}{n\epsilon} \to 0.$$

Entonces

$$\mathcal{P}\left[\left|\frac{S_n}{n} - \mu\right| \geq \epsilon\right] \to 0$$

Exercise 4.6. Sea $\{X_i\}_{i=1}^{\infty}$ una secuencia de v.a.i.id con media a y varianza b^2. Entonces para doo $\alpha, \beta \in \mathbb{R}$, con $\alpha < \beta$, entonces

$$\mathcal{P}\left(\lim_{M\to\infty} \alpha \leq \frac{\sum_{i=1}^{M} X_i - Ma}{\sqrt{M}b} \leq \beta\right) = \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\beta} \exp\left(-\frac{1}{2}x^2\right) \mathrm{d}x$$

Sea

$$Y_M = \frac{\sum_{i=1}^{M} [X_i - a]}{\sqrt{M}b},$$

Definamos

$$\overline{S_M} = \sum_{i=1}^{M} [X_i - a],$$

entonces

$$Y_M = \frac{S_M}{\sqrt{M}b}$$

demostraremos que la función generadora de momentos $\varphi_M \to \varphi$ donde $\varphi_m = \varphi_{Y_M}$ y φ es función generadora de momentos de la distribución normal estandar.

Ahora,

$$\begin{aligned}
\varphi_M(t) &= E\left[\exp\left(t\frac{S_M}{\sqrt{M}b}\right)\right] \\
&= \varphi_{SM}\left(\frac{t}{\sqrt{M}b}\right) \\
X_i \text{ v.a.i.i.d} \Rightarrow &= \left[\varphi_{(X_1-a)}\left(\frac{t}{\sqrt{M}b}\right)\right]^M \\
&= \left[E\left[\exp\left(\frac{t}{b\sqrt{M}}(X_1 - a)\right)\right]\right]
\end{aligned}$$

Recordando la serie de Taylor

$$
\begin{aligned}
\varphi_M(t) &= \left[\sum_{i=0}^{\infty}\frac{E\left[\left(\frac{t}{b\sqrt{M}}(X_1-a)\right)^i\right]}{i!}\right]^M \\
&= \left[1+\frac{1}{2}\left(\frac{t}{b\sqrt{M}}\right)^2 E\left[(X_1-a)^2\right]+\epsilon(3)\right]^M \\
&= \left[1+\frac{1}{M}\frac{t^2}{2}+\epsilon(3)\right]^M,
\end{aligned}
$$

donde

$$
\epsilon(3)=\sum_{i=3}^{\infty}\frac{E\left[\left(\frac{t}{b\sqrt{M}}(X_1-a)\right)^i\right]}{i!},
$$

Sea $s=\frac{t}{b\sqrt{M}}$, entonces $s\to 0, t\to 0$

$$
\epsilon(3)=\sum_{i=3}^{\infty}\frac{E\left[(X_1-a)^i\right]s^i}{i!}
$$

Notemos que, si φ_1 existe. Entonces

$$
\frac{\epsilon(3)}{s^2}=\sum_{i=3}^{\infty}\frac{E\left[(X_1-a)^i\right]s^{i-2}}{i!}\to 0, s\to 0.
$$

Además $s\to 0$ cuando $M\to\infty$.

$$
\Rightarrow \varphi_M(t)=\left[1+\frac{1}{M}\left[\frac{t^2}{2}+M\epsilon(3)\right]\right]^M,
$$

Entonces $\epsilon(3)s^{-2}=Me(3)b^2t^{-2}\to 0$. Como b,t estan fijas.

$$
M\epsilon(3)\to 0, M\to\infty,
$$

por lo tanto

$$\frac{t^2}{2} + M\epsilon(3) \to \frac{t^2}{2}, M \to \infty$$

$$\left[1 + \frac{1}{M}\left[\frac{t^2}{2} + M\epsilon(3)\right]\right]^M \to \exp\left(t^2\right), M \to \infty$$

$$\lim_{M\to\infty} \varphi_M(t) = \exp\left(t^2\right) = \varphi(t),$$

la función generadora de momentos de la distribución normal estándar. Por lo tanto

$$F_M(x) \to F_{N(0,1)}(x)$$

$$F_M(b) - F_M(a) \to F_N(b) - F_N(a)$$

$$\mathcal{P}\left(\lim_{M\to\infty} \alpha \le \frac{\sum_{i=1}^{M} X_i - Ma}{\sqrt{M}b} \le \beta\right) = \frac{1}{\sqrt{2\pi}}\int_\alpha^\beta \exp\left(-\frac{1}{2}x^2\right) \mathrm{d}x$$

Exercise 4.7. Sea $\{X_i\}_{i=1}^{\infty}$ una sucesión de v.a.i.i.d con media a. Entonces

$$\mathcal{P}\left[\lim_{M\to\infty} \frac{1}{M}\sum_{i=1}^{M} X_i = a\right] = 1.$$

Esto es similar a decir que

$$\lim_{M\to\infty} \frac{1}{M}\sum_{i=1}^{M} X_i \overset{\text{c.s}}{=} a$$

Sin perdida de generalidad, diremos que $X_i \ge 0, \forall i$. Definamos

$$Y_n = X_n I_{[|X_n| \le n]}, Q_n = \sum_{i=1}^{n} Y_i$$

Por la desigualdad de

$$\begin{aligned}
\sum_{n=1}^{\infty} \mathcal{P}\left[\left|\frac{Q_n - E[Q_n]}{n}\right| \ge \epsilon\right] &\le \sum_{n=1}^{\infty} \frac{\operatorname{Var}(Q_n)}{\epsilon^2 n^2} = \sum_{n=1}^{\infty} \frac{1}{\epsilon^2 n^2} \sum_{i=1}^{n} \operatorname{Var}(Y_i) \\
&\le \sum_{n=1}^{\infty} \frac{E\left(Y_n^2\right)}{\epsilon^2 n^2} = \sum_{n=1}^{\infty} \frac{1}{\epsilon^2 n^2} \int_0^n x^2 \mathrm{d}F \\
&\le \sum_{n=1}^{\infty} \frac{1}{\epsilon^2} \int_0^n x \mathrm{d}F < \infty,
\end{aligned}$$

donde F es la función de distribución de X_i. Luego

$$E\left[X_1\right] = \lim_{n\to\infty} \int_0^n x\mathrm{d}F = \lim_{n\to\infty} E\left[Y_n\right] = \lim_{n\to\infty} \frac{E\left[Q_n\right]}{n}.$$

Entonces, por el Lema de Borel Canteli. $\mathcal{P}\left[\limsup\left(\left|\frac{Q_n - E\left[Q_n\right]}{n}\right| \geq \epsilon\right)\right] = 0$

$$\lim_{n\to\infty} \frac{Q_n}{n} = E\left[X_1\right], \text{c.s}$$

Ahora, calcularemos la siguiente probabilidad

$$\sum_{i=1}^{\infty} \mathcal{P}\left[X_i \neq Y_i\right] = \sum_{i=1}^{\infty} \mathcal{P}\left[X_i > n\right]$$

como $E\left[X_i\right] < \infty$ y X_i son v.a.i.i.d.

$$\sum_{i=1}^{\infty} \mathcal{P}\left[X_i > n\right] \leq E\left[X_1\right] < \infty$$

De nuevo, por el Lema de Borel Cantelli.

$$\mathcal{P}\left[\limsup\left[X_i \neq Y_i\right]\right] = 0, \forall i$$

Entonces

$$\begin{aligned} &X_i = Y_i, \text{c.s} \\ &\Rightarrow \frac{1}{M} \sum_{i=1}^{M} X_i \to E\left[X_1\right] = \mu. \text{ c.s} \end{aligned}$$

5 Tarea 4

Exercise 5.1. Sea $W(t)$ un movimiento Browniano estándar en $[0,T]$. Pruebe que para cualquier $c>0$ fijo,

$$V(t) = \frac{1}{c}W(c^2 t)$$

es un movimiento Browniano sobre $[0,T]$.

5.0.1 Demostración

Demostraremos que V cumple las propiedades del movimiento Browniano.

5.0.1.1 Propiedad 1

Es claro que $V(0) = \dfrac{1}{c}W(c^2 0) = 0$.

5.0.1.2 Propiedad 2 (Incrementos Independientes)

Sean $s<t<u<v$ tenemos que

$$E[(V(t)-V(s))\,(V(v)-V(u))] = \frac{1}{c^2}E[(W(c^2t)-W(c^2s))\,(W(c^2v)-W(c^2u))]$$

Como el browniano tiene incrementos independientes.

$$\begin{aligned}\frac{1}{c^2}E\left[(W(c^2t)-W(c^2s))\,(W(c^2v)-W(c^2u))\right] &= \frac{1}{c^2}E\left[(W(c^2t)-W(c^2s))\right]E\left[(W(c^2v)-W(c^2u))\right]\\ &= 0\end{aligned}$$

Entonces V tiene incrementos independientes.

5.0.1.3 Propiedad 3 (Incrementos estacionarios)

Considere $s < t$.

$$V(t) - V(s) = \frac{1}{c}\left[W(c^2 t) - W(c^2 s)\right]$$

Por propiedades del movimiento Browniano.

$$\begin{aligned} E\left[V(t) - V(s)\right] &= \frac{1}{c} E\left[W(c^2 t) - W(c^2 s)\right] = 0 \\ \operatorname{Var}\left[V(t) - V(s)\right] &= \frac{1}{c^2} \operatorname{Var}\left[W(c^2 t) - W(c^2 s)\right] = \frac{1}{c^2}\left(c^2\left(t - s\right)\right) = t - s \end{aligned}$$

Entonces V tiene incrementos estacionarios.

5.0.2 Por lo tanto, V es un movimiento browniano.

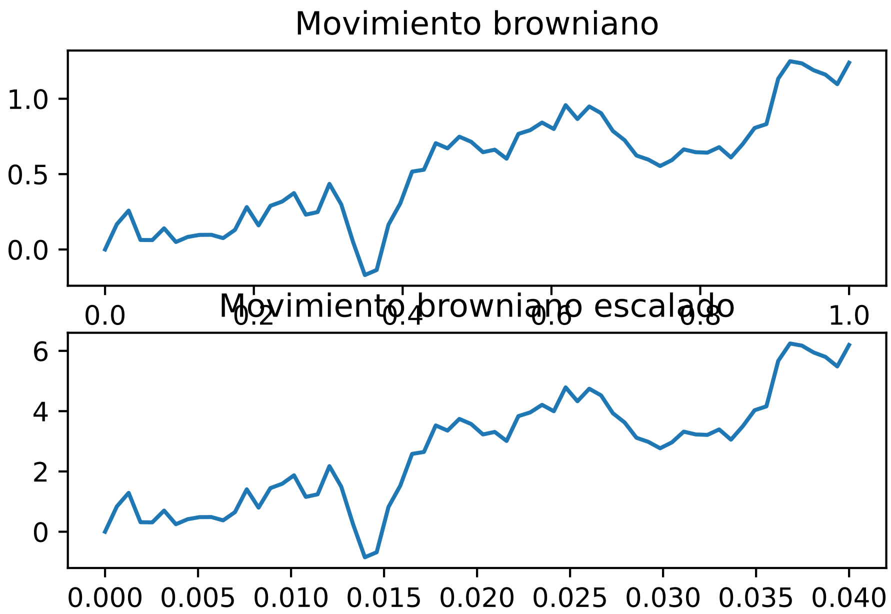
Exercise 5.2. Hacer un script para ilustrar la propiedad de escalado del movimiento Browniano para el caso de $c = \frac{1}{5}$. Estar seguro que usa el mismo camino browniano discretizado en cada subplot.

El código, se encuentra en `hw4_p2.py`. Pero aquí se muestran los resultados.

```
# Importamos las librerias necesarias
import numpy as np
import matplotlib.pyplot as plt

# Comenzaremos simulando el movimiento browniano fuerte.

prng = np.random.RandomState(10)  # Fijamos la semilla.

t_final = 1  # Extremo derecho del intervalo [0,T]
n_points = 64  # No olvidemos que aquí se incluye el 0 y t.
dt = 1 / (n_points - 1)  # Ajuste al delta t

dw = np.sqrt(dt) * prng.standard_normal(n_points - 1)  # Calculamos los incrementos.
w = np.concatenate(([0], dw.cumsum()))

time = np.linspace(0, t_final, n_points)  # Vector de tiempo.

# plt.plot(time, w)        Graficamos el browniano base.
```

```
# plt.show()

"""
Ahora, comenzamos con el browniano escalado.
"""
c = 0.2  # 1/5


"""
Esto tiene dos interpretaciones.
Sin embargo, para este ejercicio debemos partir de una trayectoria dada, entonces haremos
"""

c_time = c**2 * time  # Transformamos el intervalo del tiempo
c_w = c**(-1) * w  # Escalamos el browniano.

print("El valor de c es ",c)
fig, cbrown = plt.subplots(2)
cbrown[0].plot(time, w)
cbrown[1].plot(c_time, c_w)
cbrown[0].set_title('Movimiento browniano')
cbrown[1].set_title('Movimiento browniano escalado')
plt.show()
```

```
El valor de c es  0.2
```



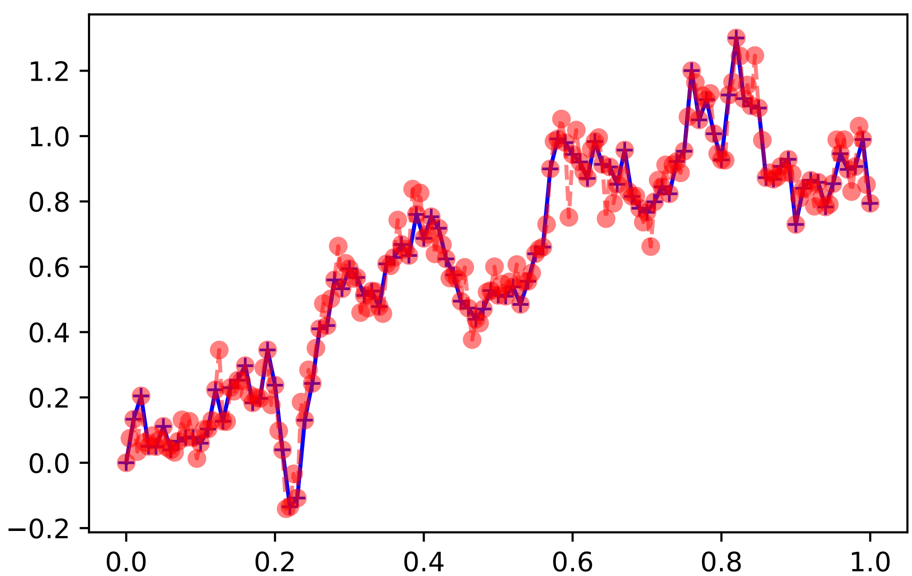
Exercise 5.3. Modifique el script `half_brownian_refinement.py` encapsulando el código en una función. Esta función deberá recibir el extremo derecho del intervalo $[0, T]$ y el número de incrementos N de un camino browniano base. El propósito es calcular los incrementos de relleno de una refinamiento con $2N$ incrementos.

```
import numpy as np
import matplotlib.pyplot as plt
prng = np.random.RandomState(10)

def refined_brownian_2n(T,L):
    dt = T / L
    W = np.zeros(L + 1)
    W_refined = np.zeros(2 * L + 1)
    xi = np.sqrt(dt) * prng.normal(size=L)
    xi_half = np.sqrt(0.5 * dt) * prng.normal(size=L)
    W[1:] = xi.cumsum()
    W_ = np.roll(W, -1)
    W_half = 0.5 * (W + W_)
    W_half = np.delete(W_half, -1) + xi_half
    W_refined[1::2] = W_half
    W_refined[2::2] = W[1:]
    t = np.arange(0, T + dt, dt)
    t_half = np.arange(0, T + 0.5 * dt, 0.5 * dt)
```

```
    return t, t_half, W, W_refined
```

Exercise 5.4. En un script separado, incluya la función de arriba y grafique una figura con la trayectoria del browniano con 100 incrementos y muestre su refinamiento correspondiente.

```
time, bi_time, w, bi_w = refined_brownian_2n(1,100)

plt.plot(time, w, 'b-+')
plt.plot(
    bi_time,
    bi_w,
    'ro--',
    alpha=0.5
)
plt.show()
```



6 Tarea 5

Exercise 6.1. Muestre que el movimiento Browniano satisface

$$E\left[\left|W\left(t\right)-W\left(s\right)\right|^{2}\right]=\left|t-s\right|$$

Si $t>s$.

$$\begin{aligned}E\left[\left|W\left(t\right)-W\left(s\right)\right|^{2}\right]&=E\left[\left(W\left(t\right)-W\left(s\right)\right)^{2}\right]\\&=t-s,\end{aligned}$$

mientras que si $t\leq s$.

$$\begin{aligned}E\left[\left(W\left(t\right)-W\left(s\right)\right)^{2}\right]&=E\left[\left(W\left(s\right)-W\left(t\right)\right)^{2}\right]\\&=s-t,\end{aligned}$$

por lo tanto

$$E\left[\left|W\left(t\right)-W\left(s\right)\right|^{2}\right]=\left|t-s\right|$$

Exercise 6.2. Dado $W\left(t_i\right)$ y $W\left(t_{i+1}\right)$, muestre que la variable aleatoria

$$W\left(t_{i+\frac{1}{2}}\right):=\frac{1}{2}\left[W\left(t_i\right)+W\left(t_{i+1}\right)\right]+\frac{1}{2}\sqrt{\Delta t}\xi,\xi\sim N\left(0,1\right)$$

es un movimiento Browniano.

6.0.0.0.1 Es claro que al ser un refinamiento del movimiento browniano.

$$W\left(0\right)=0$$

6.0.0.0.2 C_2. Notemos que

$$W_{i+\frac{i}{2}} - W_i = \frac{1}{2}\left[W\left(t_{i+1}\right) - W\left(t_i\right)\right] + \frac{1}{2}\sqrt{\Delta t}\xi,$$

Sabemos que la combinación lineal de normales es una nornal. Luego, partiendo que $t_{i+1} - t_i = \Delta t$.

$$\begin{aligned} E\left[W_{i+\frac{i}{2}} - W_i\right] &= 0, \\ \operatorname{Var}\left[W_{i+\frac{i}{2}} - W_i\right] &= \frac{1}{4}\Delta t + \frac{1}{4}\Delta t = \frac{1}{2}\Delta t, \end{aligned}$$

Por lo tanto $W_{i+\frac{1}{2}} - W_i \sim N\left(0, \dfrac{\Delta t}{2}\right)$.

6.0.0.0.3 Calculamos la esperanza.

$$E\left[\left(W_{i+1} - W_{i+\frac{1}{2}}\right)\left(W_{j+1} - W_{j+\frac{1}{2}}\right)\right] = E\left[\left(\frac{1}{2}\left[W\left(t_{i+1}\right) - W\left(t_i\right)\right] + \sqrt{\frac{\Delta t}{4}}\xi\right)\frac{1}{2}\left[W\left(t_{j+1}\right) - W\left(t_j\right)\right] + \sqrt{\frac{\Delta t}{4}}\xi\right]$$

defina $dW_i = W_{i+1} - W_i$,

$$
\begin{aligned}
E\left[\left(W_{i+1}-W_{i+\frac{1}{2}}\right)\left(W_{j+1}-W_{j+\frac{1}{2}}\right)\right] &= E\left[\left(\frac{1}{2}dW_i+\sqrt{\frac{\Delta t}{4}}\xi\right)\left(\frac{1}{2}dW_j+\sqrt{\frac{\Delta t}{4}}\xi\right)\right] \\
&= E\left[\frac{1}{4}dW_i dW_j+\frac{1}{2}dW_i\sqrt{\frac{\Delta t}{4}}\xi+\frac{1}{2}dW_j\sqrt{\frac{\Delta t}{4}}\xi+\left(\sqrt{\frac{\Delta t}{4}}\xi\right)^2\right] \\
dW_i, dW_j \text{ son independientes} &= \left(E\left[\frac{1}{2}dW_i\right]\right)\left(E\left[\frac{1}{2}dW_j\right]\right)+E\left[\frac{1}{2}dW_i\right]\sqrt{\frac{\Delta t}{4}}E\left[\xi\right] \\
&+E\left[\frac{1}{2}dW_j\right]\sqrt{\frac{\Delta t}{4}}E\left[\xi\right]+\frac{\Delta t}{4}\left(E\left[\xi\right]\right)^2 \\
&= \left(E\left[\frac{1}{2}dW_i\right]\right)\left(E\left[\frac{1}{2}dW_j+\sqrt{\frac{\Delta t}{4}}\xi\right]\right)+E\left[\frac{1}{2}dW_j\right]\sqrt{\frac{\Delta t}{4}}E\left[\xi\right]+\frac{\Delta t}{4}\left(E\right. \\
&= \left(E\left[\frac{1}{2}dW_i\right]\right)\left(E\left[\left(W_{j+1}-W_{j+\frac{1}{2}}\right)\right]\right)+E\left[\frac{1}{2}dW_j\right]\sqrt{\frac{\Delta t}{4}}E\left[\xi\right]+\frac{\Delta t}{4}\left(E\left[\xi\right]\right. \\
&= \left(E\left[\frac{1}{2}dW_i\right]\right)\left(E\left[\left(W_{j+1}-W_{j+\frac{1}{2}}\right)\right]\right)+E\left[\frac{1}{2}dW_j+\sqrt{\frac{\Delta t}{4}}\xi\right]\sqrt{\frac{\Delta t}{4}}E\left[\xi\right] \\
&= \left(E\left[\frac{1}{2}dW_i\right]\right)\left(E\left[\left(W_{j+1}-W_{j+\frac{1}{2}}\right)\right]\right)+E\left[\left(W_{j+1}-W_{j+\frac{1}{2}}\right)\right]\sqrt{\frac{\Delta t}{4}}E\left[\xi\right] \\
&= E\left[\left(W_{i+1}-W_{i+\frac{1}{2}}\right)\right]E\left[\left(W_{j+1}-W_{j+\frac{1}{2}}\right)\right],
\end{aligned}
$$

teniendo asi, que los incrementos son independientes.

Exercise 6.3. Generalice la fórmula del ejercicio anterior para en el caso donde, $W\left(t_i\right), W\left(t_{i+1}\right)$, y $\alpha \in (0,1)$ el valor

$$
W\left(t_i+\alpha\Delta t\right),
$$

es un movimiento Browniano.

Notemos que

$$
t_i+\alpha\Delta t+\left(1-\alpha\right)\Delta t=t_{i+1},
$$

entonces vamos a definir

$$
\begin{aligned}
W_{i+\alpha} &= W\left(t_i+\alpha\Delta t\right) \\
&= \left(1-\alpha\right)W_i+\alpha W_{i+1}+Y,
\end{aligned}
$$

donde Y será una v.a independiente de $W\left(t\right)$. Entonces

$$
\begin{aligned}
W_{i+\alpha}-W_i &= \left(1-\alpha\right)W_i+\alpha W_{i+1}+Y-W_i \\
&= \alpha\left(W_{i+1}-W_i\right)+Y \\
&= \alpha\left(W_{i+1}-W_i\right)+Y.
\end{aligned}
$$

Entonces
$$E\left[W_{i+\alpha}-W_i\right]=E\left[Y\right],$$

por lo tanto, $E\left[Y\right]$ tiene que ser cero. Luego
$$\operatorname{Var}\left[W_{i+\alpha}-W_i\right]=\alpha^2\Delta t+\operatorname{Var}\left[Y\right],$$

notemos que
$$\left(i+\alpha\right)\Delta t-i\Delta t=\alpha\Delta t,$$

por lo tanto tendría que cumplirse $\operatorname{Var}\left[W_{i+\alpha}-W_i\right]=\alpha\Delta t$.
$$\alpha^2\Delta t+\operatorname{Var}\left[Y\right]=\alpha\Delta t,$$

entonces
$$\operatorname{Var}\left[Y\right]=\Delta t\left(\alpha-\alpha^2\right),$$

como $Y=\sqrt{\alpha\left(1-\alpha\right)}\xi,\xi\sim N\left(0,1\right)$. Como este es un refinamiento del browniano, entonces se cumple $C1$.
$$W\left(0\right)=0.$$

Conseguimos $C2$ por construcción y de forma análoga tenemos la independiencia de los incrementos.
$$E\left[\left(W_{i+\alpha}-W_i\right)\left(W_{j+\alpha}-W_j\right)\right]=E\left[W_{i+\alpha}-W_i\right]E\left[W_{j+\alpha}-W_j\right].$$

Exercise 6.4. Suponga que $X\sim N\left(0,1\right)$. Sabemos que $E\left[X\right]=0$ y $E\left[X^2\right]=3$. Luego, de la definición, el $p-$ ésimo momento satisface
$$E\left[X^p\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}x^p\exp\left(-\frac{x^2}{2}\right)\mathrm{D}x.$$

Usando la relación, muestre que $E\left[X^3\right]=0$ y $E\left[X^4\right]=1$.

Entonces deduzca el incremento Browniano,
$$\Delta W_i=W\left(t_{i+1}\right)-W\left(t_i\right),$$

satisface $E\left[\Delta W_i^3\right]=0,E\left[\Delta W_i^4\right]=3\left(\Delta t\right)^2$. Entonces encuentre una expresión para $E\left[X^p\right]$ para un entero positivo p.\ Pista: Tu puedes usar el dato que $\int_{-\infty}^{\infty}\exp\left(-\frac{x^2}{2}\right)=\sqrt{2\pi}$.

Considere la fórmula.

$$E\left[X^p\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^p \exp\left(-\frac{x^2}{2}\right) \mathrm{D}x.$$

Partiendo la integral,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^p \exp\left(-\frac{x^2}{2}\right) \mathrm{D}x = \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} x^p \exp\left(-\frac{x^2}{2}\right) \mathrm{D}x}_{I_1} + \underbrace{\frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} x^p \exp\left(-\frac{x^2}{2}\right) \mathrm{D}x}_{I_2}$$

Entonces

$$I_1 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} x^p \exp\left(-\frac{x^2}{2}\right) \mathrm{D}x,$$

hagamos el cambio de variable. $y = -x$, tenemos que

$$\begin{aligned} I_1 &= \frac{1}{\sqrt{2\pi}} \int_{\infty}^{0} -\left(-y\right)^p \exp\left(-\frac{y^2}{2}\right) \mathrm{D}y \\ &= \frac{\left(-1\right)^p}{\sqrt{2\pi}} \int_{0}^{\infty} y^p \exp\left(-\frac{y^2}{2}\right) \mathrm{D}y = I_2 \left(-1\right)^p, \end{aligned}$$

entonces

$$E\left[X^p\right] = \left(1 + \left(-1\right)^p\right) I_2,$$

de aqui tenemos, que si p es impar $E\left[X^p\right] = 0$, entonces si p es par

$$E\left[X^p\right] = 2I_2,$$

entonces, nos concentraremos en

$$E\left[X^p\right] = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} x^p \exp\left(-\frac{x^2}{2}\right) \mathrm{D}x, p = 2k, k \in \mathbb{N}$$

Considere $y = \dfrac{x^2}{2}, \mathrm{D}y = x\mathrm{D}x.$

$$E\left[X^p\right] = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} x^{p-1} \exp\left(-y\right) \mathrm{D}y,$$

luego $\sqrt{2y} = x$, entonces

$$\begin{aligned}
E\left[X^p\right] &= \frac{2}{\sqrt{2\pi}} \int_0^\infty \left(\sqrt{2y}\right)^{p-1} \exp\left(-y\right) \mathrm{D}y \\
&= \frac{2\left(\sqrt{2}\right)^{p-1}}{\sqrt{2\pi}} \int_0^\infty y^{\frac{p-1}{2}} \exp\left(-y\right) \mathrm{D}y \\
&= \frac{2\left(\sqrt{2}\right)^{p-1}}{\sqrt{2\pi}} \int_0^\infty y^{\frac{p+1}{2}-1} \exp\left(-y\right) \mathrm{D}y,
\end{aligned}$$

recordando la función Gamma.

$$\Gamma\left(z\right) = \int_0^\infty x^{z-1} e^{-t} \mathrm{D}t,$$

entonces

$$E\left[X^p\right] = \begin{cases} 0 & p \text{ impar} \\ \dfrac{2^{\frac{p+1}{2}}}{\sqrt{2\pi}} \Gamma\left(\dfrac{p+1}{2}\right) & p \text{ par} \end{cases},$$

entonces $E\left[X^4\right] = \dfrac{4}{\sqrt{\pi}} \Gamma\left(\dfrac{5}{2}\right)$

$$\Gamma\left(\frac{5}{2}\right) = \int_0^\infty x^{\frac{3}{2}} e^{-x} \mathrm{D}x.$$

Considere

$$\begin{aligned}
u &= x^{3/2} \mathrm{D}v = e^{-x} \mathrm{D}x \\
\mathrm{D}u &= \frac{3}{2} x^{1/2} v = -e^{-x},
\end{aligned}$$

$$\begin{aligned}
\Gamma\left(\frac{5}{2}\right) &= \left[-x^{3/2} e^{-x}\right] + \frac{3}{2} \int_0^\infty x^{1/2} e^{-x} \mathrm{D}x \\
&= \frac{3}{2} \frac{\sqrt{\pi}}{2}.
\end{aligned}$$

Entonces

$$E\left[X^4\right] = 3$$

Notemos que si $\Delta W \sim N\left(0, \sigma^2\right)$, entonces

$$Z = \frac{\Delta W}{\sigma} \sim N(0,1)$$

En general,

$$E\left[(\Delta W)^p\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} w^p \exp\left(-\left(\frac{w}{\sigma}\right)^2\right) \mathrm{D}w,$$

considere $\sigma u = w$, entonces

$$\begin{aligned} E\left[(\Delta W)^p\right] &= \sigma^p \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u^p \exp\left(-u^2\right) \mathrm{D}w\right], \\ &= \sigma^p E\left[Z^p\right], \end{aligned}$$

entonces para $p = 4, \sigma^2 = \Delta t$.

$$E\left[(\Delta W)^4\right] = (\Delta t)^2 E\left[Z^4\right] = 3(\Delta t)^2$$

Exercise 6.5. Suponga que $X \sim N(0,1)$. Muestre que para $a, b \in \mathbb{R}$,

$$E\left[\exp(a + bX)\right] = \exp\left(a + \frac{1}{2}b^2\right).$$

Deduzca que

$$E\left[\exp\left(t + \frac{1}{4}W_t\right)\right] = \exp\left(\frac{33}{32}t\right)$$

Considere

$$E\left[\exp(a + bX)\right] = e^a E\left[\exp(bX)\right],$$

notemos que $bX \sim N\left(0, b^2\right)$, por lo tanto, la función generadora de momentos nos dice que

$$E\left[\exp(bX)\right] = M_{bX}(1) = \exp\left(\frac{b^2}{2}\right),$$

por lo tanto

$$E\left[\exp(a + bX)\right] = e^a \exp\left(\frac{b^2}{2}\right) = \exp\left(a + \frac{1}{2}b^2\right),$$

ahora, considere

$$E\left[\exp\left(t + \frac{1}{4}W_t\right)\right] = E\left[\exp\left(t + \frac{1}{4}\left(W_t - W_0\right)\right)\right],$$

Notemos que $W_t - W_0 \sim N(0,t)$, por lo tanto, usando la fórmula anterior

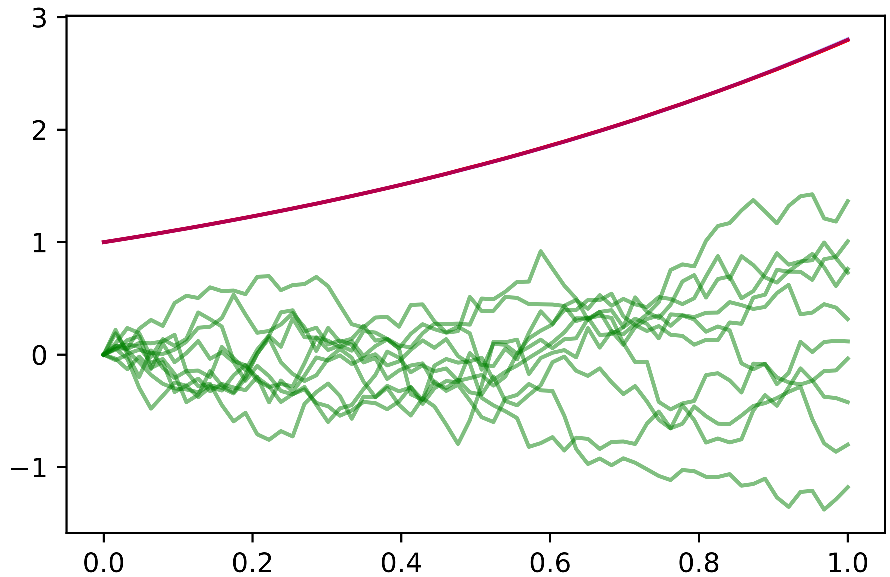
$$
\begin{aligned}
E\left[\exp\left(t+\frac{1}{4}\left(W_t-W_0\right)\right)\right] &= \exp\left(t+\frac{1}{4}\left(\sqrt{t}X\right)\right), X\sim N\left(0,1\right) \\
&= \exp\left(t+\left(\frac{1}{4}\sqrt{t}\right)X\right) = \exp\left(t+\frac{t}{32}\right) \\
&= \exp\left(\frac{33}{32}t\right)
\end{aligned}
$$

7 Tarea 6

Exercise 7.1. Hacer un script para simular 10000 trayectorias del proceso $u(t, W_t)$ definido en el Ejercicio. Grafique en una figura, 10 trayectorias y la media de las 10000 trayectorias del proceso $u(t, W_t)$.

```
import numpy as np
import matplotlib.pyplot as plt

def strong_brownian(t, n):
    dt = t / n
    dw = np.zeros(n)
    w = np.zeros(n)
    for i in np.arange(1, n):
        dw[i] = np.sqrt(dt)*np.random.standard_normal()
        w[i] = w[i - 1] + dw[i]
    time = np.linspace(0, t, n)
    return time, w



def b_function(t, a, w):
    y = np.exp(t - a * w)
    return y


n_samples = 10000
n_points = 64
t_initial = 0
t_final = 1

mean = np.zeros(n_points)
for i in range(n_samples):
    time, b_w = strong_brownian(t_final, n_points)
    y = b_function(time, 0.25, b_w)
    if i < 10:
```

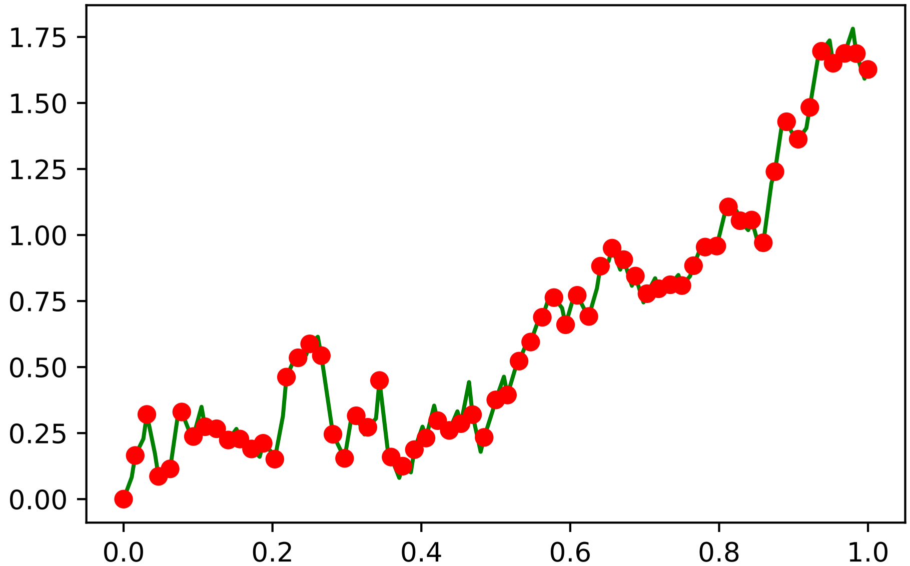
```
        plt.plot(time, b_w, 'g-', alpha=0.5)
    mean += y

mean = (n_samples)**(-1) * mean
time = np.linspace(0, t_final, n_points)

y = [np.exp(33 / 32 * t) for t in time]
plt.plot(time, mean, 'r-')
plt.plot(time, y, 'b-', alpha=0.3)
plt.show()
```



Exercise 7.2. Siguiendo las ideas del refinamiento del camino browniano. $t_{i+1/2} = t_i + \frac{1}{2}\delta t$. Hacer un código de Python para el refinamiento del Browniano para $\alpha \in (0,1)$ para el refinamiento $t_{i+ } = t_i + t $

```
import numpy as np
import matplotlib.pyplot as plt

def strong_brownian(t, n):
    dt = t / n
    dw = np.zeros(n)
    w = np.zeros(n)
```

```
    for i in np.arange(1, n):
        dw[i] = np.sqrt(dt)*np.random.standard_normal()
        w[i] = w[i - 1] + dw[i]
    time = np.linspace(0, t, n)
    return time, w

t_final = 1
n_points = 65
delta_t = 1/(n_points - 1)
alpha = 0.7

prng = np.random.RandomState(219)

time, w = strong_brownian(1, n_points)  # w_i

y = np.sqrt(delta_t * (alpha - alpha ** 2)) * prng.standard_normal(n_points - 1)

w_ = np.roll(w, -1)  # w_i+1

w_alpha = alpha * w_ + (1 - alpha) * w
w_alpha = np.delete(w_alpha, -1)
w_alpha += y
w_ref = np.zeros(2* n_points -1)

w_ref[0::2] = w
w_ref[1::2] = w_alpha

time_ref = np.zeros(2 * n_points - 1)

for i in range(2 * n_points - 1):
    if i % 2 == 0:
        time_ref[i] = time[int(i / 2)]
    else:
        time_ref[i] = time[int(i / 2)] + alpha * delta_t

plt.plot(time_ref, w_ref, 'g-')
plt.plot(time, w, 'ro')
plt.show()
```



1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
0.0
0.2
0.4
0.6
0.8
1.0

8 Tarea 7

Exercise 8.1. Sea $W(t)$ un Movimiento Browniano y Z_i una colección de variables aleatorias i.i.d, con distribución $N\left(0, \frac{\delta t}{4}\right)$.
Pruebe que la suma

$$\sum_{i=0}^{L} Z_i \left(W(t_{i+1}) - W(t_i) \right),$$

tiene valor esperado igual a cero y una varianza de $O(\delta t)$.

Sin perdida de generalidad supongamos que Z_i y $W(t_{i+1}) - W(t_i)$ son variables aleatorias independientes para cada $i = 1, \ldots L$. Entonces

$$\begin{aligned}
\mathbb{E}\left[\sum_{i=0}^{L} Z_i \left(W(t_{i+1}) - W(t_i) \right)\right] &= \sum_{i=0}^{L} \mathbb{E}\left[Z_i \left(W(t_{i+1}) - W(t_i) \right) \right] \\
&= \sum_{i=0}^{L} \mathbb{E}(Z_i) \mathbb{E}\left(W(t_{i+1}) - W(t_i) \right) \\
&= 0
\end{aligned}$$

así,

$$\begin{aligned}
Var\left[\sum_{i=0}^{L} Z_i \left(W(t_{i+1}) - W(t_i) \right)\right] &= \mathbb{E}\left[\left(\sum_{i=0}^{L} Z_i (W(t_{i+1}) - W(t_i))\right)^2\right] + \left(\mathbb{E}\left[\sum_{i=0}^{L} Z_i (W(t_{i+1}) - W(t_i))\right]\right)^2 \\
&= \mathbb{E}\left[\left(\sum_{i=0}^{L} Z_i (W(t_{i+1}) - W(t_i))\right)^2\right]
\end{aligned}$$

por el Teorema multinomial, resulta

$$\begin{aligned}
\mathbb{E}\left[\left(\sum_{i=0}^{L} Z_i (W(t_{i+1}) - W(t_i))\right)^2\right] &= \mathbb{E}\left[\sum_{i=0}^{L} \left[Z_i (W(t_{i+1}) - W(t_i))\right]^2 + 2\sum_{i \neq j}^{L} Z_i (W(t_{i+1}) - W(t_i)) Z_j (W(t_{j+1})\right. \\
&= \sum_{i=0}^{L} \mathbb{E}\left[Z_i (W(t_{i+1}) - W(t_i))\right]^2 + 2\sum_{i \neq j}^{L} \mathbb{E}\left[Z_i (W(t_{i+1}) - W(t_i)) Z_j (W(t_{j+1})\right. \\
&= \sum_{i=0}^{L} \mathbb{E}\left[Z_i (W(t_{i+1}) - W(t_i))\right]^2 + 2\sum_{i \neq j}^{L} \mathbb{E}\left[Z_i (W(t_{i+1}) - W(t_i)) Z_j (W(t_{j+1})\right.
\end{aligned}$$

observemos que, si $i < j$

$$\begin{aligned}
\mathbb{E}\left[Z_i(W(t_{i+1})-W(t_i))Z_j(W(t_{j+1})-W(t_j))\right] &= \mathbb{E}\{\mathbb{E}\left[Z_i(W(t_{i+1})-W(t_i))Z_j(W(t_{j+1})-W(t_j))|\mathcal{F}_j\right]\} \\
&= \mathbb{E}[Z_i(W(t_{i+1})-W(t_i))Z_j]\mathbb{E}\left[(W(t_{j+1})-W(t_j))|\mathcal{F}_j\right] \\
&= 0
\end{aligned}$$

y además

$$\begin{aligned}
\mathbb{E}\left[Z_i(W(t_{i+1})-W(t_i))\right]^2 &= \mathbb{E}\{\mathbb{E}\left[Z_i^2(W(t_{i+1})-W(t_i))^2|\mathcal{F}_j\right]\} \\
&= \mathbb{E}\{Z_i^2\mathbb{E}\left[(W(t_{i+1})-W(t_i))^2|\mathcal{F}_j\right]\} \\
&= \mathbb{E}[Z_i^2](t_{i+1}-t_i) \\
&= \frac{\delta t}{4}(t_{i+1}-t_i)
\end{aligned}$$

sustituyendo resulta

$$\begin{aligned}
Var\left[\sum_{i=0}^{L} Z_i\left(W(t_{i+1})-W(t_i)\right)\right] &= \sum_{i=0}^{L}\frac{\delta t}{4}(t_{i+1}-t_i) \\
&= \frac{\delta t}{4}(t_{L+1}-t_0).
\end{aligned}$$

Para un L suficientemente grande, se tiene que, $(t_{L+1}-t_0) \leq \frac{\varepsilon}{4}$, así

$$Var\left[\sum_{i=0}^{L} Z_i\left(W(t_{i+1})-W(t_i)\right)\right] \leq \varepsilon\delta t.$$

Así, la varianza es de orden δt.

La regla del punto medio de la integral de Riemann de una función $h \in C^2([a,b])$ sobre una partición de L puntos del intervalo $[a,b]$ está dada por,

$$\int_a^b h(t)dt = \lim_{\delta t\to 0,\, L\to\infty}\sum_{i=0}^{L} h\left(\frac{t_i+t_{i+1}}{2}\right)\delta t.$$

Use la relación

$$W\left(\frac{t_i+t_{i+1}}{2}\right) = \frac{1}{2}(W(t_i)+W(t_{i+1})) + \underbrace{Z_i}_{i.i.d.\sim N(0,\delta t/4)},$$

y el ejercicio anterior para demostrar que la regla del punto medio de la integral de Riemann implica que

$$\int_0^T W(t)dW(t) = \frac{1}{2}W(T)^2.$$

Sea $\Delta_L = \{0 = t_0, t_1, \dots, t_{L-1}, t_L = T\}$ una partición del intervalo $[0, T]$. De la regla del punto medio para $h(t) = W(t)$, resulta

$$\begin{aligned}
\int_0^T W(t)dW(t) &= \lim_{\delta t\to 0,\, L\to\infty} \sum_{i=0}^{L} W\left(\frac{t_i + t_{i+1}}{2}\right)(W(t_{i+1}) - W(t_i)) \\
&= \lim_{\delta t\to 0,\, L\to\infty} \sum_{i=0}^{L} \left[\frac{1}{2}(W(t_i) + W(t_{i+1})) + Z_i\right](W(t_{i+1}) - W(t_i)) \\
&= \lim_{\delta t\to 0,\, L\to\infty} \sum_{i=0}^{L} \frac{1}{2}\left(W(t_{i+1})^2 - W(t_i)^2\right) + \lim_{\delta t\to 0,\, L\to\infty} \sum_{i=0}^{L} Z_i(W(t_{i+1}) - W(t_i)) \\
&= \lim_{\delta t\to 0,\, L\to\infty} \frac{1}{2}\left(W(T)^2 - W(0)^2\right) + \lim_{\delta t\to 0,\, L\to\infty} \sum_{i=0}^{L} Z_i(W(t_{i+1}) - W(t_i)) \\
&= \frac{1}{2}W(T)^2 + \lim_{\delta t\to 0,\, L\to\infty} \sum_{i=0}^{L} Z_i(W(t_{i+1}) - W(t_i))
\end{aligned}$$

Solo falta ver que

$$\sum_{i=0}^{L} Z_i(W(t_{i+1}) - W(t_i)) \to 0 \text{ en } L^2$$

es decir,

$$\lim_{\|\Delta_L\|\to 0} E\left[\left(\sum_{i=0}^{L} Z_i(W(t_{i+1}) - W(t_i))\right)^2\right] = 0$$

Del ejercicio anterior se tiene que

$$E\left[\left(\sum_{i=0}^{L} Z_i(W(t_{i+1}) - W(t_i))\right)^2\right] = O(\delta t) \le \varepsilon \|\Delta_L\|,$$

así, tomando el limite cuando $\|\Delta_L\| \to$ se tiene que,

$$\sum_{i=0}^{L} Z_i(W(t_{i+1}) - W(t_i)) \to 0 \text{ en } L^2$$

Usando la aproximación de la suma de Riemann

$$\int_0^T h(t)dW(t) \sim \sum_{i=0}^{L} h(t_i)(W(t_{i+1}) - W(t_i)),$$

argumente que,

$$\mathbb{E}\left[\left(\int_0^T t dW(t)\right)^2\right] = \frac{T^3}{3}.$$

Por tanto, enuncie la isometría de It\^o y deduzca que esta isometría es válida para el caso $h(t) = t$.

Sea $\{0 = t_0, t_1, \dots, t_{L-1}, t_L = T\}$ una partición del intervalo $[0, T]$. De la aproximación de la suma de Riemann, resulta

$$\begin{aligned}
\int_0^T t dW(t) &\sim \sum_{i=0}^{L} t_i (W(t_{i+1}) - W(t_i)) \\
\Longrightarrow \left(\int_0^T t dW(t)\right)^2 &\sim \left(\sum_{i=0}^{L} t_i (W(t_{i+1}) - W(t_i))\right)^2,
\end{aligned}$$

por el Teorema Multinomial, resulta

$$\left(\sum_{i=0}^{L} t_i (W(t_{i+1}) - W(t_i))\right)^2 = \sum_{i=0}^{L} t_i^2 (W(t_{i+1}) - W(t_i))^2 + 2\sum_{i \neq j} t_i t_j (W(t_{i+1}) - W(t_i))(W(t_{j+1}) - W(t_j))$$

entonces sustituyendo esto, resulta

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^T t dW(t)\right)^2\right] &\sim \mathbb{E}\left[\sum_{i=0}^{L} t_i^2 (W(t_{i+1}) - W(t_i))^2 + 2\sum_{i \neq j} t_i t_j (W(t_{i+1}) - W(t_i))(W(t_{j+1}) - W(t_j))\right] \\
&= \sum_{i=0}^{L} t_i^2 \mathbb{E}(W(t_{i+1}) - W(t_i))^2 + 2\sum_{i \neq j} t_i t_j \mathbb{E}\left[(W(t_{i+1}) - W(t_i))(W(t_{j+1}) - W(t_j))\right] \\
&= \sum_{i=0}^{L} t_i^2 (t_{i+1} - t_i),
\end{aligned}$$

observemos que,

$$\lim_{L \to 0} \sum_{i=0}^{L} t_i^2 (t_{i+1} - t_i) = \int_0^T t^2 dt = \frac{T}{3}$$

entonces

$$\mathbb{E}\left[\left(\int_0^T t dW(t)\right)^2\right] = \frac{T}{3}.$$

Además, de la isometria de Itô,

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^T tdW(t)\right)^2\right] &= \mathbb{E}\left[\left(\int_0^T tdW(t)\right)\left(\int_0^T tdW(t)\right)\right] \\
&= \int_0^T \mathbb{E}(t^2)dt \\
&= \int_0^T t^2 dt \\
&= \frac{T}{3}
\end{aligned}$$

Escriba una función de Python para calcular la integral de It^o del movimiento Browniano $W(t)$ sobre $[0, T]$. La función tendría la siguiente firma.

```
import numpy as np
from PythonScripts import brownian_motion as bw

def f(x: float, t: float):
    y = x
    return y


def fB(partition: np.array, x: float, t: float):
    y = 0
    for i in range(len(partition) - 1):
        if partition[i] <= t < partition[i + 1]:
            y = f(x, t)
    return y

def ito_n(n_points: int, t: float):
    time, w = bw.u(t, n_points)
    integral = np.zeros(n_points)
    for i in range(n_points - 1):
        integral[i] = fB(time, w[i], time[i]) * (w[i + 1] - w[i])
    ito = integral.sum()
    return w, ito

b, ito = ito_n(1000, 1)
print(ito - (0.5 * (b[-1] ** 2 - 1)))
```

```
0.008139836359165209
```

9 Tarea 8

Exercise 9.1. Use la aproximación de la suma de Riemann la ecuación 6.1. Muestra la propiedad de linealidad de la integral estocástica. Es decir,

$$\int_0^T (\alpha f(t) + \beta g(t))\, dW_t = \alpha \int_0^T f(t) dW_t + \beta \int_0^T g(t) dW_t$$

Sea $\{0 = t_0, t_1, \ldots, t_{L-1}, t_L = T\}$ una particion del intervalo $[0, T]$, de la aproximación de la suma de Riemann, resulta

$$\begin{aligned}
\int_0^T (\alpha f(t) + \beta g(t))\, dW_t &\sim \sum_{i=0}^{L} (\alpha f(t_i) + \beta g(t_i))(W(t_{i+1}) - W(t_i)) \\
&= \sum_{i=0}^{L} \alpha f(t_i)(W(t_{i+1}) - W(t_i)) + \sum_{i=0}^{L} \beta g(t_i)(W(t_{i+1}) - W(t_i)) \\
&= \alpha \sum_{i=0}^{L} f(t_i)(W(t_{i+1}) - W(t_i)) + \beta \sum_{i=0}^{L} g(t_i)(W(t_{i+1}) - W(t_i))
\end{aligned}$$

tomando el limite cuando $L \to \infty$, resulta

$$\alpha \lim_{L \to \infty} \sum_{i=0}^{L} f(t_i)(W(t_{i+1}) - W(t_i)) = \alpha \int_0^T f(t) dW_t$$

y

$$\beta \lim_{L \to \infty} \sum_{i=0}^{L} g(t_i)(W(t_{i+1}) - W(t_i)) = \beta \int_0^T g(t) dW_t$$

así,

$$\int_0^T (\alpha f(t) + \beta g(t))\, dW_t = \alpha \int_0^T f(t) dW_t + \beta \int_0^T g(t) dW_t$$

Exercise 9.2. Escriba con detalle la demostración del siguiente Teorema, también incluya la demostración del Lema 5.18 del Mao.\ Teorema: Sea $f \in \mathcal{M}^2([0,T];\mathbb{R})$, sea ρ, τ dos tiempos de paro tales que $0 \leq \rho \leq \tau \leq T$. Entonces

$$\mathbb{E}\left(\int_\rho^\tau f(s)dW_s \mid \mathcal{F}_\rho\right) = 0, \tag{9.1}$$

$$\mathbb{E}\left(\left|\int_\rho^\tau f(s)dW_s\right|^2 \mid \mathcal{F}_\rho\right) = \mathbb{E}\left(\int_\rho^\tau |f(s)|^2 ds \mid \mathcal{F}_\rho\right). \tag{9.2}$$

Por el Teorema 5.14 y el teorema de paro de la martingala de Doob,

$$E(I(\tau)|\mathcal{F}_\rho) = I(\rho) \tag{9.3}$$

y

$$E(I^2(\tau) - \langle I, I\rangle_\tau|\mathcal{F}_\rho) = I^2(\rho) - \langle I, I\rangle_\rho, \tag{9.4}$$

donde $\{\langle I, I\rangle_t\}$ es definido por 5.18. Aplicando el Lema 5.18 se ve entonces de 5.22 que

$$\mathbb{E}\left(\int_\rho^\tau f(s)dB_s|\mathcal{F}_\rho\right) = \mathbb{E}(I(\tau) - I(\rho)|\mathcal{F}_\rho) = 0$$

que es (5.20). Además, por (5.22) y (5.23),

$$\mathbb{E}(|I(\tau) - I(\rho)|^2|\mathcal{F}_\rho) = \mathbb{E}(I^2(\tau)|\mathcal{F}_\rho) - 2I(\rho)\mathbb{E}(I(\tau)|\mathcal{F}_\rho) + I^2(\rho)$$

$$= \mathbb{E}(I^2(\tau)|\mathcal{F}_\rho) - I^2(\rho) = \mathbb{E}(\langle I, I\rangle_\tau - \langle I, I\rangle_\rho|\mathcal{F}_\rho) = \mathbb{E}\left(\int_\rho^\tau |f(s)|^2 ds|\mathcal{F}_\rho\right)$$

Exercise 9.3. Usando la aproximación de la suma de Riemann ecuación 6.1, la isometría de Itô y la identidad $4ab = (a+b)^2 - (a-b)^2$ pruebe que

$$\mathbb{E}\left[\left(\int_0^T g(t)dW_t\right)\left(\int_0^T f(t)dW_t\right)\right] = \int_0^T \mathbb{E}[f(t)g(t)]dt.$$

Tomemos $a = \int_0^T g(t)dW_t$ y $b = \int_0^T f(t)dW_t$, entonces usando la identidad $4ab = (a+b)^2 - (a-b)^2$

$$\begin{aligned} 4\left(\int_0^T g(t)dW_t\right)\left(\int_0^T f(t)dW_t\right) &= \left(\int_0^T g(t)dW_t + \int_0^T f(t)dW_t\right)^2 - \left(\int_0^T g(t)dW_t - \int_0^T f(t)dW_t\right)^2 \\ &= \left(\int_0^T (g(t)+f(t))dW_t\right)^2 - \left(\int_0^T (g(t)-f(t))dW_t\right)^2, \end{aligned}$$

así,

$$\begin{aligned}
4\mathbb{E}\left[\left(\int_0^T g(t)dW_t\right)\left(\int_0^T f(t)dW_t\right)\right] &= \mathbb{E}\left(\int_0^T (g(t)+f(t))dW_t\right)^2 - \mathbb{E}\left(\int_0^T (g(t)-f(t))dW_t\right)^2 \\
&= \left(\int_0^T \mathbb{E}(g(t)+f(t))^2 dt\right) - \left(\int_0^T \mathbb{E}(g(t)-f(t))^2 dt\right) \\
&= \left(\int_0^T \mathbb{E}[(g(t)+f(t))^2 - (g(t)-f(t))^2] dt\right) \\
&= 4\left(\int_0^T \mathbb{E}[g(t)f(t)]dt\right)
\end{aligned}$$

Exercise 9.4. Usando la suma de Riemann ecuación 6.1 y deduzca que,

$$\int_0^T W(t)^2 dW(t) = \frac{1}{3}W(T)^3 - \int_0^T W(t)dt.$$

Observemos que,

$$3W(t_i)^2(W(t_{i+1}) - W((t_i))) = W(t_{i+1})^3 - \left(W(t_{i+1}) - W(t_i)\right)^3 - 3\left(W(t_{i+1}) - W(t_i)\right)^2 W(t_i) - W(t_{i-1})^3,$$

aplicando la ecuación 6.1

$$\begin{aligned}
\int_0^T W(t)^2 dW(t) &\sim \sum_{i=0}^{L} W(t_i)^2(W(t_{i+1}) - W((t_i))) \\
&= \frac{1}{3}\sum_{i=0}^{L}[W(t_{i+1})^3 - W(t_{i-1})^3] - \frac{1}{3}\sum_{i=0}^{L}\left(W(t_{i+1}) - W(t_i)\right)^3 - \sum_{i=0}^{L}\left(W(t_{i+1}) - W(t_i)\right)^2 W(t_i) \\
&= \frac{1}{3}(W(T)^3 - W(t_0)^3) - \frac{1}{3}\sum_{i=0}^{L}\left(W(t_{i+1}) - W(t_i)\right)^3 - \sum_{i=0}^{L}\left(W(t_{i+1}) - W(t_i)\right)^2 W(t_i) \\
&= \frac{1}{3}W(T)^3 - \frac{1}{3}\sum_{i=0}^{L}\left(W(t_{i+1}) - W(t_i)\right)^3 - \sum_{i=0}^{L}\left(W(t_{i+1}) - W(t_i)\right)^2 W(t_i)
\end{aligned}$$

veamos que $\frac{1}{3}\sum_{i=0}^{L}\left(W(t_{i+1}) - W(t_i)\right)^3 \to 0$ en L^2.\ Ahora, calcularemos la media de la variación cuadrática. Del Teorema Multinomial

$$\frac{1}{9}\mathbb{E}\left[\left(\sum_{i=0}^{L}\left(W(t_{i+1}) - W(t_i)\right)^3\right)^2\right] = \frac{1}{9}\sum_{i=0}^{L}\mathbb{E}\left[\left(W(t_{i+1}) - W(t_i)\right)^6\right] + \frac{2}{9}\sum_{i=0}^{L}\mathbb{E}\left[\left(W(t_{i+1}) - W(t_i)\right)^3\left(W(t_{j+1}) - W\right.\right.$$

además, de la tarea 5,

$$\begin{aligned}
\mathbb{E}\left[\left(W(t_{i+1})-W(t_i)\right)^3\left(W(t_{j+1})-W(t_j)\right)^3\right] &= \mathbb{E}\{\mathbb{E}\left[\left(W(t_{i+1})-W(t_i)\right)^3\left(W(t_{j+1})-W(t_j)\right)^3\right]|\mathcal{F}_j\} \\
&= \mathbb{E}\{\left(W(t_{i+1})-W(t_i)\right)^3\mathbb{E}\left[\left(W(t_{j+1})-W(t_j)\right)^3\middle|\mathcal{F}_j\right]\} \\
&= \mathbb{E}[\left(W(t_{i+1})-W(t_i)\right)^3]\mathbb{E}\left[\left(W(t_{j+1})-W(t_j)\right)^3\right] \\
&= 0. \text{ de la tarea 5}
\end{aligned}$$

así,

$$\frac{2}{9}\sum_{i=0}^{L}\mathbb{E}\left[\left(W(t_{i+1})-W(t_i)\right)^3\left(W(t_{j+1})-W(t_j)\right)^3\right]=0$$

y también se tiene que $\mathbb{E}\left[\left(W(t_{i+1})-W(t_i)\right)^6\right]=15\left(t_{i+1}-t_i\right)^3$, así

$$\begin{aligned}
\frac{1}{9}\sum_{i=0}^{L}E\left[\left(W(t_{i+1})-W(t_i)\right)^6\right] &= \frac{5}{3}\sum_{i=0}^{L}\left(t_{i+1}-t_i\right)^3 \\
&\leq \frac{5}{3}\|\Delta_L\|^2\sum_{i=0}^{L}\left(t_{i+1}-t_i\right) \\
&\leq \frac{5}{3}\|\Delta_L\|^2 L\to 0, \|\Delta_L\|\to 0
\end{aligned}$$

Ahora veamos que

$$\sum_{i=0}^{L}\left(W(t_{i+1})-W(t_i)\right)^2 W(t_i)\to\sum_{i=0}^{L}W(t_i)\left(t_{i+1}-t_i\right)\ \text{ en } L^2$$

se tiene que,

$$\mathbb{E}\left[\left(\sum_{i=0}^{L}\left(W(t_{i+1})-W(t_i)\right)^2 W(t_i)-\sum_{i=0}^{L}W(t_i)\left(t_{i+1}-t_i\right)\right)^2\right] \quad = \quad \mathbb{E}\left[\left(\sum_{i=0}^{L}W(t_i)[\left(W(t_{i+1})-W(t_i)\right)^2-(t_{i+1}\right.\right.$$

$$\mathbb{E}\left[\sum_{i=0}^{L}W(t_i)^2[\left(W(t_{i+1})-W(t_i)\right)^2-(t_{i+1}-t_i)]^2+\sum_{i=0}^{L}W(t_i)W(t_j)[\left(W(t_{i+1})-W(t_i)\right)^2-(t_{i+1}-t_i)][\left(W(t_{j+1})\right.\right.$$

calculemos

$$\mathbb{E}\{\mathbb{E}[W(t_i)W(t_j)((W(t_{i+1})-W(t_i))^2-(t_{i+1}-t_i))((W(t_{j+1})-W(t_j))^2-(t_{j+1}-t_j))]|\mathcal{F}_j\} \quad =$$

$$\mathbb{E}\{W(t_i)W(t_j)((W(t_{i+1})-W(t_i))^2-(t_{i+1}-t_i))\mathbb{E}[((W(t_{j+1})-W(t_j))^2-(t_{j+1}-t_j))]|\mathcal{F}_j\}=0$$

Exercise 9.5. Verifique que la isometría de Itô ecuación 6.4,

$$\mathbb{E}\left[\left(\int_0^T h(t)dW(t)\right)^2\right] = \mathbb{E}\left[\int_0^T h(t)^2 dt\right],$$

se tiene cuando $h(t) := 1$.

del ejercicio 3, resulta

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^T h(t)dW(t)\right)^2\right] = \mathbb{E}\left[\left(\int_0^T 1dW(t)\right)^2\right] &= \mathbb{E}\left[\left(\int_0^T 1dW_t\right)\left(\int_0^T 1dW_t\right)\right] \\
&= \int_0^T \mathbb{E}[1]dt \\
&= \int_0^T dt \\
&= T.
\end{aligned}$$

y

$$\begin{aligned}
\mathbb{E}\left[\int_0^T h(t)^2 dt\right] &= \mathbb{E}\left[\int_0^T 1^2 dt\right] \\
&= \mathbb{E}\left[\int_0^T dt\right] \\
&= \mathbb{E}\left[T\right] \\
&= T
\end{aligned}$$

Así,

$$\mathbb{E}\left[\left(\int_0^T dW(t)\right)^2\right] = \mathbb{E}\left[\int_0^T dt\right]$$

10 Tarea 9

Exercise 10.1. El siguiente código calcula la aproximación de la Integral de Ito. Con $T = 1$, $L = 2^{13}$ correspondiente al error de $\mathcal{O}(10^{-3})$

```
import numpy as np
T = 1.0
L = 2**13
dt = T / L
dW = np.sqrt(dt) * np.random.normal(size=L)
W = np.zeros(L + 1)
W[1 :] = np.cumsum(dW)
ito_integral = np.sum(np.multiply(W[0: -1], dW))
err = np.abs(ito_integral - 0.5 * (W[-1] ** 2 - T))
```

Adapta este código para la Integral de Stratonovich correspondente y evalue el error.

```
from PythonScripts import brownian_motion as bw
import numpy as np

def u(t: float, n_points: int):
    dt = t / (n_points - 1)
    dw = np.sqrt(dt) * np.random.standard_normal(n_points - 1)
    w = np.zeros(n_points)
    w[1:] = dw.cumsum()
    time = np.linspace(0, t, n_points)
    return time, w

t_f = 1
n_p = 2** 13
t, bt = u(t_f, n_p)
y = 0.5 * np.sqrt(t[1] - t[0]) * np.random.standard_normal(n_p)
stratonovich = [(0.5 * (bt[i + 1] + bt[i]) + y[i])* (bt[i + 1] - bt[i]) for i in range(n_p
stratonovich = np.array(stratonovich).sum()
print(np.abs(stratonovich - 0.5 * bt[-1] ** 2))
```

```
0.0006194750188335957
```

Exercise 10.2. Escoja un integrando y computacionalmente verifique la Isometría de Ito.

```python
import numpy as np
from PythonScripts import brownian_motion as bw

# h(t) = Bt^2
def f(x: float, t: float):
    y = x ** 2
    return y


def fB(partition: np.array, x: float, t: float):
    y = 0
    for i in range(len(partition) - 1):
        if partition[i] <= t < partition[i + 1]:
            y = f(x, t)
    return y

def ito_n(n_points: int, t: float):
    time, w = bw.u(t, n_points)
    integral = np.zeros(n_points)
    for i in range(n_points - 1):
        integral[i] = fB(time, w[i], time[i]) * (w[i + 1] - w[i])
    ito = integral.sum()
    return ito


n = 500
t = 1

samples = 500

E_ito = np.zeros(samples)


for i in range(samples):
    E_ito[i] = ito_n(n, t) ** 2

ito_mean = E_ito.mean()
```

```
ew = np.zeros(n)
for i in range(samples):
    vector, b = bw.u(t, n)
    ws = (b**2) ** 2
    ew += ws

ew = ew * samples ** (-1)

r = np.zeros(n)
for i in range(n- 1):
    r[i] = ew[i] * (vector[i + 1] - vector[i])
riemann = r.sum()


# E[If^2] = INT_0^1 Ef^2
print(ito_mean - riemann)
```

```
1.7579967855941325
```

Exercise 10.3. Sea τ un tiempo de paro. Prueba que $W(t+\tau) - W(\tau)$ es un movimiento browniano.

Definamos

$$V_\tau(t) = W(t+\tau) - W(\tau),$$

notemos que

$$V_\tau(0) = 0,$$

Luego, considere para $s \leq t$

$$\begin{aligned} V_\tau(t) - V_\tau(s) &= W(t+\tau) - W(\tau) - [W(s+\tau) - W(\tau)] \\ &= W(t+\tau) - W(s+\tau) \sim N(0, t-s), \end{aligned}$$

esto además nos garantiza la independencia de los incrementos del Browniano.

Exercise 10.4. Sea $W_1(t), W_2(t)$ movimientos brownianos independientes con punto inicial $(W_1(0), W_2(0)) \neq (0,0)$. Defina $X_t = \ln\left(W_1^2(t) + W_2^2(t)\right)$.

10.0.1 Muestre que X_t es una martingala local.

Supongamos que X_t NO es una martingala local.

10.0.2 Muestre que $E|X_t| < \infty$ para cada $t > 0$.

Considere

$$\begin{aligned} X_t &= \ln\left(W_1^2(t) + W_2^2(t)\right), \\ \exp(X_t) &= W_1^2(t) + W_2^2(t). \end{aligned}$$

$$\begin{aligned} E[\exp(X_t)] &= E\left[W_1^2(t)\right] + E\left[W_2^2(t)\right] \\ &= 2t, \end{aligned}$$

Como $X_t \geq 0, \forall t$

$$\begin{aligned} X_t &\leq \exp(X_t) \\ E[X_t] &\leq 2t < \infty, \forall t \end{aligned}$$

10.0.3 Muestre que X_t no es una martingala.

Supongamos que existe $c \in \mathbb{R}$ tal que $E[X_t] = c, \forall t$. Entonces

$$\begin{aligned} E\left[\ln\left(W_1^2(t) + W_2^2(t)\right)\right] &= c \\ \int_0^\infty \ln\left(W_1^2(t) + W_2^2(t)\right) \mathrm{d}\mathcal{P} &= c, \end{aligned}$$

Como la integral es finita. Entonces

$$X_t \to 0, t \to \infty, \text{ c.s}$$

Luego,

$$W_1^2(t) + W_2^2(t) \to 1, t \to \infty, \text{c.s}$$

Sin embargo

$$E\left[W_1^2(t) + W_2^2(t)\right] = 2t \to \infty, t \to \infty,$$

entonces llegamos a una contradicci'on. Entonces $E[X_t]$ no es constante, por lo tanto X_t no puede ser martingala.

Considere

$$\tau_n = \inf_t \{X_t = n\}$$

Como X_t es no acotada. Entonces

$$\tau_n(\omega) \to \infty, n \to \infty, \forall n.$$

Ahora probaremos que Y_t es una martingala. Ahora, considere

$$Y_t = X_{\min\{t,\tau_n\}},$$

es adaptado con respecto a la filtración. Si $\tau_n > t$ lo tenemos por construcción. En caso contrario, para $n \in \mathbb{N}$.

$$Y_t = n$$

$$[Y_t = n] \subset [\tau_n < t] \in \mathcal{F}_t,$$

por ser tiempo de paro. Por lo tanto Y_t es adaptado a la filtración, por lo tanto nos queda probar que es una martingala.

Considere $s < t$.

$$\begin{aligned}
E[Y_t \mid \mathcal{F}_s] &= E\left[X_{\min\{t,\tau_n\}} \mid \mathcal{F}_s\right] \\
&= E\left[X_t 1_{[t<\tau_n]}(t) \mid \mathcal{F}_s\right] + E\left[X_t 1_{[\tau_n \leq t]}(t) \mid \mathcal{F}_s\right] \\
&= E\left[X_s 1_{[s<\tau_n]}(t) \mid \mathcal{F}_s\right] + E\left[X_{\tau_n} 1_{[\tau_n \leq t]}(t) \mid \mathcal{F}_s\right] \\
&= X_s 1_{[s<\tau_n]}(t) + X_{\tau_n} 1_{[\tau_n \leq s]}(s) \\
&= Y_s,
\end{aligned}$$

teniendo así que para cada n Y_t es una martingala.

11 Tarea 10

Exercise 11.1. Considere la ecuación diferencial estocástica lineal con ruido multiplicativo.

$$\mathrm{d}Y(t) = \left(\mu + \frac{1}{2}\sigma^2\right) Y(t)\,\mathrm{d}t + \sigma \mathrm{d}W(t)$$

Aplicando la Fórmula de Ito a la función

$$u(t,x) = y_0 \exp(\mu t + \sigma x).$$

Muestre que

$$Y(t) = Y(0) \exp(\mu t + \sigma W(t)),$$

resuelve la ecuación diferencial.

Considere

$$u(t,x) = y_0 \exp(\mu t + \sigma x)$$

Ahora

$$\begin{aligned}
\partial_t u &= \mu u \\
\partial_x u &= \sigma u \\
\partial_{xx} &= \sigma^2 u
\end{aligned}$$

Usando además que.

$$\mathrm{d}S_t = \mathrm{d}W_t$$

Entonces

$$\begin{aligned}
\mathrm{d}u(t, Y_t) &= \mu u \mathrm{d}t + \sigma u \mathrm{d}Y_t + \frac{1}{2}\sigma^2 u (\mathrm{d}Y_t)^2 \\
&= \mu u \mathrm{d}t + \sigma u \mathrm{d}W_t + \frac{1}{2}\sigma^2 u \mathrm{d}t \\
&= \left(\mu + \frac{1}{2}\sigma^2\right) u \mathrm{d}t + \ u \mathrm{d}W_t
\end{aligned}$$

Entonces

$$\begin{aligned} Y_t &= u\left(t, Y_t\right) \\ &= Y\left(0\right)\exp\left(\mu t + \sigma W_t\right) \end{aligned}$$

En general

$$\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t$$

Exercise 11.2. Use el hecho anterior, confirma que

$$Y\left(t\right) = Y\left(0\right)\exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W\left(t\right)\right),$$

resuelve,

$$\mathrm{d}Y\left(t\right) = \mu Y\left(t\right)\mathrm{d}t + \sigma Y\left(t\right)\mathrm{d}W\left(t\right)$$

Considere

$$u\left(t, x\right) = y_0 \exp\left(q_1 t + q_2 x\right)$$

Ahora

$$\begin{aligned} \partial_t u &= q_1 u \\ \partial_x u &= q_2 u \\ \partial_{xx} &= q_2^2 u \end{aligned}$$

Usando además que.

$$\mathrm{d}S_t = \mathrm{d}W_t$$

Entonces

$$\begin{aligned} \mathrm{d}u\left(t, Y_t\right) &= q_1 u \mathrm{d}t + q_2 u \mathrm{d}Y_t + \frac{1}{2}q_2^2 u \left(\mathrm{d}Y_t\right)^2 \\ &= q_1 u \mathrm{d}t + q_2 u \mathrm{d}W_t + \frac{1}{2}q_2^2 u \mathrm{d}t \\ &= \left(q_1 + \frac{1}{2}q_2^2\right)u\mathrm{d}t + q_2 u \mathrm{d}W_t, \end{aligned}$$

entonces,

$$q_2 = \sigma$$
$$q_1 + \frac{1}{2}\sigma^2 = \mu,$$

entonces

$$Y_t = Y(0)\exp(\mu t + \sigma W_t)$$

Exercise 11.3. Considere la siguiente ecuación diferencial estocástica lineal.

$$\mathrm{d}S(t) = (a_1 \mathrm{S}(t) + a_2)\,\mathrm{d}t + g(S(t))\,\mathrm{d}W(t),$$

donde $g : \mathbb{R} \to \mathbb{R}$ es cualquier función global de Lipschitz con crecimiento lineal, y a_1, a_2 son dos constantes diferentes de cero. Use la forma integral de la ecuación diferencial estocástica, la propiedad de martingala de la integral de Ito y la notación

$$m(t) = E[X_t],$$

para deducir que

$$m(t) - m(0) = a_1 \int_0^t m(s)\,\mathrm{d}s + a_2 t$$

Usando que $m(t)$ es la solución

$$\frac{\mathrm{d}m(t)}{\mathrm{d}t} = a_1 m(t) + a_2, m(0) = E[X_0]$$

Finalmente, muestre que

$$E[X(t)] = -\frac{a_2}{a_1} + \left(E[X(0)] + \frac{a_2}{a_1}\right)\exp(a_1 t)$$

Primero. Considerando la formula Integral.

$$\begin{aligned}
S(t) &= S(0) + \int_0^t (a_1 \mathrm{S}(t) + a_2)\,\mathrm{d}t + \int_0^t g(S(s))\,\mathrm{d}W(s) \\
&= S(0) + a_1 \int_0^t \mathrm{S}(t)\,\mathrm{d}t + a_2 t + \int_0^t g(S(s))\,\mathrm{d}W(s),
\end{aligned}$$

calculamos la esperanza.

$$
\begin{aligned}
m\left(t\right) = m\left(0\right) + a_1 \int_0^t m\left(t\right) \mathrm{d}t + a_2 t + \int_0^t E\left[g\left(S\left(s\right)\right)\right] \mathrm{d}B\left(s\right) \\
m\left(t\right) - m\left(0\right) = a_1 \int_0^t m\left(t\right) \mathrm{d}t + a_2 t + E\left[\int_0^t g\left(S\left(s\right)\right) \mathrm{d}B\left(s\right)\right]
\end{aligned}
$$

Como g es de lipschitz y con crecimiento lineal $g\left(S\right) \in L_{\mathrm{ad}}^2\left(\Omega_a^b\right)$, por lo tanto existe una constante c tal que

$$
E\left[\int_0^t g\left(S\left(s\right)\right) \mathrm{d}W\left(s\right)\right] = c, \ \forall t
$$

Luego, sabemos que

$$
\int_0^t g\left(S\left(s\right)\right) \mathrm{d}W\left(s\right) = \lim_{n\to\infty} \sum_{i=1}^{n} g\left(S\left(t_{i-1}\right)\right)\left(B_i - B_{i-1}\right),
$$

ahora, para cada i

$$
\begin{aligned}
E\left[g\left(S\left(t_{i-1}\right)\right)\left(B_i - B_{i-1}\right)\right] &= E\left[E\left[g\left(S\left(t_{i-1}\right)\right)\left(B_i - B_{i-1}\right) \mid \mathcal{F}_{i-1}\right]\right] \\
&= E\left[g\left(S\left(t_{i-1}\right)\right) E\left[\left(B_i - B_{i-1}\right) \mid \mathcal{F}_{i-1}\right]\right] \\
&= 0,
\end{aligned}
$$

por lo tanto

$$
m\left(t\right) - m\left(0\right) = a_1 \int_0^t m\left(t\right) \mathrm{d}t + a_2 t,
$$

ahora, considere su forma diferencial.

$$
\frac{\mathrm{d}m\left(t\right)}{\mathrm{d}t} = a_1 m\left(t\right) + a_2,
$$

es una ecuación diferencial lineal. Por lo tanto usaremos métodos conocidos para resolverla.

$$
\frac{\mathrm{d}m\left(t\right)}{\mathrm{d}t} - a_1 m\left(t\right) = a_2,
$$

defina

$$
\begin{aligned}
u &= \exp\left(\int a_1 \mathrm{d}t\right) \\
&= e^{-a_1 t},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left[um\left(t\right)\right] &= a_2 u \\
u\left(t\right)m\left(t\right) &= a_2\int u\left(t\right)\mathrm{d}t \\
m\left(t\right) &= \frac{a_2}{u\left(t\right)}\int u\left(t\right)\mathrm{d}t \\
&= a_2 e^{a_1 t}\int e^{-a_1 t}\mathrm{d}t \\
&= -a_2 e^{a_1 t}\left[-\frac{1}{a_1}e^{-a_1 t}+C\right] \\
m\left(t\right) &= -\frac{a_2}{a_1}-Ca_2 e^{a_1 t}
\end{aligned}
$$

Ahora, recordando la condición inicial.

$$
\begin{aligned}
m\left(0\right) &= -\frac{a_2}{a_1}-Ca_2 \\
\Rightarrow C &= -\frac{1}{a_1}-\frac{1}{a_2}m\left(0\right).
\end{aligned}
$$

Entonces

$$
E\left[X_t\right] = -\frac{a_2}{a_1}-\left[-\frac{a_2}{a_1}-E\left[X_0\right]\right]\exp\left(a_1 t\right)
$$

Exercise 11.4. Considere la siguiente ecuación diferencial estocástica lineal.

$$
\mathrm{d}S\left(t\right) = \left(\alpha\left(t\right)\mathrm{S}\left(t\right)\right)\mathrm{d}t+\beta\left(t\right)S\left(t\right)\mathrm{d}W\left(t\right), S\left(0\right)=s_0,
$$

con constantes s_0 y funciones α, β integrables. Use la formula de Ito con la fórmula

$$
u\left(t,x\right) = \ln\left(\frac{x}{S_0}\right),
$$

para deducir que

$$
S\left(t\right) = S\left(0\right)\exp\left(\int_0^t\left[\alpha\left(s\right)-\frac{1}{2}\beta^2\left(s\right)\right]\mathrm{d}s+\int_0^t\beta\left(s\right)\mathrm{d}W\left(s\right)\right)
$$

Ahora considere que

$$
\begin{aligned}
f\left(t\right) &= \beta\left(t\right)S\left(t\right) \\
g\left(t\right) &= \alpha\left(t\right)S\left(t\right),
\end{aligned}
$$

Calculamos las parciales de u.

$$u_t = 0$$
$$u_x = \frac{1}{x}$$
$$u_{xx} = -\frac{1}{x^2}$$

Entonces

$$\begin{aligned}
\mathrm{d}u\left(t, S_t\right) &= \alpha\left(t\right)\mathrm{d}t + \beta\left(t\right)\mathrm{d}W\left(t\right) - \frac{1}{2S_t^2}\left(dS_t\right)^2 \\
&= \alpha\left(t\right)\mathrm{d}t + \beta\left(t\right)\mathrm{d}W\left(t\right) - \frac{1}{2S_t^2}\beta^2 S_t^2\mathrm{d}t \\
&= \left[\alpha\left(t\right) - \frac{\beta^2\left(t\right)}{2}\right]\mathrm{d}t + \beta\left(t\right)\mathrm{d}W\left(t\right),
\end{aligned}$$

entonces

$$\begin{aligned}
\ln\left(\frac{S_t}{S_0}\right) &= \int_0^t \left[\alpha\left(t\right) - \frac{\beta^2\left(t\right)}{2}\right]\mathrm{d}t + \int_0^t \beta\left(t\right)\mathrm{d}W\left(t\right) \\
S_t &= S_0 \exp\left(\int_0^t \left[\alpha\left(t\right) - \frac{\beta^2\left(t\right)}{2}\right]\mathrm{d}t + \int_0^t \beta\left(t\right)\mathrm{d}W\left(t\right)\right)
\end{aligned}$$

Exercise 11.5. Considere la siguiente ecuación diferencial estocástica lineal.

$$\mathrm{d}S\left(t\right) = \left(\alpha\left(t\right)\mathrm{S}\left(t\right)\right)\mathrm{d}t + \beta\left(t\right)S\left(t\right)\mathrm{d}W\left(t\right), S\left(0\right) = s_0,$$

con constantes s_0 y funciones α, β integrables. Considere

$$\alpha\left(t\right) = \sin\left(t\right)$$
$$\beta\left(t\right) = \frac{t}{1+t}$$
$$s_0 = 1,$$

sobre el intervalo $[0, 5]$.

Usando el acercamiento apropiado, la salida del código reproduce 200 realizaciones de la solución con el proceso de Euler-Maruyama.

Adapta el código para obtener la media de la solución de 1000 realizaciones y comparalo con la media de la solución de la forma diferencial, usando los mismos parámetros. Ilustra la diferencia con un log-plot de

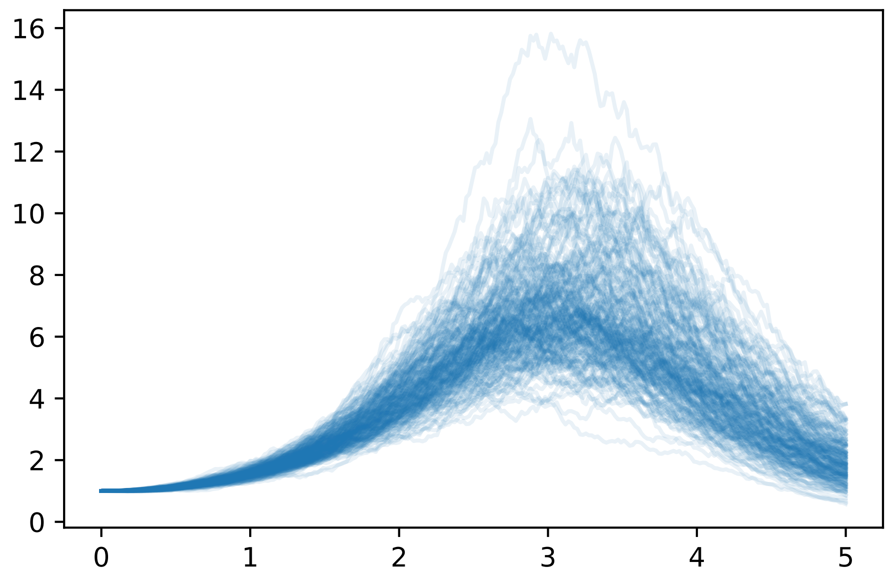
$$\ln\left|S\left(t\right)-\tilde{S}\left(t\right)\right|,$$

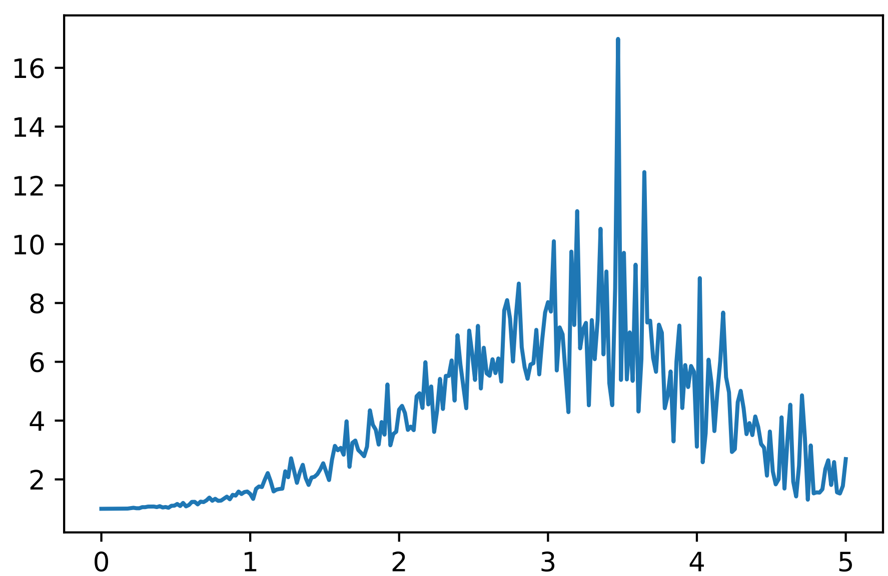
donde S es la solución de Euler y $\tilde{S}$ es la solución de la diferencial.

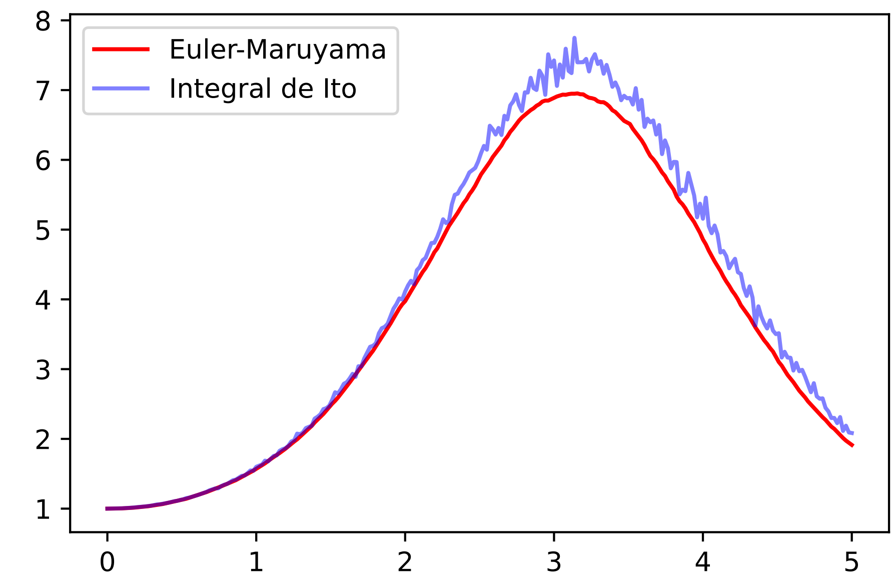
```
import numpy as np
import matplotlib.pyplot as plt
from PythonScripts import brownian_motion as bw

def alpha(t):
    y = np.sin(t)
    return y
def beta(t):
    y = t / (1.0 + t)
    return y

def drift(t, x):
    a = alpha(t) * x
    return a

def diffusion(t, x):
    b = beta(t) * x
    return b

samples = 200
sigma = 2 ** (-2)
n_p = 2 ** 8
T = 5.0
x_0 = 1.0

def get_em_solution(x_0, T, N, sigma):
    x_t = np.zeros(N)
    x_t[0] = x_0
    dt = T / N
    t, W = bw.u(T, N)
    for i in np.arange(N - 1):
        w_inc = W[i + 1] - W[i]
        f = drift(t[i], x_t[i])
        g = diffusion(t[i], x_t[i])
        x_t[i + 1] = x_t[i] + f * dt + sigma * g * w_inc  # Importante la sigma.
    return t, x_t
```

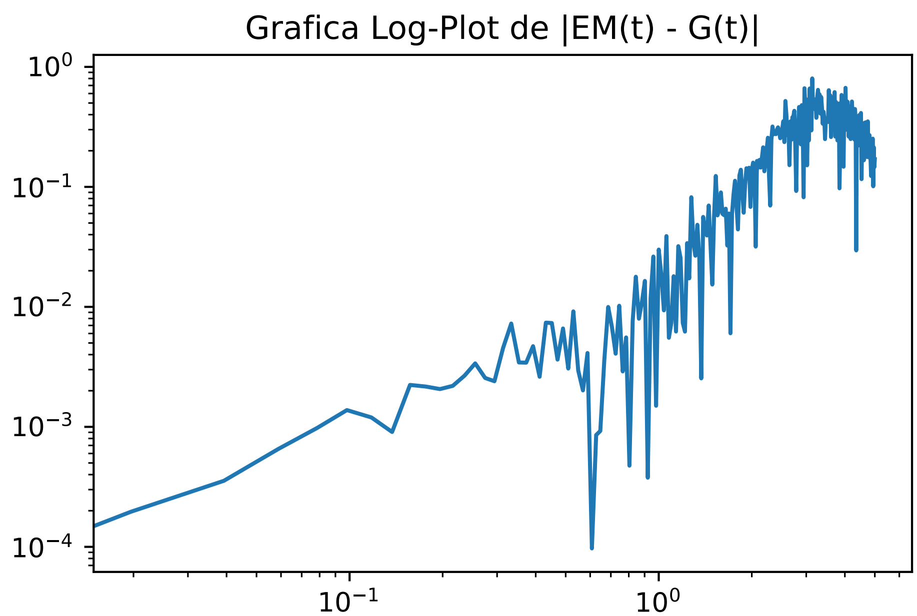
```python
fig, ax = plt.subplots()
df = []
for k in np.arange(samples):
    t, x_t = get_em_solution(x_0, T, n_p, sigma)
    df.append(x_t)
    # df.append([t, x_t])
    ax.plot(t, x_t, color="C0", alpha=0.1)
plt.show()


## Analitico

s_0 = x_0

def u(t, sigma):
    y = alpha(t) - 0.5 * (sigma * beta(t)) ** 2
    return y
def v(t,sigma):
    y = sigma * beta(t)
    return y

def ito_n(n_points: int, t: float, sigma: float):
    time, w = bw.u(t, n_points)
    integral = np.zeros(n_points)
    for i in range(n_points - 1):
        integral[i] = v(time[i], sigma) * (w[i + 1] - w[i])
    ito = integral.sum()
    return ito


def riemann_integral(a, b, n_points, sigma):
    r = np.zeros(n_points)
    time = np.linspace(a,b,n_points)
    for i in range(n_points - 1):
        r[i] = u(time[i], sigma) * (time[i + 1] - time[i])
    riemann = r.sum()
    return riemann

def St(a_0,t,n_p, sigma):
    rmnn = riemann_integral(0, t, n_p, sigma)
    ito = ito_n(n_p, t, sigma)
```

```python
    y = a_0 * np.exp(rmnn + ito)
    return y

S = []
for i in range(n_p):
    S.append(St(x_0, t[i], n_p, sigma))

plt.plot(t, S)
plt.show()

# Simulado EM

em_solutions = np.array(df)

em_mean = em_solutions.mean(axis = 0)
# Simulado, Solucion General.

general_estimation = np.zeros((samples, n_p))

for i in range(samples):
    for j in range(n_p):
        general_estimation[i, j] = St(x_0, t[j], n_p, sigma)

general_mean = general_estimation.mean(axis = 0)

#
plt.plot(t, em_mean, 'r')
plt.plot(t, general_mean, 'b', alpha = 0.5)
plt.legend(['Euler-Maruyama','Integral de Ito'])
plt.show()


##
plt.loglog(t, np.abs(em_mean - general_mean))
plt.title("Grafica Log-Plot de |EM(t) - G(t)|")
plt.show()
```


0
2
4
6
8
10
12
14
16
0
1
2
3
4
5


2
4
6
8
10
12
14
16
0
1
2
3
4
5


Euler-Maruyama
Integral de Ito
8
7
6
5
4
3
2
1
0
1
2
3
4
5


Grafica Log-Plot de |EM(t) - G(t)|
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-1}
10^{0}