restart; #*Pure duopoly capacity constraint*

$Q := q[a] + q[b]$;

$$Q := q_a + q_b \tag{1}$$

$P := 1 - Q$;

$$P := 1 - q_a - q_b \tag{2}$$

$TC[a] := m \cdot q[a] + (q[a] - x[a])^2 + F$;

$$TC_a := m\, q_a + (q_a - x_a)^2 + F \tag{3}$$

$TC[b] := m \cdot q[b] + (q[b] - x[b])^2 + F$;

$$TC_b := m\, q_b + (q_b - x_b)^2 + F \tag{4}$$

$profit[a] := q[a] * P - TC[a]$;

$$profit_a := (1 - q_a - q_b)\, q_a - m\, q_a - (q_a - x_a)^2 - F \tag{5}$$

$profit[b] := q[b] * P - TC[b]$;

$$profit_b := (1 - q_a - q_b)\, q_b - m\, q_b - (q_b - x_b)^2 - F \tag{6}$$

$SS := \frac{Q^2}{2} + profit[a] + profit[b]$;

$$SS := \frac{(q_a + q_b)^2}{2} + (1 - q_a - q_b)\, q_a - m\, q_a - (q_a - x_a)^2 - 2\, F + (1 - q_a - q_b)\, q_b - m\, q_b - (q_b - x_b)^2 \tag{7}$$

$a > 0$;

$$0 < a \tag{8}$$

$P > 0$;

$$0 < 1 - q_a - q_b \tag{9}$$

$q[a] \geq 0$;

$$0 \leq q_a \tag{10}$$

$q[b] \geq 0$;

$$0 \leq q_b \tag{11}$$

$m \geq 0$;

$$0 \leq m \tag{12}$$

#*Second stage: simultaneous quantity choice*

$FOC[q[a]] := diff(profit[a], q[a])$;

$$FOC_{q_a} := -4\, q_a + 1 - q_b - m + 2\, x_a \tag{13}$$

$q[a](q[b], q[c], x[a]) := solve(FOC[q[a]], q[a])$;

$$q_a(q_b, q_c, x_a) := \frac{1}{4} - \frac{q_b}{4} - \frac{m}{4} + \frac{x_a}{2} \tag{14}$$

$FOC[q[b]] := diff(profit[b], q[b])$;

$$FOC_{q_b} := -4\,q_b + 1 - q_a - m + 2\,x_b \tag{15}$$

$q[b](q[a], q[c], x[c]) := solve(\%, q[b]);$

$$q_b(q_a, q_c, x_c) := \frac{1}{4} - \frac{q_a}{4} - \frac{m}{4} + \frac{x_b}{2} \tag{16}$$

##SS;
##FOC[q[c]] := diff(SS, q[c]);
##FOC[q[c]] := diff(profit[c], q[c]);
##q[c](q[a], q[c], x[b]) := solve(%, q[c]);
##FOCN:=piecewise(0 ≤ %, FOC[q[c]]);
sys := { FOC[q[a]], FOC[q[b]]};

dsolve(sys, {q[a], q[b]});

$$sys := \{-4\,q_a + 1 - q_b - m + 2\,x_a,\ -4\,q_b + 1 - q_a - m + 2\,x_b\}$$

$$\left\{q_a = \frac{1}{5} - \frac{m}{5} + \frac{8\,x_a}{15} - \frac{2\,x_b}{15},\ q_b = \frac{1}{5} - \frac{m}{5} - \frac{2\,x_a}{15} + \frac{8\,x_b}{15}\right\} \tag{17}$$

$q[a](x[a], x[b], x[c]) := rhs(\%[1]);$

$$q_a(x_a, x_b, x_c) := \frac{1}{5} - \frac{m}{5} + \frac{8\,x_a}{15} - \frac{2\,x_b}{15} \tag{18}$$

$q[b](x[a], x[b], x[c]) := rhs(\%\%[2]);$

$$q_b(x_a, x_b, x_c) := \frac{1}{5} - \frac{m}{5} - \frac{2\,x_a}{15} + \frac{8\,x_b}{15} \tag{19}$$

#First stage: simultaneous capacity choice

$profit[a];$

$$\left(1 - q_a - q_b\right) q_a - m\,q_a - \left(q_a - x_a\right)^2 - F \tag{20}$$

$subs(q[a] = q[a](x[a], x[b], x[c]), q[b] = q[b](x[a], x[b], x[c]), q[c] = q[c](x[a], x[b], x[c]),$
$profit[a]);$

$$\left(\frac{3}{5} + \frac{2\,m}{5} - \frac{2\,x_a}{5} - \frac{2\,x_b}{5}\right)\left(\frac{1}{5} - \frac{m}{5} + \frac{8\,x_a}{15} - \frac{2\,x_b}{15}\right) - m\left(\frac{1}{5} - \frac{m}{5} + \frac{8\,x_a}{15} - \frac{2\,x_b}{15}\right) - \left(\frac{1}{5} - \frac{m}{5} - \frac{7\,x_a}{15} - \frac{2\,x_b}{15}\right)^2 - F \tag{21}$$

$FOC[x[a]] := diff(\%, x[a]);$

$$FOC_{x_a} := \frac{32}{75} - \frac{32\,m}{75} - \frac{194\,x_a}{225} - \frac{64\,x_b}{225} \tag{22}$$

$x[a](x[b], x[c]) := rhs(isolate(\%, x[a]));$

$$x_a(x_b, x_c) := \frac{48}{97} - \frac{48\,m}{97} - \frac{32\,x_b}{97} \tag{23}$$

$profit[b];$

$$\left(1 - q_a - q_b\right) q_b - m\, q_b - \left(q_b - x_b\right)^2 - F \tag{24}$$

$subs(q[a] = q[a](x[a], x[b], x[c]), q[b] = q[b](x[a], x[b], x[c]), q[c] = q[c](x[a], x[b], x[c]),$
$profit[b]);$

$$\left(\frac{3}{5} + \frac{2\,m}{5} - \frac{2\,x_a}{5} - \frac{2\,x_b}{5}\right)\left(\frac{1}{5} - \frac{m}{5} - \frac{2\,x_a}{15} + \frac{8\,x_b}{15}\right) - m\left(\frac{1}{5} - \frac{m}{5} - \frac{2\,x_a}{15} + \frac{8\,x_b}{15}\right) - \left(\frac{1}{5} - \frac{m}{5} - \frac{2\,x_a}{15} - \frac{7\,x_b}{15}\right)^2 - F \tag{25}$$

$FOC[x[b]] := diff(\%, x[b]);$

$$FOC_{x_b} := \frac{32}{75} - \frac{32\,m}{75} - \frac{64\,x_a}{225} - \frac{194\,x_b}{225} \tag{26}$$

$evala(Simplify(\%))$

$$\frac{32}{75} - \frac{32\,m}{75} - \frac{64\,x_a}{225} - \frac{194\,x_b}{225} \tag{27}$$

$x[b](x[a], x[c]) := rhs(isolate(\%, x[b]));$

$$x_b(x_a, x_c) := \frac{48}{97} - \frac{48\,m}{97} - \frac{32\,x_a}{97} \tag{28}$$

$evala(Simplify(\%));$

$$\frac{48}{97} - \frac{48\,m}{97} - \frac{32\,x_a}{97} \tag{29}$$

$sys := \{FOC[x[a]], FOC[x[b]]\};$

$solve(sys, \{x[a], x[b]\});$

$$sys := \left\{\frac{32}{75} - \frac{32\,m}{75} - \frac{194\,x_a}{225} - \frac{64\,x_b}{225}, \frac{32}{75} - \frac{32\,m}{75} - \frac{64\,x_a}{225} - \frac{194\,x_b}{225}\right\}$$

$$\left\{x_a = \frac{16}{43} - \frac{16\,m}{43}, x_b = \frac{16}{43} - \frac{16\,m}{43}\right\} \tag{30}$$

$x[a](x[c]) := rhs(\%[1]);$

$$x_a(x_c) := \frac{16}{43} - \frac{16\,m}{43} \tag{31}$$

$x[b](x[c]) := rhs(\%\%[2]);$

$$x_b(x_c) := \frac{16}{43} - \frac{16\,m}{43} \tag{32}$$

$q[a](x[a], x[b], x[c]);$

$$\frac{1}{5} - \frac{m}{5} + \frac{8\,x_a}{15} - \frac{2\,x_b}{15} \tag{33}$$

$q[a](x[c]) := subs(x[a] = (x[a](x[c])), x[b] = (x[b](x[c])), \%);$

$$q_a(x_c) := \frac{15}{43} - \frac{15\,m}{43} \tag{34}$$

$q[b](x[a], x[b], x[c]);$

$$\frac{1}{5} - \frac{m}{5} - \frac{2\,x_a}{15} + \frac{8\,x_b}{15} \tag{35}$$

$q[b](x[c]) := subs(x[a] = (x[a](x[c])), x[b] = (x[b](x[c])), \%);$

$$q_b(x_c) := \frac{15}{43} - \frac{15\,m}{43} \tag{36}$$

$\#\#q[c](x[a], x[b], x[c]);$
$\#\#q[c](x[c]) := subs(x[a] = (x[a](x[c])), x[b] = (x[b](x[c])), \%);$
$\#\#First\ stage$
$SS;$

$$\frac{(q_a + q_b)^2}{2} + (1 - q_a - q_b)\,q_a - m\,q_a - (q_a - x_a)^2 - 2\,F + (1 - q_a - q_b)\,q_b - m\,q_b - (q_b - x_b)^2 \tag{37}$$

$subs(q[a] = q[a](x[c]), q[b] = (q[b](x[c])), q[c] = q[c](x[c]), x[a] = (x[a](x[c])), x[b] = (x[b](x[c])), \%);$

$$\frac{\left(\frac{30}{43} - \frac{30\,m}{43}\right)^2}{2} + 2\left(\frac{13}{43} + \frac{30\,m}{43}\right)\left(\frac{15}{43} - \frac{15\,m}{43}\right) - 2\,m\left(\frac{15}{43} - \frac{15\,m}{43}\right) - 2\left(-\frac{1}{43} + \frac{m}{43}\right)^2 - 2\,F \tag{38}$$

$\#\#subs(q[a] = q[a](x[c]), q[b] = (q[b](x[c])), q[c] = q[c](x[c]), \%)$
$\#\#subs(x[a] = (x[a](x[c])), x[b] = (x[b](x[c])), \%)$
$\#\#SSx[c] := evala(Simplify(\%));$
$\#\#FOC[x[c]] := diff(\%, x[c]);$
$\#\#xstar[c] := rhs(isolate(\%, x[c]));$
$qstar[a] := subs(x[c] = xstar[c], q[a](x[c]));$

$$qstar_a := \frac{15}{43} - \frac{15\,m}{43} \tag{39}$$

$qstar[b] := subs(x[c] = xstar[c], q[b](x[c]));$

$$qstar_b := \frac{15}{43} - \frac{15\,m}{43} \tag{40}$$

$\#\#qstar[c] := subs(x[c] = xstar[c], q[c](x[c]));$
$xstar[a] := subs(x[c] = xstar[c], x[a](x[c]));$

$$xstar_a := \frac{16}{43} - \frac{16\,m}{43} \tag{41}$$

$xstar[b] := subs(x[c] = xstar[c], x[b](x[c]));$

$$xstar_b := \frac{16}{43} - \frac{16\,m}{43} \tag{42}$$

$Qstar := qstar[a] + qstar[b];$

$$Qstar := \frac{30}{43} - \frac{30\,m}{43} \tag{43}$$

$Pstar := subs(q[a] = qstar[a], q[b] = qstar[b], P);$

$$Pstar := \frac{13}{43} + \frac{30\,m}{43} \tag{44}$$

$Xstar := xstar[a] + xstar[b];$

$$Xstar := \frac{32}{43} - \frac{32\,m}{43} \tag{45}$$

```
subs(q[a] = qstar[a], q[b] = qstar[b], q[c] = qstar[c], x[a] = xstar[a], x[b] = xstar[b], x[c]
    = xstar[c], SS);
```

$$\frac{\left(\frac{30}{43} - \frac{30\,m}{43}\right)^2}{2} + 2\left(\frac{13}{43} + \frac{30\,m}{43}\right)\left(\frac{15}{43} - \frac{15\,m}{43}\right) - 2\,m\left(\frac{15}{43} - \frac{15\,m}{43}\right) - 2\left(-\frac{1}{43} + \frac{m}{43}\right)^2 - 2\,F \tag{46}$$

```
SSfinal := evala(Simplify(%));
```

$$SSfinal := \frac{838}{1849} - \frac{1676}{1849}\,m + \frac{838}{1849}\,m^2 - 2\,F \tag{47}$$

```
simplify(%, sqrt, symbolic);
```

$$\frac{838}{1849} - \frac{1676}{1849}\,m + \frac{838}{1849}\,m^2 - 2\,F \tag{48}$$

```
subs(q[a] = qstar[a], q[b] = qstar[b], q[c] = qstar[c], x[a] = xstar[a], x[b] = xstar[b], x[c]
    = xstar[c], profit[a]);
```

$$\left(\frac{13}{43} + \frac{30\,m}{43}\right)\left(\frac{15}{43} - \frac{15\,m}{43}\right) - m\left(\frac{15}{43} - \frac{15\,m}{43}\right) - \left(-\frac{1}{43} + \frac{m}{43}\right)^2 - F \tag{49}$$

```
simplify(%, sqrt, symbolic);
```

$$\frac{194}{1849} - \frac{388}{1849}\,m + \frac{194}{1849}\,m^2 - F \tag{50}$$

```
%·2;
```

$$\frac{388}{1849} - \frac{776}{1849}\,m + \frac{388}{1849}\,m^2 - 2\,F \tag{51}$$

```
CS := (Qstar)^2/2;
```

$$CS := \frac{\left(\frac{30}{43} - \frac{30\,m}{43}\right)^2}{2} \tag{52}$$

```
simplify(%, symbolic);
```

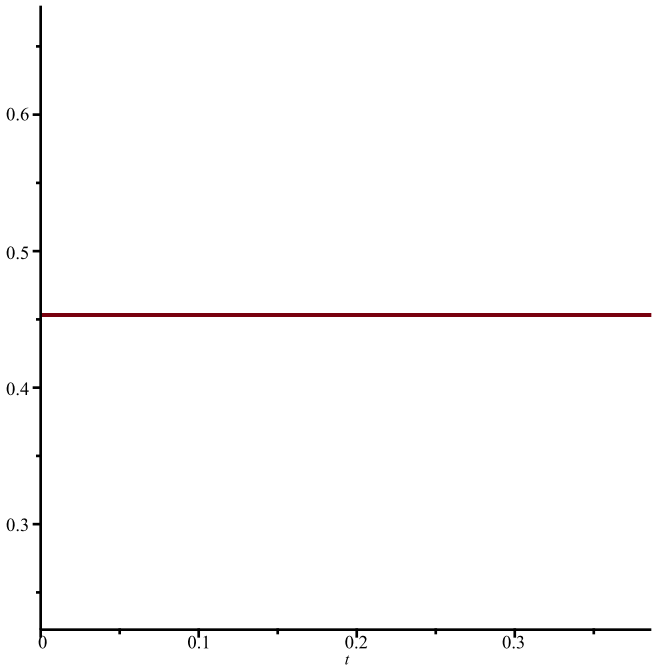
$$\frac{450\,(-1+m)^2}{1849} \tag{53}$$

```
subs(m = 0, F = 0, SSfinal); evalf(%, 10);
```

$$\frac{838}{1849}$$

$$0.4532179557 \tag{54}$$

```
plot(%%, t = 0 .. 0.386405562);
```



0.4532179557 **(55)**

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