restart; # *Mixed triopoly with capacity constraint and a transfer*

$Q := q[a] + q[b] + q[c]$;

$$Q := q_a + q_b + q_c \tag{1}$$

$P := 1 - Q$;

$$P := 1 - q_a - q_b - q_c \tag{2}$$

$TC[a] := m \cdot q[a] + (q[a] - x[a])^2 + F$;

$$TC_a := m\, q_a + (q_a - x_a)^2 + F \tag{3}$$

$TC[b] := m \cdot q[b] + (q[b] - x[b])^2 + F$;

$$TC_b := m\, q_b + (q_b - x_b)^2 + F \tag{4}$$

$TC[c] := m \cdot q[c] + t \cdot q[c] + (q[c] - x[c])^2 + F$;

$$TC_c := m\, q_c + t\, q_c + (q_c - x_c)^2 + F \tag{5}$$

$profit[a] := q[a] * P - TC[a]$;

$$profit_a := (1 - q_a - q_b - q_c)\, q_a - m\, q_a - (q_a - x_a)^2 - F \tag{6}$$

$profit[b] := q[b] * P - TC[b]$;

$$profit_b := (1 - q_a - q_b - q_c)\, q_b - m\, q_b - (q_b - x_b)^2 - F \tag{7}$$

$profit[c] := q[c] * P - TC[c]$;

$$profit_c := (1 - q_a - q_b - q_c)\, q_c - m\, q_c - t\, q_c - (q_c - x_c)^2 - F \tag{8}$$

$SS := \frac{Q^2}{2} + profit[a] + profit[b] + profit[c] + t \cdot q[c]$;

$$SS := \frac{(q_a + q_b + q_c)^2}{2} + (1 - q_a - q_b - q_c)\, q_a - m\, q_a - (q_a - x_a)^2 - 3\, F + (1 - q_a - q_b - q_c)\, q_b - m\, q_b - (q_b - x_b)^2 + (1 - q_a - q_b - q_c)\, q_c - m\, q_c - (q_c - x_c)^2 \tag{9}$$

$P > 0$;

$$0 < 1 - q_a - q_b - q_c \tag{10}$$

$q[a] > 0$;

$$0 < q_a \tag{11}$$

$q[b] > 0$;

$$0 < q_b \tag{12}$$

$m > 0$;

$$0 < m \tag{13}$$

#Fourth Stage: simultaneous quantity choice

$FOC[q[a]] := diff(profit[a], q[a])$;

$$FOC_{q_a} := -4\, q_a + 1 - q_b - q_c - m + 2\, x_a \tag{14}$$

$q[a](q[b], q[c], x[a]) := solve(FOC[q[a]], q[a])$;

$$q_a(q_b, q_c, x_a) := \frac{1}{4} - \frac{q_b}{4} - \frac{q_c}{4} - \frac{m}{4} + \frac{x_a}{2} \tag{15}$$

$FOC[q[b]] := diff(profit[b], q[b])$;

$$FOC_{q_b} := -4\,q_b + 1 - q_a - q_c - m + 2\,x_b \tag{16}$$

$q[b](q[a], q[c], x[c]) := solve(\%, q[b]);$

$$q_b(q_a, q_c, x_c) := \frac{1}{4} - \frac{q_a}{4} - \frac{q_c}{4} - \frac{m}{4} + \frac{x_b}{2} \tag{17}$$

$SS;$

$$\frac{(q_a + q_b + q_c)^2}{2} + (1 - q_a - q_b - q_c)\,q_a - m\,q_a - (q_a - x_a)^2 - 3\,F + (1 - q_a - q_b - q_c)\,q_b \tag{18}$$
$$- m\,q_b - (q_b - x_b)^2 + (1 - q_a - q_b - q_c)\,q_c - m\,q_c - (q_c - x_c)^2$$

$FOC[q[c]] := diff(SS, q[c]);$

$$FOC_{q_c} := -q_a - q_b - 3\,q_c + 1 - m + 2\,x_c \tag{19}$$

$\#FOC[q[c]] := diff(profit[c], q[c]);$
$q[c](q[a], q[c], x[b]) := solve(\%, q[c]);$

$$q_c(q_a, q_c, x_b) := -\frac{q_a}{3} - \frac{q_b}{3} + \frac{1}{3} - \frac{m}{3} + \frac{2\,x_c}{3} \tag{20}$$

$FOCN := piecewise(0 < \%, FOC[q[c]], 0 \geq \%, 0);$

$$FOCN := \begin{cases} -q_a - q_b - 3\,q_c + 1 - m + 2\,x_c & 0 < -\frac{q_a}{3} - \frac{q_b}{3} + \frac{1}{3} - \frac{m}{3} + \frac{2\,x_c}{3} \\ 0 & -\frac{q_a}{3} - \frac{q_b}{3} - \frac{m}{3} + \frac{2\,x_c}{3} \leq -\frac{1}{3} \end{cases} \tag{21}$$

$sys := \{FOC[q[a]], FOC[q[b]], FOC[q[c]]\};$

$solve(sys, \{q[a], q[b], q[c]\});$

$$sys := \{-4\,q_a + 1 - q_b - q_c - m + 2\,x_a, -q_a - q_b - 3\,q_c + 1 - m + 2\,x_c, -4\,q_b + 1 - q_a - q_c - m + 2\,x_b\}$$

$$\left\{q_a = \frac{2}{13} - \frac{2\,m}{13} + \frac{22\,x_a}{39} - \frac{4\,x_b}{39} - \frac{2\,x_c}{13}, q_b = \frac{2}{13} - \frac{2\,m}{13} - \frac{4\,x_a}{39} + \frac{22\,x_b}{39} - \frac{2\,x_c}{13}, q_c = \frac{3}{13} - \frac{3\,m}{13} - \frac{2\,x_a}{13} - \frac{2\,x_b}{13} + \frac{10\,x_c}{13}\right\} \tag{22}$$

$q[a](x[a], x[b], x[c]) := rhs(\%[1]);$

$$q_a(x_a, x_b, x_c) := \frac{2}{13} - \frac{2\,m}{13} + \frac{22\,x_a}{39} - \frac{4\,x_b}{39} - \frac{2\,x_c}{13} \tag{23}$$

$q[b](x[a], x[b], x[c]) := rhs(\%\%[2]);$

$$q_b(x_a, x_b, x_c) := \frac{2}{13} - \frac{2\,m}{13} - \frac{4\,x_a}{39} + \frac{22\,x_b}{39} - \frac{2\,x_c}{13} \tag{24}$$

$q[c](x[a], x[b], x[c]) := rhs(\%\%\%[3]);$

$$q_c(x_a, x_b, x_c) := \frac{3}{13} - \frac{3\,m}{13} - \frac{2\,x_a}{13} - \frac{2\,x_b}{13} + \frac{10\,x_c}{13} \tag{25}$$

#Third Stage: capacity choice of private firms
$profit[a];$

$$\left(1-q_a-q_b-q_c\right)q_a-m\,q_a-\left(q_a-x_a\right)^2-F \tag{26}$$

$subs(q[a]=q[a](x[a],x[b],x[c]),q[b]=q[b](x[a],x[b],x[c]),q[c]=q[c](x[a],x[b],x[c]),$
$profit[a]);$

$$\left(\frac{6}{13}+\frac{7\,m}{13}-\frac{4\,x_a}{13}-\frac{4\,x_b}{13}-\frac{6\,x_c}{13}\right)\left(\frac{2}{13}-\frac{2\,m}{13}+\frac{22\,x_a}{39}-\frac{4\,x_b}{39}-\frac{2\,x_c}{13}\right)-m\left(\frac{2}{13}-\frac{2\,m}{13}+\frac{22\,x_a}{39}-\frac{4\,x_b}{39}-\frac{2\,x_c}{13}\right)-\left(\frac{2}{13}-\frac{2\,m}{13}-\frac{17\,x_a}{39}-\frac{4\,x_b}{39}-\frac{2\,x_c}{13}\right)^2-F \tag{27}$$

$FOC[x[a]] := diff(\%, x[a]);$

$$FOC_{x_a} := \frac{176}{507}-\frac{176\,m}{507}-\frac{1106\,x_a}{1521}-\frac{352\,x_b}{1521}-\frac{176\,x_c}{507} \tag{28}$$

$x[a](x[b],x[c]) := solve(\%, x[a]);$

$$x_a\left(x_b,x_c\right) := \frac{264}{553}-\frac{264\,m}{553}-\frac{176\,x_b}{553}-\frac{264\,x_c}{553} \tag{29}$$

$profit[b];$

$$\left(1-q_a-q_b-q_c\right)q_b-m\,q_b-\left(q_b-x_b\right)^2-F \tag{30}$$

$subs(q[a]=q[a](x[a],x[b],x[c]),q[b]=q[b](x[a],x[b],x[c]),q[c]=q[c](x[a],x[b],x[c]),$
$profit[b]);$

$$\left(\frac{6}{13}+\frac{7\,m}{13}-\frac{4\,x_a}{13}-\frac{4\,x_b}{13}-\frac{6\,x_c}{13}\right)\left(\frac{2}{13}-\frac{2\,m}{13}-\frac{4\,x_a}{39}+\frac{22\,x_b}{39}-\frac{2\,x_c}{13}\right)-m\left(\frac{2}{13}-\frac{2\,m}{13}-\frac{4\,x_a}{39}+\frac{22\,x_b}{39}-\frac{2\,x_c}{13}\right)-\left(\frac{2}{13}-\frac{2\,m}{13}-\frac{4\,x_a}{39}-\frac{17\,x_b}{39}-\frac{2\,x_c}{13}\right)^2-F \tag{31}$$

$FOC[x[b]] := diff(\%, x[b]);$

$$FOC_{x_b} := \frac{176}{507}-\frac{176\,m}{507}-\frac{352\,x_a}{1521}-\frac{1106\,x_b}{1521}-\frac{176\,x_c}{507} \tag{32}$$

$x[b](x[a],x[c]) := solve(\%, x[b]);$

$$x_b\left(x_a,x_c\right) := \frac{264}{553}-\frac{264\,m}{553}-\frac{176\,x_a}{553}-\frac{264\,x_c}{553} \tag{33}$$

$\#\#subs(q[a]=q[a](x[a],x[b],x[c]),q[b]=q[b](x[a],x[b],x[c]),q[c]=q[c](x[a],x[b],x[c]),$
$SS);$
$\#\#tert:=diff(\%, x[c]);$
$\#\#solve(\%, x[c]);$
$x[a](x[b],x[c])+x[b](x[a],x[c]);$

$$\frac{528}{553}-\frac{528\,m}{553}-\frac{176\,x_b}{553}-\frac{528\,x_c}{553}-\frac{176\,x_a}{553} \tag{34}$$

$solve(\%, x[c]);$

$$1-m-\frac{x_b}{3}-\frac{x_a}{3} \tag{35}$$

$\#\#ses := \{FOC[x[a]], FOC[x[b]], tert\};$
$\#\#solve(\ ses, \{x[a], x[b], x[c]\});$
$sys := \{\ FOC[x[a]], FOC[x[b]]\};$

$solve(\ sys,\ \{x[a], x[b]\});$

$$sys := \left\{ \frac{176}{507} - \frac{176\,m}{507} - \frac{1106\,x_a}{1521} - \frac{352\,x_b}{1521} - \frac{176\,x_c}{507},\ \frac{176}{507} - \frac{176\,m}{507} - \frac{352\,x_a}{1521} - \frac{1106\,x_b}{1521} - \frac{176\,x_c}{507} \right\}$$

$$\left\{ x_a = \frac{88}{243} - \frac{88\,m}{243} - \frac{88\,x_c}{243},\ x_b = \frac{88}{243} - \frac{88\,m}{243} - \frac{88\,x_c}{243} \right\} \tag{36}$$

$x[a](x[c]) := rhs(\%[1]);$

$$x_a(x_c) := \frac{88}{243} - \frac{88\,m}{243} - \frac{88\,x_c}{243} \tag{37}$$

$x[b](x[c]) := rhs(\%\%[2]);$

$$x_b(x_c) := \frac{88}{243} - \frac{88\,m}{243} - \frac{88\,x_c}{243} \tag{38}$$

$q[a](x[a], x[b], x[c]);$

$$\frac{2}{13} - \frac{2\,m}{13} + \frac{22\,x_a}{39} - \frac{4\,x_b}{39} - \frac{2\,x_c}{13} \tag{39}$$

$q[a](x[c]) := subs(x[a] = (x[a](x[c])), x[b] = (x[b](x[c])), \%);$

$$q_a(x_c) := \frac{26}{81} - \frac{26\,m}{81} - \frac{26\,x_c}{81} \tag{40}$$

$q[b](x[a], x[b], x[c]);$

$$\frac{2}{13} - \frac{2\,m}{13} - \frac{4\,x_a}{39} + \frac{22\,x_b}{39} - \frac{2\,x_c}{13} \tag{41}$$

$q[b](x[c]) := subs(x[a] = (x[a](x[c])), x[b] = (x[b](x[c])), \%);$

$$q_b(x_c) := \frac{26}{81} - \frac{26\,m}{81} - \frac{26\,x_c}{81} \tag{42}$$

$q[c](x[a], x[b], x[c]);$

$$\frac{3}{13} - \frac{3\,m}{13} - \frac{2\,x_a}{13} - \frac{2\,x_b}{13} + \frac{10\,x_c}{13} \tag{43}$$

$q[c](x[c]) := subs(x[a] = (x[a](x[c])), x[b] = (x[b](x[c])), \%);$

$$q_c(x_c) := \frac{29}{243} - \frac{29\,m}{243} + \frac{214\,x_c}{243} \tag{44}$$

#Second Stage: capacity choice of public firm
SS;

$$\frac{(q_a + q_b + q_c)^2}{2} + (1 - q_a - q_b - q_c)\,q_a - m\,q_a - (q_a - x_a)^2 - 3\,F + (1 - q_a - q_b - q_c)\,q_b - m\,q_b - (q_b - x_b)^2 + (1 - q_a - q_b - q_c)\,q_c - m\,q_c - (q_c - x_c)^2 \tag{45}$$

$subs(q[a] = q[a](x[c]), q[b] = (q[b](x[c])), q[c] = q[c](x[c]), x[a] = (x[a](x[c])), x[b] = (x[b](x[c])), \%);$

$$\frac{\left(\frac{185}{243}-\frac{185\,m}{243}+\frac{58\,x_c}{243}\right)^2}{2}+2\left(\frac{58}{243}+\frac{185\,m}{243}-\frac{58\,x_c}{243}\right)\left(\frac{26}{81}-\frac{26\,m}{81}-\frac{26\,x_c}{81}\right)-2\,m\left(\frac{26}{81}-\frac{26\,m}{81}-\frac{26\,x_c}{81}\right)-2\left(-\frac{10}{243}+\frac{10\,m}{243}+\frac{10\,x_c}{243}\right)^2-3\,F+\left(\frac{58}{243}+\frac{185\,m}{243}-\frac{58\,x_c}{243}\right)\left(\frac{29}{243}-\frac{29\,m}{243}+\frac{214\,x_c}{243}\right)-m\left(\frac{29}{243}-\frac{29\,m}{243}+\frac{214\,x_c}{243}\right)-\left(\frac{29}{243}-\frac{29\,m}{243}-\frac{29\,x_c}{243}\right)^2 \tag{46}$$

#subs(q[a]=q[a](x[c]), q[b]=(q[b](x[c])), q[c]=q[c](x[c]), %)
#subs(x[a]=(x[a](x[c])), x[b]=(x[b](x[c])), %)
SSx[c] := evala(Simplify(%));

$$SSx_c := \frac{5446}{59049}\,x_c-3\,F-\frac{53603}{59049}\,m+\frac{53603}{118098}+\frac{53603}{118098}\,m^2-\frac{5446}{59049}\,m\,x_c-\frac{2723}{59049}\,x_c^2 \tag{47}$$

FOC[x[c]] := diff(%, x[c]);

$$FOC_{x_c} := \frac{5446}{59049}-\frac{5446\,m}{59049}-\frac{5446\,x_c}{59049} \tag{48}$$

xstar[c] := rhs(isolate(%, x[c]));

$$xstar_c := 1-m \tag{49}$$

qstar[a] := subs(x[c]=xstar[c], q[a](x[c]));

$$qstar_a := 0 \tag{50}$$

qstar[b] := subs(x[c]=xstar[c], q[b](x[c]));

$$qstar_b := 0 \tag{51}$$

qstar[c] := subs(x[c]=xstar[c], q[c](x[c]));

$$qstar_c := 1-m \tag{52}$$

xstar[a] := subs(x[c]=xstar[c], x[a](x[c]));

$$xstar_a := 0 \tag{53}$$

xstar[b] := subs(x[c]=xstar[c], x[b](x[c]));

$$xstar_b := 0 \tag{54}$$

Qstar := qstar[a]+qstar[b]+qstar[c];

$$Qstar := 1-m \tag{55}$$

Pstar := subs(q[a]=qstar[a], q[b]=qstar[b], q[c]=qstar[c], P);

$$Pstar := m \tag{56}$$

Xstar := xstar[a]+xstar[b]+xstar[c];

$$Xstar := 1-m \tag{57}$$

subs(q[a]=qstar[a], q[b]=qstar[b], q[c]=qstar[c], x[a]=xstar[a], x[b]=xstar[b], x[c]=xstar[c], SS);

$$\frac{(1-m)^2}{2}-3\,F \tag{58}$$

SSfinal := evala(Simplify(%));

$$SSfinal := \frac{1}{2}\, m^2 - m + \frac{1}{2} - 3\, F \tag{59}$$

$profitfinal := subs(q[a] = qstar[a], q[b] = qstar[b], q[c] = qstar[c], x[a] = xstar[a], x[b] = xstar[b], x[c] = xstar[c], profit[a])$;

$$profitfinal := -F \tag{60}$$

$profitfinal := subs(q[a] = qstar[a], q[b] = qstar[b], q[c] = qstar[c], x[a] = xstar[a], x[b] = xstar[b], x[c] = xstar[c], profit[c])$;

$$profitfinal := -t\,(1 - m) - F \tag{61}$$

$subs(m = 0, SSfinal)$

$$\frac{1}{2} - 3\, F \tag{62}$$