Exercises: Path Independence of Line Integral 2

Judge if the following line integrals are path independent. If so, calculate the integral on a curve from point $(0,0)$ to point $(1,1)$ in 2d, or from point $(0,0,0)$ to point $(1,1,1)$ in 3d.

Problem 1. $\int_C 2e^{x^2}(x\cos(2y)\,dx - \sin(2y)\,dy)$.

Solution: Let $f_1(x,y) = 2e^{x^2}\cdot x\cos(2y)$ and $f_2(x,y) = -2e^{x^2}\cdot\sin(2y)$. Thus, $\frac{\partial f_1}{\partial y} = -4xe^{x^2}\sin(2y)$ and $\frac{\partial f_2}{\partial x} = -4xe^{x^2}\sin(2y)$. Hence, the integral is path independent.

Next, we evaluate the integral. If you can observe quickly that $g(x,y) = e^{x^2}\cos(2y)$ satisfies $\frac{\partial g}{\partial x} = f_1$ and $\frac{\partial g}{\partial y} = f_2$, then you can directly give the answer $g(1,1) - g(0,0) = e\cos(2) - 1$.

Suppose that you cannot observe the above $g(x,y)$ directly. Here would be another way of solving the line integral. Choose a curve C on which the integral is easy to evaluate. Let C be the concatenation of two curves: C_1 from $(0,0)$ to $(1,0)$, and C_2 from $(1,0)$ to $(1,1)$. We first evaluate

$$\begin{aligned}
\int_{C_1} 2e^{x^2}(x\cos(2y)\,dx - \sin(2y)\,dy) &= \int_{C_1} 2e^{x^2}x\cos(2y)\,dx \\
&= \int_0^1 2e^{x^2}x\cos(2\cdot 0)\,dx \\
&= \int_0^1 2e^{x^2}x\,dx \\
&= \int_0^1 e^{x^2}\,d(x^2) = e - 1
\end{aligned}$$

Then evaluate

$$\begin{aligned}
\int_{C_2} 2e^{x^2}(x\cos(2y)\,dx - \sin(2y)\,dy) &= -\int_{C_2} 2e^{x^2}\sin(2y)\,dy \\
&= -\int_0^1 2e\sin(2y)\,dy = e\cos(2) - e
\end{aligned}$$

Hence, $\int_C 2e^{x^2}(x\cos(2y)\,dx - \sin(2y)\,dy)$ equals $e - 1 + e\cos(2) - e = e\cos(2) - 1$.

Problem 2. $\int_C (x^2y\,dx - 4xy^2\,dy + 8z^2x\,dz)$.

Solutions: Let $f_1 = x^2y$, $f_2 = -4xy^2$, and $f_3 = 8z^2x$. Hence, $\frac{\partial f_1}{\partial y} = x^2$ and $\frac{\partial f_2}{\partial x} = -4y^2$. Since $\frac{\partial f_1}{\partial y} \neq \frac{\partial f_2}{\partial x}$, we conclude that the integral is not path independent.

Problem 3. $\int_C (e^y\,dx + (xe^y - e^z)\,dy - ye^z\,dz)$.

Solutions: Let $f_1 = e^y$, $f_2 = xe^y - e^z$, and $f_3 = -ye^z$. Hence, $\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x} = e^y$, $\frac{\partial f_1}{\partial z} = \frac{\partial f_3}{\partial x} = 0$, and $\frac{\partial f_2}{\partial z} = \frac{\partial f_3}{\partial y} = -e^z$. Hence, the integral is path independent.

Next, we evaluate the integral. If you can observe quickly that $g(x,y,z) = xe^y - ye^z$ satisfies $\frac{\partial g}{\partial x} = f_1$, $\frac{\partial g}{\partial y} = f_2$, and $\frac{\partial g}{\partial z} = f_3$, then you can directly give the answer $g(1,1,1) - g(0,0,0) = 0$.

Suppose that you cannot observe the above $g(x,y)$ directly. Choose a curve C on which the integral is easy to evaluate. Let C be the concatenation of three curves: C_1 from $(0,0,0)$ to $(0,0,1)$,

C_2 from $(0,0,1)$ to $(0,1,1)$, and C_3 from $(0,1,1)$ to $(1,1,1)$. We first evaluate

$$\begin{aligned}\int_{C_1}(e^y\,dx + (xe^y - e^z)\,dy - ye^z\,dz) &= -\int_{C_1} ye^z\,dz \\ &= -\int_0^1 0e^z\,dz = 0.\end{aligned}$$

Then evaluate:

$$\begin{aligned}\int_{C_2}(e^y\,dx + (xe^y - e^z)\,dy - ye^z\,dz) &= \int_{C_2}(xe^y - e^z)\,dy \\ &= \int_0^1 -e\,dy = -e.\end{aligned}$$

Finally evaluate:

$$\begin{aligned}\int_{C_3}(e^y\,dx + (xe^y - e^z)\,dy - ye^z\,dz) &= \int_{C_3} e^y\,dx \\ &= \int_0^1 e\,dx = e.\end{aligned} \tag{1}$$

Hence, $\int_C(e^y\,dx + (xe^y - e^z)\,dy - ye^z\,dz) = 0 - e + e = 0$.

Problem 4. $\int_C(4y\,dx + (4x+z)\,dy + (y-2z)\,dz)$.

Solutions: Let $f_1 = 4y$, $f_2 = 4x+z$, and $f_3 = y-2z$. Hence, $\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x} = 4$, $\frac{\partial f_1}{\partial z} = \frac{\partial f_3}{\partial x} = 0$, and $\frac{\partial f_2}{\partial z} = \frac{\partial f_3}{\partial y} = 1$. Hence, the integral is path independent.

Next, we evaluate the integral. If you can observe quickly that $g(x,y,z) = 4xy + yz - z^2$ satisfies $\frac{\partial g}{\partial x} = f_1$, $\frac{\partial g}{\partial y} = f_2$, and $\frac{\partial g}{\partial z} = f_3$, then you can directly give the answer $g(1,1,1) - g(0,0,0) = 4$.

Suppose that you cannot observe the above $g(x,y)$ directly. Choose a curve C on which the integral is easy to evaluate. Let C be the line segment given by $\boldsymbol{r}(t) = [x(t), y(t), z(t)]$ with $x(t) = y(t) = z(t) = t$, and $t \in [0,1]$. Then

$$\begin{aligned}\int_C(4y\,dx + (4x+z)\,dy + (y-2z)\,dz) &= \int_0^1 (4t\,\frac{dx}{dt} + (4t+t)\,\frac{dy}{dt} + (t-2t)\,\frac{dz}{dt})dt \\ &= \int_0^1 (4t + 5t - t)\,dt = 4.\end{aligned}$$