Sequential Equilibrium

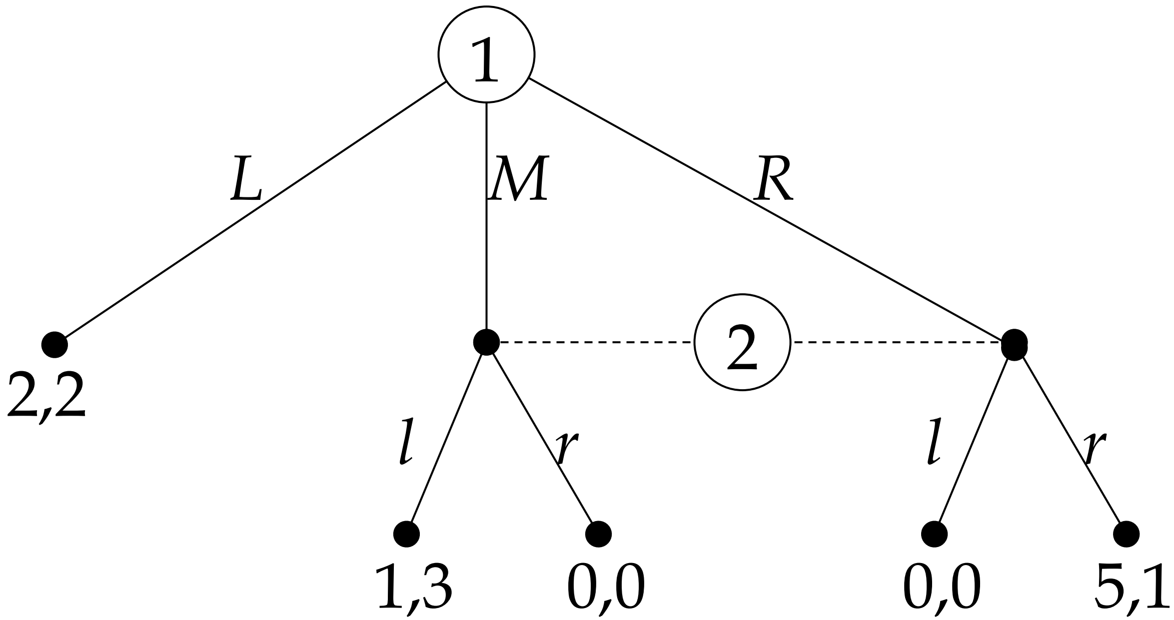
DEFINITION. An **assessment** in an extensive game is a pair (β, μ), where β is a profile of behavioural strategies and μ is a function that assigns to every information set a probability measure on the set of histories in the information set.

DEFINITION. An assessment is a **sequential equilibrium** of an extensive game with perfect recall if it is sequentially rational and consistent.

Sequential Equilibrium

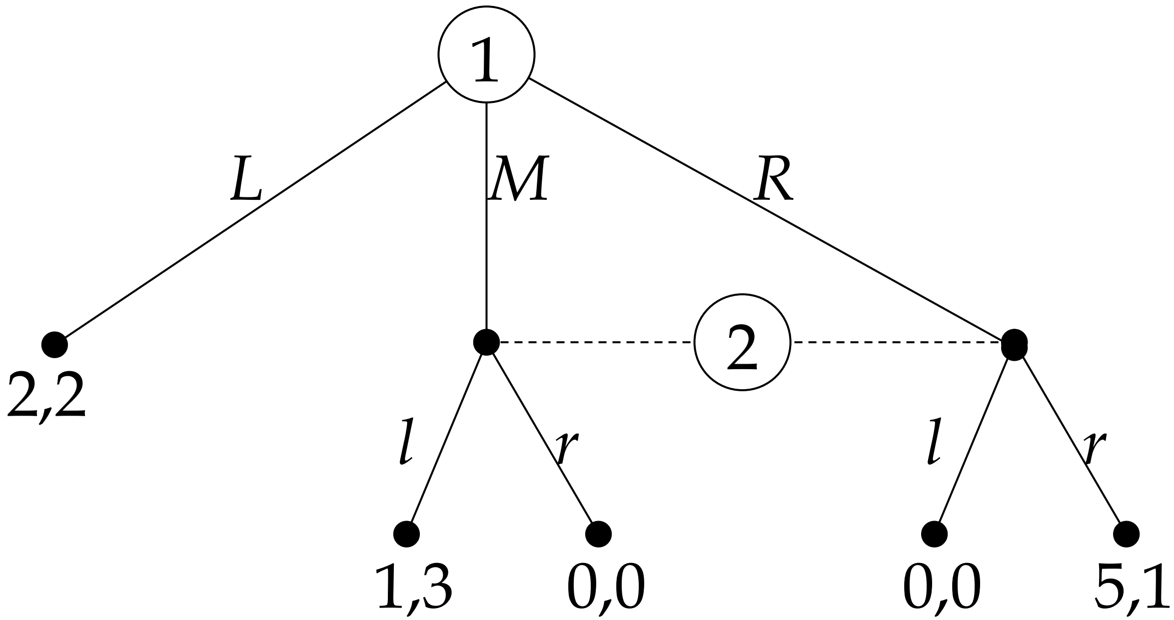
The concept of sequential equilibrium permits great freedom regarding the beliefs that players hold when they observe actions that are not consistent with the equilibrium strategies.

Beliefs as part of the specification of an equilibrium ➔Further restrictions on these beliefs ➔refinements of sequential equilibrium.



A sequential equilibrium: (R, r).

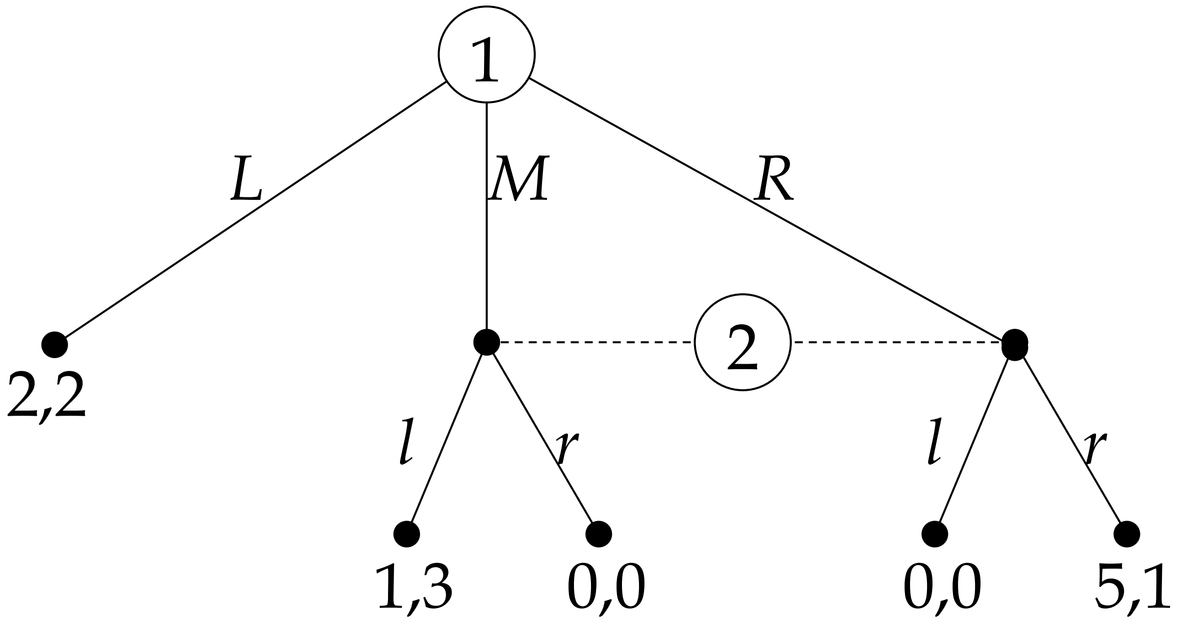
Question: what are β and μ?



Another sequential equilibrium: (L, l).

$\mu(\{M, R\})(M) > \mu(\{M, R\})(R)$.

Question: what are β and μ?



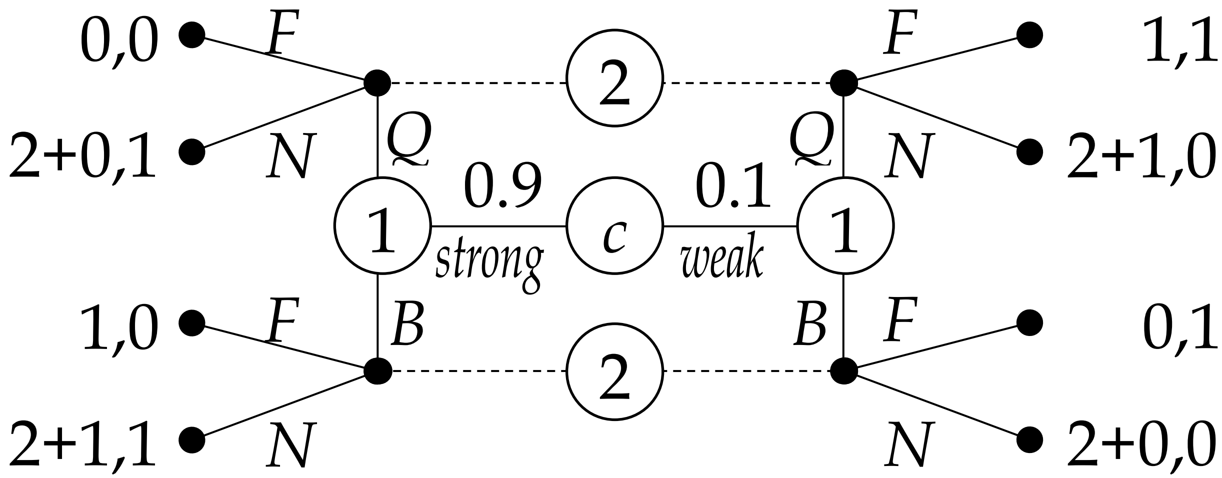
Another sequential equilibrium: (L, l).

$\mu(\{M, R\})(M) > \mu(\{M, R\})(R)$.

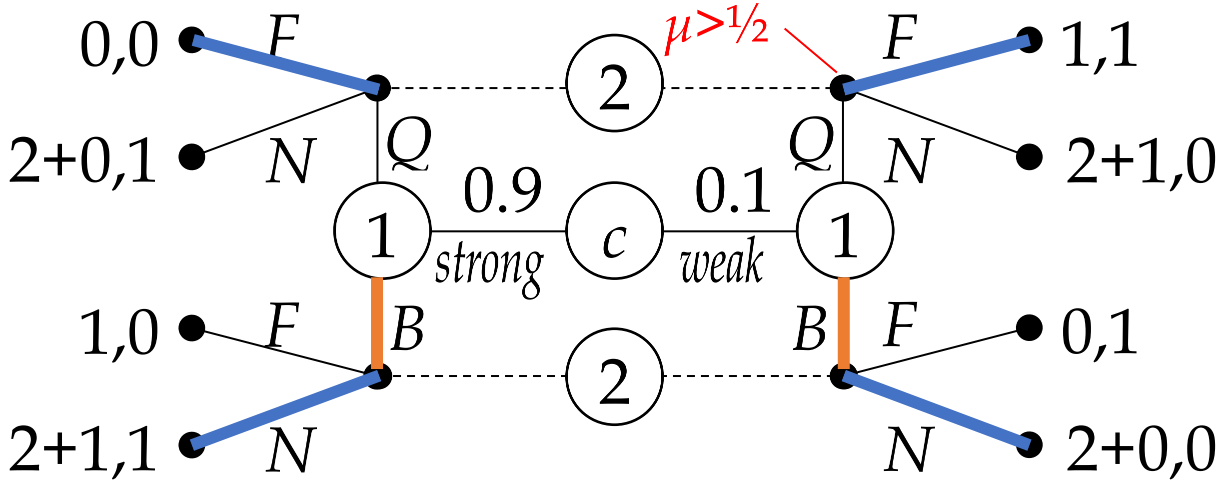
<u>**BUT**</u> if $\{M, R\}$ is reached, player 1 should have chosen R, because M is dominated by L!

Player 1 could not have chosen M!

Do we miss anything?

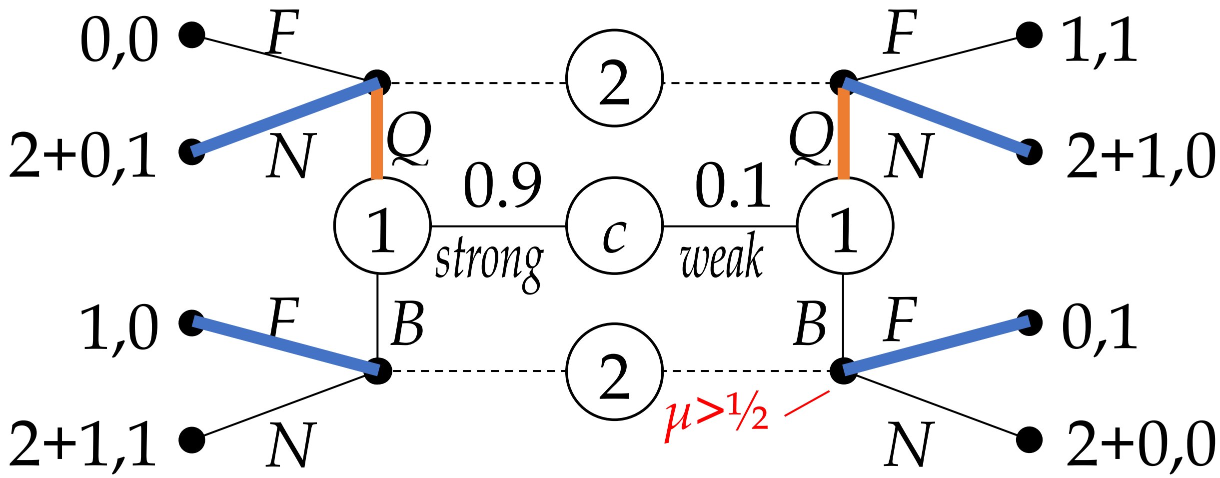


Player 1: *Q*uiche or *B*eer.   Player 2: *F*ight or *N*ot.



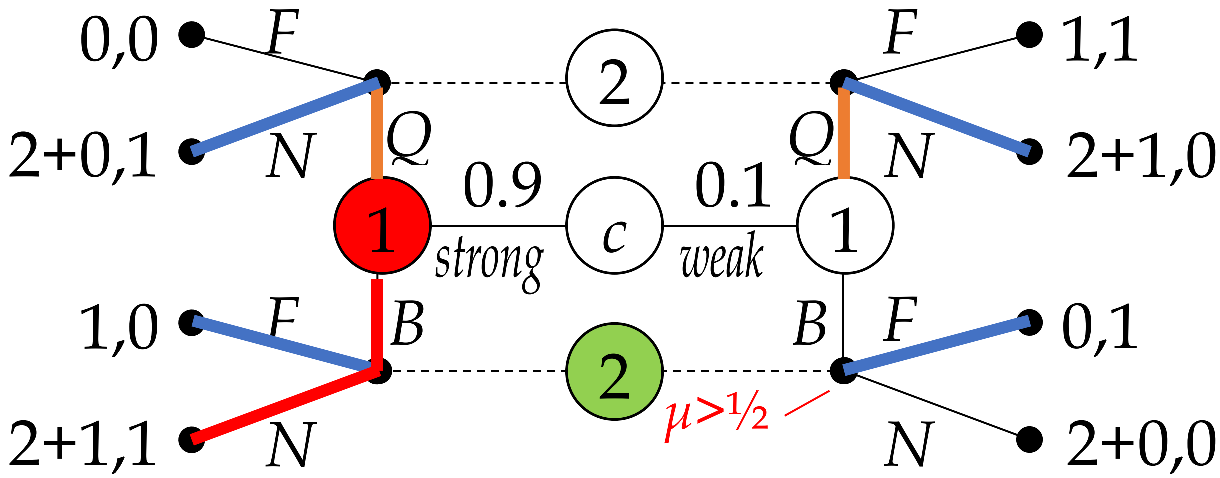
Sequential Equilibrium 1:

$$\beta = (BB, FN),\ \mu(\{(s,Q),(w,Q)\})(w,Q) > \frac{1}{2}$$



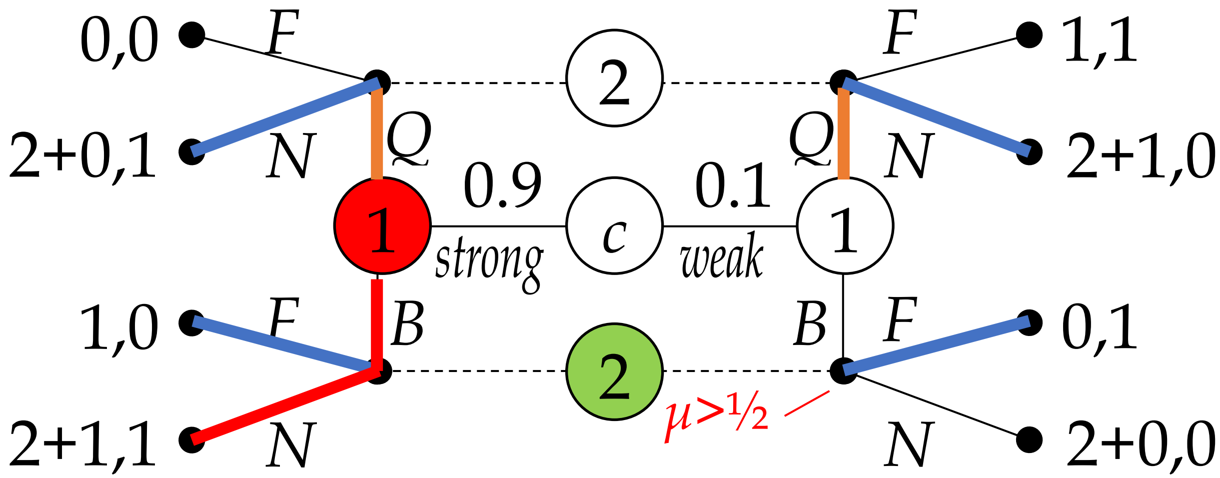
Sequential Equilibrium 2:

$$\beta = (QQ, NF),\ \ \mu(\{(s,B),(w,B)\})(w,B) > \frac{1}{2}$$



But μ in Sequential Equilibrium 2 is unreasonable!

(1) If Player 1 is weak, he should follow the equilibrium and use Q.



But μ in Sequential Equilibrium 2 is unreasonable!

(1) If Player 1 is weak, he should use Q.

(2) Is Player 1 strong? Definitely! *He plays B!*

So, Player 2 should conclude that player 1 is strong and Player 2 should choose N. (Player 1 is indeed better off than in the equilibrium.)

Therefore, the belief system is not reasonable.

Sequential Equilibrium

Beliefs as part of the specification of an equilibrium
➔Further restrictions on these beliefs ➔refinements of sequential equilibrium.

The refinements of sequential equilibrium introduce new strategic considerations.

Trembling Hand Perfect Equilibrium

Now we consider that the players have trembling hands that other players could make uncorrelated mistakes leading to some unexpected events.

The basic idea is that each player’s actions be optimal not only given his equilibrium beliefs but also given a perturbed belief that allows for the possibility of slight mistakes.

Trembling Hand Perfect Equilibrium

What are the Nash equilibria?

(σ_{-i}, σ_i)

	A	B	C
A	$0,0$	$0,0$	$0,0$
B	$0,0$	$1,1$	$2,0$
C	$0,0$	$0,2$	$2,2$

Trembling Hand Perfect Equilibrium

What are the Nash equilibria?

σ^1_{-i} σ^3_{-i} σ^5_{-i}

σ^2_{-i} (σ_{-i}, σ_i)

σ^0_{-i} σ^6_{-i} σ^4_{-i}

	A	B	C
A	$0,0$	$0,0$	$0,0$
B	$0,0$	$1,1$	$2,0$
C	$0,0$	$0,2$	$2,2$

$$\sigma^0_{-i}, \sigma^1_{-i}, \sigma^2_{-i}, \sigma^3_{-i}, \ldots \to \sigma_{-i}$$

Trembling Hand Perfect Equilibrium

DEFINITION. A **trembling hand perfect equilibrium** of a finite strategic game is a mixed strategy profile σ with the property that there exists a sequence $(\sigma^k)_{k=0}^{\infty}$ of completely mixed strategy profiles that converges to σ such that for each player i the strategy σ_i is a best response to σ_{-i}^k for all values of k.

Trembling Hand Perfect Equilibrium

$$\sigma^0 = (\sigma_1^0, \sigma_2^0, \sigma_3^0, \ldots \sigma_n^0)$$
$$\sigma^1 = (\sigma_1^1, \sigma_2^1, \sigma_3^1, \ldots \sigma_n^1)$$
$$\sigma^2 = (\sigma_1^2, \sigma_2^2, \sigma_3^2, \ldots \sigma_n^2)$$
$$\downarrow$$
$$\sigma = (\sigma_1, \sigma_2, \sigma_3, \ldots \sigma_n) \quad \text{N.E.}$$

A sequence $(\sigma^k)_{k=0}^{\infty}$ of completely mixed strategy profiles that converges to σ.

For each player i the strategy σ_i is a best response to the sequence (*not all*) $\sigma_{-i}^0, \sigma_{-i}^1, \sigma_{-i}^2, \sigma_{-i}^3, \sigma_{-i}^4, \cdots$.

Trembling Hand Perfect Equilibrium

$$\sigma^0 = (\sigma_1^0, \sigma_2^0, \sigma_3^0, \ldots \sigma_n^0)$$
$$\sigma^1 = (\sigma_1^1, \sigma_2^1, \sigma_3^1, \ldots \sigma_n^1)$$
$$\sigma^2 = (\sigma_1^2, \sigma_2^2, \sigma_3^2, \ldots \sigma_n^2)$$
$$\downarrow$$
$$\sigma = (\sigma_1, \sigma_2, \sigma_3, \ldots \sigma_n) \quad \text{N.E.}$$

As a player's expected payoff is continuous in the vector of the other players' mixed strategies, hence the strategy σ_i is a best response to σ_{-i}, every trembling hand perfect equilibrium is a Nash equilibrium.

Trembling Hand Perfect Equilibrium

Consider this example.

	A	B	C
A	$0, 0$	$0, 0$	$0, 0$
B	$0, 0$	$1, 1$	$2, 0$
C	$0, 0$	$0, 2$	$2, 2$

Among the Nash equilibria

(A, A)
(B, B)
(C, C)

which one(s) is/ are trembling hand perfect?

Trembling Hand Perfect Equilibrium

	A	B	C
A	$0,0$	$0,0$	$0,0$
B	$0,0$	$1,1$	$2,0$
C	$0,0$	$0,2$	$2,2$

Consider (B, B).

Does there exist a sequence $((\sigma_1^0, \sigma_2^0), (\sigma_1^1, \sigma_2^1), (\sigma_1^2, \sigma_2^2), \dots)$ of completely mixed strategy profiles that

1. converges to (B, B) such that
2. for each player i the strategy $\sigma_i = B$ is a best response to $\sigma_{-i}^0, \sigma_{-i}^1, \sigma_{-i}^2, \sigma_{-i}^3, \dots$?

Trembling Hand Perfect Equilibrium

Consider (A, A).

Does there exist a sequence $((\sigma_1^0, \sigma_2^0), (\sigma_1^1, \sigma_2^1), (\sigma_1^2, \sigma_2^2), \ldots)$ of completely mixed strategy profiles that

1. converges to (A, A) such that
2. for each player i the strategy $\sigma_i = A$ is a best response to $\sigma_{-i}^0, \sigma_{-i}^1, \sigma_{-i}^2, \sigma_{-i}^3, \ldots$?

	A	B	C
A	$0,0$	$0,0$	$0,0$
B	$0,0$	$1,1$	$2,0$
C	$0,0$	$0,2$	$2,2$

Trembling Hand Perfect Equilibrium

	A	B	C
A	$0,0$	$0,0$	$0,0$
B	$0,0$	$1,1$	$2,0$
C	$0,0$	$0,2$	$2,2$

Consider (C, C).

Does there exist a sequence $((\sigma_1^0, \sigma_2^0), (\sigma_1^1, \sigma_2^1), (\sigma_1^2, \sigma_2^2), \dots)$ of completely mixed strategy profiles that

1. converges to (C, C) such that
2. for each player i the strategy $\sigma_i = C$ is a best response to $\sigma_{-i}^0, \sigma_{-i}^1, \sigma_{-i}^2, \sigma_{-i}^3, \dots$?

Trembling Hand Perfect Equilibrium

Actions A and C are weakly dominated actions in this game.

	A	B	C
A	$0, 0$	$0, 0$	$0, 0$
B	$0, 0$	$1, 1$	$2, 0$
C	$0, 0$	$0, 2$	$2, 2$

Weakly Dominated Actions

Definition. The action $a_i \in A_i$ of player i in the strategic game $\langle N, (A_i), (u_i) \rangle$ is **weakly dominated** if there is a mixed strategy α_i of player i such that $U_i(a_{-i}, \alpha_i) \geq U_i(a_{-i}, a_i)$ for all $a_{-i} \in A_{-i}$ and $U_i(a_{-i}, \alpha_i) > U_i(a_{-i}, a_i)$ for some $a_{-i} \in A_{-i}$, where $U_i(a_{-i}, \alpha_i)$ is the payoff of player i if he uses the mixed strategy α_i and the other players' vector of actions is $a_{-i} \in A_{-i}$.

Hence a weakly dominated strategy is never a best response to a vector of completely mixed strategies.

Trembling Hand Perfect Equilibrium

Actions A and C are weakly dominated actions. They are not best responses to any vector of completely mixed strategies.

	A	B	C
A	$0,0$	$0,0$	$0,0$
B	$0,0$	$1,1$	$2,0$
C	$0,0$	$0,2$	$2,2$

Hence the notion of trembling hand perfect equilibrium rules out the use of all weakly dominated actions!

Trembling Hand Perfect Equilibrium

	A	B	C
A	$0,0$	$0,0$	$0,0$
B	$0,0$	$1,1$	$2,0$
C	$0,0$	$0,2$	$2,2$

Hence a strategy profile (*e.g.,* (B, B)) in a finite *two-player* strategic game is a trembling hand perfect equilibrium **only if** it is a mixed strategy Nash equilibrium and the strategy of neither player is weakly dominated.

Trembling Hand Perfect Equilibrium

PROPOSITION. A strategy profile in a finite *two-player* strategic game is a trembling hand perfect equilibrium **if, and only if,** it is a mixed strategy Nash equilibrium and the strategy of neither player is weakly dominated.

Trembling Hand Perfect Equilibrium

ℓ:

	L	R
T	$1, 1, 1$	$1, 0, 1$
B	$1, 1, 1$	$0, 0, 1$

r:

	L	R
T	$1, 1, 0$	$0, 0, 0$
B	$0, 1, 0$	$1, 0, 0$

Question: What is the Nash equilibrium of this 3-player strategic game?

Question: Is the Nash equilibrium dominated?

	L	R
T	$1,1,1$	$1,0,1$
B	$1,1,1$	$0,0,1$

ℓ

	L	R
T	$1,1,0$	$0,0,0$
B	$0,1,0$	$1,0,0$

r

Question: Is the undominated Nash equilibrium (B, L, ℓ) a trembling hand perfect equilibrium?

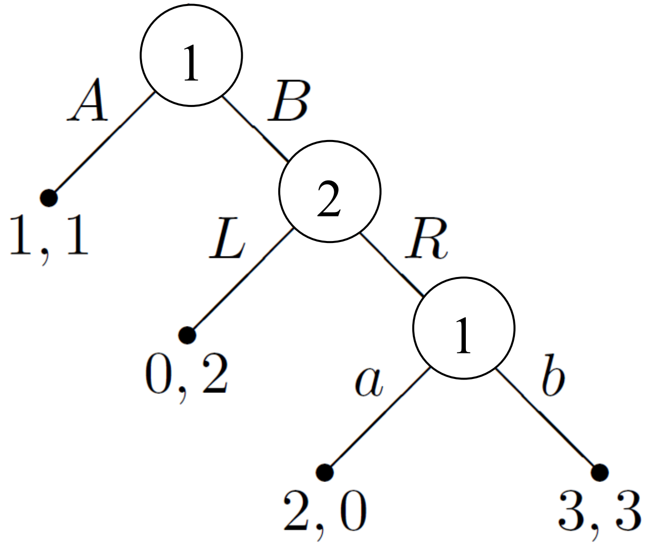
Answer: Player 1′s payoff to T exceeds his payoff to B whenever players 2 and 3 assign small enough positive probability to R and r respectively.

Trembling Hand Perfect Equilibrium

PROPOSITION. Every finite strategic game has a trembling hand perfect equilibrium.

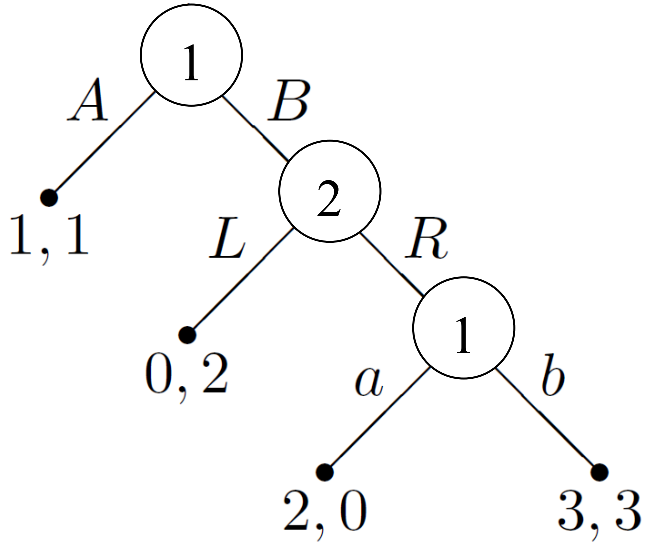
Trembling Hand Perfect Equilibrium

How about extensive games?



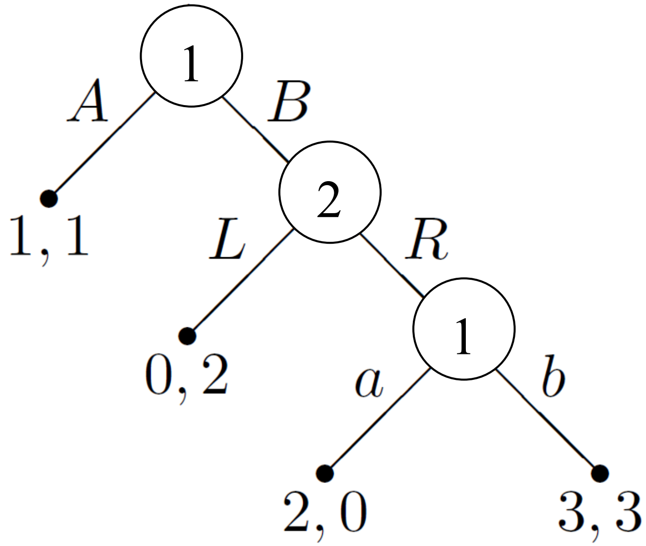
	L	R
A, a	$1, 1$	$1, 1$
A, b	$1, 1$	$1, 1$
B, a	$0, 2$	$2, 0$
B, b	$0, 2$	$3, 3$

What are the subgame perfect equilibria?



	L	R
A, a	$1, 1$	$1, 1$
A, b	$1, 1$	$1, 1$
B, a	$0, 2$	$2, 0$
B, b	$0, 2$	$3, 3$

Unique subgame perfect equilibrium: $((B, b), R)$.
But $((A, a), L)$ is also a trembling hand perfect equilibrium of the *strategic form* of the game.



	L	R
A, a	$1, 1$	$1, 1$
A, b	$1, 1$	$1, 1$
B, a	$0, 2$	$2, 0$
B, b	$0, 2$	$3, 3$

Trembling hand perfect equilibrium: $((A, a), L)$.

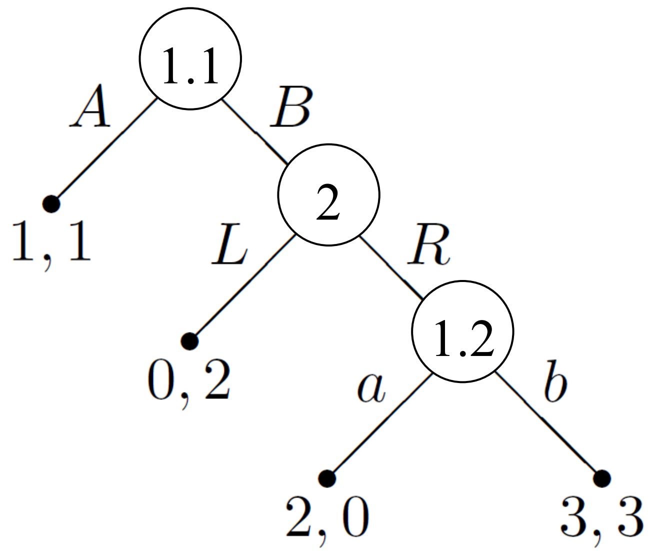
- (A, a) is a best response if $\sigma_2(L)$ is close to 1.
- L is a best response if $\sigma_1((A, a))$ is close to 1, and $\sigma_1\big((B, a)\big) \gg \sigma_1\big((B, b)\big)$.

BUT, what if player 1 uses B by mistake…

Agent Strategic Form

The **agent strategic form** of an extensive game is one in which there is one player for each information set in the extensive game: each player in the extensive game is split into a number of *agents*, one for each of his information sets, all agents of a given player having the same payoffs.

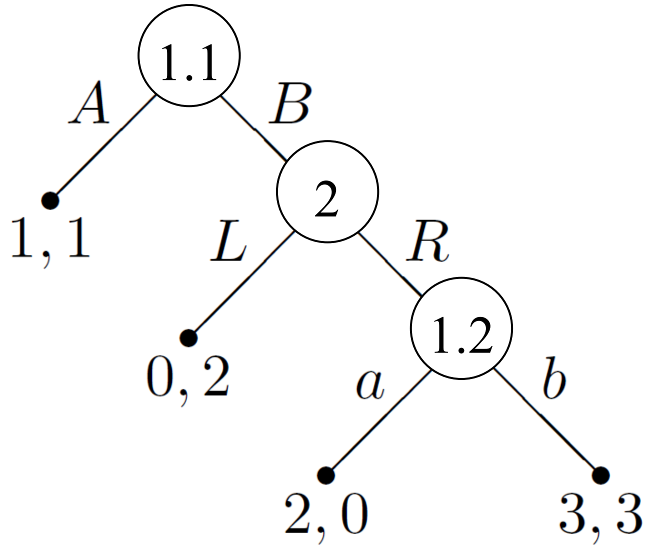
Agent Strategic Form



$$u_1 = u_{1.1} = u_{1.2}$$

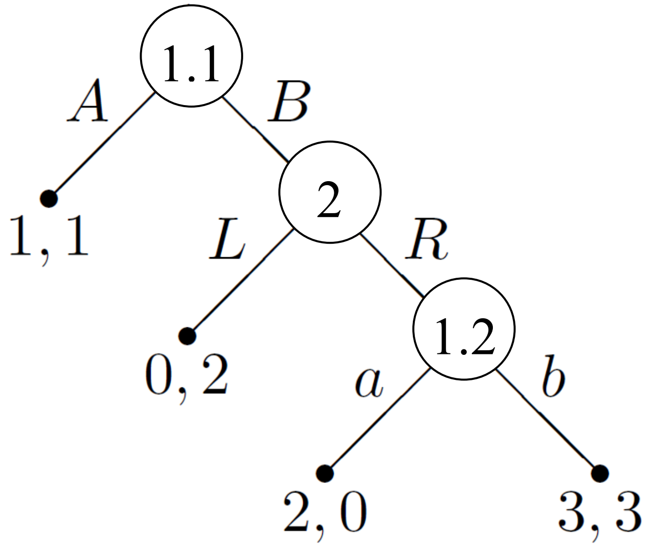
Agent Strategic Form

Any mixed strategy profile σ in the agent strategic form corresponds to the behavioural strategy profile β in which $\beta_i(I_i)$ is the mixed strategy of player i's agent at the information set I_i.



Trembling Hand Perfect Equilibrium

DEFINITION. A **trembling hand perfect equilibrium** of a finite extensive game is a behavioural strategy profile that corresponds to a trembling hand perfect equilibrium of the agent strategic form of the game.



	L	R
A, a	$1, 1$	$1, 1$
A, b	$1, 1$	$1, 1$
B, a	$0, 2$	$2, 0$
B, b	$0, 2$	$3, 3$

The behavioural $((A, a), L)$ is NOT a trembling hand perfect equilibrium. This is because for any pair of completely mixed strategies of player 1.1 and player 2 the unique best response of player 1.2 is b.

Trembling Hand Perfect Equilibrium

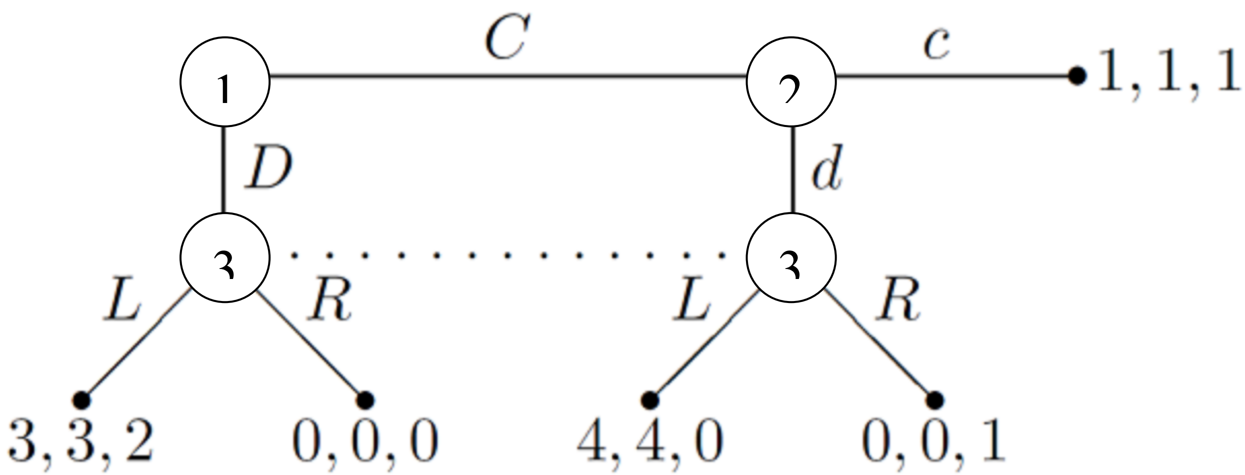
PROPOSITION. For every trembling hand perfect equilibrium β of a finite extensive game with perfect recall there is a belief system μ such that (β, μ) is a sequential equilibrium of the game.

Note: The converse of this result does not hold since in a game with simultaneous moves every Nash equilibrium is the strategy profile of a sequential equilibrium, but only those Nash equilibria in which no player's strategy is weakly dominated can be trembling hand perfect.

Trembling Hand Perfect Equilibrium

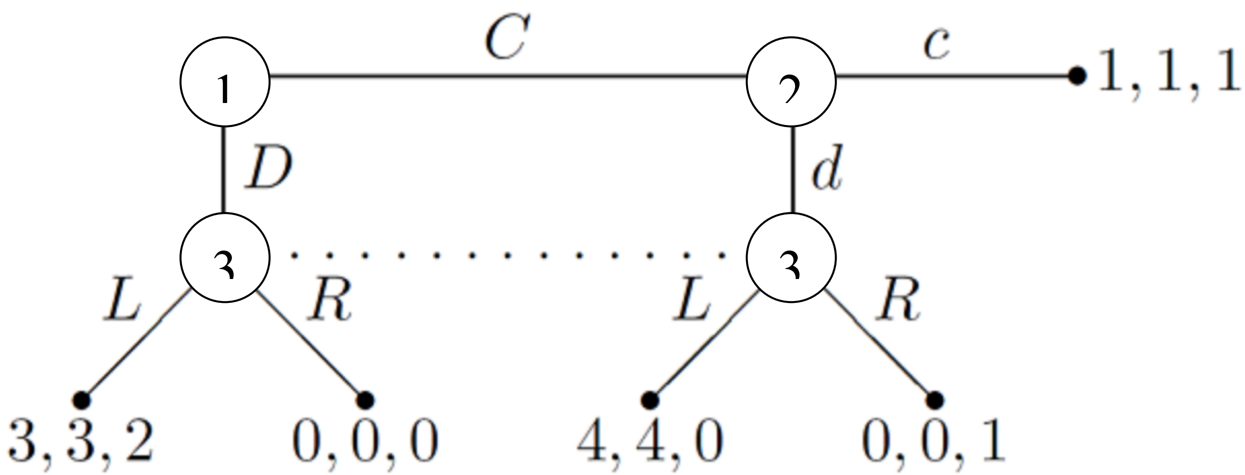
PROPOSITION. For every trembling hand perfect equilibrium β of a finite extensive game with perfect recall there is a belief system μ such that (β, μ) is a sequential equilibrium of the game.

Note: The converse of this result is 'almost' true: for almost every game the strategy profile of almost every sequential equilibrium is a trembling hand perfect equilibrium.



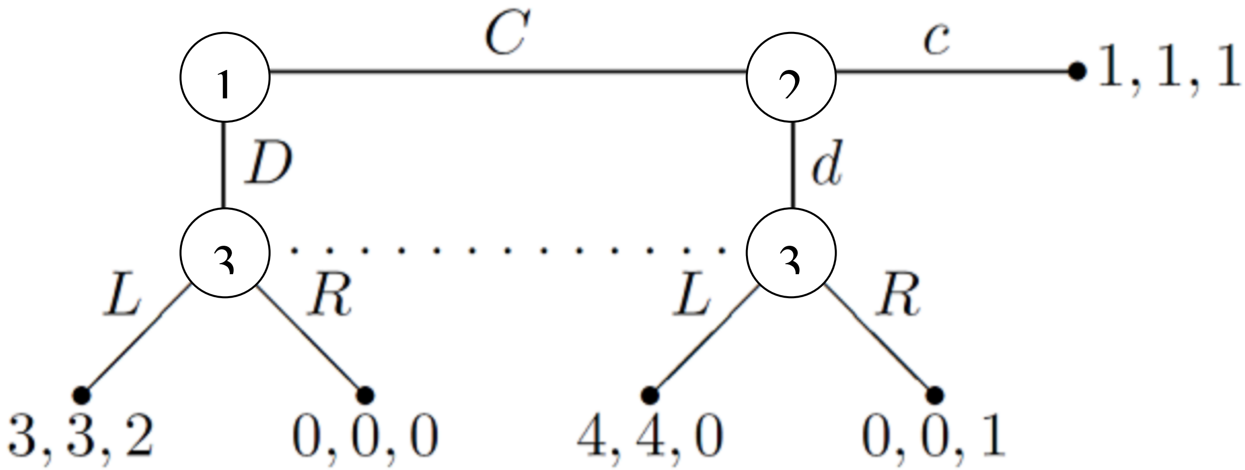
Consider $(\underbrace{D}_{\text{player 1's strategy}}, \underbrace{(c(\geq \tfrac{1}{3}), d(< \tfrac{2}{3}))}_{\text{player 2's strategy}}, \underbrace{L}_{\text{player 3's strategy}})$.

- Is it a Nash equilibrium?
- Is it a sequential equilibrium? No.
- Is it trembling hand perfect? …



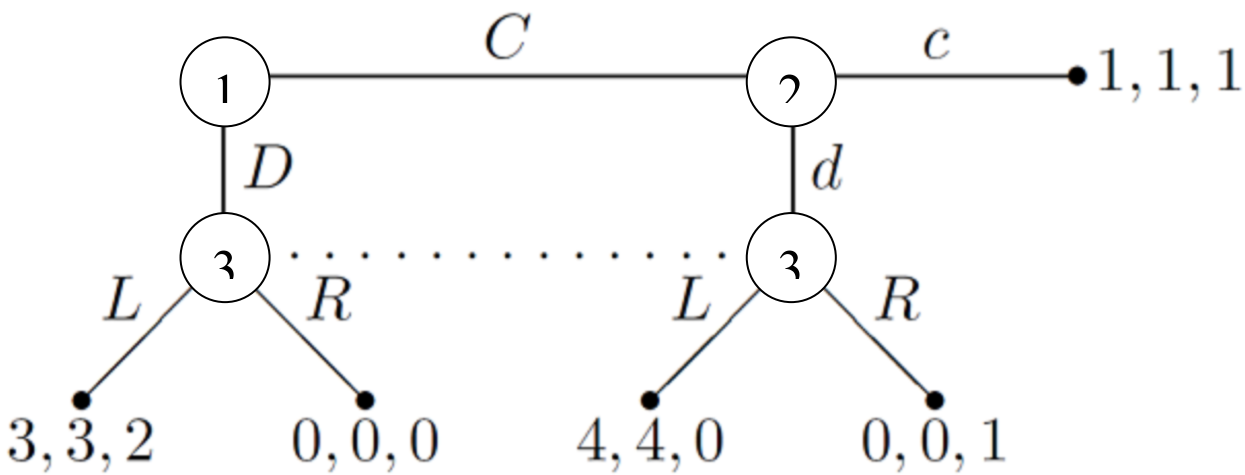
Consider $(D, (c(\geq \frac{1}{3}), d(< \frac{2}{3})), L)$.

- Is it a Nash equilibrium?
- Is it a sequential equilibrium? No.
- Is it trembling hand perfect?
  No, because if $\sigma_1(C) > 0$ and $\sigma_3(L) \approx 1$, then d is better than c for player 2.



Consider $(C, c, (R(\geq \frac{3}{4}), L(< \frac{1}{4})))$.

- Is it a Nash equilibrium?
- Is it a sequential equilibrium? Yes.
- Is it trembling hand perfect?



Consider $(C, c, (R(\geq \frac{3}{4}), L(< \frac{1}{4})))$.

- Is it a Nash equilibrium?
- Is it a sequential equilibrium? Yes.
- Is it trembling hand perfect?
  Yes.

PROPOSITION. Every finite strategic game has a trembling hand perfect equilibrium.

PROPOSITION. For every trembling hand perfect equilibrium β of a finite extensive game with perfect recall there is a belief system μ such that (β, μ) is a sequential equilibrium of the game.

COROLLARY. Every finite extensive game with perfect recall has a trembling hand perfect equilibrium and thus also a sequential equilibrium.