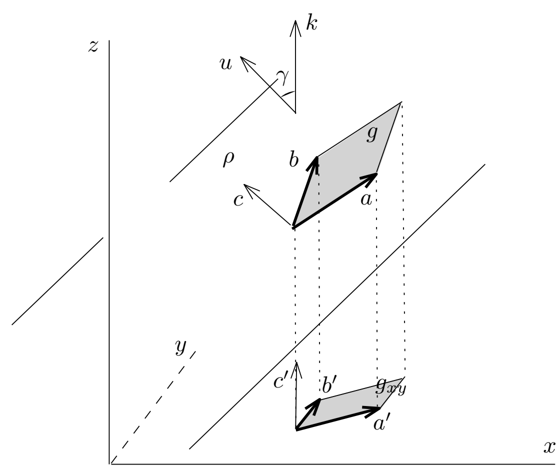
Lecture Notes: Areas of Planar Projections

Yufei Tao
Department of Computer Science and Engineering
Chinese University of Hong Kong
taoyf@cse.cuhk.edu.hk

In this lecture, we will pave a stepping stone for our subsequent discussion on surface integrals by discussing a topic that is interesting in its own right: the relationship between the area of a planar region embedded in $\mathbb{R}^3$ and the area of its projection onto the xy-plane.

1 Projection of a Parallelogram

Let us start with the following problem. In $\mathbb{R}^3$, we are given a parallelogram g that is in a plane ρ with a normal vector $\boldsymbol{u}$. Now, project g onto the xy-plane, which gives us another parallelogram g_{xy}; see the figure below. Denote by A the area of g, and by A_{xy} the area of g_{xy}. We want to explore the relationship between A and A_{xy}.



Denote by γ the angle between the directions of $\boldsymbol{u}$ and $\boldsymbol{k}$ (i.e., the positive z-direction). Next, we prove a very neat result:

Lemma 1. $A_{xy} = A \cdot |\cos\gamma|$.

Proof. If $A = 0$ (i.e., g degenerates into a point), then A_{xy} is trivially 0, in which case the lemma is obviously true. Next, we consider that $A \neq 0$.

Consider first $\gamma \in [0, \pi/2]$. Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be the vectors corresponding to the two directed segments as shown in the above figure. Let $\boldsymbol{a}'$ and $\boldsymbol{b}'$ be the projections of $\boldsymbol{a}$ and $\boldsymbol{b}$ onto the xy-plane, respectively. If we write out the components of $\boldsymbol{a}$ and $\boldsymbol{b}$ as:

$$\begin{aligned}\boldsymbol{a} &= [x_1, y_1, z_1] \\ \boldsymbol{b} &= [x_2, y_2, z_2]\end{aligned}$$

then we have

$$\begin{aligned} \boldsymbol{a'} &= [x_1, y_1, 0] \\ \boldsymbol{b'} &= [x_2, y_2, 0]. \end{aligned}$$

We know that the areas of g and g_{xy} are

$$\begin{aligned} A &= |\boldsymbol{a} \times \boldsymbol{b}| \\ A_{xy} &= |\boldsymbol{a}' \times \boldsymbol{b}'|. \end{aligned}$$

Define $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$ and $\boldsymbol{c} = \boldsymbol{a}' \times \boldsymbol{b}'$. By definition of cross product, we know:

$$\begin{aligned} \boldsymbol{c} &= [y_1 z_2 - z_1 y_2, z_1 x_2 - x_1 z_2, x_1 y_2 - y_1 x_2] \\ \boldsymbol{c}' &= [0, 0, x_1 y_2 - y_1 x_2]. \end{aligned}$$

The directions of $\boldsymbol{c}$ and $\boldsymbol{c'}$ are shown in the above figure. Note that γ is also the angle between $\boldsymbol{c}$ and $\boldsymbol{c'}$.

We thus have:

$$\begin{aligned} A &= |\boldsymbol{c}| \\ A_{xy} &= |\boldsymbol{c}'|. \end{aligned}$$

If $\boldsymbol{c'} = \boldsymbol{0}$, then it means that $\boldsymbol{a'}$ and $\boldsymbol{b'}$ have exactly the same or opposite directions, which further implies that $\gamma = \pi/2$. In this case, we trivially have $A_{xy} = 0 = A \cos \gamma$. If $c' \neq \boldsymbol{0}$, we have

$$\cos \gamma = \frac{\boldsymbol{c} \cdot \boldsymbol{c'}}{|\boldsymbol{c}||\boldsymbol{c'}|} = \frac{(x_1 y_2 - y_1 x_2)^2}{|\boldsymbol{c}||\boldsymbol{c'}|} = \frac{|\boldsymbol{c'}|^2}{|\boldsymbol{c}||\boldsymbol{c'}|} = \frac{|\boldsymbol{c'}|}{|\boldsymbol{c}|} = \frac{A_{xy}}{A}$$

which is precisely what we want to prove.

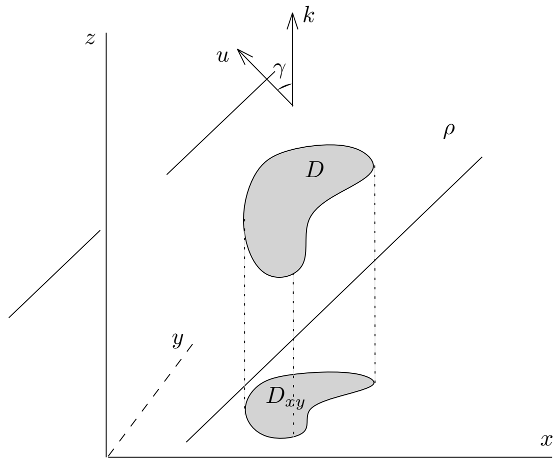
For the case where $\gamma \in [\pi/2, \pi]$, let $\boldsymbol{v} = -\boldsymbol{u}$. The angle between the directions of $\boldsymbol{v}$ and $\boldsymbol{k}$ is within $[\pi/2, \pi]$. Now we can apply the above argument with respect to the normal vector $\boldsymbol{v}$ to establish the lemma. □

Example 1. Consider the plane ρ given by $x + 2y + 3z = 4$. Let D a rectangle on the xy-plane with area 1, and D' the projection of D onto ρ. What is the area of D'?

Solution. A normal vector of ρ is $\boldsymbol{u} = [1, 2, 3]$. Let γ be the angle between $\boldsymbol{u}$ and $\boldsymbol{k} = [0, 0, 1]$. We know that $\cos \gamma = \frac{\boldsymbol{u} \cdot \boldsymbol{k}}{|\boldsymbol{u}||\boldsymbol{k}|} = \frac{3}{\sqrt{14}}$. Hence, by Lemma 1, the area of D' equals $\mathit{area}(D)/\cos \gamma = \sqrt{14}/3$. □

2 Projection of Any Planar Region

We now generalize Lemma 1. In $\mathbb{R}^3$, we are given an arbitrary region D that is in a plane ρ with a normal vector $\boldsymbol{u}$. Suppose that the boundary of D is a smooth curve. Now, project D onto the xy-plane, which gives us another region D_{xy}; see the figure below.



Denote by γ the angle between the directions of $\boldsymbol{u}$ and $\boldsymbol{k}$. In general, we still have:

Lemma 2. $area(D_{xy}) = area(D) \cdot |\cos\gamma|$.

We will not prove the lemma formally, but its key idea is easy to grasp. Imagine that we approximate D as the union of a huge number of very small disjoint parallelograms, and project all those parallelograms onto the xy-plane. The union of those parallelograms' projections approximates D_{xy}. Then, by Lemma 1, there is a $\cos\gamma$ factor between the areas of each parallelogram and its projection, which thus gives Lemma 2.

Example 2. Consider the plane ρ given by $x + 2y + 3z = 4$. Let D be a circle on the xy-plane with radius 1, and D' the projection of D onto ρ. What is the area of D'?

Solution. A normal vector of ρ is $\boldsymbol{u} = [1, 2, 3]$. Let γ be the angle between $\boldsymbol{u}$ and $\boldsymbol{k} = [0, 0, 1]$. We know that $\cos\gamma = \frac{\boldsymbol{u}\cdot\boldsymbol{k}}{|\boldsymbol{u}||\boldsymbol{k}|} = \frac{3}{\sqrt{14}}$. Hence, by Lemma 2, the area of D' equals $area(D)/\cos\gamma = \sqrt{14}\pi/3$. $\square$