CSCI3320: Homework 1, Spring (2019-2020) **Deadline: Feb 2, 2020, 5:00 pm.**
Teacher: John C.S. Lui

**1. Sampling Distribution of means.** *A population consists of the five numbers 2, 3, 6, 8, and 11. Consider all possible samples of size 2 that can be drawn with replacement from this population. Find (a) the mean of the population, (b) the standard deviation of the population, (c) the mean of the sampling distribution of means, and (d) the standard deviation of the sampling distribution of means (i.e., the standard error of means).*

**Answer:**

(*a*)
$$\mu = \frac{2+3+6+8+11}{5} = \frac{30}{5} = 6.0$$

(*b*)
$$\sigma^2 = \frac{(2-6)^2+(3-6)^2+(6-6)^2+(8-6)^2+(11-6)^2}{5} = \frac{16+9+0+4+25}{5} = 10.8$$

and $\sigma = 3.29$.

(*c*) There are $5(5) = 25$ samples of size 2 that can be drawn with replacement (since any one of the five numbers on the first draw can be associated with any one of the five numbers on the second draw). These are

| | | | | |
|---|---|---|---|---|
| (2, 2) | (2, 3) | (2, 6) | (2, 8) | (2, 11) |
| (3, 2) | (3, 3) | (3, 6) | (3, 8) | (3, 11) |
| (6, 2) | (6, 3) | (6, 6) | (6, 8) | (6, 11) |
| (8, 2) | (8, 3) | (8, 6) | (8, 8) | (8, 11) |
| (11, 2) | (11, 3) | (11, 6) | (11, 8) | (11, 11) |

The corresponding sample means are

| | | | | |
|---|---|---|---|---|
| 2.0 | 2.5 | 4.0 | 5.0 | 6.5 |
| 2.5 | 3.0 | 4.5 | 5.5 | 7.0 |
| 4.0 | 4.5 | 6.0 | 7.0 | 8.5 |
| 5.0 | 5.5 | 7.0 | 8.0 | 9.5 |
| 6.5 | 7.0 | 8.5 | 9.5 | 11.0 |

(*8*)

and the mean of sampling distribution of means is

$$\mu_{\bar{X}} = \frac{\text{sum of all sample means in } (8)}{25} = \frac{150}{25} = 6.0$$

illustrating the fact that $\mu_{\bar{X}} = \mu$.

(*d*) The variance $\sigma^2_{\bar{X}}$ of the sampling distribution of means is obtained by subtracting the mean 6 from each number in (*8*), squaring the result, adding all 25 numbers thus obtained, and dividing by 25. The final result is $\sigma^2_{\bar{X}} = 135/25 = 5.40$, and thus $\sigma_{\bar{X}} = \sqrt{5.40} = 2.32$. This illustrates the fact that for finite populations involving sampling with replacement (or infinite populations), $\sigma^2_{\bar{X}} = \sigma^2/N$ since the right-hand side is $10.8/2 = 5.40$, agreeing with the above value.

**2.** *Assume that the heights of 3000 male students at a university are normally distributed with mean 68.0 inches (in) and standard deviation 3.0 in. If 80 samples consisting of 25 students each are obtained, what would be the expected mean and standard deviation of the resulting sampling distribution of means if the sampling were done (a) with replacement and (b) without replacement?*

**Answer:**

The numbers of samples of size 25 that could be obtained theoretically from a group of 3000 students with and without replacement are $(3000)^{25}$ and $\binom{3000}{25}$, which are much larger than 80. Hence we do not get a true sampling distribution of means, but only an *experimental* sampling distribution. Nevertheless, since the number of samples is large, there should be close agreement between the two sampling distributions. Hence the expected mean and standard deviation would be close to those of the theoretical distribution. Thus we have:

(*a*) $$\mu_{\bar{X}} = \mu = 68.0\,\text{in} \qquad \text{and} \qquad \sigma_{\bar{X}} = \frac{\sigma}{\sqrt{N}} = \frac{3}{\sqrt{25}} = 0.6\,\text{in}$$

(*b*) $$\mu_{\bar{X}} = 68.0\,\text{in} \qquad \text{and} \qquad \sigma_{\bar{X}} = \frac{\sigma}{\sqrt{N}}\sqrt{\frac{N_p - N}{N_p - 1}} = \frac{3}{\sqrt{25}}\sqrt{\frac{3000-25}{3000-1}}$$

which is only very slightly less than 0.6 in and can therefore, for all practical purposes, be considered the same as in sampling with replacement.

Thus we would expect the experimental sampling distribution of means to be approximately normally distributed with mean 68.0 in and standard deviation 0.6 in.

**3.** *For problem (2), how many samples would you expect to find the mean (a) between 66.8 and 68.3 ? (b) Less than 66.4 in?*

**Answer:**

The mean $\bar{X}$ of a sample in standard units is here given by

$$z = \frac{\bar{X} - \mu_{\bar{X}}}{\sigma_{\bar{X}}} = \frac{\bar{X} - 68.0}{0.6}$$

(*a*) $$66.8 \text{ in standard units} = \frac{66.8 - 68.0}{0.6} = -2.0$$

$$68.3 \text{ in standard units} = \frac{68.3 - 68.0}{0.6} = 0.5$$

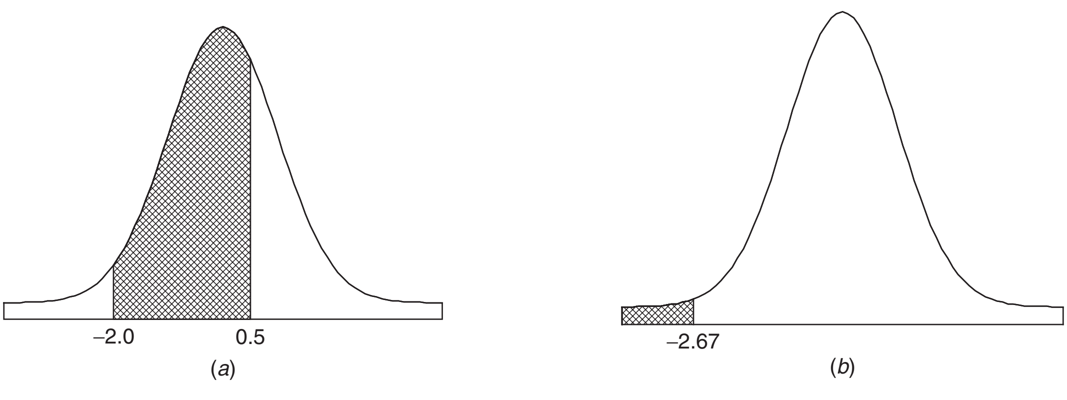



**Fig. 8-2** Areas under the standard normal curve. (*a*) Standard normal curve showing the area between $z = -2$ and $z = 0.5$; (*b*) Standard normal curve showing the area to the left of $z = -2.67$.

As shown in Fig. 8-2($a$),

Proportion of samples with means between 66.8 and 68.3 in

$$\begin{aligned}&= (\text{area under normal curve between } z = -2.0 \text{ and } z = 0.5)\\ &= (\text{area between } z = -2 \text{ and } z = 0) + (\text{area between } z = 0 \text{ and } z = 0.5)\\ &= 0.4772 + 0.1915 = 0.6687\end{aligned}$$

Thus the expected number of samples is $(80)(0.6687) = 53.496$, or 53.

($b$)

$$66.4 \text{ in standard units} = \frac{66.4 - 68.0}{0.6} = -2.67$$

As shown in Fig. 8.2($b$),

$$\begin{aligned}\text{Proportion of samples with means less than 66.4 in} &= (\text{area under normal curve to left of } z = -2.67)\\ &= (\text{area to left of } z = 0)\\ &\quad -(\text{area between } z = -2.67 \text{ and } z = 0)\\ &= 0.5 - 0.4962 = 0.0038\end{aligned}$$

Thus the expected number of samples is $(80)(0.0038) = 0.304$, or zero.

**4.** Find the probability that in 120 tosses of a fair coin (a) less than 40% or more than 60% will be heads and (b) 5/8 or more will be heads.

**Answer:**

$$\mu_P = p = \tfrac{1}{2} = 0.50 \qquad \sigma_P = \sqrt{\frac{pq}{N}} = \sqrt{\frac{(\frac{1}{2})(\frac{1}{2})}{120}} = 0.0456$$

$$40\% \text{ in standard units} = \frac{0.40 - 0.50}{0.0456} = -2.19$$

$$60\% \text{ in standard units} = \frac{0.60 - 0.50}{0.0456} = 2.19$$

$$\begin{aligned}\text{Required probability} &= (\text{area to the left of } -2.19 \text{ plus area to the right of } 2.19)\\ &= (2(0.0143) = 0.0286)\end{aligned}$$

Although this result is accurate to two significant figures, it does not agree exactly since we have not used the fact that the proportion is actually a discrete variable. To account for this, we subtract

$1/2N = 1/2(120)$ from 0.40 and add $1/2N = 1/2(120)$ to 0.60; thus, since $1/240 = 0.00417$, the required proportions in standard units are

$$\frac{0.40 - 0.00417 - 0.50}{0.0456} = -2.28 \qquad \text{and} \qquad \frac{0.60 + 0.00417 - 0.50}{0.0456} = 2.28$$

so that agreement with the first method is obtained.

Note that $(0.40 - 0.00417)$ and $(0.60 + 0.00417)$ correspond to the proportions $47.5/120$ and $72.5/120$ in the first method.

(b)

$$(0.6250 - 0.00417) \text{ in standard units} = \frac{0.6250 - 0.00417 - 0.50}{0.0456} = 2.65$$

$$\begin{aligned}\text{Required probability} &= (\text{area under normal curve to right of } z = 2.65) \\ &= (\text{area to right of } z = 0) - (\text{area between } z = 0 \text{ and } z = 2.65) \\ &= 0.5 - 0.4960 = 0.0040\end{aligned}$$

## Estimation Theory

**5.** In a sample of five measurements, the diameter of a sphere was recorded by a scientist as 6.33, 6.37, 6.36, 6.32, and 6.37 centimeters (cm). Determine unbiased and efficient estimates of (a) the true mean and (b) the true variance.

**Answer:**

(*a*) The unbiased and efficient estimate of the true mean (i.e., the populations mean) is

$$\bar{X} = \frac{\sum X}{N} = \frac{6.33 + 6.37 + 6.36 + 6.32 + 6.37}{5} = 6.35\,\text{cm}$$

(*b*) The unbiased and efficient estimate of the true variance (i.e., the population variance) is

$$\begin{aligned}\hat{s}^2 &= \frac{N}{N-1}\,s^2 = \frac{\sum (X - \hat{X})^2}{N-1} \\ &= \frac{(6.33 - 6.35)^2 + (6.37 - 6.35)^2 + (6.36 - 6.35)^2 + (6.32 - 6.35)^2 + (6.37 - 6.35)^2}{5 - 1} \\ &= 0.00055\,\text{cm}^2\end{aligned}$$

Note that although $\hat{s} = \sqrt{0.00055} = 0.023\,\text{cm}$ is an estimate of the true standard deviation, this estimate is neither unbiased nor efficient.

**6.** In measuring reaction time, a psychologist estimates that the standard deviation is 0.05 seconds (s). How large a sample of measurements must he take in order to be (a) 95% and (b) 99% confident that the error of his estimate will not exceed 0.01 s?

**Answer:**

(*a*) The 95% confidence limits are $\bar{X} \pm 1.96\sigma/\sqrt{N}$, the error of the estimate being $1.96\sigma/\sqrt{N}$. Taking $\sigma = s = 0.05\,\text{s}$, we see that this error will be equal to 0.01 s if $(1.96)(0.05)/\sqrt{N} = 0.01$; that is, $\sqrt{N} = (1.96)(0.05)/0.01 = 9.8$, or $N = 96.04$. Thus we can be 95% confident that the error of the estimate will be less than 0.01 s if $N$ is 97 or larger.

**Another method**

$$\frac{(1.96)(0.05)}{\sqrt{N}} \leq 0.01 \qquad \text{if} \qquad \frac{\sqrt{N}}{(1.96)(0.05)} \geq \frac{1}{0.01} \qquad \text{or} \qquad \sqrt{N} \geq \frac{(1.96)(0.05)}{0.01} = 9.8$$

Then $N \geq 96.04$, or $N \geq 97$.

(*b*) The 99% confidence limits are $\bar{X} \pm 2.58\sigma/\sqrt{N}$. Then $(2.58)(0.05)/\sqrt{N} = 0.01$, or $N = 166.4$. Thus we can be 99% confident that the error of the estimate will be less than 0.01 s only if $N$ is 167 or larger.

**7.** *A random sample of 50 mathematics grades out of a total of 200 showed a mean of 75 and a standard deviation of 10. (a) What are the 95% confidence limits for estimates of the mean of the 200 grades? (b) With what degree of confidence could we say that the mean of all 200 grades is 75 +/- 1?*

## Answer:

(*a*) Since the population size is not very large compared with the sample size, we must adjust for it. Then the 95% confidence limits are

$$\bar{X} \pm 1.96\sigma_{\bar{X}} = \bar{X} \pm 1.96\,\frac{\sigma}{\sqrt{N}}\sqrt{\frac{N_p - N}{N_p - 1}} = 75 \pm 1.96\,\frac{10}{\sqrt{50}}\sqrt{\frac{200-50}{200-1}} = 75 \pm 2.4$$

(*b*) The confidence limits can be represented by

$$\bar{X} \pm z_c\,\sigma_{\bar{X}} = \bar{X} \pm z_c\,\frac{\sigma}{\sqrt{N}}\sqrt{\frac{N_p - N}{N_p - 1}} = 75 \pm z_c\,\frac{10}{\sqrt{50}}\sqrt{\frac{200-50}{200-1}} = 75 \pm 1.23z_c$$

Since this must equal $75 \pm 1$, we have $1.23z_c = 1$, or $z_c = 0.81$. The area under the normal curve from $z = 0$ to $z = 0.81$ is 0.2910; hence the required degree of confidence is $2(0.2910) = 0.582$, or 58.2%.

**8.** *The voltages of 50 batteries of the same type have a mean of 18.2 volts (V) and a standard deviation of 0.5 V. Find (a) the probable error of the mean and (b) the 50% confidence limits.*

## Answer:

(*a*)
$$\text{Probable error of the mean} = 0.674\sigma_{\bar{X}} = 0.6745\frac{\sigma}{\sqrt{N}} = 0.6745\frac{s}{\sqrt{N}}$$
$$= 0.6745\frac{s}{\sqrt{N-1}} = 0.6745\frac{0.5}{\sqrt{49}} = 0.048\,\text{V}$$

Note that if the standard deviation of 0.5 V is computed as $\hat{s}$, the probable error is $0.6745(0.5/\sqrt{50}) = 0.048$ also, so that either estimate can be used if $N$ is large enough.

(*b*) The 50% confidence limits are $18 \pm 0.048$ V.

## Hypothesis Testing

**9.** (a) *Find the probability of getting between 40 and 60 heads inclusive in 100 tosses of a fair coin.*
*(b) To test the hypothesis that a coin is fair, adopt the following decision rule:*
"Accept the hypothesis if the number of heads in a single sample of 100 tosses is between 40 and 60 inclusive. Reject the hypothesis otherwise. "
*Find the probability of rejecting the hypothesis when it is actually correct.*
*(c) Graph the decision rule and the result of part (b)*
*(d) What conclusions would you draw if the sample of 100 tosses yielded 53 heads? And if it yielded 60 heads?*

*(e) Could you be wrong in your conclusions about part (d)? Explain.*

**Answer:**

(a)

According to the binomial distribution, the required probability is

$$\binom{100}{40}\left(\frac{1}{2}\right)^{40}\left(\frac{1}{2}\right)^{60}+\binom{100}{41}\left(\frac{1}{2}\right)^{41}\left(\frac{1}{2}\right)^{59}+\cdots+\binom{100}{60}\left(\frac{1}{2}\right)^{60}\left(\frac{1}{2}\right)^{40}$$

Since $Np = 100(\frac{1}{2})$ and $Nq = 100(\frac{1}{2})$ are both greater than 5, the normal approximation to the binomial distribution can be used in evaluating this sum. The mean and standard deviation of the number of heads in 100 tosses are given by

$$\mu = Np = 100(\tfrac{1}{2}) = 50 \qquad \text{and} \qquad \sigma = \sqrt{Npq} = \sqrt{(100)(\tfrac{1}{2})(\tfrac{1}{2})} = 5$$

On a continuous scale, between 40 and 60 heads inclusive is the same as between 39.5 and 60.5 heads. We thus have

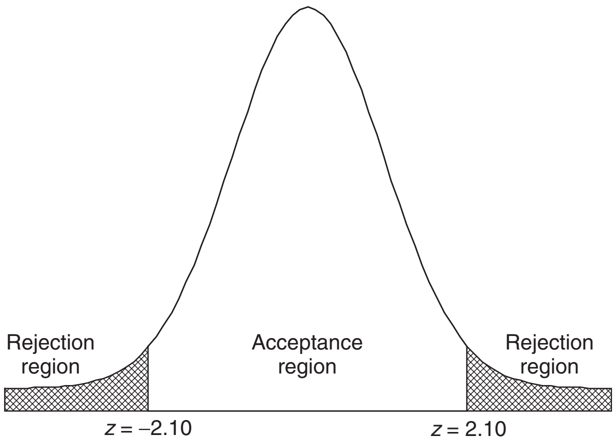
$$39.5 \text{ in standard units} = \frac{39.5-50}{5} = -2.10 \qquad 60.5 \text{ in standard units} = \frac{60.5-50}{5} = 2.10$$

$$\begin{aligned}\text{Required probability} &= \text{area under normal curve between } z=-2.10 \text{ and } z=2.10\\ &= 2(\text{area between } z=0 \text{ and } z=2.10) = 2(0.4821) = 0.9642\end{aligned}$$

(b) The probability of not getting between 40 and 60 heads inclusive if the coin is fair is 1-0.9642=0.0358. Thus the probability of rejecting the hypothesis when it is correct is 00358.

(c) The following figure shows the probability distribution of heads in 100 tosses of a fair coin. If a single sample of 100 tosses yields a $z$ score between -2.10 and 2.10, we accept the hypothesis; others, we reject the hypothesis and decide that the coin is not fair.

The error made in rejecting the hypothesis when it should be accepted is the *Type 1 error* of the decision rule; and the probability of making this error, equal to 0.0358 from part (b), is represented by the total shaded area of the figure. If a single sample of 100 tosses yields a number of heads whose $z$ score (or $z$ statistic) lies in the shaded regions, we would say that this $z$ score differed *significantly* from what would be expected if the hypothesis were true. For this reason, the total shaded area (i.e., the probability of a Type I error) is called the *significance level* of the decision rule and equals 0.0358 in this case. Thus we speak of rejecting the hypothesis at the 0.0358 (or 3.58%) significance level.




(d) According to the decision rule, we would have to accept the hypothesis that the coin is fair in both cases. One might argue that if only one more head had been obtained, we would have rejected the hypothesis. This is what one must face when any sharp line of division is used in making decisions.

(e) Yes. We could accept the hypothesis when it actually should be rejected—as would be the case, for example, when the probability of heads is really 0.7 instead of 0.5. The error made in accepting the hypothesis when it should be rejected is the Type II error of the decision.