Exercises: Linear Systems and Matrix Inverse

Problem 1. Consider the following linear system:

$$\begin{cases} x_1 + x_2 + x_3 + x_4 & = & 1 \\ 3x_1 + x_2 + x_3 + x_4 & = & a \\ x_2 + 2x_3 + 2x_4 & = & 3 \\ 5x_1 + 4x_2 + 3x_3 + 3x_4 & = & a \end{cases}$$

Depending on the value of a, when does the system have no solution, a unique solution, and infinitely many solutions?

Solution. Consider the augmented matrix $\tilde{\boldsymbol{A}}$:

$$\tilde{\boldsymbol{A}} = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 1 & a \\ 0 & 1 & 2 & 2 & 3 \\ 5 & 4 & 3 & 3 & a \end{array}\right]$$

Note that the part of $\tilde{\boldsymbol{A}}$ to the left of the vertical bar is the coefficient matrix $\boldsymbol{A}$. We will discuss the ranks of $\boldsymbol{A}$ and $\tilde{\boldsymbol{A}}$. For this purpose, we apply elementary row operations to convert $\tilde{\boldsymbol{A}}$ into row echelon form:

$$\begin{aligned} \tilde{\boldsymbol{A}} \Rightarrow & \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & -2 & a-3 \\ 0 & 1 & 2 & 2 & 3 \\ 0 & -1 & -2 & -2 & a-5 \end{array}\right] \\ \Rightarrow & \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & 3 \\ 0 & -2 & -2 & -2 & a-3 \\ 0 & -1 & -2 & -2 & a-5 \end{array}\right] \\ \Rightarrow & \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & 3 \\ 0 & 0 & 2 & 2 & a+3 \\ 0 & 0 & 0 & 0 & a-2 \end{array}\right] \end{aligned}$$

Now we can analyze the solutions of the linear system:

- If $a \neq 2$, then *rank* $\tilde{\boldsymbol{A}} = 4$ whereas *rank* $\boldsymbol{A} = 3$. In this case, the system has no solution.
- If $a = 2$, then *rank* $\boldsymbol{A} =$ *rank* $\tilde{\boldsymbol{A}} = 3$, which is smaller than the number 4 of variables. Hence, the system has infinitely many solutions.

It is worth mentioning that, regardless of the value of a, the linear system never has a unique solution.

Problem 2. Consider the following linear system:

$$\begin{cases} 2x_1 + x_2 + bx_3 & = & 0 \\ x_1 + x_2 + bx_3 & = & 0 \\ bx_1 + x_2 + 2x_3 & = & 0 \end{cases}$$

Depending on the value of b, when does the system have no solution, a unique solution, and infinitely many solutions?

Solution. Consider the augmented matrix $\tilde{\boldsymbol{A}}$:

$$\tilde{\boldsymbol{A}} = \left[\begin{array}{ccc|c} 2 & 1 & b & 0 \\ 1 & 1 & b & 0 \\ b & 1 & 2 & 0 \end{array}\right]$$

Again, the part of $\tilde{\boldsymbol{A}}$ to the left of the vertical bar is the coefficient matrix $\boldsymbol{A}$.

If $b = 0$, then

$$\begin{aligned} \tilde{\boldsymbol{A}} &= \left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \\ &\Rightarrow \left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \\ &\Rightarrow \left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \end{aligned}$$

Hence, the system has a unique solution.

Next we consider that $b \neq 0$.

$$\begin{aligned} \tilde{\boldsymbol{A}} &\Rightarrow \left[\begin{array}{ccc|c} b & 1 & 2 & 0 \\ 2 & 1 & b & 0 \\ 1 & 1 & b & 0 \end{array}\right] \\ &\Rightarrow \left[\begin{array}{ccc|c} b & 1 & 2 & 0 \\ b & b/2 & b^2/2 & 0 \\ b & b & b^2 & 0 \end{array}\right] \\ &\quad \text{(Note that we multiplied the 2nd row by } b/2\text{, and the 3rd one by } b. \\ &\quad \text{These are elementary row operations because } b \neq 0.) \\ &\Rightarrow \left[\begin{array}{ccc|c} b & 1 & 2 & 0 \\ 0 & b/2-1 & b^2/2-2 & 0 \\ 0 & b-1 & b^2-2 & 0 \end{array}\right] \\ &\Rightarrow \left[\begin{array}{ccc|c} b & 1 & 2 & 0 \\ 0 & b-2 & b^2-4 & 0 \\ 0 & b-1 & b^2-2 & 0 \end{array}\right] \qquad (1) \end{aligned}$$

If $b = 2$, then

$$(1) \Rightarrow \left[\begin{array}{ccc|c} 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 2 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Hence, the system has infinitely many solutions.

If, on the other hand, $b = 1$, then

$$(1) \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Hence, the system has a unique solution.

Next, we consider that $b \neq 0, 1, 2$. In this case:

$$\begin{aligned}
(1) \quad \Rightarrow & \left[\begin{array}{ccc|c} b & 1 & 2 & 0 \\ 0 & 1 & b+2 & 0 \\ 0 & b-1 & b^2-2 & 0 \end{array}\right] \\
\Rightarrow & \left[\begin{array}{ccc|c} b & 1 & 2 & 0 \\ 0 & b-1 & (b+2)(b-1) & 0 \\ 0 & b-1 & b^2-2 & 0 \end{array}\right] \\
= & \left[\begin{array}{ccc|c} b & 1 & 2 & 0 \\ 0 & b-1 & b^2+b-2 & 0 \\ 0 & b-1 & b^2-2 & 0 \end{array}\right] \\
\Rightarrow & \left[\begin{array}{ccc|c} b & 1 & 2 & 0 \\ 0 & b-1 & b^2+b-2 & 0 \\ 0 & 0 & -b & 0 \end{array}\right]
\end{aligned}$$

Clearly, (as $b \neq 0$) the above matrix has rank 3; therefore, the linear system has a unique solution.

In summary, when $b = 2$, the original linear system has infinitely many solutions. For any other value of b, the system has a unique solution.

Problem 3. Use Cramer's rule to solve the following linear system:

$$\begin{cases} 2x - 4y &= -24 \\ 5x + 2y &= 0 \end{cases}$$

Solution. The coefficient matrix equals

$$\boldsymbol{A} = \begin{bmatrix} 2 & -4 \\ 5 & 2 \end{bmatrix}$$

Since $det(\boldsymbol{A}) = 24 \neq 0$, the system has a unique solution. Define:

$$\boldsymbol{A}_1 = \begin{bmatrix} -24 & -4 \\ 0 & 2 \end{bmatrix}, \boldsymbol{A}_2 = \begin{bmatrix} 2 & -24 \\ 5 & 0 \end{bmatrix}$$

By Cramer's rule, we have:

$$\begin{aligned}
x &= \frac{det(\boldsymbol{A}_1)}{det(\boldsymbol{A})} = \frac{-48}{24} = -2 \\
y &= \frac{det(\boldsymbol{A}_2)}{det(\boldsymbol{A})} = \frac{120}{24} = 5.
\end{aligned}$$

Problem 4. Compute the inverse of

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Solution. We apply Gauss-Jordan elimination. Specifically, we start with

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \tag{2}$$

and convert the left hand side of the vertical bar into an identity matrix using elementary row operations.

$$(2) \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

Now, what remains on the right hand side of the bar is the inverse of $\boldsymbol{A}$, namely:

$$\boldsymbol{A}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Problem 5. Compute the inverse of

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 5 & 9 & 1 \end{bmatrix}$$

Solution. We apply Gauss-Jordan elimination:

$$\begin{aligned}
\boldsymbol{A} &\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ -2 & -3 & 1 & 0 & 1 & 0 \\ 5 & 9 & 1 & 0 & 0 & 1 \end{array}\right] \\
&\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 2 & 1 & 0 \\ 0 & -1 & -4 & -5 & 0 & 1 \end{array}\right] \\
&\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 2 & 1 & 0 \\ 0 & 0 & -1 & -3 & 1 & 1 \end{array}\right] \\
&\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 2 & 1 & 0 \\ 0 & 0 & 1 & 3 & -1 & -1 \end{array}\right] \\
&\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -2 & 1 & 1 \\ 0 & 1 & 0 & -7 & 4 & 3 \\ 0 & 0 & 1 & 3 & -1 & -1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 12 & -7 & -5 \\ 0 & 1 & 0 & -7 & 4 & 3 \\ 0 & 0 & 1 & 3 & -1 & -1 \end{array}\right]
\end{aligned}$$

Now, what remains on the right hand side of the bar is the inverse of $\boldsymbol{A}$, namely:

$$\boldsymbol{A}^{-1} = \begin{bmatrix} 12 & -7 & -5 \\ -7 & 4 & 3 \\ 3 & -1 & -1 \end{bmatrix}$$

Problem 6. Let $\boldsymbol{A}$ be an $n \times n$ matrix. Also, let $\boldsymbol{I}$ be the $n \times n$ identity matrix. Prove: if $\boldsymbol{A}^3 = \boldsymbol{0}$, then

$$(\boldsymbol{I} - \boldsymbol{A})^{-1} = \boldsymbol{I} + \boldsymbol{A} + \boldsymbol{A}^2.$$

Proof.

$$(\boldsymbol{I} - \boldsymbol{A})(\boldsymbol{I} + \boldsymbol{A} + \boldsymbol{A}^2) = \boldsymbol{I}^2 - \boldsymbol{A} + \boldsymbol{A} - \boldsymbol{A}^{\mathbf{2}} + \boldsymbol{A}^2 - \boldsymbol{A}^3 = \boldsymbol{I}$$

which completes the proof. $\square$

Problem 7. Consider:

$$\boldsymbol{A} = \begin{bmatrix} 2 & 1 & b \\ 1 & 1 & b \\ b & 1 & 2 \end{bmatrix}$$

Under what values of b does $\boldsymbol{A}^{-1}$ exist?

Solution. We know that $\boldsymbol{A}^{-1}$ exists if and only if $det(\boldsymbol{A}) \neq 0$.

$$\begin{aligned} det(\boldsymbol{A}) &= \begin{vmatrix} 2 & 1 & b \\ 1 & 1 & b \\ b & 1 & 2 \end{vmatrix} \\ &= 2\begin{vmatrix} 1 & b \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & b \\ b & 2 \end{vmatrix} + b\begin{vmatrix} 1 & 1 \\ b & 1 \end{vmatrix} \\ &= 2(2-b) - (2-b^2) + b(1-b) \\ &= 2 - b. \end{aligned}$$

Therefore, $\boldsymbol{A}^{-1}$ exists if and only if $b \neq 2$.