Multivariable Calculus

Week 7 NOTES – Lagrange Multipliers

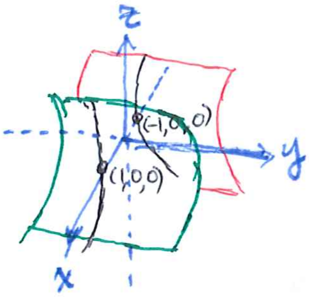
Prepared by Hugo MAK

Ref.: Ch 13.3 of Adams and Essex

Ch 13.10 of Larson and Edwards

Re-visit: Constrained Optimization

Find the points on the hyperbolic cylinder $x^2 - z^2 = 1$ that are closest to the origin.



Hyperbolic cylinder

$x^2 - z^2 = 1$

Idea: We want to seek for the points whose coordinates (x, y, z) will minimize the value of the following function $f(x,y,z) = (x-0)^2 + (y-0)^2 + (z-0)^2$

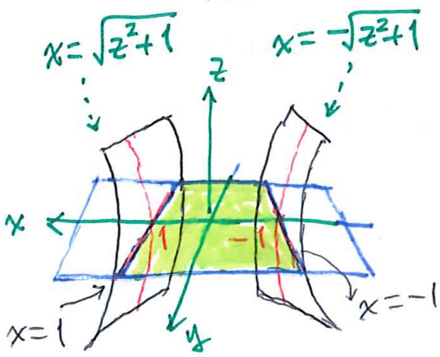
subject to the constraint $x^2 - z^2 = 1$.

Now, we regard x and y as independent variables, then $z^2 = x^2 - 1$

i.e. we wish to find the points in the xy-plane whose coordinates will minimize

$$h(x,y) = (x-0)^2 + (y-0)^2 + (x^2-1)$$
$$= 2x^2 + y^2 - 1$$

Excluding the band $-1 < x < 1$ in xy-plane.



Now, consider $h_x(x,y) = 4x$ and $h_y(x,y) = 2y$ and we set $h_x = h_y = 0 \Rightarrow \boxed{(x,y) = (0,0)}$

However such point does NOT lie on the cylinder. What went wrong?

Reason: The first derivative test finds out the point in the domain of h where h has a minimum value. However, we want the points on the cylinder where h has a minimum value.

i.e. It does not include the band between $x = -1$ and $x = 1$

Notice that the domain of h is the entire xy-plane, however the domain from which we can select the first two coordinates of the points (x,y,z) on the cylinder is restricted to the "shadow" of the cylinder on the xy-plane.

Correct Approach: Treat y and z as independent variables (instead of x and y)

i.e. $x^2 = z^2 + 1$

$\Rightarrow f(x,y,z) = x^2+y^2+z^2$ will become $g(y,z) = (z^2+1)+y^2+z^2$
$= 1+y^2+2z^2$

We look for the points where g takes on the least value.

Domain of g in the yz-plane will be in line with the domain from which we select the y and z coordinates of the points (x,y,z) on the cylinder.

Consider $g_y(y,z) = 2y = 0$ and $g_z(y,z) = 4z = 0 \Rightarrow y = z = 0$

Hence, $x^2 = z^2+1 = 1 \Rightarrow x = \pm 1$

i.e. Corresponding points on the cylinder are $(\pm 1, 0, 0)$.

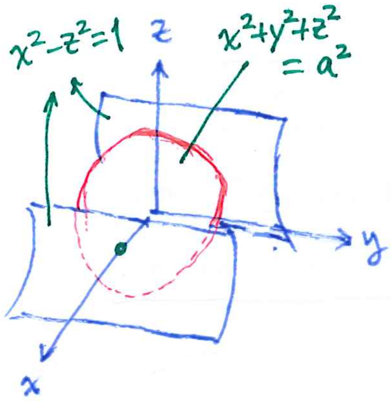
Also, $g(y,z) \geq 1$ for every $(y,z) \in \mathbb{R}^2$.

i.e. The points $(\pm 1, 0, 0)$ give a minimum value for g.

$\Rightarrow$ minimum distance from the origin to a point on the cylinder

$=$ 1 unit

Alternative Solution: Imagine we have a small sphere centered at the origin, and we keep on expanding it like soap bubble until it just touches the cylinder.



At every point of contact, the cylinder and sphere have the same tangent plane and normal line.

Therefore, we represent the sphere and cylinder as two level surfaces: $f(x,y,z) = x^2+y^2+z^2-a^2 = 0$ (sphere)

$g(x,y,z) = x^2-z^2-1 = 0$ (hyperbolic cylinder)

Then ∇f and ∇g will be parallel where the surfaces touch. At all points of contact, $\nabla f = \lambda \nabla g$, $\lambda \in \mathbb{R}$.

i.e. $\langle 2x, 2y, 2z\rangle = \lambda \langle 2x, 0, -2z\rangle$

Comparing component by component, $\begin{cases} 2x = 2\lambda x \\ 2y = 0 \\ 2z = -2\lambda z \end{cases}$

We wish to find values of λ such that the corresponding (x,y,z) also satisfies the surface equation $x^2-z^2-1=0$.

Assume $x \neq 0$ ($\because$ no point on the surface has a zero x-coordinate). Then

$$2 = 2\lambda \Rightarrow \lambda = 1$$

For $\lambda = 1$, $2z = -2z \Rightarrow z = 0$. Also $y = 0$ $\Rightarrow$ Our desired solution $(x,0,0)$ such that $x^2 - z^2 = 1$.

i.e. $x^2 = 1 \Rightarrow x = \pm 1$

The points on the cylinder closest to the origin are the points $(\pm 1, 0, 0)$.

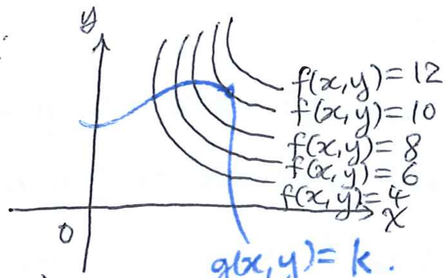
In solution 2, we use the method of Lagrange Multipliers, i.e. The local extreme values of a function $f(x,y,z)$ whose variables are subject to certain constraint $\underbrace{g(x,y,z)=0}$ are to be found on the surface $g=0$ among the points where

$$\nabla f = \lambda \nabla g \text{ , where } \lambda \in \mathbb{R}.$$

i.e. the point (x,y,z) is restricted to lie on the level surface $g(x,y,z)=0$.

Let us go to 2D case first:

To maximize (minimize) $f(x,y)$ subject to $g(x,y)=k$



Several level curves of f, where $c = 4, 6, 8, 10, 12$

$\Leftrightarrow$ To find the largest (smallest) value of c such that the level curve $f(x,y)=c$ intersects $g(x,y)=k$.

Happens when these curves just touch each other, i.e. when they have a common tangent line. (otherwise, the value of c will be increased (decreased) further)

$\downarrow$

normal lines at the point (x_0,y_0) where they touch are identical, i.e. Gradient vectors are parallel, i.e. $\nabla f(x_0,y_0) = \lambda \nabla g(x_0,y_0)$ for some $\lambda \in \mathbb{R}$.

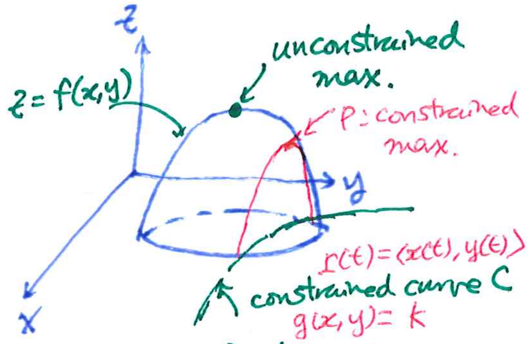
Extension to 3D cases

Finding extreme values of $f(x,y,z)$ subject to the constraint $g(x,y,z)=k$.

(x,y,z) is restricted to lie on the level surface S with equation $g(x,y,z)=k$

Instead of level curves, we consider the level surfaces $f(x,y,z)=c$ and argue that if the maximum value of f is $f(x_0,y_0,z_0)=c$, then the level surface $f(x,y,z)=c$ is tangent to the level surface $g(x,y,z)=k$, i.e. corresponding gradient vectors are parallel.

Suppose that a function f has an extreme value at a point $P(x_0, y_0, z_0)$ on the surface S. Let C be a curve with vector equation $\underline{r}(t) = \langle x(t), y(t), z(t) \rangle$ that lies on S and passes through P.



Simplified version (2D)

If t_0 is the parameter value corresponding to P, then $\underline{r}(t_0) = \langle x(t_0), y(t_0), z(t_0) \rangle = \langle x_0, y_0, z_0 \rangle$.

The composite function $h(t) = f(x(t), y(t), z(t))$ represents the values that f takes on the curve C.

Since f has an extreme value at $(x_0, y_0, z_0) \Rightarrow h$ has an extreme value at t_0.

i.e. $h'(t_0) = 0$.

Assume f is differentiable,

$$
\begin{aligned}
0 = h'(t_0) &= f_x(x_0, y_0, z_0)\, x'(t_0) + f_y(x_0, y_0, z_0)\, y'(t_0) \\
&\quad + f_z(x_0, y_0, z_0)\, z'(t_0) \\
&= \underline{\nabla} f(x_0, y_0, z_0) \cdot \underline{r}'(t_0)
\end{aligned}
$$

i.e. $\underline{\nabla} f(x_0, y_0, z_0)$ is orthogonal to the tangent vector $\underline{r}'(t_0)$ to every such curve C.

But we know that $\underline{\nabla} g(x_0, y_0, z_0)$ is also orthogonal (Gradient Thm) to $\underline{r}'(t_0)$ for every such curve $\Rightarrow \underline{\nabla} f(x_0, y_0, z_0)$ is parallel to $\underline{\nabla} g(x_0, y_0, z_0)$.

i.e. If $\underline{\nabla} g(x_0, y_0, z_0) \neq \underline{0}$, $\exists \lambda \in \mathbb{R}$ such that

λ: Lagrange multiplier

$$\underline{\nabla} f(x_0, y_0, z_0) = \lambda \underline{\nabla} g(x_0, y_0, z_0)$$

Method of Lagrange Multipliers: (a) Find all values of x, y, z and λ such that

$$\underline{\nabla} f(x, y, z) = \lambda \underline{\nabla} g(x, y, z)$$

and $g(x, y, z) = k$

(b) Evaluate f at all points (x, y, z) resulting from step (a). The largest of these values is the maximum value of f; the smallest of these values is the minimum value of f.

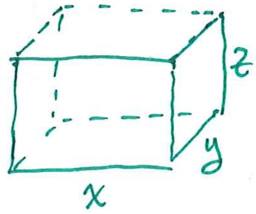
3D	$f_x = \lambda g_x$ $f_y = \lambda g_y$ $f_z = \lambda g_z$ $g(x,y,z) = k$
2D	$f_x = \lambda g_x$ $f_y = \lambda g_y$ $g(x,y) = k$

Example ① A rectangular box without a lid is to be made from $12 m^2$ of cardboard paper. Find the maximum volume of such box.

Sol^n Let x, y and z be the length, width and height of the box (in m).

we wish to max. $V = xyz$

subject to $g(x,y,z) = 2xz + 2yz + xy = 12$



Approach: $\begin{cases} \nabla V = \lambda \nabla g \\ g(x,y,z) = 12 \end{cases} \Leftrightarrow \begin{cases} V_x = \lambda g_x \\ V_y = \lambda g_y \\ V_z = \lambda g_z \\ g(x,y,z) = 12 \end{cases}$

$\Leftrightarrow \begin{cases} yz = \lambda(2z + y) & \text{— ①} \\ xz = \lambda(2z + x) & \text{— ②} \\ xy = \lambda(2x + 2y) & \text{— ③} \\ 2xz + 2yz + xy = 12 & \text{— ④} \end{cases}$

Multiply x to ①, y to ②, z to ③,

we have $\begin{cases} xyz = \lambda(2xz + xy) \\ xyz = \lambda(2yz + xy) \\ xyz = \lambda(2xz + 2yz) \end{cases}$

Here $\lambda \neq 0$ ($\because \lambda = 0$ implies $xy = xz = yz = 0$!!!)

$\therefore 2xz + xy = 2yz + xy \Rightarrow xz = yz$

But $z \neq 0$ ($\because z = 0 \Rightarrow V = 0$!!!), so $\boxed{x = y}$.

Now, $2yz + xy = 2xz + 2yz \Rightarrow \boxed{y = 2z}$ ($\because x \neq 0$).

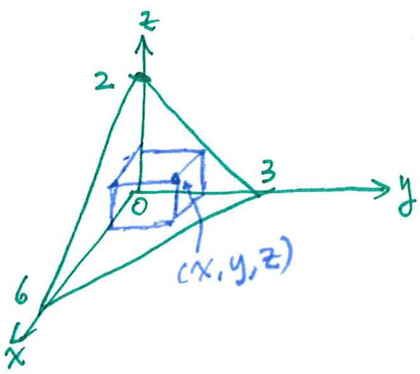
Put $x = y = 2z$ into ④ $\Rightarrow 4z^2 + 4z^2 + 4z^2 = 12 \Rightarrow \boxed{z = 1}$ ($\because z > 0$)

$\therefore \boxed{x = 2}$ and $\boxed{y = 2}$

Maximum Volume $= \underline{\underline{4 m^3}}$

---

② Find the volume of the largest rectangular box in the first octant with three faces in the coordinate planes and one vertex in the plane $x + 2y + 3z - 6 = 0$.



Goal: max $V = f(x,y,z) = xyz$

s.t. $g(x,y,z) = x + 2y + 3z = 6$

$\nabla f = \lambda \nabla g \Leftrightarrow \langle yz, xz, xy \rangle = \lambda \langle 1, 2, 3 \rangle$ for some $\lambda \in \mathbb{R}$.

$\lambda = yz = \frac{1}{2}xz = \frac{1}{3}xy$

$\Rightarrow x = 2y, \; z = \frac{2}{3}y$

Put this to $g(x,y,z) = 6 \Rightarrow x = 2, y = 1, z = \frac{2}{3}$

$\therefore$ Volume $= f(2, 1, \frac{2}{3}) = \underline{\underline{\frac{4}{3} \text{ cubic units}}}$

③ The Cobb–Douglas production function for a manufacturer is given by

$$f(x,y) = 100x^{\frac{3}{4}}y^{\frac{1}{4}}$$

Here x represents the units of labor (at \$150 per unit),
y represents the units of capital (at \$250 per unit).

Given that the total cost of labor and capital is limited to \$50,000, find the maximum production level for this manufacturer.

Sol^n We wish to maximize $f(x,y) = 100x^{\frac{3}{4}}y^{\frac{1}{4}}$
subject to $g(x,y) = 150x + 250y = 50{,}000$

$\Rightarrow \nabla f = \lambda \nabla g$

$\langle 75x^{-\frac{1}{4}}y^{\frac{1}{4}}, 25x^{\frac{3}{4}}y^{-\frac{3}{4}} \rangle = \lambda \langle 150, 250 \rangle$ for some $\lambda \in \mathbb{R}$

i.e. $\begin{cases} 75x^{-\frac{1}{4}}y^{\frac{1}{4}} = 150\lambda & \text{— ①} \\ 25x^{\frac{3}{4}}y^{-\frac{3}{4}} = 250\lambda & \text{— ②} \\ 150x + 250y = 50{,}000 & \text{— ③} \end{cases}$

From ①, $\lambda = \dfrac{x^{-\frac{1}{4}}y^{\frac{1}{4}}}{2}$

Sub into ②: $x = 5y$ (Why?)

Sub into ③: $1000y = 50{,}000$
$y = 50$
$\therefore x = 250$.

Maximum production level
$= f(250, 50) = 100(250)^{\frac{3}{4}}(50)^{\frac{1}{4}}$
≈ 16.7 units of products

Remarks: $\lambda = \dfrac{x^{-\frac{1}{4}}y^{\frac{1}{4}}}{2} \approx 0.334 \rightarrow$ marginal productivity of money at $x = 250$ and $y = 50$.

For each additional dollar spent on production, an additional 0.334 unit of the product can be further produced.

Generalization: minimize or maximize $f(x_1, x_2, \ldots, x_n)$ subject to

$$\begin{cases} \phi_1(x_1, x_2, \ldots, x_n) = c_1 \\ \phi_2(x_1, x_2, \ldots, x_n) = c_2 \\ \vdots \\ \phi_k(x_1, x_2, \ldots, x_n) = c_k. \end{cases} \quad \text{— } (*)$$

Our question will become: Find $x_1, x_2, \ldots, x_n$ and $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that

$\nabla f = \sum_{i=1}^{k} \lambda_i \nabla \phi_i$ and the constraint equations $(*)$.

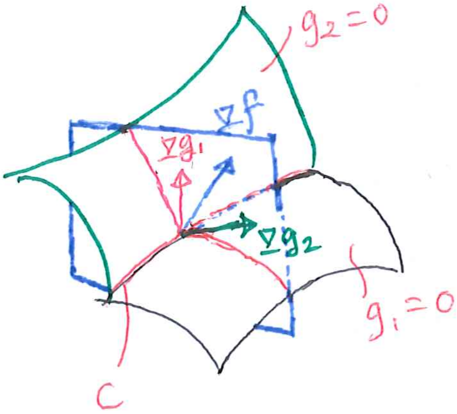
$\lambda_1, \lambda_2, \ldots, \lambda_k$: Lagrange Multipliers.

Example (2 constraints) Find the extreme values of a differentiable function $f(x,y,z)$ whose variables are subject to 2 constraints:

$$\begin{cases} g_1(x,y,z)=0 \\ g_2(x,y,z)=0 \end{cases}$$

(g_1, g_2 are differentiable)

∇g_1 <u>not</u> parallel to ∇g_2



$$\boxed{\begin{aligned} &\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \\ &g_1(x,y,z)=0 \\ &g_2(x,y,z)=0 \end{aligned}}$$

∇g_1 and ∇g_2 lie in a plane perpendicular to the curve C because ∇g_1 is normal to the surface $g_1=0$ and ∇g_2 is normal to the surface $g_2=0$.

C: intersection of the surfaces $g_1=0$ and $g_2=0$

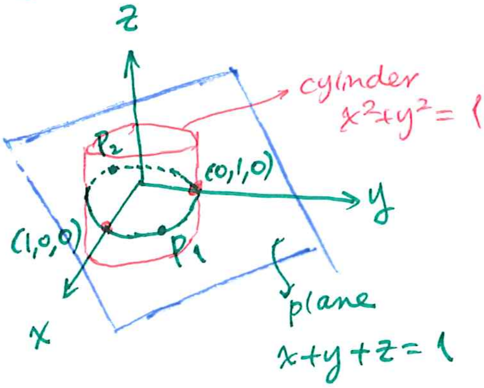
Along this curve, we seek the points where f has local max. and min. values relative to other points on the curve.

These are the points where ∇f is normal to C.

But ∇g_1 and ∇g_2 are also normal to C at these points ($\because$ C lies in the surfaces $g_1=0$ and $g_2=0$).

$\Rightarrow$ ∇f lies in the plane determined by ∇g_1 and ∇g_2.

Example The plane $x+y+z=1$ cuts the cylinder $x^2+y^2=1$ in an ellipse.
(Thomas Calculus) Find the points on the ellipse that lie closest to and farthest from origin.



We want to max. and min. $f_1(x,y,z)=\sqrt{x^2+y^2+z^2}$

Subject to $g_1(x,y,z)=x^2+y^2-1=0$

and $g_2(x,y,z)=x+y+z-1=0$

Now, max. and min. $f_1(x,y,z)$ is equivalent to max. and min. $f(x,y,z)=x^2+y^2+z^2$.

Hence $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$

$$\langle 2x, 2y, 2z\rangle = \langle 2\lambda x+\mu,\ 2\lambda y+\mu,\ \mu\rangle$$

i.e. $\begin{cases} 2x = 2\lambda x+\mu \\ 2y = 2\lambda y+\mu \\ 2z=\mu \end{cases}$ and $\begin{cases} x^2+y^2-1=0 \\ x+y+z-1=0 \end{cases}$

Now, $2x = 2\lambda x + 2z \Rightarrow (1-\lambda)x = z$

$2y = 2\lambda y + 2z \Rightarrow (1-\lambda)y = z$

$$\lambda = 1 \text{ and } z = 0 \quad \text{OR} \quad \lambda \neq 1 \text{ and } x = y = \frac{z}{1-\lambda}$$

If $z = 0$, then we have the two points $(1,0,0)$ and $(0,1,0)$

If $x = y$, then $x^2 + x^2 - 1 = 0$ and $x + x + z - 1 = 0$

$$x = \pm\frac{\sqrt{2}}{2} \qquad z = 1 - 2x$$

$$z = 1 \mp \sqrt{2}$$

Corresponding points on ellipse are

$$P_1 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1-\sqrt{2}\right) \text{ and } P_2 = \left(\frac{-\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}, 1+\sqrt{2}\right)$$

Both P_1 and P_2 give local maxima of f on the ellipse,

P_2 is farther from origin than P_1.

Conclusion: The points on the ellipse closest to the origin are $(1,0,0)$ and $(0,1,0)$.

The point on the ellipse farthest from the origin is P_2

$$= \left(\frac{-\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}, 1+\sqrt{2}\right)$$