Exercises: Vector Spaces

Problem 1. Let V be the set of vectors $[2x - 3y, x + 2y, -y, 4x]$ with $x, y \in \mathbb{R}^2$. Addition and scalar multiplication are defined in the same way as on vectors. Prove that V is a vector space. Also, point out a basis of it.

Problem 2. For each of the following sets, indicate whether it is a vector space. If so, point out a basis of it; otherwise, point out which vector-space property is violated.

1. The set V of vectors $[2x, x^2]$ with $x \in \mathbb{R}^2$. Addition and scalar multiplication are defined in the same way as on vectors.

2. The set V of vectors $[x, y, z] \in \mathbb{R}^3$ satisfying $x + y + z = 3$ and $x - y + 2z = 6$. Addition and scalar multiplication are defined in the same way as on vectors.

3. The set V of symmetric 2×2 matrices. Addition and scalar multiplication are defined in the same way as on matrices.

4. The set V of 2×2 matrices $[a_{ij}]$ with $a_{11} + a_{22} = 0$. Addition and scalar multiplication are defined in the same way as on matrices.

Problem 3. Determine if the following transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ has a reverse transformation. If so, give the reverse transformation.

$$\begin{aligned} y_1 &= 3x_1 + 2x_2 \\ y_2 &= 4x_1 + x_2 \end{aligned}$$

Problem 4. Determine if the following transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ has a reverse transformation. If so, give the reverse transformation.

$$\begin{aligned} y_1 &= 3x_1 + 2x_2 + x_3 \\ y_2 &= x_1 + x_2 - x_3 \\ y_3 &= 5x_1 + 4x_2 - x_3 \end{aligned}$$

Problem 5. Consider the following linear system about $\boldsymbol{x}$

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$$

where $\boldsymbol{A}$ is an $m \times n$ coefficient matrix, and $\boldsymbol{x}$ an $n \times 1$ matrix. Let V be the set of all such $\boldsymbol{x}$ satisfying the system. Suppose that the rank of $\boldsymbol{A}$ is r. Prove that V is a vector space of dimension $n - r$ (addition and scalar multiplication are defined in the same way as on vectors).