Lecture Notes: Path Independence of Certain Line Integrals (Part 2)

Yufei Tao
Department of Computer Science and Engineering
Chinese University of Hong Kong
taoyf@cse.cuhk.edu.hk

1 2D

Define S to be the set of line integrals of the form

$$\int_C f_1\,dx + \int_C f_2\,dy.$$

In the previous lecture, we learned:

Theorem 1 (Proved Previously). *S is path independent if and only if we can find a function $g(x, y)$ such that*

$$f_1(x,y) = \frac{\partial g}{\partial x}, \text{ and } f_2(x,y) = \frac{\partial g}{\partial y}. \tag{1}$$

Example 1. Prove that the set of integrals of the form

$$\int_C y^2(\sin(x) + x \cdot \cos(x))\,dx + \int_C 2xy\sin(x)\,dy. \tag{2}$$

is path independent.

Proof. Let $g(x, y) = x\sin(x) \cdot y^2$. We have that $\frac{\partial g}{\partial x} = y^2(\sin(x) + x\cos(x))$ and $\frac{\partial g}{\partial y} = 2xy\sin(x)$. Hence, by Theorem 1, the set of integrals (2) is path independent. □

Proving path independence by Theorem 1 demands the ability of observing $g(x, y)$. What if such $g(x, y)$ is difficult to observe (as may be the case in the previous example)? Fortunately, it is often easy to determine whether S is path independent *without* deriving $g(x, y)$. This is shown in the following Theorem:

Theorem 2. *Suppose that $\frac{\partial f_1}{\partial y}$ and $\frac{\partial f_2}{\partial x}$ are both continuous in $\mathbb{R}^2$. S is path independent if and only if*

$$\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x}. \tag{3}$$

Proof. *The Only-If Direction.* Suppose that S is path independent. We want to prove that (3) must hold. Since S is path independent, by Theorem 1, there is a function $g(x, y)$ satisfying (1). Therefore, $\frac{\partial f_1}{\partial y} = \frac{\partial^2 g}{\partial x \partial y}$ and $\frac{\partial f_2}{\partial x} = \frac{\partial^2 g}{\partial y \partial x}$. The continuity of $\frac{\partial f_1}{\partial y}$ and $\frac{\partial f_2}{\partial x}$ determines that $\frac{\partial^2 g}{\partial x \partial y} = \frac{\partial^2 g}{\partial y \partial x}$, which leads to (3).

The If-Direction. The proof requires the Green's Theorem, which will be introduced later in the course. □

Example 1 (Revisited). Prove that the set of integrals of the form

$$\int_C y^2(\sin(x) + x \cdot \cos(x))\, dx + \int_C 2xy \sin(x)\, dy. \tag{4}$$

is path independent.

Proof. Let $f_1(x, y) = y^2(\sin(x) + x \cdot \cos(x))$ and $f_2(x, y) = 2xy \sin(x)$. We have that $\frac{\partial f_1}{\partial y} = 2y(\sin(x) + x \cos(x))$ and $\frac{\partial f_2}{\partial x} = 2y(\sin(x) + x \cos(x))$. Hence, by Theorem 2, the set of integrals (4) is path independent. $\square$

2 3D

The discussion in the previous section can be generalized to $\mathbb{R}^3$. Let $f_1(x, y, z)$, $f_2(x, y, z)$, and $f_3(x, y, z)$ be scalar functions. Define S to be the set of all possible line integrals of the form

$$\int_C f_1\, dx_1 + \int_C f_2\, dx_2 + \int_C f_3\, dx_3. \tag{5}$$

Theorem 3. *Suppose that $\frac{\partial f_1}{\partial y}$, $\frac{\partial f_1}{\partial z}$, $\frac{\partial f_2}{\partial x}$, $\frac{\partial f_2}{\partial z}$, $\frac{\partial f_3}{\partial x}$, and $\frac{\partial f_3}{\partial y}$ are all continuous in $\mathbb{R}^3$. S is path independent <u>if and only if</u> all the following hold:*

$$\begin{aligned} \frac{\partial f_1}{\partial y} &= \frac{\partial f_2}{\partial x} \\ \frac{\partial f_2}{\partial z} &= \frac{\partial f_3}{\partial y} \\ \frac{\partial f_3}{\partial x} &= \frac{\partial f_1}{\partial z}. \end{aligned}$$

Proof. The only-if direction is a direct extension of the proof of Theorem 2. The if-direction requires the Stokes's Theorem, which will be introduced later in the course. $\square$

Example 2. Prove that the set of integrals of the form:

$$\int_C 2xy^2z\, dx + \int_C 2x^2yz\, dy + \int_C x^2y^2\, dz \tag{6}$$

is path independent.

Proof. Let $f_1(x, y, z) = 2xy^2z$, $f_2(x, y, z) = 2x^2yz$, and $f_3(x, y, z) = x^2y^2$. We have that

$$\begin{aligned} \frac{\partial f_1}{\partial y} &= \frac{\partial f_2}{\partial x} = 4xyz \\ \frac{\partial f_2}{\partial z} &= \frac{\partial f_1}{\partial z} = 2xy^2 \\ \frac{\partial f_3}{\partial x} &= \frac{\partial f_3}{\partial y} = 2x^2y. \end{aligned}$$

Hence, by Theorem 3, the set of integrals (6) is path independent. $\square$

It is worth mentioning that Theorem 3 is closely related to a concept of *curl* defined as follows:

Definition 1. *Let $\boldsymbol{f}(x, y, z)$ be a vector function defined as*

$$\boldsymbol{f}(x, y, z) \quad = \quad [f_1(x, y, z), f_2(x, y, z), f_3(x, y, z)].$$

Then, the **curl** *of $\boldsymbol{f}(x, y, z)$ is defined as:*

$$\text{curl } \boldsymbol{f} \quad = \quad [h_1(x, y, z), h_2(x, y, z), h_3(x, y, z)]$$

where

$$\begin{aligned} h_1(x, y, z) &= \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \\ h_2(x, y, z) &= \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \\ h_3(x, y, z) &= \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}. \end{aligned}$$

Hence, by Theorem 3, the set of integrals (5) is path independent if and only if curl $\boldsymbol{f} = \mathbf{0}$.

Example 3. Define $\boldsymbol{f}(x, y, z) = [xyz, x^2, y^2 z]$. Then,

$$\text{curl } \boldsymbol{f} \quad = \quad [2yz - 2x, xy, 2x - xz].$$

By Theorem 3, we know that the set of integrals of the form:

$$\int_C xyz\,dx + \int_C x^2\,dy + \int_C y^2 z\,dz$$

is *not* path independent. □