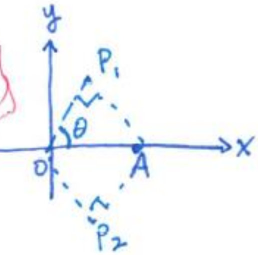
Multivariable Calculus HW 1 Solution

Due Date: 24 Jan 2020

Question 1

① Total (8 marks)



Let $\vec{P}$ and $\vec{A}$ be the position vectors of the points P and A respectively, i.e. $\vec{OA} := \vec{A}$, $\vec{OP} := \vec{P}$

$$\vec{PA} = \vec{OA} - \vec{OP} := \vec{A} - \vec{P}$$

$$\vec{PO} = -\vec{OP} := -\vec{P}$$

Since $\angle OPA = 90^\circ$, $-\vec{P}\cdot(\vec{A}-\vec{P}) = 0$

$$\vec{P}\cdot(\vec{P}-\vec{A}) = 0$$

$$\|\vec{P}\|^2 - \|\vec{P}\|\|\vec{A}\|\cos\theta = 0 \quad (*)$$

where θ is the angle between $\vec{P}$ and the x-axis.

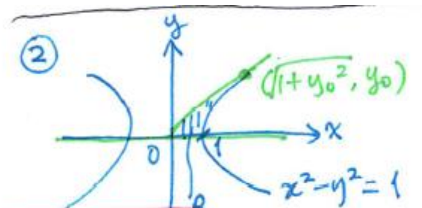
We let $r = \|\vec{P}\|$ and $a = \|\vec{A}\|$, then $(*)$ is equivalent to $r^2 = ra\cos\theta$

$$\boxed{r = a\cos\theta}$$

Description: A standard polar equation for a circle with diameter a.

Question 2

②



Total: (8 marks)

(a) Area of $R = \dfrac{y_0\sqrt{1+y_0^2}}{2} - \displaystyle\int_1^{\sqrt{1+y_0^2}} \sqrt{x^2-1}\,dx$.

Let $x = \sec\theta$, $dx = \sec\theta\tan\theta\,d\theta$

$$\Rightarrow \text{Area of } R = \frac{y_0\cdot\sqrt{1+y_0^2}}{2} - \int_0^{\tan^{-1}y_0} \sec\theta\tan^2\theta\,d\theta$$

$$= \frac{y_0}{2}\sqrt{1+y_0^2} - \int_0^{\tan^{-1}y_0} \sec^3\theta\,d\theta + \int_0^{\tan^{-1}y_0}\sec\theta\,d\theta$$

$$= \frac{y_0}{2}\sqrt{1+y_0^2} + \left(\frac{-1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta+\tan\theta| + \ln|\sec\theta+\tan\theta|\right)\Big|_0^{\tan^{-1}y_0}$$

$$= \frac{y_0}{2}\sqrt{1+y_0^2} - \frac{y_0}{2}\sqrt{1+y_0^2} + \frac{1}{2}\ln(y_0+\sqrt{1+y_0^2})$$

$$= \underline{\frac{1}{2}\ln(y_0+\sqrt{1+y_0^2}) \text{ sq units}}$$

(b) $\text{Area} = \frac{1}{2}\ln(\sinh t + \cosh t)$

$$= \frac{1}{2}\ln(e^t)$$

$$= \underline{\frac{t}{2} \text{ sq units}}$$

Question 3

(3) (a) $\underline{u} \cdot \underline{v} = (2)(1) + (1)(2) + (-2)(2) = 0 \Rightarrow \underline{u} \perp \underline{v}$

3 marks By definition of cross product, $\underline{u} \times \underline{v}$ is a vector orthogonal to both $\underline{u}$ and $\underline{v}$

$\Rightarrow \underline{w} \perp \underline{u}$ and $\underline{w} \perp \underline{v}$ as well.

Hence, $\underline{u}$, $\underline{v}$ and $\underline{w}$ are mutually orthogonal vectors.

(b) Let $\underline{r} = a\underline{u} + b\underline{v} + c\underline{w}$

5 marks Taking dot product with $\underline{u}$ on both sides,

$$\underline{r} \cdot \underline{u} = (a\underline{u} + b\underline{v} + c\underline{w}) \cdot \underline{u}$$

$$\underline{r} \cdot \underline{u} = a\underline{u} \cdot \underline{u} + b\underline{v} \cdot \underline{u} + c\underline{w} \cdot \underline{u}$$

$$\underline{r} \cdot \underline{u} = a\|\underline{u}\|^2 + b(0) + c(0) \quad (\because \underline{v} \perp \underline{u} \text{ and } \underline{w} \perp \underline{u})$$

$$a = \frac{\underline{r} \cdot \underline{u}}{\|\underline{u}\|^2}$$

Similarly, taking dot product with $\underline{v}$ and $\underline{w}$ on both sides of

$\underline{r} = a\underline{u} + b\underline{v} + c\underline{w} \Rightarrow b = \dfrac{\underline{r} \cdot \underline{v}}{\|\underline{v}\|^2}$, $c = \dfrac{\underline{r} \cdot \underline{w}}{\|\underline{w}\|^2}$, result follows

(c) The result in (b) applies to any arbitrary vector $\underline{r} = x\hat{\underline{i}} + y\hat{\underline{j}} + z\hat{\underline{k}}$, including the vector $\hat{\underline{i}}$.

4 marks

$$\Rightarrow \hat{\underline{i}} = \frac{\hat{\underline{i}} \cdot \underline{u}}{\|\underline{u}\|^2}\underline{u} + \frac{\hat{\underline{i}} \cdot \underline{v}}{\|\underline{v}\|^2}\underline{v} + \frac{\hat{\underline{i}} \cdot \underline{w}}{\|\underline{w}\|^2}\underline{w}$$

$\dfrac{\hat{\underline{i}} \cdot \underline{u}}{\|\underline{u}\|^2} = \dfrac{2}{9}$, $\dfrac{\hat{\underline{i}} \cdot \underline{v}}{\|\underline{v}\|^2} = \dfrac{1}{9}$ and $\underline{w} = \underline{u} \times \underline{v} = 6\hat{\underline{i}} - 6\hat{\underline{j}} + 3\hat{\underline{k}}$

$$\frac{\hat{\underline{i}} \cdot \underline{w}}{\|\underline{w}\|^2} = \frac{6}{81} = \frac{2}{27}$$

$$\therefore \hat{\underline{i}} = \frac{2}{9}\underline{u} + \frac{1}{9}\underline{v} + \frac{2}{27}\underline{w}$$

Question 4

④ (a) Distance from $(1,2,0)$ to $3x-4y-5z=2$ is

$$\frac{|3-8-0-2|}{\sqrt{3^2+4^2+5^2}}=\frac{7}{5\sqrt{2}}=\frac{7\sqrt{2}}{10}\text{ units}$$

(b) The line $\begin{cases} x+2y=3 \\ y+2z=3 \end{cases}$ contains the points $(1,1,1)$ and $(3,0,\frac{3}{2})$, so is parallel

to the vector $2\hat{\underline{i}}-\hat{\underline{j}}+\frac{1}{2}\hat{\underline{k}}$ (i.e. $4\hat{\underline{i}}-2\hat{\underline{j}}+\hat{\underline{k}}$)

The line $\begin{cases} x+y+z=6 \\ x-2z=-5 \end{cases}$ contains the points $(-5,11,0)$ and $(-1,5,2)$, so is parallel

to the vector $4\hat{\underline{i}}-6\hat{\underline{j}}+2\hat{\underline{k}}$ (i.e. $2\hat{\underline{i}}-3\hat{\underline{j}}+\hat{\underline{k}}$)

using the values; $\underline{r}_1=\hat{\underline{i}}+\hat{\underline{j}}+\hat{\underline{k}}$, $\underline{a}_1=4\hat{\underline{i}}-2\hat{\underline{j}}+\hat{\underline{k}}$

$\underline{r}_2=-\hat{\underline{i}}+5\hat{\underline{j}}+2\hat{\underline{k}}$, $\underline{a}_2=2\hat{\underline{i}}-3\hat{\underline{j}}+\hat{\underline{k}}$

Distance between 2 lines $=\dfrac{|(\underline{r}_1-\underline{r}_2)\cdot(\underline{a}_1\times\underline{a}_2)|}{\|\underline{a}_1\times\underline{a}_2\|}=\dfrac{|(2\hat{\underline{i}}-4\hat{\underline{j}}-\hat{\underline{k}})\cdot(\hat{\underline{i}}-2\hat{\underline{j}}-8\hat{\underline{k}})|}{\|\hat{\underline{i}}-2\hat{\underline{j}}-8\hat{\underline{k}}\|}$

$$=\frac{18\sqrt{69}}{69}=\frac{6}{23}\sqrt{69}$$

$\boxed{\text{Total: (10 marks)}}$

Question 5

(5) (a) The line $x-2=\frac{y+3}{2}=\frac{z-1}{4}$ passes through the point $(2,-3,1)$, and is parallel to $\underline{a}=\hat{\underline{i}}+2\hat{\underline{j}}+4\hat{\underline{k}}$.

The plane $2y-z=1$ has normal $\underline{n}=2\hat{\underline{j}}-\hat{\underline{k}}$

$\because \underline{a}\cdot\underline{n}=4-4=0 \Rightarrow$ the line is parallel to the plane

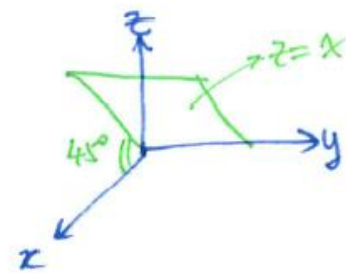
(b) The required distance = distance from $(2,-3,1)$ to the plane $2y-z=1$

Distance $=\frac{|-6-1-1|}{\sqrt{2^2+1^2}}=\underline{\underline{\frac{8\sqrt{5}}{5}}}$

$\boxed{\text{Total: } (\underline{8} \text{ marks})}$

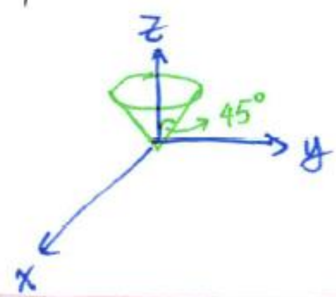
Question 6

(6) (a) $z=x$ is a plane containing the y-axis and making 45° angles with the positive directions of the x and z-axes.



(6) (b) $z \geq \sqrt{x^2+y^2}$ represents every point whose distance above the xy-plane is NOT less than its horizontal distance from the z-axis.

It therefore consists of all points inside and on a circular cone with axis along the positive z-axis, vertex at the origin, and semi-vertical angle 45°.



(c) $x^2+y^2+z^2=4$, $x+y+z=3$ together represent the circle in which the sphere of radius 2 centered at origin intersects the plane through $(1,1,1)$ with normal $\hat{\underline{i}}+\hat{\underline{j}}+\hat{\underline{k}}$.

Since this plane lies at distance $\sqrt{3}$ from the origin, the circle has radius $\sqrt{4-3}=1$.

$\boxed{\text{Total: (10 marks)}}$

Question 7

(7) $z=\frac{y^2}{b^2}-\frac{x^2}{a^2}$, $z=bx+ay$

$$bx+ay=\frac{y^2}{b^2}-\frac{x^2}{a^2}$$

$$\frac{1}{a^2}\left(x^2+a^2bx+\frac{a^4b^2}{4}\right)=\frac{1}{b^2}\left(y^2-ab^2y+\frac{a^2b^4}{4}\right)$$

$$\frac{\left(x+\frac{a^2b}{2}\right)^2}{a^2}=\frac{\left(y-\frac{ab^2}{2}\right)^2}{b^2}\Rightarrow y=\pm\frac{b}{a}\left(x+\frac{a^2b}{2}\right)+\frac{ab^2}{2}$$

In particular
We let $x=at$, we obtain the two intersecting lines:

① $x=at$, $y=-bt$, $z=0$

② $x=at$, $y=bt+ab^2$, $z=2abt+a^2b^2$

(Just for reference)

$\boxed{\text{Total: (4 marks)}}$

Question 8

(8) (a) $\because \underline{u} \times (\underline{v} \times \underline{w})$ is perpendicular to $\underline{v} \times \underline{w}$, it must be in the plane of $\underline{v}$ and $\underline{w}$.

(b) We let the x-axis lie in the direction of $\underline{v}$, let the y-axis be such that $\underline{w}$ lies in the xy-plane, i.e. $\underline{v} = v_1 \hat{\underline{i}}$, $\underline{w} = w_1 \hat{\underline{i}} + w_2 \hat{\underline{j}}$

Then $\underline{v} \times \underline{w} = v_1 w_2 \hat{\underline{i}} \times \hat{\underline{j}} = v_1 w_2 \hat{\underline{k}}$

i.e.
$$\begin{aligned} \underline{u} \times (\underline{v} \times \underline{w}) &= (u_1 \hat{\underline{i}} + u_2 \hat{\underline{j}} + u_3 \hat{\underline{k}}) \times (v_1 w_2 \hat{\underline{k}}) \\ &= u_1 v_1 w_2 \hat{\underline{i}} \times \hat{\underline{k}} + u_2 v_1 w_2 \hat{\underline{j}} \times \hat{\underline{k}} \\ &= u_2 v_1 w_2 \hat{\underline{i}} - u_1 v_1 w_2 \hat{\underline{j}}. \end{aligned}$$

$$\begin{aligned} RHS = (\underline{u} \cdot \underline{w}) \underline{v} - (\underline{u} \cdot \underline{v}) \underline{w} &= (u_1 w_1 + u_2 w_2) v_1 \hat{\underline{i}} - u_1 v_1 (w_1 \hat{\underline{i}} + w_2 \hat{\underline{j}}) \\ &= u_2 v_1 w_2 \hat{\underline{i}} - u_1 v_1 w_2 \hat{\underline{j}} \end{aligned}$$

$$\therefore \underline{u} \times (\underline{v} \times \underline{w}) = (\underline{u} \cdot \underline{w}) \underline{v} - (\underline{u} \cdot \underline{v}) \underline{w}$$

(c) From (b), $(\underline{u} \times \underline{v}) \times (\underline{w} \times \underline{x}) = [(\underline{u} \times \underline{v}) \cdot \underline{x}] \underline{w} - [(\underline{u} \times \underline{v}) \cdot \underline{w}] \underline{x}$

(swap $\underline{u}$ as $\underline{u} \times \underline{v}$, $\underline{v}$ as $\underline{w}$, $\underline{w}$ as $\underline{x}$)

Similarly, we write
$$\begin{aligned} (\underline{u} \times \underline{v}) \times (\underline{w} \times \underline{x}) &= -(\underline{w} \times \underline{x}) \times (\underline{u} \times \underline{v}) \\ &= -[(\underline{w} \times \underline{x}) \cdot \underline{v}] \underline{u} \\ &\quad + [(\underline{w} \times \underline{x}) \cdot \underline{u}] \underline{v} \end{aligned}$$

(d) Let $\underline{w} = \underline{u}$, then
$$(\underline{u} \times \underline{v}) \times (\underline{u} \times \underline{x}) = [(\underline{u} \times \underline{v}) \cdot \underline{x}] \underline{u} - [(\underline{u} \times \underline{v}) \cdot \underline{u}] \underline{x}$$

But $(\underline{u} \times \underline{v}) \cdot \underline{u} = 0 \Rightarrow (\underline{u} \times \underline{v}) \times (\underline{u} \times \underline{x}) = [(\underline{u} \times \underline{v}) \cdot \underline{x}] \underline{u}$

Replacing $\underline{x}$ by $\underline{w}$, $\quad (\underline{u} \times \underline{v}) \times (\underline{u} \times \underline{w}) = [(\underline{u} \times \underline{v}) \cdot \underline{w}] \underline{u}$

$\boxed{\text{Total: (15 marks)}}$

Note: Hard Expansion is allowed!
(Students can compare LHS with RHS)

Question 9

(9)(a) First, we try to express A_n and B_n in terms of A_{n-1} and B_{n-1}.

(10marks) $$A_n = \int_0^{\pi/2} \cos^{2n-1}x \, d(\sin x) = \left[\sin x \cdot \cos^{2n-1}x\right]_0^{\pi/2} - \int_0^{\pi/2} \sin x \cdot (2n-1)\cos^{2n-2}x \cdot (-\sin x)\,dx$$

$$= \underset{(\because \cos\frac{\pi}{2} = \sin 0 = 0)}{0} + \int_0^{\pi/2} (2n-1)\cos^{2n-2}x \sin^2 x \, dx$$

$$= (2n-1)\int_0^{\pi/2} \cos^{2n-2}x\,dx - (2n-1)\int_0^{\pi/2}\cos^{2n}x\,dx$$

$$= (2n-1)A_{n-1} - (2n-1)A_n$$

$$\Rightarrow \boxed{A_n = \frac{2n-1}{2n}A_{n-1} \quad \forall n \geq 1} \quad (*)$$

$$B_n = \int_0^{\pi/2} x^2 \cos^{2n-1}x \, d(\sin x) = \left[x \sin x \cos^{2n-1}x\right]_0^{\pi/2}$$

$$- \int_0^{\pi/2} \sin x \cdot \underbrace{\left(2x\cos^{2n-1}x + (2n-1)x^2\cos^{2n-2}x \cdot (-\sin x)\right)}_{\frac{d}{dx}(x^2\cos^{2n-1}x)} dx$$

$$= 0 - 2\int_0^{\pi/2} x \sin x \cos^{2n-1}x\,dx$$

$$+ (2n-1)\int_0^{\pi/2} x^2\cos^{2n-2}x \cdot \sin^2 x\,dx$$

$$= -2\underbrace{\int_0^{\pi/2} x\sin x \cos^{2n-1}x\,dx}_{:= I_n} + \underbrace{(2n-1)\int_0^{\pi/2} x^2\cos^{2n-2}x\,(1-\cos^2 x)\,dx}_{(2n-1)(B_{n-1}-B_n)}$$

To find I_n, $$I_n = \int_0^{\frac{\pi}{2}} x\sin x\cos^{2n-1}x\,dx = -\int_0^{\pi/2} x\cos^{2n-1}x \, d(\cos x)$$

$$= -\left[x\cos^{2n}x\right]_0^{\frac{\pi}{2}} + \int_0^{\pi/2}\cos x \cdot \left(\cos^{2n-1}x + x\cdot(2n-1)\cos^{2n-2}x\cdot(-\sin x)\right)dx$$

$$= 0 + \int_0^{\frac{\pi}{2}}\cos^{2n}x\,dx - (2n-1)\int_0^{\pi/2} x\sin x\cdot\cos^{2n-1}x\,dx$$

$$= A_n - (2n-1)I_n \Rightarrow I_n = \frac{A_n}{2n}$$

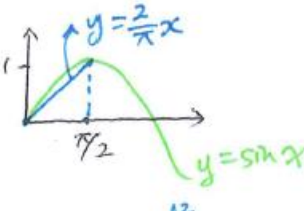
$$\Rightarrow B_n = \frac{-1}{n}A_n + (2n-1)(B_{n-1} - B_n)$$

$$\boxed{B_n = \frac{2n-1}{2n}B_{n-1} - \frac{1}{2n^2}A_n \quad \forall n \geq 1} \quad (**)$$

$$(**): \frac{B_n}{A_n} = \frac{2n-1}{2n}\frac{B_{n-1}}{A_n} - \frac{1}{2n^2} = \frac{2n-1}{2n}\frac{B_{n-1}}{\underbrace{\frac{2n-1}{2n}A_{n-1}}_{\text{from } (*)}} - \frac{1}{2n^2} = \frac{B_{n-1}}{A_{n-1}} - \frac{1}{2n^2}$$

$$\Rightarrow 2\frac{B_{n-1}}{A_{n-1}} - 2\frac{B_n}{A_n} = \frac{1}{n^2}$$

⑨ (b) By concavity of $\sin x$ on $x \in [0, \frac{\pi}{2}]$



$\frac{d^2}{dx^2}\sin x = -\sin x \le 0$ on $x \in [0, \frac{\pi}{2}]$

(8 marks) $\frac{2}{\pi}x \le \sin x$ on $x \in [0, \frac{\pi}{2}]$

i.e. $x \le \frac{\pi}{2}\sin x$ on $x \in [0, \frac{\pi}{2}]$

$$B_n = \int_0^{\frac{\pi}{2}} x^2 \cos^{2n} x\,dx \le \int_0^{\frac{\pi}{2}} \left(\frac{\pi}{2}\sin x\right)^2 \cos^{2n} x\,dx$$

$$= \left(\frac{\pi}{2}\right)^2 \int_0^{\pi/2} (1-\cos^2 x)\cos^{2n} x\,dx$$

$$\le \left(\frac{\pi}{2}\right)^2 (A_n - A_{n+1})$$

$$= \left(\frac{\pi}{2}\right)^2 \left(A_n - \frac{2n+1}{2(n+1)}A_n\right) \longrightarrow \text{from } (*)$$

$$= \left(\frac{\pi}{2}\right)^2 \left(\frac{1}{2(n+1)}A_n\right)$$

We let $\boxed{C = \frac{\pi^2}{8}}$, then $B_n \le \frac{C}{n+1}A_n$ as desired.

(c) From (a), $2\sum_{n=1}^{N}\left(\frac{B_{n-1}}{A_{n-1}} - \frac{B_n}{A_n}\right) = \sum_{n=1}^{N}\frac{1}{n^2}$

(7 marks)

$$2\left(\frac{B_0}{A_0} - \frac{B_N}{A_N}\right) = \sum_{n=1}^{N}\frac{1}{n^2}$$

From (b), $0 \le \frac{B_N}{A_N} \le \frac{C}{N+1}$

as $\cos^{2n} x \ge 0$ $\frac{C}{N+1} \to 0$ as $N \to \infty$

By sandwich Theorem,

$$\lim_{N\to\infty}\frac{B_N}{A_N} = 0$$

$$\therefore \sum_{n=1}^{\infty}\frac{1}{n^2} = \lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{n^2} = \lim_{N\to\infty} 2\left(\frac{B_0}{A_0} - \frac{B_N}{A_N}\right) = \frac{2B_0}{A_0}$$

$$= 2\cdot\frac{\int_0^{\pi/2} x^2\,dx}{\int_0^{\frac{\pi}{2}} dx} = \frac{2\left(\frac{\pi}{2}\right)^3/3}{\pi/2} = \frac{\pi^2}{6}$$

END

Total: (25 marks)