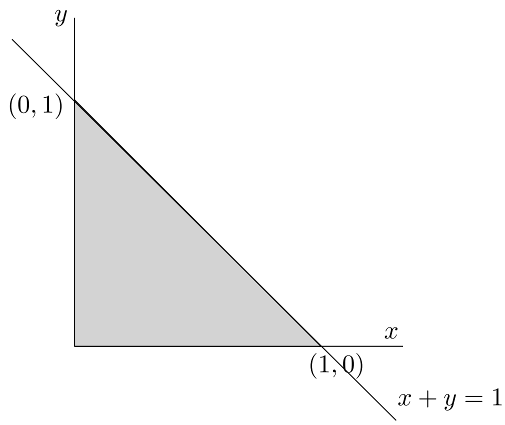
Exercises: Surface Integral by Coordinate

Problem 1. Let S be the upper side of the plane $x + y + z = 1$ with $x \geq 0$ and $y \geq 0$. Calculate $\iint_S z\, dxdy$.

Solution: Let D be the projection of S onto the xy-plane. In other words, D is the shaded triangle as shown below:



Hence:

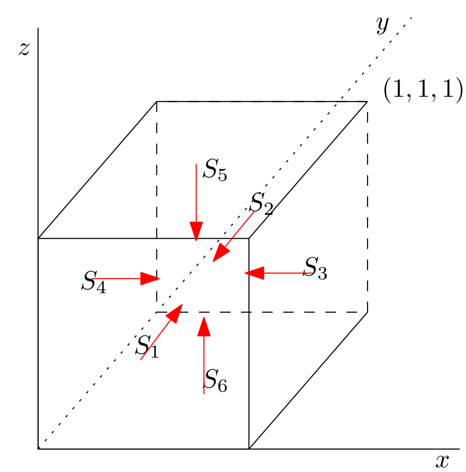
$$\iint_S z\, dxdy \quad = \quad \iint_D 1 - x - y\, dxdy. \tag{1}$$

$\iint_D dxdy$ is simply the area of the triangle, namely, 1/2. Regarding the second term of (1):

$$\begin{aligned}\iint_D x\, dxdy &= \int_0^1 \left(\int_0^{1-x} x\, dy\right) dx \\ &= \int_0^1 x(1-x) dx = 1/6.\end{aligned}$$

By symmetry, we also have $\iint_D y\, dxdy = 1/6$. Therefore, (1) equals $1/2 - 1/6 - 1/6 = 1/6$.

Problem 2. Let S be the inner side of the cube that has the origin and the point $(1, 1, 1)$ as the opposite corners (see below). Calculate $\iint_S (z^2\, dxdy + xy\, dzdx)$.



Solution: We can break S into 6 oriented surfaces $S_1, S_2, ..., S_6$ as shown in the above figure. Each S_i ($1 \le i \le 6$) corresponds to a face of the cube. Hence:

$$\iint_S (z^2\, dxdy + xy\, dzdx) \quad = \quad \sum_{i=1}^{6} \iint_{S_i} (z^2\, dxdy + xy\, dzdx). \tag{2}$$

We have:

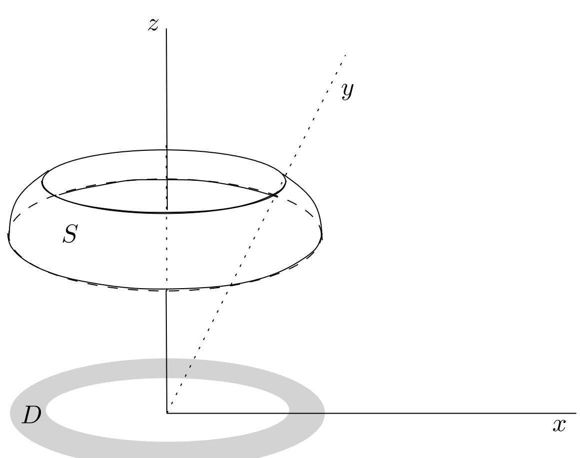
$$\begin{aligned}
\sum_{i=1}^{6} \iint_{S_i} z^2\, dxdy &= \iint_{S_5} z^2\, dxdy + \iint_{S_6} z^2\, dxdy \\
&= \iint_{S_5} 1\, dxdy + \iint_{S_6} 0\, dxdy \\
&= -\int_0^1 \int_0^1 dxdy = -1.
\end{aligned}$$

Also:

$$\begin{aligned}
\sum_{i=1}^{6} \iint_{S_i} xy\, dzdx &= \iint_{S_1} xy\, dzdx + \iint_{S_2} xy\, dzdx \\
&= \iint_{S_1} x \cdot 0\, dzdx + \iint_{S_2} x\, dzdx \\
&= -\int_0^1 \left(\int_0^1 x\, dz \right) dx = -1/2.
\end{aligned}$$

Therefore, (2) equals $-1 - 1/2 = -3/2$.

Problem 3. Let S be the upper side of the surface $x^2 + y^2 + z^2 = 1$ with $\sqrt{2}/2 \le z \le \sqrt{3}/2$. Calculate $\iint_S \frac{1}{z}\, dxdy$.



Solution 1: Let D be the projection of S onto the xy-plane. D is the annulus $1/4 \le x^2 + y^2 \le 1/2$. Hence:

$$\iint_S \frac{1}{z}\, dxdy \quad = \quad \iint_D \frac{1}{z}\, dxdy. \tag{3}$$

Let us represent S in a parametric form $\boldsymbol{r}(u,v) = [x(u,v), y(u,v), z(u,v)]$ where

$$\begin{aligned} x(u,v) &= \cos u \sin v \\ y(u,v) &= \sin u \sin v \\ z(u,v) &= \cos v \end{aligned}$$

where $u \in [0, 2\pi]$ and $v \in [\pi/6, \pi/4]$. The Jacobian J equals:

$$\begin{aligned} J &= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \\ &= -\sin u \cdot \sin v \cdot \sin u \cdot \cos v - \cos u \cdot \cos v \cdot \cos u \cdot \sin v \\ &= -\sin v \cdot \cos v. \end{aligned}$$

Now we can change the variables x, y in (3) to u, v as:

$$\begin{aligned} \iint_D \frac{1}{z}\, dxdy &= \iint_D \frac{1}{z} \cdot |J|\, dudv \\ &= \iint_D \frac{1}{\cos v} \cdot |\sin v \cdot \cos v|\, dudv \\ &= \int_0^{2\pi} \left(\int_{\pi/6}^{\pi/4} \sin v\, dv \right) du \\ &= (\sqrt{3} - \sqrt{2})\pi. \end{aligned}$$

Solution 2: We can also represent S in another parametric form $\boldsymbol{r}(u,v) = [x(u,v), y(u,v), z(u,v)]$ where

$$\begin{aligned} x(u,v) &= u \cos v \\ y(u,v) &= u \sin v \\ z(u,v) &= \sqrt{1-u^2} \end{aligned}$$

where $u \in [1/2, \sqrt{2}/2]$ and $v \in [0, 2\pi]$. The Jacobian J equals:

$$\begin{aligned} J &= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \\ &= \cos v \cdot u \cos v - u(-\sin v) \cdot \sin v \\ &= u. \end{aligned}$$

Now we can change the variables x, y in (3) to u, v as:

$$\begin{aligned} \iint_D \frac{1}{z}\, dxdy &= \iint_D \frac{1}{z} \cdot |J|\, dudv \\ &= \iint_D \frac{1}{\sqrt{1-u^2}} \cdot |u|\, dudv \\ &= \int_0^{2\pi} \left(\int_{1/2}^{\sqrt{1/2}} \frac{u}{\sqrt{1-u^2}}\, du \right) dv \\ &= (\sqrt{3} - \sqrt{2})\pi. \end{aligned}$$