Subgame Perfect Equilibrium

We recall the definition of subgame perfect equilibrium for extensive games with perfect information.

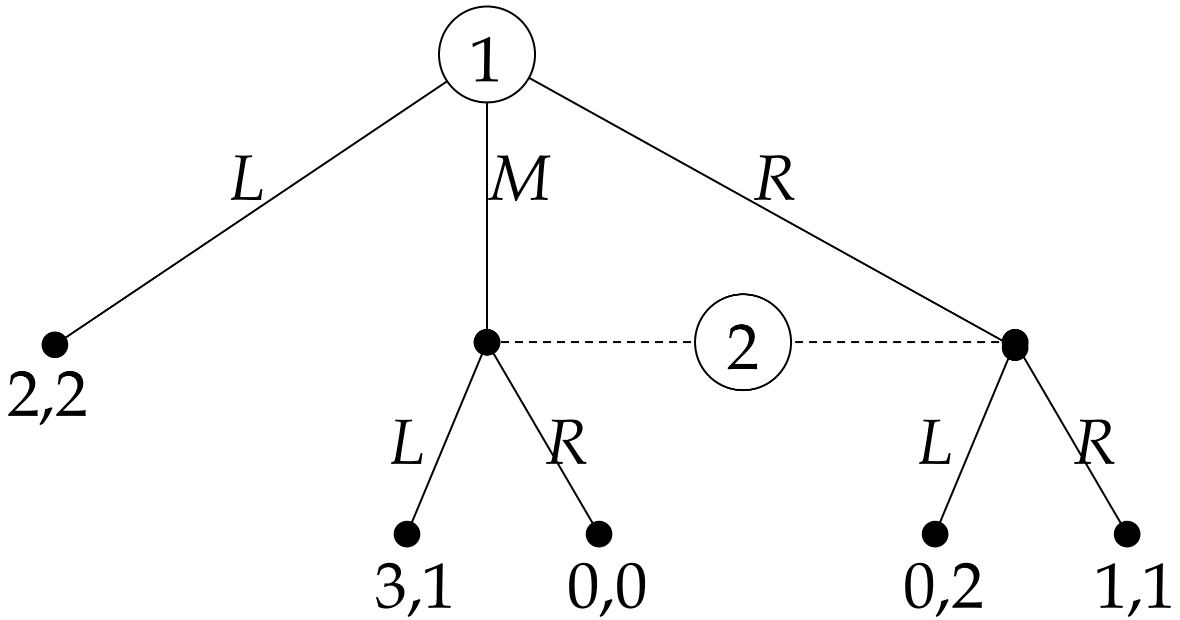
DEFINITION. *The* ***subgame perfect equilibrium of an extensive game with perfect information*** $\Gamma = \langle N, H, P, (\succsim_i) \rangle$ *is the strategy profile* s^* *such that for every player* $i \in N$ *and* *<u>every</u>* *nonterminal history* $h \in H \backslash Z$ *for which* $P(h) = i$ *we have* $O_h(s^*_{-i}|_h, s^*_i|_h) \succsim_i|_h O_h(s^*_{-i}|_h, s_i)$ *for every strategy* s_i *of player* i *in the subgame* $\Gamma(h)$.

How does this concept of subgame perfect equilibrium of extensive games with perfect information extend to extensive games *with imperfect information*?

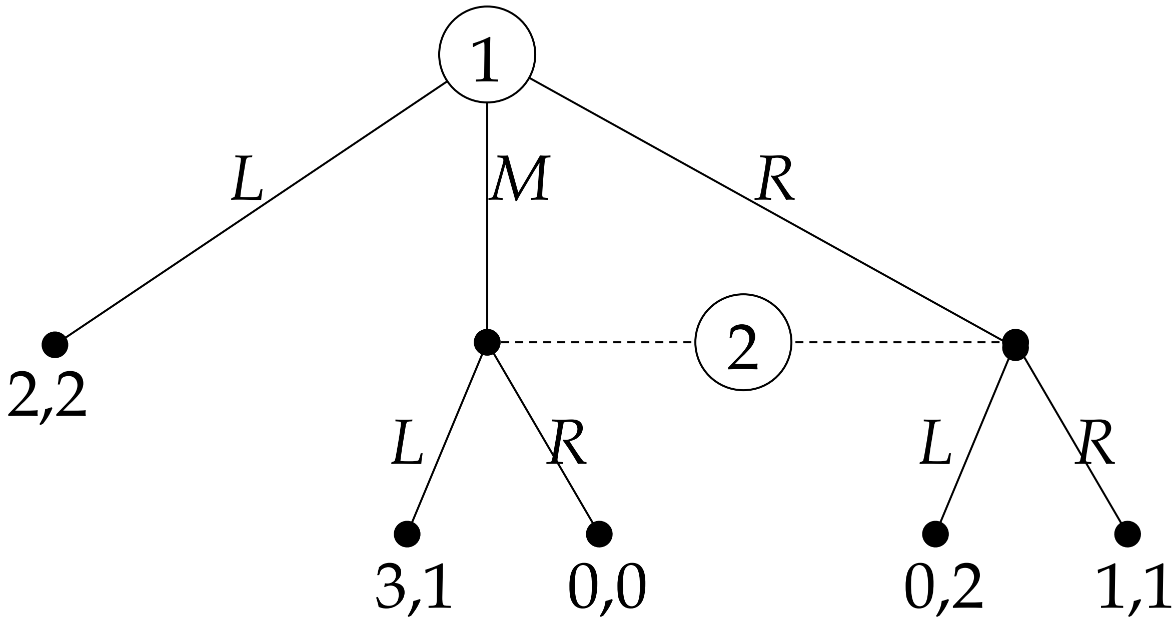
A suggestion…

In the definition, we require that each player's strategy be optimal at every nonterminal history.

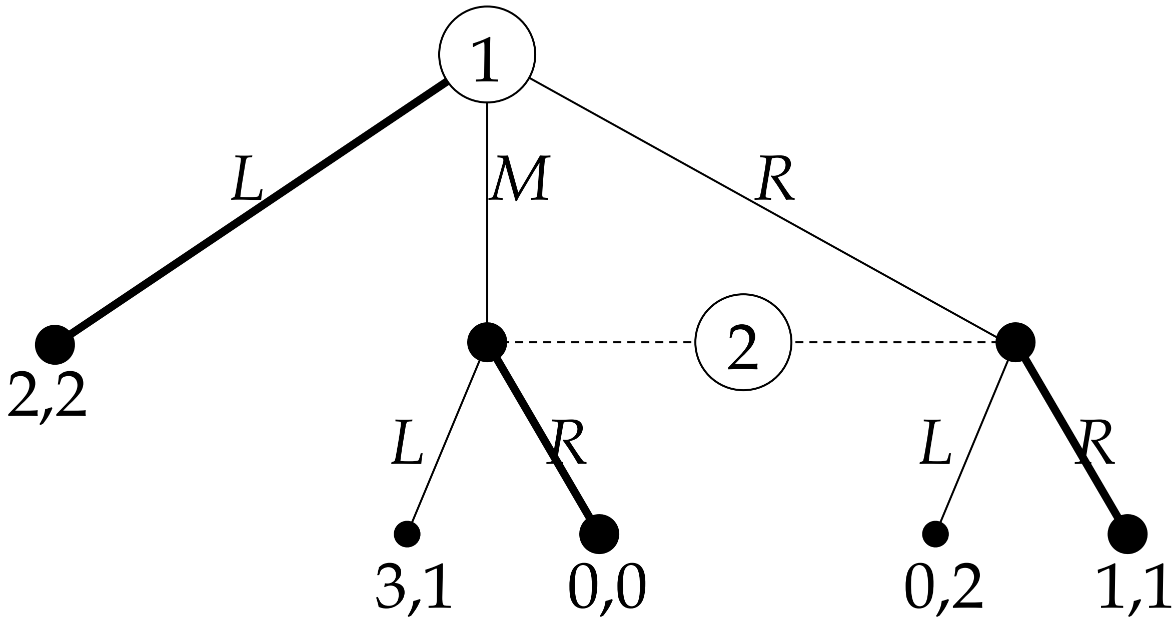
So, for extensive games *with imperfect information*, shall we require that each player's strategy be optimal at each of his information sets?



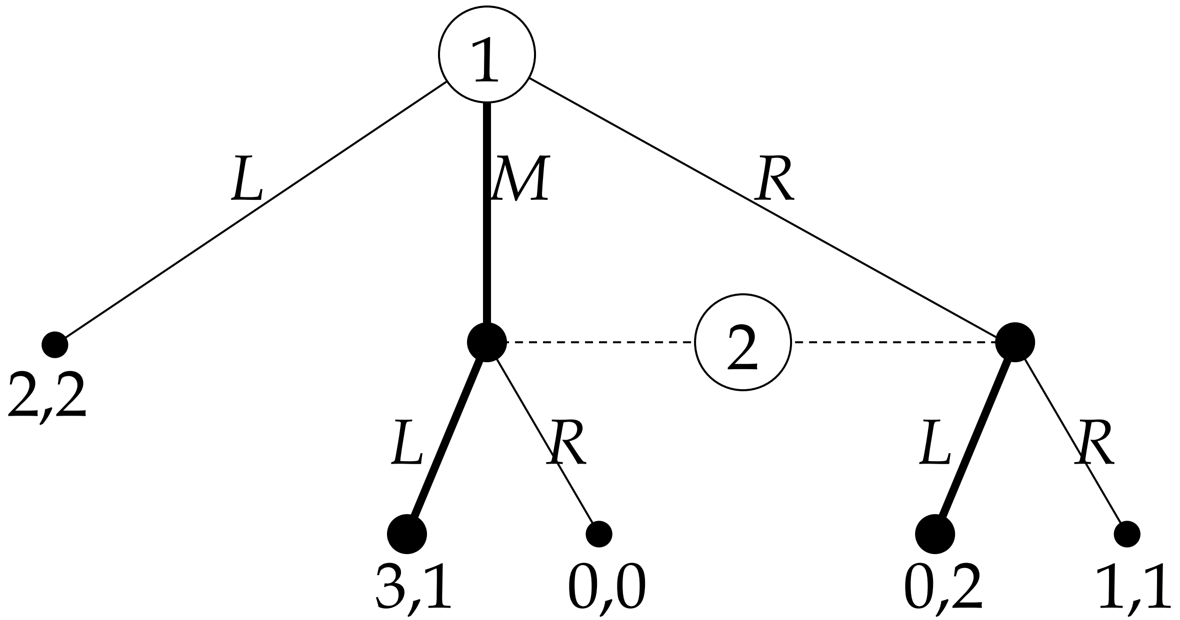
- A Nash equilibrium is (L, R).
- So L must be the best response to R, and R must be the best response to L.



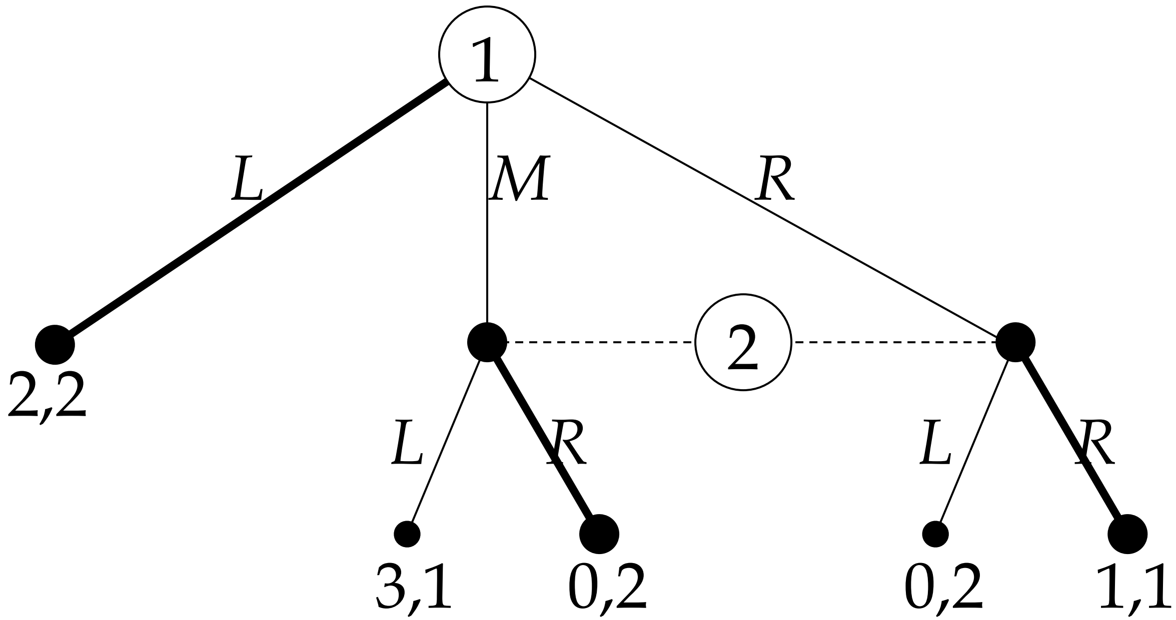
- *Really*? So L must be the best response to R, and R must be the best response to L.
- But why is R a good strategy at all!



The Nash equilibrium (L, R) ***<u>is not</u>*** **'subgame perfect.'**



This Nash equilibrium (M, L) <u>*is*</u> 'subgame perfect.'



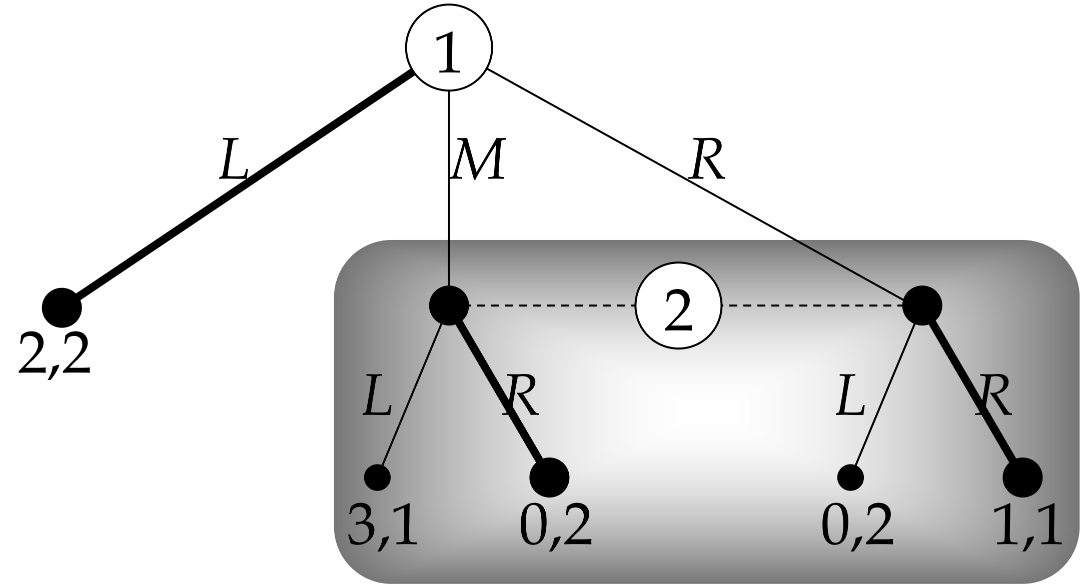
A more common situation:

The Nash equilibrium (L, R) is 'subgame perfect'

***if and only if* it is more probable that player 1 plays**

M than R.

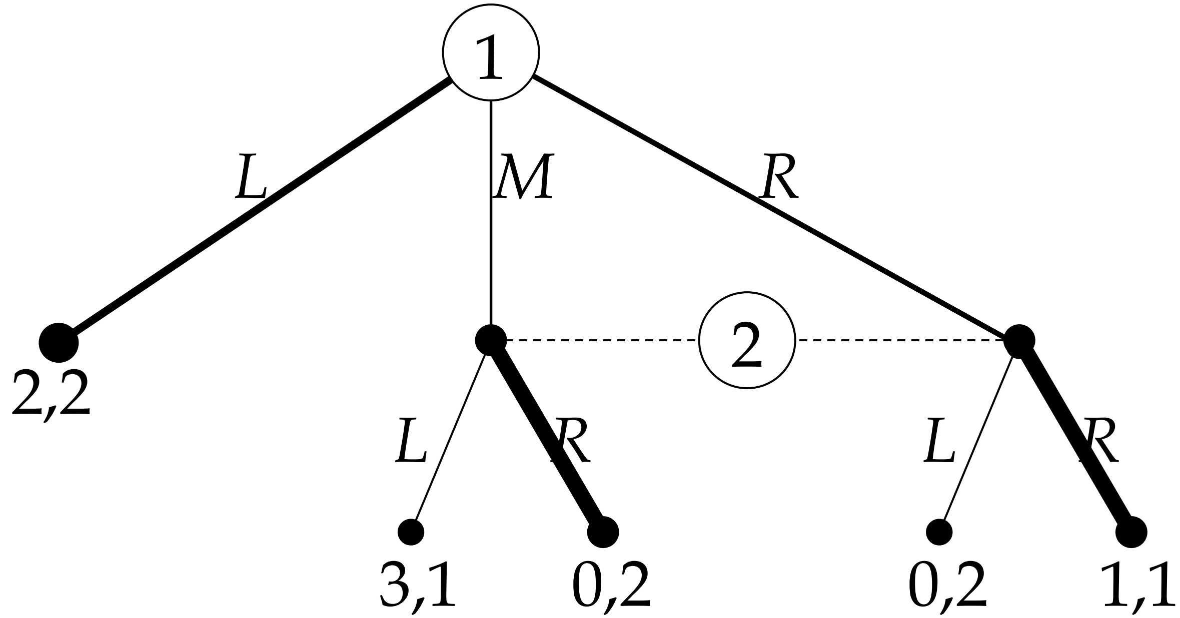
Sequential Equilibrium



A **sequential equilibrium** consists of a <u>strategy profile</u> and a <u>belief system</u> (*e.g.*, the probability of history M is at least ½, and that of R is at most ½).

In the following, we restrict attention to games with perfect recall, in which every information set contains a finite number of histories.

Assessments



Strategy Profile β*:*

$((L(\frac{1}{2}), M(\frac{1}{8}), R(\frac{3}{8})),$

$(L(0), R(1)))$.

Belief system μ*:*

$\{\, \{\emptyset\} \mapsto \emptyset(1),$

$\{M, R\} \mapsto (M(\frac{1}{3}), R(\frac{2}{3}))\ \}$

This pair (β, μ) is an example of **assessments**.

Assessments

An **assessment** consists of

(i) a profile of *behavioural strategies* and

(ii) a belief system consisting of a collection of probability measures, one for each information set.

Notations:

If

$$\mu = \{\{\emptyset\} \mapsto \emptyset(1), \{M, R\} \mapsto (M(\frac{1}{3}), R(\frac{2}{3}))\}$$

then

$$\mu(\{\emptyset\})(\emptyset) = 1,$$

$$\mu(\{M, R\})(M) = \frac{1}{3},\ \mu(\{M, R\})(R) = \frac{2}{3}.$$

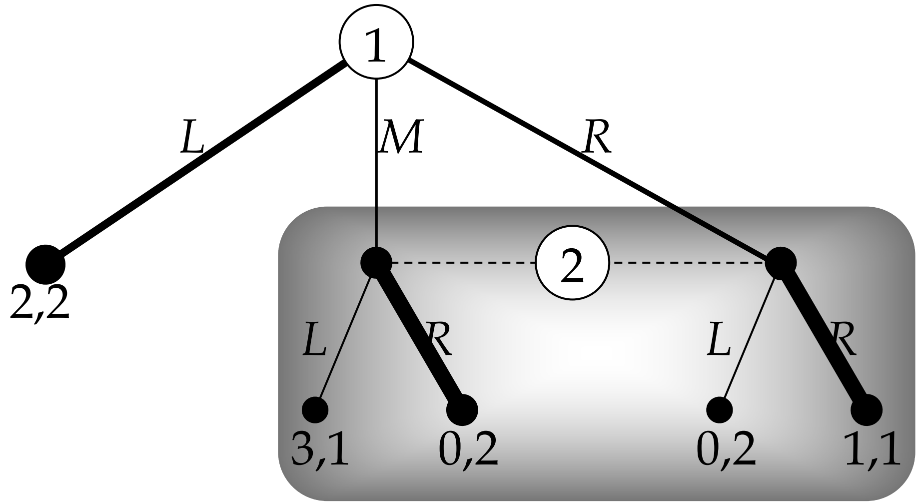
Assessments

DEFINITION. An **assessment** in a extensive game is a pair (β, μ), where β is a profile of behavioural strategies and μ is a function that assigns to every information set a probability measure on the set of histories in the information set.

Class Discussion

What is an ‘assessment’ if the game is an extensive game with perfect information?

Class Discussion



Strategy Profile β:

$\beta = ((L(\frac{1}{2}), M(\frac{1}{8}), R(\frac{3}{8})),$

$(L(0), R(1)))$.

Belief system μ:

$\mu = \{\{\emptyset\} \mapsto \emptyset(1), \{M, R\} \mapsto (M(\frac{1}{3}), R(\frac{2}{3}))\}$

Q: What is the **outcome** of this assessment if the game reaches information set $\{M, R\}$?

Strategy Profile β*:*

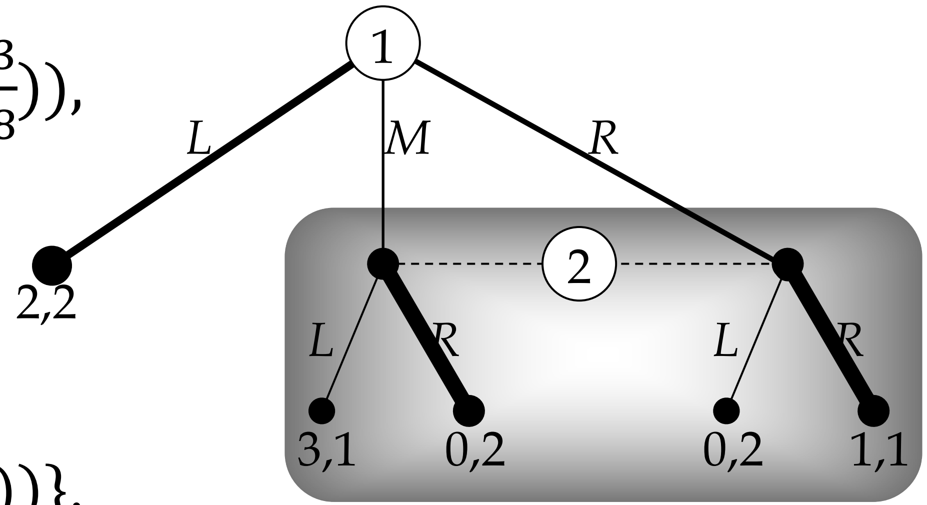
$\beta = ((L(\frac{1}{2}), M(\frac{1}{8}), R(\frac{3}{8})),$

$(L(0), R(1))).$

Belief system μ*:*

$\mu = \{\{\emptyset\} \mapsto \emptyset(1),$

$\{M, R\} \mapsto (M(\frac{1}{3}), R(\frac{2}{3}))\}.$



Formally, $O(\beta, \mu | \{M, R\})$ is the distribution $\{L \mapsto 0, (M, L) \mapsto 0, (M, R) \mapsto \frac{1}{3}, (R, L) \mapsto 0, (R, R) \mapsto \frac{2}{3}\}$.

Strategy Profile β:

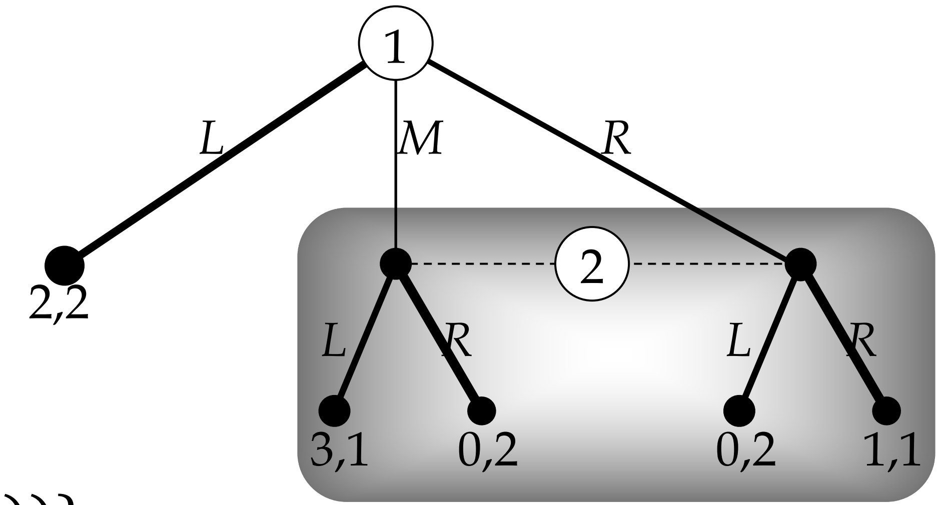
$$((L(\tfrac{1}{2}), M(\tfrac{1}{8}), R(\tfrac{3}{8})),$$

$$(L(\tfrac{2}{5}), R(\tfrac{3}{5}))).$$

Belief system μ:

$$\{\{\emptyset\} \mapsto \emptyset(1),$$

$$\{M, R\} \mapsto (M(\tfrac{1}{3}), R(\tfrac{2}{3}))\}.$$

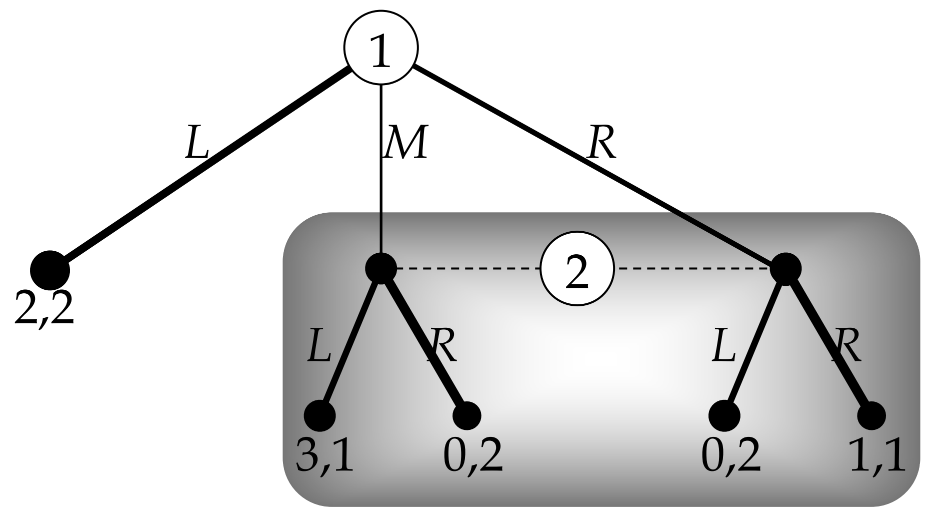


Q: What if player 2 uses a behavioural strategy $(L(\frac{2}{5}), R(\frac{3}{5}))$ and the game reaches $\{M, R\}$?

Outcomes of Assessments

The **outcome** $O(\beta, \mu|I)$ **of** (β, μ) **conditional on** I is a distribution over terminal histories determined by β and μ conditional on I being reached.

$$O(\beta, \mu|\{M, R\}) = \{L \mapsto 0, (M, L) \mapsto 0, (M, R) \mapsto \frac{1}{3},$$
$$(R, L) \mapsto 0, (R, R) \mapsto \frac{2}{3}\}$$



Outcomes of Assessments

The **outcome** $O(\beta, \mu | I)$ **of** (β, μ) **conditional on** I is the distribution over terminal histories determined by β and μ conditional on I being reached, is defined as follows.

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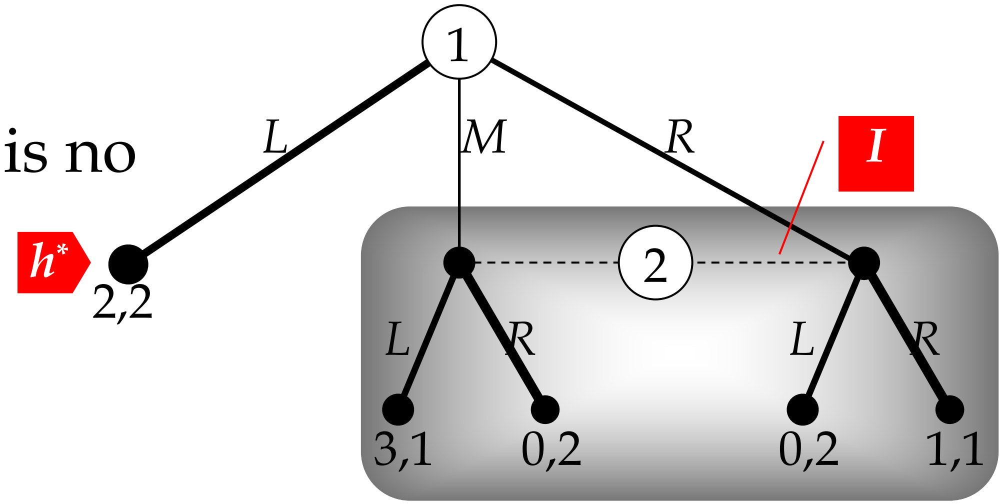
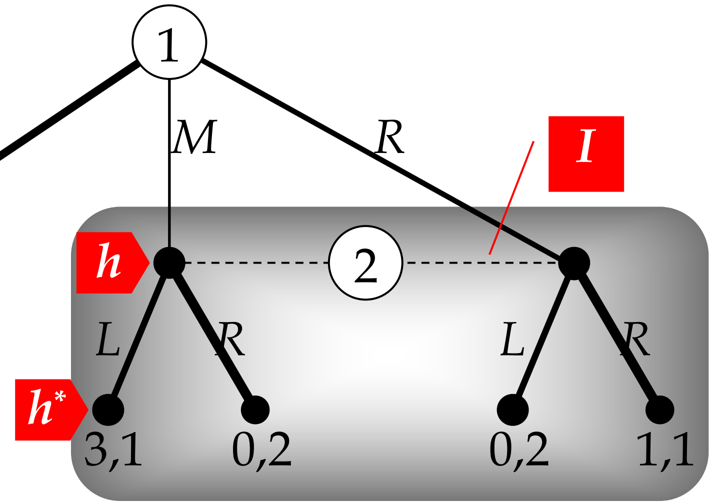
- $O(\beta, \mu|I)(h^*) = 0$ if there is no subhistory of h^* in I. (I is reached, so h^* is impossible.)



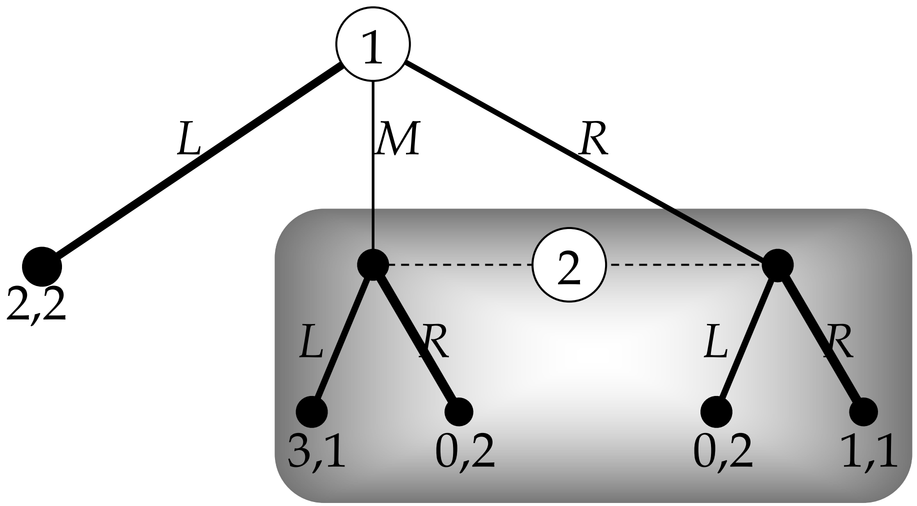
- $O(\beta, \mu|I)(h^*)$
$= \mu(I)(h) \cdot \prod_{k=L}^{K-1} \beta_{P(a^1,\ldots,a^k)}(a^1, \ldots, a^k)(a^{k+1})$
if $h^* = (a^1, \ldots, a^K)$, $h = (a^1, \ldots, a^L) \in I$, $L < K$.



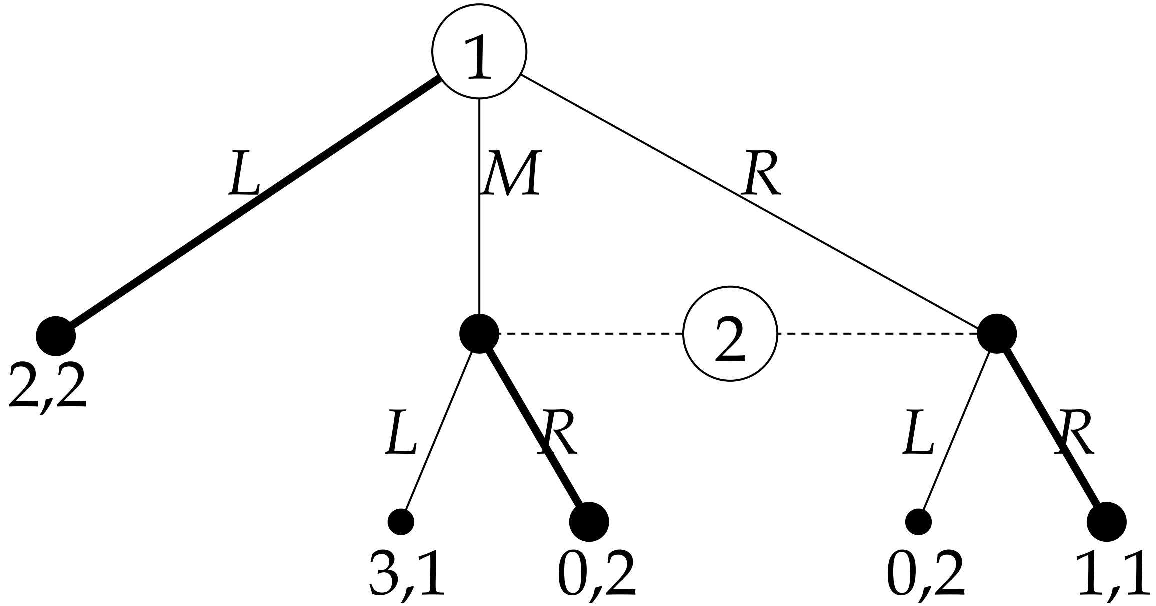
Class Discussion

- $O(\beta,\mu|I)(h^*) = 0$ if there is no subhistory of h^* in I. (I is reached, so h^* is impossible.)
- $O(\beta,\mu|I)(h^*)$
  $= \mu(I)(h) \cdot \prod_{k=L}^{K-1} \beta_{P(a^1,\ldots,a^k)}(a^1,\ldots,a^k)(a^{k+1})$
  if $h^* = (a^1,\ldots,a^K)$, $h = (a^1,\ldots,a^L) \in I$, $L < K$.

Q: What is $O(\beta,\mu|\emptyset)$?



Class Discussion



If $\alpha \geq \frac{1}{2}$, then the assessment $\beta_1 = L$, $\beta_2 = R$, and $\mu(\{M, R\})(M) = \alpha$ is '*sequentially rational,*' an extension of the concept 'subgame-perfect.'

Sequential Rationality

An assessment is **sequentially rational** if for every player i and every information set $I_i \in \mathcal{I}_i$ the (behavioural) strategy of player i is a best response to the other players' strategies, given player i's beliefs at that information set I_i.

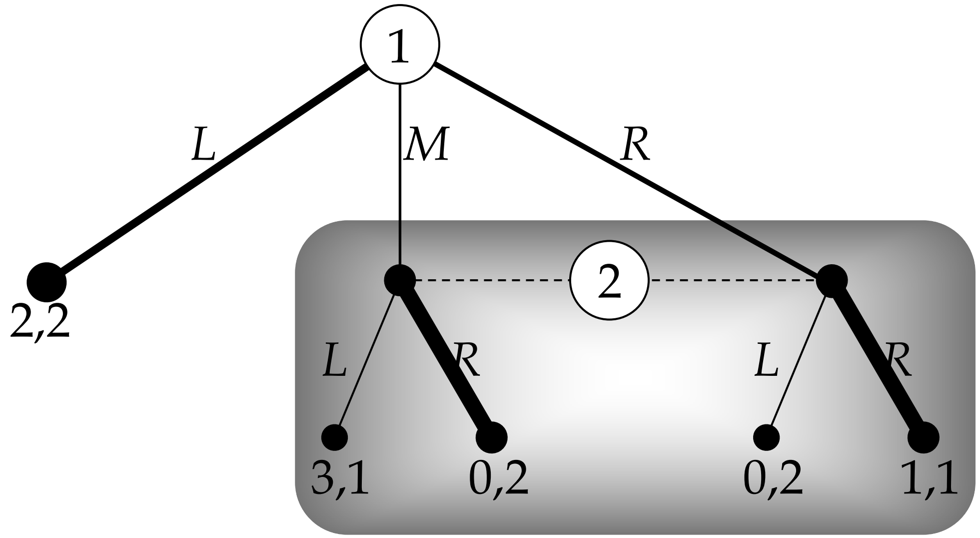
Sequential Rationality of Assessments

DEFINITION. Let $\langle N, H, P, f_c, (\mathcal{I}_i), (\succsim_i)\rangle$ be an extensive game with perfect recall. The assessment (β, μ) is **sequentially rational** if for every player $i \in N$ and every information set $I_i \in \mathcal{I}_i$, we have

$$O(\beta, \mu | I_i) \succsim_i O((\beta_{-i}, \beta_i'), \mu | I_i)$$

for every behavioural strategy β_i' of player i.

Class Discussion



Strategy Profile β*:*

$((L(\frac{1}{2}), M(\frac{1}{8}), R(\frac{3}{8})),$

$(L(0), R(1)))$.

Belief system μ*:*

$\{\{\emptyset\} \mapsto \emptyset(1),$

$\{M, R\} \mapsto (M(\frac{1}{3}), R(\frac{2}{3}))\}$

Is this assessment (β, μ) **consistent**? Why?

Class Discussion

Strategy Profile β*:*

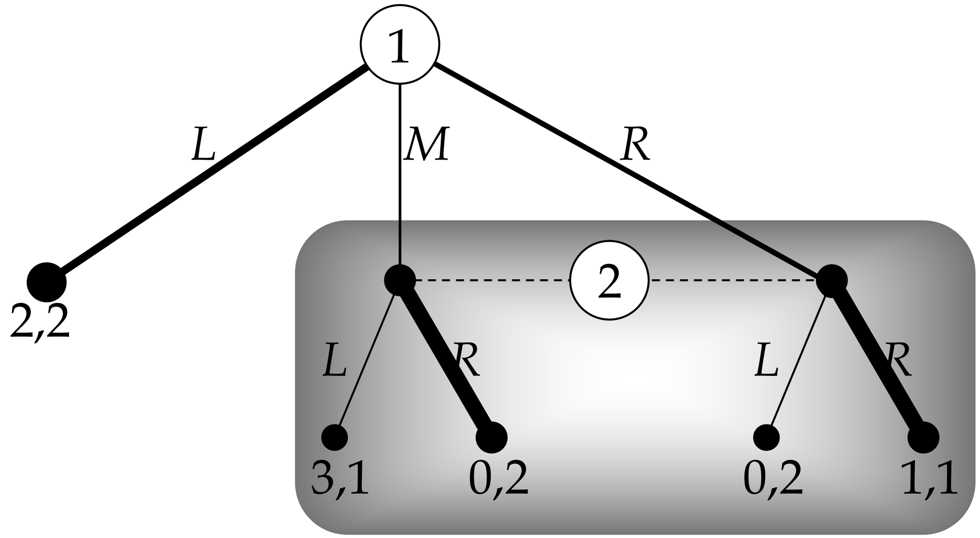
$((L(\frac{1}{2}), M(\frac{1}{6}), R(\frac{2}{6})),$

$(L(0), R(1))).$

Belief system μ*:*

$\{\{\emptyset\} \mapsto \emptyset(1),$

$\{M, R\} \mapsto (M(\frac{1}{3}), R(\frac{2}{3}))\}$



Is this assessment (β, μ) **consistent**? Why?

Class Discussion

Strategy Profile β*:*

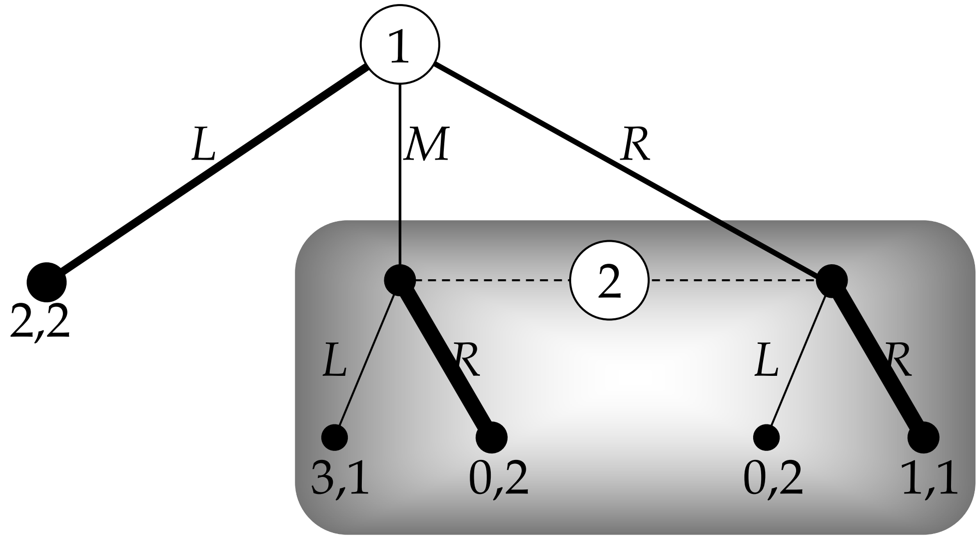
$((L(\frac{9}{10}), M(\frac{1}{30}), R(\frac{2}{30})),$

$(L(0), R(1))).$

Belief system μ*:*

$\{\{\emptyset\} \mapsto \emptyset(1),$

$\{M, R\} \mapsto (M(\frac{1}{3}), R(\frac{2}{3}))\}$



Is this assessment (β, μ) **consistent**? Why?

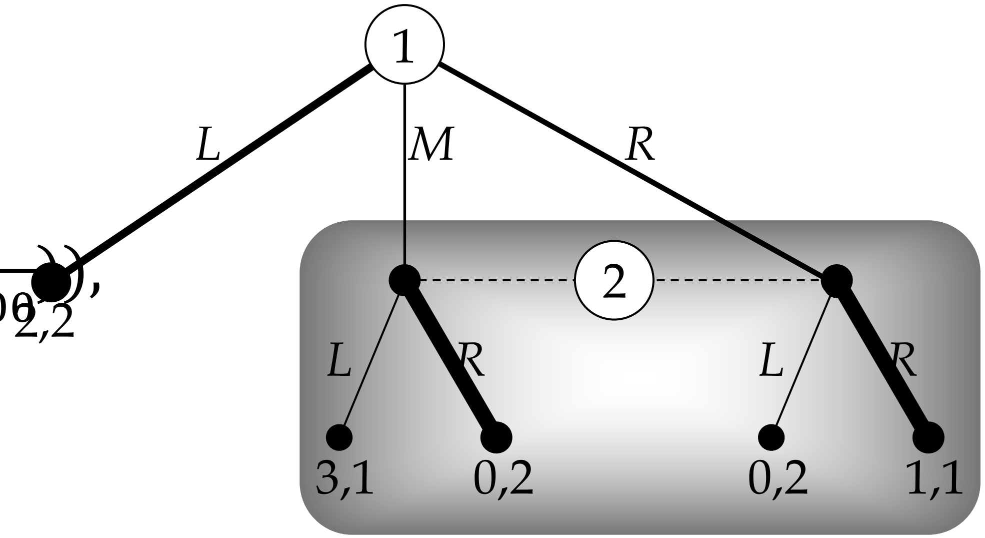
Class Discussion

Strategy Profile β*:*

$$((L(\tfrac{9999}{10000}), M(\tfrac{1}{30000}), R(\tfrac{2}{30000})),$$
$$(L(0), R(1))).$$

Belief system μ*:*

$$\{\{\emptyset\} \mapsto \emptyset(1),$$
$$\{M, R\} \mapsto (M(\tfrac{1}{3}), R(\tfrac{2}{3}))\}$$



Is this assessment (β, μ) **consistent**? Why?

Class Discussion

Strategy Profile β*:*

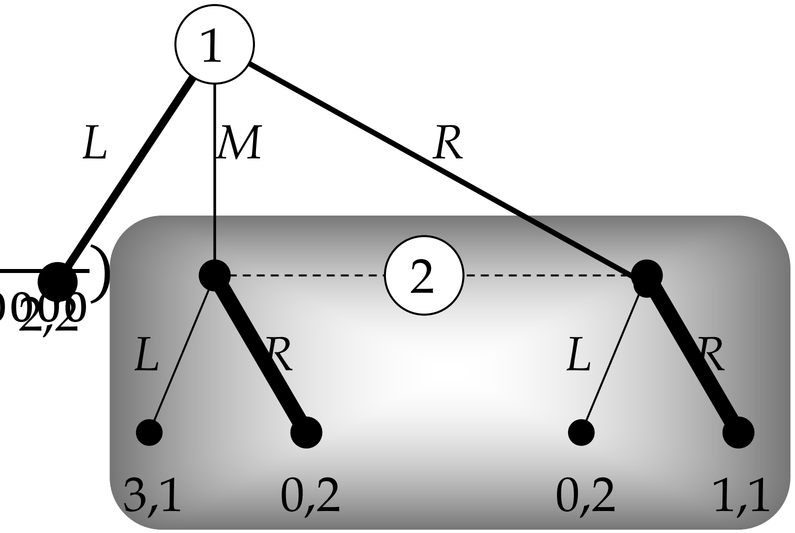
$((L(\frac{9999999}{10000000}), M(\frac{1}{30000000}), R(\frac{2}{30000000})),$

$(L(0), R(1)))$.

Belief system μ*:*

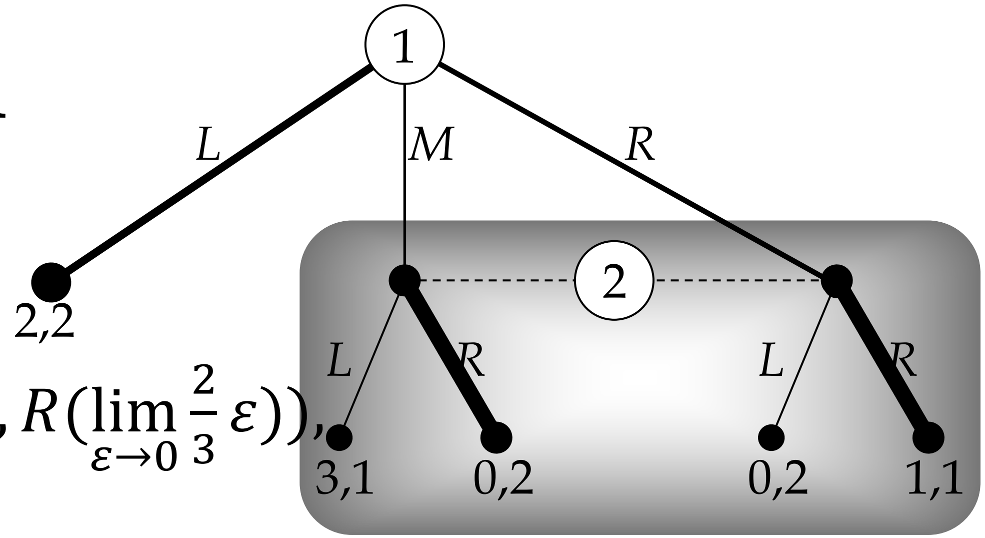
$\{\{\emptyset\} \mapsto \emptyset(1),$

$\{M, R\} \mapsto (M(\frac{1}{3}), R(\frac{2}{3}))\}$



Is this assessment (β, μ) **consistent**? Why?

Class Discussion



Strategy Profile β*:*

$((L(\lim_{\varepsilon \to 0}(1-\varepsilon)), M(\lim_{\varepsilon \to 0}\frac{1}{3}\varepsilon), R(\lim_{\varepsilon \to 0}\frac{2}{3}\varepsilon)),$

$(L(0), R(1)))$.

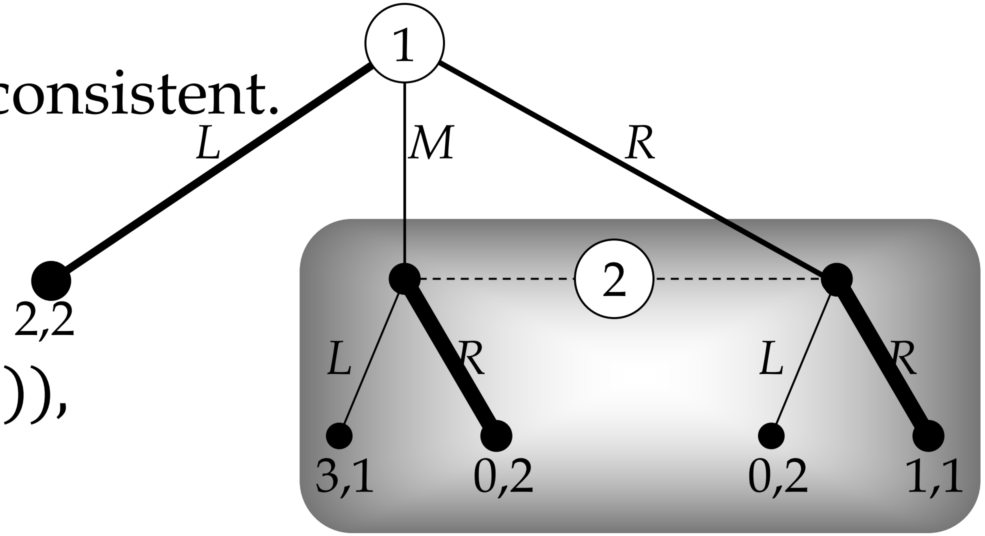
Belief system μ*:*

$\{\{\emptyset\} \mapsto \emptyset(1),$

$\{M, R\} \mapsto (M(\frac{1}{3}), R(\frac{2}{3}))\}$

Is this assessment (β, μ) **consistent**? Why?

The other beliefs are also consistent.



Strategy Profile β:

$\lim_{\varepsilon \to 0}((L(1-\varepsilon), M(\frac{1}{3}\varepsilon), R(\frac{2}{3}\varepsilon)),$

$(L(\varepsilon), R(1-\varepsilon)))$.

Belief system μ:

$\{\{\emptyset\} \mapsto \emptyset(1),$

$\{M, R\} \mapsto (M(\frac{1}{3}), R(\frac{2}{3}))\}$

This assessment (β, μ) is **consistent** as the limit of a sequence of assessments.

Class Discussion

Strategy Profile β:

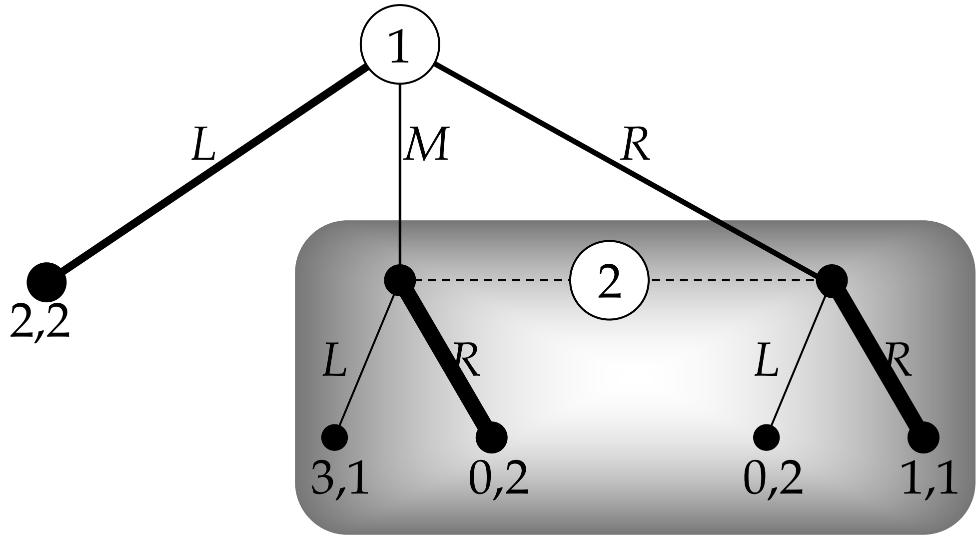
$((L(1), M(0), R(0)),$

$(L(0), R(1)))$.

Belief system μ:

$\{\{\emptyset\} \mapsto \emptyset(1),$

$\{M, R\} \mapsto (M(\frac{1}{3}), R(\frac{2}{3}))\}$



Consistency of Assessments

An assessment (β, μ) is **consistent** if there is a sequence of assessments that converges to (β, μ), and has the properties that ***each*** strategy profile in the sequence is completely mixed and that ***each*** belief system is derived from the corresponding strategy profile.

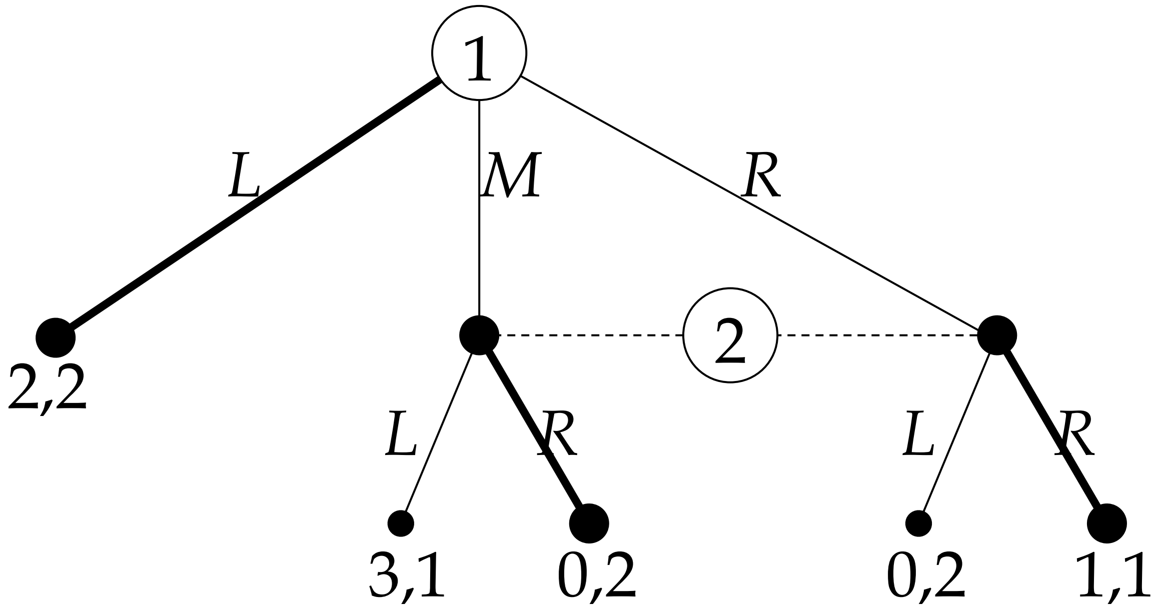
$$\beta = ((L(1-\varepsilon), M(\tfrac{1}{3}\varepsilon), R(\tfrac{2}{3}\varepsilon)), (L(\varepsilon), R(1-\varepsilon)))$$

$$\mu = (\dots, \{M, R\} \mapsto (M(\tfrac{1}{3}), R(\tfrac{2}{3})), \dots)$$

Consistency of Assessments

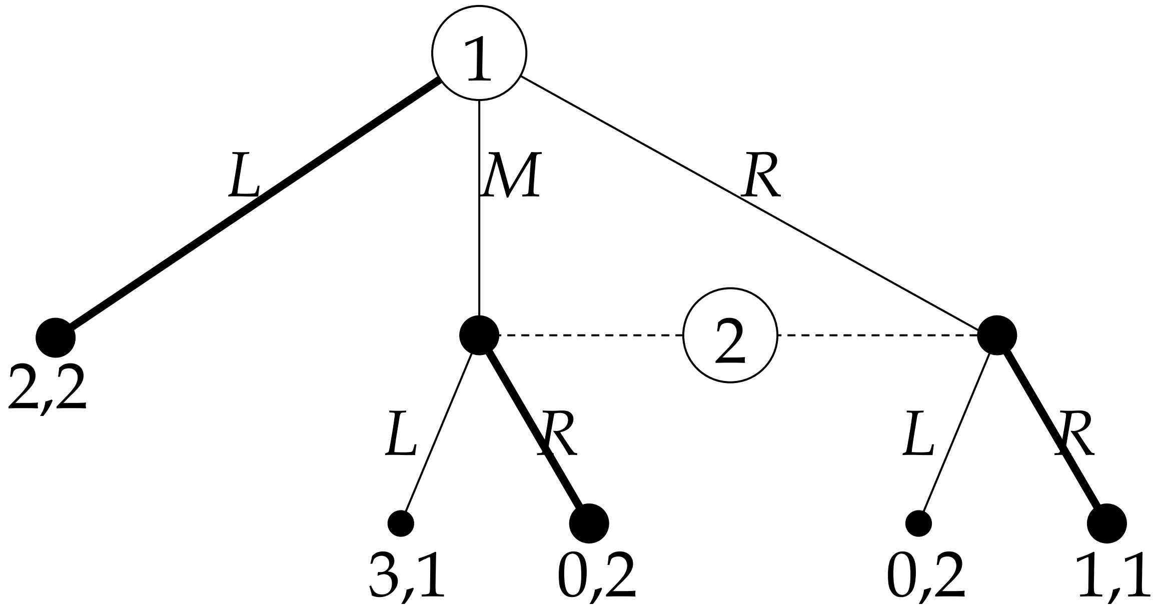
DEFINITION. Let $\Gamma = \langle N, H, P, f_c, (\mathcal{I}_i), (\succsim_i) \rangle$ be a finite extensive game with perfect recall. An assessment (β, μ) is **consistent** if there is a sequence $((\beta^n, \mu^n))_{n=1}^{\infty}$ of assessments that converges to (β, μ) in Euclidian space and has the properties that ***each*** strategy profile β^n is completely mixed and that ***each*** belief system μ^n is derived from β^n using Bayes' rule.

EXAMPLE.



The following assessment is consistent: $\beta_1 = L$, $\beta_2 = R$, and $\mu(\{M,R\})(M) = \frac{1}{3}$, because it is the limit $\varepsilon \to 0$ as of $\beta_1^\varepsilon = (1-\varepsilon, \frac{1}{3}\varepsilon, \frac{2}{3}\varepsilon)$, $\beta_2^\varepsilon = (\varepsilon, 1-\varepsilon)$ and $\mu^\varepsilon(\{M,R\})(M) = \frac{1}{3}$ for every ε.

EXAMPLE.

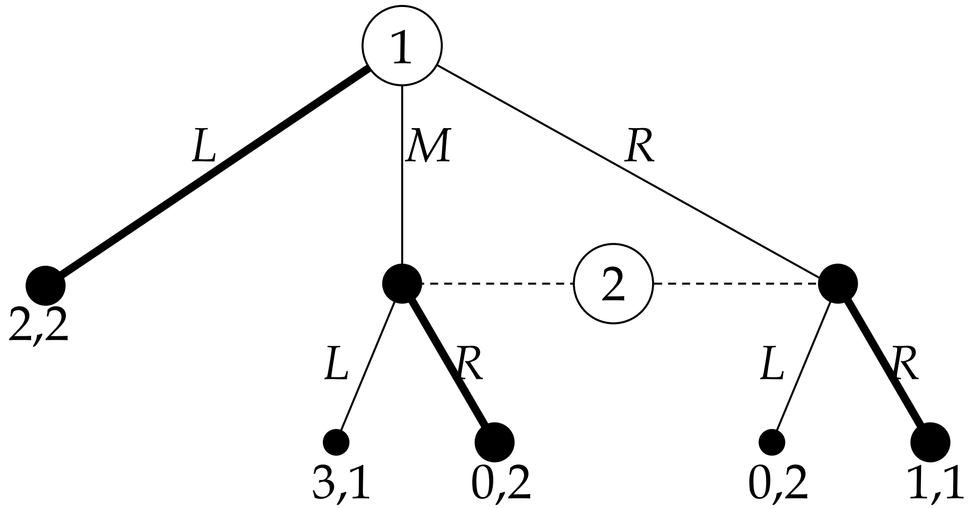


If $\alpha \geq \frac{1}{2}$, the assessment $\beta_1 = L$, $\beta_2 = R$, and $\mu(\{M, R\})(M) = \alpha$ is also sequentially rational.

Sequential Equilibrium

DEFINITION. An assessment is a **sequential equilibrium** of an extensive game with perfect recall if it is sequentially rational and consistent.

EXAMPLE.



If $\alpha \geq \frac{1}{2}$, the assessment $\beta_1 = L$, $\beta_2 = R$, and $\mu(\{M, R\})(M) = \alpha$ is both

1. consistent, and
2. sequentially rational.

So it is a sequential equilibrium.

Class Discussion

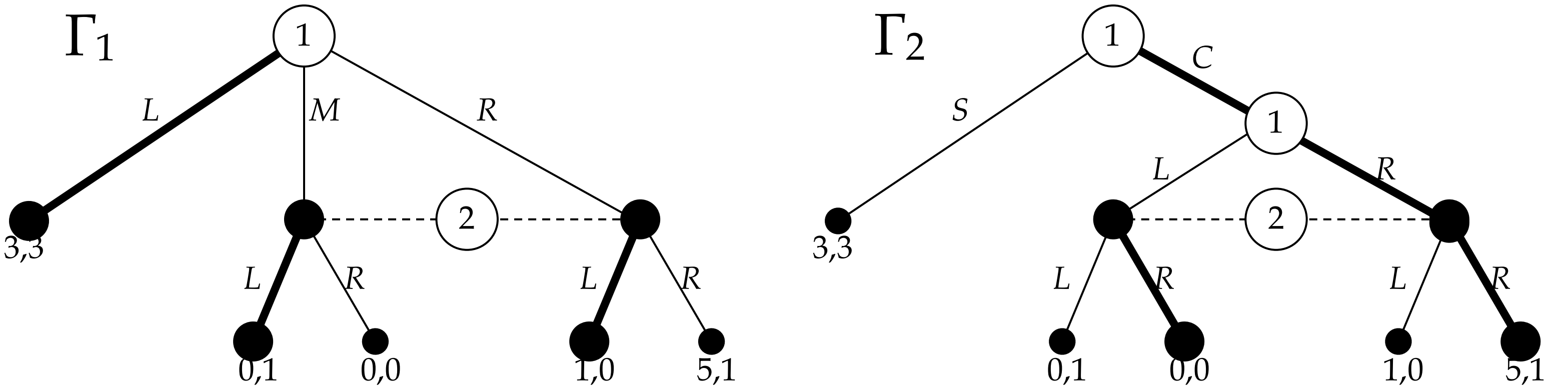
Q: If (β, μ) is a **sequential equilibrium**, then is β a Nash equilibrium?

Y

Q: Consider an extensive game with perfect information. If (β, μ) is a **sequential equilibrium**, then is β a subgame perfect equilibrium? And *vice versa*?

Y

Class Discussion



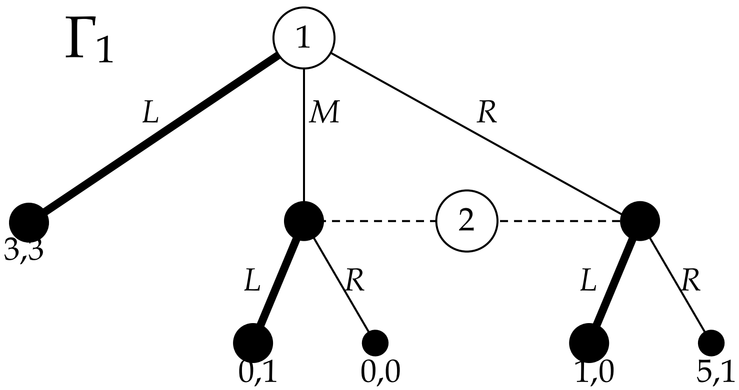
A Sequential Equilibrium

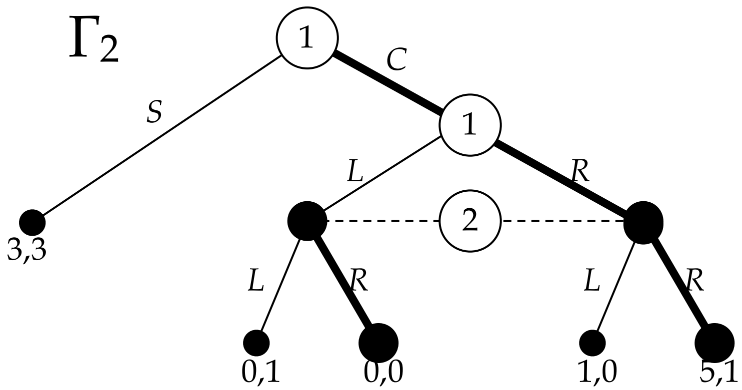
$$\beta_1 = L,\ \ \beta_2 = L$$

$$\mu(\{M, R\})(R) = 0$$

Q: $\beta = \lim_{\varepsilon \to 0}((L(1-\varepsilon), M(\varepsilon(1-\varepsilon)), R(\varepsilon\varepsilon)), (L(1-\varepsilon), R(\varepsilon)))$.

Class Discussion





<u>A Sequential Equilibrium</u>

$\beta_1 = L, \ \beta_2 = L$

$\mu(\{M, R\})(R) = 0$

Q:

$\beta_1(C) = \_\_$

$\beta_1 = (\_\_, \_\_)$

$\therefore \beta_2 = \_\_$