THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1020
Exercise 11
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(Properties of the Dot Product)

(i) $\mathbf{a} \cdot \mathbf{b} = 0$ if $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$

(ii) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ commutative law

(iii) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b}) + (\mathbf{a} \cdot \mathbf{c})$ distributive law

(iv) $\mathbf{a} \cdot (k\mathbf{b}) = (k\mathbf{a}) \cdot \mathbf{b} = k(\mathbf{a} \cdot \mathbf{b}), k$ a scalar

(v) $\mathbf{a} \cdot \mathbf{a} \geq 0$

(vi) $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$

Exercise 1 Show that $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b}) + (\mathbf{a} \cdot \mathbf{c})$.

Solution: Let $\mathbf{a} = a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$ and $\mathbf{c} = c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}$. Then

$$\begin{aligned}
\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \cdot ((b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) + (c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k})) \\
&= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \cdot ((b_1 + c_1)\mathbf{i} + (b_2 + c_2)\mathbf{j} + (b_3 + c_3)\mathbf{k}) \\
&= a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3) \\
&= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3 \\
&= (a_1b_1 + a_2b_2 + a_3b_3) + (a_1c_1 + a_2c_2 + a_3c_3) \\
&= \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}.
\end{aligned}$$

Exercise 2 Let $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$ and $\mathbf{i} + \mathbf{j} + 2\mathbf{k}$. Find
(a)$\text{comp}_{\mathbf{b}}\mathbf{a}$ and **(b)**$\text{comp}_{\mathbf{a}}\mathbf{b}$.

Solution:

(a) We first form a unit vector in the direction of $\mathbf{b}$:

Then using

$$\text{comp}_{\mathbf{b}}\mathbf{a} = \mathbf{a} \cdot \left(\frac{\mathbf{b}}{|\mathbf{b}|}\right),$$

we have

$$\mathrm{comp}_{\mathbf{b}}\mathbf{a} = (2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) \cdot \frac{1}{\sqrt{6}}(\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = -\frac{3}{\sqrt{6}}.$$

(b) By modifying

$$\mathrm{comp}_{\mathbf{b}}\mathbf{a} = \mathbf{a} \cdot \left(\frac{\mathbf{b}}{|\mathbf{b}|}\right),$$

accordingly, we have

$$\mathrm{comp}_{\mathbf{a}}\mathbf{b} = \mathbf{b} \cdot \left(\frac{\mathbf{a}}{|\mathbf{a}|}\right).$$

Then

$$|\mathbf{a}| = \sqrt{29} \text{ so } \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{1}{\sqrt{29}}(2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}),$$

and

$$\mathrm{comp}_{\mathbf{a}}\mathbf{b} = (\mathbf{i} + \mathbf{j} + 2\mathbf{k}) \cdot \frac{1}{\sqrt{29}}(2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) = -\frac{3}{\sqrt{29}}.$$