Exercises: Green's Theorem

For Problems 1-3, use the Green's Theorem to evaluate the following line integrals as double integrals. The curve C in each case is always in the positive direction.

Problem 1. $\int_C \boldsymbol{f}(\boldsymbol{r}) \cdot d\boldsymbol{r}$, where $\boldsymbol{f} = [y, -x]$ and C is the circle $x^2 + y^2 = 1$.

Solution: Let $f_1(x, y) = y$ and $f_2(x, y) = -x$. Let D be the region enclosed by C. By the Green's theorem, we know

$$\begin{aligned}
\int_C \boldsymbol{f}(\boldsymbol{r}) \cdot d\boldsymbol{r} &= \int_C (f_1\, dx + f_2 dy) \\
&= \iint_D \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\, dxdy \\
&= \iint_D -1 - 1\, dxdy = -2\pi.
\end{aligned}$$

Problem 2. $\int_C \boldsymbol{f}(\boldsymbol{r}) \cdot d\boldsymbol{r}$, where $\boldsymbol{f} = [6y^2, 2x - 2y^4]$, and C is the boundary of the square with $(0, 0)$ and $(1, 1)$ as the opposite corners.

Solution: Let $f_1(x, y) = 6y^2$ and $f_2(x, y) = 2x - 2y^4$. Let D be the region enclosed by C. By the Green's theorem, we know

$$\begin{aligned}
\int_C \boldsymbol{f}(\boldsymbol{r}) \cdot d\boldsymbol{r} &= \int_C (f_1\, dx + f_2 dy) \\
&= \iint_D \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\, dxdy \\
&= \iint_D 2 - 12y\, dxdy \\
&= 2 - 12 \int_0^1 y \left(\int_0^1 dx \right) dy \\
&= 2 - 12 \int_0^1 y\, dy = -4
\end{aligned}$$

Problem 3. $\int_C \boldsymbol{f}(\boldsymbol{r}) \cdot d\boldsymbol{r}$, where $\boldsymbol{f} = [x^2 e^y, y^2 e^x]$, and C is the boundary of the square with $(0, 0)$ and $(1, 1)$ as the opposite corners.

Solution: Let $f_1(x, y) = x^2 e^y$ and $f_2(x, y) = y^2 e^x$. Let D be the region enclosed by C. By the

Green's theorem, we know

$$\begin{aligned}
\int_C \boldsymbol{f}(\boldsymbol{r}) \cdot d\boldsymbol{r} &= \int_C (f_1\, dx + f_2 dy) \\
&= \iint_D \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\, dxdy \\
&= \iint_D y^2 e^x - x^2 e^y\, dxdy \\
&= \int_0^1 \left(\int_0^1 y^2 e^x - x^2 e^y\, dx \right) dy \\
&= \int_0^1 \left(\left(y^2 e^x - \frac{e^y}{3} x^3 \right) \Big|_{x=0}^{x=1} \right) dy \\
&= \int_0^1 y^2 e - \frac{e^y}{3} - y^2\, dy = 0.
\end{aligned}$$

Problem 4. Consider the set S of line integrals of the form $\int_C (f_1\, dx + f_2\, dy)$. Prove that if (i) $\frac{\partial f_1}{\partial y}$ and $\frac{\partial f_2}{\partial x}$ are continuous in $\mathbb{R}^2$ and (ii) $\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x}$, then S is path independent.

Proof: We will prove that when $\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x}$, then $\int_{C'} (f_1\, dx + f_2\, dy) = 0$ for any closed curve C'. This property implies that S is path independent (see Prob 5 of Ex List 8).

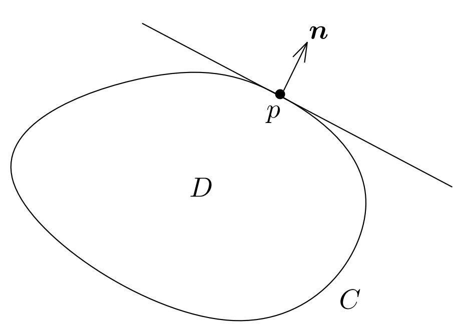
Let D be the region enclosed by C'. By Green's theorem, we know that

$$\begin{aligned}
\int_{C'} (f_1\, dx + f_2\, dy) &= \iint_D \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\, dxdy \\
&= \iint_D 0\, dxdy = 0.
\end{aligned}$$

$\square$

Problem 5* (Hard). Let C be a closed piecewise smooth curve such that the region D enclosed by C is monotone. Consider an arbitrary point p on C. We call $\boldsymbol{n}(x, y)$ a *unit outer normal vector* at $p = (x, y)$ if it satisfies all the following conditions:

- $|\boldsymbol{n}| = 1$;
- the direction of $\boldsymbol{n}$ is perpendicular to the tangent line of C at p;
- the direction of $\boldsymbol{n}$ points towards the outer area of D at p.



Define $\boldsymbol{f}(x, y) = [f_1(x, y), f_2(x, y)]$ such that $\frac{\partial f_1}{\partial x}$ and $\frac{\partial f_2}{\partial y}$ are continuous in D. Prove:

$$\iint_D \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}\, dxdy \quad = \quad \int_C \boldsymbol{f} \cdot \boldsymbol{n}\, ds.$$

Proof. As p travels for a full round along C, we can view its unit tangent vector as $\boldsymbol{u} = [\cos\theta, \sin\theta]$, where θ goes from 0 to 2π. Hence, $\boldsymbol{n} = [\cos(\theta - \pi/2), \sin(\theta - \pi/2)] = [\sin\theta, -\cos\theta]$.

We thus have

$$\int_C \boldsymbol{f} \cdot \boldsymbol{n}\, ds \quad = \quad \int_C f_1 \sin\theta + f_2(-\cos\theta)\, ds \tag{1}$$

On the other hand, we have (think: why?):

$$\begin{aligned} \frac{dx}{ds} &= \cos\theta \\ \frac{dy}{ds} &= \sin\theta. \end{aligned}$$

Therefore:

$$(1) \quad = \quad \int_C (-f_2\, dx + f_1\, dy)$$

which by Green's theorem equals $\iint_D \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}\, dxdy$. □