of equations (12.3) and equation

$$\xi_1+\xi_2+\xi_3=1 \tag{12.5}$$

consists of nonnegative numbers.

Consider for example the system consisting of equations

$$\xi_1=1-c_1, \qquad \xi_2=1-c_2$$

and equation (12.5). [Inequalities (12.1) imply that ξ_1 and ξ_2 are nonnegative.] The solution of the system yields

$$\xi_3=1-c_1-c_2,$$

and since $\xi_3 \geqslant 0$ we have

$$c_1+c_2 \leqslant 1. \tag{12.6}$$

If this condition is not satisfied, then the lower vertex of the shaded triangle falls outside the simplex of imputations (Figure 45) and the core becomes a quadrangle.

A similar role is played by the inequalities

$$c_1+c_3 \leqslant 1, \qquad c_2+c_3 \leqslant 1. \tag{12.7}$$

Depending on the possible variations in the sign between the left- and right-hand side of inequalities (12.6) and (12.7) (there are eight possible cases), the core will either be a line segment, triangle, quadrangle, pentagon, hexagon, or a single point. For example, if neither of the inequalities (12.6) and (12.7) is satisfied, the core is a hexagon (see Figure 46).

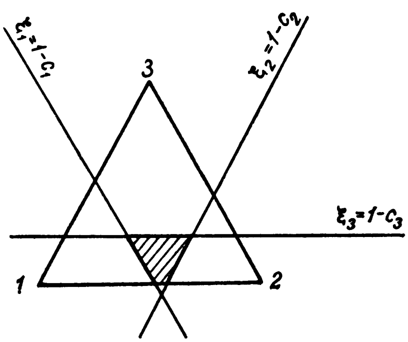

Figure 45

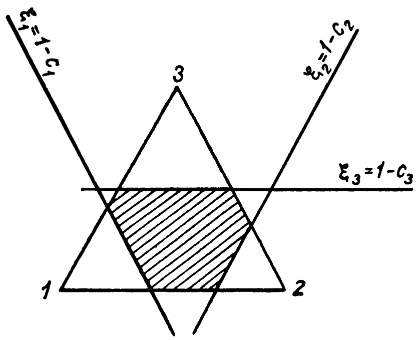

Figure 46

4.13 von Neumann–Morgenstern solutions

4.13.1. The core, discussed in the preceding section, consists of imputations that are stable but in a somewhat negative or passive sense: the state of affairs does not stimulate us to use an imputation that does not belong to the core. However, the properties of imputations in the core contain no recommendation that these imputations should be used, and moreover these properties do not help us to set them off against other suggested (or recommended) imputations. It would be ideal to have an imputation that, besides being nondominated by any other imputation, in turn dominates any other imputation. Unfortunately, there is no essential cooperative game that possesses such an imputation. It is also impossible to find imputations that possess even a reasonably weakened property of this kind. Therefore, the solution to our problem should be sought along the lines of extending the class of objects to be compared in cooperative games, i.e. along the route of extending the class of imputations. Such a procedure, as have seen before, was fruitful in the case of a noncooperative game: the introduction of mixed strategies there made it possible to solve the problem of the existence of an equilibrium situation for arbitrary finite (and also many infinite) noncooperative games. We shall therefore seek—for cooperative games—a solution in the form of a set of imputations rather than a single "ideal" imputation.

In their classical treatise, J. von Neumann and O. Morgenstern [3A] suggested that a set of imputations that serves as a solution for a cooperative game should satisfy the following two properties: internal stability, which means that imputations belonging to the solution could not "contrast" one another, and external stability, which implies that for any deviation from the solution one can offer a "better" imputation belonging to the solution.

Formally these two properties lead to the following definition:

4.13.2 **Definition.** The von Neumann–Morgenstern solution (or the vN–M *solution* or the *stable set*) of a cooperative game is called the set R of imputations satisfying the following conditions:

(1) No imputation in R dominates any other imputation in R (internal stability).

(2) If s is an imputation not in R, then an imputation r exists belonging to R, which dominates s (external stability).

An intuitive interpretation of a vN–M *solution* of a cooperative game is the existence of a system of "behavioral norms" such that no social force (coalition) can contrapose the consequences of two behaviors consistent with these norms, while any deviation from the norm will result in the appearance of some force (i.e. a coalition) in the society (i.e. in the set of all the players I) that will strive to restore the norm.

4.13.3. A certain relationship between the core and a vN–M solution of a game is given by the following theorem.

Theorem. *If a cooperative game possesses both the core C and a* vN–M *solution R, then $C \subset R$.*

Proof. If imputation x belongs to C, it cannot be dominated by any other imputation. If it does not belong to the solution, it should be dominated by some imputation belonging to the solution. Consequently, any imputation belonging to the core also belongs to each of the vN–M solutions of the game. □

4.13.4. A vN–M solution of a cooperative game cannot consist of a single imputation.

Theorem. *If a* vN–M *solution of a cooperative game* $\langle I,v\rangle$ *consists of a single imputation* x, *then the characteristic function* v *is inessential.*

PROOF. Assume that the characteristic function v is essential. In view of Section 4.10.5, we may assume that it is in the 0-1 reduced form. Let x_i be a positive component of x. In the case of an essential characteristic function $|I|=n>1$, and therefore one can construct an imputation

$$y=(y_1,\dots,y_n)$$

by setting

$$y_j=\begin{cases} x_j+\dfrac{x_i}{n-1}, & \text{if } j\neq i,\\ 0, & \text{if } j=i.\end{cases}$$

By the definition of domination, imputation x does not dominate y, hence y, being different from x, either belongs with x to the vN–M solution of the game, or implies the existence in the vN–M solution of a third imputation z that differs from x (and dominates y). □

4.13.5. Notwithstanding its numerous advantages, the notion of a vN–M solution possesses certain defects. We shall mention three of them.

First, examples of cooperative games are known for which there is no vN–M solution. Moreover, at the present time, no criteria for the existence of a vN–M solution of a cooperative game are known. Therefore, the optimality principle inherited in a vN–M solution is not universally realized, and the region of its "realizability" so far is undetermined.

Second, a large number of cooperative games possess more than one solution. Hence the optimality principle that leads to a vN–M solution is not complete: in general it does not generate a unique scheme for distributing payoffs.

Finally, as we have seen, solutions for essential cooperative games consist of more than one imputation. Therefore, even if we agree on a particular vN–M solution, we still have not determined (uniquely) the payoff for each of the players.

The defects mentioned above should not be considered as faults to cause revisions of this principle, but as defects that we would like to rectify. Unfortunately, for the games under consideration, this is hardly possible except by drastic reduction of the cases for which this principle is realizable. Moreover, this state of affairs actually reflects the facts of life:

the majority of economic and social problems admit multiple solutions and these solutions are not always directly and consistently comparable with respect to their preferability.

4.14 vN–M solutions for three-person constant sum games

4.14.1. A vN–M solution for inessential games as well as for arbitrary two-person games is easily determined (as in the case of determination of the core for such games; cf. Section 4.11.3). Namely, there is no dominance of imputations in the case of two-person games, while every inessential game possesses only one imputation. Therefore, the only vN–M solutions for one of these games is the set of all the imputations.

Consequently, a detailed analysis of vN–M solutions of cooperative games should start with the study of three-person essential games. In view of the remarks in Section 4.10.5, we shall confine ourselves to the games in the 0-1 reduced form.

4.14.2. The set of imputations for an essential three-person game in its 0-1 reduced form is a triangle. The internal stability of a vN–M solution means no two imputations belonging to the solution dominate each other. Therefore, any two imputations belonging to a single solution should be located on a straight line parallel to one of the sides of the triangle of all the imputations (cf. Section 4.10.7). Consequently, the intervals joining, pairwise, imputations belonging to a vN–M solution should be parallel to the three directions (corresponding to the sides of the triangle of imputations).

We consider separately the following two cases:

(1) The imputations belonging to a vN–M solution are all located on a single straight line.

(2) Not all the imputations belonging to a vN–M solution are situated on a single line.

4.14.3. Let the whole vN–M solution R be located on a single line. For definiteness, let the line AB be parallel to the side *12* of the triangle of all the imputations (Figure 47.). No two imputations in R dominate each other so that the internal stability of R is automatically valid.

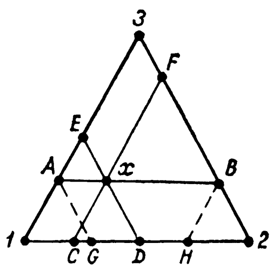

Figure 47

As far as external stability, observe that no imputation on the interval AB is dominated by all the remaining imputations on this interval. Therefore, the external stability of R requires that the interval AB be entirely in R. Next, the imputation x (cf. Figure 47) dominates through coalition $\{1,2\}$ all the imputations that form the parallelogram $xF3E$, and all the imputations on this interval dominate through $\{1,2\}$ the union of all these parallelograms, i.e. the triangle $AB3$. Consider now the domination of imputations located below the line AB. The imputation x dominates through coalition $\{1,3\}$ the whole parallelogram $xD2B$, and all the imputations on AB the union of such parallelograms, i.e. the parallelogram $AB2G$. Analogously the imputations on AB dominate the parallelogram $BH1A$ through the coalition $\{2,3\}$. Clearly, in order that the imputations on AB dominate all the imputations belonging to the trapezoid $A12B$, it is necessary that the intercept of the lines AG and BH be located strictly below the basis 12 of the triangle of imputations. We shall express this necessary condition algebraically.

The equation of the line AB is $\xi_3 = x_3$. Hence the points A and B have the barycentric coordinates $(1-x_3, 0, x_3)$ and $(0, 1-x_3, x_3)$, respectively. Therefore, in these coordinates the equations of the lines AG and BH are $\xi_1 = 1 - x_3$ and $\xi_2 = 1 - x_3$. The condition that the point of intersection of these lines lies outside the triangle 123 implies that $\xi_1 + \xi_2 > 1$, i.e. $x_3 < \frac{1}{2}$. This means that in order that the set R be externally stable (and therefore constitute a vN–M solution) it is necessary and sufficient that this interval be situated below the median of the triangle. Each interval of this kind represents a vN–M solution of the game under consideration.

The vN–M solutions described above can be "physically" interpreted as follows: Player three obtains a certain fraction α (of the total sum) and the remaining fraction $1-\alpha$ is arbitrarily subdivided between players one and two. We observe that for all imputations in this solution the amount obtained by player three is the same. Such a solution is usually called a *discriminating* solution in the theory of cooperative games and the player who receives the same payoff in all the imputations belonging to this solution is referred to as the *discriminated* one.

Analogously, one can construct the sequences of vN–M solutions of a game in which players one or two, respectively, are *discriminated.*

4.14.4. Consider now the case when not all the imputations in a vN–M solution are situated on a straight line.

Let x, y, and z be three imputations not on the same line. We shall try to "adjoin" to these three imputations a fourth one u that is not dominated by any one of them.

Join u with x. The interval ux should be either parallel to one of the sides of the triangle xyz or coincide with it. If the direction of xu coincides with the direction of xy (Figure 48), then the imputation u is on the line xy, but differs from both x and y. In this case, the interval zu is parallel to neither zx nor zy nor xy, which is impossible. Analogously, the case when

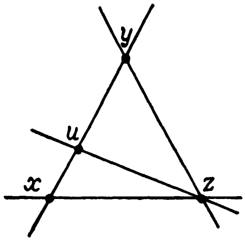

Figure 48

the interval xu is situated along the line xz should be rejected as impossible.

Now, let the interval xu be parallel to yz (Figure 49). In this case, the line joining z with u ought to be parallel to xy and the quadrangle $xyzu$ becomes a parallelogram. The diagonal joining y with u is not parallel to either side of this parallelogram, nor is it parallel to the second diagonal. This yields a fourth direction, which is impossible.

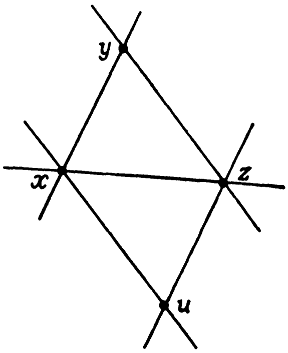

Figure 49

Therefore, if not all the imputations in a vN–M solution are located on a single straight line, then the solution cannot contain four imputations. Therefore it consists of exactly three imputations that form the vertices of a triangle. The sides of this triangle should be parallel to the sides of the triangle of all imputations. Consequently, the small triangle should be situated within the large one in a "similar" (Figure 50) or "anti-similar" fashion (Figure 51).

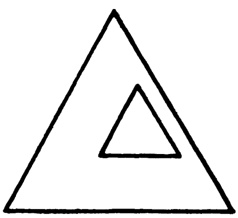
Figure 50

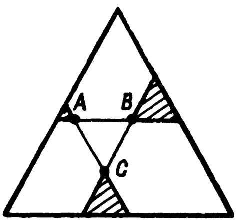

Figure 51

4.14.5. Consider first the "similar" case. It is easy to verify that in this case no point of the interior triangle dominates either of its vertices. Hence the triple of the vertices of this triangle cannot form a vN–M solution.

In the second case, as it is easy to verify, all the imputations located in the three closed shaded triangles in Figure 51 remain undominated. Consequently, in order that the set $\{A,B,C\}$ form a vN–M solution, it is necessary that each of these shaded triangles shrink to a single vertex, i.e. that the points A, B, and C belong to the sides of the basic triangle of all imputations. This is, however, possible only when the points A, B, and C are located at the middles of the corresponding sides of the basic triangle.

So far our arguments were to establish the necessary conditions for a vN–M solution. We deduced that if a vN–M solution exists in a three-person essential game with imputations not located on the same straight line, then it may consist of only three points, namely, the middles of the sides of the basic triangle (Figure 52). Now we shall verify that the set $\{A,B,C\}$ is indeed a vN–M solution.

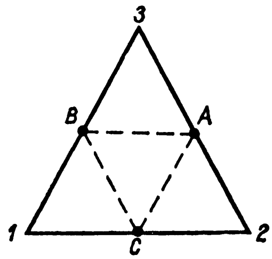

Figure 52

The internal stability of this set up to now was one of the conditions in its construction. Let us check the external stability of the set. Imputation A dominates all the imputations belonging to the parallelogram *1BAC* except for the interior points of the intervals BA and AC. Note that it dominates all the interior points of the interval BC. Next, imputation B dominates the whole parallelogram *2CBA*, except for the points of the intervals BC and BA, while imputation C dominates the parallelogram *CB3A* except for the points of BC and AC. However, the interior points of BC are dominated by A, those of AC by B, and those of BA by C. Thus all the imputations except the imputations A,B,C are dominated by these three imputations. Therefore, the set $\{A,B,C\}$ is indeed externally stable and thus it is a vN–M solution.

The unique solution obtained consists of imputations

$$A=\left\{0,\tfrac{1}{2},\tfrac{1}{2}\right\},\qquad B=\left\{\tfrac{1}{2},0,\tfrac{1}{2}\right\},\qquad C=\left\{\tfrac{1}{2},\tfrac{1}{2},0\right\}.$$

It is called a *symmetric* solution.

4.15 vN–M solutions for general three-person cooperative games

4.15.1. Let us return to the general three-person game considered in Section 4.12. We shall use the notation of that section.

Consider first the case when the *core of the game is empty*. Geometrically it means that the points of pairwise intersections of the straight lines

$$\xi_1 = 1 - c_1, \qquad \xi_2 = 1 - c_2, \qquad \xi_3 = 1 - c_3$$

form a triangle as is shown in Figure 53.

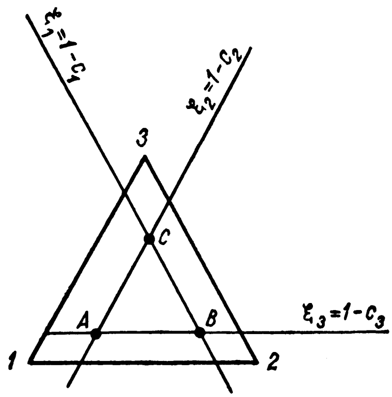

Figure 53

4.15.2. It follows from the discussion in Section 4.10.8 that the imputations located within the triangle ABC cannot be dominated by any imputation outside of this triangle. Therefore, every subset of the triangle ABC that is internally and externally stable with respect to domination within the confines of this small triangle will also be stable with respect to domination within the confines of the larger triangle and conversely. For convenience, we shall refer to such a subset of the triangle ABC as "a vN–M solution in small." Thus if a certain vN–M solution exists in the game under consideration, then its intersection with the triangle ABC should be a vN–M solution in small. The latter, however, were described in the previous section.

To obtain a vN–M solution for the whole game from a vN–M solution in small, it is necessary to add certain imputations located outside the triangle ABC. We shall consider separately the cases when the solution in small is discriminating or symmetric.

4.15.3. Let a vN–M solution in small be discriminating (Figure 54). The set of all imputations not dominated by the set AB is shaded in this figure. We shall restrict ourselves to consideration of a necessary addition to the

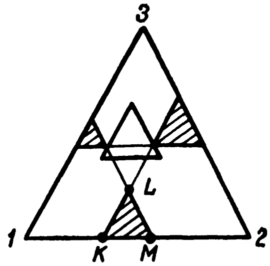

Figure 54

interval AB which will assure that all the imputations in the triangle KLM be dominated. (Additions for the other two triangles are described analogously.) Clearly, such an addition should be located in the triangle KLM. The equations of the oblique sides of this triangle are of the form $\xi_1 = \alpha$ and $\xi_2 = \beta$ (for some α and β). Consider in the triangle KLM a point x with coordinates (ξ_1, ξ_2) (Figure 55). Imputations dominated by this point form two parallelograms (indicated by dotted lines). Consequently, if two imputations belong to the same addition needed to achieve a vN–M solution, then the interval joining them should form at most a 30° angle with the vertical line. In particular, every horizontal line can intersect this addition in at most one point. This means that the whole addition must be situated in a curvilinear interval joining the point L with the basis of the initial triangle of imputations and, moreover, this interval is within 30° of the vertical line. Assume that one of the points of the curvilinear interval does not belong to the addition. In view of its particular location, the point cannot be dominated by any other imputations in the addition. Clearly no imputation outside the addition can dominate this point as well. Therefore, if a point of the curvilinear interval does not belong to the addition, the latter is insufficient to yield a vN–M solution. Thus each point of the curvilinear interval must belong to the addition.

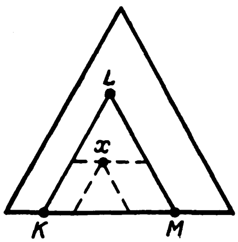

Figure 55

Analogous curves form additions towards a vN–M solution for the two remaining triangles and the whole vN–M solution becomes of the form presented in Figure 56. Clearly, the curves adjoined to a vN–M solution in small should emanate from the corresponding points and deviate from the

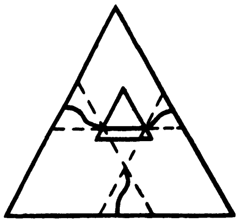

Figure 56

perpendicular direction to the corresponding side of the basic triangle of imputations by at most 30°, and finally these curves should go as far as the side of the basic triangle. Otherwise, these additions are completely arbitrary.

4.15.4. Now let a vN–M solution in small be symmetric. In this case, one can also supplement it by means of curvilinear intervals as indicated in Figure 57. The restrictions on these curves are the same as those stated in the previous item; otherwise they are arbitrary.

Comparing the discrimination solution with the symmetric one in the general case of a three-person cooperative game, we observe that the latter can be viewed as a limiting case of the first.

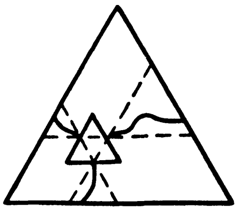

Figure 57

4.15.5. Finally we discuss the case when the game possesses the *nonempty core*. The collection of imputations that form the core dominates all the imputations except those in the triangles shaded in Figure 58. In order to supplement the core up to a vN–M solution, it is necessary to adjoin to it a curvilinear interval (Figure 59) in each of the triangles of the nondominated imputations.

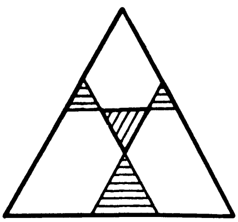

Figure 58

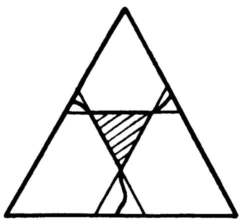

Figure 59

Clearly, if the core is either a quadrangle, pentagon, or hexagon, some of the nondominated triangles will disappear and there will be no need in the corresponding additional curves (Figure 60). In particular, if the core is a hexagon (Figure 46 in Section 4.12), then it coincides with a vN–M solution.

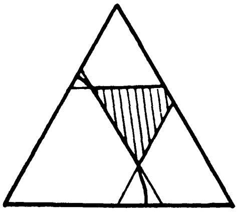

Figure 60

4.16 Shapely's vector; axiomatization

4.16.1. Game theory studies a variety of optimality principles. Some of these reflect directly our intuitive notions of optimality. Such are, for example, the principle of admissible situations in noncooperative games (cf. Section 1.2)—and its most important particular case the maximin principle—as well as the principles of optimality discussed in the preceding sections that lead to the notions of the core and a vN–M solution. However, in addition to these "natural" principles, game theory derives its own principle of optimality by stipulating the conditions that these principles should satisfy. This is essentially an axiomatic approach to the problem.

In this section we shall consider an axiomatic description of a principle of optimality that is quite interesting from both the theoretical and practical points of view. If we try to delineate the principles of optimality informally, then this principle can be characterized as the *principle* of a *fair subdivision of payoffs*.

Our purpose is to indicate a procedure that corresponds to each cooperative game (i.e. each characteristic function) v over the set of players $I=\{1,2,\ldots,n\}$ a vector

$$\Phi(v)=(\Phi_1(v),\Phi_2(v),\ldots,\Phi_n(v)),$$

whose components describe fair payoffs in a certain sense, to each of the players participating in the game. It is also desirable that the vector $\Phi(v)$ be an *imputation* under the conditions of the characteristic function v.

4.16.2. We now state the natural conditions that a fair subdivision of "wealth" should satisfy:

Definition. Player i, in a cooperative game with characteristic function v, is called a *dummy* if for each coalition K not containing i,

$$v(K\cup i)=v(K)+v(i). \tag{16.1}$$