# CSCI2100 Data Structures
# Hashing

Irwin King

king@cse.cuhk.edu.hk
http://www.cse.cuhk.edu.hk/~king

Department of Computer Science & Engineering
The Chinese University of Hong Kong

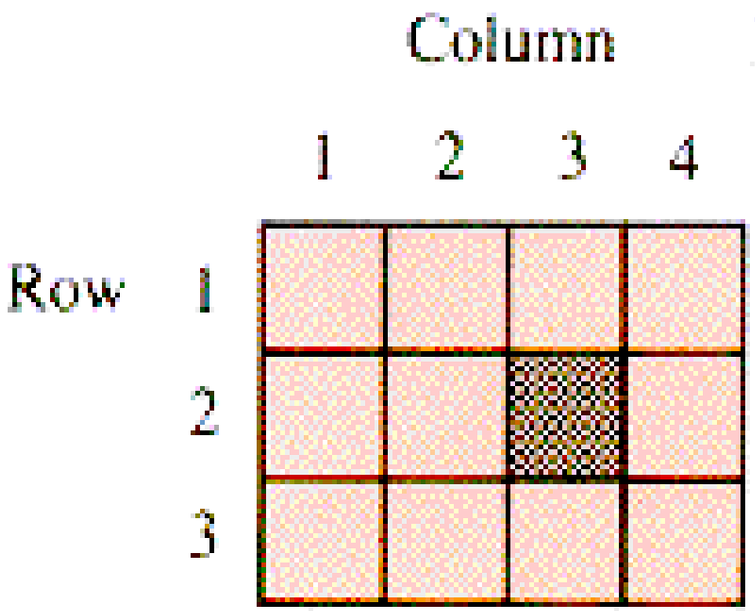


# Introduction

- Hashing is a technique used for performing insertions, deletions and finds in constant **average** time.
- Tree operations that require any ordering information among the elements are not supported efficiently.
- See several methods of implementing the hash table.
- Compare these methods analytically.
- Show numerous applications of hashing.
- Compare hash tables with binary search trees.



# Rectangular Arrays




| | col 1 | col 2 | ... | col $u_2$ |
|---|---|---|---|---|
| row 1 | X | X | ... | X |
| row 2 | X | X | ... | X |
| row 3 | X | X | ... | X |
| | | ⋮ | | |
| row $u_1$ | X | X | ... | X |

(a)



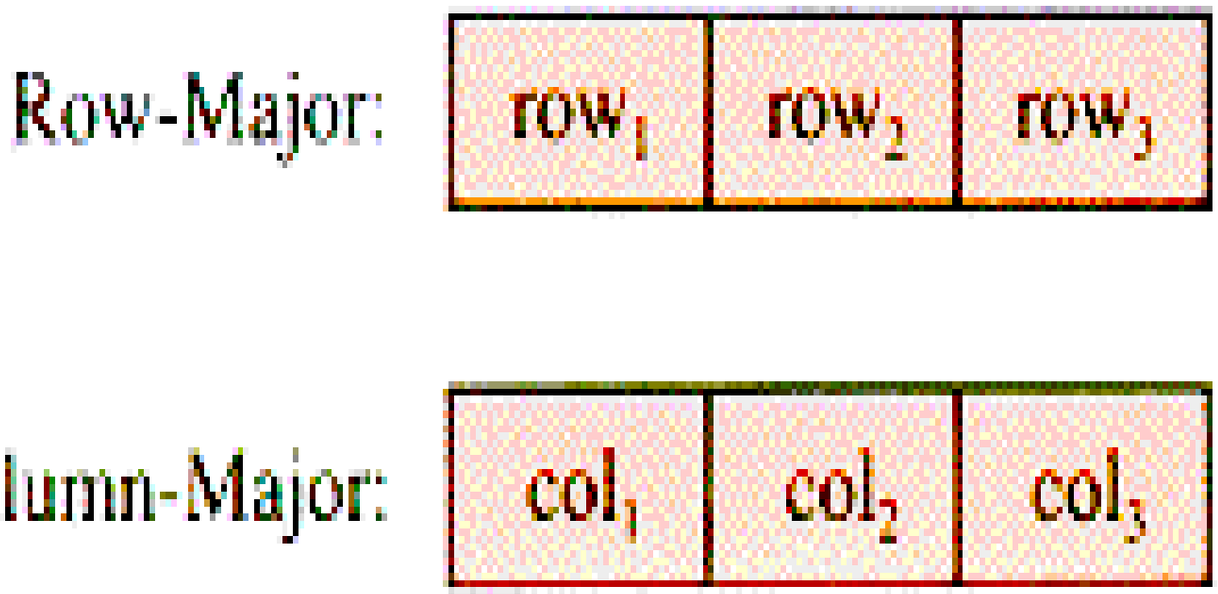
# Row- and Column-Major Ordering

- How does one index an entry in an array?

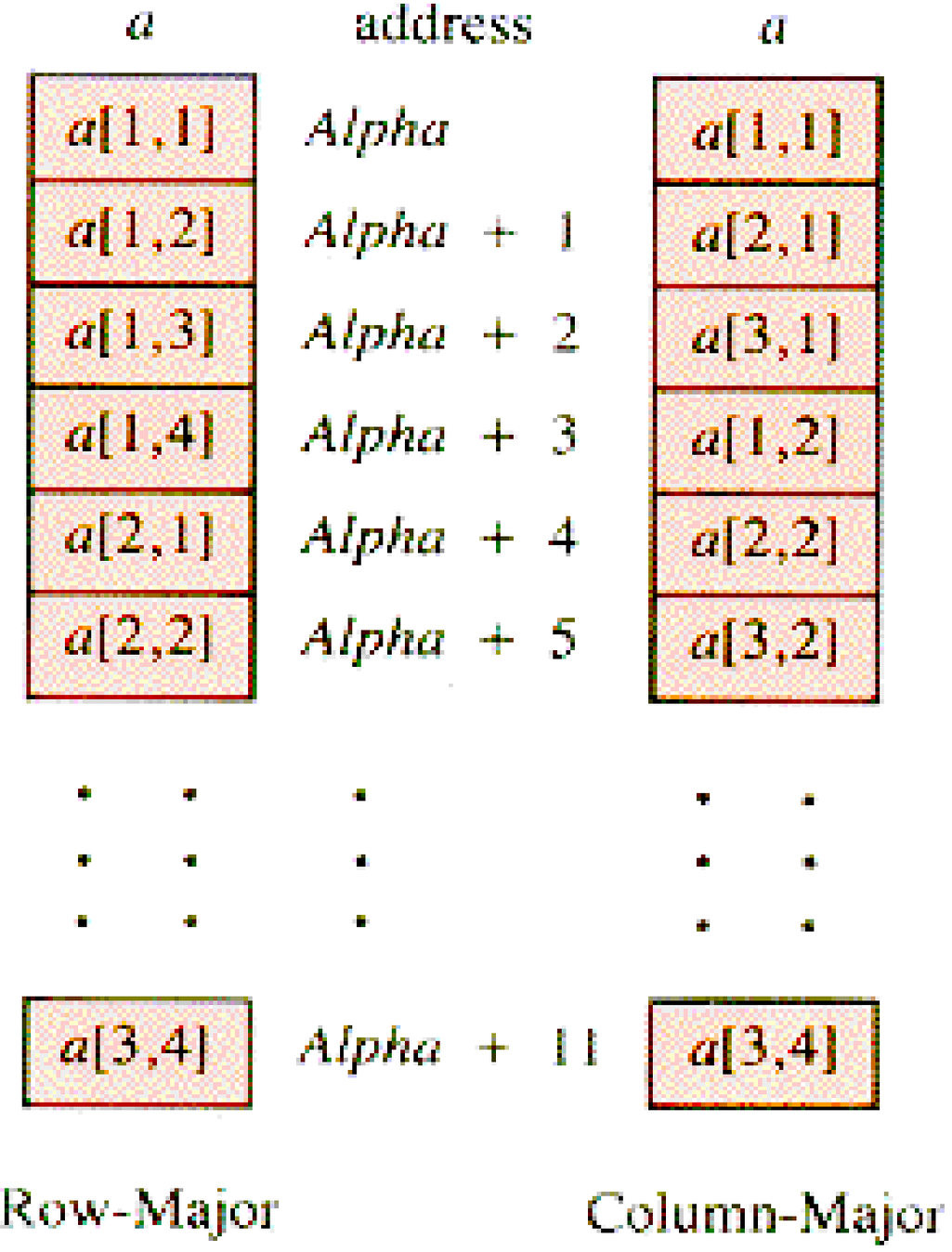
- Entry [*i*,*j*] goes to position *ni* + *j* for row-major ordering and *i* + *jm* for column-major ordering when the rows are numbered from 0 to *m*-1 and the columns from 0 to *n*-1 and entry [0,0] is at position 0.



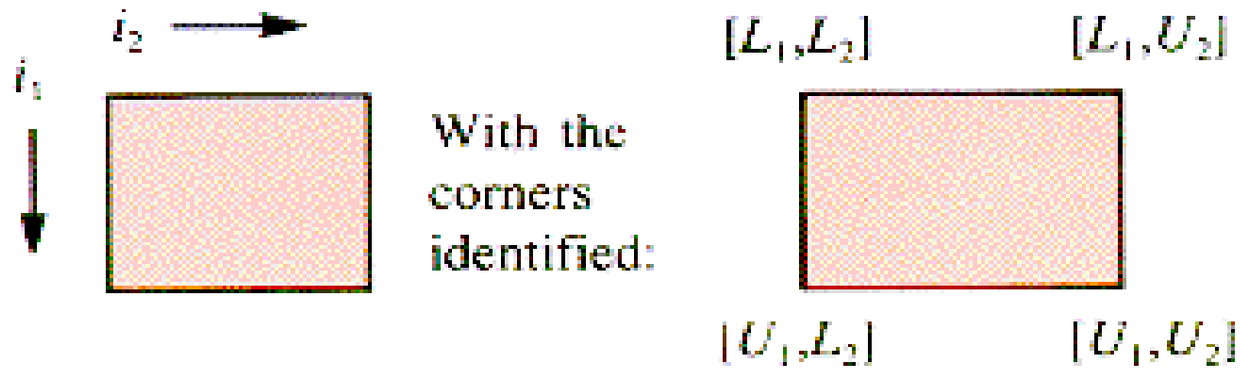
# Row- and Column-Major Example

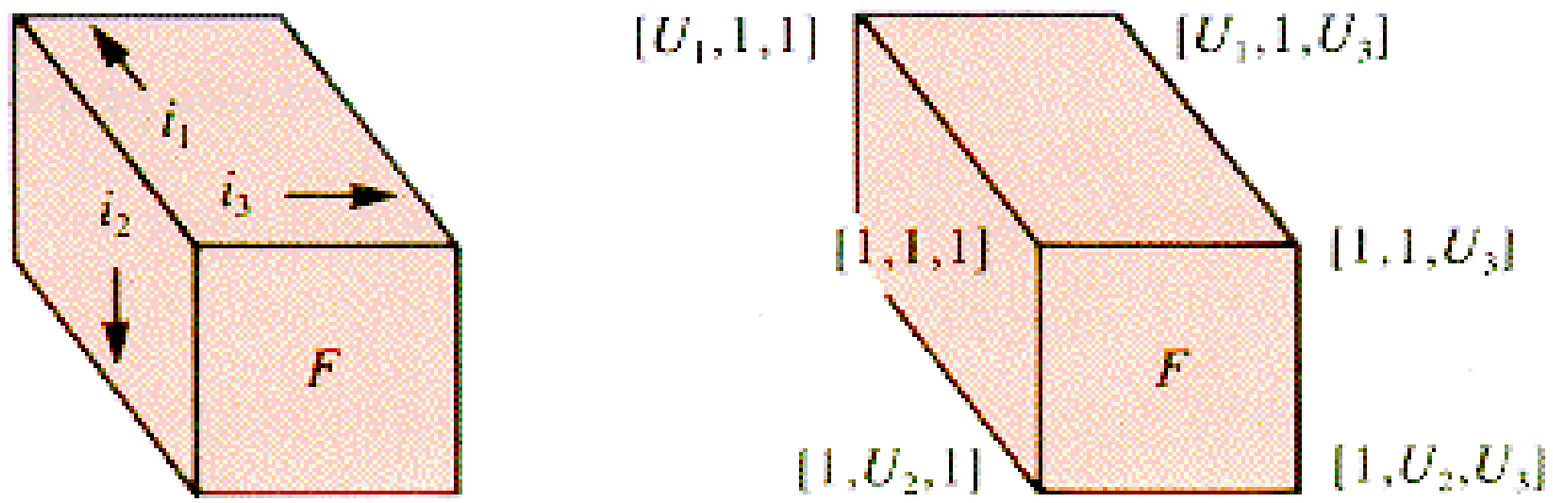


# Example

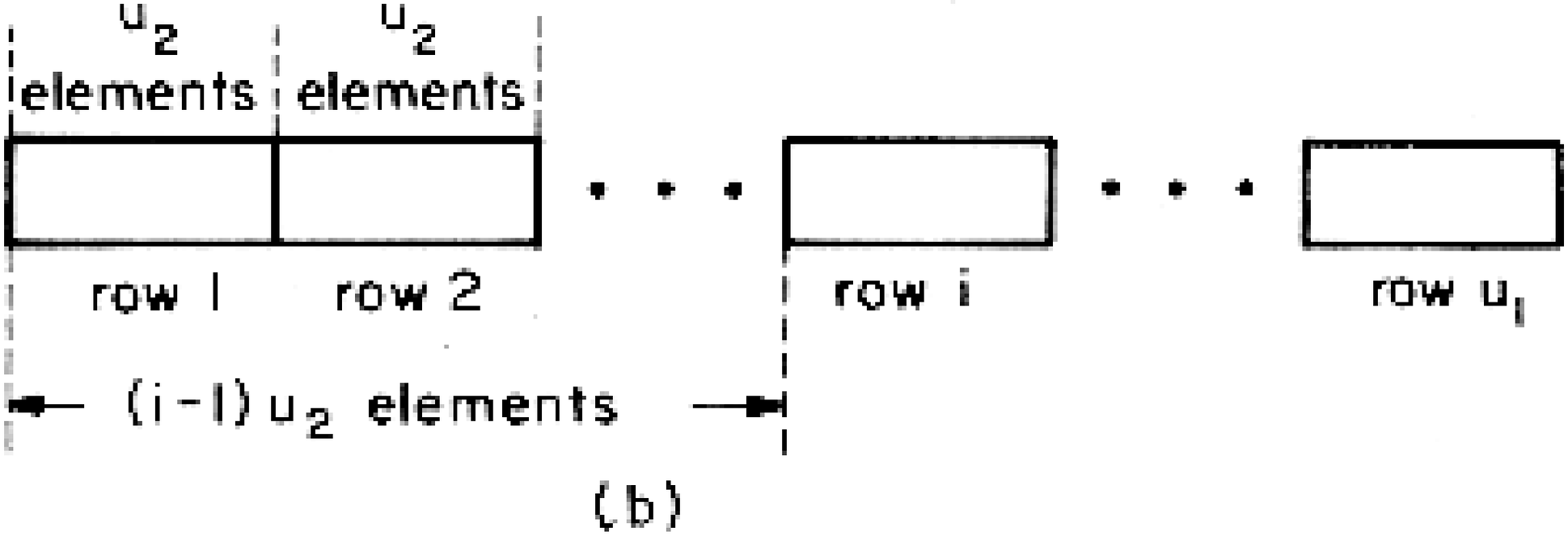


# Implementation Example



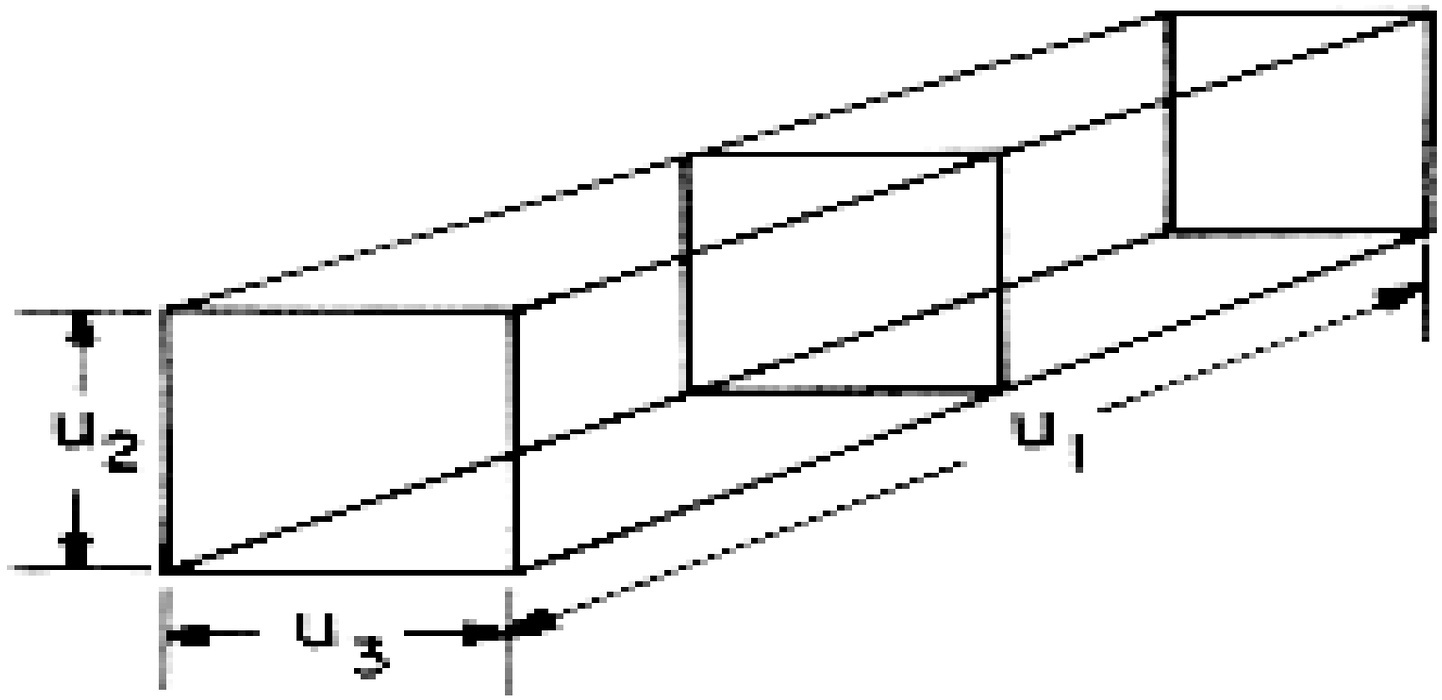
# More Implementation Example



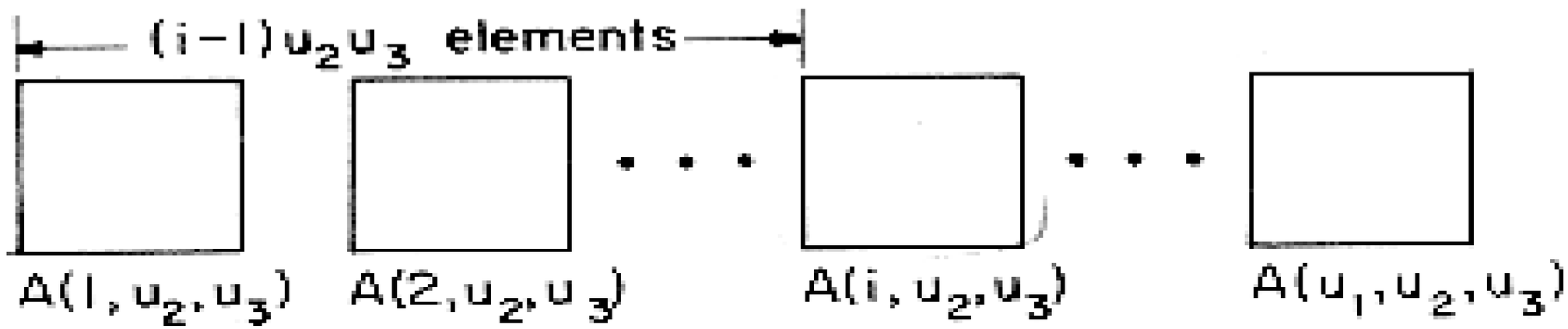

(b)



# 3D Array Implementation



# 3-D Array Implementation



# Problem

- In some applications the full use of the whole array is seldom.
- This leads to sparse array or matrix representation.
- For example, population count in a 2-D grid map.

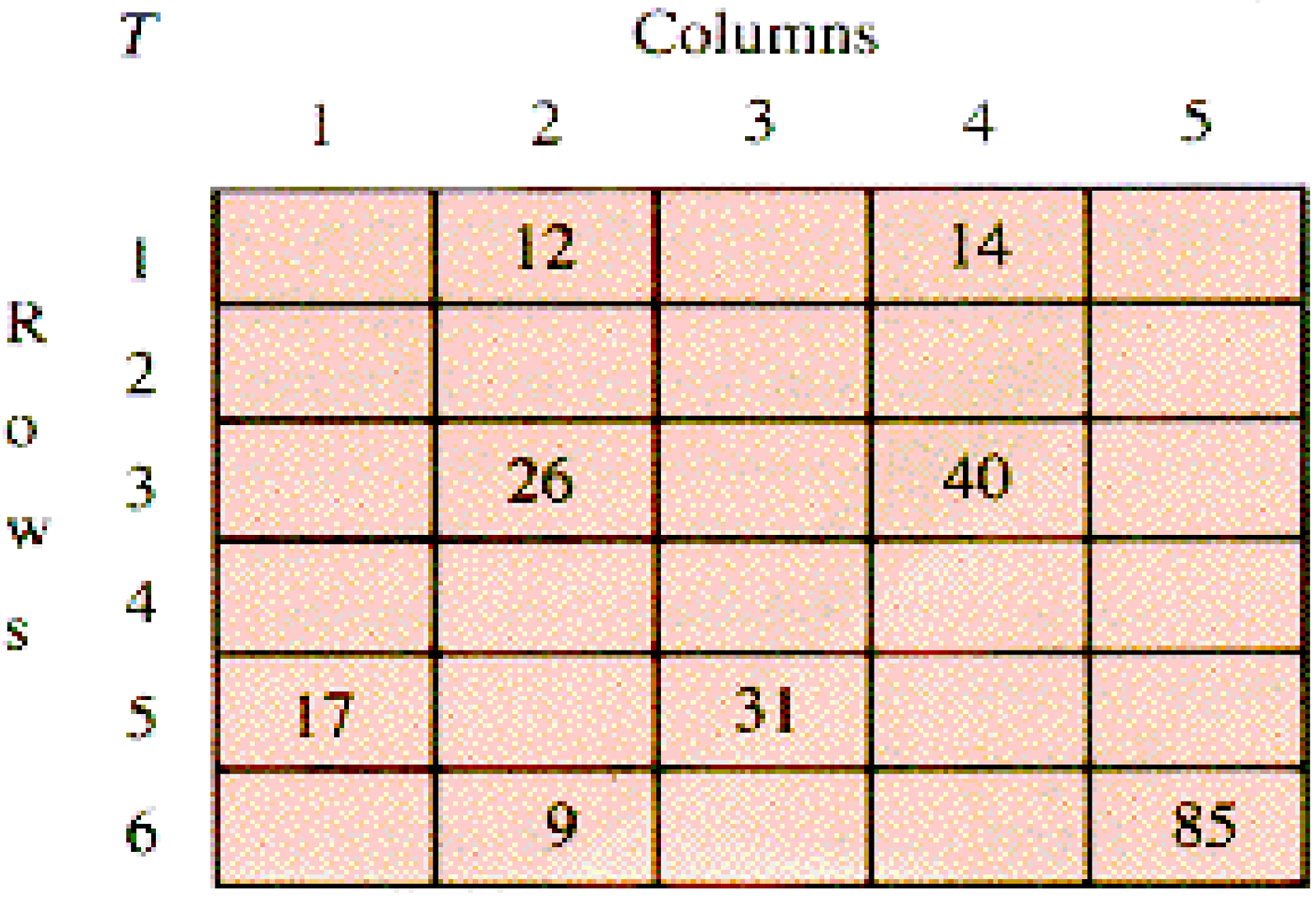


# An Access Table

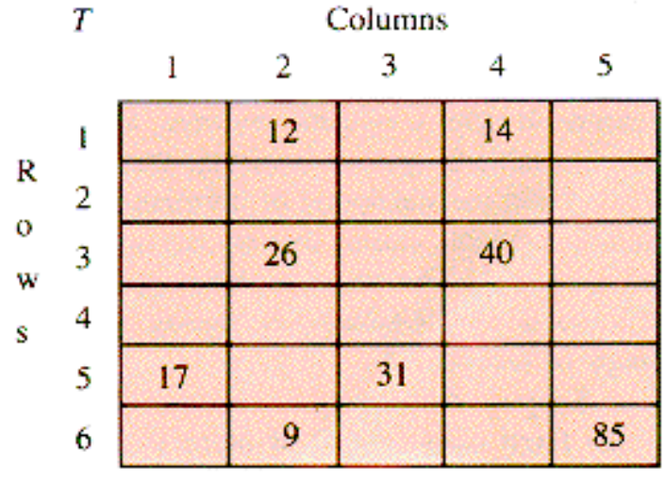
- One method to eliminate the multiplications needed in calculating the index to an entry is to use an access table.
- The array will contain the values 0, *n*, 2 *n*, 3 *n*, ..., (*m*-1) *n*.
- Then for all references to the rectangular array, the index for [*i*,*j*] is calculated by taking the entry in position i of the auxiliary table, adding j, and going to the resulting position.
- Again we see a trade-off between space used and execution speed.



# Example



# Example




Columns

1

| | |
|---|---|
| 5 | 17 |

2

| | |
|---|---|
| 1 | 12 |
| 3 | 26 |
| 6 | 9 |

3

| | |
|---|---|
| 5 | 31 |

4

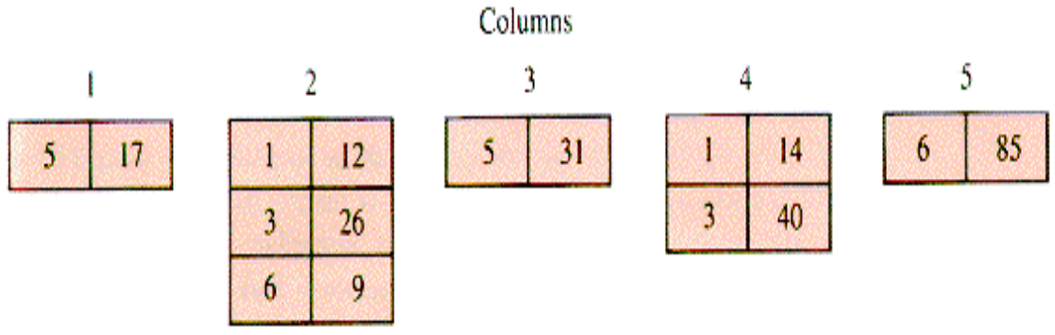
| | |
|---|---|
| 1 | 14 |
| 3 | 40 |

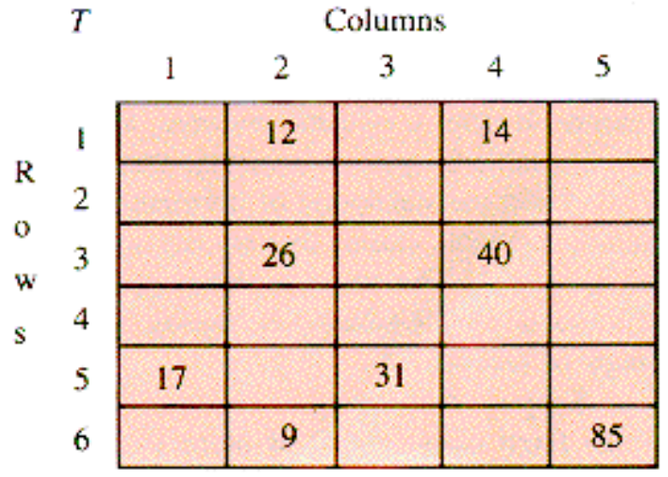
5

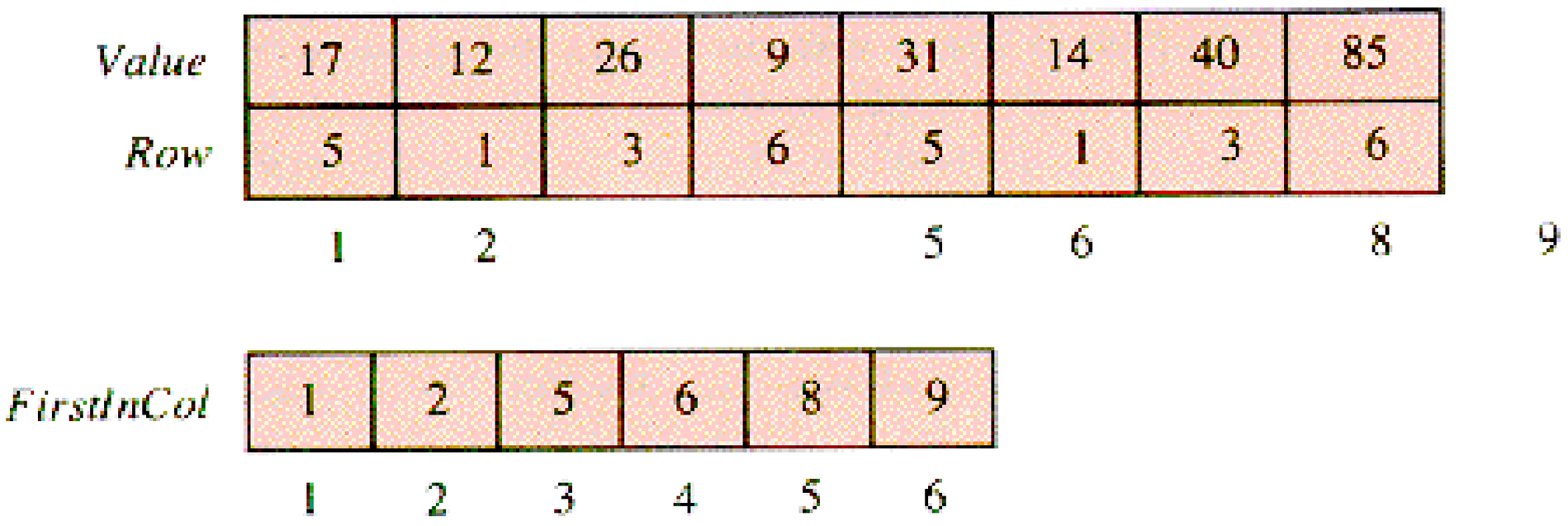
| | |
|---|---|
| 6 | 85 |



# Example



# Tables: A New Abstract Data Type

- Functions: a function is defined in terms of two sets and a correspondence from elements of the first set to elements of the second.

- If *f* is a function from a set A to a set B, then f assigns to each element of A a unique elements of B.

- The set A is called the domain of *f*, and the set B is called the codomain of *f*. The subset of B containing just those element that occur as values of *f* is called the range of *f*.



# Example

- For a table, we call the domain the **index** set, and we call the codomain the **base type** or **value type**.
- For example, to index into the cell [2,3] the offset value may be 13 if the matrix size is [10,10].



# An Abstract Data Type

- A table with index set I and base type T is a function from I into T together with the following operations.
  - Table access: Evaluate the function at any index in I.
  - Table assignment: Modify the function by changing its value at a specified index in I to the new value specified in the assignment.



# An Abstract Data Type

- Insertion: Adjoin a new element *x* to the index set I and define a corresponding value of the function at *x*.
- Deletion: Delete an element *x* from the index set I and restrict the function to the resulting smaller domain.



# Why Hash Table?

- Often, array indices are not natural identifiers for items that are to be stored, accessed, and retrieved.
- For example, let's try to store the list in an array.

```
beef    bellpepper   blackpepper   dillweed

onion   potato       olive         salt

cumin   carrot       mushroom      tomatopaste
```



# Problem

- While it is true that STORE and RETRIEVE are O(1) operations for arrays, that is only so if the indices are **known** and the value in the target of a STORE can be discarded.
- Without a complete set in hand it cannot be known that potato has index 10 in the sorted list of items.



# Solution

- Use item as a **KEY**--Because an index integer is not known on the entry of one of the items, it would be helpful if the item itself could be used as a key to index the cell where it will be stored.



# Solution

- A solution would be to convert the keys (the list items, here) into unique integers and use them as array indices.
- A function that does so is called a hash function.
- The conversion process is called hashing
- The storage structure is called a hash table or scatter-storage.



# Example Solution

- We may sum up the ASCII value from each character in the key, e.g., *a* = 1, *b*=2, …, *z* = 26, so beef = 2+5+5+6=18.

| Item | HF1(Item) | Item | HF1(Item) |
|---|---|---|---|
| beef | 18 | carrot | 75 |
| onion | 67 | salt | 52 |
| cumin | 60 | blackpepper | 105 |
| dillweed | 74 | olive | 63 |
| bellpepper | 107 | tomatopaste | 145 |
| potato | 87 | mushroom | 122 |



# Introduction to Hashing

- Hashing is a technique used for performing insertions, deletions and finds in constant average time.
- Tree operations that require any ordering information among the elements are not supported efficiently.
- See several methods of implementing the hash table.
- Compare these methods analytically.
- Show numerous applications of hashing.
- Compare hash tables with binary search trees.



# General Idea

- Hash table data structure is merely an array of some fixed size, containing the keys.
- A key is a string with an associated value (for instance, salary information).
- Each key is mapped into some number in the range 0 to H_SIZE - 1 and placed in the appropriate cell.

$$f : key \rightarrow value$$



# General Idea

- The mapping is called a **hash function**, which ideally should be **simple** to compute and should ensure that any two distinct keys get **different** cells.
- This is difficult to achieve in reality since there are a finite number of cells and a virtually inexhaustible supply of keys.
- We seek a hash function that distributes the keys **evenly** among the cells.



# Issues

- Choosing the hashing function
  - How to make sure that one has selected a good function for the application
- Collision handling
  - How to handle conflict when two keys have the same location
- Deletion handling
  - How to deal with the table when items are being removed



# Example

| | |
|---|---|
| 0 | |
| 1 | |
| 2 | |
| 3 | john 25000 |
| 4 | phil 31250 |
| 5 | |
| 6 | dave 27500 |
| 7 | mary 28200 |
| 8 | |
| 9 | |



# Hash Tables

- We can continue to exploit table lookup even in situations where the key is no longer an index that can be used directly as in array indexing.
- What we can do is to set up a **one-to-one** correspondence between the keys by which we wish to retrieve information and indices that we can use to access an array.



# Hash Tables

- The idea of a **hash table** is to allow many of the different possible keys that might occur to be mapped to the same location in an array under the action of the index function.
- Others have called **scatter-storage** or **key-transformation**.



# Hash Function

- A **hash function** that takes a key and maps it to some index in the array.
- Often, two records may want to go to the same location.
- Therefore, a **collision** may occur and a **collision procedure** must be devised to handle this.



# Choosing a Hash Function

- The two principal criteria in selecting a hash function are that
  - it should be **easy** and **quick** to compute and that
  - it should achieve an **even distribution** of the keys that actually occur across the range of indices.



# Hash Function

- If the input keys are integers, then simply returning key mod H_SIZE is generally a reasonable strategy.
- For example, student ID mod 10000 would be a reasonable strategy.
- It is usually a good idea to ensure that the table size is **prime**.
- When the input keys are random integers, then this function is **simple** to compute and also distributes the keys **evenly**.



# A Simple Hash Function

```
INDEX

hash( char *key, unsigned int H_SIZE )

{

unsigned int hash_val = 0;


/*1*/       while( *key != '\0' )

/*2*/            hash_val += *key++;


/*3*/       return( hash_val % H_SIZE );

}
```



# Another Hash Function

```
INDEX
hash( char *key, unsigned int H_SIZE )

{

return ( ( key[0] + 27*key[1] + 729*key[2] ) %
H_SIZE );

}
```



# Notes

- Assuming key has at least two characters plus the NULL terminator.
  - 27 represents the number of letters in the English alphabet, plus the blank.
  - 729 is $27^2$.
- This function only examines the first three characters, but if these are random, and the table size is 10,007, as before, then we would expect a reasonably equitable distribution.



# Quick Analysis

- Unfortunately, English is not random.
- Although there are 26x26x26 = **17,576** possible combinations of three characters (ignoring blanks), a check of a reasonably large on-line dictionary reveals that the number of different combinations is actually only **2,851**.
- Even if none of these combinations collide, only 28% of the table can actually be hashed to.



# A Good Hash Function

```
INDEX

hash( char *key, unsigned int H_SIZE )

{

unsigned int hash_val = 0;

/*1*/       while( *key != '\0' )

/*2*/            hash_val = ( hash_val << 5 ) + *key++;


/*3*/       return( hash_val % H_SIZE );

}
```



# Notes

- This hash function involves all characters in the key .
- It computes $\sum_{i=0}^{Key\_Size-1} Key[Key\_Size - i] \cdot 32^i$
- The code computes a polynomial function (of 32) by use of Horner's rule.
- For instance, another way of computing $h_k = k_1 + 27\ k_2 + 272\ k_3$ is by the formula $h_k = ((k_3) * 27 + k_2) * 27 + k_1$.
- Horner's rule extends this to an nth degree polynomial.



# Notes

- It is common to not use all the characters.
- The length and properties of the keys would influence the choice.
- The hash function might include a couple of characters from each field.



# Truncation

- Ignore part of the key, and use the remaining part directly as the index (considering non-numeric fields as their numerical codes).
  - Example: If the keys are eight-digit integers and the hash table has 1000 locations, then the first, second, and fifth digits from the right make the hash function, so that 62538194 maps to 394.
- Truncation is a very fast method, but it often fails to distribute the keys evenly through the table.



# Folding

- Partition the key into several parts and combine the parts in a convenient way (often using addition or multiplication) to obtain the index.
- For example, 62538194 maps to 625+381+94 = 1100, which is then truncated to 100.



# Modular Arithmetic

- Convert the key to an integer (using the preceding devices as desired), divide by the size of the index range, and take the reminder as the result.
- For example, 'abcd' = 64+65+66+67 mod 100 = 62.



# Collision Resolution

- Open Hashing (Separate Chaining)
- Closed Hashing (Open Addressing)
  - Linear probing
  - Quadratic probing
  - Double hashing
- Rehashing



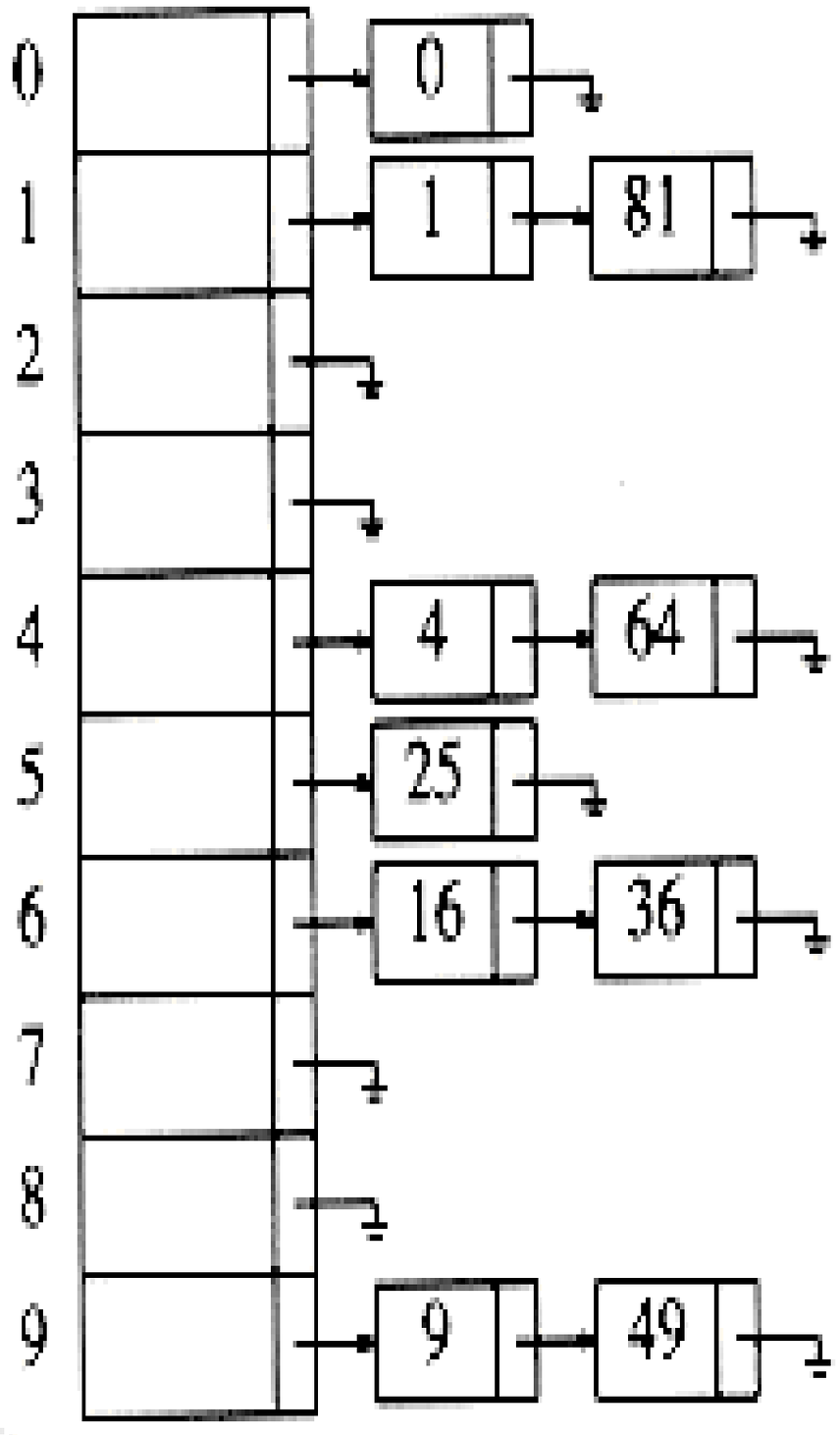
# Open Hashing

- The first strategy, commonly known as either **open hashing**, or **separate chaining**, is to keep a **list** of all elements that hash to the same value.

- We assume for this section that the keys are the first 10 perfect squares and that the hashing function is simply hash($x$) = $x$ mod 10. (The table size is not prime, but is used here for simplicity.)



# Open Hashing Example



# Find in Open Hashing

- Find
  - We use the hash function to determine which list to traverse.
  - We then traverse this list in the normal manner, returning the position where the item is found.



# Insert in Open Hashing

- Insert
- we traverse down the appropriate list to check whether the element is already in place.
- If the element turns out to be new, it is inserted either at the front of the list or at the end of the list.
- Sometimes new elements are inserted at the **front** of the list, since it is convenient and also because frequently it happens that recently inserted elements are the most likely to be accessed in the near future.



# Deletion in Open Hashing

- Deletion
  - The deletion routine is a straightforward implementation of deletion in a linked list.
  - First perform a FIND operation and then perform a delete operation of an item in a linked list.



# Advantages of Linked Storage

- Considerable space may be saved.
- It allows simple and efficient collision handling.
- It is no longer necessary that the size of the hash table exceed the number of records.
- Deletion becomes a quick and easy task in a chained hash table.



# Disadvantage of Linked Storage

- All the links require space.
- If the records are small this space usage is large when compared with the records.



# Closed Hashing (Open Addressing)

- Open hashing has the disadvantage of requiring pointers.
- This tends to slow the algorithm:
  - The time required to allocate new cells.
  - It requires the implementation of a second data structure.
- In a **closed hashing** system, if a collision occurs, **alternate cells** are tried until an empty cell is found.



# Closed Hashing

- For example, cells $h_0(x), h_1(x), h_2(x), \ldots$ are tried in succession where $h_i(x)$ = (hash($x$) + $f(i)$) mod H_SIZE, with $f(0) = 0$.
- The function, f, is the **collision resolution** strategy.
- Because all the data goes inside the table, a bigger table is needed for closed hashing than for open hashing.
- Generally, the load factor should be below l = 0.5 for closed hashing.



# Insertion Operation Outline

- An array must be declared that will hold the hash table.
- Initializing all locations in the array to show that they are empty.
- To insert a record into the hash table, the hash function for the key is first calculated.
- If the corresponding location is **empty**, then the record can be inserted, or else
- if the keys are **equal**, then insertion of the new record would not be allowed. In this case, it becomes necessary to resolve the collision.



# Find Operation Outline

- To retrieve the record with a given key is entirely similar. The hash function for the key is computed.
- If the desired record is in the corresponding location, then the retrieval has succeeded;
- otherwise,
- while the location is **nonempty** and not all locations have been examined, follow the same steps used for collision resolution.
- If an **empty** position is found, or $h_0$ have been considered, then no record with the given key is in the table, and the search in unsuccessful.



# Linear Probing

- The simplest method to resolve a collision is to start with the hash address (the location where the collision occurred) and do a **sequential search** for the desired key or an empty location.
- The problem with the above method is that the data become **clustered**:
- Records start to appear in long strings of adjacent positions with gaps between the strings.



# Example

| | Empty Table | After 89 | After 18 | After 49 | After 58 | After 69 |
|---|---|---|---|---|---|---|
| 0 | | | | 49 | 49 | 49 |
| 1 | | | | | 58 | 58 |
| 2 | | | | | | 69 |
| 3 | | | | | | |
| 4 | | | | | | |
| 5 | | | | | | |
| 6 | | | | | | |
| 7 | | | | | | |
| 8 | | | 18 | 18 | 18 | 18 |
| 9 | | 89 | 89 | 89 | 89 | 89 |



# Problems

- The time to search for an empty cell may be long.
- The problem of **primary clustering** is essentially one of **instability**.
- If a few keys happen randomly to be near each other, then it becomes more and more likely that other keys will join in the cluster.
- Furthermore, the distribution will become progressively more unbalanced.



# Analysis

- It can be shown that the expected number of probes using linear probing is roughly
  - Insertions and unsuccessful searches

$$\frac{1}{2}(1+1/(1-\lambda)^2)$$

- Successful searched

$$\frac{1}{2}(1+1/(1-\lambda))$$

- $\lambda$, of a hash table is the ratio of the number of elements in the hash table to the table size.

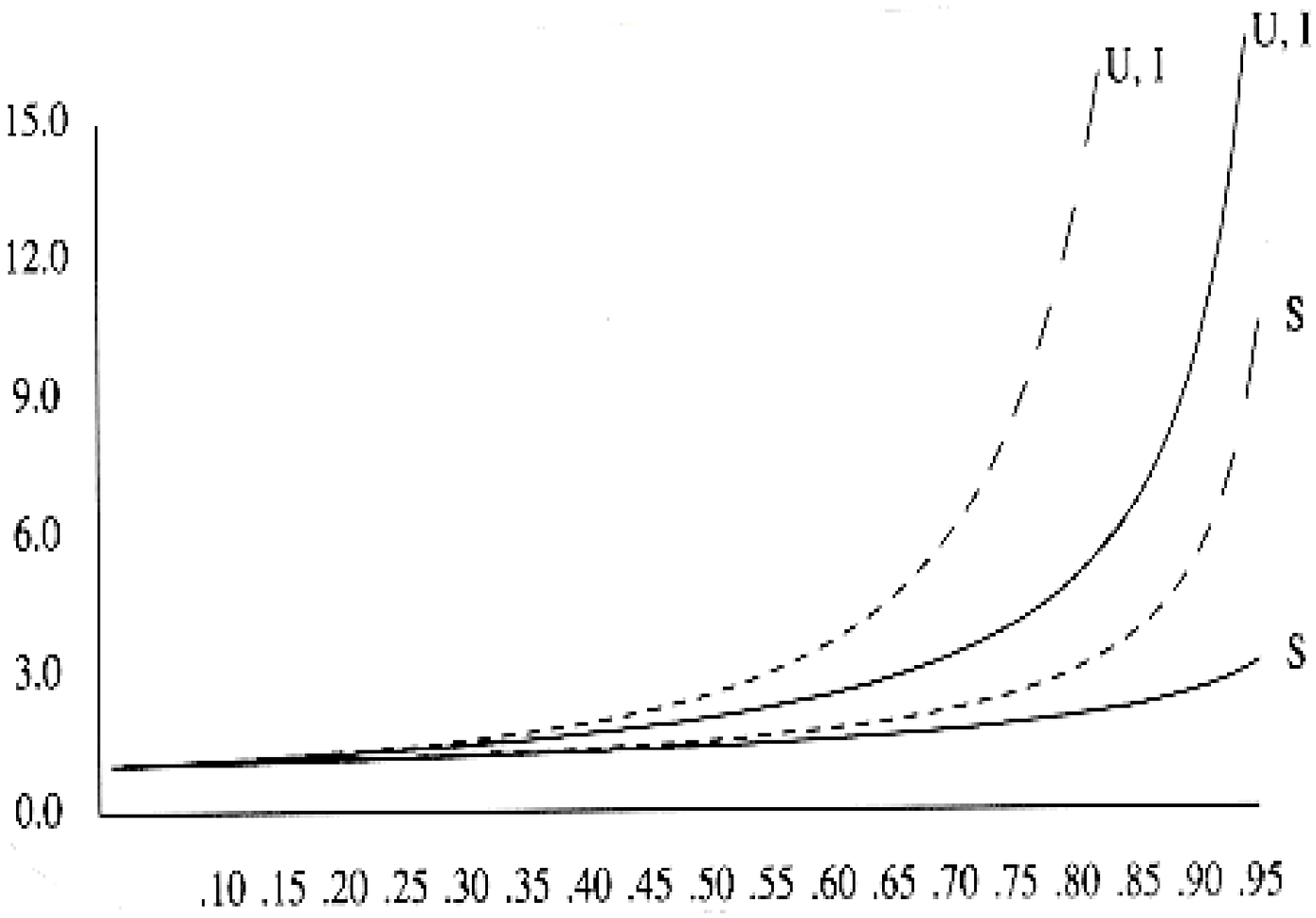


# Analysis

- We assume a very large table and that each probe is independent of the previous probes.
- The expected number of probes in an unsuccessful search.
    - The number of probes for a successful search = the number of probes required when the particular element was inserted.
    - When an element is inserted, it is done as a result of an unsuccessful search.
    - We can use the cost of an unsuccessful search to compute the average cost of a successful search.
    - Since the fraction of empty cells is $1 - \lambda$, the number of cells we expect to probe is $1/(1- \lambda)$.



# Probes vs. Load Factor




- **Dashed curves**-- linear probing
- **Solid curves**- random collision resolution
- **S**-successful
- **U**-unsuccessful
- **I**-insert

- What it is saying is that the linear probing is not a very good method to handle collision.



# Notes

- If $\lambda = 0.75$, then the formula above indicates that 8.5 probes are expected for an insertion in linear probing.
- If $\lambda = 0.9$, then 50 probes are expected.
- This compares with 4 and 10 probes for the respective load factors if clustering were not a problem.
- We see from these formulas that linear probing can be a bad idea if the table is expected to be more than half full.
- If $\lambda = 0.5$, however, only 2.5 probes for insertion and only 1.5 probes are required for a successful search.



# Quadratic Probing

- Quadratic probing avoid the primary clustering problem of linear probing.
- If there is a collision at hash address H, the method call **quadratic probing** looks in the table at locations $h$+1, $h$+4, $h$+9, …, that is, at locations $h + i^2$ (mod hashsize) for $i$=1, 2, ....
- This reduces clustering, but it is not obvious that it will probe all locations in the table, and in fact it does not.



# Observation

- Theorem-- If quadratic probing is used and the table size is prime, then a new element can always be inserted if the table is at least half empty. (see book for more details)



# Example

| | Empty Table | After 89 | After 18 | After 49 | After 58 | After 69 |
|---|---|---|---|---|---|---|
| 0 | | | | 49 | 49 | 49 |
| 1 | | | | | | |
| 2 | | | | | 58 | 58 |
| 3 | | | | | | 69 |
| 4 | | | | | | |
| 5 | | | | | | |
| 6 | | | | | | |
| 7 | | | | | | |
| 8 | | | 18 | 18 | 18 | 18 |
| 9 | | 89 | 89 | 89 | 89 | 89 |



# Notes

- If the table is even one more than half full, the insertion could fail (although this is extremely unlikely).
- It is also crucial that the table size be prime.
- If the table size is not prime, the number of alternate locations can be severely reduced.
- Standard deletion cannot be performed in a closed hash table, because the cell might have caused a collision to go past it.
- Closed hash tables require lazy deletion.



# About Lazy Deletion

- Deletion in a hash table is not an easy task. One method to delete an entry is to invent another **special key**, to be placed in any deleted position.
- This special key would indicate that this position is free to receive an insertion when desired but that is **should not** be used to terminate the search for some other item in the table.



# Key-Dependent Increments

- Rather than having the increment depend on the number of probes already made, we can let it be some simple function of the key itself.
- For example, we could truncate the key to a single character and use its code as the increment.



# Random Probing

- Use a pseudorandom number generator to obtain the increment.
- The generator used should be one that always generates the same sequence provided it starts with the same seed.
- This method is excellent in avoiding clustering, but is likely to be slower than the others.



# Double Hashing

- For double hashing, one popular choice is $f(i) = i * h_2(x)$.
- We apply a second hash function to $x$ and probe at a distance $h_2(x)$, 2 $h_2(x)$, . . ., and so on.
- A poor choice of $h_2(x)$ would be disastrous.
- The function must never evaluate to zero.
- Make sure all cells can be probed.



# Double Hashing

- For instance, the obvious choice $h_2(x) = x \bmod 9$ would not help if 99 were inserted into the input in the previous examples.
- A function such as $h_2(x) = R - (x \bmod R)$, with R a prime smaller than H_SIZE, will work well.
- One may continue to perform triple hashing, and so on.



# Rehashing

- When the table gets too full, the running time for the operations will deteriorate, specially when there are too many removals intermixed with insertions.
- Solution
  - Build another table that is about twice as big (with associated new hash function).
  - Scan down the entire original hash table, computing the new hash value for each element and inserting it in the new table.



# Example

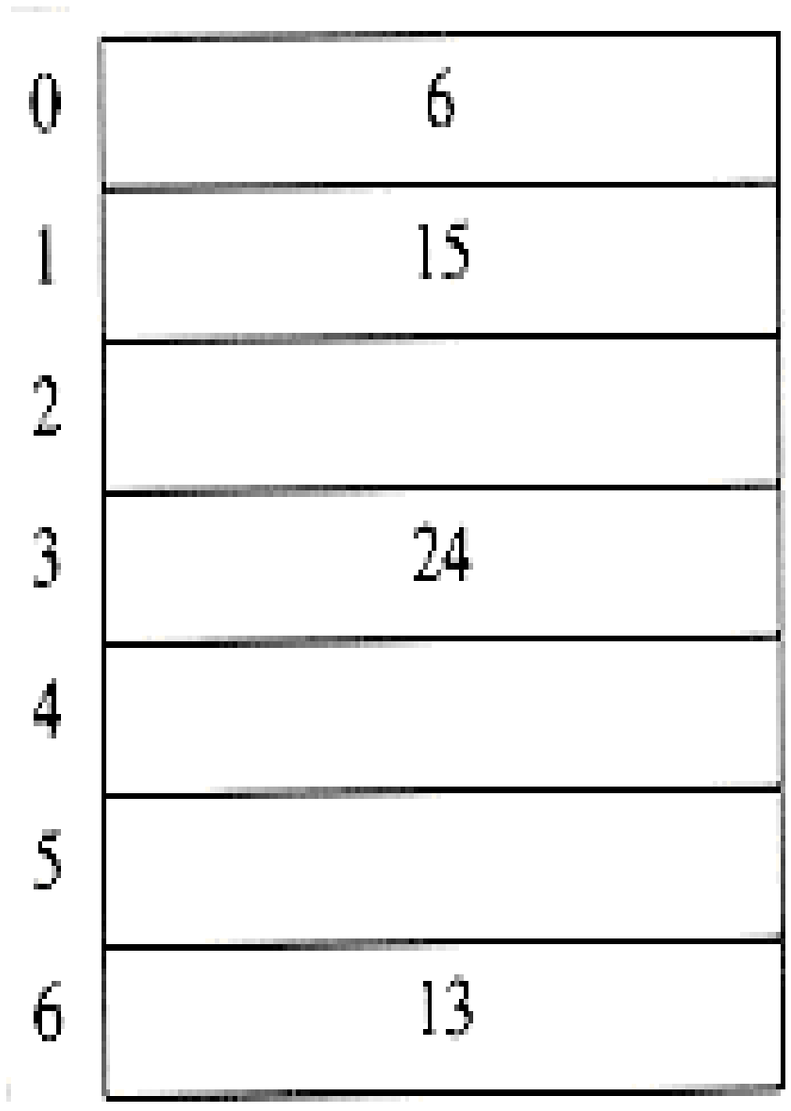
$h_2(x) = R - (x \bmod R)$, R = 7

| | Empty Table | After 89 | After 18 | After 49 | After 58 | After 69 |
|---|---|---|---|---|---|---|
| 0 | | | | | | 69 |
| 1 | | | | | | |
| 2 | | | | | | |
| 3 | | | | | 58 | 58 |
| 4 | | | | | | |
| 5 | | | | | | |
| 6 | | | | 49 | 49 | 49 |
| 7 | | | | | | |
| 8 | | | 18 | 18 | 18 | 18 |
| 9 | | 89 | 89 | 89 | 89 | 89 |



# Example

- Table size = 7
- Insert 13, 15, 24, and 6.
- $h(x) = x \bmod 7$.



# Closed Hash Table, Insert 23

| | |
|---|---|
| 0 | 6 |
| 1 | 15 |
| 2 | 23 |
| 3 | 24 |
| 4 | |
| 5 | |
| 6 | 13 |

- After 23 is inserted, the resulting table will be over 70% full.
- A new table is created with size = 17 since this is the first prime that is twice as large as the old table size.
- The new hash function is then $h(x) = x \bmod 17$.



# After Rehashing

The old table is scanned, and elements 6, 15, 23, 24, and 13 are inserted into the new table.

- The running time is O(n).
- It is expensive.
- It should not be done so frequently.

| | |
|---|---|
| 0 | |
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| 6 | 6 |
| 7 | 23 |
| 8 | 24 |
| 9 | |
| 10 | |
| 11 | |
| 12 | |
| 13 | 13 |
| 14 | |
| 15 | 15 |
| 16 | |



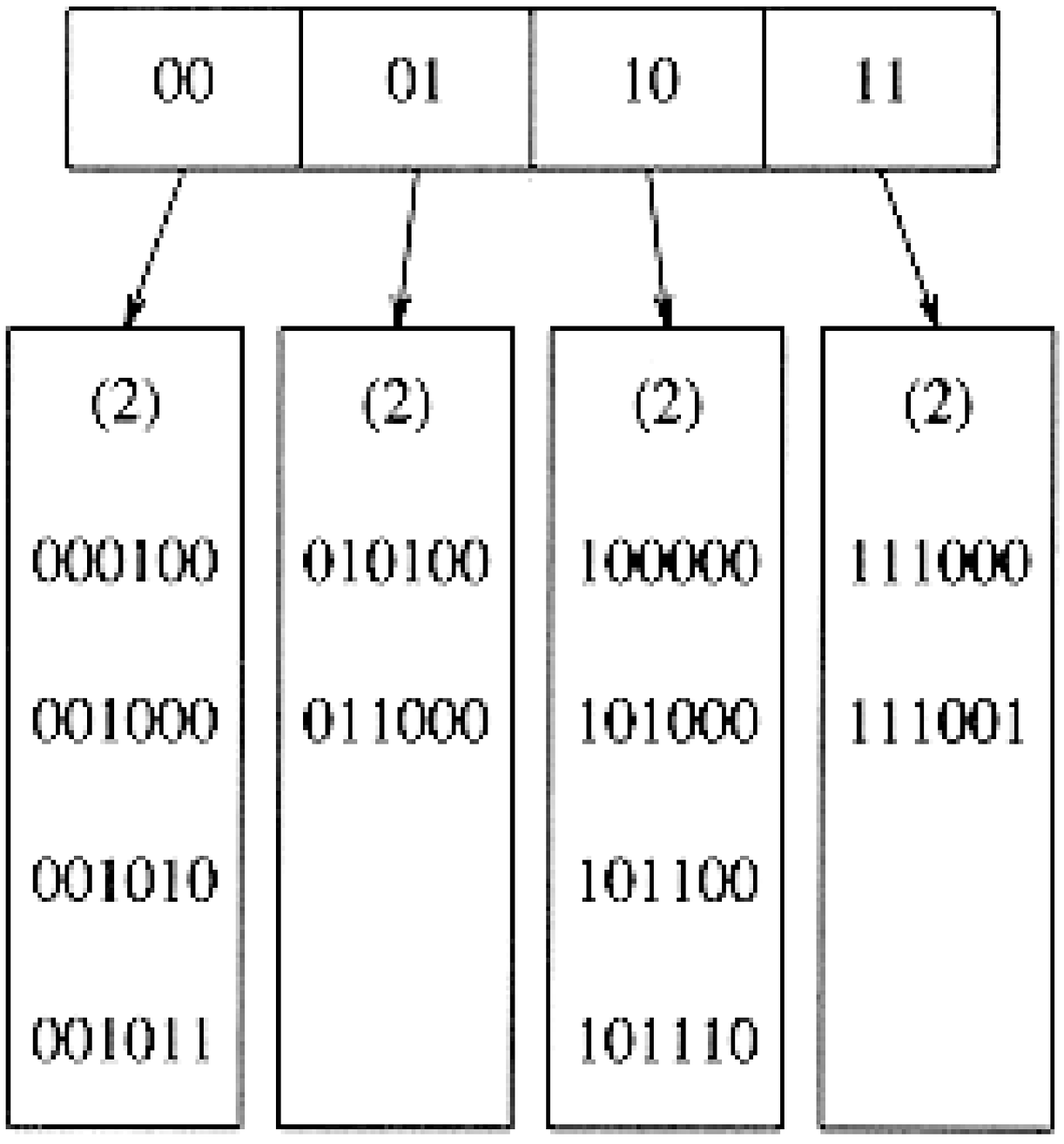
# Extendible Hashing

- What happens when the amount of data is too large to fit in main memory and must be stored on the disk?
- How can we minimize the disk access?

- Suppose that our data consists of several 6 bit integers.
- The root of the tree contains 4 pointers determined by the leading two bits of the data.



# Example



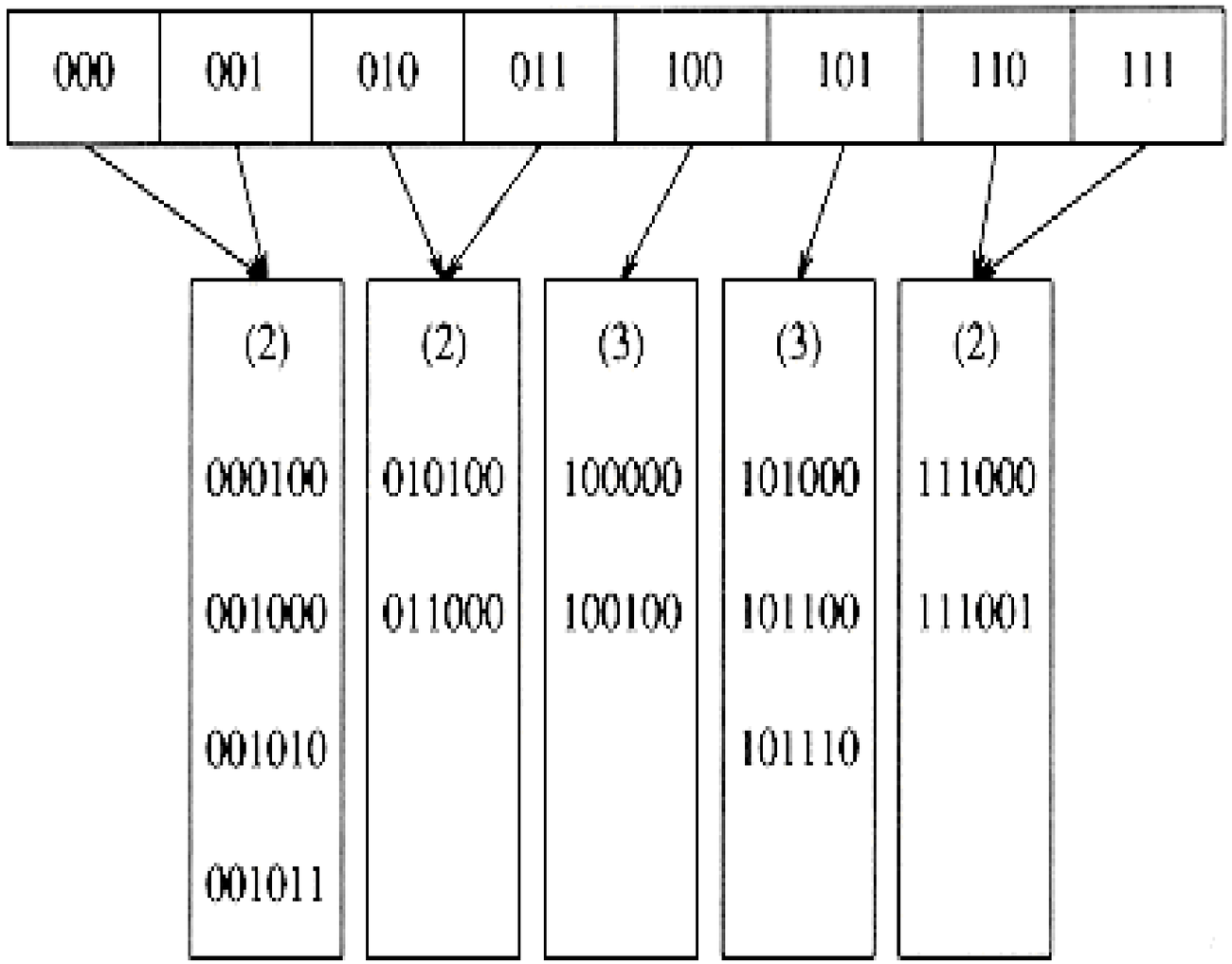
# Extendible Hashing

- Each leaf has up to $m = 4$ elements.
- D represents the number of bits used by the root, which is sometimes known as the **directory**.
- The number of entries in the directory is thus 2D *dl* (the number of leading bits that all the elements of some leaf l have in common.
- *dl* will depend on the particular leave.

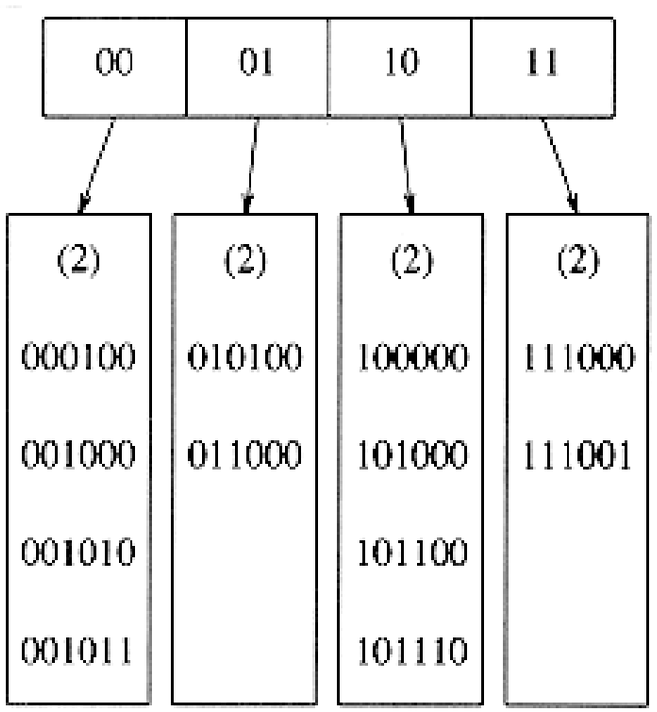


# Example, insert 100100

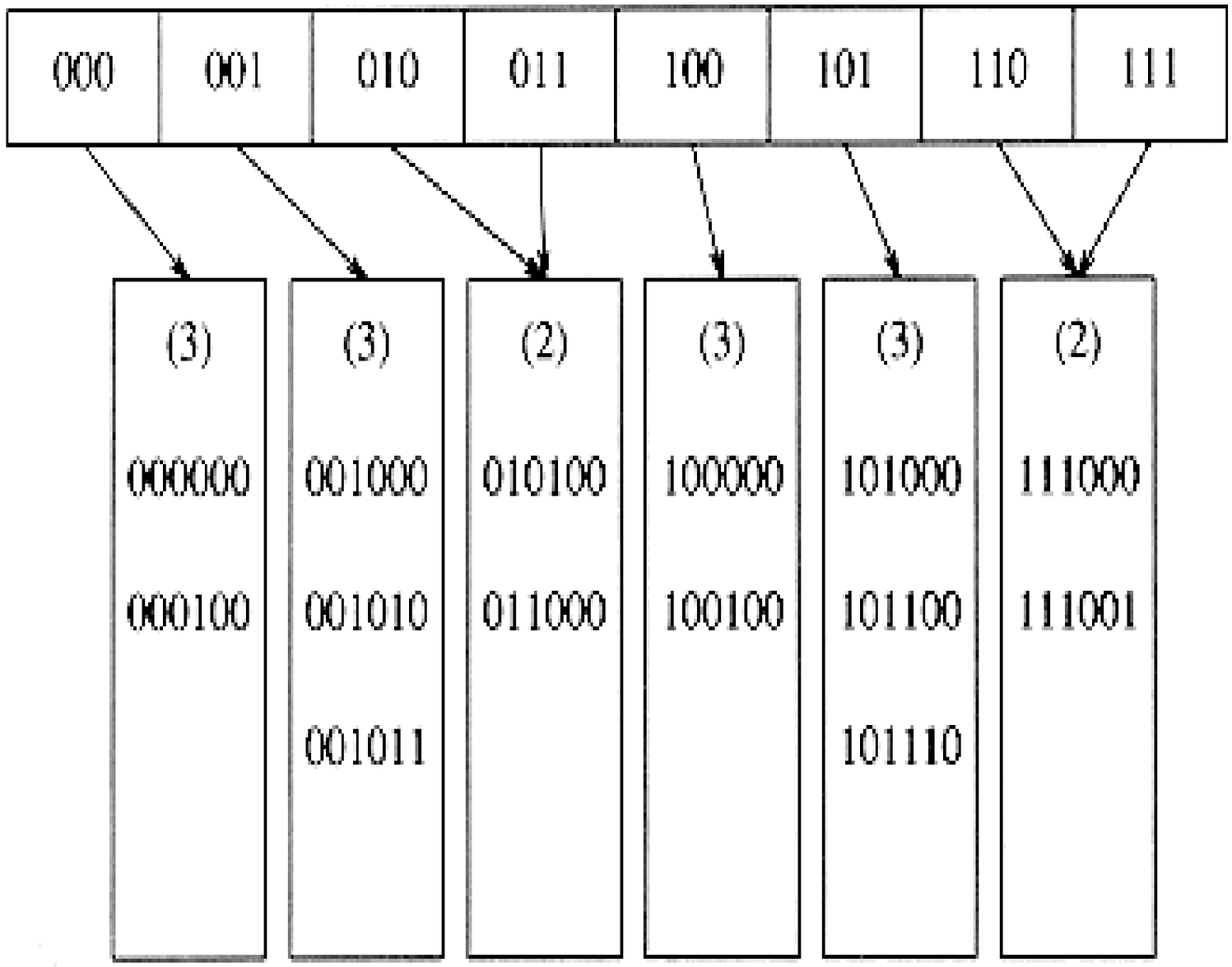
This will go into the third leaf and cause a split.




Now, the leaves are now determined by the first 3 bits.



# Example, insert 000000



# Notes

- It is possible that several directory splits will be required if the elements in a leaf agree in more than D+1 leading bits.
  - For example, 111010, 111011, and 111100 are inserted, the directory size must be increased to 4.
- The possibility of duplicate keys. This algorithm does not work when there are more than m duplicates.
- It is important for the bits to be fairly random.



# Summary

- Hash tables can be used to implement the **insert** and find operations in constant average time.
- It is especially important to pay attention to details such as load factor when using hash tables.
- It is also important to choose the hash function carefully when the key is not a short string or integer.



# Summary

- For open hashing, the load factor should be close to 1.
- For closed hashing, the load factor should not exceed 0.5, unless this is completely unavoidable.
- Using a hash table, it is not possible to find the minimum element.
- It is not possible to search efficiently for a string unless the exact string is known.



# Summary

- Compilers use hash tables to keep track of declared variables in source code. The data structure is known as a symbol table.
- A hash table is useful for any graph theory problem where the nodes have real names instead of numbers.
- A third common use of hash tables is in programs that play games.
- Another use of hashing is in online spelling checkers.