Exercises: Vector Spaces

Problem 1. Let V be the set of vectors $[2x - 3y, x + 2y, -y, 4x]$ with $x, y \in \mathbb{R}^2$. Addition and scalar multiplication are defined in the same way as on vectors. Prove that V is a vector space. Also, point out a basis of it.

Solution. To show a set V is a vector space, we need to prove all the following properties: (i) (for addition) closeness, addition commutativity, addition associativity, zero element, existence of opposites; (ii) (for scalar multiplication) closeness, distributivity on elements, distributivity on scalars, scalar associativity, product with 1, and product with 0.

1. Addition closeness: For any $\boldsymbol{v_1}, \boldsymbol{v_2} \in V$, we need to show that $\boldsymbol{v_1} + \boldsymbol{v_2}$ is also in V. Suppose that $\boldsymbol{v_1} = [2x_1 - 3y_1, x_1 + 2y_1, -y_1, 4x_1]$ and $\boldsymbol{v_2} = [2x_2 - 3y_2, x_2 + 2y_2, -y_2, 4x_2]$. Then:

$$\boldsymbol{v_1} + \boldsymbol{v_2} = [2(x_1 + x_2) - 3(y_1 + y_2), (x_1 + x_2) + 2(y_1 + y_2), -(y_1 + y_2), 4(x_1 + x_2)]$$

   In other words, $\boldsymbol{v_1} + \boldsymbol{v_2}$ is produced by real values $x = x_1 + x_2$ and $y = y_1 + y_2$. Hence, $\boldsymbol{v_1} + \boldsymbol{v_2} \in V$.

2. We skip the proofs of addition commutativity and addition associativity because they are trivial.

3. Zero element: $[0, 0, 0, 0]$.

4. Existence of opposites: The opposite of vector $\boldsymbol{v_1} \in V$ given by $x = x_1, y = y_1$ is the vector $\boldsymbol{v_2} \in V$ given by $x = -x_1, y = -y_1$.

5. Scalar multiplication closeness: For any $\boldsymbol{v_1} \in V$, we need to show that $c\boldsymbol{v_1}$ is also in V for any $c \in \mathbb{R}$. Suppose that $\boldsymbol{v_1} = [2x_1 - 3y_1, x_1 + 2y_1, -y_1, 4x_1]$. Then:

$$c\boldsymbol{v_1} = [2(cx_1) - 3(cy_1), (cx_1) + 2(cy_1), -(cy_1), 4(cx_1)]$$

   In other words, $c\boldsymbol{v_1}$ is produced by real values $x = cx_1$ and $y = cy_1$. Hence, $c\boldsymbol{v_1} \in V$.

6. We skip the proofs of distributivity on elements, distributivity on scalars, scalar associativity, product with 1, and product with 0 because they are trivial.

The dimension of V is 2. Here is a basis: $[2, 1, 0, 4]$ (given by $x = 1, y = 0$) and $[-3, 2, -1, 0]$ (given by $x = 0, y = 1$).

Remark. It can be quite tedious to prove that a set is a vector space. In the final exam, you are required to explicitly prove 3 only properties: addition closeness, scalar multiplication closeness, and zero element; and you are allowed to omit the proof of the other properties.

Problem 2. For each of the following sets, indicate whether it is a vector space. If so, point out a basis of it; otherwise, point out which vector-space property is violated.

1. The set V of vectors $[2x, x^2]$ with $x \in \mathbb{R}^2$. Addition and scalar multiplication are defined in the same way as on vectors.

2. The set V of vectors $[x, y, z] \in \mathbb{R}^3$ satisfying $x + y + z = 3$ and $x - y + 2z = 6$. Addition and scalar multiplication are defined in the same way as on vectors.

3. The set V of symmetric 2×2 matrices. Addition and scalar multiplication are defined in the same way as on matrices.

4. The set V of 2×2 matrices $[a_{ij}]$ with $a_{11} + a_{22} = 0$. Addition and scalar multiplication are defined in the same way as on matrices.

Solution.

1. No. Addition is not closed on V. For example, $[2, 1]$ and $[4, 4]$ are in V, but $[6, 5]$ is not.

2. No. V is essentially the set of vectors $[-3t, t, 3 + 2t]$ with $t \in \mathbb{R}$. This set is not closed under scalar multiplication. For example, $t = 1$ gives $[-3, 1, 5]$, but $2[-3, 1, 5] = [-6, 2, 10]$ is not in V.

3. Yes. A basis: $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$.

4. Yes. A basis: $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$.

Problem 3. Determine if the following transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ has a reverse transformation. If so, give the reverse transformation.

$$\begin{aligned} y_1 &= 3x_1 + 2x_2 \\ y_2 &= 4x_1 + x_2 \end{aligned}$$

Solution. The transformation can be written as:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The coefficient matrix $\boldsymbol{A} = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$ has rank 2. Hence, the transformation has a reverse transformation. From $\boldsymbol{A}^{-1} = \begin{bmatrix} -1/5 & 2/5 \\ 4/5 & -3/5 \end{bmatrix}$, we obtain the reverse transformation as:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1/5 & 2/5 \\ 4/5 & -3/5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Problem 4. Determine if the following transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ has a reverse transformation. If so, give the reverse transformation.

$$\begin{aligned} y_1 &= 3x_1 + 2x_2 + x_3 \\ y_2 &= x_1 + x_2 - x_3 \\ y_3 &= 5x_1 + 4x_2 - x_3 \end{aligned}$$

Solution. The transformation can be written as:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & -1 \\ 5 & 4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The coefficient matrix $\boldsymbol{A} = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & -1 \\ 5 & 4 & -1 \end{bmatrix}$ has rank 2, which is smaller than the number 3 of coordinates. Hence, the transformation has no reverse transformation.

Problem 5. Consider the following linear system about $\boldsymbol{x}$

$$\boldsymbol{A}\boldsymbol{x} \quad = \quad \boldsymbol{0}$$

where $\boldsymbol{A}$ is an $m \times n$ coefficient matrix, and $\boldsymbol{x}$ an $n \times 1$ matrix. Let V be the set of all such $\boldsymbol{x}$ satisfying the system. Suppose that the rank of $\boldsymbol{A}$ is r. Prove that V is a vector space of dimension $n - r$ (addition and scalar multiplication are defined in the same way as on vectors).

Solution. First, we need to prove that V is a vector space:

- Addition closeness: Let $\boldsymbol{x_1}, \boldsymbol{x_2} \in V$, namely, $\boldsymbol{A}\boldsymbol{x_1} = \boldsymbol{A}\boldsymbol{x_2} = \boldsymbol{0}$. This means that $\boldsymbol{A}(\boldsymbol{x_1}+\boldsymbol{x_2}) = \boldsymbol{0}$, namely, $\boldsymbol{x_1} + \boldsymbol{x_2}$ is also in V.
- Scalar multiplication closeness: Let $\boldsymbol{x} \in V$, namely, $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$. This means that, for any real value c, $c\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$, namely, $c\boldsymbol{x}$ is also in V.
- Zero element: the $n \times 1$ zero-matrix.
- We omit the proofs of the other vector space properties.

Next, we prove that the dimension of V is $n - r$. Let $\boldsymbol{B}$ be a row echelon form of $\boldsymbol{A}$. We know that $\boldsymbol{B}$ has exactly r non-zero rows. Now, fix $(x_{r+1}, x_{r+2}, ..., x_n)$ to an arbitrary point in $\mathbb{R}^{n-r}$. Then, those r non-zero rows of $\boldsymbol{B}$ give a linear system with respect to $x_1, x_2, ..., x_r$. This linear system has a unique solution. In other words, V is the set of all outputs of a vector function $\boldsymbol{f}(x_{r+1}, x_{r+2}, ..., x_n)$ where (i) each output of $\boldsymbol{f}$ is an n-dimensional vector $\boldsymbol{v}$, and (ii) each component of $\boldsymbol{v}$ is a linear combination of $x_{r+1}, x_{r+2}, ..., x_n$, and (iii) $(x_{r+1}, x_{r+2}, ..., x_n)$ can be any point in $\mathbb{R}^{n-r}$. It thus follows that the dimension of V is $n - r$. $\square$