

1. Suppose $P(A) = 3/4$ and $P(B) = 1/3$. Show that $1/12 \leq P(A \cap B) \leq 1/3$.

Solution: By the difference rule, $P(A \cap B) \leq P(B) \leq \frac{1}{3}$. By inclusion-exclusion, we have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{3}{4} + \frac{1}{3} - P(A \cap B) = \frac{13}{12} - P(A \cap B).$$

Because $P(A \cup B) \leq 1$, we have $\frac{13}{12} - P(A \cap B) \leq 1$, so $P(A \cap B) \geq \frac{13}{12} - 1 = \frac{1}{12}$.

2. A six-sided die is loaded in a way that each even face is twice as likely as each odd face. All even faces are equally likely, as are all odd faces. What is the probability that the outcome is less than 4? (*Textbook problem 1.6*)

Solution: The sample space consists of the face values $\{1, 2, 3, 4, 5, 6\}$. If outcomes 1, 3, and 5 are each assigned probability p then outcomes 2, 4, and 6 must each be assigned probability $2p$. Since the probabilities of all outcomes must add up to one, $3 \cdot p + 3 \cdot 2p$ must equal 1, so $p = 1/9$. Therefore the desired probability is

$$P(\{1, 2, 3, 4\}) = P(\{1\}) + P(\{2\}) + P(\{3\}) + P(\{4\}) = p + 2p + p + 2p = 6p = \frac{2}{3}.$$

3. A bin contains 50 black balls and 50 white balls. You draw two balls without replacement. What is the probability that at least one of them is white?

Solution: The sample space Ω consists of all $\binom{100}{50}$ arrangements of the balls. We assume equally likely outcomes. The event A consists of those arrangements in which a white ball appears among the first two positions. The complementary event A^c consists of those arrangements that start with two black balls, so A^c has size $\binom{98}{50}$ and

$$P(A) = 1 - P(A^c) = 1 - \frac{|A^c|}{|\Omega|} = 1 - \frac{\binom{98}{50}}{\binom{100}{50}} = 1 - \frac{49 \cdot 50}{99 \cdot 100} \approx 0.7525.$$

4. A six-sided die is rolled three times. Which is more likely: A sum of 11 or a sum of 12? (*Textbook problem 1.49*)

Solution: Let A and B be the events of a sum of 11 and a sum of 12, respectively. As the outcomes are equally likely, the probabilities of the two sums are $|A|/6^3$ and $|B|/6^3$ so we need to determine which of the sets A and B is bigger. The set A can be partitioned into A_1 up to A_6 depending on the first die roll. Similarly, B can be partitioned into B_1 up to B_6 . Now A_1 has the same size as B_2 as they both share the same pairs of values for the second and third dice. By the same argument, $|A_2| = |B_3|$, $|A_3| = |B_4|$, $|A_4| = |B_5|$, and $|A_5| = |B_6|$. Comparing $|A|$ and $|B|$ therefore amounts to comparing $|A_6|$ and $|B_1|$. For an outcome to be in A_6 the remaining two rolls must add up to 5, while for B_1 they must add up to 11. Therefore $|A_6| = 4$ and $|B_1| = 2$, so A is the larger set and a sum of 11 is more likely.

5. A bin contains 3 white balls and 5 black balls. Alice and Bob take turns drawing balls from the bin without replacement until a white ball is drawn. Assuming Alice goes first, what is the probability that she gets the white ball?

Solution: Our sample space will consist of all $\binom{8}{3}$ arrangements of 3 white balls and 5 black balls. The first ball in the arrangement is drawn by Alice, the second by Bob (if necessary), the third by Alice (if necessary), and so on. We assume equally likely outcomes.

Let E_i be the event that Alice selects the white ball in the i -th turn, and therefore all balls drawn in previous turns are black. This event consists of all arrangements that start with $i - 1$ blacks followed by a white. We are interested in the probability of the event E that Alice selects the white ball at *some* turn, that is:

$$E = E_1 \cup E_3 \cup E_5.$$

(E_7 and so on are empty because there are not enough black balls.) The three events are pairwise disjoint so

$$|E| = |E_1| + |E_3| + |E_5|.$$

Since the arrangement of the first i balls is fixed and contains exactly one white, the size of E_i is the number of arrangements of the remaining $8 - i$ balls out of which two are white. Therefore $|E_i| = \binom{8-i}{2}$, and by the equally likely outcomes formula

$$P(E) = \frac{\binom{7}{2} + \binom{5}{2} + \binom{3}{2}}{\binom{8}{3}} \approx 0.607.$$