Exercises: Vector Derivative

Problem 1. Let $f(x) = 3x + 5$. Clearly, $\lim_{x \to 2} f(x) = 11$. Answer the following questions:

1. Set $\delta = 1$. By definition of limit, we know that we can find an $\epsilon > 0$, such that for any x satisfying $|x - 2| < \epsilon$, it holds that $|f(x) - 11| < \delta$. Give such an ϵ.
2. Repeat the above for $\delta = 0.001$.
3. Repeat the above for $\delta = 0.000001$.

Solutions

1. $|f(x) - 11| < \delta = 1$ means $-1 < f(x) - 11 < 1$, and hence $10 < 3x + 5 < 12$, namely, $5/3 < x < 7/3$. Therefore, $\epsilon = 1/3$ suffices.
2. $|f(x) - 11| <= 0.001$ means $-0.001 < f(x) - 11 < 0.001$, and hence $10.999 < 3x + 5 < 11.001$, namely, $5.999/3 < x < 6.001/3$. Therefore, $\epsilon = 0.001/3$ suffices.
3. $\epsilon = 0.000001/3$ suffices.

Problem 2. Solve the following limits:

1. $\lim_{t \to 3} \boldsymbol{f}(t)$, where $\boldsymbol{f}(t) = [5t + 3, \frac{\sin(t-3)}{t-3}]$.
2. $\lim_{t \to 0} \boldsymbol{f}(t)$, where $\boldsymbol{f}(t) = [5t^2 + 3t, t^2, \frac{e^t - 1}{t}]$.
3. $\lim_{t \to 0} \boldsymbol{f}(t)$, where

$$\boldsymbol{f}(t) = \begin{cases} [5t^2 + 3t, t^2, \frac{e^t-1}{t}] & \text{if } t \neq 0 \\ [10, 10, 10] & \text{otherwise} \end{cases}$$

Solutions

1. Since $\lim_{t \to 3}(5t + 3) = 18$ and $\lim_{t \to 3} \frac{\sin(t-3)}{t-3} = 1$, we know that $\lim_{t \to 3} \boldsymbol{f}(t) = [18, 1]$.
2. $\lim_{t \to 0} \boldsymbol{f}(t) = [0, 0, 1]$.
3. $\lim_{t \to 0} \boldsymbol{f}(t) = [0, 0, 1]$. Note that $f(0)$ is irrelevant to the limit.

Problem 3. Discuss the continuity of $\boldsymbol{f}(t)$ at $t = 0$.

1. $\boldsymbol{f}(t) = [5t^2 + 3t, t^2, \frac{e^t-1}{t}]$.
2. $\boldsymbol{f}(t) = [5t^2 + 3t, t^2, \frac{e^t-1}{t}]$ if $t \neq 0$; otherwise, $\boldsymbol{f}(t) = [10, 10, 10]$.
3. $\boldsymbol{f}(t) = [5t^2 + 3t, t^2, \frac{e^t-1}{t}]$ if $t \neq 0$; otherwise, $\boldsymbol{f}(t) = [0, 0, 1]$.

Solutions

1. No. The function is not defined at $t = 0$.

2. No because $\lim_{t\to 0} \boldsymbol{f}(t) = [0,0,1] \neq \boldsymbol{f}(0)$.

3. Yes because $\lim_{t\to 0} \boldsymbol{f}(t) = [0,0,1] = \boldsymbol{f}(0)$.

Problem 4. Suppose that $\boldsymbol{f}(t) = [\sin(t), \cos(t^3), 5t^2]$. Answer the following questions:

1. Give the function $\boldsymbol{f}'(t)$.

2. Give the function $\boldsymbol{f}''(t)$ (which is the derivative of $\boldsymbol{f}'(t)$).

3. Give the function $\boldsymbol{f}'''(1)$ (where $\boldsymbol{f}'''(t)$ is the derivative of $\boldsymbol{f}''(t)$).

Solutions

1. To compute $\boldsymbol{f}'(t)$, simply take the derivative of each component function of $\boldsymbol{f}(t)$. We thus obtain $\boldsymbol{f}'(t) = [\cos(t), -3t^2\sin(t^3), 10t]$.

2. To compute $\boldsymbol{f}''(t)$, simply take the derivative of each component function of $\boldsymbol{f}'(t)$. We thus obtain $\boldsymbol{f}''(t) = [-\sin(t), -6t\sin(t^3) - 9t^4\cos(t^3), 10]$.

3. To compute $\boldsymbol{f}'''(t)$, simply take the derivative of each component function of $\boldsymbol{f}''(t)$. Doing so and then plugging in $t = 1$ gives $\boldsymbol{f}'''(1) = [-\cos(1), -54\cos(1) + 21\sin(1), 0]$.

Problem 5. Suppose that $\boldsymbol{f}(t) = [t^2, \sin(t), 2t]$ and $\boldsymbol{g}(t) = 2t\boldsymbol{i} + \frac{1}{\sin(t)}\boldsymbol{j} + 3t^2\boldsymbol{k}$.

1. Give the function $h(t) = \boldsymbol{f}(t) \cdot \boldsymbol{g}(t)$.

2. Give the function $h'(t)$.

3. Give the function $\boldsymbol{f}'(t)$ and $\boldsymbol{g}'(t)$.

4. Verify that $h'(t) = \boldsymbol{f}'(t) \cdot \boldsymbol{g}(t) + \boldsymbol{g}'(t) \cdot \boldsymbol{f}(t)$.

Solutions

1. $h(t) = t^2 \cdot 2t + \sin(t)\frac{1}{\sin(t)} + 2t \cdot 3t^2 = 8t^3 + 1$.

2. $h'(t) = 24t^2$.

3. $\boldsymbol{f}'(t) = [2t, \cos(t), 2]$ and $\boldsymbol{g}'(t) = [2, -\frac{\cos(t)}{\sin^2(t)}, 6t]$.

4.
$$\begin{aligned}\boldsymbol{f}'(t) \cdot \boldsymbol{g}(t) + \boldsymbol{g}'(t) \cdot \boldsymbol{f}(t) &= 2t \cdot 2t + \frac{\cos(t)}{\sin(t)} + 2 \cdot 3t^2 + 2 \cdot t^2 - \sin(t)\frac{\cos(t)}{\sin^2(t)} + 2t \cdot 6t \\ &= 24t^2\end{aligned}$$

Problem 6. Suppose that $\boldsymbol{f}(t) = [t, t^2, 1]$ and $\boldsymbol{g}(t) = [1, t, t^2]$.

1. Give the function $\boldsymbol{h}(t) = \boldsymbol{f}(t) \times \boldsymbol{g}(t)$.

2. Give the function $\boldsymbol{h}'(t)$.

Solutions

1. $\boldsymbol{h}(t) = [x(t), y(t), z(t)]$ where

$$\begin{aligned} x(t) &= t^2 \cdot t^2 - 1 \cdot t = t^4 - t \\ y(t) &= 1 \cdot 1 - t \cdot t^2 = 1 - t^3 \\ z(t) &= t \cdot t - t^2 \cdot 1 = 0 \end{aligned}$$

2. $\boldsymbol{h}'(t) = [4t^3 - 1, -3t^2, 0]$.