Lecture Notes: Surface Integral by Area

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In this lecture, we follow the convention that if $f(x, y, z)$ is a scalar function in $\mathbb{R}^3$, then we use the notation $f(p)$ to denote the value of the function at the point $p = (x, y, z)$.

1 Definition of Surface Integral by Area

Now we define another form of surface integral:

Definition 1. *Let S be a smooth surface, and $h(x, y, z)$ be a scalar function in $\mathbb{R}^3$. Break S arbitrarily into a set of n surfaces $S_1, S_2, ..., S_n$. Define A_i $(1 \le i \le n)$ to be the area of S_i. Also let ρ_{max} be the largest diameter[1] of $S_1, S_2, ..., S_n$. For each $i \in [1, n]$, pick a point p_i arbitrarily within S_i. If the following limit exists:*

$$\lim_{\rho_{max} \to 0} \sum_{i=1}^{n} h(p_i) \cdot A_i, \tag{1}$$

then we define

$$\iint_S h(x, y, z)\, dA \tag{2}$$

to be the value of the limit.

Integral (2) is called *surface integral by area.* Note that S does not need to be oriented in the above definition (namely, no need to choose a side)—orientation is needed for surface integral by coordinate, but not for by area.

We state the next lemma without proof:

Lemma 1. *If $h(x, y, z)$ is continuous on S, then the limit* (1) *always exists.*

As usual, we can extend Definition 1 to piecewise smooth surfaces. Let S be a surface that can be divided into a finite number of smooth surfaces $S_1, S_2, ..., S_k$. Then, we define:

$$\iint_S h(x, y, z)\, dA \quad = \quad \sum_{i=1}^{k} \iint_{S_i} h(x, y, z)\, dA.$$

The following sections explain the evaluation of surface integrals of the above form.

[1] The diameter of S_i is the maximum distance between two points on S_i.

2 Evaluating Surface Integral by Area: Method 1

When S is xy-monotone, we can convert (2) into surface integral by coordinate. For this purpose, suppose that S is represented by equation $f(x, y, z) = 0$. Recall that $\nabla f = [\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}]$ is a normal vector of S at point $p = (x, y, z)$. If ∇f points upwards, we take the upper side of S; otherwise, we take the lower side of S. Let γ be the angle between ∇f and the positive direction of the z-axis. Finally, let D be the projection of S onto the xy-plane. Then, we have:

Lemma 2. *When S is smooth, it holds that:*

$$\begin{aligned}
\iint_S h(x, y, z)\, dA &= \iint_D h(x, y, z) \cdot \frac{1}{|\cos\gamma|}\, dxdy \\
&= \iint_S h(x, y, z) \cdot \frac{1}{\cos\gamma}\, dxdy \\
&= \iint_S h(x, y, z) \frac{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2}}{\frac{\partial f}{\partial z}}\, dxdy.
\end{aligned}$$

We will not give a formal proof of the lemma, but its idea is easy to grasp. Recall that

$$dA = \frac{1}{|\cos\gamma|}\, dxdy$$

is exactly the rationale we used to calculate the area of a surface. This explains the first equality in the lemma. The second equality is due to how the orientation of S affects the sign of the surface integral by coordinate. Finally, when discussing surface areas, we explicitly derived

$$\cos\gamma = \frac{\frac{\partial f}{\partial z}}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2}}.$$

This explains the third equality of the lemma.

Example 1. Let S be the hemisphere $x^2 + y^2 + z^2 = 1$ with $z \le 0$. Calculate $\iint_S z dA$.

Solution. Let D be the projection of S onto the xy-plane, namely, the disc $x^2 + y^2 \le 1$. Introduce $f(x, y, z) = x^2 + y^2 + z^2 - 1$. S can be described by $f(x, y, z) = 0$ with $z \le 0$. We thus have: $\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = 2y$, and $\frac{\partial f}{\partial z} = 2z$. Since $\nabla f = [\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}]$ points downwards, we take the lower side of S.

We know from Lemma 2:

$$\begin{aligned}
\iint_S z\, dA &= \iint_S z \frac{\sqrt{(2x)^2 + (2y)^2 + (2z)^2}}{2z}\, dxdy. \\
&= \iint_S dxdy. \\
\text{(integration on the lower side of } S\text{)} &= -\iint_D dxdy = -\pi.
\end{aligned}$$

□

3 Evaluating Surface Integral by Area: Method 2

Recall that another way to represent S is to use a parametric form $\boldsymbol{r}(u,v) = [x(u,v), y(u,v), z(u,v)]$ with (u,v) defined in a certain region R.

$$\begin{aligned}
\boldsymbol{r}_u &= \left[\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right] \\
\boldsymbol{r}_v &= \left[\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right] \\
\boldsymbol{N} &= \boldsymbol{r}_u \times \boldsymbol{r}_v.
\end{aligned}$$

Note that $\boldsymbol{r}_u$, $\boldsymbol{r}_v$, and $\boldsymbol{N}$ are all functions of u and v. From our discussion in earlier lectures, we know that $\boldsymbol{N}$ is a normal vector of S at the point $p = (x(u,v), y(u,v), z(u,v))$. We have:

Lemma 3. *When S is smooth, it holds that:*

$$\iint_S h(x,y,z)\, dA = \iint_R h(x,y,z) \cdot |\boldsymbol{N}|\, dudv.$$

Proof. From Lemma 2, we know:

$$\iint_S h(x,y,z)\, dA = \iint_D h(x,y,z) \cdot \frac{1}{|\cos\gamma|}\, dxdy$$

where D is the projection of S onto the xy-plane. Then, the lemma follows from changing the integral variables x, y to u, v by Jocobian (see the detailed steps in the lecture on "surface areas"). $\square$

Example 2. Let S be the hemisphere $x^2 + y^2 + z^2 = 1$ with $z \le 0$. Calculate $\iint_S z\, dA$.

Solution. Let D be the projection of S onto the xy-plane, namely, the disc $x^2 + y^2 \le 1$. Represent S as a parametric form $\boldsymbol{r}(u,v) = [x(u,v), y(u,v), z(u,v)]$:

$$\begin{aligned}
x(u,v) &= \cos u \cdot \sin v \\
y(u,v) &= \sin u \cdot \sin v \\
z(u,v) &= \cos v
\end{aligned}$$

for $u \in [0, 2\pi]$ and $v \in [\pi/2, \pi]$. Define R to be the set of all such (u,v). Therefore:

$$\begin{aligned}
\boldsymbol{r}_u &= \left[\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right] = [-\sin u \sin v, \cos u \sin v, 0] \\
\boldsymbol{r}_v &= \left[\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right] = [\cos u \cos v, \sin u \cos v, -\sin v].
\end{aligned}$$

Hence:

$$\begin{aligned}
\boldsymbol{N} &= \boldsymbol{r}_u \times \boldsymbol{r}_v \\
&= [-\cos u \sin^2 v, -\sin u \sin^2 v, -\sin^2 u \sin v \cos v - \cos^2 u \sin v \cos v] \\
&= [-\cos u \sin^2 v, -\sin u \sin^2 v, -\sin v \cos v]
\end{aligned}$$

which means

$$|\boldsymbol{N}| \quad = \quad \sqrt{\sin^4 v + \sin^2 v \cos^2 v} = \sin v.$$

Therefore, by Lemma 3

$$\begin{aligned} \iint_S z\, dA &= \iint_R \cos v \cdot |\sin v|\, dudv \\ &= \int_0^{2\pi} \left(\int_{\pi/2}^{\pi} \cos v \cdot \sin v\, dv \right) du = -\pi. \end{aligned}$$

□

4 A Vector Form of Surface Integral by Area

In this section, we will introduce a few short forms to represent the surface integrals we have already discussed. Keep in mind that we are not defining any new integrals, but instead are giving alternative forms for the same type of integrals.

Consider that S is given in a parametric form $\boldsymbol{r}(u,v) = [x(u,v), y(u,v), z(u,v)]$ with (u,v) defined in a certain region R. As before, define:

$$\begin{aligned} \boldsymbol{r}_u &= \left[\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right] \\ \boldsymbol{r}_v &= \left[\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right] \\ \boldsymbol{N} &= \boldsymbol{r}_u \times \boldsymbol{r}_v. \end{aligned}$$

Define:

- α as the angle between the direction of $\boldsymbol{N}$ and the positive direction of the x-axis;
- β as the angle between the direction of $\boldsymbol{N}$ and the positive direction of the y-axis;
- γ as the angle between the direction of $\boldsymbol{N}$ and the positive direction of the z-axis.

Define

$$\boldsymbol{n} \quad = \quad [\cos\alpha, \cos\beta, \cos\gamma].$$

Note that $\boldsymbol{n}$ is the normalized vector of $\boldsymbol{N}$, namely, it holds that:

$$\boldsymbol{n} \quad = \quad \frac{\boldsymbol{N}}{|\boldsymbol{N}|}.$$

Given a vector function $\boldsymbol{f}(x,y,z) = [f_1(x,y,z), f_2(x,y,z), f_3(x,y,z)]$, we can concisely write

$$\iint_S \boldsymbol{f}(x,y,z) \cdot \boldsymbol{n}\, dA \quad = \quad \iint_R \boldsymbol{f}(x(u,v), y(u,v), z(u,v)) \cdot \boldsymbol{N}\, dudv. \tag{3}$$

To see why this is true, note that by Lemma 3, we have:

$$\begin{aligned} \iint_S \boldsymbol{f}(x,y,z) \cdot \boldsymbol{n}\, dA &= \iint_R (\boldsymbol{f}(x,y,z) \cdot \boldsymbol{n}) \cdot |\boldsymbol{N}|\, dudv \\ &= \iint_R \boldsymbol{f}(x,y,z) \cdot (\boldsymbol{n} \cdot |\boldsymbol{N}|)\, dudv \\ &= \iint_R \boldsymbol{f}(x(u,v), y(u,v), z(u,v)) \cdot \boldsymbol{N}\, dudv. \end{aligned}$$