ENGG 1130 Multivariable Calculus for Engineers

Assignment 4 (Term 2, 2019-2020)

Assigned Date: 1 Mar 2020 (Sunday) 10:00 am

Deadline: **13 Mar 2020 (Friday) 12 noon**

- Show **ALL** your steps and details for each question unless otherwise specified.
- Make a copy of your homework before submitting the original!
- Please submit the **soft copy of your HW 4, TOGETHER WITH THE "DECLARATION FORM" to Blackboard system** on or before the prescribed deadline.
- Feel free to discuss with your friends, but make sure you all present your answers in different manners. **NO** citation (reference) is needed if only discussion takes place.

Notation:

In differential calculus, suppose f is a function of two variables u and v, then both the notations f_u and $\frac{\partial f}{\partial u}$ represent the first derivative of f with respect to u, while f_v and $\frac{\partial f}{\partial v}$ represent the first derivative of f with respect to v.

Full score: 100

1. **(15 marks)** Given that $g(x,y) = x - 2y$ and $h(x,y) = 3x^5 - 4xy^3 + e^{3x^2y}$.

   Find g_x , g_y , h_{xx} and h_{yxy} respectively, then evaluate each of the previous partial derivatives at $(x,y) = (2,5)$.

2. **(15 marks)** Given that $F(x,y) = y^4 + 2xy^3 + x^2y^2$.

   (a) Find the gradient of F.

   (b) Find the rate of change of $F(x,y)$ at $(0, 1)$ measured in each of the following directions.

   (i) $\mathbf{j} - 2\mathbf{i}$ (ii) $2020\mathbf{i}$

3. **(10 marks)** Suppose the radius of a right-circular cone is changed from 10 cm to 10.2 cm, and the height is changed from 1 m to 0.99 m, approximate the change in its volume. Also, find the percentage of such change comparing with its original volume.

4. **(10 marks)** Suppose that $y = f(u)$ and $u = g(a,b)$, and we assume all partial derivatives of f and g are continuous. Draw a tree diagram, and use it to find $\frac{\partial^2 y}{\partial a^2}$.

5. **(15 marks)**

Find all local extrema (a.k.a. critical points) of the given functions. Also, determine the nature of each of these critical points with the aid of the Second Derivative Test (if applicable). If the Second Derivative Test is inconclusive, provide sufficient explanations that can help to determine its nature. Show all your steps clearly.

(a) $F_1(x,y) = 2xy - \frac{1}{2}(x^4 + y^4) + 1$

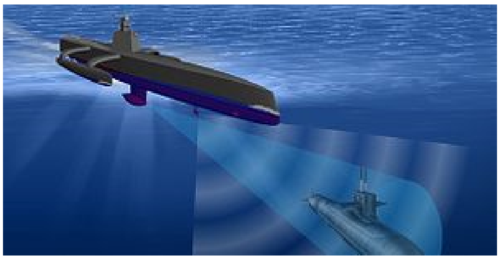
(b) $F_2(x,y) = x^4 + y^4$

6. **(15 marks) – Recovery of an object via the use of sonar technology**

A team of oceanographers is mapping the ocean floor to recover the position of a vessel underwater. They develop the following model with the aid of sonar technology.

$$D(x,y) = 300 + 20x^2 + 60\sin\left(\frac{\pi y}{2}\right) \quad , \qquad x \in [0,2], y \in [0,2]$$

Here D is the depth (in m), and x and y are distances (in km).



Credits: U.S. Defense Advanced Research Projects Agency (DARPA)

(a) The graph $D(x,y)$ is only showing depth of the vessel, but does not provide a map of the ocean floor.

Write down the function of the graph $G(x,y)$ that shows the ocean floor.

(b) Find the steepness of the ocean floor in the positive x-direction from the position of the vessel, provided that the vessel is located at the spatial position with coordinates $x = 0.75$ and $y = 1.25$.

(c) Find the steepness of the ocean floor in the positive y-direction from the position of the vessel, provided that the vessel is located at the spatial position with coordinates $x = 1$ and $y = 0.3$.

(d) Find the direction of the greatest rate of change of depth from the position of the vessel.

7. **(20 marks) – Laplace's Equation and Harmonic Functions**

In mathematics, a **harmonic function** is a twice continuously differentiable function $f: D \to \mathbf{R}$ that satisfies the **Laplace's equation**, i.e.

$$\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = 0$$

Here we may assume D to be an open subset of $\mathbf{R}^n$, and the input variables of f are $x_1, x_2, x_3, \dots, x_n$.

In particular, the 2-dimensional **Laplace's equation** (with x and y as variables) is as follows:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

The **Laplace 's equation** describes the heat distribution within a thin metal plate at steady state.

(a) Show that $P: \mathbf{R}^2 \backslash \{(1,0)\} \to \mathbf{R}$, with

$$P(x, y) = \frac{1}{2\pi}\left[\frac{1 - x^2 - y^2}{(x-1)^2 + y^2}\right]$$

is a **harmonic function.**

(**Note:** $P(x, y)$ with suitable domain is also called the *Poisson kernel* for the open unit disk, with centre at origin and radius 1.)

(b) Let $Q_n: \mathbf{R}^2 \backslash \{(0,0)\} \to \mathbf{R}$ be defined as

$$Q_n(x, y) = (x^2 + y^2)^n$$

where n is a non-zero constant.

(i) Suppose

$$\frac{\partial^2 Q_n(x, y)}{\partial x^2} + \frac{\partial^2 Q_n(x, y)}{\partial y^2} = G(n, x, y) Q_n(x, y)$$

Find $G(n, x, y)$, where G is a function of three variables, namely n, x and y respectively.

(ii) Explain whether $Q_n(x, y)$ is a **harmonic function**. Show your arguments.

END OF ASSIGNMENT 4