Multivariable Calculus (Week 4)

Topics: More on Vector-valued Functions

Functions of Several Variables

Limits and Continuity

(**Reference Chapters:** Chapter 11.3 and 12.1-12.2 of Adams and Essex; Chapter 12.5,13.1-13.2 of Larson and Edwards)

Key References of this file:

Larson, R. and Edwards, B., Multivariable Calculus (Metric Version – 11th Edition), CENGAGE Learning, 2016.

Adams, R.A. and Essex, C., Calculus – A Complete Course, 9th Edition, Pearson, 2018.

Prepared and Revised by Dr. Hugo MAK

HW 2 Question 5

We define the curvature of a path by $\|\mathbf{r}''(s)\|$, where $\mathbf{r}(s)$ is the arc-length parametrization of the path. Given a path $\mathbf{r}(t)$, we let $\mathbf{r}(s)$ be its arc-length parametrization so that $s = \int_0^t \|\mathbf{r}'(\tau)\| d\tau$.

(a) Show that $\mathbf{r}'(t) \times \mathbf{r}''(t) = \left(\frac{ds}{dt}\right)^3 \mathbf{r}'(s) \times \mathbf{r}''(s)$.

By chain rule, $\frac{d\underline{r}}{dt} = \frac{d\underline{r}}{ds}\frac{ds}{dt} = \underline{r}'(s)\frac{ds}{dt}$ — ①

$\frac{d^2\underline{r}}{dt^2} = \frac{d}{dt}\left(\frac{d\underline{r}}{dt}\right) = \frac{d}{dt}\left(\underline{r}'(s)\frac{ds}{dt}\right) = \frac{d\underline{r}'(s)}{dt}\frac{ds}{dt} + \underline{r}'(s)\frac{d^2s}{dt^2}$ — (*)

By chain rule again, $\frac{d\underline{r}'(s)}{dt} = \frac{d\underline{r}'(s)}{ds}\frac{ds}{dt} = \underline{r}''(s)\frac{ds}{dt}$ — (**)

Put (**) to (*), $\frac{d^2\underline{r}}{dt^2} = \underline{r}''(s)\left(\frac{ds}{dt}\right)^2 + \underline{r}'(s)\frac{d^2s}{dt^2}$ — (***)

Taking cross product of ① and (***),

$$\frac{d\underline{r}}{dt} \times \frac{d^2\underline{r}}{dt^2} = \left(\frac{ds}{dt}\right)^3 \underline{r}'(s) \times \underline{r}''(s) + \underbrace{\frac{d^2s}{dt^2}\frac{ds}{dt}\,\underline{r}'(s) \times \underline{r}'(s)}_{=0}$$

$$= \left(\frac{ds}{dt}\right)^3 \underline{r}'(s) \times \underline{r}''(s)$$

HW 2 Question 5

We define the curvature of a path by $\|\mathbf{r}''(s)\|$, where $\mathbf{r}(s)$ is the arc-length parametrization of the path. Given a path $\mathbf{r}(t)$, we let $\mathbf{r}(s)$ be its arc-length parametrization so that $s = \int_0^t \|\mathbf{r}'(\tau)\| d\tau$.

(b) Hence, or otherwise, show that the curvature can be expressed in terms of t. Give the explicit form of the curvature function.

As $\underline{r}(t)$ travels a constant speed, and $\underline{r}'(s)$ and $\underline{r}''(s)$ are two orthogonal vectors, we have

$$\|\underline{r}'(s) \times \underline{r}''(s)\| = \underbrace{\|\underline{r}'(s)\|}_{1} \, \|\underline{r}''(s)\| \sin\frac{\pi}{2} = \|\underline{r}''(s)\|$$

Taking the magnitude on both sides of the expression in (a),

$$\|\underline{r}'(t) \times \underline{r}''(t)\| = \|\underline{r}''(s)\| \left\|\frac{ds}{dt}\right\|^3$$

$$\|\underline{r}''(s)\| = \frac{\|\underline{r}'(t) \times \underline{r}''(t)\|}{\left\|\frac{ds}{dt}\right\|^3}$$

Now, since $s = \int_0^t \|\underline{r}'(\tau)\| d\tau$

$$\Rightarrow \frac{ds}{dt} = \|\underline{r}'(t)\|,$$

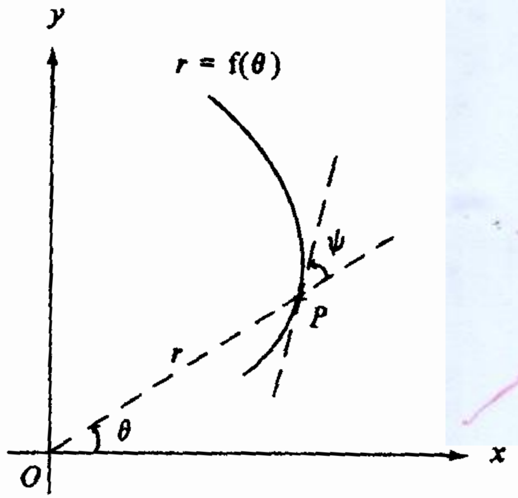
which is in terms of t as well

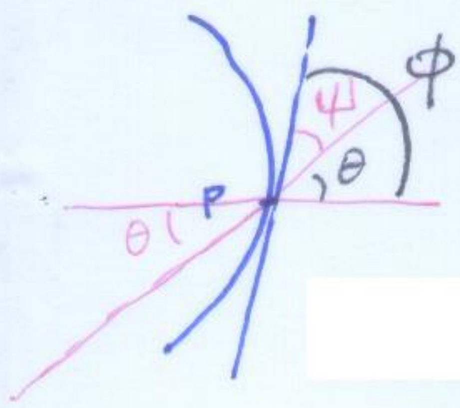
$$\|\mathbf{r}''(s)\| = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$$

HW 2 Question 6

Given a curve *C* on the *xy*-plane with equation $r = f(\theta)$, where *r* and θ are the polar coordinates used to describe any point lying on *C*, in particular point *P* in the following figure. Let *O* be the origin and ψ be the angle from the line *OP* to the tangent line at point *P*. We assume *f* is continuously differentiable and non-negative.

(a) Express $\tan\psi$ in terms of r and derivative of r with respect to θ.





(a) Let (x, y) be the Cartesian coordinates of P, $\tan\phi$ be the gradient of the curve at P.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Rightarrow \begin{aligned} \frac{dx}{d\theta} &= \cos\theta\frac{dr}{d\theta} - r\sin\theta \\ \frac{dy}{d\theta} &= \sin\theta\frac{dr}{d\theta} + r\cos\theta \end{aligned}$$

$$\tan\phi = \frac{dy}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta} = \frac{\sin\theta\frac{dr}{d\theta} + r\cos\theta}{\cos\theta\frac{dr}{d\theta} - r\sin\theta} = \frac{\tan\theta\frac{dr}{d\theta} + r}{\frac{dr}{d\theta} - r\tan\theta}$$

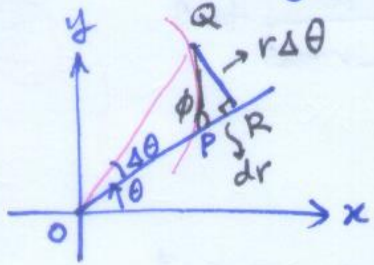
Now $\tan\psi = \tan(\phi - \theta)$

$$= \left(\frac{\tan\theta\frac{dr}{d\theta} + r}{\frac{dr}{d\theta} - r\tan\theta} - \tan\theta\right) \times \frac{1}{1 + \frac{\tan\theta\frac{dr}{d\theta} + r}{\frac{dr}{d\theta} - r\tan\theta}\tan\theta}$$

$$= \frac{r(1 + \tan^2\theta)}{\frac{dr}{d\theta}(1 + \tan^2\theta)} = \frac{r}{\frac{dr}{d\theta}} \;/\!/$$

Alternatively,

6(a) In the diagram



$PR \approx \Delta r$

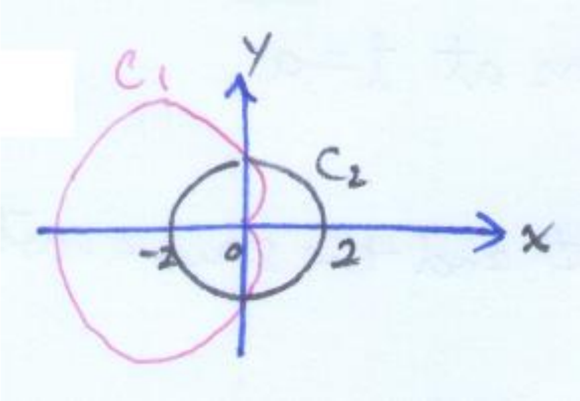
$QR \approx r\Delta\theta$

$\therefore \tan\phi \approx \frac{r\Delta\theta}{\Delta r}$ important

$\Rightarrow \tan\psi = \lim_{\Delta\theta \to 0} r\frac{\Delta\theta}{\Delta r} = \frac{r}{dr/d\theta}$

HW 2 Question 6

Given a curve *C* on the *xy*-plane with equation $r = f(\theta)$, where *r* and θ are the polar coordinates used to describe any point lying on *C*, in particular point *P* in the following figure. Let *O* be the origin and ψ be the angle from the line *OP* to the tangent line at point *P*. We assume *f* is continuously differentiable and non-negative.



(c) Given two curves C_1: $r = 2 - 2\cos\theta$ (where $0 \le \theta < 2\pi$) and C_2: $r = 2$.

Some part of C_1 is inside C_2, find the arc length of such part.

For any point (r, θ) of C_1 lying inside C_2

$r = 2(1-\cos\theta)$, $r < 2$

$\therefore 2(1-\cos\theta) < 2 \Rightarrow 0 < \theta < \frac{\pi}{2}$ or $\frac{3\pi}{2} < \theta < 2\pi$

$\therefore$ Length of C_1 inside C_2

$$= \int_0^{\pi/2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta + \int_{3\pi/2}^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= 2\int_0^{\pi/2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta \quad \text{(by symmetry)}$$

$$= 2\int_0^{\pi/2} \sqrt{\left(\cos\theta \frac{dr}{d\theta} - r\sin\theta\right)^2 + \left(\sin\theta \frac{dr}{d\theta} + r\cos\theta\right)^2}\, d\theta$$

$$= 2\int_0^{\pi/2} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \sqrt{4\sin^2\theta + 4(1-\cos\theta)^2}\, d\theta$$

$$= 4\sqrt{2}\int_0^{\pi/2} \sqrt{1-\cos\theta}\, d\theta$$

$$= 8\int_0^{\frac{\pi}{2}} \sin\frac{\theta}{2}\, d\theta = 16\left[-\cos\frac{\theta}{2}\right]_0^{\frac{\pi}{2}} = \underline{\underline{8(2-\sqrt{2})}}$$

Alternatively,

$$ds^2 = (r\,d\theta)^2 + (dr)^2$$

Length of C_1 inside $C_2 = \int ds = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \; \ldots\ldots = 8(2-\sqrt{2})$

Example 1 : $\frac{d}{dt}\|\mathbf{r}(t)\|$

Method 1: Let $\mathbf{r}(t)= \langle x(t), y(t), z(t)\rangle$

Then $\|\mathbf{r}(t)\| = \sqrt{x(t)^2 + y(t)^2 + z(t)^2}$

Hence,

$$\frac{d}{dt}\|\mathbf{r}(t)\|$$
$$= \frac{1}{2}\left[x(t)^2 + y(t)^2 + z(t)^2\right]^{-\frac{1}{2}} \cdot \left(2x(t)x'(t) + 2y(t)y'(t) + 2z(t)z'(t)\right)$$
$$= \frac{1}{\|\mathbf{r}(t)\|}\langle x(t), y(t), z(t)\rangle \cdot \langle x'(t), y'(t), z'(t)\rangle$$
$$= \frac{\mathbf{r}(t)\cdot\mathbf{r}'(t)}{\|\mathbf{r}(t)\|}$$

<u>Example 1</u> : $\frac{d}{dt}\|\mathbf{r}(t)\|$

Method 2: Consider $\|\mathbf{r}(t)\|^2 = \mathbf{r}(t) \cdot \mathbf{r}(t)$

Therefore, $\frac{d}{dt}\|\mathbf{r}(t)\|^2 = \frac{d}{dt}(\mathbf{r}(t) \cdot \mathbf{r}(t))$

Thus, $2\|\mathbf{r}(t)\|\frac{d}{dt}\|\mathbf{r}(t)\| = \mathbf{r}(t) \cdot \mathbf{r}'(t) + \mathbf{r}'(t) \cdot \mathbf{r}(t)$

Re-arranging, $\frac{d}{dt}\|\mathbf{r}(t)\| = \frac{\mathbf{r}(t) \cdot \mathbf{r}'(t)}{\|\mathbf{r}(t)\|}$

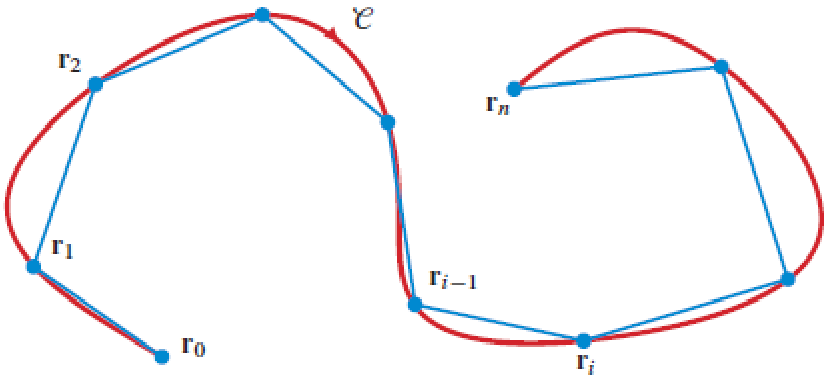
Take-Home Exercise: Show that if **r** is a vector-valued function, then

$$\frac{d}{dt}[\mathbf{r}(t) \times \mathbf{r}'(t)] = \mathbf{r}(t) \times \mathbf{r}''(t)$$

Arc Length

- Let C be a bounded, continuous curve specified by
$$\mathbf{r} = \mathbf{r}(t), \qquad t \in [a, b]$$
- We sub-divide the closed interval $[a, b]$ into n sub-intervals by points:
$$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b$$
- The points $\mathbf{r}_i = \mathbf{r}(t_i)$ $(0 \leq i \leq n)$ sub-divide C into n arcs. We use $\|\mathbf{r}_i - \mathbf{r}_{i-1}\|$ as an approximation to the arc length between $\mathbf{r}_{i-1}$ and $\mathbf{r}_i$, then $\sum_{i=1}^{n} \|\mathbf{r}_i - \mathbf{r}_{i-1}\|$ approximates the length C by the length of a polygonal line.
- We let $\Delta t_i = t_i - t_{i-1}$ and $\Delta \mathbf{r}_i = \mathbf{r}_i - \mathbf{r}_{i-1}$, then

$$s_n := \sum_{i=1}^{n} \|\mathbf{r}_i - \mathbf{r}_{i-1}\| \approx \sum_{i=1}^{n} \left\| \frac{\Delta \mathbf{r}_i}{\Delta t_i} \right\| \Delta t_i$$



(by completeness axiom, and we assume C is rectifiable, i.e. smooth in a certain measure-theoretic sense)

Ref: Adams, R.A. and Essex, C., Calculus – A Complete Course, 9th Edition, Pearson, 2018.

Arc Length (Formula)

- $s_n := \sum_{i=1}^{n} \|\mathbf{r}_i - \mathbf{r}_{i-1}\| \approx \sum_{i=1}^{n} \left\| \frac{\Delta \mathbf{r}_i}{\Delta t_i} \right\| \Delta t_i$
- If **r**(t) has a continuous derivative **v**(t), then

$$s = \lim_{\substack{n \to \infty \\ \max \Delta t_i \to 0}} s_n = \int_a^b \left\| \frac{d\mathbf{r}}{dt} \right\| dt = \int_a^b \|\mathbf{v}(t)\| dt$$

In mechanics, the interpretation is as follows:

The distance travelled by a moving particle is the integral of its speed.

Arc Length (Examples)

Ref:
Adams, R.A. and Essex, C., Calculus – A Complete Course, 9th Edition, Pearson, 2018.

Example 1: Find the length of the part of the circular helix

$$\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + bt\,\mathbf{k}$$

between the points $(a, 0,0)$ and $(a, 0, 2\pi b)$. Give a description of the part of the circular helix as well.

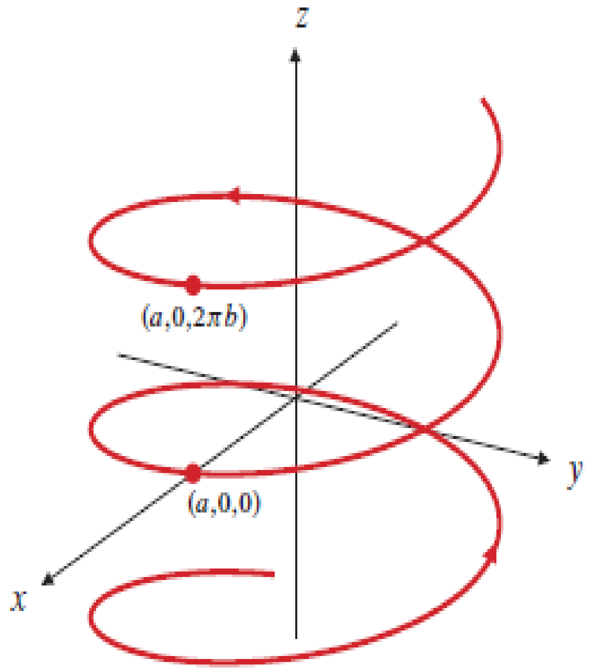
Solution:

The curve spirals around the z-axis, increase as z-value as it turns.

It lies on the surface of the circular cylinder $x^2 + y^2 = a^2$.

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -a\ \sin t\,\mathbf{i} + a\cos t\,\mathbf{j} + b\mathbf{k}$$

$$v = \sqrt{a^2 + b^2},$$



Length:

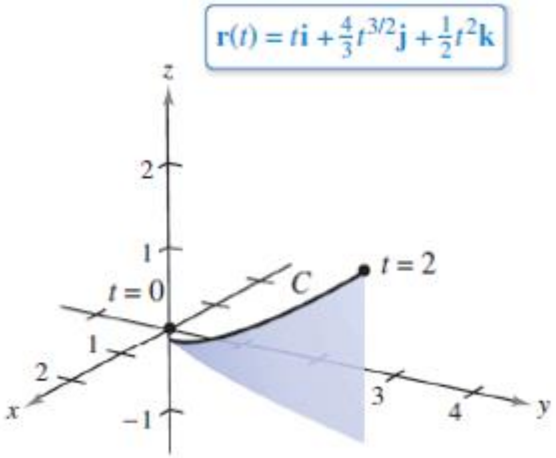
$$s = \int_0^{2\pi} v(t)\,dt = \int_0^{2\pi} \sqrt{a^2 + b^2}\,dt = 2\pi\sqrt{a^2 + b^2}$$

Arc Length (Example 2)

Find the arc length of the curve given by

$$\mathbf{r}(t) = t\mathbf{i} + \frac{4}{3}t^{3/2}\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$$

from $t = 0$ to $t = 2$.



As t increases from 0 to 2, the vector $\mathbf{r}(t)$ traces out a curve.

Using $x(t) = t$, $y(t) = \frac{4}{3}t^{3/2}$, and $z(t) = \frac{1}{2}t^2$, you obtain $x'(t) = 1$ $y'(t) = 2t^{1/2}$, and $z'(t) = t$.

So, the arc length from $t = 0$ and $t = 2$ is given by

$$s = \int_0^2 \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt \qquad \text{Formula for arc length}$$

$$= \int_0^2 \sqrt{1 + 4t + t^2}\, dt$$

$$= \int_0^2 \sqrt{(t+2)^2 - 3}\, dt$$

$$= \left[\frac{t+2}{2}\sqrt{(t+2)^2 - 3} - \frac{3}{2}\ln\left|(t+2) + \sqrt{(t+2)^2 - 3}\right|\right]_0^2$$

$$= 2\sqrt{13} - \frac{3}{2}\ln\left(4 + \sqrt{13}\right) - 1 + \frac{3}{2}\ln 3 \approx 4.816.$$

Ref.:
Larson, R. and Edwards, B., Multivariable Calculus (Metric Version – 11th Edition), CENGAGE Learning, 2016.

Arc Length Parametrization

- Considering the two curves

$$\mathbf{r}_1(t) = \langle \cos t, \sin t, 2t \rangle, \ \ t \in [0, 2\pi]$$

$$\mathbf{r}_2(t) = \langle \cos 3t, \sin 3t, 6t \rangle, \ \ t \in \left[0, \frac{2\pi}{3}\right]$$

- These two curves are the same, but with different speeds. The curve $\mathbf{r}_2(t)$ is obtained by replacing every t in $\mathbf{r}_1(t)$ by $3t$. The initial and final times are adjusted such that the end points of both graphs remain the same. We say $\mathbf{r_2}$ is a **re-parametrization** of $\mathbf{r_1}$.

Definition

- If **r**(s) is a parametric curve such that $\|\mathbf{r}'(s)\| = 1$ for any s, we say the curve is parametrized by arc-length.

Goal

- Given a parametric curve **r**(t), we want to reparametrize the curve by arc-length, such that with the new parameter s, **r**(s) travels at **unit speed**.

Arc Length Parametrization

Process and Steps of Re-parametrization:

1. Given a curve $\mathbf{r}(t)\colon [a, b] \to \mathbf{R}^3$, we compute $s = \int_a^t \|\mathbf{r}'(\tau)\| d\tau$.
2. Since the upper limit of the above integral is t , the function s is a function of t.
3. Express t as $t = t(s)$.
4. Replace all t by this function of s in the curve $\mathbf{r}(t)$.

The new parametrization $\mathbf{r}(s)$ will be arc-length parametrized.

Proof: It suffices to show that $\|\mathbf{r}'(s)\| = 1$.

Consider $\|\mathbf{r}'(s)\| = \left\|\frac{d\mathbf{r}}{dt}\frac{dt}{ds}\right\| = \|\mathbf{r}'(t)\| \left\|\frac{dt}{ds}\right\|$

Recall that $s = \int_a^t \|\mathbf{r}'(\tau)\| d\tau$, hence $\frac{ds}{dt} = \|\mathbf{r}'(t)\|$ **(Why? By what theorem?)**

Therefore, $\frac{dt}{ds} = \frac{1}{\frac{ds}{dt}} = \frac{1}{\|\mathbf{r}'(t)\|}$, and it follows that $\|\mathbf{r}'(s)\| = 1$.

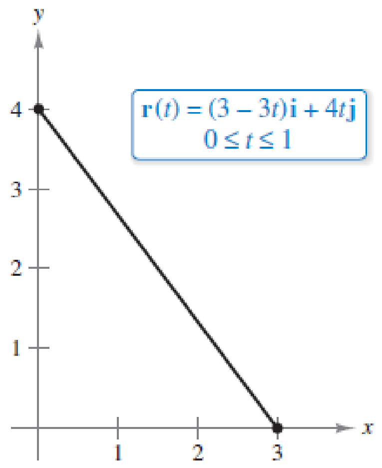
Conclusion: The parametrization $\mathbf{r}(s)$ will be arc-length parametrized, and has unit speed.

Example 1

Find the arc length function $s(t)$ for the line segment given by

$$\mathbf{r}(t) = (3 - 3t)\mathbf{i} + 4t\,\mathbf{j}, \qquad 0 \le t \le 1$$

and write **r** as a function of the parameter *s*.



The line segment from (3, 0) to (0, 4) can be parametrized using the arc length parameter *s*.

$$\|\mathbf{r}'(t)\| = \sqrt{(-3)^2 + 4^2} = 5$$

$$s(t) = \int_0^t \|\mathbf{r}'(u)\|\,du$$

$$= \int_0^t 5\,du \quad = 5t.$$

Using $s = 5t$ (or $t = s/5$), we can rewrite **r** using the arc length parameter as follows.

$$\mathbf{r}(s) = \left(3 - \tfrac{3}{5}s\right)\mathbf{i} + \tfrac{4}{5}s\,\mathbf{j}, \quad 0 \le s \le 5$$

Ref.:

Larson, R. and Edwards, B., Multivariable Calculus (Metric Version – 11th Edition), CENGAGE Learning, 2016.

Example 2

Re-parametrize the given curve in the same orientation in terms of the arc length measured from the point where $t = 0$.

$$\mathbf{r}(t) = 3t\cos t\ \mathbf{i} + 3t\sin t\ \mathbf{j} + 2\sqrt{2}\ t^{\frac{3}{2}}\ \mathbf{k}$$

Ref.: Adams, R.A. and Essex, C., Calculus – A Complete Course, 9th Edition, Pearson, 2018.

Solution:

$$\mathbf{r} = 3t\cos t\,\mathbf{i} + 3t\sin t\,\mathbf{j} + 2\sqrt{2}t^{3/2}\mathbf{k}, \quad (t \geq 0)$$

$$\mathbf{v} = 3(\cos t - t\sin t)\mathbf{i} + 3(\sin t + t\cos t)\mathbf{j} + 3\sqrt{2}\sqrt{t}\,\mathbf{k}$$

$$v = |\mathbf{v}| = 3\sqrt{1 + t^2 + 2t} = 3(1 + t)$$

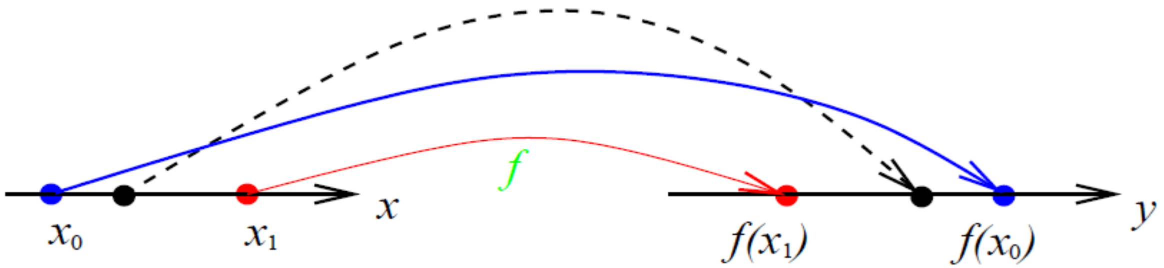
$$s = \int_0^t 3(1 + u)\,du = 3\left(t + \frac{t^2}{2}\right)$$

Thus $t^2 + 2t = \dfrac{2s}{3}$, so $t = -1 + \sqrt{1 + \dfrac{2s}{3}}$ since $t \geq 0$.

The required parametrization is the given one with t replaced by $-1 + \sqrt{1 + (2s)/3}$.

Review of Functions

Function of one variable: $f : \mathbb{R} \to \mathbb{R}$, $y = f(x)$. This is a curve in a plane.



domain of f is the set of allowable values for independent variable x

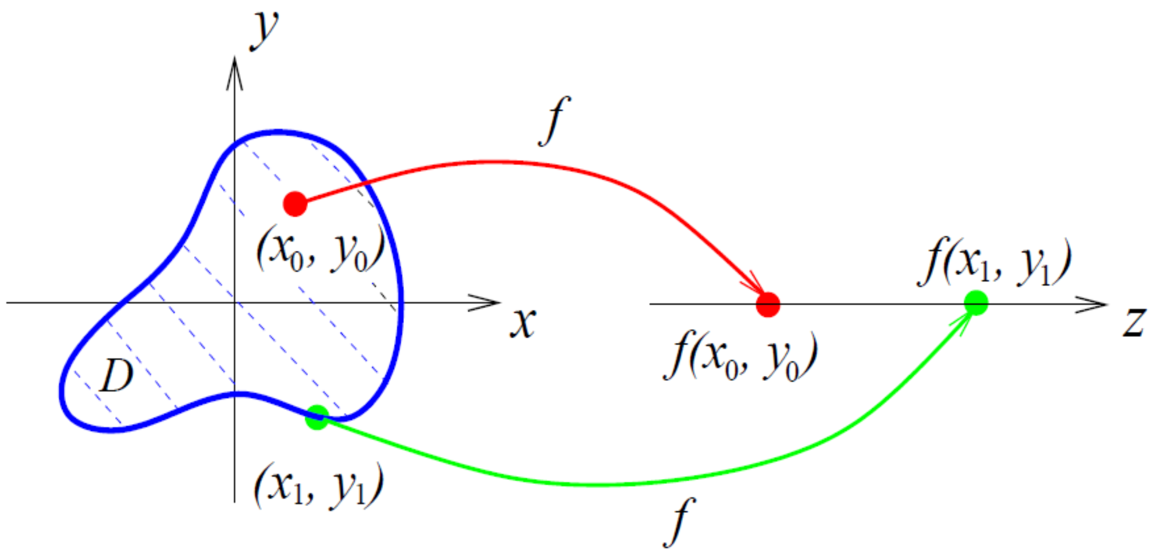
range: the set of all possible values of f

Function of two variables

$f : \mathbb{R}^2 \to \mathbb{R}$, $z = f(x, y)$. This is a surface in 3D-space.

(x_0, y_0) – interior point

(x_1, y_1) – boundary point



Definition of a Function of Two Variables

Let D be a set of ordered pairs of real numbers. If to each ordered pair (x, y) in D there corresponds a unique real number $f(x, y)$, then f is a **function of x and y.** The set D is the **domain** of f, and the corresponding set of values for $f(x, y)$ is the **range** of f. For the function

$$z = f(x, y)$$

x and y are called the **independent variables** and z is called the **dependent variable.**

Domain of Functions of Several Variables

Ref.:
Larson, R. and Edwards, B., Multivariable Calculus (Metric Version – 11th Edition), CENGAGE Learning, 2016.

Find the domain of each function.

a. $f(x, y) = \dfrac{\sqrt{x^2 + y^2 - 9}}{x}$

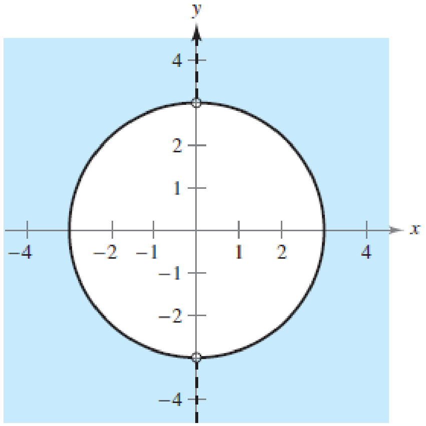
b. $g(x, y, z) = \dfrac{x}{\sqrt{9 - x^2 - y^2 - z^2}}$

Solution to (a)

The function f is defined for all points (x, y) such that $x \neq 0$ and

$$x^2 + y^2 \geq 9.$$

Therefore, the domain is the set of all points lying on or outside the circle $x^2 + y^2 = 9$ ***except*** those points on the y-axis.



Domain of $f(x, y) = \dfrac{\sqrt{x^2 + y^2 - 9}}{x}$

Solution to (b) The function g is defined for all points (x, y, z) such that

$$x^2 + y^2 + z^2 < 9.$$

Consequently, the domain is the set of all points (x, y, z) lying inside a sphere of radius 3 that is centered at the origin.

More Examples

Domain of a function: set of allowable inputs
Domain of a two-variable function: each input can be represented by a point $(x, y) \in \mathbf{R}^2$.

Examples
(1) $f(x, y) = \frac{3}{xy}$ is undefined when $xy = 0$, i.e. when at least one of x and y = 0.
Domain: $\{(x, y): x \neq 0 \text{ and } y \neq 0\}$
Geometrical representation: whole $\mathbf{R}^2$ plane with x-axis and y-axis removed.

(2) $g(x, y) = \sqrt{3y - 5x^2}$ is defined only when $3y \geq 5x^2$.
Domain: $\left\{(x, y): y \geq \frac{5}{3}x^2\right\}$
Geometric representation: Region above the parabola $y = \frac{5}{3}x^2$ in $\mathbf{R}^2$.

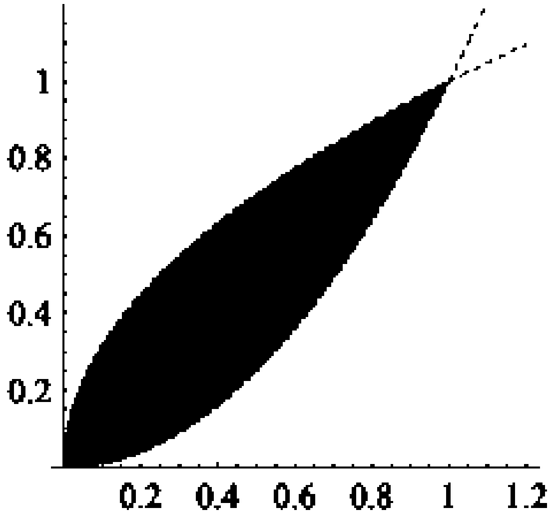
(3) $h(x, y) = \frac{1}{x+y}$ is defined everywhere on $\mathbf{R}^2$ except on the line $x = -y$.
Domain: $\{(x, y): x \neq -y\}$

$f : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ is given by $f(x, y, z) = \left(\dfrac{xyz}{\sqrt{2x^2 + y^2 + z^2 - 1}}, zx, xy \right)$

In this example, $\Omega = \{ (x, y, z) : 2x^2 + y^2 + z^2 > 1 \}$, which is the outside of an ellipsoid.

$f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ is given by $f(x, y) = (\ln(x - y^2), \ln(y - x^2))$.

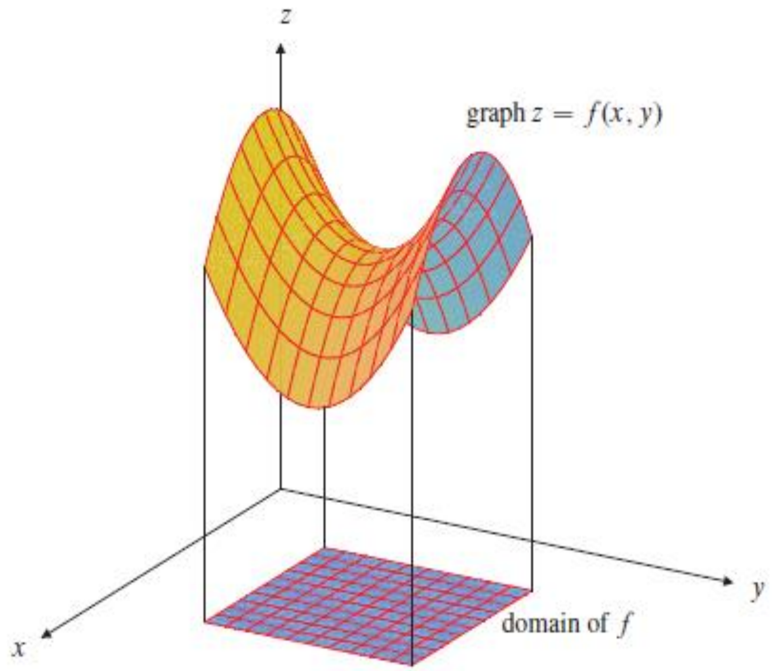
The domain for this f, $\Omega = \{ (x, y) : x - y^2 > 0 \} \cap \{ (x, y) : y - x^2 > 0 \}$. It is shown in the picture below.



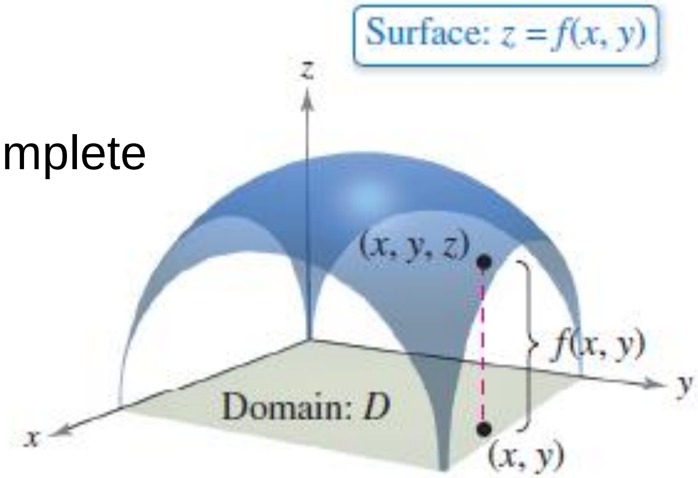
Graph of a Function of two variables

The **graph** of a function *f* of two variables is the set of all points (*x*, *y*, *z*) for which *z* = *f*(*x*, *y*) and (*x*, *y*) is in the domain of *f*.

This graph can be interpreted geometrically as a *surface in space*. In the figure, note that the graph of *z* = *f*(*x*, *y*) is a surface whose projection onto the *xy*-plane is *D*, the domain of *f*.



To each point (*x*, *y*) in *D* there corresponds a point (*x*, *y*, *z*) on the surface, and, conversely, to each point (*x*, *y*, *z*) on the surface there corresponds a point (*x*, *y*) in *D*.



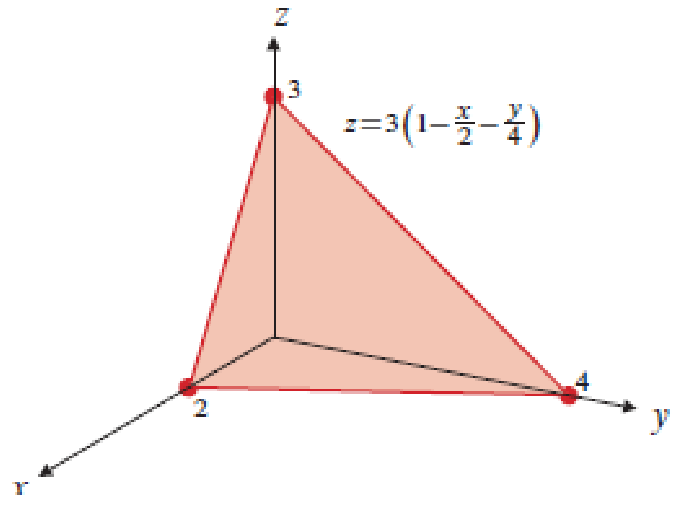
Ref.:
Adams, R.A. and Essex, C., Calculus – A Complete Course, 9th Edition, Pearson, 2018.

Examples

Ref.: Adams, R.A. and Essex, C., Calculus – A Complete Course, 9th Edition, Pearson, 2018.

(1) $f(x,y) = 3\left(1 - \frac{x}{2} - \frac{y}{4}\right), \quad x \in [0,2], \; y \in [0, 4-2x]$

The graph of *f* is the plane: triangular surface with vertices at (2, 0, 0), (0, 4, 0) and (0, 0, 3).

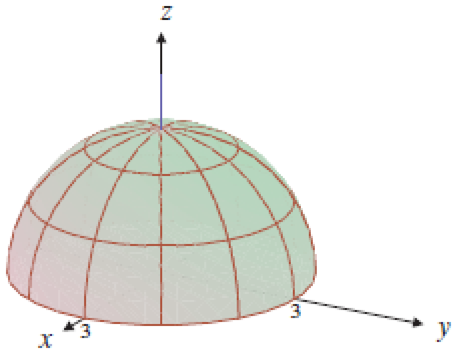


(2) $g(x,y) = \sqrt{9 - x^2 - y^2}$

Domain: the disk $x^2 + y^2 \leq 9$ in the *xy* –plane.

Squaring the equation $z = \sqrt{9 - x^2 - y^2}$, we obtain $x^2 + y^2 + z^2 = 9$, which is a sphere of radius 3 centered at the origin.

The graph of g is only the upper hemisphere where $z \geq 0$



Examples

Ref.:
Larson, R. and Edwards, B., Multivariable Calculus (Metric Version – 11th Edition), CENGAGE Learning, 2016.

(3) $h(x, y) = \sqrt{16 - 4x^2 - y^2}$

The domain implied by the equation of *h* is the set of all points (x, y) such that $16 - 4x^2 - y^2 \geq 0$

Domain: the set of all points lying on or inside the ellipse

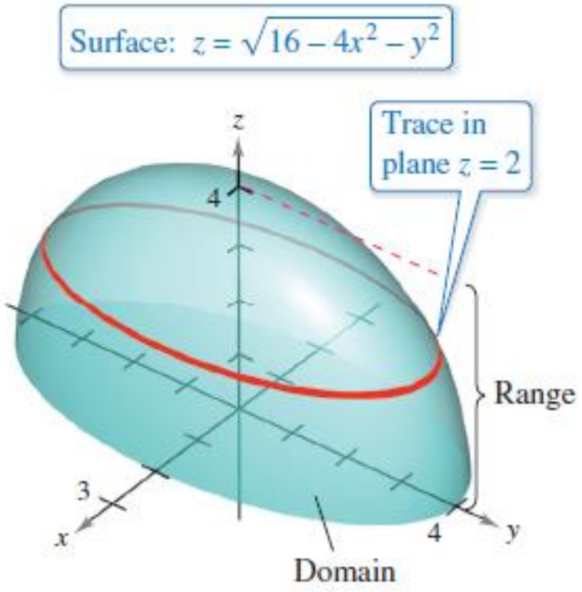
$$\frac{x^2}{4} + \frac{y^2}{16} = 1.$$

Ellipse in the *xy*-plane

Range: all values *z* = *h*(*x*, *y*) such that $0 \leq z \leq 4$

A point (*x*, *y*, *z*) is on the graph of *h* if and only if

$$z = \sqrt{16 - 4x^2 - y^2}$$
$$z^2 = 16 - 4x^2 - y^2$$
$$\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{16} = 1, \qquad 0 \leq z \leq 4.$$



The graph of $f(x, y) = \sqrt{16 - 4x^2 - y^2}$ is the upper half of an ellipsoid.

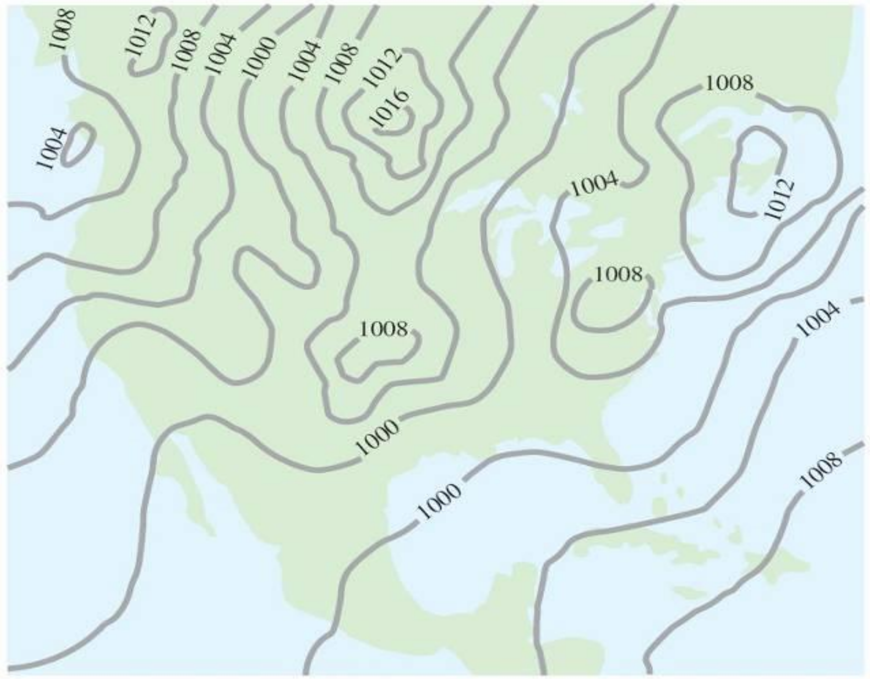
Level Curves

Ref.:
Larson, R. and Edwards, B., Multivariable Calculus (Metric Version – 11th Edition), CENGAGE Learning, 2016.

A second way to visualize a function of two variables is to use a **scalar field** in which the scalar $z = f(x, y)$ is assigned to the point (x, y).

A scalar field can be characterized by **level curves** (or **contour lines**) along which the value of $f(x, y)$ is constant.

For instance, the weather map in the figure shows level curves of equal pressure called **isobars.**



Level curves show the lines of equal pressure (isobars) measured in millibars.

Level Curves

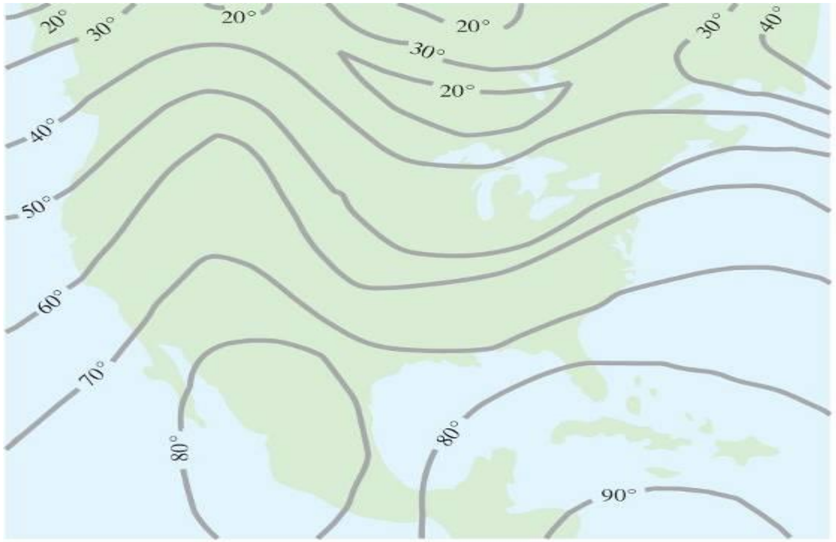
Ref.:

Larson, R. and Edwards, B., Multivariable Calculus (Metric Version – 11^{th} Edition), CENGAGE Learning, 2016.

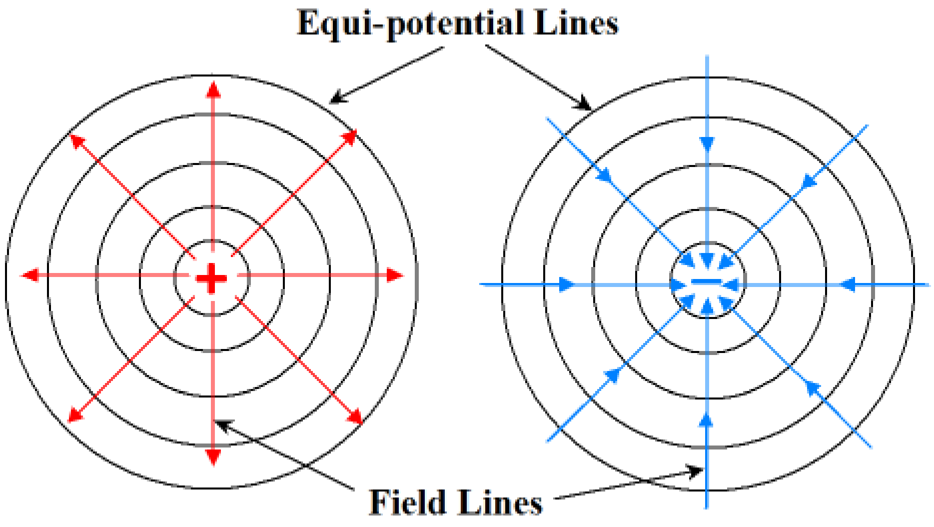
In weather maps for which the level curves represent points of equal temperature, the level curves are called **isotherms.**

Another common use of level curves is in representing electric potential fields.

In this type of map, the level curves are called **equipotential lines.**



Level curves show the lines of equal temperature (isotherms) measured in degrees Fahrenheit.



Level Curves

Ref.:
Larson, R. and Edwards, B., Multivariable Calculus (Metric Version – 11th Edition), CENGAGE Learning, 2016.

Contour maps are commonly used to show regions on Earth's surface, with the level curves representing the height above sea level. This type of map is called a **topographic map.**

The mountain shown in Figure 1 is represented by the topographic map in Figure 2.

Figure 1

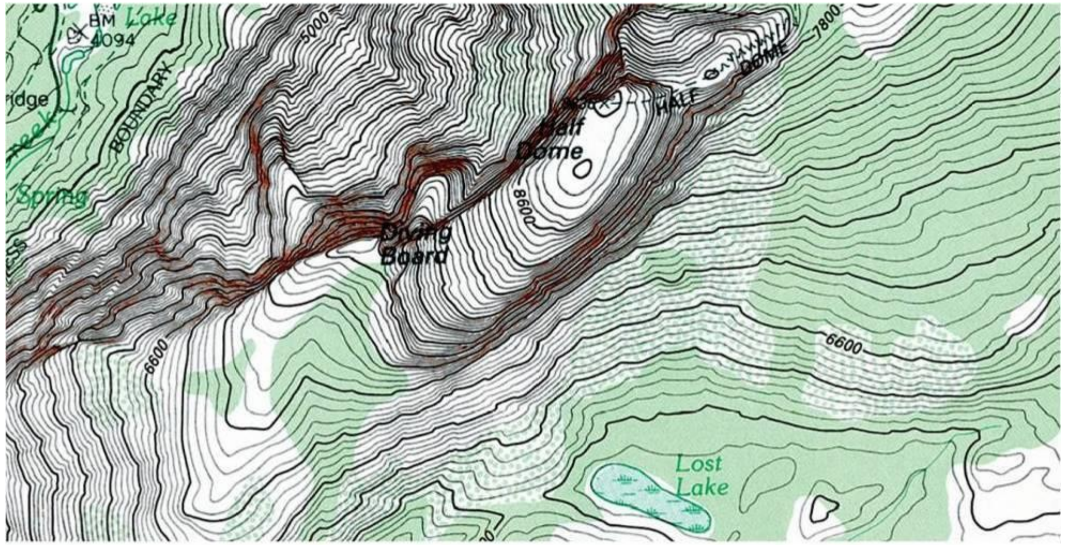


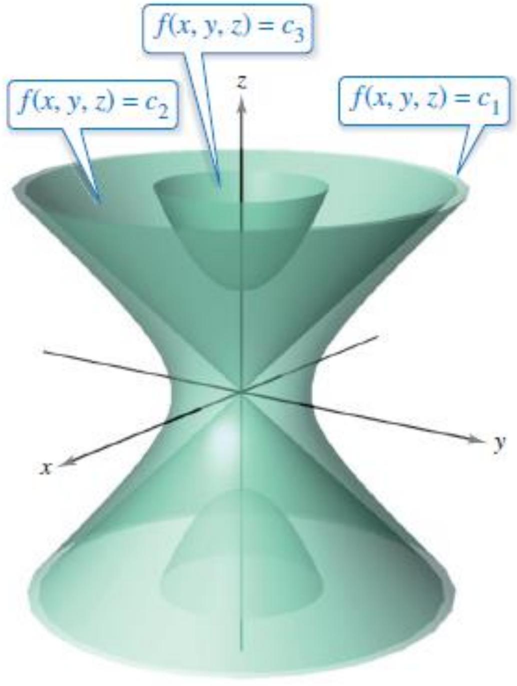
Figure 2

Ref.:
Larson, R. and Edwards, B., Multivariable Calculus (Metric Version – 11th Edition), CENGAGE Learning, 2016.

Level Surfaces

The concept of a level curve can be extended by one dimension to define a **level surface.**

If f is a function of three variables and c is a constant, then the graph of the equation $f(x, y, z) = c$ is a **level surface** of f, as shown in the figure.



Level surfaces of f

Definition (Level curves (Surfaces))

The level set with constant c for a function $f: \mathbf{R}^n \to \mathbf{R}$ is the set

$$\{(x_1, x_2, \dots, x_n) | f(x_1, x_2, \dots, x_n) = c\}$$

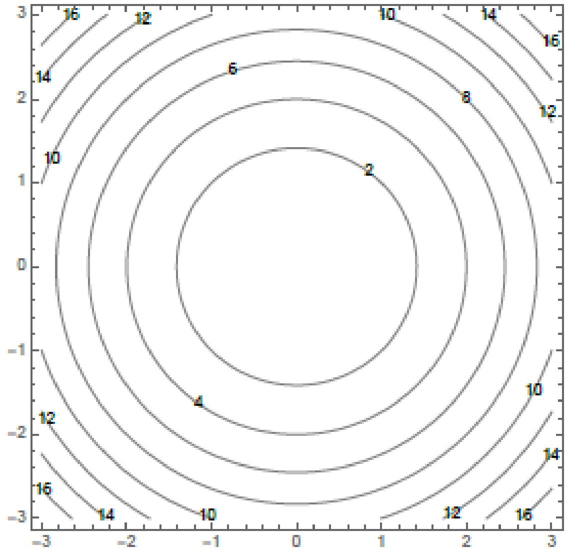
When $n = 2$: a level set is also called a **level curve** (e.g. $f(x, y) = x^2 + y^2 = c$)

When $n = 3$: a level set is also called a **level surface** (e.g. $f(x, y, z) = x^2 + y^2 + z^2 = c$)

$$f: \mathbf{R}^2 \to \mathbf{R}, \text{with } f(x, y) = x^2 + y^2$$

We try to substitute different values of c , then we get several level sets on the 2-dimensional plane.

These are circles at the origin with different radii (which depends on the selected values of c). Notice that when $c = 0$, we have a point circle.



Level set diagram of $f(x, y) = x^2 + y^2$

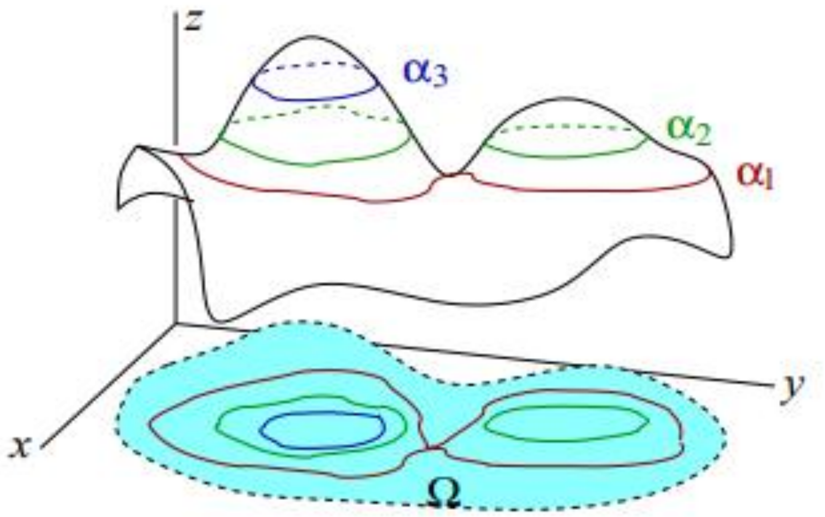
Functions	Graph	Level Sets
$f(x)$	Curve in $\mathbf{R}^2$	$f(x) = c$ is a point on $\mathbf{R}$
$f(x,y)$	Surface in $\mathbf{R}^3$	$f(x,y) = c$ is a curve on $\mathbf{R}^2$
$f(x,y,z)$	4-dimensional surface	$f(x,y,z) = c$ is a surface in $\mathbf{R}^3$
$f(x_1, x_2, \dots, x_n)$	(n+1)-dimensional surface	$f(x_1, x_2, \dots, x_n) = c$ is a surface in $\mathbf{R}^n$

Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ and a number $\alpha \in \mathbb{R}$ be given. Then the level set at α is

$$L_\alpha = \{\mathbf{x} \in \Omega \;:\; f(\mathbf{x}) = \alpha\} \subset \mathbb{R}^n .$$

Note that L_α is a subset of the domain Ω. It is actually the same as $f^{-1}(\alpha)$, the pre-image of α. For $n = 2$, level sets is often called level curves.

From the example above, when we wish to visualize Graph(f), we often consider the vertical cross-sections and also the level sets, which are the horizontal cross-sections.

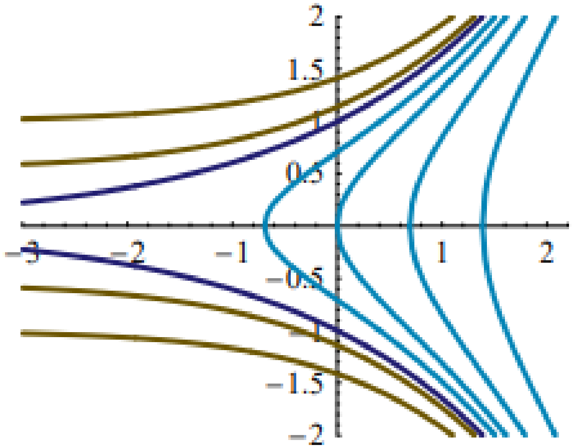


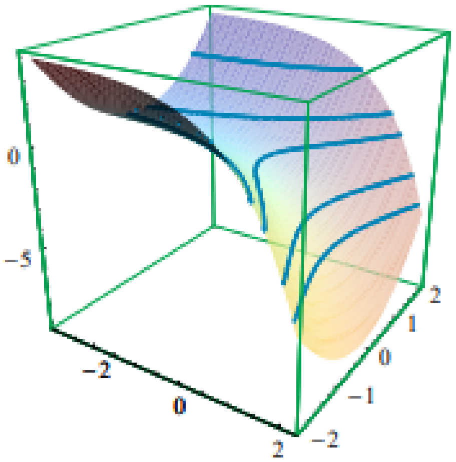
EXAMPLE. Determine the level sets of $f(x,y) = y^2 - e^x$ for $(x,y) \in \Omega = \mathbb{R}^2$.

Let $L_\alpha = \{ (x,y) \in \mathbb{R}^2 : y^2 - e^x = \alpha \}$. In order to understand L_α, we may need to rewrite the equation, i.e., $(x,y) \in L_\alpha$ if and only if

$$y^2 = e^x + \alpha .$$

The easy case occurs when $\alpha = 0$, i.e., $y = \pm\sqrt{e^x} = \pm e^{x/2}$, which is the blue curve below. We can easily draw the curve in the xy-plane. The cases for $\alpha < 0$ versus > 0 are also shown below.





With the above family of level curves, one may be able to figure out the surface of Graph(f) as above.

Ref.: MATH 2010 Notes in CUHK

More Examples

Ref.:
Larson, R. and Edwards, B., Multivariable Calculus (Metric Version – 11th Edition), CENGAGE Learning, 2016.

Describe the level surfaces of $f(x, y, z) = 4x^2 + y^2 + z^2.$

Solution:

Each level surface has an equation of the form $4x^2 + y^2 + z^2 = c.$ Equation of level surface

Therefore, the level surfaces are ellipsoids (whose cross sections parallel to the *yz*-plane are circles).

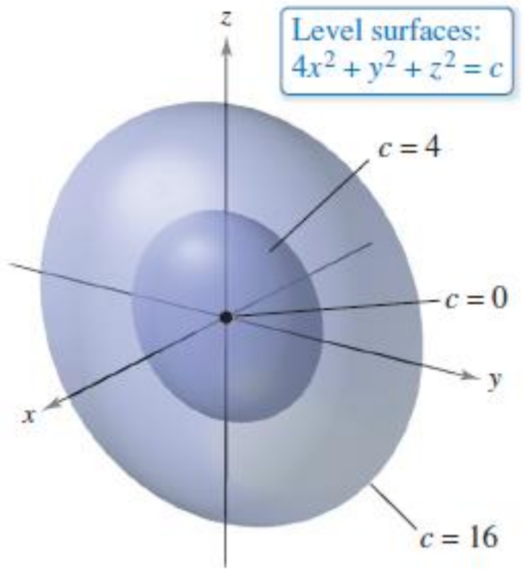
As *c* increases, the radii of the circular cross sections increase according to the square root of *c*.

For example, the level surfaces corresponding to the values *c* = 0, *c* = 4, and *c* = 16 are as follows.

$4x^2 + y^2 + z^2 = 0$ Level surface for $c = 0$ (single point)

$\frac{x^2}{1} + \frac{y^2}{4} + \frac{z^2}{4} = 1$ Level surface for $c = 4$ (ellipsoid)

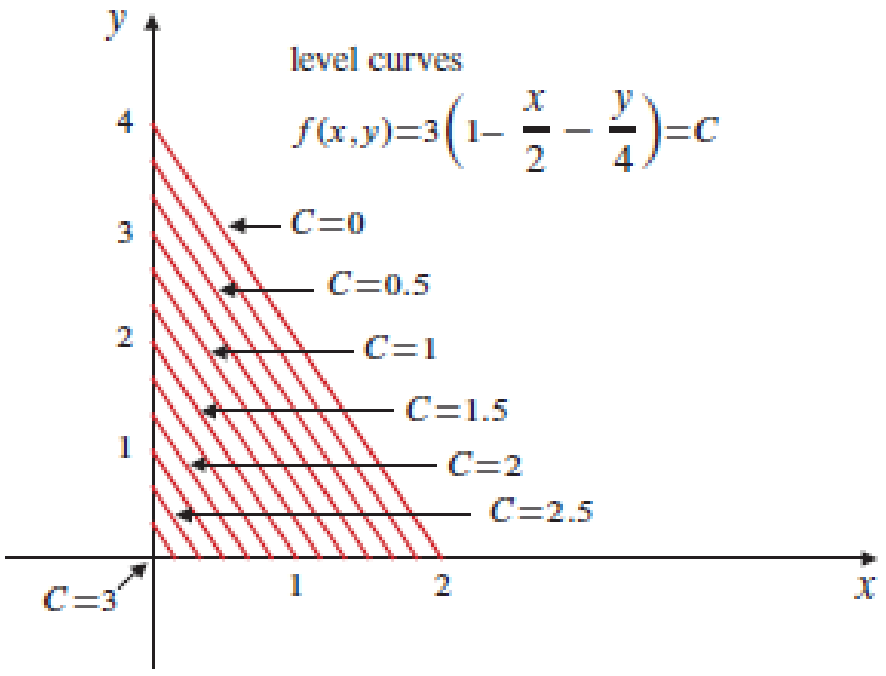
$\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{16} = 1$ Level surface for $c = 16$ (ellipsoid)



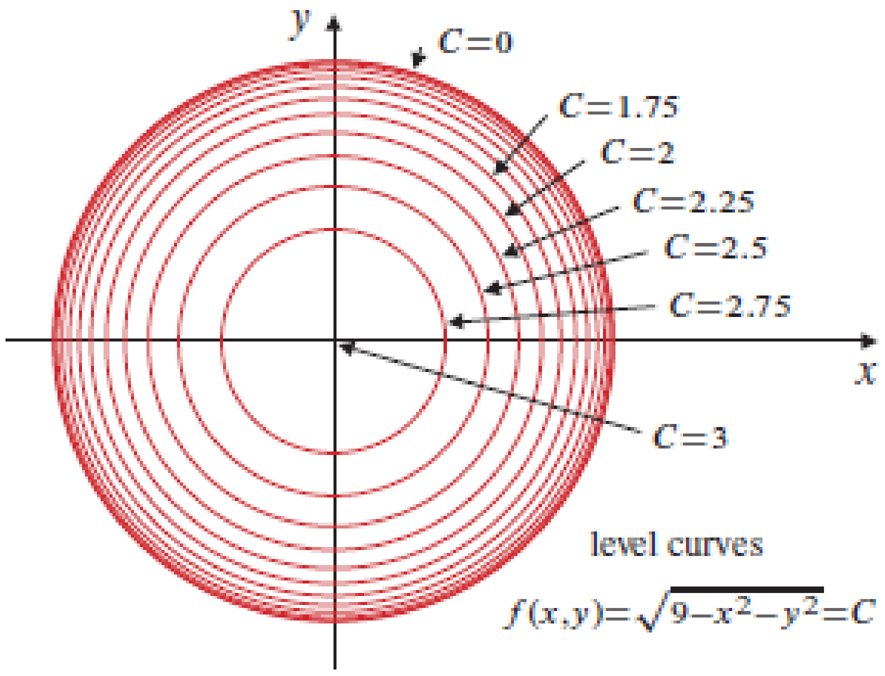
Level Curves of the graphs

$f(x,y) = 3\left(1 - \frac{x}{2} - \frac{y}{4}\right)$ and $f(x,y) = \sqrt{9 - x^2 - y^2}$

Ref.: Adams, R.A. and Essex, C., Calculus – A Complete Course, 9th Edition, Pearson, 2018.



(a)



(b)

$$3\left(1 - \frac{x}{2} - \frac{y}{4}\right) = C \qquad \text{or} \qquad \frac{x}{2} + \frac{y}{4} = 1 - \frac{C}{3}, \qquad (0 \le C \le 3),$$

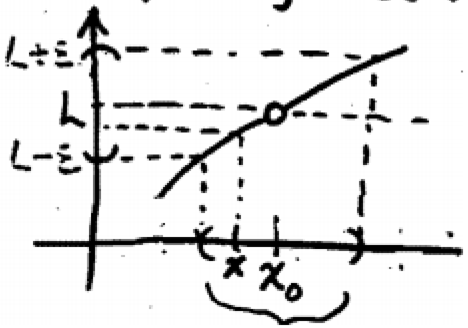
which lie in the first quadrant. Several such level curves are shown

$$\sqrt{9 - x^2 - y^2} = C \qquad \text{or} \qquad x^2 + y^2 = 9 - C^2, \qquad (0 \le C \le 3).$$

The bunching of the circles as $C \to 0^+$ indicates the steepness of the hemispherical surface, i.e. the graph of f.

Limits of Functions (Revision on Epsilon-Delta Definition of 1 variable function) – Figures based on K.Y. LI (Optional)

Let S be an interval (more generally a set), and let $f: S \to \mathbf{R}$ be a function.
$\lim_{x \to x_0} f(x) = L$ roughly means for any desired distance $\varepsilon > 0$, with $x \in S$ $(x \neq x_0)$ sufficiently close to x_0, we can obtain $d(f(x), L) < \varepsilon$, i.e. the distance between $f(x)$ and L is sufficiently small.

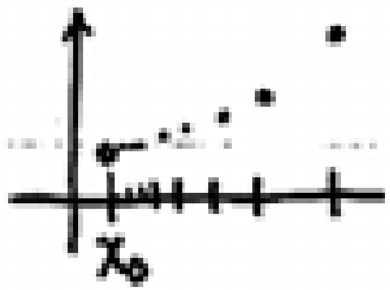


Definition (Limit)
$\lim_{x \to x_0} f(x) = L$ if and only if $\forall \varepsilon > 0, \exists \delta > 0$ such that

$$\forall x \in S,\ x \in (x_0 - \delta, x_0 + \delta) \text{ and } x \neq x_0 \text{ implies } |f(x) - L| < \varepsilon .$$

Equivalently, $\forall \varepsilon > 0, \exists \delta > 0$ such that

$$\forall x \in S, 0 < |x - x_0| < \delta \text{ implies } |f(x) - L| < \varepsilon .$$



$\mathrm{S} = \{s_1, s_2, s_3, \dots, s_n\}$ is a sequence, x_0 is a limit
Definition (Accumulation point): x_0 is an accumulation point of S if and only if there exists $s_n \in S$ (with $s_n \neq x_0$) such that $\lim_{n \to \infty} s_n = x_0$.

Note: Accumulation points may or may **NOT** be in S
Example, take $S = \left\{\frac{1}{n} : n \text{ is a natural number}\right\}$

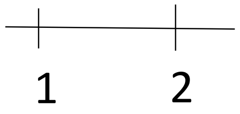
Examples (Optional)

(1) Let $f: \mathbf{R} \setminus \{0\} \to \mathbf{R}$ be defined by $f(x) = \frac{1}{7x}$. Show, by epsilon-delta approach that $\lim_{x \to 2} f(x) = \frac{1}{14}$.

(Scratch work) x is close to 2, then $\left|f(x) - \frac{1}{14}\right| = \left|\frac{1}{7x} - \frac{1}{14}\right| = \frac{|x-2|}{14x} \le \frac{|x-2|}{14} < \varepsilon$

$|x - 2| < 14\varepsilon$ and $x \ge 1$ are enough…

if $x \ge 1$ (i.e. $\delta < 1$)



Proof: $\forall \varepsilon > 0$, we let $\delta = \min(1, 14\varepsilon)$, then $\forall x \in \mathbf{R} \setminus \{0\}$,

$0 < |x - 2| < \delta$ implies $\begin{cases} |x - 2| < 1 \\ |x - 2| < 14\varepsilon \end{cases}$, which further implies to

$\begin{cases} x \in (1,3) \\ |x - 2| < 14\varepsilon \end{cases}$.

Thus, $\left|f(x) - \frac{1}{14}\right| = \frac{|x-2|}{14x} \le \frac{|x-2|}{14} < \varepsilon$ (by scratch)

Examples (Optional)

(2) Let $f: [0, +\infty) \to \mathbf{R}$ be defined by $f(x) = \frac{x}{1+2x} + \frac{2}{2+\sqrt{x}}$. Show, by epsilon-delta approach that $\lim_{x \to 1} f(x) = 1$.

(Scratch) $|f(x) - 1| = \left|\frac{x}{1+2x} + \frac{2}{2+\sqrt{x}} - 1\right| = \left|\left(\frac{x}{1+2x} - \frac{1}{3}\right) + \left(\frac{2}{2+\sqrt{x}} - \frac{2}{3}\right)\right|$

$$\leq \left|\frac{x}{1+2x} - \frac{1}{3}\right| + \left|\frac{2}{2+\sqrt{x}} - \frac{2}{3}\right| = \left|\frac{x-1}{3(1+2x)}\right| + \left|\frac{2-2\sqrt{x}}{3(2+\sqrt{x})}\right| \leq \frac{|x-1|}{3} + \frac{2\sqrt{|1-x|}}{6} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$|x-1| < \frac{3\varepsilon}{2}$ $and \sqrt{|1-x|} < \frac{3\varepsilon}{2}$

$\longrightarrow |x-1| < \frac{9\varepsilon^2}{4}$

Solution

$\forall \varepsilon > 0$, let $\delta = \min\{\frac{3\varepsilon}{2}, \frac{9\varepsilon^2}{4}\} > 0$, then

$x \in [0, +\infty)$ and $0 < |x-1| < \delta$ implies $|x-1| < \frac{3\varepsilon}{2}$ and

$|x-1| < \frac{9\varepsilon^2}{4}$, which further implies that $|f(x) - 1| < \varepsilon$.

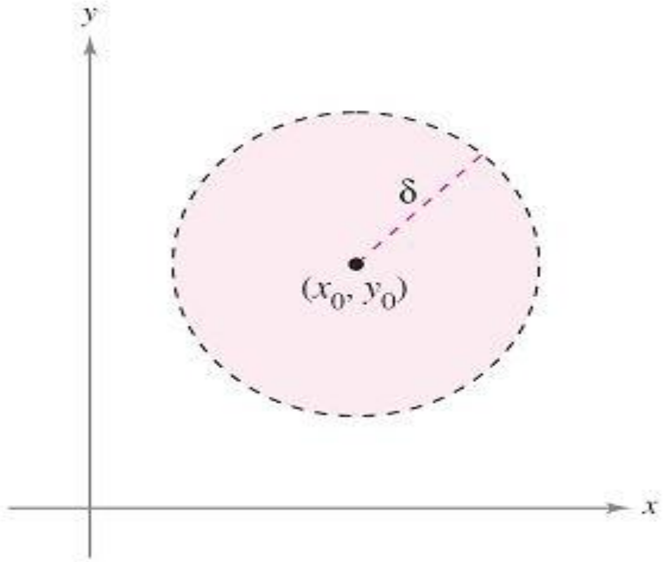
(By the scratch above)

Generalization to Functions of two variables

Using the formula for the distance between two points (x, y) and (x_0, y_0) in the plane, you can define the δ-**neighborhood** about (x_0, y_0) to be the **disk** centered at (x_0, y_0) with radius $\delta > 0$.

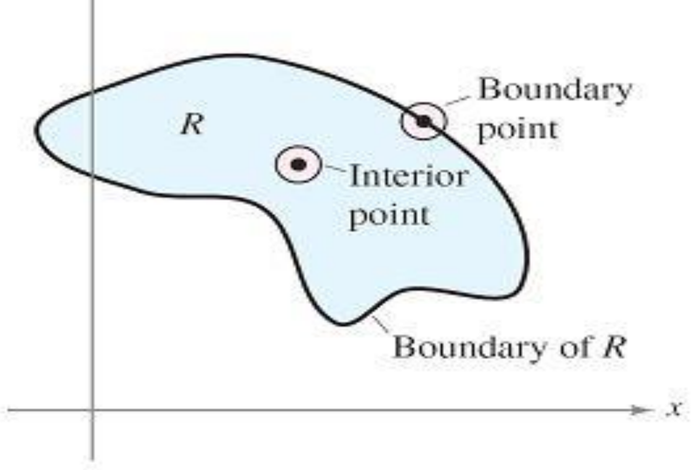
$$\{(x, y):\ \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta\}$$

Open disk



An open disk

Ref.:
Larson, R. and Edwards, B., Multivariable Calculus (Metric Version – 11th Edition), CENGAGE Learning, 2016.



The boundary and interior points of a region R

Let the region R be a set of points in the plane. A point (x_0, y_0) in R is an **interior point** of R if there exists a δ-neighborhood about (x_0, y_0) that lies entirely in R

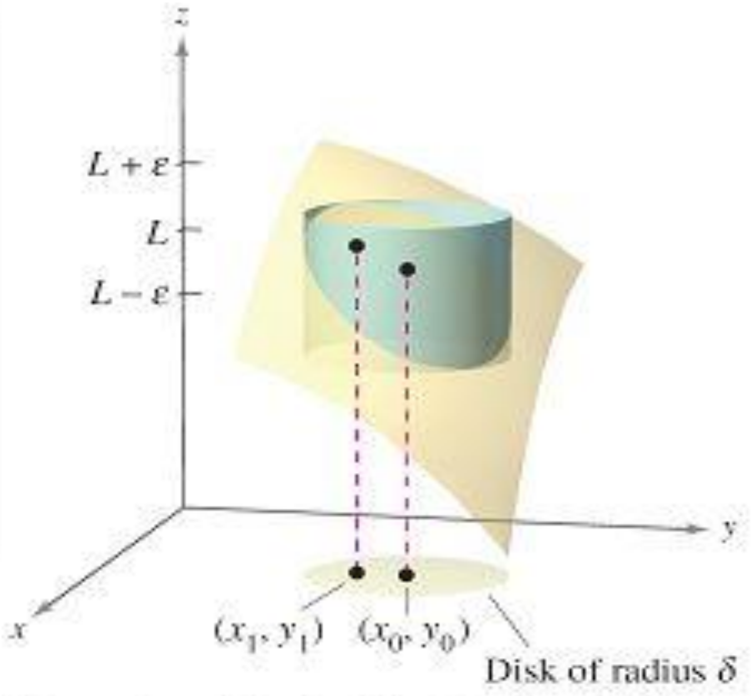
Definition of the Limit of a Function of Two Variables

Let f be a function of two variables defined, except possibly at (x_0, y_0), on an open disk centered at (x_0, y_0), and let L be a real number. Then

$$\lim_{(x, y)\to(x_0, y_0)} f(x, y) = L$$

if for each $\varepsilon > 0$ there corresponds a $\delta > 0$ such that

$$|f(x, y) - L| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$



For any (x, y) in the disk of radius δ, the value $f(x, y)$ lies between $L + \varepsilon$ and $L - \varepsilon$.

General Case

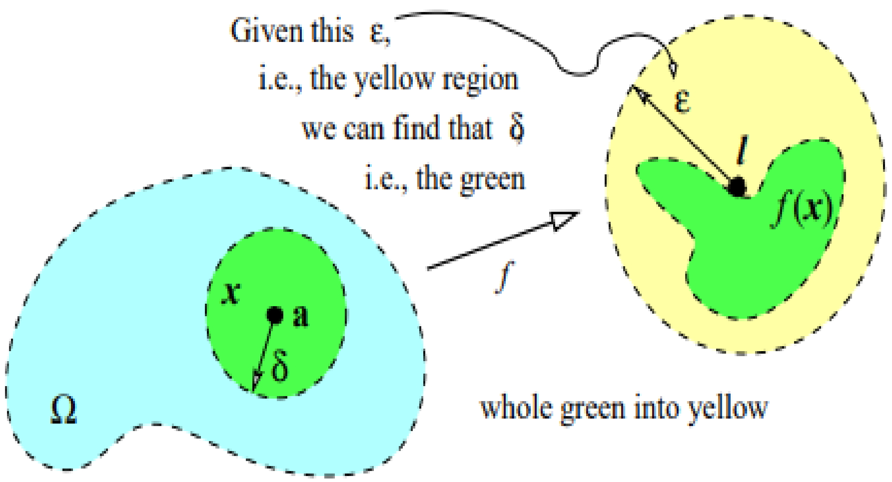
Given $f:\ \Omega \subset \mathbb{R}^n \ \to\ \mathbb{R}$ and a point $\mathbf{a} \in \Omega$ (always interior), we say $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = \ell$ if

for every $\varepsilon > 0$, there is $\delta > 0$ such that for all $0 \neq \|\mathbf{x} - \mathbf{a}\| < \delta$, $|f(\mathbf{x}) - \ell| < \varepsilon$.

From the definition, $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = \ell$ if and only if $\lim_{\mathbf{x}\to\mathbf{a}} |f(\mathbf{x}) - \ell| = 0$. In the language of distance, when the distance $0 \neq \|\mathbf{x} - \mathbf{a}\| \to 0$, we have $|f(x) - \ell| \to 0$. In set language, we have

$$f\left(B(\mathbf{a}, \delta) \setminus \{\mathbf{a}\}\right) \subset B(\ell, \varepsilon),$$

where $B(p, r) = \{x \ :\ \|x - p\| < r\}$.



Examples

(1) Show that $\lim_{(x,y)\to(1,0)} 1 + 2y + xy = 1$

Solution:

Given ε, we can pick $\delta = \min\left\{1, \frac{\varepsilon}{4}\right\}$, so from $(x-1)^2 + y^2 < \delta^2$, we have $|y| < \delta$ and $|x-1| < \delta$, then

$|x+2| = |x-1+3| \le |x-1| + 3 < \delta + 3 \le 4$ (for $\delta \le 1$)

Then, $|1 + 2y + xy - 1| = |y||2+x| < 4\delta \le \varepsilon$.

Hence, $\lim_{(x,y)\to(1,0)} 1 + 2y + xy = 1$.

(2) Find the limit: $\lim_{(x,y)\to(0,0)} \frac{2020x^2y}{x^2+y^2}$

Ref. of Figure:
Adams, R.A. and Essex, C., Calculus – A Complete Course, 9th Edition, Pearson, 2018.

In this case, the limits of the numerator and denominator are both 0, so we cannot determine whether limit exists by simply taking limits of the numerator and the denominator separately, followed by dividing the respective values.

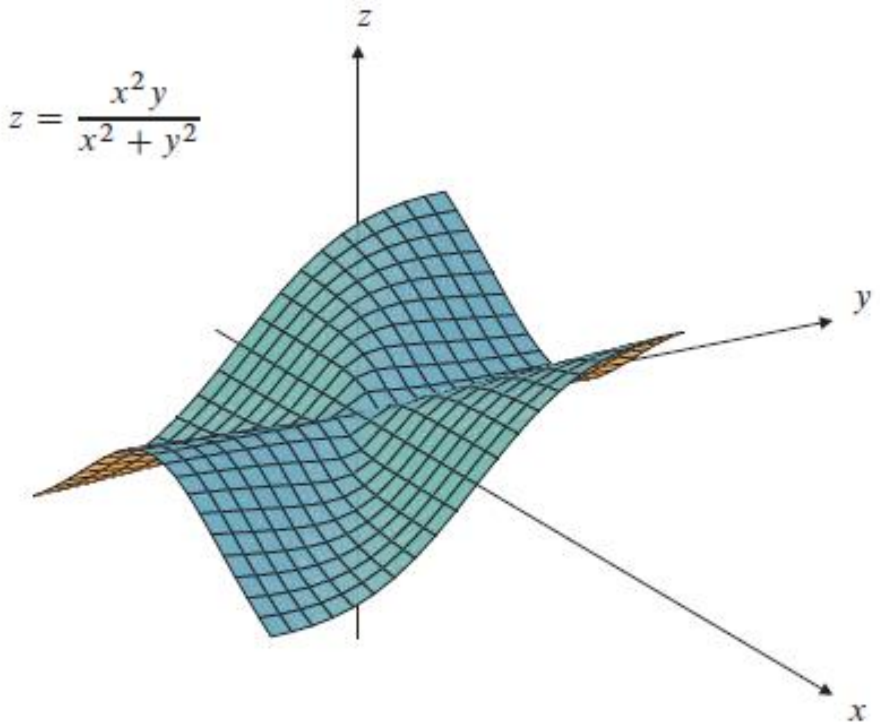
Note that $|y| \leq \sqrt{x^2+y^2}$ and $\frac{x^2y}{x^2+y^2} \leq 1$.

Then, in a δ-neighborhood about (0, 0), we have $0 < \sqrt{x^2+y^2} < \delta$, and it follows that for $(x, y) \neq (0, 0)$,

$$\left|\frac{2020x^2y}{x^2+y^2} - 0\right| = 2020|y|\frac{x^2}{x^2+y^2} \leq 2020|y|$$
$$\leq 2020\sqrt{x^2+y^2} < 2020\delta$$

Therefore, we can choose $\delta = \frac{\varepsilon}{2020}$ and conclude that $\left|\frac{2020x^2y}{x^2+y^2} - 0\right| < \varepsilon$, thus

$$\lim_{(x,y)\to(0,0)} \frac{2020x^2y}{x^2+y^2} = 0$$



Rules: If $\lim_{(x,y)\to(a,b)} f(x,y) = L$, $\lim_{(x,y)\to(a,b)} g(x,y) = M$, and every neighborhood of (a,b) contains points in $Dom(f) \cap Dom(g)$ other than (a,b), then we have the following rules.

$$\lim_{(x,y)\to(a,b)} \big(f(x,y) \pm g(x,y)\big) = L \pm M,$$

$$\lim_{(x,y)\to(a,b)} f(x,y)\,g(x,y) = LM,$$

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}, \qquad \text{provided } M \neq 0.$$

Also, if $F(t)$ is continuous at $t = L$, then

$$\lim_{(x,y)\to(a,b)} F\big(f(x,y)\big) = F(L).$$

Ref.:
Adams, R.A. and Essex, C., Calculus – A Complete Course, 9th Edition, Pearson, 2018.

Example 2

The limit of $f(x, y) = 5x^2y/(x^2 + y^2)$ as $(x, y) \to (1, 2)$ can be evaluated by direct substitution, i.e. compute limits of numerator and denominator respectively.

That is, the limit is $f(1, 2) = 2$.

Example 1

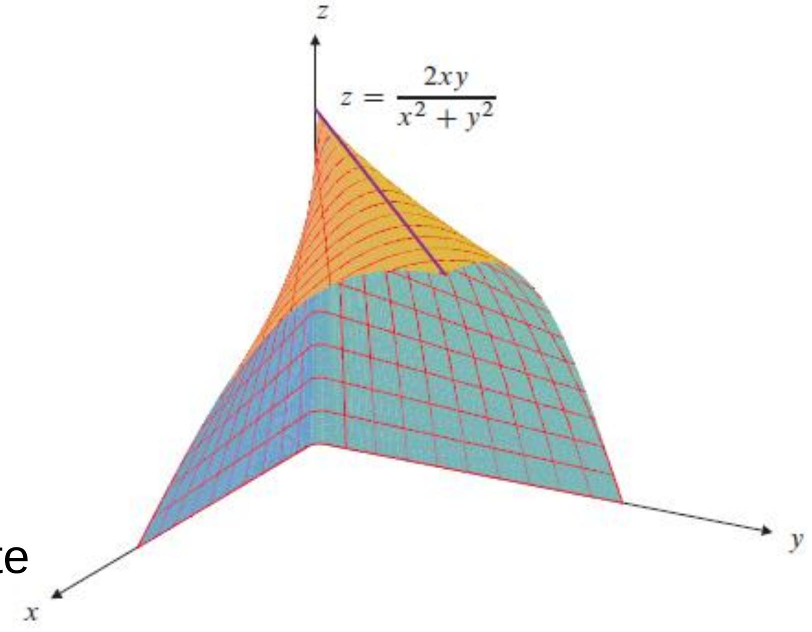
Let $f(x,y) = e^{xy} \sin \frac{\pi y}{4} + xy \ln \sqrt{y - x}$.

In a small ball of $(1,2)$, everything is continuously defined, so

$$\lim_{(x,y)\to(1,2)} f(x,y) = f(1,2) = \cdots = e^2.$$

Example 3: Investigate the limiting behavior of $f(x,y) = \frac{2xy}{x^2+y^2}$ as (x,y) approaches the origin.

- Note that $f(x,y)$ is well-defined at all points on the xy-plane except at origin.
- **Path 1:** Let $(x,y) \to (0,0)$ along the $\boldsymbol{x}$**-axis**, then $f(x,y) = f(x,0) = 0$, thus $\lim_{(x,y)\to(0,0)} f(x,y)$ must be 0 if it exists.
- **Path 2:** Let $(x,y) \to (0,0)$ along the $\boldsymbol{y}$**-axis**, then $f(x,y) = f(0,y) = 0$,
- **Path 3:** Let $(x,y) \to (0,0)$ along the line $\boldsymbol{y = kx}$, then $f(x,y) = \frac{2kx^2}{(1+k^2)x^2} = \frac{2k}{1+k^2}$, which depends on the value of *k*. For different values of *k* (i.e. different paths), the limit will be different. In particular, for $k = 1$, $f(x,y) = 1$; for $k = 3, f(x,y) = \frac{3}{5}$.
- Therefore, $\lim_{(x,y)\to(0,0)} f(x,y)$ does **NOT** exist.
- The first octant part of the graph is the "hood-shaped" surface.



$f(x,y)$ has different limits as (*x*,*y*) tends to (0, 0) along different straight lines.

Ref.: Adams, R.A. and Essex, C., Calculus – A Complete Course, 9th Edition, Pearson, 2018.

Continuity of Multivariable Functions

- A function $f(x, y)$ is continuous at (x_0, y_0) if the following two conditions hold:

  (1) $(x_0, y_0) \in Dom\left(f(x, y)\right)$

  (2) $\forall \varepsilon > 0, \exists \delta > 0$ such that whenever $(x, y) \in Dom(f)$, with $\sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$, then we have $|f(x, y) - f(x_0, y_0)| < \varepsilon$.

Definition of Continuity of a Function of Two Variables

A function f of two variables is **continuous at a point** (x_0, y_0) in an open region R if $f(x_0, y_0)$ is defined and is equal to the limit of $f(x, y)$ as (x, y) approaches (x_0, y_0). That is,

$$\lim_{(x, y) \to (x_0, y_0)} f(x, y) = f(x_0, y_0).$$

The function f is **continuous in the open region** R if it is continuous at every point in R.

Ref.:
Larson, R. and Edwards, B., Multivariable Calculus (Metric Version – 11th Edition), CENGAGE Learning, 2016.

THEOREM 13.1 Continuous Functions of Two Variables

If k is a real number and $f(x, y)$ and $g(x, y)$ are continuous at (x_0, y_0), then the following functions are also continuous at (x_0, y_0).

1. Scalar multiple: kf
2. Sum or difference: $f \pm g$
3. Product: fg
4. Quotient: f/g, $g(x_0, y_0) \neq 0$

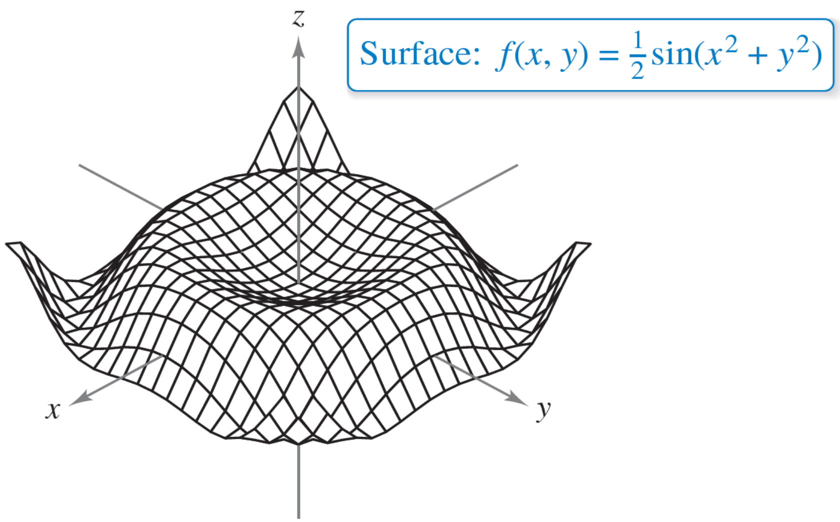
THEOREM 13.2 Continuity of a Composite Function

If h is continuous at (x_0, y_0) and g is continuous at $h(x_0, y_0)$, then the composite function given by $(g \circ h)(x, y) = g(h(x, y))$ is continuous at (x_0, y_0). That is,

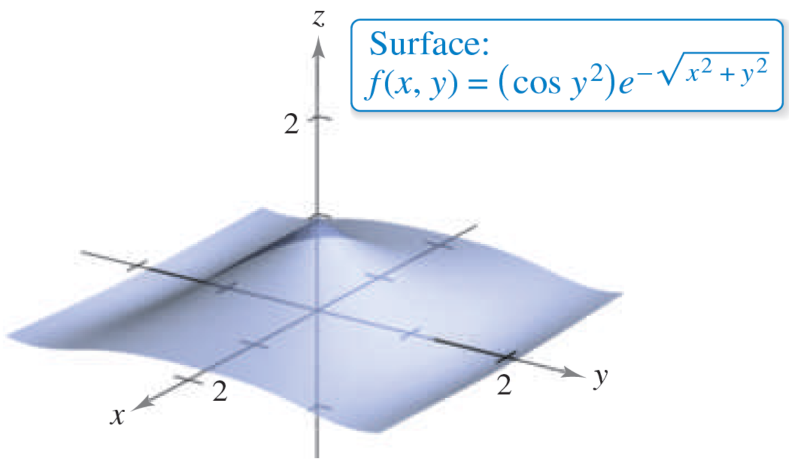
$$\lim_{(x, y) \to (x_0, y_0)} g(h(x, y)) = g(h(x_0, y_0)).$$

Ref.:
Larson, R. and Edwards, B., Multivariable Calculus (Metric Version – 11th Edition), CENGAGE Learning, 2016.

In the above theorem, note that *h* is a function of two variables and *g* is a function of one variable.



The function f is continuous at every point in the plane.



The function f is continuous at every point in the plane.

For function of single variable:

If $\lim_{x \to x_0} f(x) = L$ and $f(x_0) = L$, then $f(x)$ is *continuous* at $x = x_0$.

For function of n variables:

If $\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = L$ and $f(\mathbf{x}_0) = L$, then $f(\mathbf{x})$ is *continuous* at $\mathbf{x} = \mathbf{x}_0$.

Note also that if $f(x)$ and $g(x)$ are both *continuous* functions, then the composite function $f[g(x)]$ is also *continuous*.

If $f(x)$ is a *continuous* function of one variable and $g(x, y)$ is a *continuous* function of two variables, then

$$h(x, y) = f[g(x, y)]$$

is a *continuous* function of x and y.

As for functions of one variable, the existence of a limit of a function at a point does not imply that the function is continuous at that point. The function

$$f(x, y) = \begin{cases} 0 & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0) \end{cases}$$

satisfies $\lim_{(x,y)\to(0,0)} f(x, y) = 0$, which is not equal to $f(0, 0)$, so f is not continuous at $(0, 0)$. Of course, we can *make* f continuous at $(0, 0)$ by redefining its value at that point to be 0.

Example:

$f(x) = \dfrac{\sin x}{x}$ itself is **NOT** continuous, however

$$g(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

is continuous.

Ref.:
Adams, R.A. and Essex, C., Calculus – A Complete Course, 9th Edition, Pearson, 2018.

Example:

How can the function

$$f(x, y) = \frac{x^2 + y^2 - x^3 y^3}{x^2 + y^2}, \qquad (x, y) \neq (0, 0),$$

be defined at the origin so that it becomes continuous at all points of the xy-plane?

Ref.:
Adams, R.A. and Essex, C., Calculus – A Complete Course, 9th Edition, Pearson, 2018.

Solution:

$$f(x, y) = \frac{x^2 + y^2 - x^3 y^3}{x^2 + y^2} = 1 - \frac{x^3 y^3}{x^2 + y^2}.$$ But

$$\left|\frac{x^3 y^3}{x^2 + y^2}\right| = \left|\frac{x^2}{x^2 + y^2}\right| |xy^3| \leq |xy^3| \to 0$$

as $(x, y) \to (0, 0)$. Thus $\lim_{(x,y)\to(0,0)} f(x, y) = 1 - 0 = 1.$

Define $f(0, 0) = 1.$