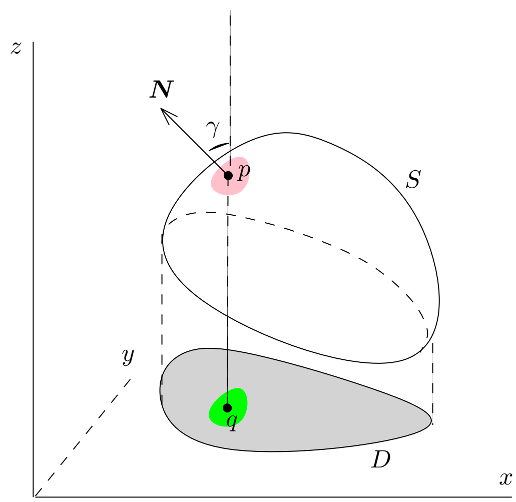
# Lecture Notes: Surface Areas

Yufei Tao
Department of Computer Science and Engineering
Chinese University of Hong Kong
*taoyf@cse.cuhk.edu.hk*

## 1 Area of a Smooth xy-Monotone Surface

Let $S$ be a smooth xy-monotone surface, and $D$ the projection of $S$ onto the xy-plane. Next, we will define its *area*. Note that this currently still remains undefined—you may have a clear understanding of the area of a *two-dimensional* region, but we are talking about a *three*-dimensional surface here.

We need to introduce some notions that will be helpful along the way. Consider an arbitrary point $q = (x, y)$ in $D$. This point *uniquely* corresponds to a point $p = (x, y, z)$ on $S$ by the xy-monotonicity of $S$. Let $\boldsymbol{N}$ be a normal vector of $S$ at $p$, and $\gamma$ be the angle between the direction of $\boldsymbol{N}$ and the positive direction of the z-axis (namely, of $\boldsymbol{k} = [0, 0, 1]$). Note that $\gamma$ is a function of $x, y$, because of which we will denote it by $\gamma(x, y)$.




We are now ready to define the area $A$ of $S$ as the value of a double integral:

$$A = \iint_D \frac{1}{|\cos(\gamma(x, y))|} \, dxdy. \tag{1}$$

The above definition is in fact much less mysterious than it may appear. Let $D'$ be a small region in $D$ that contains $q$, and $S'$ be the part of $S$ such that $D'$ is precisely the projection of $S'$ onto the xy-plane. In the figure above, $D'$ is shown in green, and $S'$ in pink. When $D'$ is sufficiently small, we can regard $S'$ approximately as a planar region embedded in $\mathbb{R}^3$. Thus, $area(D') \approx area(S') \cdot |\cos \gamma|$. Now, imagine partitioning $D$ into a huge number of such small "green" pieces, and sum up the areas of all the "pink" pieces on $S$ that correspond to those green pieces, respectively. The sum is an approximation of the area of $S$. When the number of "green" pieces tends to infinity, the sum equals the value of the double integral in (1).

The key to evaluating (1) is to work out $\cos(\gamma(x, y))$. The rest of the lecture notes gives two ways to do so, depending on the representation of $S$.

## 2 Area Calculation: Method 1

Suppose that $S$ is given as an equation $f(x, y, z) = 0$. We know that $\nabla f = [\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}]$ is a normal vector at point $p = (x, y, z)$. We thus know that:

$$\begin{aligned}
\left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] \cdot \boldsymbol{k} &= \left|\left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]\right| |\boldsymbol{k}| \cos\gamma \\
\Rightarrow \cos\gamma &= \frac{\left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] \cdot \boldsymbol{k}}{\left|\left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]\right|} \\
&= \frac{\frac{\partial f}{\partial z}}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2}}. \qquad (2)
\end{aligned}$$

**Example 1.** Calculate the area of the hemisphere $x^2 + y^2 + z^2 = 1$ with $z \geq 0$.

*Solution.* Let $S$ be the hemisphere, and $D$ be the projection of $S$ onto the xy-plane, namely, the disc $x^2 + y^2 \leq 1$.

Introduce $f(x, y, z) = x^2 + y^2 + z^2 - 1$. $S$ can be described by $f(x, y, z) = 0$ with $z \geq 0$. We thus have: $\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = 2y$, and $\frac{\partial f}{\partial z} = 2z$. By (2), we know:

$$\cos(\gamma(x, y, z)) = \frac{2z}{\sqrt{(2x)^2 + (2y)^2 + (2z)^2}} = z.$$

Therefore, from (2), we know that the area of $S$ equals

$$\iint_D \frac{1}{z}\, dxdy = \iint_D \frac{1}{\sqrt{1 - x^2 - y^2}}\, dxdy = 2\pi.$$

□

## 3 Area Calculation: Method 2

Suppose that $S$ is given in a parametric form with parameters $u, v$, namely, the x-, y-, and z-coordinates of each point on $S$ are given by functions $x(u, v)$, $y(u, v)$, and $z(u, v)$, respectively.

Define vectors $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$ as follows:

$$\begin{aligned}
\boldsymbol{r}_u &= \left[\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right] \\
\boldsymbol{r}_v &= \left[\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right].
\end{aligned}$$

We have:

**Lemma 1.** *If $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$ neither have the same nor have the opposite directions, then*

$$\boldsymbol{N} = \boldsymbol{r}_u \times \boldsymbol{r}_v$$

*is a normal vector of $S$ at the point $(x(u, v), y(u, v), z(u, v))$.*

*Proof.* Notice that $\boldsymbol{r}_u$ is a tangent vector of the curve $\boldsymbol{r}(u) = [x(u,v), y(u,v), z(u,v)]$ by fixing $v$ to be a constant. Likewise, $\boldsymbol{r}_v$ is a tangent vector of the curve $\boldsymbol{r'}(v) = [x(u,v), y(u,v), z(u,v)]$ by fixing $u$ to be a constant. Hence, $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$ determine the tangent plane of $S$ at the point $(x(u,v), y(u,v), z(u,v))$. The lemma then follows from the geometric interpretation of cross product. $\square$

Therefore:

$$\begin{aligned} \boldsymbol{N} \cdot \boldsymbol{k} &= |\boldsymbol{N}||\boldsymbol{k}| \cos\gamma \\ \Rightarrow \cos\gamma &= \frac{\boldsymbol{N} \cdot \boldsymbol{k}}{|\boldsymbol{N}|} \\ \text{(by definition of cross product)} &= \frac{\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}}{|\boldsymbol{N}|}. \end{aligned}$$

Next, we convert (1) into a double integral on $u, v$. For this purpose, we calculate the Jarcobian:

$$J = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

Denoting $R$ as the set of $(u,v)$ defining $D$, we know:

$$\begin{aligned} \iint_D \frac{1}{|\cos(\gamma(x,y))|}\, dxdy &= \iint_R \frac{|\boldsymbol{N}|}{\left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right|} \cdot |J|\, dudv \\ &= \iint_R |\boldsymbol{N}|\, dudv. \end{aligned} \tag{3}$$

**Example 2.** Calculate the area of the hemisphere $x^2 + y^2 + z^2 = 1$ with $z \geq 0$.

*Solution.* Let $S$ be the hemisphere, and $D$ be the projection of $S$ onto the xy-plane, namely, the disc $x^2 + y^2 \leq 1$.

Represent $S$ as a parametric form:

$$\begin{aligned} x(u,v) &= \cos u \cdot \sin v \\ y(u,v) &= \sin u \cdot \sin v \\ z(u,v) &= \cos v \end{aligned}$$

for $u \in [0, 2\pi]$ and $v \in [0, \pi/2]$. Let $R$ be the set of such $(u,v)$. Therefore:

$$\begin{aligned} \boldsymbol{r}_u &= \left[\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right] = [-\sin u \sin v, \cos u \sin v, 0] \\ \boldsymbol{r}_v &= \left[\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right] = [\cos u \cos v, \sin u \cos v, -\sin v]. \end{aligned}$$

Hence:

$$\begin{aligned} \boldsymbol{N} &= \boldsymbol{r}_u \times \boldsymbol{r}_v \\ &= [-\cos u \sin^2 v, -\sin u \sin^2 v, -\sin^2 u \sin v \cos v - \cos^2 u \sin v \cos v] \\ &= [-\cos u \sin^2 v, -\sin u \sin^2 v, -\sin v \cos v] \end{aligned}$$

which means

$$|\boldsymbol{N}| = \sqrt{\sin^4 v + \sin^2 v \cos^2 v} = \sin v.$$

Therefore, the area of the hemisphere is

$$\begin{aligned}\iint_R |\boldsymbol{N}|\, dudv &= \iint_R \sin v\, dudv \\ &= \int_0^{2\pi} \left( \int_0^{\pi/2} \sin v\, dv \right) du = 2\pi.\end{aligned}$$

□