Multivariable Calculus (Week 5)

Topics: Review of Continuity

Partial Derivatives, Chain Rule, Differentials

Gradients

(**Reference Chapters:** Chapter 12.3 – 12.8 of Adams and Essex; Chapter 13.3 – 13.6 of Larson and Edwards)

Key References of this file:

Larson, R. and Edwards, B., Multivariable Calculus (Metric Version – 11th Edition), CENGAGE Learning, 2016.

Adams, R.A. and Essex, C., Calculus – A Complete Course, 9th Edition, Pearson, 2018.

Prepared and Revised by Dr. Hugo MAK

Q & A (from Week 4)

HW 2 Question 5

We define the curvature of a path by $\|\mathbf{r}''(s)\|$, where $\mathbf{r}(s)$ is the arc-length parametrization of the path. Given a path $\mathbf{r}(t)$, we let $\mathbf{r}(s)$ be its arc-length parametrization so that $s = \int_0^t \|\mathbf{r}'(\tau)\| d\tau$.

Students' Questions: (1) Why are $\mathbf{r}'(s)$ and $\mathbf{r}''(s)$ orthogonal vectors?

(2) Why is $\|\mathbf{r}'(s)\| = 1$?

Responses: For Question (2), you may refer to the definition of "arc-length parametrization"

If $\mathbf{r}(s)$ is a parametric curve such that $\|\mathbf{r}'(s)\| = 1$ for any s, we say the curve is parametrized by arc-length.

As for Question (1), here you are the claim:

Statement: Let $\mathbf{r}(t)$ be a twice continuously differentiable vector-valued function (which describes the curve), then the speed $\|\mathbf{r}'(t)\| = C$ (where C is a constant) **if and only if** the velocity $\mathbf{r}'(t)$ and the acceleration vector $\mathbf{r}''(t)$ are perpendicular for all t. (In the question, our $\|\mathbf{r}'(s)\| = 1$.)

Proof: Consider $\frac{d}{dt}[\mathbf{r}'(t) \cdot \mathbf{r}'(t)] = \mathbf{r}''(t) \cdot \mathbf{r}'(t) + \mathbf{r}'(t) \cdot \mathbf{r}''(t) = 2\mathbf{r}'(t) \cdot \mathbf{r}''(t)$

If $\|\mathbf{r}'(t)\| = C$ for all t, then $0 = \frac{d(C^2)}{dt} = \frac{d}{dt}[\mathbf{r}'(t) \cdot \mathbf{r}'(t)] = 2\mathbf{r}'(t) \cdot \mathbf{r}''(t)$ implies the two vectors are perpendicular to each other.

On the other hand, if $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = 0$, then $0 = 2\mathbf{r}'(t) \cdot \mathbf{r}''(t) = \frac{d}{dt}[\mathbf{r}'(t) \cdot \mathbf{r}'(t)]$, thus the speed $\|\mathbf{r}'(t)\| = \sqrt{\mathbf{r}'(t) \cdot \mathbf{r}'(t)}$ is a constant.

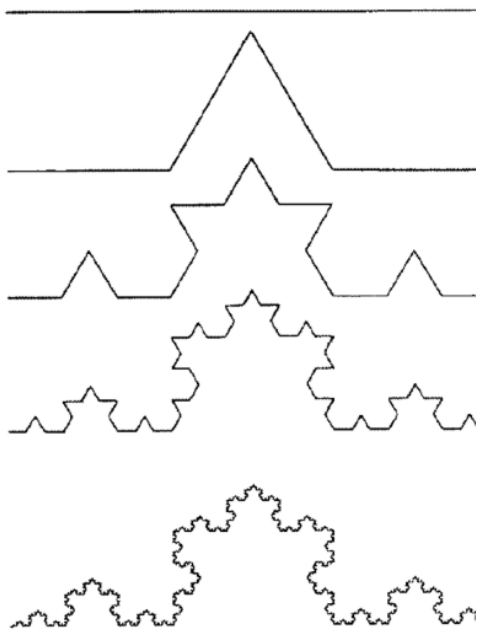
Q & A (from Week 4)

Remarks or Q&A from students:

- $\varepsilon - \delta$ approach (Slide 33-35) is **NOT** required in exam, just for your reference.
- Rectification / Rectifiable (**NOT** a key word in our content, will **NOT** be included in exam)

1. Rectification of curve: Find the length of an irregular arc segment of that curve
2. Rectifiable curve: There exists a smallest number *L* that can serve as an upper bound on the length of any polygonal approximation of that curve. *L:* arc-length
3. For further details, please refer to https://en.wikipedia.org/wiki/Arc_length#Definition_for_a_smooth_curve
4. Non-rectifiable curve: Length of that curve can be made arbitrarily large.

   Example: Koch curve (also known as Koch snowflake)



- Recall an example of Continuity

$f(x) = \frac{\sin x}{x}$ itself is **NOT** continuous, however

$$g(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

is continuous.

1 more example on Continuity

Example: Show that

$$f(x,y)=\begin{cases}\dfrac{3xy}{x^2+y^2} & \text{when } (x,y)\neq(0,0)\\ 0 & \text{when } (x,y)=(0,0)\end{cases}$$

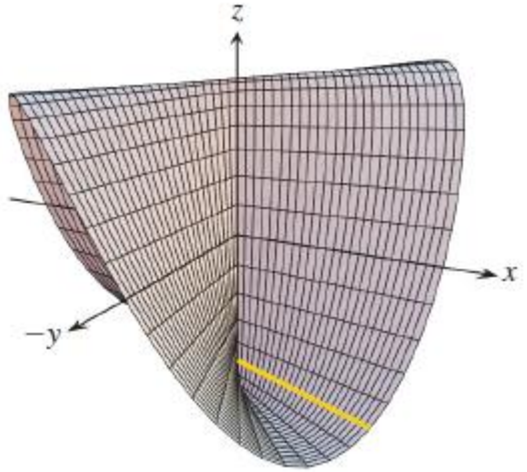
is continuous at every point in $\mathbf{R}^2$ except at the origin.

DEFINITION A function $f(x, y)$ is **continuous at the point** (x_0, y_0) if

1. f is defined at (x_0, y_0),
2. $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists,
3. $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$.

A function is **continuous** if it is continuous at every point of its domain.

<u>Ref.: From Thomas Calculus</u>



- f is well-defined at all points in $\mathbf{R}^2$ (including the origin).
- The function *f* is continuous at any point $(x,y)\neq(0,0)$, because both $3xy$ and x^2+y^2 are polynomials (rational functions of x and y), therefore the limit values can be obtained by substituting respective (x,y) into the functional expression.
- <u>At origin, the value of *f* is well-defined, but *f* has no limit as $(x,y)\to(0,0)$.</u>

Reason: Different paths directed towards the origin will lead to different limits.

1 more example on Continuity

Example: Show that

$$f(x,y) = \begin{cases} \frac{3xy}{x^2+y^2} & \text{when } (x,y) \neq (0,0) \\ 0 & \text{when } (x,y) = (0,0) \end{cases}$$

is continuous at every point in $\mathbf{R}^2$ except at the origin.

Note that the function f has a constant value on the "punctured" line $y = mx$ (straight line with slope m) for all m, where $x \neq 0$.

$$f(x,y)\Big|_{y=mx} = \frac{3xy}{x^2+y^2}\Bigg|_{y=mx} = \frac{3mx^2}{x^2+(mx)^2} = \frac{3m}{1+m^2}$$

Thus, f will have this number as its limit as $(x,y) \to (0,0)$ along the line.

$$\lim_{\substack{(x,y)\to(0,0) \\ along\ y=mx}} f(x,y) = \frac{3m}{1+m^2}$$

Such expression (i.e. the limit) changes when m changes.

The limit fails to exists, thus conditions (2) and (3) of "continuity" do not hold, therefore the function is **NOT** continuous.

<u>1 more example on Continuity</u>

Example: Show that

$f(x,y) = \begin{cases} \frac{3xy}{x^2+y^2} & \text{when } (x,y) \neq (0,0) \\ 0 & \text{when } (x,y) = (0,0) \end{cases}$ is continuous at every point in $\mathbf{R}^2$ except at the origin.

<u>Alternative approach (by Polar coordinates)</u>

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{3xy}{x^2+y^2} = \lim_{r\to 0^+} \frac{3(r\cos\theta)(r\sin\theta)}{r^2} = \lim_{r\to 0^+} 3\cos\theta\sin\theta = \lim_{r\to 0^+} \frac{3}{2}\sin 2\theta$$

(independent of *r*)

Then, we take two different directions towards the origin (i.e. two values of θ) as follows:

Path 1 ($\theta = \frac{\pi}{4}$ (straight line with slope 1)): The limit will be $\frac{3}{2}$.

Path 2 ($\theta = 0$ (the x-axis)): The limit will be 0.

Therefore, the above limit does <u>**NOT**</u> exist, thus violating conditions (2) and (3).

Partial Derivatives

In 1-variable calculus, for $y = f(x)$, we have $\frac{dy}{dx}$ or $\frac{df}{dx}$ (rate of change of *f* with respect to *x)*

In multivariable calculus, one can determine the **rate of change of a function *f*** with respect to one of its several independent variables.

This process is called **partial differentiation,** and the result is referred to as the **partial derivative** of *f* with respect to the chosen independent variable.

Idea:

When we differentiate a function of several variables with respect to one of the independent variables, we have to keep all other independent variables constant, then apply product rule, quotient rule, chain rule etc.

e.g. $z = f(x, y, t)$, we can have $z_x = f_x = \frac{\partial f}{\partial x}$; $z_y = f_y = \frac{\partial f}{\partial y}$ and $z_t = f_t = \frac{\partial f}{\partial t}$ respectively.

When we compute $\frac{\partial f}{\partial x}$, we keep both y and t constant.

Partial Derivatives

Motivation: Extreme humidity makes us think that the temperature is higher than actual temperature, whereas when the air is very dry, we perceive the temperature to be lower than the indicator on thermometer.

Daily Life Example:

The National Weather Service has devised the **heat index** (also called the **temperature-humidity index,** or **humidex**) to describe the combined effects of temperature and humidity.

The **heat index** I is the perceived air temperature when the **actual temperature** is T and the **relative humidity** is H. Thus, I is a function of T and H and we can write $I = f(T, H)$.

Relative humidity (%)

Actual temperature (°F)

T \ H	50	55	60	65	70	75	80	85	90
90	96	98	100	103	106	109	112	115	119
92	100	103	105	108	112	115	119	123	128
94	104	107	111	114	118	122	127	132	137
96	109	113	116	121	125	130	135	141	146
98	114	118	123	127	133	138	144	150	157
100	119	124	129	135	141	147	154	161	168

Figure from
Stewart Calculus
(National Weather Service)

Partial Derivatives (Example due to Stewart Calculus)

- We now concentrate on the highlighted column in the table, where $H = 70\%$.
- Thus, we are now considering the heat index as a function of the single variable *T* for a fixed value of *H*. Therefore, *g*(*T*) = *f*(*T*, 70). *g*(*T*) describes how the heat index *I* increases as the actual temperature *T* increases when the relative humidity is 70%.
- The derivative of *g* when *T* = 96°F is the rate of change of *I* with respect to *T* when *T* = 96°F.

$$g'(96) = \lim_{h\to 0} \frac{g(96+h) - g(96)}{h} = \lim_{h\to 0} \frac{f(96+h, 70) - f(96,70)}{h}$$

Relative humidity (%)

Actual temperature (°F)

T \ H	50	55	60	65	70	75	80	85	90
90	96	98	100	103	106	109	112	115	119
92	100	103	105	108	112	115	119	123	128
94	104	107	111	114	118	122	127	132	137
96	109	113	116	121	125	130	135	141	146
98	114	118	123	127	133	138	144	150	157
100	119	124	129	135	141	147	154	161	168

We try to approximate $g'(96)$ by taking $h = 2$ and $h = -2$ respectively.

$$g'(96) \approx \frac{g(98) - g(96)}{2} = \frac{133 - 125}{2} = 4$$

$$g'(96) \approx \frac{g(94) - g(96)}{-2} = \frac{118 - 125}{-2} = 3.5$$

Averaging these values, we say that the derivative *g*′(96) is **<u>approximately</u>** 3.75.

When the actual temperature is 96°F and the relative humidity is 70%, the apparent temperature (heat index) rises by about 3.75°F for every degree that the actual temperature rises.

General Definition of Partial Derivatives

- In general, if *f* is a function of two variables *x* and *y*, suppose we only vary *x* while keeping *y* fixed, say *y* = *b*, where *b* is a constant.
- Then, we are actually considering a function of *x*, namely, *g*(*x*) = *f*(*x*, *b*). If *g* has a derivative at the point *a*, then we denote it as the **partial derivative of *f* with respect to *x* at (*a*, *b*)** - $f_x(a, b)$. Thus, we have

$$f_x(a,b) = g'(a), \text{ with } g(x) = f(x,b)$$

- By definition, we have $g'(a) = \lim_{h\to 0} \frac{g(a+h)-g(a)}{h}$, hence

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$$

- Similarly, the **partial derivative of *f* with respect to *y* at (*a*, *b*), i.e.** $f_y(a, b)$, is obtained by keeping *x* fixed (at *x* = *a*), then we find the ordinary derivative of the function *G*(*y*) = *f*(*a*, *y*) at b.

$$f_y(a,b) = \lim_{k\to 0} \frac{f(a,b+k)-f(a,b)}{k}$$

General Definition of Partial Derivatives

Definition of Partial Derivatives of a Function of Two Variables

If $z = f(x, y)$, then the **first partial derivatives** of f with respect to x and y are the functions f_x and f_y defined as follows, provided that the limits exist:

$$\frac{\partial f(x,y)}{\partial x} := f_x(x,y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}$$

$$\frac{\partial f(x,y)}{\partial y} := f_y(x,y) = \lim_{k \to 0} \frac{f(x, y+k) - f(x,y)}{k}$$

Example: If $f(x, y) = x^3 + x^2y^3 - 2y^2$, find $f_x(3, 1)$ and $f_y(1, 7)$.

Solution: Holding *y* constant and differentiating with respect to x, we have

$$f_x(x, y) = 3x^2 + 2xy^3$$

Hence, $f_x(3, 1) = 3(3)^2 + 2(3)(1)^3 = 33$.

Similarly, holding x constant and differentiating with respect to y, we have $f_y(x, y) = 3x^2y^2 - 4y$, thus $f_y(1, 7) = 3(1)^2(7)^2 - 4(7) = 119$

General Definition of Partial Derivatives

Definition of Partial Derivatives of a Function of *n* Variables

If $f: \mathbf{R}^n \to \mathbf{R}$, i.e. $z = f(x_1, x_2, \ldots, x_n)$, then

$$\frac{\partial f}{\partial x_i} = \lim_{\Delta x_i \to 0} \frac{f(x_1, x_2, \ldots, x_i + \Delta x_i, x_{i+1}, \ldots, x_n) - f(x_1, x_2, \ldots, x_i, \ldots, x_n)}{\Delta x_i}$$

Example: $z = f(x, y, t, w, a, b, c, d) = 3x^2yabc + 2xwty - \ln(ac) + e^{axy}$

$$\frac{\partial f}{\partial x} = 6xyabc + 2wty + aye^{axy} \quad ; \quad \frac{\partial f}{\partial y} = 3x^2abc + 2xwt + axe^{axy}$$

$$\frac{\partial f}{\partial t} = 2xwy \quad ; \quad \frac{\partial f}{\partial w} = 2xty$$

$$\frac{\partial f}{\partial a} = 3x^2ybc - \frac{1}{a} + xye^{axy} \quad ; \quad \frac{\partial f}{\partial b} = 3x^2yac$$

$$\frac{\partial f}{\partial c} = 3x^2yab - \frac{1}{c} \quad ; \quad \frac{\partial f}{\partial d} = 0$$

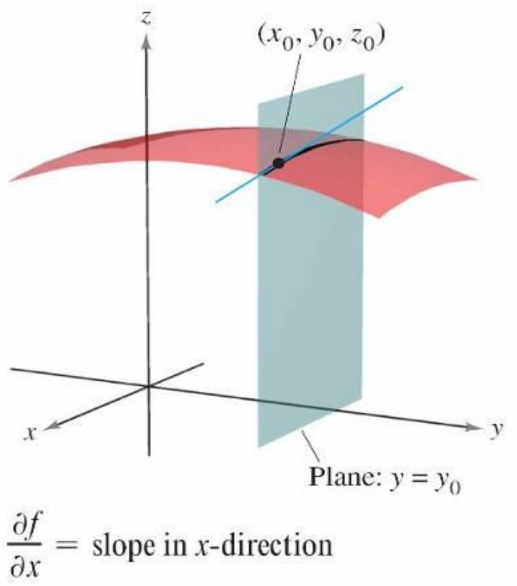
Geometric Interpretation of Partial Derivatives (2 variables)

Given a function of two variables $z = f(x, y)$, we can interpret the partial derivatives of f geometrically.

If $y = y_0$, then $z = f(x, y_0)$ represents the curve formed by intersecting the surface $z = f(x, y)$ with the plane $y = y_0$.

$$f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

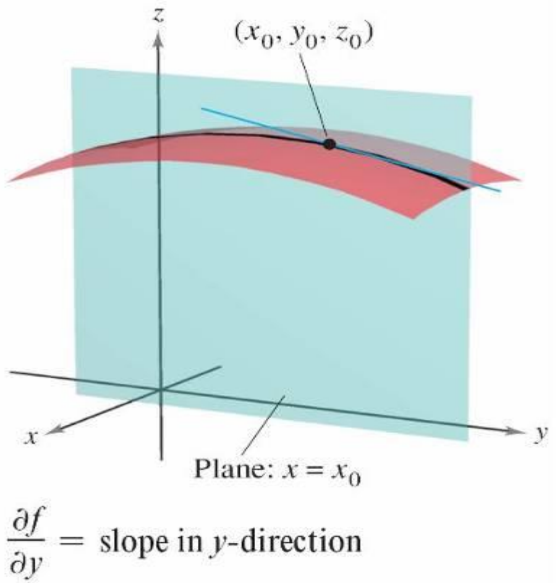
represents the **slope of this curve** at the point $(x_0, y_0, f(x_0, y_0))$.



Remarks: Both the curve and the tangent line lie in the plane $y = y_0$.

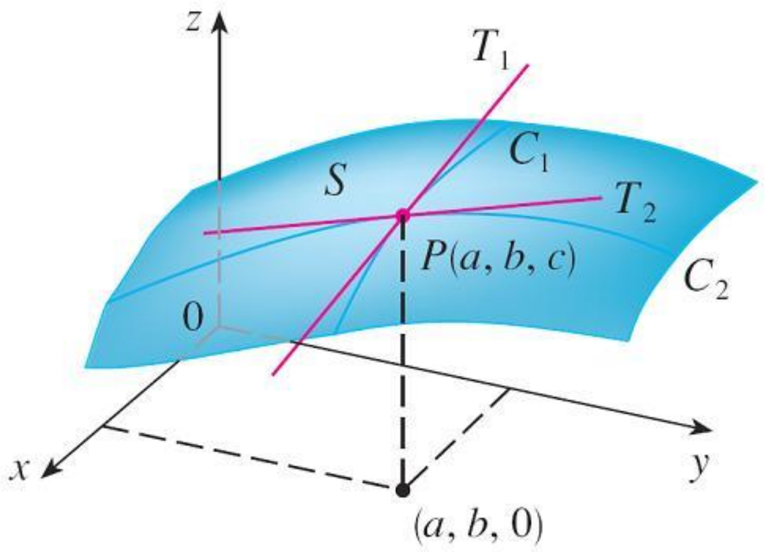
Similarly, $f_y(x_0, y_0) = \lim_{k \to 0} \frac{f(x_0, y_0 + k) - f(x_0, y_0)}{k}$

represents the **slope of the curve** given by the intersection of $z = f(x, y)$ and the plane $x = x_0$ at $(x_0, y_0, f(x_0, y_0))$.



Figures due to Larson and Edwards

Geometric Interpretation of Partial Derivatives (2 variables)



The partial derivatives of f at (a, b) are **the slopes** of the tangents to C_1 and C_2.

Figure due to Stewart

- The curve C_1 is the graph of the function $g(x) = f(x, b)$, therefore the **slope of its tangent** T_1 at P is $g'(a) = f_x(a, b)$.

- The curve C_2 is the graph of the function $G(y) = f(a, y)$, therefore the **slope of its tangent** T_2 at P is $G'(b) = f_y(a, b)$.

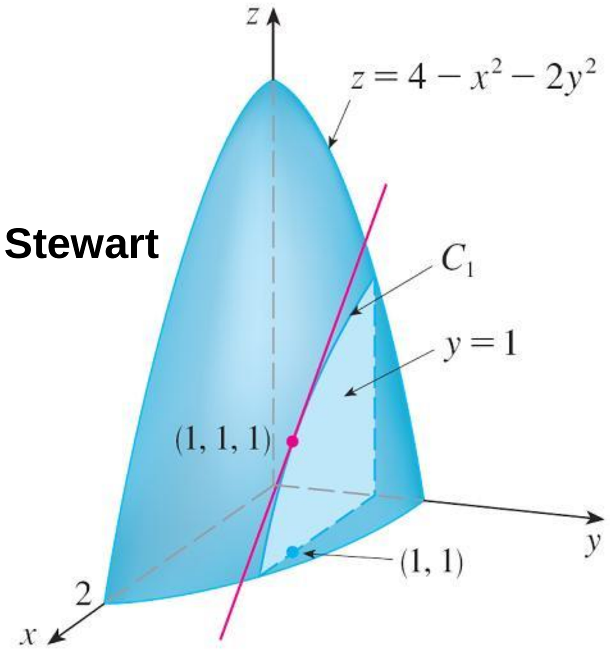
- The partial derivatives $f_x(a, b)$ and $f_y(a, b)$ can be interpreted geometrically as the **slopes of the tangent lines** at $P(a, b, c)$ to the traces C_1 and C_2 of surface S in the planes $y = b$ and $x = a$.

Geometric Interpretation of Partial Derivatives (2 variables)

Given that $f(x, y) = 4 - x^2 - 2y^2$, find $f_x(1, 1)$ and $f_y(1, 1)$ and interpret these numbers as slopes.

Solution:

Figures due to Stewart



$f_x(x, y) = -2x$ and $f_y(x, y) = -4y$, hence $f_x(1, 1) = -2$ and $f_y(1,1) = -4$

- The graph of f is the **paraboloid** $z = 4 - x^2 - 2y^2$ and the vertical plane $y = 1$ intersects with the paraboloid in the parabola

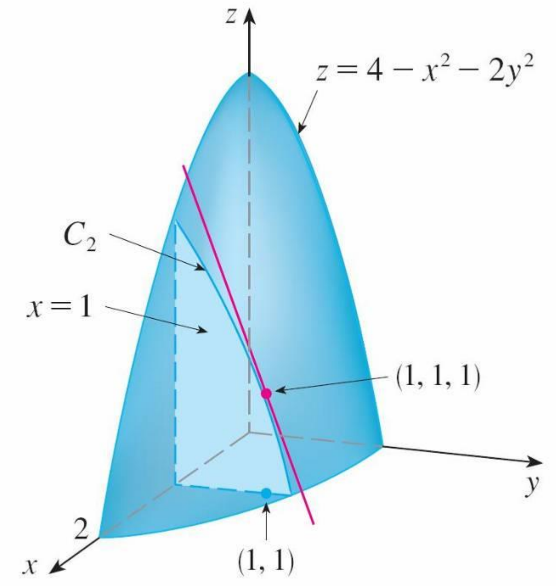
  $z = 2 - x^2$, with $y = 1$.

- The slope of the tangent line to this parabola at (1, 1, 1) is

  $f_x(1, 1) = -2$.

- Similarly, the curve in which the plane $x = 1$ intersects the paraboloid is the parabola $z = 3 - 2y^2$ with $x = 1$.
- The slope of the tangent line to this parabola at (1, 1, 1) is

  $f_y(1, 1) = -4$.



More Examples

Example 1: Given that $yz - \ln z = x^2 + y$ defines z as a function of two independent variables x and y, and all partial derivatives exist. Find $z_x \coloneqq \frac{\partial z}{\partial x}$.

Solution: Differentiate both sides with respect to x (but keeping y constant), and notice that z is a differentiable function of x, then we have:

$$yz_x - \frac{1}{z}z_x = 2x + 0$$

$$\left(y - \frac{1}{z}\right)z_x = 2x$$

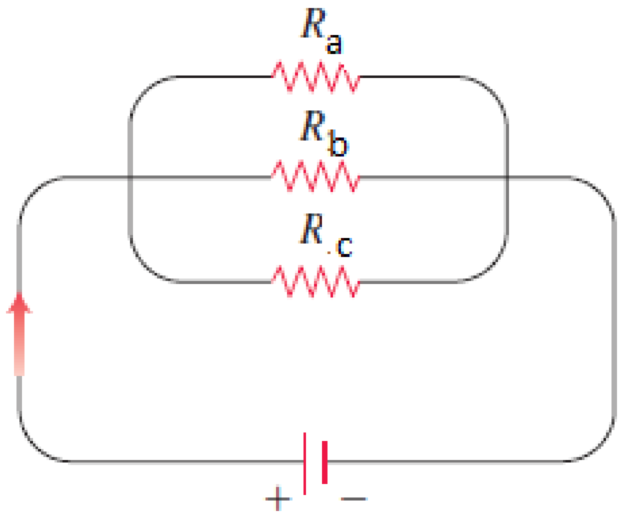
$$z_x = \frac{2xz}{yz - 1}$$

Take Home Exercise: How about z_y ?

(Hint: x is a constant this time, and for yz, you will need some rules.)

More Examples

Example 2: If resistors of R_a, R_b and R_c ohms are connected in parallel to make an R-ohm resistor. Find the value of $\frac{\partial R}{\partial R_b}$ when $R_a = 30, R_b = 60$ and $R_c = 90$ ohms.



Since $\frac{1}{R} = \frac{1}{R_a} + \frac{1}{R_b} + \frac{1}{R_c}$, to find $\frac{\partial R}{\partial R_b}$, we treat R_a, R_c as constants, and we use implicit differentiation.
Differentiate both sides with respect to R_b :

$$-\frac{1}{R^2}\frac{\partial R}{\partial R_b} = 0 - \frac{1}{R_b^2} + 0$$

$$\frac{\partial R}{\partial R_b} = \left(\frac{R}{R_b}\right)^2$$

Consider $\frac{1}{R} = \frac{1}{30} + \frac{1}{60} + \frac{1}{90} = \frac{11}{180}$, hence $R = \frac{180}{11}$.

Therefore, $\frac{\partial R}{\partial R_b} = \left(\frac{\frac{180}{11}}{60}\right)^2 = \frac{9}{121}$.

Hence, at the given values, a small change in resistance R_b leads to a change in R about $\frac{9}{121}$ as large.

Higher order Derivatives

- If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables, hence we can consider the four partial derivatives $(f_x)_x$, $(f_x)_y$, $(f_y)_x$, and $(f_y)_y$, which are called the **second partial derivatives** of f.

If $z = f(x, y)$, we use the following notations:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}. \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}. \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}. \quad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}.$$

The third and fourth cases are called **mixed partial derivatives**.

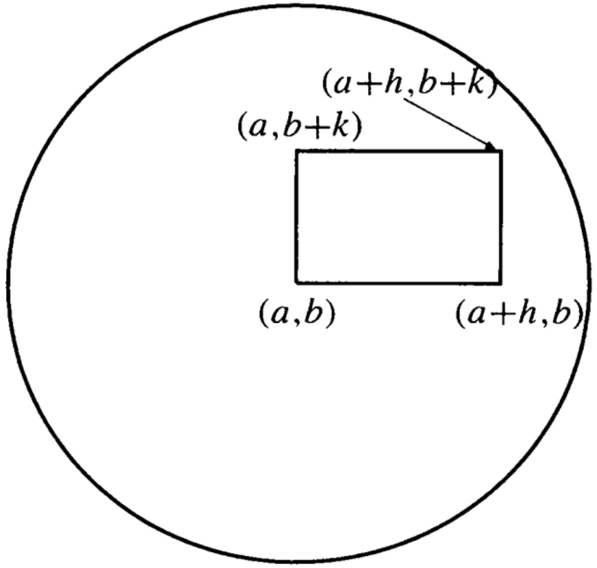
Theorem (Clairaut's Theorem): Let f be a function of two variables, if f_x, f_y, f_{xy}, f_{yx} are continuous on an open set (in a neighborhood of (x, y)), then $f_{xy} = f_{yx}$ at each point of the set, i.e. the order of differentiation can be swapped.

Note: The theorem is named after the French mathematician, Alexis Clairaut. To calculate a mixed second-order derivative, we may differentiate in either order, provided that continuity conditions are satisfied.

Higher order Derivatives

Proof of Clairaut's Theorem

Let *h* and *k* be sufficiently small values such that the point $(a+h, b+k)$ lies in this disk, then so do all points of the rectangle with sides parallel to the coordinate axes and diagonally opposite corners at (a, b) and $(a+h, b+k)$.



Let $Q = f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a, b)$ and define single-variable functions $u(x)$ and $v(y)$ as follows:

$$u(x) = f(x, b+k) - f(x, b)$$
$$v(y) = f(a+h, y) - f(a, y)$$

Now, we may write $Q = u(a+h) - u(a) = v(b+k) - v(b)$.

By **Mean-Value Theorem**, there exists a number θ_x (where $0 < \theta_x < 1$) such that $a + \theta_x h$ lies between a and $a+h$, then

$$\begin{aligned} Q &= u(a+h) - u(a) = hu'(a+\theta_x h) \\ &= h[f_x(a+\theta_x h, b+k) - f_x(a+\theta_x h, b)] \end{aligned}$$

Now, we apply the **Mean-Value Theorem** to f_x (considered as a function of its second variable), and obtain another number θ_y (where $0 < \theta_y < 1$) such that

$$f_x(a+\theta_x h, b+k) - f_x(a+\theta_x h, b) = kf_{xy}(a+\theta_x h, b+\theta_y k)$$

Thus, $Q = hkf_{xy}(a+\theta_x h, b+\theta_y k)$.

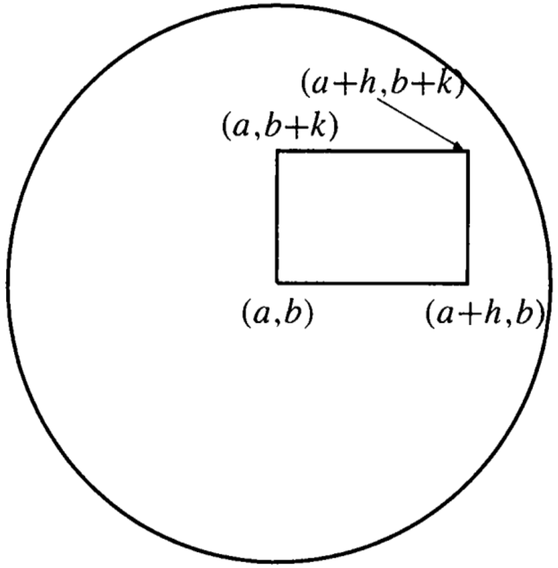
Similar as above, by applying **Mean-Value Theorem** to $Q = v(b+k) - v(b)$ yields
$Q = hkf_{yx}(a + \theta_{x'}h, b + \theta_{y'}k)$, where $\theta_{x'}, \theta_{y'}$ are two numbers, each between 0 and 1.

Equating these two expressions of Q, and cancelling out the common factor hk, we get

$$f_{xy}\left(a + \theta_x h, b + \theta_y k\right) = f_{yx}(a + \theta_{x'}h, b + \theta_{y'}k)$$

Since f_{xy} and f_{yx} are continuous at (a, b), we may simply let $h, k \to 0$ to obtain $f_{xy}(a, b) = f_{yx}(a, b)$.

Mean-value Theorem
Let $f(x)$ be differentiable on (a, b) and continuous on $[a, b]$, then there exists at least one point $c \in (a, b)$, where $f'(c) = \frac{f(b)-f(a)}{b-a}$.
If $b = a + h$, then $\frac{f(a+h)-f(a)}{h} = f'(c) = f'(a + \theta h)$, where $\theta \in (0,1)$.



Higher order Derivatives

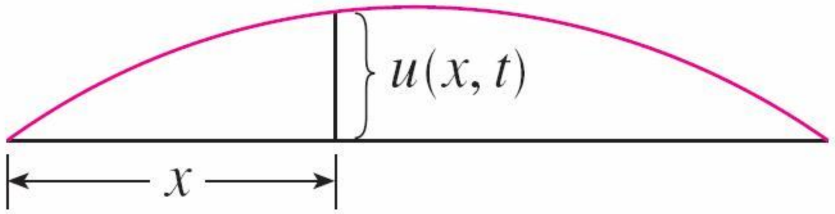
Example 1: Find f_{yxyz} if $f(x, y, z) = 1 - 2xy^2z^2 + x^2y^4$.

$$f_y = -4xyz^2 + 4x^2y^3$$
$$f_{yx} = -4yz^2 + 8xy^3$$
$$f_{yxy} = -4z^2 + 24xy^2$$
$$f_{yxyz} = -8z$$

Example 2 (Wave Equation – 2nd order PDE):

The **wave equation** $u_{tt} = c^2u_{xx}$ describes the motion of a waveform, which could be an ocean wave, a sound wave, a light wave, or a wave traveling along a vibrating string. Here the constant *c* depends on the density of the string and the tension in the string.

u(*x*, *t*) represents the displacement of a vibrating violin string at time *t* and at a distance *x* from one end of the string.

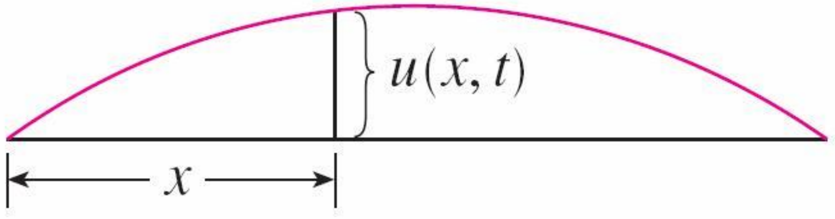


Higher order Derivatives

Show that any function of the form $u(x,t) = f(x+ct) + g(x-ct)$ is a solution of the wave equation

$$u_{tt} = c^2 u_{xx}$$

Here we assume both f and g are twice-differentiable functions of one variable.



Solution:

Using Chain Rule for functions of one variable, we have

$$u_t = cf'(x+ct) - cg'(x-ct)$$
$$u_x = f'(x+ct) + g'(x-ct)$$
$$u_{tt} = c^2 f''(x+ct) + c^2 g''(x-ct)$$
$$u_{xx} = f''(x+ct) + g''(x-ct)$$

Hence, $u_{tt} = c^2 u_{xx}$.

Remark: If *t* measures time, then
$f(x+ct)$ represents a waveform travelling to the left with speed *c.*
$g(x-ct)$ represents a waveform travelling to the right with speed *c.*

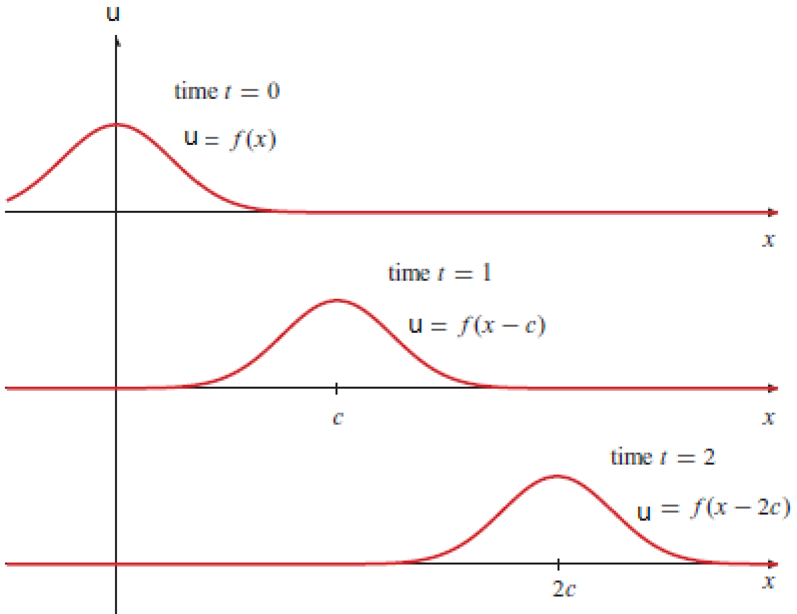


Figure from Adams and Essex

Chain Rule (Case 1)

Motivation: Imagine you are hiking in a mountainous region. Let (x, y) be the coordinates of your position on the map. Let $z = f(x, y)$ be the height above sea level at position (x, y). Suppose you are walking along a trail such that your position at time t is given by $\begin{cases} x = u(t) \\ y = v(t) \end{cases}$ (Parametric equations of the trail). Then the altitude above sea level at time t is : $z = f\big(u(t), v(t)\big) = g(t)$, i.e. a function of only one variable.

Question: How fast does your altitude change with respect to time at time t?

Answer: Simply the derivative of $g(t)$**.**

Consider $g'(t) = \lim_{h \to 0} \frac{g(t+h) - g(t)}{h} = \lim_{h \to 0} \frac{f\big(u(t+h), v(t+h)\big) - f\big(u(t), v(t)\big)}{h}$

$$= \lim_{h \to 0} \frac{f\big(u(t+h), v(t+h)\big) - f\big(u(t), v(t+h)\big)}{h} + \lim_{h \to 0} \frac{f\big(u(t), v(t+h)\big) - f\big(u(t), v(t)\big)}{h}$$

By single-variable Chain Rule, we have $g'(t) = f_x\big(u(t), v(t)\big)u'(t) + f_y\big(u(t), v(t)\big)v'(t)$.

Using Leibniz notation, if z is a function of x and y with continuous first partial derivatives, and if x and y are differentiable functions of t, then $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \left\langle \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$.

Chain Rule (Case 2)

Now, consider a function f of two variables (x and y), each of them is a function of two other variables, s and t, then we can form the composite function

$$z = f(x, y) = f\big(x(s, t), y(s, t)\big) = g(s, t)$$

Assume that f, x, y all have first partial derivatives with respect to respective variables, and those partial derivatives are continuous. Then

$$z_s = z_x x_s + z_y y_s \text{ and } z_t = z_x x_t + z_y y_t$$

Quick Example: If $f(x, y) = x^6 + 3y$, where $x(s, t) = st^2$ and $y(s, t) = s + t$, then

$$g(s, t) = s^6 t^{12} + 3s + 3t$$

$$f_s = f_x x_s + f_y y_s = 6x^5(t^2) + 3(1) = 6s^5 t^{12} + 3$$
$$f_t = f_x x_t + f_y y_t = 6x^5(2st) + 3(1) = 12s^6 t^{11} + 3$$

Question: How do you draw out the tree?

In general, if $f(x_1, x_2, x_3, \dots, x_n)$ and $x_i = x_i(t_1, t_2, \dots, t_m)$ are differentiable, then

$$\frac{\partial f}{\partial t_j} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial t_j}$$

where $j = 1,2,3, \dots, m$.

Chain Rule (Case 2)

In general, if $f(x_1, x_2, x_3, \dots, x_n)$ and $x_i = x_i(t_1, t_2, \dots, t_m)$ are differentiable, then

$$\frac{\partial f}{\partial t_j} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\frac{\partial x_i}{\partial t_j}$$

where $j = 1,2,3, \dots, m$.

Particular case: If $w = f(x(t), y(t), z(t))$, then

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = \langle f_x, f_y, f_z \rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$$

Examples

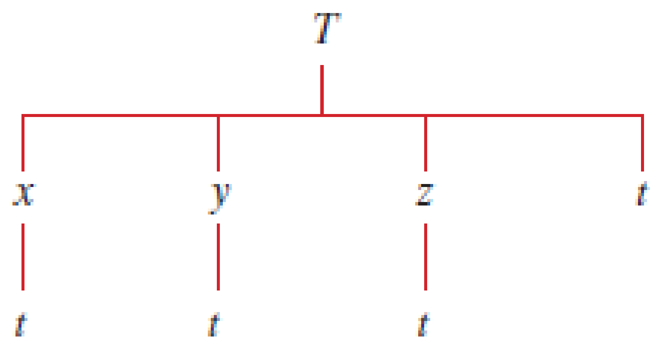
(1) Let $w = x^2y - y^2$, where $x = \sin t$ and $y = e^t$. Find dw/dt when $t = 0$.

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} = 2xy(\cos t) + (x^2 - 2y)(e^t) = 2\sin t \cos t\, e^t + (\sin^2 t - 2e^t)e^t$$

At t = 0, the answer will be – 2.

Chain Rule

Example 2. Atmospheric temperature depends on position and time. Denote the spatial position as (x, y, z) (in km) and time by t (in hrs), then the temperature is a function of four variables, $T(x, y, z, t)$. If a thermometer is attached to a weather balloon that moves through the atmosphere along the curve $x = t, y = 2t, z = t - t^2$, and $T(x, y, z, t) = \frac{xy}{1+z} + \frac{xy}{1+z}t$, find the rate of change of the recorded temperature at $t = 3$.



Solution:

The rate of change of the thermometer reading will depend on the change of position of the thermometer, as well as increasing time.

Rate = $\frac{dT}{dt} = T_x \frac{dx}{dt} + T_y \frac{dy}{dt} + T_z \frac{dz}{dt} + T_t$ ....(*)

The first three terms are arising from the balloon motion, while the last term refers to the rate of change of the temperature with respect to time at a fixed position in atmosphere.

At $t = 3, x = 3, y = 6, z = -6$

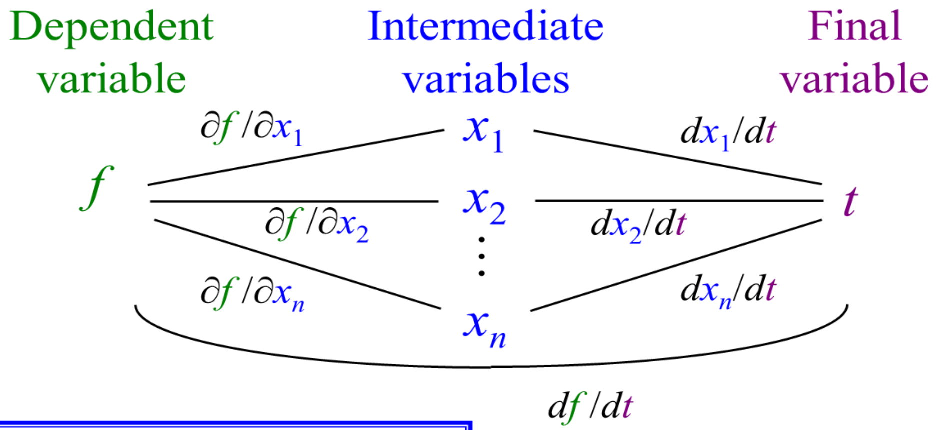
$$T_x = \frac{y}{1+z}(1+t) = -\frac{24}{5}; \;\; T_y = \frac{x}{1+z}(1+t) = -\frac{12}{5};$$

$$T_z = -\frac{xy}{(1+z)^2}(1+t) = -\frac{72}{25}; \;\; T_t = \frac{xy}{1+z} = -\frac{18}{5}$$

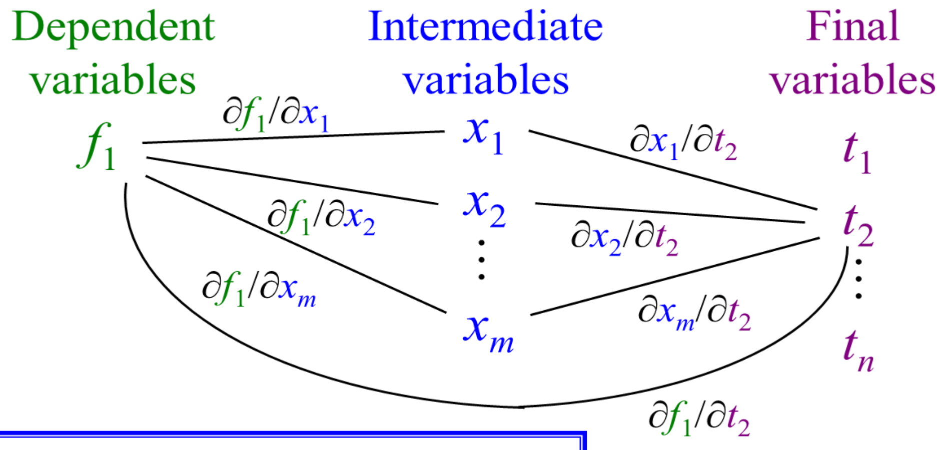
Also, $\frac{dx}{dt} = 1, \frac{dy}{dt} = 2, \frac{dz}{dt} = 1 - 2t = -5$

Substitute everything into (*) and calculate the answer…

Schematic Diagram of Chain Rule

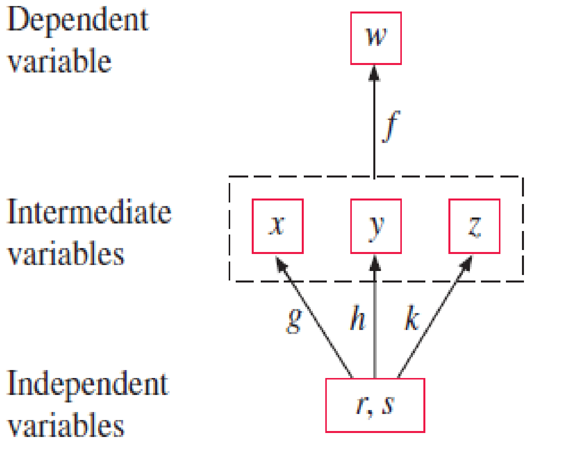


$$\boxed{\frac{df}{dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dt} + \cdots + \frac{\partial f}{\partial x_n}\frac{dx_n}{dt}}$$

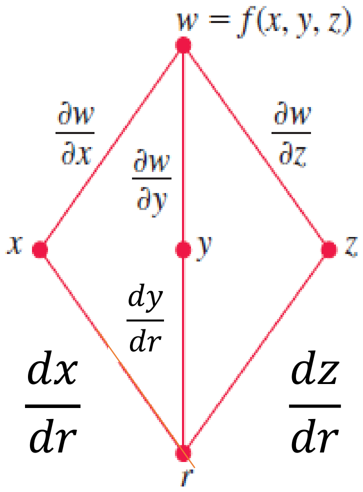


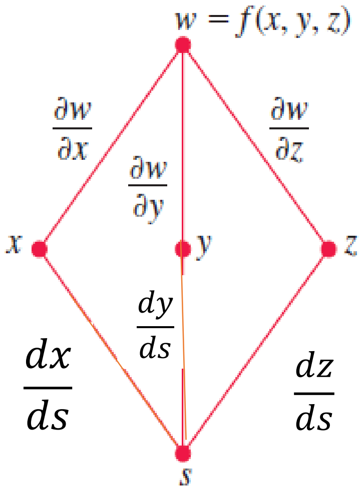
$$\boxed{\frac{\partial f_1}{\partial t_2} = \frac{\partial f_1}{\partial x_1}\frac{\partial x_1}{\partial t_2} + \frac{\partial f_1}{\partial x_2}\frac{\partial x_2}{\partial t_2} + \ldots + \frac{\partial f_1}{\partial x_m}\frac{\partial x_m}{\partial t_2}}$$

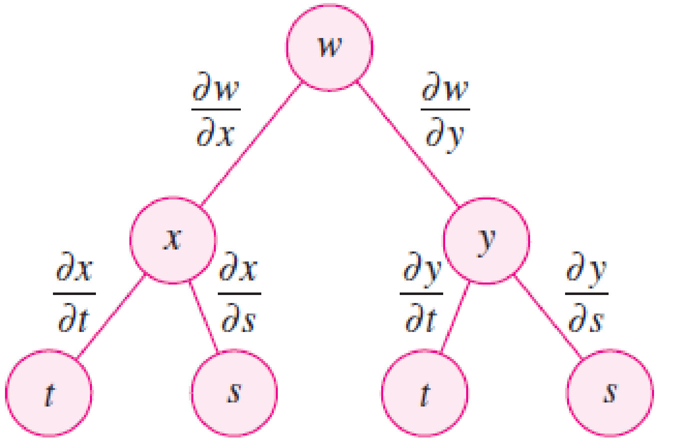
Due to Page 813, Thomas Calculus



$w = f(g(r, s), h(r, s), k(r, s))$







Chain Rule: two independent variables

Implicit Partial Differentiation

Let x and y be related by the equation $F(x, y) = 0$, where $y = f(x)$ is a differentiable function of x.

We try to adopt the approach of **Chain Rule**.

Consider the function $w = F(x, y) = F(x, f(x))$, we have

$$\frac{dw}{dx} = F_x(x,y)\frac{dx}{dx} + F_y(x,y)\frac{dy}{dx}$$

Since $w = F(x, y) = 0$ for all x in the domain of f, we have $\frac{dw}{dx} = 0$. Also, if $F_y(x, y) \neq 0$, we can use the fact that $dx/dx = 1$ to conclude that $\frac{dy}{dx} = -\frac{F_x(x,y)}{F_y(x,y)}$.

A similar procedure can be used to find the partial derivatives of functions of three variables that are defined implicitly.

If $F(x, y, z) = 0$ defines z implicitly as a differentiable function of x and y, we have

$$\frac{\partial z}{\partial x} = -\frac{F_x(x,y,z)}{F_z(x,y,z)} \ \ ; \ \ \frac{\partial z}{\partial y} = -\frac{F_y(x,y,z)}{F_z(x,y,z)}$$

Here, we assume $F_z(x, y, z)$ is non-zero.

Examples on Implicit Partial Differentiation

Example 1 (2-variable case)

Find $\frac{dy}{dx}$ for $y^3 + 2y^2 - 7y - x^2 + 12 = 3$.

Solution

We let $F(x, y) = y^3 + 2y^2 - 7y - x^2 + 9$.

Then $F_x(x, y) = -2x$ and $F_y(x, y) = 3y^2 + 4y - 7$.

Hence, $\frac{dy}{dx} = -\frac{F_x(x,y)}{F_y(x,y)} = \frac{2x}{3y^2+4y-7}$ holds for all values of x and y except when $3y^2 + 4y - 7 = 0$.

Example 2 (3-variable case)

Find z_x and z_y if $x^2z - x^2y^2 + 2z^3 = 1$.

Solution

We let $F(x, y, z) = x^2z - x^2y^2 + 2z^3 - 1$, then

$$F_x(x, y, z) = 2xz - 2xy^2, \qquad F_y = -2x^2y, \qquad F_z = x^2 + 6z^2$$

Hence, $z_x = -\frac{F_x(x,y,z)}{F_z(x,y,z)} = \frac{2xy^2-2xz}{x^2+6z^2}$ and $z_y = -\frac{F_y(x,y,z)}{F_z(x,y,z)} = \frac{2x^2y}{x^2+6z^2}$.

Differentials

Increments and Differentials

For a differentiable function of one variable, $y = f(x)$, we define the differential *dx* to be an independent variable; i.e. dx can be given the value of any real number. The differential of y is then defined as

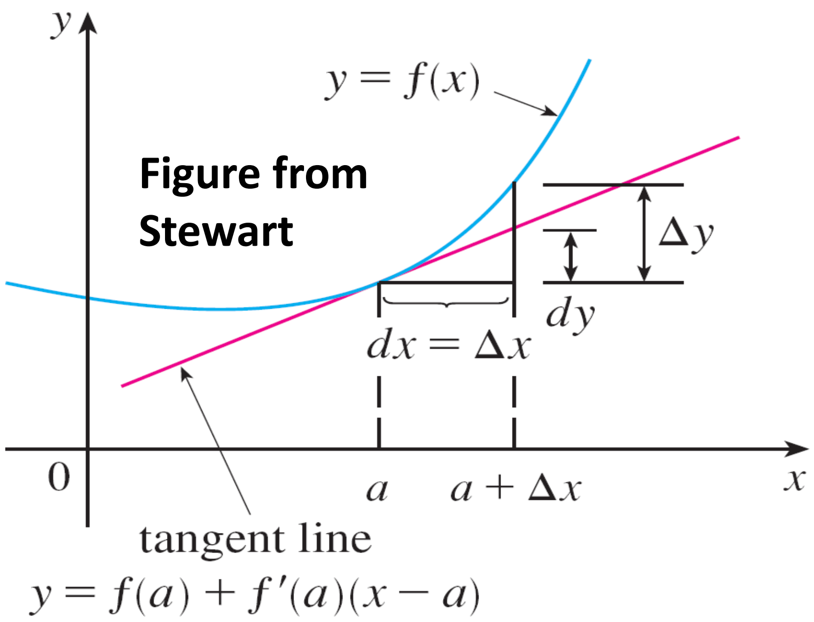
$$dy = f'(x)\,dx$$

Relationship between the increment Δ*y* and the differential *dy*: Δ*y* represents the change in height of the curve *y* = *f*(*x*) and *dy* represents the change in height of the tangent line when *x* changes by an amount *dx* = Δ*x*.

For a differentiable function of two variables, *z* = *f*(*x*, *y*), we define the **differentials** *dx* and *dy* to be independent variables; that is, we can assign any real values to them. Then the **differential** *dz*, also called the **total differential**, is defined as follows:

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = f_x(x,y)dx + f_y(x,y)dy$$

Now, take *dx* = Δ*x* = *x* – *a* and *dy* = Δ*y* = *y* – *b*, then the differential of *z* is $dz = f_x(a, b)(x - a) + f_y(a, b)(y - b)$.



Differentials

Geometric Interpretation

Differential *dz* and the **increment** Δz:
dz represents the change in height of the tangent plane, whereas Δz represents the change in height of the surface $z = f(x, y)$ when (x, y) changes from (a, b) to $(a + \Delta x, b + \Delta y)$.

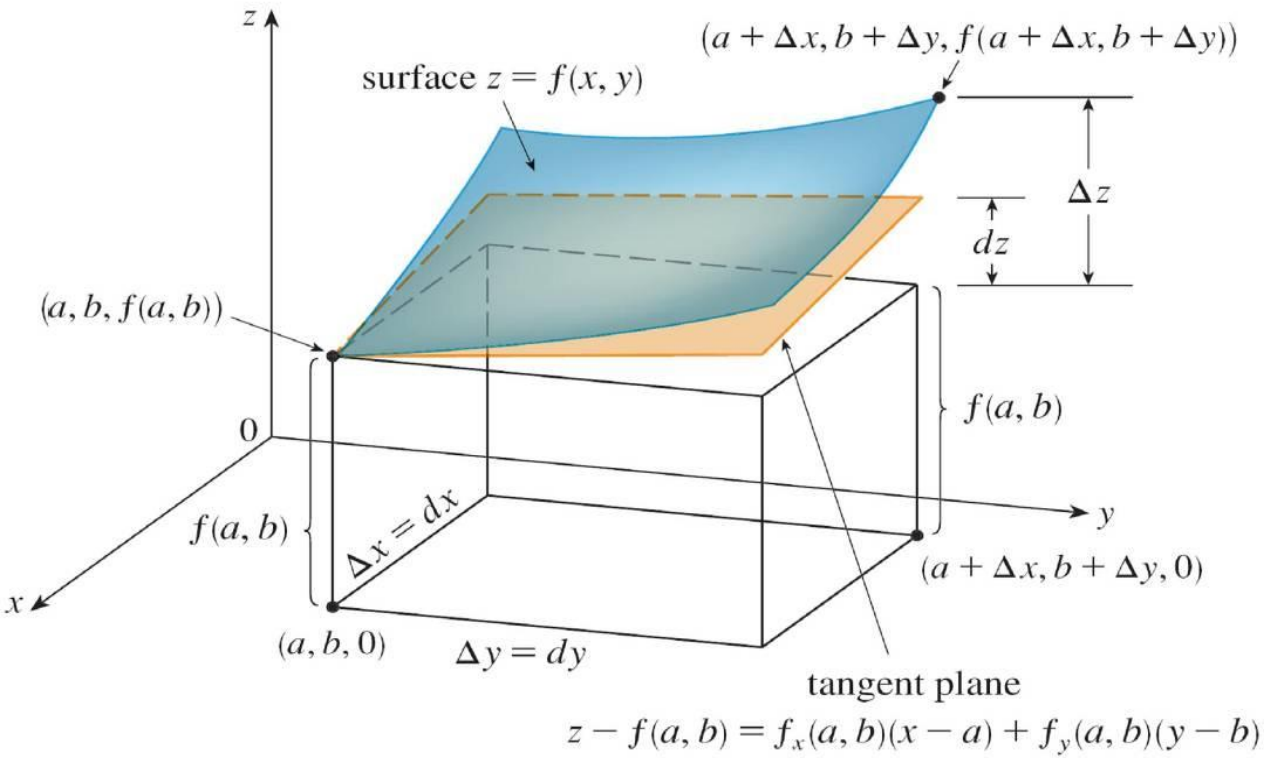


Figure from Stewart

General Expression of Differential

If $w = f(x_1, x_2, \dots, x_n)$, then the **increment** of w is

$$\Delta w = f(x_1 + \Delta x_1, x_2 + \Delta x_2, \dots, x_n + \Delta x_n) - f(x_1, x_2, \dots, x_n)$$

The **differential** *dw* is defined in terms of the differentials $dx_1, dx_2, \dots, dx_n$ of the independent variables by the following expression:

$$dw = w_{x_1} dx_1 + w_{x_2} dx_2 + \cdots + w_{x_n} dx_n, \text{ where } w_{x_i} := \frac{\partial w}{\partial x_i}$$

Examples of Differentials

Example 1

The dimensions of a rectangular box are measured to be 75 cm, 60 cm, and 40 cm, and each measurement is correct to be within 0.2 cm. Estimate the largest possible error when the volume of the box is calculated based on all these measurements.

Solution

Let the dimensions of the box be *x*, *y*, and *z*, then its volume is $V = xyz$.

$$dV = V_x\, dx + V_y\, dy + V_z\, dz = yz\, dx + xz\, dy + xy\, dz$$

Since $|\Delta x| \leq 0.2$, $|\Delta y| \leq 0.2$, and $|\Delta z| \leq 0.2$, and our aim is to estimate the largest error in the volume, we choose $dx = 0.2, dy = 0.2, dz = 0.2$.

Hence $\Delta V \approx dV = (60)(40)(0.2) + (75)(40)(0.2) + (75)(60)(0.2) = 1980$

From the above calculation, we notice that an error of only 0.2 cm in each dimension could lead to an error of approximately 1980 cm^3 in the calculated volume.

However, we notice that $V = (75)(60)(40) = 180000\ cm^3$, hence the error is only about 1%.

Examples of Differentials

Example 2

Find the approximate value of the function $f(x, y) = x^2y^3$ at (3.1, 0.9) using the concepts of differential.

Solution:

Let $x = 3, y = 1, \text{then } \Delta x = 0.1, \Delta y = -0.1.$

Consider $f(x + \Delta x, y + \Delta y) \approx f(x, y) + f_x(\Delta x) + f_y(\Delta y)$

$$= x^2y^3 + (2xy^3)(\Delta x) + (3x^2y^2)(\Delta y)$$

Therefore, $f(3.1, 0.9) \approx 3^2(1)^3 + (2 \cdot 3 \cdot 1^3)(0.1) + (3 \cdot 3^2 \cdot 1^2)(-0.1) = 6.9$

Exact answer $= f(3.1,0.9) = 7.00569$

Error = $6.9 - 7.00569 = -0.10569$

Differentiability

Definition: In 1-variable calculus, f is differentiable at x_0 if there exists a number $f'(x_0)$ such that the following holds:

$$\Delta f = f(x+\Delta x) - f(x) = f'(x)\Delta x + \varepsilon \Delta x, \text{ with } \varepsilon \to 0 \text{ as } \Delta x \to 0$$

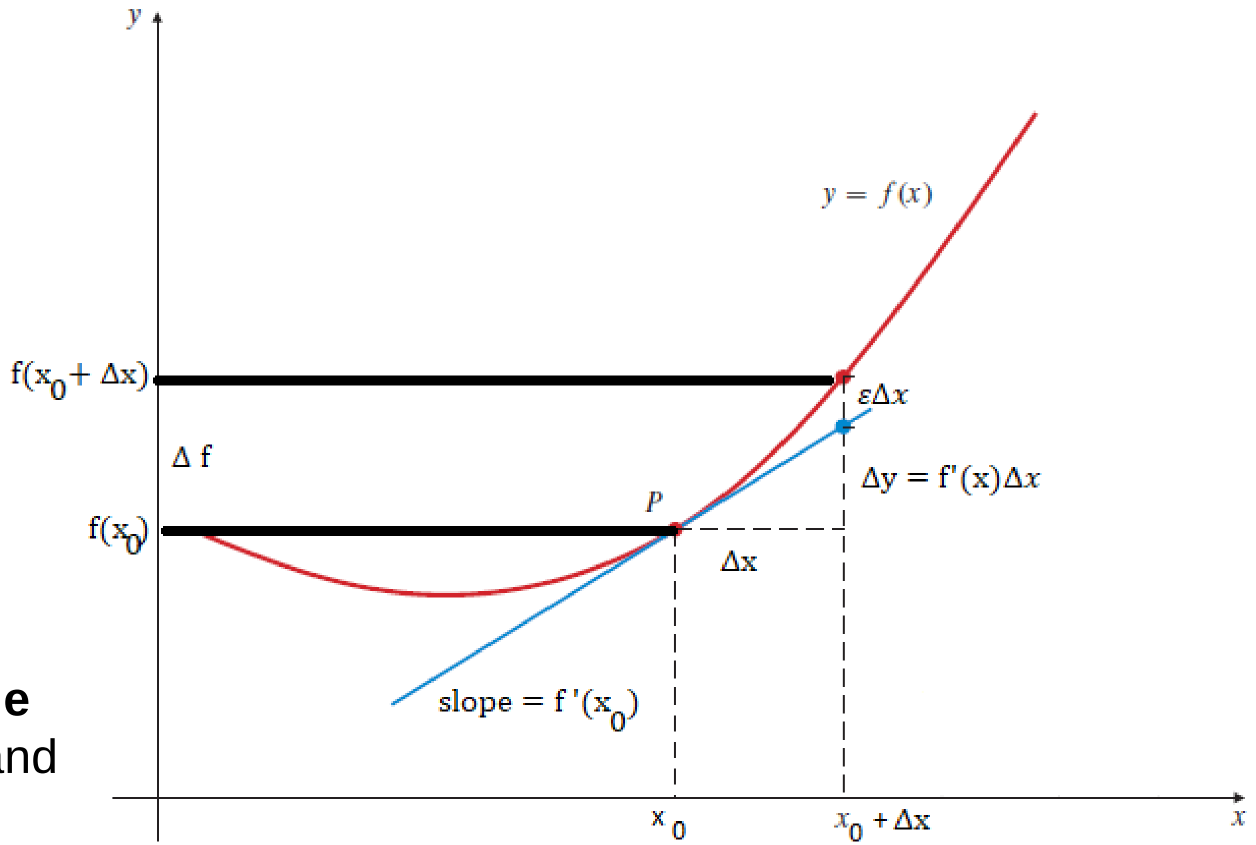
For 2-variable functions, $z = f(x, y)$ is differentiable at (x_0, y_0) if both $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist, then the change

$$\Delta f = f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0)$$

satisfies the following equation:

$$\Delta f = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

Here $\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.



Alternative, we say the function $f(x, y)$ is **differentiable** at the point (x_0, y_0) if the partial derivatives $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist, and

$$\lim_{(\Delta x, \Delta y)\to(0,0)} \frac{f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0) - \Delta x f_x(x_0, y_0) - \Delta y f_y(x_0, y_0)}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0$$

Differentiability

Corollary : If the partial derivatives f_x and f_y of a function $f(x, y)$ are continuous throughout an open region D, then f is **differentiable** at every point of D.

If $z = f(x, y)$ is differentiable, then the **definition of differentiability** gives us the following result:
$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) \to 0$ as Δx and Δy both tend to 0.

Theorem (Differentiability implies Continuity): If a function $f(x, y)$ is differentiable at a point (x_0, y_0), then f is continuous at that point.

Example (**Show that a function is Differentiable**) Show that the function $f(x, y) = x^2 + 5y$ is differentiable at every point in the plane.

Solution:

Let $z = f(x, y)$, the increment of z at an arbitrary point (x, y) in the plane is

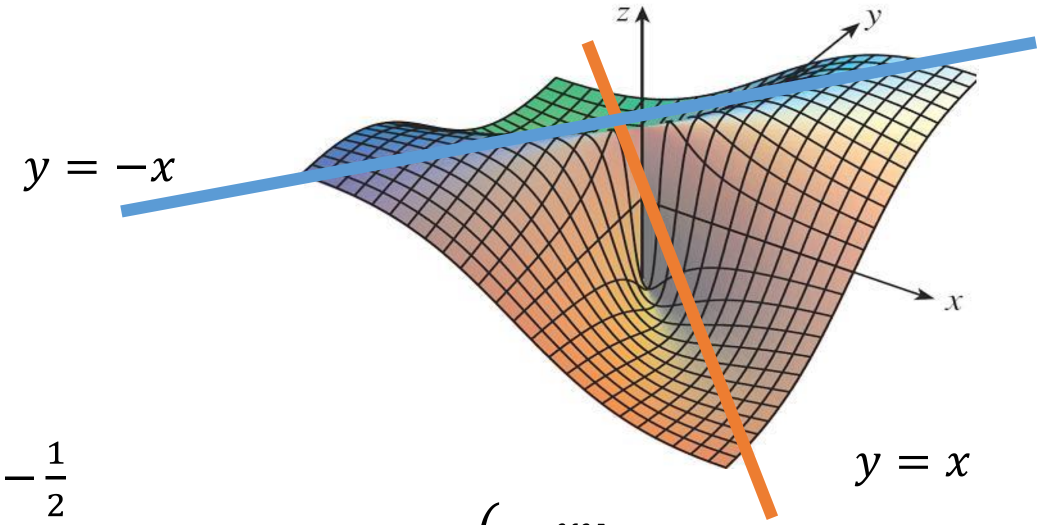
$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y) = (x + \Delta x)^2 + 5(y + \Delta y) - (x^2 + 5y)$$

$$\begin{aligned} &= x^2 + 2x\Delta x + (\Delta x)^2 + 5y + 5\Delta y - x^2 - 5y \\ &= 2x\Delta x + (\Delta x)^2 + 5\Delta y \\ &= 2x(\Delta x) + 5(\Delta y) + \Delta x(\Delta x) + 0(\Delta y), \text{ which is in the form } f_x(x, y)\,\Delta x + f_y(x, y)\,\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y \end{aligned}$$

Here $\varepsilon_1 = \Delta x$ and $\varepsilon_2 = 0$. Since $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$, f is differentiable at every point in the plane.

Differentiability

Consider $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$



$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

(1) f is not differentiable at (0, 0)

Along the line $y = x$, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{(x,x)\to(0,0)} \frac{x^2}{2x^2} = \frac{1}{2}$

Along the line $y = -x$, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{(x,-x)\to(0,0)} \frac{-x^2}{2x^2} = -\frac{1}{2}$

Hence, limit of $f(x,y)$ as $(x,y) \to (0,0)$ does not exist.
Therefore, *f* is not continuous at (0, 0), thus not differentiable.

(2) $f_x(0,0)$ and $f_y(0,0)$ both exist

By definition, $f_x(0,0) = \lim\limits_{\Delta x\to 0} \frac{f(\Delta x,0)-f(0,0)}{\Delta x} = 0$

and $f_y(0,0) = \lim\limits_{\Delta y\to 0} \frac{f(0,\Delta y)-f(0,0)}{\Delta y} = 0$

i.e. Partial derivatives at (0, 0) exist.

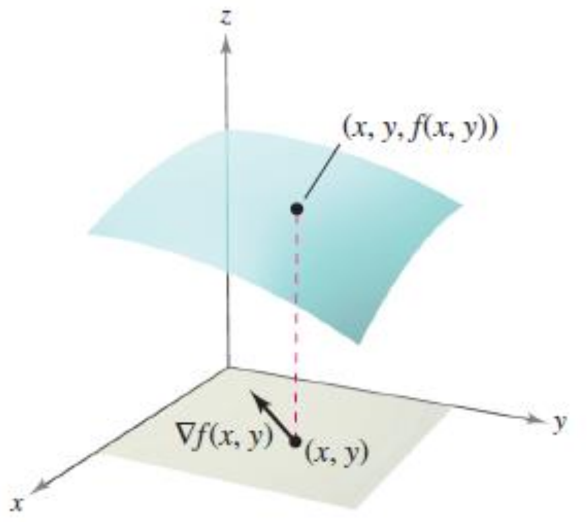
Gradients of Functions

(**1-variable function**): Let $y = f(x)$, then the **gradient** of the function at $x = x_0$ is $f'(x_0)$, i.e. the slope of $f(x)$ at $x = x_0$.

(**2-variable function**): Let $z = f(x, y)$ be a function of x and y such that both f_x and f_y exist. Then the **gradient** of f is the vector:

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$$

Note that for each (x, y) , the gradient is a vector in the plane (**not in space**).



The gradient of f is a vector in the xy-plane.

Due to Larson and Edwards

(***n*-variable function**): Let $z = f(x_1, x_2, \ldots, x_n)$ be a continuously differentiable function. Then the gradient of this function is:

$$\nabla f(x_1, x_2, \ldots, x_n) = \langle f_{x_1}, f_{x_2}, \ldots, f_{x_n} \rangle$$

Here $\nabla = \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n} \right\rangle$

Gradients of Functions

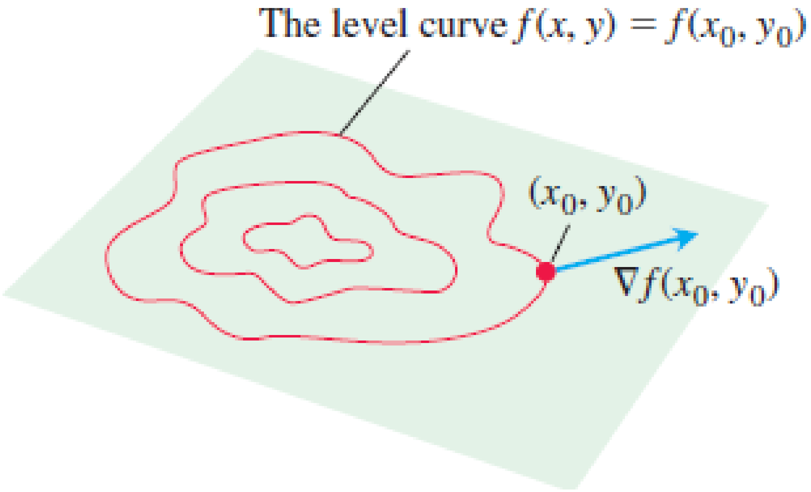
Examples

(1) Let $f(x, y, z) = x^3 y + y^2 z$, then $\nabla f = \langle f_x, f_y, f_z \rangle = \langle 3x^2 y, x^3 + 2yz, y^2 \rangle$.
At the point (1, 1, 1), $\nabla f(1, 1, 1) = \langle 3,3,1 \rangle$

(2) For $f(x, y) = y \ln x + xy^3$, then $\nabla f = \langle f_x, f_y \rangle = \left\langle \frac{y}{x} + y^3, \ln x + 3xy^2 \right\rangle$
At the point (1, 2), ∇f(1, 2) = $\langle 10, 12 \rangle$.

Note: In general, $\vec{\nabla} f$ is a vector with n components.

Gradient of a differentiable function of two variables at a point (x_0, y_0) is always normal to the function's level curve passing through it.



Gradients of Functions

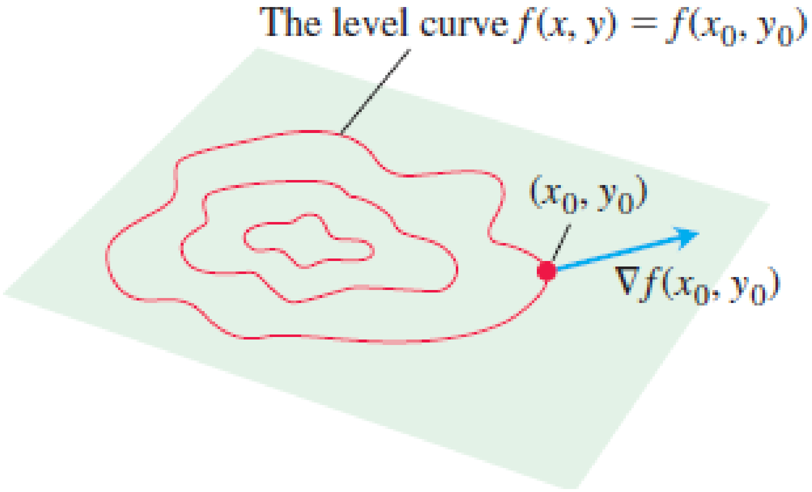
Theorem: At every point (x_0, y_0) in the domain of a **differentiable** function $f(x, y)$, the gradient of f is normal to the level curve of f through the point (x_0, y_0).

Proof: Suppose $z = f(x, y) = C$ is the level curve (C is a constant).
Let $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ be a parametric curve of the level curve, then $f\big(x(t), y(t)\big) = C$.

Differentiating with respect to t, we have $\frac{d}{dt}\Big(f\big(x(t), y(t)\big)\Big) = \frac{d}{dt}(C)$.

$$f_x \frac{dx}{dt} + f_y \frac{dy}{dt} = 0$$

$$\langle f_x, f_y \rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = 0$$



Hence, gradient of *f* is normal to the tangent vector $\frac{d\mathbf{r}}{dt}$, so it is normal to the curve.

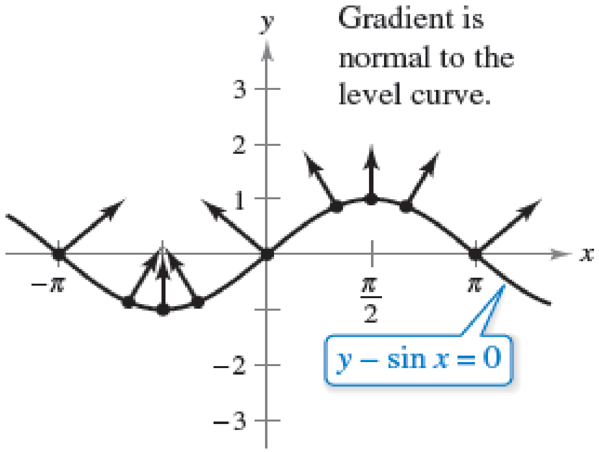
Similarly, if f is a function of three variables, then $\nabla f(x_0, y_0, z_0)$ is normal to the level surface of f through the point (x_0, y_0, z_0).

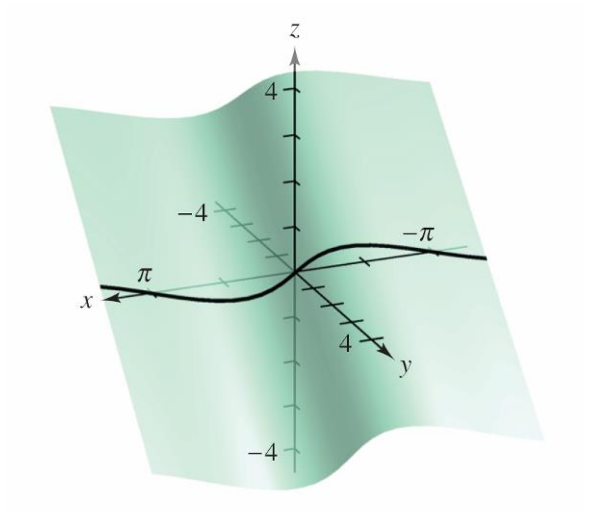
Gradients of Functions

Example (**Ref.:** Pg. 926 of Larson and Edwards)
Sketch the level curve corresponding to $c = 0$ for the function given by $f(x, y) = y - \sin x$ and find a normal vector at several points on the curve.

Solution
The level curve for $c = 0$ is given by

$$0 = y - \sin x \quad \Rightarrow \quad y = \sin x$$





The surface is given by $f(x, y) = y - \sin x$.

The gradient of f at (x, y) is

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = -\cos x\mathbf{i} + \mathbf{j}$$

Therefore, $\nabla f(x, y)$ is normal to the level curve at the point (x, y).

Examples of Gradients:

$$\nabla f(-\pi, 0) = \mathbf{i} + \mathbf{j} \qquad \nabla f\left(-\frac{2\pi}{3}, -\frac{\sqrt{3}}{2}\right) = \frac{1}{2}\mathbf{i} + \mathbf{j} \qquad \nabla f\left(-\frac{\pi}{2}, -1\right) = \mathbf{j} \qquad \nabla f\left(-\frac{\pi}{3}, -\frac{\sqrt{3}}{2}\right) = -\frac{1}{2}\mathbf{i} + \mathbf{j}$$

$$\nabla f(0, 0) = -\mathbf{i} + \mathbf{j} \qquad \nabla f\left(\frac{2\pi}{3}, \frac{\sqrt{3}}{2}\right) = \frac{1}{2}\mathbf{i} + \mathbf{j} \qquad \nabla f\left(\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right) = -\frac{1}{2}\mathbf{i} + \mathbf{j} \qquad \nabla f(\pi, 0) = \mathbf{i} + \mathbf{j}.$$

Gradient Rules

(**Ref**.: Page 824, Thomas' Calculus)

Algebra Rules for Gradients

1. *Sum Rule:*	$\nabla(f+g) = \nabla f + \nabla g$	
2. *Difference Rule:*	$\nabla(f-g) = \nabla f - \nabla g$	
3. *Constant Multiple Rule:*	$\nabla(kf) = k\nabla f$	(any number k)
4. *Product Rule:*	$\nabla(fg) = f\nabla g + g\nabla f$	Scalar multipliers on left of gradients
5. *Quotient Rule:*	$\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$	

END OF WEEK 5

Revised by Dr. Hugo MAK