Lecture Notes: Divergence Theorem and Stokes' Theorem

Yufei Tao
Department of Computer Science and Engineering
Chinese University of Hong Kong
taoyf@cse.cuhk.edu.hk

In this lecture, we will discuss two useful theorems on surface integrals.

1 Divergence Theorem

Theorem 1 (Divergence Theorem). *Let T be a closed region in $\mathbb{R}^3$ that is bounded by a surface S, which is the union of a finite number of smooth surfaces $S_1, S_2, .., S_k$. Let f_1, f_2, and f_3 be functions of x, y, z that have continuous partial derivatives on each S_i ($1 \le i \le k$). If we orient S by taking its outer side, then it holds that*

$$\iiint_T \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\, dxdydz \quad = \quad \iint_S f_1\, dydz + f_2\, dxdz + f_3\, dxdy.$$

We omit a proof for the theorem (which follows the same idea as our proof of the Green's theorem, and is a good exercise for you). The theorem is also called Gauss' Theorem.

Example 1. Calculate the volume of the ball $x^2 + y^2 + z^2 \le 1$.

Solution. Let T be the target ball, and S be its boundary $x^2 + y^2 + z^2 = 1$, oriented by having its outer side taken. Introduce $f_1 = 0$, $f_2 = 0$, and $f_3 = z$. Then, by Theorem 1, we know that

$$\iiint_T 1\, dxdydz \quad = \quad \iint_S z\, dxdy. \tag{1}$$

Denote by S_1 the upper half of S satisfying $z \ge 0$, and S_2 the lower half of S satisfying $z \le 0$. We thus have:

$$\iint_S z\, dxdy \quad = \quad \iint_{S_1} z\, dxdy + \iint_{S_2} z\, dxdy. \tag{2}$$

Let us first calculate $\iint_{S_1} z\, dxdy$. Note that S_1 is oriented with its upper side taken. Hence:

$$\iint_{S_1} z\, dxdy \quad = \quad \iint_D z\, dxdy. \tag{3}$$

Let us represent S_1 in a parametric form: $\boldsymbol{r}(u, v) = [x(u, v), y(u, v), z(u, v)]$ where

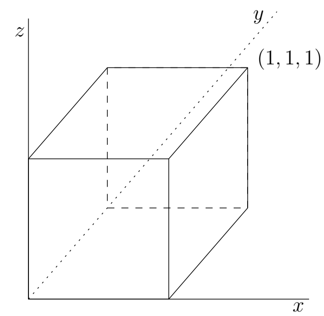
$$\begin{aligned} x(u, v) &= \cos u \cdot \sin v \\ y(u, v) &= \sin u \cdot \sin v \\ z(u, v) &= \cos v \end{aligned}$$

where $u \in [0, 2\pi]$ and $v \in [0, \pi/2]$. Let R be the collection of all such (u, v). Now we change the integral variables in (3) to u and v by the Jacobian rule:

$$\iint_D z\, dxdy \quad = \quad \iint_R \cos v \cdot \sin v \cos v\, dudv = 2\pi/3.$$

Similarly, we also have $\iint_{S_2} z\, dxdy = 2\pi/3$. Thus, we know from (1) and (2) that $\iiint_T 1\, dxdydz = 4\pi/3$. □

Example 2. Let T be the cube as shown below, and S be its boundary surface with its outer side taken.



Calculate

$$\iint_S y(x - z)\, dydz + x^2\, dzdx + (y^2 + xz)\, dxdy.$$

Solution. By Theorem 1, we have:

$$\iint_S y(x - z)\, dydz + x^2\, dzdx + (y^2 + xz)\, dxdy \quad = \quad \iiint_T x + y\, dxdydz. \tag{4}$$

We know

$$\iiint_T x\, dxdydz \quad = \quad \int_0^1 \left(\int_0^1 \left(\int_0^1 x\, dx \right) dy \right) dz = 1/2.$$

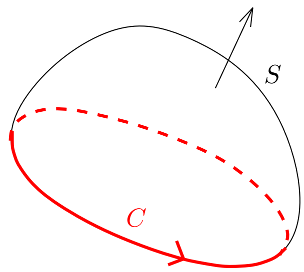
Similarly, $\iiint_T y\, dxdydz = 1/2$. Hence, (4) equals 1. □

Remark 1. You may be wondering why Theorem 1 is called the "Divergence Theorem". In fact, if we define a vector function $\boldsymbol{f}(x, y, z) = [f_1, f_2, f_3]$, then $\text{div}\boldsymbol{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$. Hence, the left hand side of the equation in Theorem 1 can also be written as $\iiint_T \text{div}\boldsymbol{f}\, dxdydz$.

Remark 2. Recall that surface integrals are inherently 2d. Hence, the divergence theorem essentially reveals a relationship between a 2d integral and a 3d integral. Also recall that, in contrast, the Green's theorem reveals a relationship between a 1d integral and a 2d integral.

2 Stokes' Theorem

Consider S to be a piecewise-smooth surface with a closed boundary curve C. Note that, not all surfaces have a boundary curve, e.g., a sphere $x^2 + y^2 + z^2 = 1$ does not have a boundary curve. Intuitively, S has a boundary curve C if it is not "closed", namely, it does not separate $\mathbb{R}^3$ into an interior part and an exterior part. The following is an example of such an S:



Fix a direction of C. Let us then *orient* S by choosing a side of it as shown in the above figure. Formally, imagine that a point moves around C along the direction we have decided. Now, watch the point's movement from the side of S we have chosen (i.e., allowing normal vectors of that side to shoot into our eyes). We should see that the point is moving in the counterclockwise direction.

Theorem 2. *Let S and C be as described earlier (with the direction of C fixed, and S oriented). Suppose that S is the union of a finite number of smooth surfaces $S_1, S_2, .., S_k$. Let f_1, f_2, and f_3 be functions of x, y, z that have continuous partial derivatives on each S_i ($1 \le i \le k$). Then:*

$$\int_C f_1\,dx + f_2\,dy + f_3\,dz \quad = \quad \iint_S \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\,dydz + \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\,dzdx + \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\,dxdy.$$

Proof. We first prove the theorem for the special case where S is *all monotone*, namely, it is simultaneously xy-monotone, xz-monotone, and yz-monotone. In particular, we will show that

$$\begin{aligned} \int_C f_1 dx &= \iint_S \frac{\partial f_1}{\partial z}\,dzdx - \frac{\partial f_1}{\partial y}\,dxdy \qquad (5) \\ \int_C f_2 dy &= \iint_S \frac{\partial f_2}{\partial x}\,dxdy - \frac{\partial f_2}{\partial z}\,dydz \\ \int_C f_3 dz &= \iint_S \frac{\partial f_3}{\partial y}\,dydz - \frac{\partial f_3}{\partial x}\,dzdx. \end{aligned}$$

Then, the theorem will follow from summing up both sides of the above three equations.

Due to symmetry, it suffices to prove (5). For this purpose, suppose that S is described by $z = g(x, y)$, and that we oriented S by taking its upper side. Introduce $h(x, y, z) = z - g(x, y)$. Hence, S is also described by $h(x, y, z) = 0$. Let C_{xy} be the projection of C onto the xy-plane, and D_{xy} be the region in the xy-plane enclosed by C_{xy}.

$$\begin{aligned} \int_C f_1 dx &= \int_{C_{xy}} f_1(x, y, z)\,dx \\ \text{(by Green's Theorem)} &= \iint_{D_{xy}} -\frac{\partial f_1(x, y, z)}{\partial y}\,dxdy \\ &= -\iint_{D_{xy}} \frac{\partial f_1}{\partial y} + \frac{\partial f_1}{\partial z}\frac{\partial z}{\partial y}\,dxdy. \qquad (6) \end{aligned}$$

Note that $\nabla h = [\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}, \frac{\partial h}{\partial z}]$ is a normal vector of S. Let γ be the angle between the directions of ∇h and $\boldsymbol{k} = [0, 0, 1]$, and β be the angle between the directions of ∇h and $\boldsymbol{j} = [0, 1, 0]$. We have:

$$\begin{aligned} \cos\beta &= \frac{\frac{\partial h}{\partial y}}{|\nabla g|} = \frac{-\frac{\partial g}{\partial y}}{|\nabla g|} = \frac{-\frac{\partial z}{\partial y}}{|\nabla g|} \\ \cos\gamma &= \frac{\frac{\partial h}{\partial z}}{|\nabla g|} = \frac{\frac{\partial z}{\partial z}}{|\nabla g|} = \frac{1}{|\nabla g|}. \end{aligned}$$

Therefore:

$$\frac{\cos\beta}{\cos\gamma} = -\frac{\partial z}{\partial y}.$$

We thus have:

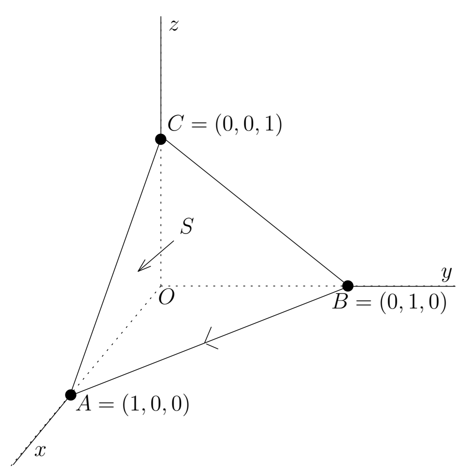
$$\begin{aligned}
(6) &= -\iint_{D_{xy}} \frac{\partial f_1}{\partial y}\, dxdy + \iint_{D_{xy}} \frac{\partial f_1}{\partial z}\frac{\cos\beta}{\cos\gamma}\, dxdy. \\
&= -\iint_S \frac{\partial f_1}{\partial y}\, dxdy + \iint_S \frac{\partial f_1}{\partial z}\cos\beta\, dA \\
&= -\iint_S \frac{\partial f_1}{\partial y}\, dxdy + \iint_S \frac{\partial f_1}{\partial z} dzdx. \\
&= \text{right hand side of (5).}
\end{aligned}$$

If S is not all monotone, we can always cut S into a set of disjoint surfaces each of which is all monotone. Then, we can obtain Theorem 2 by applying what we have proved to each of those surfaces. The details are omitted and serve as a good exercise for you. $\square$

Example 3. Use Stokes' Theorem to calculate

$$\int_C y\, dx + z\, dy + x\, dz$$

where C is the sequence of line segments: $ACBA$ with points $A = (1, 0, 0)$, $B = (0, 1, 0)$, and $C = (0, 0, 1)$.



Solution. Let S be the triangle ABC, oriented with its lower side taken. Introduce $f_1(x, y, z) = y$, $f_2(x, y, z) = z$, and $f_3(x, y, z) = x$. We have:

$$\begin{aligned}
\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} &= 0 - 1 = -1 \\
\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} &= 0 - 1 = -1 \\
\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} &= 0 - 1 = -1.
\end{aligned} \tag{7}$$

Therefore, Stokes' Theorem gives:

$$\int_C y\,dx + z\,dy + x\,dz \quad = \quad \int_S (-1)\,dydz + (-1)\,dzdx + (-1)\,dxdy. \tag{8}$$

Letting D be the projection of S onto the xy-plane, we know:

$$\int_S 1\,dxdy \quad = \quad -\int_D 1\,dxdy = -1/2.$$

Similarly, we know $\int_S 1\,dydz = \int_S 1\,dzdx = -1/2$. Therefore, (8) equals $3/2$. □

The Stokes' Theorem has another useful form. Let $\boldsymbol{n} = [\cos\alpha, \cos\beta, \cos\gamma]$ be a unit normal vector of S, coming out from the side of S we have chosen. Specifically:

- α is the angle between the directions of $\boldsymbol{n}$ and $\boldsymbol{i} = [1, 0, 0]$;
- β is the angle between the directions of $\boldsymbol{n}$ and $\boldsymbol{j} = [0, 1, 0]$;
- γ is the angle between the directions of $\boldsymbol{n}$ and $\boldsymbol{k} = [0, 0, 1]$.

From our earlier discussion on surface integral by area, we know:

$$\begin{aligned}
\iint_S \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\cos\alpha\,dA &= \iint_S \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\,dydz \\
\iint_S \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\cos\beta\,dA &= \iint_S \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\,dzdx \\
\iint_S \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\cos\gamma\,dA &= \iint_S \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\,dxdy.
\end{aligned}$$

Therefore, from Theorem 2, we get:

$$\begin{aligned}
&\int_C f_1\,dx + f_2\,dy + f_3\,dz \\
= \quad &\iint_S \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\cos\beta + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\cos\gamma\,dA.
\end{aligned}$$

Let us introduce a vector function $\boldsymbol{f} = [f_1, f_2, f_3]$. In an earlier lecture, we have defined:

$$\mathrm{curl}\boldsymbol{f} \quad = \quad \left[\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}, \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}, \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right].$$

We thus obtain the following concise form of Theorem 2:

$$\int_C f_1\,dx + f_2\,dy + f_3\,dz \quad = \quad \iint_S \mathrm{curl}\boldsymbol{f} \cdot \boldsymbol{n}\,dA. \tag{9}$$

Example 4. Let S be the hemisphere $x^2 + y^2 + z^2 = 1$ with $z \geq 0$. Orient S by taking its upper side. Let $\boldsymbol{f} = [-y, x, z^2]$. Calculate $\iint_S \mathrm{curl}\boldsymbol{f} \cdot \boldsymbol{n}\,dA$.

Solution. We could solve this question by using the methods we learned in the lecture "surface integral by area". However, due to the special nature of the integrand function, we can apply Stokes' Theorem to convert the surface integral into a line integral, which simplifies calculation significantly. The boundary curve C of S is the circle $x^2 + y^2 = 1$ in the xy-plane, directed counterclockwise. Hence, by (9), we have:

$$\begin{aligned} \iint_S \operatorname{curl}\boldsymbol{f} \cdot \boldsymbol{n}\, dA &= \int_C -y\, dx + x\, dy + z^2\, dz. \\ \text{(as } C \text{ is in the xy-plane)} &= \iint_C -y\, dx + x\, dy \\ \text{(by Green's Theorem)} &= 2\iint_D dxdy = 2\pi \end{aligned}$$

where D is the disc $x^2 + y^2 \leq 1$ in the xy-plane. $\square$

Remark 3. Note that the Stokes' theorem reveals a relationship between a 1d integral and a 2d integral. Namely, it has the same nature as the Green's theorem (which in fact is a special case of the Stokes' theorem).