Exercises: Dot Product and Cross Product

Problem 1. Give the result of $\boldsymbol{a} \cdot \boldsymbol{b}$ for each of the following:

1. $\boldsymbol{a} = [1, 2], \boldsymbol{b} = [2, 5]$.
2. $\boldsymbol{a} = [1, 2, 3], \boldsymbol{b} = [2, 5, -7]$.

Solution:

1. $\boldsymbol{a} \cdot \boldsymbol{b} = 1 \cdot 2 + 2 \cdot 5 = 12$.
2. -9.

Problem 2. Give the result of $\boldsymbol{a} \times \boldsymbol{b}$ for each of the following:

1. $\boldsymbol{a} = [1, 2, 3], \boldsymbol{b} = [3, 2, 1]$.
2. $\boldsymbol{a} = \boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}, \boldsymbol{b} = [3, 2, 1]$.

Solution:

1. $\boldsymbol{a} \times \boldsymbol{b} = [2 \cdot 1 - 3 \cdot 2, 3 \cdot 3 - 1 \cdot 1, 1 \cdot 2 - 2 \cdot 3] = [-4, 8, -4]$.
2. $[-3, 2, 5]$.

Problem 3. In each of the following, you are given two vectors $\boldsymbol{a} \cdot \boldsymbol{b}$. Let γ be the angle between the two vectors' directions. Give the value of $\cos \gamma$.

1. $\boldsymbol{a} = [1, 2], \boldsymbol{b} = [2, 5]$
2. $\boldsymbol{a} = [1, 2, 3], \boldsymbol{b} = [3, 2, 1]$

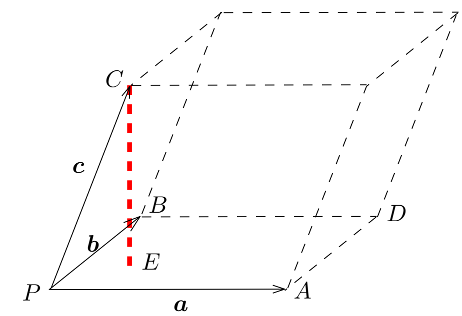
Solution:

1. From the notes of Lecture 2, we know $\cos \gamma = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|} = \frac{12}{\sqrt{5} \cdot \sqrt{29}} = \frac{12}{\sqrt{145}}$.
2. $\frac{5}{7}$.

Problem 4. This exercise explores the usage of dot product for calculation of projection lengths. Consider points $P(1, 2, 3), A(2, -1, 4), B(3, 2, 5)$. Let ℓ be the line passing P and A. Now, let us project point B onto ℓ; denote by C the projection. Calculate the distance between P and C.

Solution: Let $\overrightarrow{P, A}$ and $\overrightarrow{P, B}$ be the instantiations of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively. Let γ be the angle between the directions of $\boldsymbol{a}$ and $\boldsymbol{b}$. We have $|\overrightarrow{P, C}| = |\boldsymbol{b}||\cos \gamma| = |\boldsymbol{b}| \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}|} = \frac{\boldsymbol{a}}{|\boldsymbol{a}|} \cdot \boldsymbol{b} = \frac{4}{\sqrt{11}}$.

Problem 5. Let $\overrightarrow{P, A}$, $\overrightarrow{P, B}$, and $\overrightarrow{P, C}$ be directed segments that are not in the same plane. They determine a parallelepiped as shown below:



Suppose that $\overrightarrow{P,A}$, $\overrightarrow{P,B}$, and $\overrightarrow{P,C}$ are instantiations of vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$, respectively. Prove that the volume of the parallelepiped equals $|(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}|$.

Proof: Let E be the projection of point C onto the plane defined by P, A, B (see the above figure). Denote by $\overline{CE}$ the segment connecting C and E, and by $\overline{CE}$ its length. Clearly, the volume of the parallelepiped equals $area(PADB) \cdot |\overline{CE}|$. From the notes of Lecture 2, we know that $|\boldsymbol{a} \times \boldsymbol{b}|$ is exactly $area(PADB)$. So to complete the proof, we need to show:

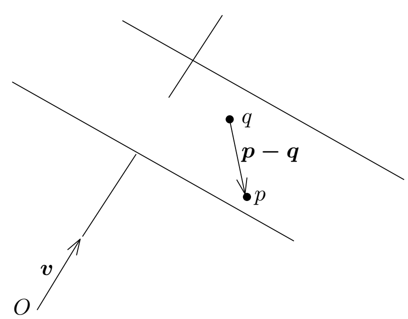
$$\begin{aligned} |(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}| &= |\boldsymbol{a} \times \boldsymbol{b}||\overline{CE}| \quad \Leftrightarrow \\ |(\boldsymbol{a} \times \boldsymbol{b})||\boldsymbol{c}| \cos\gamma &= |\boldsymbol{a} \times \boldsymbol{b}||\overline{CE}| \end{aligned} \tag{1}$$

where γ is the angle between the directions of $\boldsymbol{a} \times \boldsymbol{b}$ and $\boldsymbol{c}$. To prove Equation 1, it suffices to prove

$$|\boldsymbol{c}||\cos\gamma| = |\overline{CE}|$$

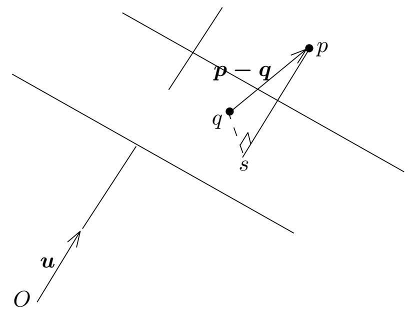
which is true because γ is also the angle between $\overrightarrow{P,C}$ and $\overline{CE}$.

Problem 6. Given a point $p(x, y, z)$ in $\mathbb{R}^3$, we use $\boldsymbol{p}$ to denote the corresponding vector $[x, y, z]$. Let q be a point in $\mathbb{R}^3$, and $\boldsymbol{v}$ be a non-zero 3d vector. Denote by ρ the plane passing q that is perpendicular to the direction of $\boldsymbol{v}$. Prove that for any p on ρ, it holds that $(\boldsymbol{p} - \boldsymbol{q}) \cdot \boldsymbol{v} = 0$.



Proof: The equation obviously holds if $q = p$. Now consider the case where $q \neq p$, as shown in the above figure. We know that the directions of $\boldsymbol{v}$ and $\boldsymbol{p} - \boldsymbol{q}$ are orthogonal. Therefore, $(\boldsymbol{p} - \boldsymbol{q}) \cdot \boldsymbol{v} = 0$. $\square$

Problem 7. Given a point $p(x, y, z)$ in $\mathbb{R}^3$, we use $\boldsymbol{p}$ to denote the corresponding vector $[x, y, z]$. Let q be a point in $\mathbb{R}^3$, and $\boldsymbol{u}$ be a unit 3d vector (i.e., $|\boldsymbol{u}| = 1$). Denote by ρ the plane passing q that is perpendicular to the direction of $\boldsymbol{u}$. Prove that for any p in $\mathbb{R}^3$, its distance to ρ equals $|(\boldsymbol{p} - \boldsymbol{q}) \cdot \boldsymbol{u}|$.



Proof: If p falls on ρ, then the equation follows from the result of Problem 6. Otherwise, let s be the projection of p onto ρ. See the above figure. Let γ be the angle between the two segments $\overline{pq}$ and $\overline{ps}$. Hence:

$$|ps| = |pq||\cos\gamma|$$

It suffices to prove that

$$\begin{aligned} |pq||\cos\gamma| &= |(\boldsymbol{p}-\boldsymbol{q})\cdot\boldsymbol{u}| \\ &= |(\boldsymbol{p}-\boldsymbol{q})||\boldsymbol{u}||\cos\theta| \end{aligned}$$

where θ is the angle between the directions of $\boldsymbol{u}$ and $\boldsymbol{p}-\boldsymbol{q}$. The above is true because (i) $|pq| = |(\boldsymbol{p}-\boldsymbol{q})|$ and (ii) either $\theta = \gamma$ or $\theta = 180° - \gamma$. We thus complete the proof. $\square$

Problem 8. Consider the plane $x + 2y + 3z = 4$ in $\mathbb{R}^3$. Calculate the distance from point $(0, 0, 0)$ to the plane.

Solution: We can re-write the plane's equation as

$$1\cdot(x-0) + 2\cdot(y-0) + 3\cdot(z-4/3) = 0.$$

Hence, $q(0, 0, 4/3)$ is a point on the plane. Also, we know that the direction of $\boldsymbol{v} = [1, 2, 3]$ is perpendicular to the plane. Let $\boldsymbol{u} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = [\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}]$. Note that the direction of $\boldsymbol{u}$ is also perpendicular to the plane, and that $|\boldsymbol{u}| = 1$. Therefore, we can now apply the result of the previous problem to compute the distance from $p(0, 0, 0)$ to the plane as:

$$\left|([0,0,0]-[0,0,4/3])\cdot[\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}]\right| = \left|-\frac{4}{3}\cdot\frac{3}{\sqrt{14}}\right| = \frac{4}{\sqrt{14}}$$

Problem 9. Consider the line $x + 2y = 4$ in $\mathbb{R}^2$. Calculate the distance from point $(0, 0)$ to the line.

Solution: This problem follows the same idea as the previous one, and gives you another chance to practice on the new technique we have developed.

We re-write the line's equation as

$$1\cdot(x-0) + 2(y-2) = 0.$$

Hence, $q(0, 2)$ is a point on the line. Also, we know that the direction of $\boldsymbol{v} = [1, 2]$ is perpendicular to the line. Let $\boldsymbol{u} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = [\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}]$. Note that the direction of $\boldsymbol{u}$ is also perpendicular to the line, and that $|\boldsymbol{u}| = 1$. The distance from $p(0,0)$ to the line is:

$$\begin{aligned}
|(\boldsymbol{p} - \boldsymbol{q}) \cdot \boldsymbol{u}| &= \left|([0,0] - [0,2]) \cdot [\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}]\right| \\
&= \left|-2 \cdot \frac{2}{\sqrt{5}}\right| = \frac{4}{\sqrt{5}}
\end{aligned}$$

Problem 10. Given a point $p(x, y, z)$ in $\mathbb{R}^3$, we use $\boldsymbol{p}$ to denote the corresponding vector $[x, y, z]$. Let q be a fixed point in $\mathbb{R}^3$, and $\boldsymbol{v}$ a non-zero 3d vector. Given a real value s, $f(s)$ gives a point p in $\mathbb{R}^3$ such that $\boldsymbol{p} = \boldsymbol{q} + s \cdot \boldsymbol{v}$. As s goes from $-\infty$ to ∞, what is the locus of $f(s)$?

Solution: It is the line that passes p, and is parallel to the direction of $\boldsymbol{v}$. The figure below illustrates the points of $f(1), f(2), f(3)$. You can easily extend the idea to obtain $f(s)$ of arbitrary s.