ENGG 1130 Multivariable Calculus for Engineers

Assignment 1 (Term 2, 2019-2020)

Assigned Date: 6 Jan 2020 (Monday)

Deadline: due on **24 Jan 2020 (Friday) 5:00 pm**

- Show **ALL** your steps and details for each question unless otherwise specified.
- Make a copy of your homework before submitting the original!
- Please submit the hard copy of your HW to **your TA** before the prescribed deadline.
- **Submit it to the TAs of ENGG 1130A – from Department of Systems Engineering and Engineering**

* Harder questions

Notation: $\langle a, b, c\rangle$ represents the vector $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

1. Consider the two-dimensional xy-plane. Let O be the origin and A be a point lying on the x-axis. Let P be an arbitrary point lying on a curve. Given that $\angle OPA$ (with P being the vertex) is always equal to 90°, describe the geometrical object traced by P (the curve). Show your derivation and explain briefly.

2. (a) Given a hyperbola with equation $x^2 - y^2 = 1$, find the area of the region bounded by the x-axis, the hyperbola, and the straight line from (0, 0) to the point $(\sqrt{1+y_0^2}, y_0)$.
   (**Note:** Here we assume y_0 is positive, and notice that the point $(\sqrt{1+y_0^2}, y_0)$ actually lies on the given hyperbola.)

   (b) Suppose $y_0 = \sinh t = \frac{e^t - e^{-t}}{2}$. Express the area calculated in (a) in terms of t only.

3. Let $\mathbf{u} = \langle 2,1,-2\rangle$, $\mathbf{v} = \langle 1,2,2\rangle$ and $\mathbf{w} = \mathbf{u} \times \mathbf{v}$.

   (a) Show that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are mutually orthogonal vectors.

   (b) Given any vector $\mathbf{r} = \langle x, y, z\rangle$ in $\mathbf{R}^3$, show that
   $$\mathbf{r} = \frac{\mathbf{r}\cdot\mathbf{u}}{\|\mathbf{u}\|^2}\mathbf{u} + \frac{\mathbf{r}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} + \frac{\mathbf{r}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w}$$

   (c) Using the result of (b), express the vector $\mathbf{i}$ as a linear combination of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

   (**Hint:** You may use the fact that since $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are mutually orthogonal and non-zero, the vector $\mathbf{r}$ can be expressed as a linear combination of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.)

4. (a) Find the distance from the point (1, 2, 0) to the plane $3x - 4y - 5z = 2$.

   (b) Find the distance between the lines $\begin{cases} x + 2y = 3 \\ y + 2z = 3 \end{cases}$ and $\begin{cases} x + y + z = 6 \\ x - 2z + 5 = 0 \end{cases}$.

5. (a) Show that the line $x-2=\frac{y+3}{2}=\frac{z-1}{4}$ is parallel to the plane $2y-z-1=0$.

(b) Find the distance between the line and the plane.

6. (a) Describe the set of points in $\mathbf{R}^3$ that satisfy the equation $z=x$. Provide your descriptions in <u>words</u>.

(b) Describe the set of points in $\mathbf{R}^3$ that satisfy the inequality $z \geq \sqrt{x^2+y^2}$.

Provide your descriptions in <u>words</u>.

(c) Describe the set of points in $\mathbf{R}^3$ that satisfy the simultaneous equation $\begin{cases} x^2+y^2+z^2=4 \\ x+y+z=3 \end{cases}$

Provide your descriptions in <u>words</u>.

*7. Determine the intersection between the hyperbolic paraboloid $z=\frac{y^2}{b^2}-\frac{x^2}{a^2}$ and the plane $bx+ay-z=0$.

Here we may assume both a, b are positive values.

*8. The cross product of two vectors $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})$ is called a **vector triple product**.

In this question, we assume $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors in 3-space.

(a) Explain briefly why $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})$ must lie in the plane of $\mathbf{v}$ and $\mathbf{w}$.

(b) Show that $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})=(\mathbf{u}\cdot\mathbf{w})\mathbf{v}-(\mathbf{u}\cdot\mathbf{v})\mathbf{w}$.

(**Hint:** The proof will be much easier if you select the coordinate axes such that $\mathbf{v}$ lies along the x-axis and $\mathbf{w}$ lies in the xy-plane. If you are not certain of this hint, just expand everything and compare L.H.S. with R.H.S..)

(c) Using the result of (b), show that

$$(\mathbf{u}\times\mathbf{v})\times(\mathbf{w}\times\mathbf{x})=\big((\mathbf{u}\times\mathbf{v})\cdot\mathbf{x}\big)\mathbf{w}-\big((\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}\big)\mathbf{x}$$

and

$$(\mathbf{u}\times\mathbf{v})\times(\mathbf{w}\times\mathbf{x})=\big((\mathbf{w}\times\mathbf{x})\cdot\mathbf{u}\big)\mathbf{v}-\big((\mathbf{w}\times\mathbf{x})\cdot\mathbf{v}\big)\mathbf{u}$$

for any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ and $\mathbf{x}$ in 3-space.

(d) Using the result of (c), show that $(\mathbf{u}\times\mathbf{v})\times(\mathbf{u}\times\mathbf{w})=\big((\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}\big)\mathbf{u}$ for any vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ in 3-space.

*9. For each non-negative integer n, we define the following expressions:

$$A_n := \int_0^{\frac{\pi}{2}} \cos^{2n} x \, dx$$

$$B_n := \int_0^{\frac{\pi}{2}} x^2 \cos^{2n} x \, dx$$

(a) Show that for any positive integer n, $2\frac{B_{n-1}}{A_{n-1}} - 2\frac{B_n}{A_n} = \frac{1}{n^2}$.

(b) By considering the concavity of $\sin x$ on $x \in \left[0, \frac{\pi}{2}\right]$, show that there exists a constant $C > 0$ (independent of n), such that $B_n \leq \frac{C}{n+1} A_n$ for any positive integer n. Show your choice of C explicitly in your proof.

(c) Hence, show that $\zeta(2) := \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. You are allowed to use the above results **ONLY**, but not other methods.

END OF ASSIGNMENT 1