Lecture Notes: Determinant of a Square Matrix

Yufei Tao
Department of Computer Science and Engineering
Chinese University of Hong Kong
taoyf@cse.cuhk.edu.hk

1 Determinant Definition

Let $\boldsymbol{A} = [a_{ij}]$ be an $n \times n$ matrix (i.e., $\boldsymbol{A}$ is a square matrix). Given a pair of (i, j), we define $\boldsymbol{M}_{ij}$ to be the $(n-1) \times (n-1)$ matrix obtained by removing the i-th row and j-th column of $\boldsymbol{A}$. For example, suppose that

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & -2 \\ -1 & -1 & 2 \end{bmatrix}$$

Then:

$$\boldsymbol{M}_{21} = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}, \boldsymbol{M}_{22} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}, \boldsymbol{M}_{32} = \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix}$$

We are now ready to define determinants:

Definition 1. *Let $\boldsymbol{A} = [a_{ij}]$ be an $n \times n$ matrix. If $n = 1$, its* **determinant***, denoted as $det(\boldsymbol{A})$, equals a_{11}. If $n > 1$, we first choose an arbitrary $i^* \in [1, n]$, and then define the determinant of $\boldsymbol{A}$ recursively as:*

$$det(\boldsymbol{A}) = \sum_{j=1}^{n} (-1)^{i^*+j} \cdot a_{i^*j} \cdot det(\boldsymbol{M}_{i^*j}). \tag{1}$$

Besides $det(\boldsymbol{A})$, we may also denote the determinant of $\boldsymbol{A}$ as $|\boldsymbol{A}|$. Henceforth, if we apply (1) to compute $det(\boldsymbol{A})$, we say that we *expand* $\boldsymbol{A}$ *by row* i^*. It is important to note that the value of $det(\boldsymbol{A})$ does *not* depend on the choice of i^*. We omit a proof of this fact, but illustrate it in the following examples.

Example 1 (Second-Order Determinants). In general, if $\boldsymbol{A} = [a_{ij}]$ is a 2×2 matrix, then

$$det(\boldsymbol{A}) = a_{11}a_{22} - a_{12}a_{21}.$$

For instance:

$$\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 2 \times 2 - 1 \times (-1) = 5.$$

We may verify the above by definition as follows. Choosing $i^* = 1$, we get:

$$\begin{aligned} \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} &= (-1)^{1+1} \cdot 2 \cdot det(\boldsymbol{M}_{11}) + (-1)^{1+2} \cdot 1 \cdot det(\boldsymbol{M}_{12}) \\ &= 2 \times 2 + (-1) \times (-1) = 5. \end{aligned}$$

Alternatively, choosing $i^* = 2$, we get:

$$\begin{vmatrix} 2 & 1 \\ \text{-1} & 2 \end{vmatrix} = (-1)^{2+1} \cdot (-1) \cdot det(\boldsymbol{M}_{21}) + (-1)^{2+2} \cdot 2 \cdot det(\boldsymbol{M}_{22})$$
$$= 1 \times 1 + 2 \times 2 = 5.$$

$\square$

Example 2 (Third-Order Determinants). Suppose that

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & \text{-2} \\ \text{-1} & \text{-1} & 2 \end{bmatrix}$$

Choosing $i^* = 1$, we get:

$$\begin{aligned} det(\boldsymbol{A}) &= 1 \begin{vmatrix} 0 & -2 \\ -1 & 2 \end{vmatrix} - 2 \begin{vmatrix} 3 & -2 \\ -1 & 2 \end{vmatrix} + 1 \begin{vmatrix} 3 & 0 \\ -1 & -1 \end{vmatrix} \\ &= 1(0-2) - 2(6-2) + 1(-3-0) = -13. \end{aligned}$$

Alternatively, choosing $i^* = 2$, we get:

$$\begin{aligned} det(\boldsymbol{A}) &= -3 \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} + 0 \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} - (-2) \begin{vmatrix} 1 & 2 \\ -1 & -1 \end{vmatrix} \\ &= (-3)(4+1) + 0(2+1) + 2(-1+2) = -13. \end{aligned}$$

$\square$

2 Properties of Determinants

Expansion by a Column. Definition 1 allows us to compute the determinant of a matrix by row expansion. We may also achieve the same purpose by column expansion.

Lemma 1. *Let $\boldsymbol{A} = [a_{ij}]$ be an $n \times n$ matrix with $n > 1$. Choose an arbitrary $j^* \in [1, n]$. The determinant of $\boldsymbol{A}$ equals:*

$$det(\boldsymbol{A}) = \sum_{i=1}^{n} (-1)^{i+j^*} \cdot a_{ij^*} \cdot det(\boldsymbol{M}_{ij^*}).$$

The value of the above equation does not depend on the choice of j^*.

We omit a proof but illustrate the lemma with an example below. Henceforth, if we compute $det(\boldsymbol{A})$ by the above lemma, we say that we *expand $\boldsymbol{A}$ by column j^*.*

Example 3. Suppose that

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & \text{-2} \\ \text{-1} & \text{-1} & 2 \end{bmatrix}$$

Choosing $j^* = 1$, we get:

$$
\begin{aligned}
det(\boldsymbol{A}) &= 1\begin{vmatrix} 0 & -2 \\ -1 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 2 & 1 \\ 0 & -2 \end{vmatrix} \\
&= 1(0-2) - 3(4+1) - 1(-4-0) = -13.
\end{aligned}
$$

□

Corollary 1. *Let $\boldsymbol{A}$ be a square matrix. Then, $det(\boldsymbol{A}) = det(\boldsymbol{A}^T)$.*

Proof. Note that expanding $\boldsymbol{A}$ by column k is equivalent to expanding $\boldsymbol{A}^T$ by row k. □

Corollary 2. *If $\boldsymbol{A}$ has a zero row or a zero column, then $det(\boldsymbol{A}) = 0$.*

Proof. If $\boldsymbol{A}$ has a zero row, then $det(\boldsymbol{A}) = 0$ follows from expanding $\boldsymbol{A}$ by that row. The case where $\boldsymbol{A}$ has a zero column is similar. □

Determinant of a Row-Echelon Matrix. The next lemma shows that the determinant of a matrix in row-echelon form is simply the product of the values on the main diagonal.

Lemma 2. *Let $\boldsymbol{A} = [a_{ij}]$ be an $n \times n$ matrix in row-echelon form. Then, $det(\boldsymbol{A}) = \Pi_{i=1}^{n} a_{ii}$.*

Proof. We can prove the lemma by induction. First, correctness is obvious for $n = 1$. Assuming correctness for $n \leq t-1$ (for $t \geq 2$), consider $n = t$. Expanding $\boldsymbol{A}$ by the first row gives:

$$
det(\boldsymbol{A}) = \sum_{j=1}^{n} (-1)^{1+j} \cdot a_{1j} \cdot det(\boldsymbol{M}_{1j}). \tag{2}
$$

From induction we know that $det(\boldsymbol{M}_{11}) = \Pi_{i=2}^{n} a_{ii}$. Furthermore, for $j > 1$, $det(\boldsymbol{M}_{1j}) = 0$ because the first column of $\boldsymbol{M}_{1j}$ contains all 0's. It thus follows that (2) equals $\Pi_{i=1}^{n} a_{ii}$. □

Determinants under Elementary Row Operations. Let $\boldsymbol{A} = [a_{ij}]$ be an $n \times n$ matrix. Define the elementary row operations on $\boldsymbol{A}$:

1. Switch two rows of $\boldsymbol{A}$.
2. Multiply all numbers of a row of $\boldsymbol{A}$ by the same non-zero value c.
3. Let $\boldsymbol{r}_i$ and $\boldsymbol{r}_j$ be two row vectors of $\boldsymbol{A}$. Update row $\boldsymbol{r}_i$ to $\boldsymbol{r}_i + c \cdot \boldsymbol{r}_j$, where c can be any real value.

Next, we refer to the above as Operation 1, 2, and 3, respectively.

Lemma 3. *The determinant of $\boldsymbol{A}$*

1. *should be multiplied by -1 after Operation 1;*
2. *should be multiplied by c after Operation 2;*

3. has no change after Operation 3.

Again, we omit a proof, but illustrate the lemma with an example.

Example 4.

$$\begin{vmatrix} 1 & 2 & 1 \\ 3 & 0 & -2 \\ -1 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 0 & -2 \\ 0 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 0 & -6 & -5 \\ 0 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 0 & -6 & -5 \\ 0 & 0 & 13/6 \end{vmatrix} = -13.$$

Here is another derivation giving the same result:

$$\begin{vmatrix} 1 & 2 & 1 \\ 3 & 0 & -2 \\ -1 & -1 & 2 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & 1 \\ -1 & -1 & 2 \\ 3 & 0 & -2 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -6 & -5 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 13 \end{vmatrix} = -13.$$

$\square$

Corollary 3. *If $\boldsymbol{A}$ has two identical rows or columns, then $det(\boldsymbol{A}) = 0$.*

Proof. We prove only the row case. Switching the two rows gets back the same matrix. However, by Lemma 3, the determinant of the matrix should be multiplied by -1. Therefore, we get $det(\boldsymbol{A}) = -det(\boldsymbol{A})$, meaning $det(\boldsymbol{A}) = 0$. $\square$

Determinant under Matrix Multiplication. We state the following result without proof:

Lemma 4. *Let $\boldsymbol{A}, \boldsymbol{B}$ be $n \times n$ matrices. It holds that $det(\boldsymbol{AB}) = det(\boldsymbol{A}) \cdot det(\boldsymbol{B})$.*

Example 5.

$$\begin{vmatrix} 1 & 2 & 1 \\ 3 & 0 & -2 \\ -1 & -1 & 2 \end{vmatrix} = -13$$

$$\begin{vmatrix} -2 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & -2 & 0 \end{vmatrix} = -3$$

$$\left| \begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & -2 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} -2 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & -2 & 0 \end{bmatrix} \right| = \begin{vmatrix} -1 & 1 & 2 \\ -8 & 7 & 0 \\ 4 & -6 & -1 \end{vmatrix} = 39.$$

$\square$

Relationships with Ranks. We close the lecture notes by giving two lemmas that relate determinants to ranks:

Lemma 5. *Let $\boldsymbol{A}$ be an $n \times n$ matrix. $\boldsymbol{A}$ has rank n if and only if $det(\boldsymbol{A}) \neq 0$.*

Proof. We can first apply elementary row operations to convert $\boldsymbol{A}$ into row-echelon form $\boldsymbol{A}^*$. Thus, $\boldsymbol{A}$ has rank n if and only if $\boldsymbol{A}^*$ has rank n. Since $\boldsymbol{A}^*$ is a square matrix, that it has rank n is equivalent to saying that all the numbers on its main diagonal are non-zero. Thus, by Lemma 2, we know that $\boldsymbol{A}^*$ has rank n if and only if $det(\boldsymbol{A}^*) \neq 0$. Finally, by Lemma 3, $det(\boldsymbol{A}) \neq 0$ if and only if $det(\boldsymbol{A}^*) \neq 0$. We thus complete the proof. $\square$

Let $\boldsymbol{A}$ be an $m \times n$ matrix. We say that $\boldsymbol{B}$ is a *submatrix* of $\boldsymbol{A}$ if we can obtain $\boldsymbol{B}$ is by discarding some rows and columns of $\boldsymbol{A}$. For instance, suppose that

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 3 & 2 & 1 & 0 \end{bmatrix} \tag{3}$$

Then all the following are submatrices of $\boldsymbol{A}$:

$$\begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 3 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 6 & 7 \\ 2 & 1 \end{bmatrix}, [0].$$

We state the next lemma without proof:

Lemma 6. *Let $\boldsymbol{A}$ be an $m \times n$ matrix. $\boldsymbol{A}$ has rank r if and only if both the following hold simultaneously:*

- *A has an $r \times r$ submatrix $\boldsymbol{B}$ such that $det(\boldsymbol{B}) \neq 0$.*
- *all $(r+1) \times (r+1)$ submatrices of $\boldsymbol{A}$ have determinant 0.*

Example 6. Let $\boldsymbol{A}$ be as shown in (3). It can be verified that rank $\boldsymbol{A} = 2$. Indeed, you can easily find a 2×2 submatrix of $\boldsymbol{A}$ with a non-zero determinant. On the other hand, any 3×3 submatrix of $\boldsymbol{A}$ always has determinant 0. $\square$