Exercises: Planar-Region Projection, Surface Areas, and Surface Integral by Area

Problem 1. Let g be a region (bounded by a continuous curve) in the plane $x + y + z = 1$. Let g_{xy} be the projection of g onto the xy-plane. If we know that the area of g is 1, what is the area of g_{xy}.

Solution: We know from the equation $x + y + z = 1$ that $\boldsymbol{N} = [1, 1, 1]$ is a normal vector of the plane. Let γ be the angle between $\boldsymbol{N}$ and $\boldsymbol{k} = [0, 0, 1]$. Thus:

$$\cos\gamma = \frac{\boldsymbol{N} \cdot \boldsymbol{k}}{|\boldsymbol{N}||\boldsymbol{k}|} = 1/\sqrt{3}.$$

Hence, the area of g_{xy} equals $1 \cdot \cos\gamma = 1/\sqrt{3}$.

Problem 2. Consider the surface $S : z = x^2 + y^2$ with $0 \le z \le 1$. Compute the area of S.

Solution. Let D be the projection of the surface; note that D is the disc $x^2 + y^2 \le 1$. Introduce $f(x, y, z) = x^2 + y^2 - z$. We know that S can be described by $f(x, y, z) = 0$.

S is xy-monotone. Each point (x, y) in D uniquely corresponds to a point $p = (x, y, z(x, y))$ on S. Let $\boldsymbol{N}$ be a normal vector of S at p, and γ the angle between $\boldsymbol{N}$ and $\boldsymbol{k} = [0, 0, 1]$. We know from the definition of surface area that the area of S equals:

$$A = \iint_D \frac{1}{|\cos\gamma|} \, dxdy.$$

We know that $\nabla f = [\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}] = [2x, 2y, -1]$ is a normal vector of S. Let us choose this normal vector as our $\boldsymbol{N}$. Hence, we have:

$$\cos\gamma = \frac{\frac{\partial f}{\partial z}}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2}} = \frac{-1}{\sqrt{4x^2 + 4y^2 + 1}}.$$

Therefore:

$$A = \iint_D \sqrt{4x^2 + 4y^2 + 1} \, dxdy.$$

To evaluate the above double integral, we represent the points of D using polar coordinates: $x = r\cos\theta$ and $y = r\sin\theta$, where $r \in [0, 1]$ and $\theta \in [0, 2\pi]$. This leads to:

$$\begin{aligned} A &= \iint_D \sqrt{4x^2 + 4y^2 + 1} \, dxdy \\ &= \int_0^{2\pi} \left(\int_0^1 \sqrt{4r^2 + 1} \cdot r \, dr \right) d\theta = (5^{3/2} - 1)\pi/6. \end{aligned}$$

Problem 3. Consider the surface S in a parametric form $\boldsymbol{r}(u, v) = [x(u, v), y(u, v), z(u, v)]$ where

$$\begin{aligned} x(u, v) &= u + v \\ y(u, v) &= u - v \\ z(u, v) &= uv \end{aligned}$$

with (u, v) in the disc $u^2 + v^2 \leq 1$. Compute the area of S.

Solution. Define:

$$\begin{aligned}
\boldsymbol{r}_u &= \left[\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right] = [1, 1, v] \\
\boldsymbol{r}_v &= \left[\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right] = [1, -1, u].
\end{aligned}$$

Define:

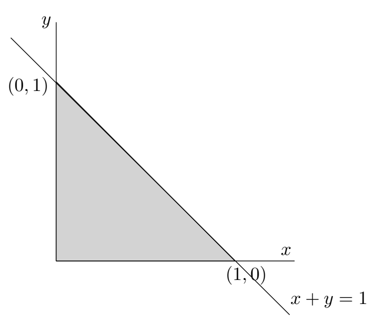
$$\boldsymbol{N} = \boldsymbol{r}_u \times \boldsymbol{r}_v = [u + v, v - u, -2].$$

Let R be the disc $u^2 + v^2 \leq 1$. Therefore:

$$\begin{aligned}
A &= \iint_R |\boldsymbol{N}|\, dudv \\
&= \iint_R \sqrt{2u^2 + 2v^2 + 4}\, dudv \\
&= (6^{3/2} - 8)\pi/3.
\end{aligned}$$

Problem 4. Let S be the surface $x + y + z = 1$ with $x \in [0, 1]$, $y \in [0, 1]$, and $z \in [0, 1]$. Compute $\iint_S x\, dA$.

Solution. Introduce $f(x, y, z) = x + y + z - 1$. Hence, S can be described as $f(x, y, z) = 0$. Take the gradient of f: $\nabla f = [\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}] = [1, 1, 1]$, which points upwards. Let us orient S by taking its upper side. Let D be the projection of S onto the xy=plane. The figure below illustrates D (the shaded triangle).



$$\begin{aligned}
\iint_S x\, dA &= \iint_S x \frac{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2}}{\frac{\partial f}{\partial z}}\, dxdy \\
&= \iint_D x\sqrt{3}\, dxdy = \sqrt{3}/6.
\end{aligned}$$

Problem 5. Let S be the surface $\boldsymbol{r}(u, v) = [x(u, v), y(u, v), z(u, v)]$ where $x(u, v) = u, y(u, v) = v, z(u, v) = u^3$ with $u \in [0, 1]$ and $v \in [-2, 2]$. Compute $\iint_S (1 + 9xz)^{1/2}\, dA$.

Solution. Define:

$$\begin{aligned} \boldsymbol{r}_u &= \left[\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right] = [1, 0, 3u^2] \\ \boldsymbol{r}_v &= \left[\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right] = [0, 1, 0]. \end{aligned}$$

Define:

$$\boldsymbol{N} = \boldsymbol{r}_u \times \boldsymbol{r}_v = [-3u^2, 0, 1].$$

Let R be the set of (u, v) $u \in [0, 1]$ and $v \in [-2, 2]$. Therefore:

$$\begin{aligned} \iint_S (1 + 9xz)^{1/2} \, dA &= \iint_R (1 + 9xz)^{1/2} |\boldsymbol{N}| \, dudv \\ &= \iint_R (1 + 9u^4)^{1/2} \sqrt{9u^4 + 1} \, dudv = 56/5. \end{aligned}$$

Problem 6. Define $\boldsymbol{f}(x, y, z) = [-x^2, y^2, 0]$. Let S be the surface $\boldsymbol{r}(u, v) = [x(u, v), y(u, v), z(u, v)]$ where $x(u, v) = u, y(u, v) = v, z(u, v) = 3u - 2v$ with $0 \le u \le 1$ and $0 \le v \le 1$. Calculate $\iint_S \boldsymbol{f} \cdot \boldsymbol{n} \, dA$.

Solution. Let R be the set of all (u, v) with $0 \le u \le 1$ and $0 \le v \le 1$. Define:

$$\begin{aligned} \boldsymbol{r}_u &= \left[\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right] = [1, 0, 3] \\ \boldsymbol{r}_v &= \left[\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right] = [0, 1, -2]. \end{aligned}$$

Define:

$$\boldsymbol{N} = \boldsymbol{r}_u \times \boldsymbol{r}_v = [-3, 2, 1].$$

Recall (from the vector representation of surface integral by area as discussed in the class) that $\boldsymbol{n} = \boldsymbol{N}/|\boldsymbol{N}|$. Also, as discussed in the class:

$$\begin{aligned} \iint_S \boldsymbol{f} \cdot \boldsymbol{n} \, dA &= \iint_R \boldsymbol{f} \cdot \boldsymbol{N} \, dudv \\ &= \iint_R [-x^2, y^2, 0] \cdot [-3, 2, 1] \, dudv \\ &= \iint_R 3x^2 + 2y^2 \, dudv \\ &= \iint_R 3u^2 + 2v^2 \, dudv = 5/3. \end{aligned}$$