Lecture Notes: Tangent and Gradient

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Let $p(x_1, x_2, ..., x_d)$ be a point in $\mathbb{R}^d$. We will often view it as a d-dimensional vector $[x_1, x_2, ..., x_d]$. As a convention, if it has been clear from the context that p is a point, then $\boldsymbol{p}$ represents this corresponding vector.

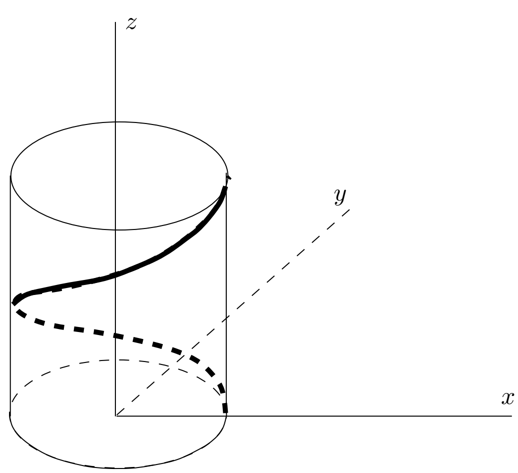
1 Curves and Tangent Vectors

Curves. Imagine that you move a point around in $\mathbb{R}^d$. The locus of the point forms a *curve*. Intuitively, a curve is a 1d geometric object. Indeed, we can represent a curve using a vector function $\boldsymbol{r}(t)$:

$$\boldsymbol{r}(t) = [x_1(t), x_2(t), ..., x_d(t)]$$

where t is a real value in a certain range, and each function $x_i(t)$ (with $i \in [1, d]$) returns a real value. For each t, $(x_1(t), ..., x_d(t))$ defines a point, and $\boldsymbol{r}(t)$ gives the corresponding vector.

For example, $\boldsymbol{r}(t) = [\cos t, \sin t]$ for $t \in [0, 2\pi)$ defines a circle in $\mathbb{R}^2$, whereas $\boldsymbol{r}(t) = [\cos t, \sin t, t]$ for $t \in [0, 2\pi)$ defines a circular helix in $\mathbb{R}^3$ as shown below:

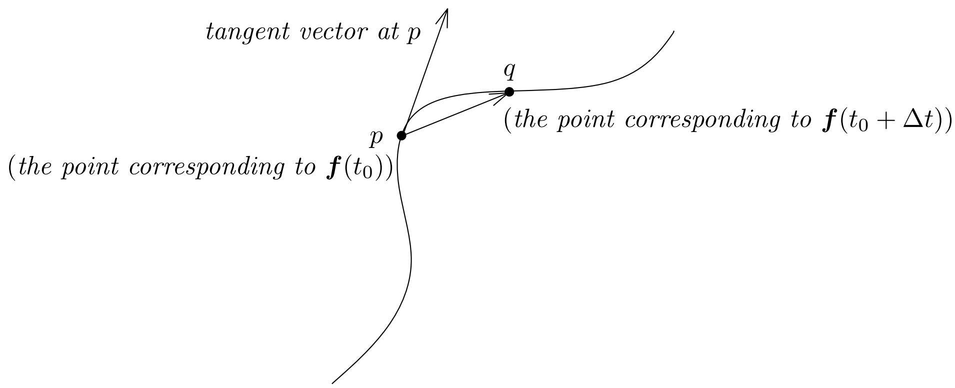


As yet another example, given constant d-dimensional vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ with $\boldsymbol{q} \neq \boldsymbol{0}$, function $\boldsymbol{r}(t) = \boldsymbol{p} + t\boldsymbol{q}$ for $t \in (-\infty, \infty)$ gives a line in $\mathbb{R}^d$.

Tangent Vectors. We are ready to introduce:

Definition 1. *Let $\boldsymbol{r}(t)$ be a curve, t_0 be a value of t, and p be the point corresponding to $\boldsymbol{r}(t_0)$. If $\boldsymbol{r}(t)$ is differentiable at t_0, then the vector $\boldsymbol{r}'(t_0)$ is the* **tangent vector** *of the curve at p.*

The tangent vector has an intuitive geometric interpretation. Let q be the point that corresponds to $\boldsymbol{f}(t_0 + \Delta t)$; see the figure below. Let us focus on the direction of the directed segment $\overrightarrow{p,q}$. Now, imagine q moving along the curve towards p (namely, Δt tends to 0). The direction of the directed segment gradually converges to the direction of the tangent vector at p.



We will refer to

$$\boldsymbol{u}(t_0) \quad = \quad \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|}$$

as the *unit tangent vector* of the curve at p. Note that $|\boldsymbol{u}(t_0)| = 1$.

As an example, consider the helix mentioned earlier: $\boldsymbol{r}(t) = [\cos t, \sin t, t]$ for $t \in [0, 2\pi)$. Let p be the point corresponding to $\boldsymbol{r}(1)$. Then, the tangent vector of the curve at p is $\boldsymbol{r}'(1) = [-\sin(1), \cos(1), 1]$. The unit tangent vector at p is therefore $[-\frac{\sin(1)}{\sqrt{2}}, \frac{\cos(1)}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$.

2 Gradient

Let $f(x_1, x_2, ..., x_d)$ be a scalar function of real-valued parameters $x_1, ..., x_d$. In other words, for each point $p(x_1, ..., x_d)$ of $\mathbb{R}^d$, $f(x_1, x_2, ..., x_d)$ returns a real value, if it is defined at p. For simplicity, sometimes we may write $f(x_1, x_2, ..., x_d)$ simply as $f(p)$. Next, we introduce a concept called *gradient* for such functions:

Definition 2. *Let $f(x_1, ..., x_d)$ be a function defined as above. Consider a point $(t_1, t_2, ..., t_d)$ at which the partial derivative $\frac{\partial f}{\partial x_i}(t_1, ..., t_d)$ exists for all $i \in [1, d]$. Then, the* **gradient** *of $f(x_1, ..., x_d)$ at $(t_1, t_2, ..., t_d)$ is the vector:*

$$\nabla f(t_1, ..., t_d) \quad = \quad \left[\frac{\partial f}{\partial x_1}(t_1, ..., t_d), \frac{\partial f}{\partial x_2}(t_1, ..., t_d), ..., \frac{\partial f}{\partial x_d}(t_1, ..., t_d)\right].$$

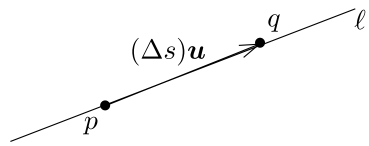
For example, suppose that $f(x, y, z) = x^3 + 2xy + 3xz^2$. We know that $\frac{\partial f}{\partial x} = 3x^2 + 2y + 3z^2$, $\frac{\partial f}{\partial y} = 2x$, and $\frac{\partial f}{\partial z} = 6x$. Therefore,

$$\nabla f(t_1, t_2, t_3) \quad = \quad \left[3t_1^2 + 2t_2 + 3t_3^2, 2t_1, 6t_1\right].$$

We can as well just write the gradient as $\nabla f(x, y, z) = \left[3x^2 + 2y + 3z^2, 2x, 6x\right]$ by renaming the variables.

The gradient $\nabla f(t_1, ..., t_d)$ has an important geometric interpretation. Imagine that we are standing at the point $p(t_1, ..., t_d)$. Then the gradient points to the direction we should move in order to increase the value of function $f(x_1, ..., x_d)$ the *most*. Next, we will formalize the intuition.

Suppose that we decide to move from p towards the direction of a unit vector $\boldsymbol{u}$ by a distance Δs. Let q be the point we will reach, as shown below:



We now prove an important lemma:

Lemma 1.

$$\lim_{\Delta s \to 0} \frac{f(q) - f(p)}{\Delta s} = \Big(\nabla f(p)\Big) \cdot \boldsymbol{u}. \tag{1}$$

Proof. Suppose that $\boldsymbol{u} = [u_1, u_2, ..., u_d]$, and the coordinates of p are $(t_1, t_2, ..., t_d)$.

Let ℓ be the line that passes p and q. We know that we can represent any point on ℓ as $(x_1(s), x_2(s), ..., x_d(s))$, where for all $i \in [1, d]$:

$$x_i(s) = t_i + s \cdot u_i.$$

In particular, if $s = 0$, the above representation gives p, whereas if $s = \Delta s$, the above representation gives q.

Define $g(s)$ to be the $f(x_1(s), ..., x_d(s))$. We can re-write the left hand side of (1) as:

$$\begin{aligned} \lim_{\Delta s \to 0} \frac{f(q) - f(p)}{\Delta s} &= \lim_{\Delta s \to 0} \frac{g(\Delta s) - g(0)}{\Delta s} \\ \text{(by def. of derivative)} &= g'(0). \end{aligned}$$

On the other hand, applying the chain rule, we know:

$$\begin{aligned} g'(s) &= \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}(x_1(s), ..., x_d(s)) \frac{dx_i}{ds} \\ &= \left[\frac{\partial f}{\partial x_1}(x_1(s), ..., x_d(s)), ..., \frac{\partial f}{\partial x_d}(x_1(s), ..., x_d(s))\right] \cdot \left[x_1'(s), ..., x_d'(s)\right] \\ &= (\nabla f(x_1(s), ..., x_d(s))) \cdot [u_1, ..., u_d] \\ &= (\nabla f(x_1(s), ..., x_d(s))) \cdot \boldsymbol{u}. \end{aligned}$$

Therefore, $g'(0) = (\nabla f(x_1(0), ..., x_d(0))) \cdot \boldsymbol{u} = (\nabla f(p)) \cdot \boldsymbol{u}$. □

As a corollary of the above lemma, we obtain

$$\lim_{\Delta s \to 0} \frac{f(q) - f(p)}{\Delta s} = \Big|\nabla f(p)\Big| |\boldsymbol{u}| \cos \gamma.$$

where γ is the angle between the directions of $\nabla f(p)$ and $\boldsymbol{u}$. Hence, the limit is maximized if $\gamma = 0$, namely, $\boldsymbol{u}$ has the same direction as $\nabla f(p)$.

It is worth mentioning that the limit on the left hand side of (1) is called the *directional derivative* in the direction of $\boldsymbol{u}$, and is denoted as $D_{\boldsymbol{u}} f$. Note that this is a function of p. In other words, $D_{\boldsymbol{u}} f(p)$ gives the directional derivative in the direction of $\boldsymbol{u}$ at point p.