CSCI3320: Homework 4, Spring 2020 Deadline: March 8th, 11:59 pm
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1. In class, we discussed about the multivariate Gaussian distribution. Assume now we have a bivariate Gaussian distribution with mean vector and covariance matrix as :

mean vector is $\boldsymbol{\mu}^T = [\mu_1, \mu_2]$ $\quad\quad$ $\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$

Show that the joint bivariate density function is:

$$p(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left(z_1^2 - 2\rho z_1 z_2 + z_2^2\right)\right]$$

where $z_i = (x_i - \mu_i)/\sigma_i, i = 1, 2,$ are standardized variables

Answer:

Given that

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

we have

$$\begin{aligned} |\boldsymbol{\Sigma}| &= \sigma_1^2\sigma_2^2 - \rho^2\sigma_1^2\sigma_2 = \sigma_1^2\sigma_2(1-\rho^2) \\ |\boldsymbol{\Sigma}|^{1/2} &= \sigma_1\sigma_2\sqrt{1-\rho^2} \\ \boldsymbol{\Sigma}^{-1} &= \frac{1}{\sigma_1^2\sigma_2(1-\rho^2)} \begin{bmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{bmatrix} \end{aligned}$$

and $(\boldsymbol{x} - \boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x} - \boldsymbol{\mu})$ can be expanded as

$$[x_1 - \mu_1 \ x_2 - \mu_2] \begin{bmatrix} \frac{\sigma_2^2}{\sigma_1^2\sigma_2(1-\rho^2)} & -\frac{\rho\sigma_1\sigma_2}{\sigma_1^2\sigma_2(1-\rho^2)} \\ -\frac{\rho\sigma_1\sigma_2}{\sigma_1^2\sigma_2(1-\rho^2)} & \frac{\sigma_1^2}{\sigma_1^2\sigma_2(1-\rho^2)} \end{bmatrix} \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix}$$

$$= \frac{1}{1-\rho^2}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]$$

2.

Let us say we have two variables x_1 and x_2 and we want to make a quadratic fit using them, namely,

$$f(x_1, x_2) = w_0 + w_1 x_1 + w_2 x_2 + w_3 x_1 x_2 + w_4 (x_1)^2 + w_5 (x_2)^2$$

How can we find $w_i, i = 0, \ldots, 5$, given a sample of $\mathcal{X} = \{x_1^t, x_2^t, r^t\}$?

Answer:

We write the fit as

$$f(x_1, x_2) = w_0 + w_1 z_1 + w_2 z_2 + w_3 z_3 + w_4 z_4 + w_5 z_5$$

where $z_1 = x_1$, $z_2 = x_2$, $z_3 = x_1 x_2$, $z_4 = (x_1)^2$, and $z_5 = (x_2)^2$. We can then use linear regression to learn $w_i, i = 0, \ldots, 5$. The linear fit in the five-dimensional $(z_1, z_2, z_3, z_4, z_5)$ corresponds to a quadratic fit in the two-dimensional (x_1, x_2) space. We discuss such generalized linear models in more detail (and other nonlinear basis functions) in chapter 10.