Multivariable Calculus

HW 4 Suggested Solution

① $g_x = \underline{\underline{1}}$

$g_y = \underline{\underline{-2}}$

$h_x = 3(5x^4) - 4y^3 + e^{3x^2y}(6xy)$

$= 15x^4 - 4y^3 + 6xye^{3x^2y}$

$\therefore h_{xx} = 15(4x^3) + 6ye^{3x^2y} + 6xye^{3x^2y}(6xy)$

$= \underline{\underline{60x^3 + (6y + 36x^2y^2)e^{3x^2y}}}$

$h_y = -4x(3y^2) + 3x^2e^{3x^2y}$

$= -12xy^2 + 3x^2e^{3x^2y}$

$h_{yx} = -12y^2 + 6xe^{3x^2y} + 3x^2e^{3x^2y}(6xy)$

$= -12y^2 + 6xe^{3x^2y} + 18x^3ye^{3x^2y}$

$h_{yxy} = -24y + 18x^3e^{3x^2y} + 18x^3e^{3x^2y} + 54x^5ye^{3x^2y}$

$= \underline{\underline{-24y + 36x^3e^{3x^2y} + 54x^5ye^{3x^2y}}}$

At $(x,y) = (2,5)$, $\underline{\underline{g_x(2,5) = 1}}$; $\underline{\underline{g_y(2,5) = -2}}$,

$h_{xx}(2,5) = 60(8) + (30 + 36\cdot 4\cdot 25)e^{3(4)(5)}$

$= \underline{\underline{480 + 3630e^{60}}}$

$h_{yxy}(2,5) = -24(5) + 36(2)^3e^{60} + 54(2)^5(5)e^{60}$

$= -120 + 288e^{60} + 8640e^{60}$

$= \underline{\underline{-120 + 8928e^{60}}}$

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② (a) $\nabla F = \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right\rangle = \langle 2y^3 + 2xy^2,\ 4y^3 + 6xy^2 + 2x^2y \rangle$

$= (2y^3 + 2xy^2)\,\hat{i} + (4y^3 + 6xy^2 + 2x^2y)\,\hat{j}$ //

② (b) (i) From (a), all partial derivatives of F are continuous, thus F is differentiable.

At $(0,1)$, $\nabla F = \langle 2, 4\rangle$

The unit vector in the direction of $\hat{j} - 2\hat{i}$ is $\frac{-2}{\sqrt{5}}\hat{i} + \frac{1}{\sqrt{5}}\hat{j}$. Denote it as $\hat{v}$

$\therefore$ Rate of change required $= D_{\hat{v}} F(0,1) = \nabla F(0,1) \cdot \hat{v} = 2\left(-\frac{2}{\sqrt{5}}\right) + 4\left(\frac{1}{\sqrt{5}}\right) = 0$

$= \underline{\underline{0}}$

(ii) From (a), all partial derivatives of F are continuous, thus F is differentiable.

Unit vector in the direction of $2020\hat{i}$ is $\hat{i}$. Denote it as $\hat{u}$.

$\therefore$ Rate of change required $= D_{\hat{u}} F(0,1) = \nabla F(0,1) \cdot \hat{u} = 2(1) + 4(0)$

$= \underline{\underline{2}}$

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③ Volume of the cone $V(r, h) = \frac{1}{3}\pi r^2 h$

Differential of this function $dV = V_r\,dr + V_h\,dh$

$$= \frac{2}{3}\pi rh\,dr + \frac{1}{3}\pi r^2\,dh$$

Here $r = 10$, $dr = 0.2$, $h = 100$, $dh = -1$,

$\therefore\ dV = \frac{2}{3}\pi(10)(100)(0.2) + \frac{1}{3}\pi(100)(-1)$

$= \frac{400}{3}\pi - \frac{100}{3}\pi = \underline{\underline{100\pi \text{ cm}^3}}$

Percentage of such change $= \dfrac{100\pi}{\frac{1}{3}\pi(10)^2(100)} \times 100\% = \dfrac{3}{100} \times 100\% = \underline{\underline{3\%}}$

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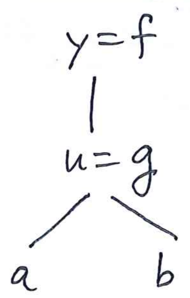
④ Tree diagram



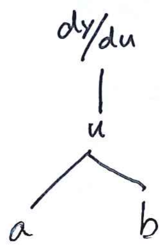
$$\frac{\partial y}{\partial a} = \frac{dy}{du}\frac{\partial u}{\partial a}$$

$$\frac{\partial^2 y}{\partial a^2} = \frac{\partial}{\partial a}\left(\frac{dy}{du}\frac{\partial u}{\partial a}\right)$$

$$= \frac{dy}{du}\frac{\partial^2 u}{\partial a^2} + \frac{\partial u}{\partial a}\left[\frac{\partial}{\partial a}\left(\frac{dy}{du}\right)\right]$$

$$= \frac{dy}{du}\frac{\partial^2 u}{\partial a^2} + \frac{\partial u}{\partial a}\left[\frac{d}{du}\left(\frac{dy}{du}\right)\frac{\partial u}{\partial a}\right]$$

$$= \frac{dy}{du}\frac{\partial^2 u}{\partial a^2} + \left(\frac{\partial u}{\partial a}\right)^2\frac{d^2 y}{du^2}$$



or in this form

$$\boxed{f'(g(a,b))\,g_{aa} + (g_a)^2 f''(g(a,b))}$$

⑤ (a) $F_1(x,y) = 2xy - \frac{1}{2}(x^4+y^4)+1$

$F_{1x}(x,y) = 2y - 2x^3$ Set $2y-2x^3 = 2x-2y^3 = 0$, then $x=y^3$

$F_{1y}(x,y) = 2x - 2y^3$ $\Rightarrow 2y - 2y^9 = 0$

$2y(1-y^8)=0$

$2y(1+y^4)(1+y^2)(1+y)(1-y) = 0$

$y = 0$ or 1 or -1 if we assume y is real.

When $y=0$, $x=0$, $F_1(0,0) = 1$

$y=1$, $x=1$, $F_1(1,1) = 2$

$y=-1$, $x=-1$, $F_1(-1,-1) = 2$

$\therefore$ There are 3 critical points, namely $(0,0,1)$, $(-1,-1,2)$ and $(1,1,2)$

Accept $(0,0)$, $(-1,-1)$, $(1,1)$

Now, $F_{1xx}(x,y) = -6x^2$, $F_{1xy}(x,y) = F_{1yx}(x,y) = 2$, $F_{1yy}(x,y) = -6y^2$

Hessian matrix $\underline{H} = \begin{pmatrix} -6x^2 & 2 \\ 2 & -6y^2 \end{pmatrix} \Rightarrow \det \underline{H} = 36x^2y^2 - 4$

① When $(x,y) = (0,0)$, $\det \underline{H} = -4 < 0 \Rightarrow (0,0,1)$ is a saddle point

② When $(x,y) = (-1,-1)$, $\det \underline{H} = 32 > 0$, $F_{1xx} = -6 < 0 \Rightarrow (-1,-1,2)$ is a relative (local) maximum

③ When $(x,y) = (1,1)$, $\det \underline{H} = 32 > 0$, $F_{1xx} = -6 < 0 \Rightarrow (1,1,2)$ is a relative (local) maximum.

(b) $F_2(x,y) = x^4 + y^4$

Accept $(0,0)$

$F_{2x}(x,y) = 4x^3$ Set $4x^3 = 4y^3 = 0$, we get $x=y=0 \Rightarrow (0,0,0)$ is a critical point

$F_{2y}(x,y) = 4y^3$

Consider $F_{2xx} = 12x^2$, $F_{2xy} = F_{2yx} = 0$, $F_{2yy} = 12y^2$

Hessian matrix $\underline{H} = \begin{pmatrix} 12x^2 & 0 \\ 0 & 12y^2 \end{pmatrix} \Rightarrow \det \underline{H} = 144x^2y^2$

When $(x,y) = (0,0)$, $\det \underline{H} = 0$, hence the 2nd Derivative Test is inconclusive.

Now, consider $F_2(x,y) = x^4+y^4$ is always non-negative, i.e.

$F_2(x,y) \geq F_2(0,0) = 0$ for all $(x,y) \in \mathbb{R}^2$.

In other words, $(0,0,0)$ is a local (relative) minimum. (Accept $(0,0)$)

Remarks: $(0,0,0)$ is also a global minimum.

⑥ (a) $G(x,y) = -D(x,y) = -300 - 20x^2 - 60\sin\left(\frac{\pi y}{2}\right)$ //

(b) Consider $\frac{\partial D(x,y)}{\partial x} = 40x$

Steepness $= \left.\frac{\partial D(x,y)}{\partial x}\right|_{x=0.75} = 40(0.75) = \underline{\underline{30}}$

(c) Consider $\frac{\partial D(x,y)}{\partial y} = 60\cos\left(\frac{\pi y}{2}\right)\cdot\frac{\pi}{2} = 30\pi\cos\left(\frac{\pi y}{2}\right)$

Steepness $= \left.\frac{\partial D(x,y)}{\partial y}\right|_{y=0.3} = \underline{\underline{30\pi\cos\left(\frac{3\pi}{20}\right)}}$

(d) Direction of greatest rate of change of depth $= \nabla D(x,y)$

$= \underline{\underline{\langle 40x, 30\pi\cos\left(\frac{\pi y}{2}\right)\rangle}}$

(or $40x\hat{i} + 30\pi\cos\left(\frac{\pi y}{2}\right)\hat{j}$)
or any scalar multiple of such vector.

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⑦ (a) Consider $p(x,y) = \frac{1}{2\pi}\left[\frac{1-x^2-y^2}{(x-1)^2+y^2}\right]$

Our goal is to show $\underline{P_{xx} + P_{yy} = 0 \quad \forall (x,y) \in \mathbb{R}^2 \setminus \{(1,0)\}}$

Consider $P_x(x,y) = \frac{1}{2\pi}\left[\frac{((x-1)^2+y^2)\cdot(-2x) - 2(1-x^2-y^2)(x-1)}{((x-1)^2+y^2)^2}\right]$

$= \frac{1}{2\pi}\left\{\frac{-2x^3+4x^2-2x-2xy^2-2x+2x^3+2xy^2+2-2x^2-2y^2}{[(x-1)^2+y^2]^2}\right\}$

$= \frac{1}{2\pi}\left\{\frac{2x^2-4x-2y^2+2}{[(x-1)^2+y^2]^2}\right\} = \frac{1}{\pi}\left[\frac{(x-1)^2-y^2}{((x-1)^2+y^2)^2}\right]$

$P_{xx}(x,y) = \frac{1}{\pi}\left\{\frac{[(x-1)^2+y^2]^2\cdot 2(x-1) - [(x-1)^2-y^2]\cdot 2[(x-1)^2+y^2]\cdot 2(x-1)}{((x-1)^2+y^2)^4}\right\}$

$= \frac{2}{\pi}\left\{\frac{[(x-1)^2+y^2](x-1)\{[(x-1)^2+y^2] - 2[(x-1)^2-y^2]\}}{[(x-1)^2+y^2]^4}\right\}$

$= \frac{2}{\pi}\left\{\frac{(x-1)(-x^2+2x-1+3y^2)}{[(x-1)^2+y^2]^3}\right\} = \frac{2}{\pi}\frac{(x-1)(3y^2-(x-1)^2)}{[(x-1)^2+y^2]^3}$

Next, consider $P_y(x,y) = \frac{1}{2\pi}\left[\frac{[(x-1)^2+y^2]\cdot(-2y) - 2y(1-x^2-y^2)}{[(x-1)^2+y^2]^2}\right]$

$$= \frac{1}{2\pi}\left\{\frac{-2x^2y+4xy-2y-2y^3-2y+2x^2y+2y^3}{[(x-1)^2+y^2]^2}\right\}$$

$$= \frac{1}{\pi}\frac{2(x-1)y}{[(x-1)^2+y^2]^2}$$

$$P_{yy}(x,y) = \frac{1}{\pi}\left\{\frac{[(x-1)^2+y^2]^2\cdot 2(x-1) - 2(x-1)y\cdot 2[(x-1)^2+y^2]\cdot 2y}{[(x-1)^2+y^2]^4}\right\}$$

$$= \frac{2}{\pi}\frac{(x-1)[(x-1)^2+y^2]}{[(x-1)^2+y^2]^4}\left\{[(x-1)^2+y^2]-4y^2\right\}$$

$$= \frac{2}{\pi}\frac{(x-1)}{[(x-1)^2+y^2]^3}\left[(x-1)^2-3y^2\right]$$

Now, $P_{xx}(x,y) + P_{yy}(x,y) = \frac{2}{\pi}\frac{(x-1)}{[(x-1)^2+y^2]^3}\left[3y^2-(x-1)^2+(x-1)^2-3y^2\right]$

$$= 0$$

$\therefore$ P is a harmonic function.

(b) Consider $Q_n(x,y) = (x^2+y^2)^n$

(i) $Q_{nx}(x,y) = n(x^2+y^2)^{n-1}(2x)$

$$Q_{nxx}(x,y) = 2n(x^2+y^2)^{n-1} + 2n(n-1)x(x^2+y^2)^{n-2}(2x)$$

$$= 2n(x^2+y^2)^{n-1} + 4n(n-1)x^2(x^2+y^2)^{n-2}$$

By symmetry, $Q_{nyy}(x,y) = 2n(x^2+y^2)^{n-1} + 4n(n-1)y^2(x^2+y^2)^{n-2}$

Consider $\frac{\partial^2 Q_n(x,y)}{\partial x^2} + \frac{\partial^2 Q_n(x,y)}{\partial y^2}$

$$= 4n(x^2+y^2)^{n-1} + 4n(n-1)(x^2+y^2)^{n-2}(x^2+y^2)$$

$$= 4n(x^2+y^2)^{n-1} + 4n(n-1)(x^2+y^2)^{n-1}$$

$$= \frac{4n^2}{x^2+y^2}\cdot(x^2+y^2)^n = \frac{4n^2}{x^2+y^2}Q_n(x,y)$$

$\Rightarrow G(n,x,y) = \frac{4n^2}{x^2+y^2}$

(ii) For $Q_n(x,y)$ to be harmonic, RHS of previous eq. $= 0$

i.e. $\frac{4n^2}{x^2+y^2}(x^2+y^2)^n = 0$

$\Rightarrow \frac{4n^2}{x^2+y^2} = 0$ or $(x^2+y^2)^n = 0$

As $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$, $x^2+y^2 > 0 \Rightarrow (x^2+y^2)^n \neq 0$.

$\because x^2+y^2 > 0$

$\therefore \frac{4n^2}{x^2+y^2} = 0$ if and only if $4n^2 = 0$

i.e. $n = 0$,

but n is non-zero.

$\therefore \frac{\partial^2 Q_n(x,y)}{\partial x^2} + \frac{\partial^2 Q_n(x,y)}{\partial y^2} \neq 0 \Rightarrow Q_n(x,y)$ is <u>NOT</u> a harmonic function.

<u>END of Sol'n</u>