Lecture Note 6

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MATH1020
General Mathematics

Logarithmic Functions

Recall that a one–to–one function $y = f(x)$ has an inverse function that is defined (implicitly) by the equation $x = f(y)$. In particular, the exponential function

$$y = f(x) = a^x,\ a > 0,\ a \neq 1,$$

is one-to-one and hence has an inverse function that is defined implicitly by the equation

$$x = a^y,\ a > 0,\ a \neq 1.$$

This inverse function is so important that it is given a name, the **logarithmic function.**

Exercises 1 Relating Logarithms to Exponents

(a) If $y = \log_3 x$, then $x = 3^y$. For example, $2 = \log_3 9$ is equivalent to $9 = 3^2$.

(b) If $y = \log_7 x$, then $x = 7^y$. For example, $-1 = \log_7 \left(\frac{1}{7}\right)$ is equivalent to $\frac{1}{7} = 7^{-1}$.

Definition 1 The logarithmic function to the base a, where $a > 0$ and $a \neq 1$ is denoted by $y = log_a x$ (read as "y is the logarithmic to the base a of x") and is defined by

$$\boxed{y = \log_a x \quad \text{if and only if} \quad x = a^y}$$

The domain of the logarithmic function $y = \log_a x$ is $x > 0$.

Exercises 2 **Changing Exponential Statements to Logarithmic Statements**

Change each exponential statement to an equivalent statement involving a logarithm.

(a) $1.4^3 = k$ (b) $e^m = 9$ (c) $a^4 = 25$

Exercises 3 Changing Logarithmic Statements to Exponential Statements

Change each logarithmic statement to an equivalent statement involving an exponent.

(a) $\log_a 4 = 5$ (b) $\log_e b = -3$ (c) $\log_3 5 = c$

Evaluate Logarithmic Expression

To find the extra value of a logarithm, we write the logarithm in exponential notation and use the fact that if $a^u = a^v$, then $u = v$.

Exercises 4 Finding the Exact Value of a Logarithmic Expression

Find the exact value of:

(a) $\log_2 16$ (b) $\log_3 \left(\dfrac{1}{27} \right)$

Determine the Domain of a Logarithmic Function

The logarithmic function $y = \log_a x$ has been defined as the inverse of the exponential function $y = a^x$. That is, if $f(x) = a^x$, then $f^{-1}(x) = \log_a x$. According to the discussion given in Lecture 4 on inverse functions, for a function f and its inverse f^{-1}, we have

Domain of f^{-1} =Range of f and Range of f^{-1} =Domain of f

Consequently, it follows that

Domain of the logarithmic function$=$ Range of the exponential function $= (0, \infty)$.

Range of the logarithmic function $=$ Domain of the exponential function $= (-\infty, \infty)$.

We summarize some properties of the logarithmic function:

$$y = \log_a x \qquad \text{(defining equation:} \qquad x = a^y);$$

$$\text{Domain: } 0 < x < \infty \qquad \text{Range: } -\infty < y < \infty.$$

The domain of a logarithmic function consists of the *positive* real numbers, so the argument of a logarithmic function must be greater than zero.

Exercises 5 Finding the Domain of a Logarithmic Function

Find the domain of each logarithmic function.

(a) $F(x) = \log_2(x + 3)$

(b) $g(x) = \log_5 \left(\dfrac{1 + x}{1 - x} \right)$

(c) $h(x) = \log_{1/2} |x|$

Graph Logarithmic Functions

Since exponential functions and logarithmic functions are inverse of each other, the graph of the logarithmic function $y = \log_a x$ is the reflection about the line $y = x$ of the graph of the exponential function $y = a^x$, as shown in Figure 1.

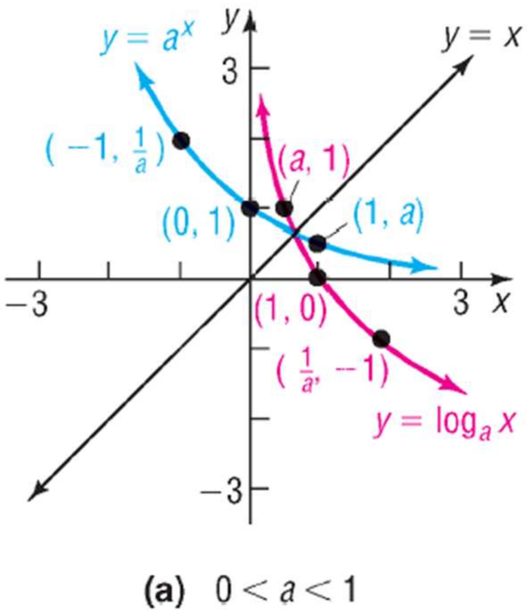

Figure 1:

For example, to graph $y = \log_2 x$ (in red), graph $y = 2^x$ (in blue) and reflect it about the line $y = x$. See Figure 2.

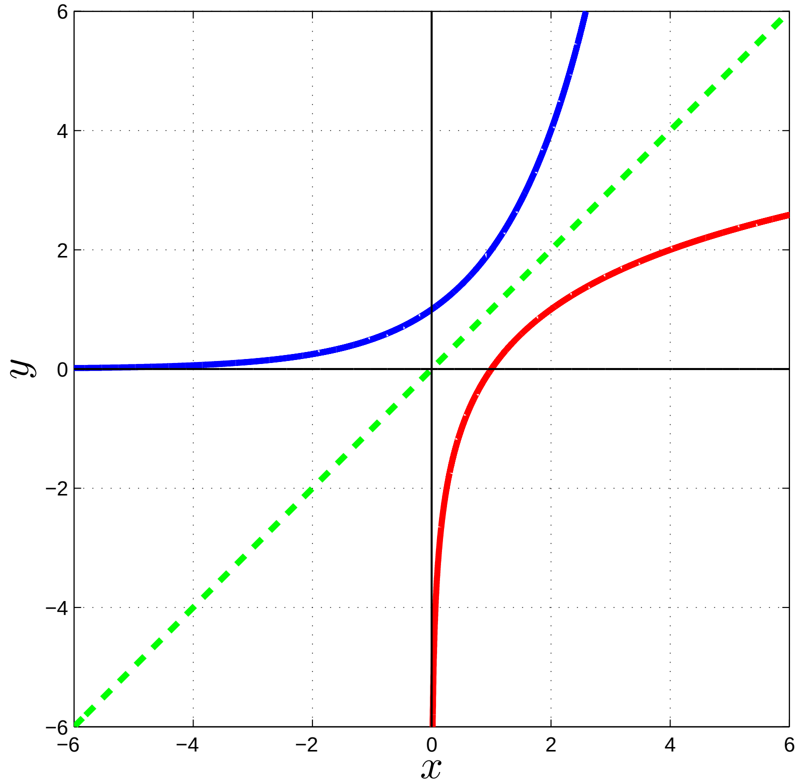

Figure 2:

To graph $y = \log_{1/3} x$ (in red), graph $y = \left(\frac{1}{3}\right)^x$ (in blue) and reflect it about the line $y = x$. See Figure 3.

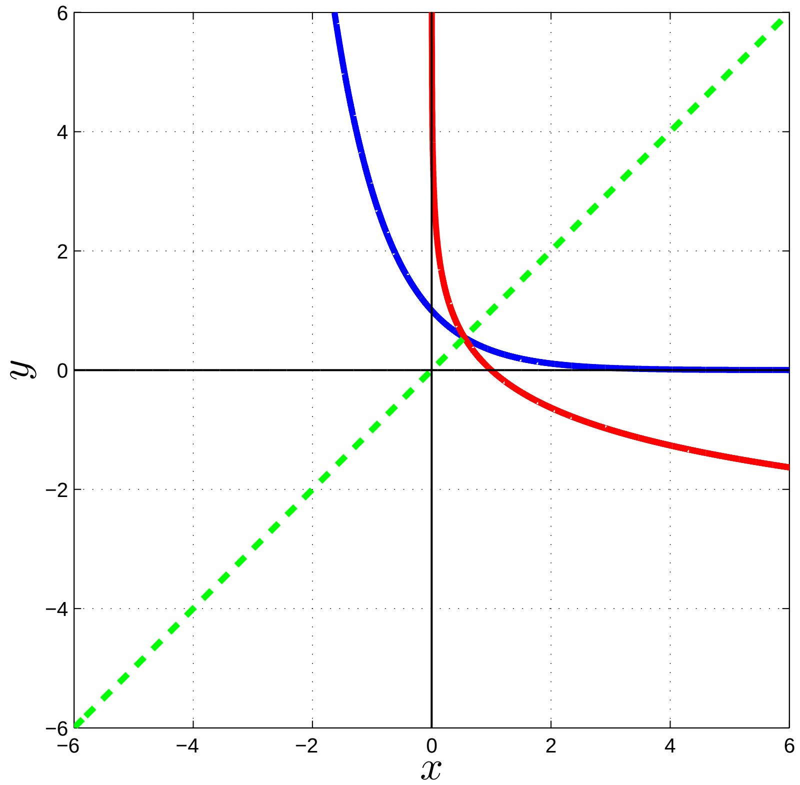

Figure 3:

The graphs of $y = \log_a x$ in Figure 1(a) and 1(b) lead to the following properties.

Properties of the Logarithmic Function $f(x) = \log_a x$

1. The domain is the set of positive real numbers; the range is the set of all real numbers.

2. The $x-$intercept of the graph is 1. There is no $y-$intercept.

3. The $y-$axis $(x = 0)$ is a vertical asymptote of the graph.

4. A logarithmic function is decreasing if $0 < a < 1$ and increasing if $a > 1$.

5. The graph of f contains the point $(1, 0)$, $(a, 1)$, and $\left(\frac{1}{a}, -1\right)$.

6. The graph is smooth and continuous, with no corners or gaps.

If the base of a logarithmic function is the number e, then we have the **natural logarithm function**. This function occurs so frequently in applications that it is given a special symbol, $\ln$ (from the Latin, logarithmus naturalis). That is,

$$\boxed{y = \ln x \quad \text{if and only if} \quad x = e^y \qquad (1)}$$

Since $y = \ln x$ and the exponential function $y = e^x$ are inverse functions, we can obtain the graph of $y = \ln x$ by reflecting the graph of $y = e^x$ about the line $y = x$. See Figure 4.

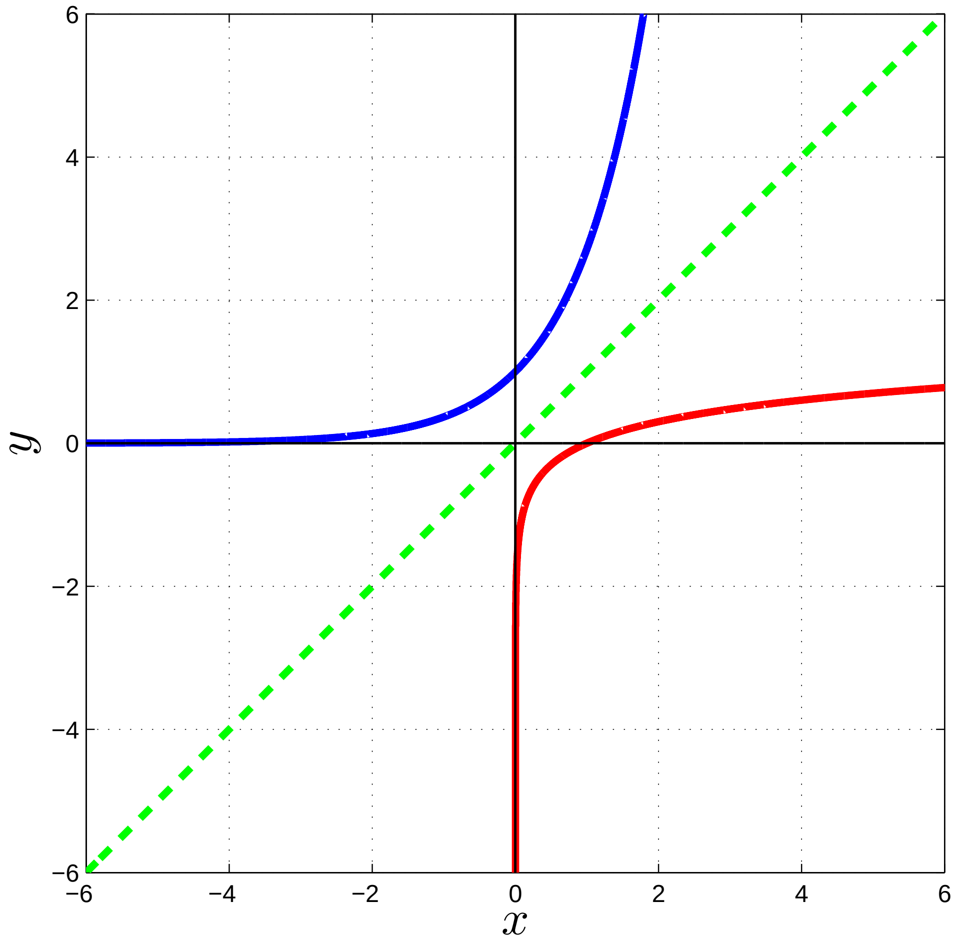

Figure 4:

Graphing a Logarithmic Function and Its Inverse

Exercises 6 (a) Find the domain of the logarithmic function $f(x) = -\ln(x-2)$.

(b) Graph f.

(c) From the graph, determine the range and vertical asymptote of f.

(d) Find f^{-1}, the inverse of f.

(e) Use f^{-1} to confirm the range of f found in part (c). From the domain of f, find the range of f^{-1}.

(f) Graph f^{-1}.

Exercises 7 **Graphing a Logarithmic Function and Its Inverse**

(a) Find the domain of the logarithmic function $f(x) = 3\log(x-1)$.

(b) Graph f.

(c) From the graph, determine the range and vertical asymptote of f.

(d) Find f^{-1}, the inverse of f.

(e) Use f^{-1} to confirm the range of f found in part (c). From the domain of f, find the range of f^{-1}.

(f) Graph f^{-1}.

Solve Logarithmic Equations

Equations that contain logarithms are called logarithmic equations. Care must be taken when solving logarithmic equations algebraically. In the expression $\log_a M$, remember that a and M are positive and $a \neq 1$. Be sure to check each apparent solution in the original equation and discard any that are extraneous.

Some logarithmic equation can be solved by changing from a logarithmic expression to an exponential expression.

Exercises 8 Solving a Logarithmic Equation

Solve: (a) $\log_3(4x - 7) = 2$ (b) $\log_x 64 = 2$

Exercises 9 **Using Logarithms to Solve Exponential Equations**

Solve: $e^{2x} = 5$