ENGG 2430A: Midterm Exam Answers

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Problem 1.

(a) Let random variable X denote the number showed up in a roll.

$$\begin{cases} \mathbb{P}(X=1) = 2\mathbb{P}(X=2) \\ \mathbb{P}(X=2) = 2\mathbb{P}(X=3) \\ \mathbb{P}(X=3) = 2\mathbb{P}(X=4) \\ \mathbb{P}(X=1) + \mathbb{P}(X=2) + \mathbb{P}(X=3) + \mathbb{P}(X=4) = 1 \end{cases}$$

Solve the above equations and get the following solution.

$$\begin{cases} \mathbb{P}(X=1) = 8/15 \\ \mathbb{P}(X=2) = 4/15 \\ \mathbb{P}(X=3) = 2/15 \\ \mathbb{P}(X=4) = 1/15 \end{cases}$$

(b) A={the outcome is strictly less than 4}

$$\begin{aligned} \mathbb{P}(A) &= \mathbb{P}(X=1) + \mathbb{P}(X=2) + \mathbb{P}(X=3) \\ &= 14/15 \end{aligned} \tag{1}$$

Problem 2.

$$\begin{aligned} \mathbb{E}[X] &= \mathbb{E}[X_1 + X_2 + X_3] \\ &= \mathbb{E}[X_1] + \mathbb{E}[X_2] + \mathbb{E}[X_3] \\ &= 1 + 2 + 3 \\ &= 6 \\ Var[X] &= Var[X_1 + X_2 + X_3] \\ &= Var[X_1] + Var[X_2] + Var[X_3] \quad (\because\ X_1,\ X_2,\ X_3 \text{ are independent}) \\ &= 1 + 4 + 9 \\ &= 14 \end{aligned} \tag{2}$$

Problem 3. Let i_j denote the person who receives the book after the j-th pass and let i_0 denote the first person who passes out the book. The event (denoted by X) "by the k-th time that the book has been passed, it has not com back to someone who has already received it before" is exactly the same as "$i_0, i_1, \cdots, i_k$ are all distinct". Thus if $1 \leq k \leq n-1$,

$$\mathbb{P}(X) = (n-1)(n-2)\cdots(n-k)\left(\frac{1}{n-1}\right)^k = \frac{(n-1)!}{(n-1-k)!(n-1)^k}.$$

Otherwise, $\mathbb{P}(X) = 0$.

Problem 4. Let L denote the event "a person is lying". Let $\widehat{L}$ denote the event "the polygraph indicates a person is lying". Let $\overline{A}$ denote the complement of an event A. Then according to the problem setting,

$$\mathbb{P}(\widehat{L}|L) = 0.9,\ \mathbb{P}(\widehat{L}|\overline{L}) = 0.15,\ \mathbb{P}(\overline{L}) = 0.8.$$

Then we can calculate the joint PMF shown in Table 1.

	$\widehat{L}$	$\overline{\widehat{L}}$
L	0.18	0.02
$\overline{L}$	0.12	0.68

Table 1: Joint PMF

$$\mathbb{P}(\overline{L}, \widehat{L}) = \mathbb{P}(\widehat{L}|\overline{L})\mathbb{P}(\overline{L}) = 0.15 \times 0.8 = 0.12,$$

$$\mathbb{P}(L, \widehat{L}) = \mathbb{P}(\widehat{L}|L)(1 - \mathbb{P}(\overline{L})) = 0.9 \times (1 - 0.8) = 0.18.$$

Thus

$$\mathbb{P}(L|\widehat{L}) = \frac{\mathbb{P}(L, \widehat{L})}{\mathbb{P}(\widehat{L})} = \frac{\mathbb{P}(L, \widehat{L})}{\mathbb{P}(L, \widehat{L}) + \mathbb{P}(\overline{L}, \widehat{L})} = \frac{0.18}{0.18 + 0.12} = \frac{3}{5}.$$

Problem 5.

(a).Since X and Y are independent, the PMF of (X, Y) is

$$p_{X,Y}(x,y) = \begin{cases} 0.15, & x = 50,\ y = 50 \\ 0.15, & x = 50,\ y = 100 \\ 0.2, & x = 100,\ y = 50 \\ 0.2, & x = 100,\ y = 100 \\ 0.15, & x = 200,\ y = 50 \\ 0.15, & x = 200,\ y = 100 \\ 0, & \text{otherwise} \end{cases}.$$

(b). Let $Z = Y - X$, then

$$\begin{aligned}
&\mathbb{P}(Z=-150) = 0.3 \times 0.5 = 0.15, && \text{(when X=200 and Y=50)}\\
&\mathbb{P}(Z=-100) = 0.3 \times 0.5 = 0.15, && \text{(when X=200 and Y=100)}\\
&\mathbb{P}(Z=-50) = 0.4 \times 0.5 = 0.2, && \text{(when X=100 and Y=50)}\\
&\mathbb{P}(Z=0) = 0.3 \times 0.5 + 0.4 \times 0.5 = 0.35, && \text{(when X=Y=50 or X=Y=100)}\\
&\mathbb{P}(Z=50) = 0.3 \times 0.5 = 0.15. && \text{(when X=50 and Y=100)}
\end{aligned}$$

Therefore, the PMF of Z is

$$p_Z(z) = \begin{cases} 0.15, & z = -150 \\ 0.15, & z = -100 \\ 0.2, & z = -50 \\ 0.35, & z = 0 \\ 0.15, & z = 50 \\ 0, & \text{otherwise} \end{cases}.$$

Problem 6. Since X follows the exponential distribution, then

$$\mathbb{E}[X] = \frac{1}{\lambda} = 4.$$

Therefore, we have $\lambda = \frac{1}{4}$ and the distribution of X is

$$f_X(x) = \begin{cases} \frac{1}{4}e^{-\frac{1}{4}x}, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}.$$

Then

$$\begin{aligned}
\mathbb{P}_X(X \geq 6) &= \int_6^{\infty} \frac{1}{4}e^{-\frac{1}{4}x}dx = e^{-\frac{3}{2}},\\
\mathbb{P}_X(X \geq 8) &= \int_8^{\infty} \frac{1}{4}e^{-\frac{1}{4}x}dx = e^{-2}.
\end{aligned}$$

So the conditional probability of $\mathbb{P}_X(X \geq 8 | X \geq 6)$ is

$$\mathbb{P}_X(X \geq 8 | X \geq 6) = \frac{e^{-2}}{e^{-\frac{3}{2}}} = e^{-\frac{1}{2}}.$$