Lecture Notes: Line (Curve) Integral by Length

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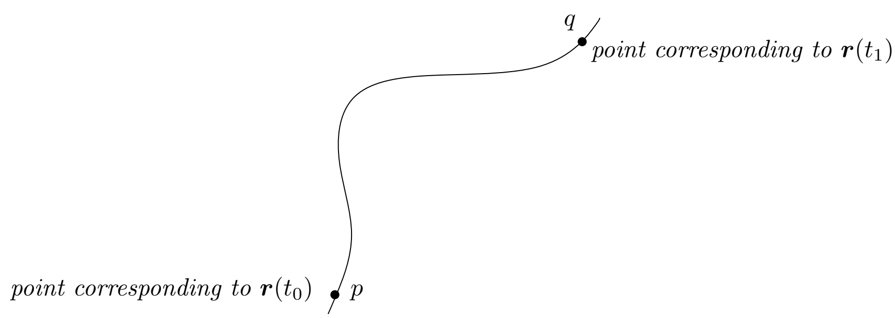
1 Length of a Curve

Recall that we can represent a curve in $\mathbb{R}^d$ using a vector function $\boldsymbol{r}(t) = [x_1(t), x_2(t), ..., x_d(t)]$, where $x_1(t), x_2(t), ..., x_d(t)$ give the coordinates of the point on the curve corresponding to a value of t. Henceforth, we will always take the view that the curve has a *starting point* corresponding to $\boldsymbol{r}(t_0)$ where t_0 is a fixed constant. Given any $t_1 \geq t_0$, when we say *the curve from* t_0 *to* t_1, we refer to the locus of the point corresponding to $\boldsymbol{r}(t)$ as t goes from t_0 to t_1. In the example below, the curve from t_0 to t_1 is the part of the curve between p and q.

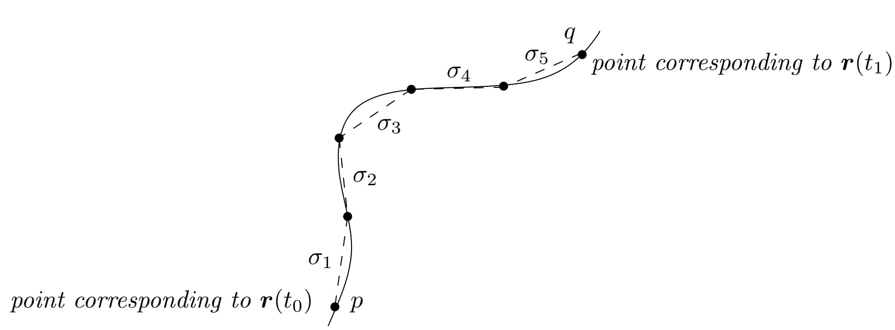


Intuitively, since a curve is a 1d object, it should have a length. Next, we will formalize define this notion as a limit:

Definition 1. *Let C be a curve given by $\boldsymbol{r}(t)$ with t from t_0 to t_1. Evenly divide the interval $[t_0, t_1]$ by inserting $n+1$ break points $\tau_0, \tau_1, \tau_2, ..., \tau_n$ where $\tau_0 = t_0$, $\tau_1 = t_1$, and $\tau_i - \tau_{i-1} = (t_1 - t_0)/n$ for each $i \in [1, n]$. Define σ_i to be the (straight) line segment connecting the points corresponding to $\boldsymbol{r}(\tau_{i-1})$ and $\boldsymbol{r}(\tau_i)$, and denote by $|\sigma_i|$ the length of σ_i. Then, if the following limit exists:*

$$\lim_{n \to \infty} \sum_{i=1}^{n} |\sigma_i| \tag{1}$$

we say that the limit is the **length** *of C.*



The figure above shows an example with $n = 5$. Note how we approximate the length of the curve by the total length of a sequence of segments.

In this course, we will restrict our attention to *smooth curves*. Intuitively, these are curves that (i) do not degenerate into a point, and (ii) do not have "corners" (e.g., the boundary of a triangle is not smooth). Mathematically, we formalize the notion as follows:

Definition 2. *Let C be a curve given by $\boldsymbol{r}(t)$ with t from t_0 to t_1. C is* **smooth** *if (i) $\boldsymbol{r}'(t)$ is continuous in $[t_0, t_1]$, and (ii) $\boldsymbol{r}'(t) \neq \boldsymbol{0}$ at any $t \in [t_0, t_1]$.*

We will state without proof the following lemma:

Lemma 1. *Let C be as described in Definition 1. If C is smooth, then the limit* (1) *always exists.*

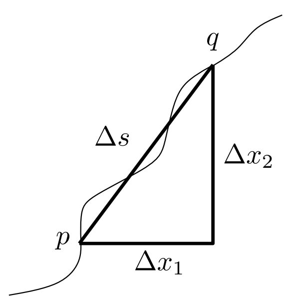
Now imagine that t increases from t_0. Let C be the curve from t_0 to t. Note that C extends as t grows, which means that the length s of C is a function of t. Sometimes, we will emphasize this by writing the length explicitly as $s(t)$.

The following is an important lemma:

Lemma 2. *If C is smooth, then it holds that:*

$$\frac{d(s(t))}{dt} = \sqrt{\sum_{i=1}^{d} \left(\frac{d(x_i(t))}{dt} \right)^2}.$$

We will not present a rigorous proof of the lemma (which is rather involved), but the lemma is very intuitive, as we illustrate using the figure below in 2d space. Imagine that we increase t by a tiny amount Δt. By doing so, we have traveled on the curve a little from point p to point q. Δx_1 and Δx_2 give the coordinate differences of p and q on the two dimensions, respectively. When Δt is extremely small, the length of the curve from p to q should be very close to the length of the segment connecting p and q, which equals $\sqrt{(\Delta x_1)^2 + (\Delta x_2)^2}$.



2 Line Integral by Length

An important message from the previous section is that we can use the length parameter s to uniquely refer to a point on a smooth curve. Specifically, let $\boldsymbol{r}(t) = [x_1(t), ..., x_d(t)]$ be a curve with starting point $p(0)$ corresponding to $\boldsymbol{r}(t_0)$. Then, given a value of s, let us define $p(s)$ as the point such that the point from $p(0)$ to $p(s)$ has length exactly s. More specifically, let $x_1, ..., x_d$ be the coordinates of $p(s)$; then each coordinate is a function of s. Sometimes, we may emphasize this by writing the i-th $(1 \leq i \leq d)$ coordinate of $p(s)$ explicitly as $x_i(s)$.

Let C be the curve from point p to q. Without loss of generality, let s_p and s_q be the lengths that define p and q, respectively. Let $f(s)$ be a scalar function of s. We now introduce a notation:

$$\int_C f(s)ds \tag{2}$$

as a shortform for $\int_{s_p}^{s_q} f(s)ds$. We refer to this shortform as a *line integral.*[1] For example, $\int_C (x_1^2 + \sqrt{x_2 x_3})\, ds$ is merely just a notation for $\int_{s_p}^{s_q} (x_1(s)^2 + \sqrt{x_2(s)x_3(s)})\, ds$.

As an important special case, when $f(s) = 1$, we have:

$$\begin{aligned}\int_C ds &= \int_{s_p}^{s_q} ds \\ &= s_q - s_p\end{aligned}$$

which is exactly the length of C.

A line integral is almost always evaluated by changing the integral variable s to t, using Lemma 2. Next, we illustrate this using two examples.

Example 1. Consider the circle $x^2 + y^2 = 1$. Let p be the point $(1, 0)$ and q the point $(-1, 0)$. Let C be the curve on the circle from p to q. Calculate $\int_C ds$.

Solution. First of all, we need to represent the circle using a single parameter. One way of doing so is to define:

$$\begin{aligned}x(t) &= \cos(t) \\ y(t) &= \sin(t).\end{aligned}$$

Then C is essentially the curve from $t = 0$ (point p) to $t = \pi$ (point q).

Next, we need to choose an arbitrary point p^* on the circle as the starting point. Where p^* is does not matter, but just to be specific, let it be the point given by $t = -\pi/2$. Let s_p be the length of the curve from p^* to p, and s_q be the length the curve from p^* to q. Then, we have:

$$\begin{aligned}\int_C ds &= \int_{s_p}^{s_q} ds \\ &= \int_0^{\pi} \frac{ds}{dt}\, dt \\ \text{(by Lemma 2)} \quad &= \int_0^{\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_0^{\pi} \sqrt{(-\sin(t))^2 + (\cos(t))^2}\, dt \\ &= \int_0^{\pi} 1\, dt = \pi.\end{aligned}$$

Note that, as a nice trick, we never had to be bothered with the values of s_p and s_q in the above derivation. It suffices to obtain the t values corresponding to p and q, respectively. □

[1] The name "line integral" may be a bit confusing because, after all, C is a curve, instead of a line. "Curve integral" would probably have been a better fit. However, for some reason, people have got used to the somewhat inaccurate name of "line" integral.

In general, let C be a smooth curve given by $\boldsymbol{r}(t) = [x_1(t), ..., x_d(t)]$ with t from t_0 to t_1. By applying the idea illustrated in the above example, we have that

$$\begin{aligned}
\text{length of } C &= \int_C ds \\
&= \int_{t_0}^{t_1} \sqrt{\sum_{i=1}^{d} \left(\frac{dx_i(t)}{dt}\right)^2} \, dt.
\end{aligned}$$

Example 2. Consider the helix $\boldsymbol{r}(t) = [x(t), y(t), z(t)]$ where

$$\begin{aligned}
x(t) &= \cos(t) \\
y(t) &= \sin(t) \\
z(t) &= t.
\end{aligned}$$

Let p be the point corresponding to $t = 0$, and q be the point corresponding to $t = \pi$. Let C be the curve from p to q. Calculate:

$$\int_C x + y + z^3 \, ds.$$

Solution. As before, find an arbitrary point p^* on the curve of $\boldsymbol{r}(t)$. Let s_p be the length the curve from p^* to p, and s_q be the length the curve from p^* to q. Then, we have:

$$\begin{aligned}
\int_C x^2 + y + z^3 \, ds &= \int_{s_p}^{s_q} x(s) + y(s) + z(s)^3 \, ds \\
&= \int_0^{\pi} \left(x(t) + y(t) + z(t)^3\right) \frac{ds}{dt} dt \\
\text{(by Lemma 2)} &= \int_0^{\pi} \left(x(t) + y(t) + z(t)^3\right) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt \\
&= \int_0^{\pi} \left(\cos(t) + \sin(t) + t^3\right) \sqrt{(-\sin(t))^2 + (\cos(t))^2 + 1^2} \, dt \\
&= \sqrt{2} \int_0^{\pi} \cos(t) + \sin(t) + t^3 \, dt \\
&= \sqrt{2}(2 + \pi^4/4).
\end{aligned}$$

$\square$