Exercises: Line Integral by Length

Problem 1. Let C be the curve from point $p(0,0)$ to point $q(1,1)$ on the parabola $y = x^2$. Calculate $\int_C x\,ds$.

Solution: First, write C into its parametric form: $\boldsymbol{r}(t) = [x(t), y(t)]$ where $x(t) = t$, and $y(t) = t^2$. Points p and q are given by $t = 0$ and 1, respectively. Thus:

$$\begin{aligned}
\int_C x\,ds &= \int_0^1 x(t)\frac{ds}{dt}dt \\
&= \int_0^1 x(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\
&= \int_0^1 t\sqrt{1+4t^2}\,dt \\
&= \frac{1}{12}(1+4t^2)^{3/2}\Big|_0^1 = \frac{5\sqrt{5}-1}{12}.
\end{aligned}$$

Problem 2. Let C be the line segment from point $p(1,2,3)$ to point $q(8,7,6)$. Calculate $\int_C x+z^2\,ds$.

Solution: Vector $\boldsymbol{q}-\boldsymbol{p} = [8,7,6]-[1,2,3] = [7,5,3]$ gives the direction of the line segment. Hence, C can be written into its parametric form: $\boldsymbol{r}(t) = [x(t), y(t), z(t)]$ where $x(t) = 1+7t$, $y(t) = 2+5t$, and $z(t) = 3+3t$. Points p and q are given by $t = 0$ and $t = 1$, respectively. Thus:

$$\begin{aligned}
\int_C x+z^2\,ds &= \int_0^1 (x(t)+(z(t))^2)\,\frac{ds}{dt}dt \\
&= \int_0^1 (1+7t+(3+3t)^2)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt \\
&= \int_0^1 (10+25t+9t^2)\sqrt{7^2+5^2+3^2}dt \\
&= \sqrt{83}\int_0^1 (10+25t+9t^2)dt \\
&= \frac{51\sqrt{83}}{2}.
\end{aligned}$$

Problem 3. Let C be the circle $x^2+y^2 = 1$. Calculate $\int_C y\,ds$.

Solution: Note that C is a closed circle. Next, we give two methods to solve the problem, which illustrate two different ways to deal with closed curves.

Method 1. Choose two arbitrary points on C, e.g., $p(1,0)$ and $q(-1,0)$. Break C into two curves: (i) C_1 from p counterclockwise to q, and (ii) C_2 from q counterclockwise to p. We will calculate $\int_{C_1} y\,ds$ and $\int_{C_2} y\,ds$ separately.

Introduce the parametric form of C: $\boldsymbol{r}(t) = [x(t), y(t)]$ where $x(t) = \cos(t)$ and $y(t) = \sin(t)$.

For C_1, p and q are given by $t = 0$ and π, respectively. Thus:

$$\begin{aligned}
\int_{C_1} y\,ds &= \int_0^{\pi} \sin(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\
&= \int_0^{\pi} \sin(t)\sqrt{(-\sin(t))^2 + (\cos(t))^2}dt \\
&= \int_0^{\pi} \sin(t)\,dt \\
&= -\cos(t)\Big|_0^{\pi} = 2
\end{aligned}$$

For C_2, q and p are given by $t = \pi$ and 2π, respectively. Thus:

$$\begin{aligned}
\int_{C_2} y\,ds &= \int_{\pi}^{2\pi} \sin(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\
&= \int_{\pi}^{2\pi} \sin(t)\sqrt{(-\sin(t))^2 + (\cos(t))^2}dt \\
&= \int_{\pi}^{2\pi} \sin(t)\,dt \\
&= -\cos(t)\Big|_{\pi}^{2\pi} = -2
\end{aligned}$$

Hence:

$$\int_C y\,ds = \int_{C_1} y\,ds + \int_{C_2} y\,ds = 0.$$

Method 2. Introduce the parametric form of C: $\boldsymbol{r}(t) = [x(t), y(t)]$ where $x(t) = \cos(t)$ and $y(t) = \sin(t)$. Pick an arbitrary point on C, e.g., $p(1, 0)$. Let $p' = p$ (i.e., another copy of the same point). View p as being given by $t = 0$, and q as being given by $t = 2\pi$.

$$\begin{aligned}
\int_C y\,ds &= \int_0^{2\pi} \sin(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\
&= \int_0^{2\pi} \sin(t)\sqrt{(-\sin(t))^2 + (\cos(t))^2}dt \\
&= \int_0^{2\pi} \sin(t)\,dt \\
&= -\cos(t)\Big|_0^{2\pi} = 0.
\end{aligned}$$

Problem 4. Let C be the intersection of two surfaces: sphere $x^2 + y^2 + z^2 = 3$ and plane $x = y$. Calculate $\int_C x^2 ds$.

Solution: The main difficulty of the problem is that the curve is given as the intersection of two surfaces. It is important to observe that the intersection is a closed curve. Introduce $x(t) = y(t) = \frac{\sqrt{3}}{\sqrt{2}}\cos(t)$ and $z(t) = \sqrt{3}\sin(t)$. Pick a point on C by setting $t = 0$, which gives $p(\sqrt{3/2}, \sqrt{3/2}, 0)$.

What is the smallest t that will give the same p? Clearly, the answer is $t = 2\pi$. Let $p' = p$, and view p' as being given by $t = 2\pi$.

$$
\begin{aligned}
\int_C x^2\,ds &= \int_0^{2\pi} \frac{3}{2}(\cos(t))^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt \\
&= \int_0^{2\pi} \frac{3}{2}(\cos(t))^2 \sqrt{\left(-\frac{\sqrt{3}}{\sqrt{2}}\sin(t)\right)^2 + \left(-\frac{\sqrt{3}}{\sqrt{2}}\sin(t)\right)^2 + \left(\sqrt{3}\cos(t)\right)^2}\,dt \\
&= \frac{3\sqrt{3}}{2}\int_0^{2\pi} (\cos(t))^2\,dt \\
&= \frac{3\sqrt{3}}{2}\left(\frac{t}{2} + \frac{\sin(2t)}{4}\right)\Big|_0^{2\pi} \\
&= \frac{3\sqrt{3}}{2}\pi.
\end{aligned}
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