Lecture Notes: Path Independence of Certain Line Integrals

Yufei Tao
Department of Computer Science and Engineering
Chinese University of Hong Kong
taoyf@cse.cuhk.edu.hk

Let C be a piecewise-smooth curve from point $p = (2, 0)$ to $q = (1, 1)$ in $\mathbb{R}^2$. Now, calculate the following line integral:

$$\int_C y\,dx + \int_C x\,dy. \tag{1}$$

At this moment, you may have sensed something is missing: the details of C have not been given yet! It turns out that *we do not need those details* to evaluate the integral. In other words, the result of the integral will *always* be the same regardless of C. Furthermore, introducing $g(x, y) = xy$, we know that the result of (1) is definitely $g(1, 1) - g(2, 0) = 1$. In this case, we say that (1) is *path independent* (i.e., independent of the path from p to q).

In this lecture, we will give the if- and only-if conditions for a line integral to be path independent.

1 Path Independence in $\mathbb{R}^2$

Let $f_1(x, y)$ and $f_2(x, y)$ be two continuous scalar functions with real-valued parameters x, y. Define S to be the set of all possible line integrals of the form

$$\int_C f_1\,dx + \int_C f_2\,dy.$$

Definition 1. *We say that S is* **path independent** *in $\mathbb{R}^2$ if, for any two piecewise-smooth curves C_1, C_2 with the same starting and ending points, it holds that*

$$\int_{C_1} f_1\,dx + \int_{C_1} f_2\,dy \quad = \quad \int_{C_2} f_1\,dx + \int_{C_2} f_2\,dy.$$

We now prove an important theorem:

Theorem 1. *S is path independent <u>if and only if</u> we can find a function $g(x, y)$ such that*

$$f_1(x, y) = \frac{\partial g}{\partial x}, \text{ and } f_2(x, y) = \frac{\partial g}{\partial y} \tag{2}$$

Proof. <u>*The If-Direction.*</u> Given the equations in (2), we will prove that S is path independent. Fix any point $p = (x_p, y_p)$ and $q = (x_q, y_q)$. Consider any curve C from p to q. Let $[x(t), y(t)]$ be a parametric form of C. Denote by t_p and t_q the values of t for p and q, respectively.

We have

$$
\begin{aligned}
\int_C f_1\,dx + \int_C f_2\,dy &= \int_C \frac{\partial g}{\partial x}\,dx + \int_C \frac{\partial g}{\partial y}\,dy \\
&= \int_{t_p}^{t_q} \frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dt}\,dt \\
&= \int_{t_p}^{t_q} \frac{dg}{dt}\,dt \\
&= g(x(t), y(t))\Big|_{t_p}^{t_q} \\
&= g(x_q, y_q) - g(x_p, y_p)
\end{aligned}
$$

which is a value that does not depend on C.

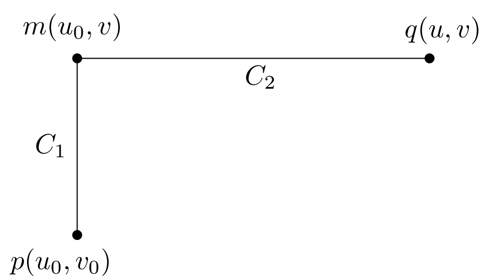
The Only-If Direction. Suppose that S is path independent. We will prove that there is a function $g(x, y)$ satisfying (2).

In fact, we will give such a function directly. First, fix any point $p(u_0, v_0)$. Given any point $q(u, v)$ in $\mathbb{R}^2$, define:

$$
g(u, v) = \int_{C_{pq}} f_1\,dx + \int_{C_{pq}} f_2\,dy
$$

where C_{pq} is an arbitrary curve from p to q. Note that $g(u, v)$ does not depend on the choice of C_{pq} because S is path independent. We argue that $g(u, v)$ is a function fulfilling (2), namely, $\frac{\partial g}{\partial u} = f_1(u, v)$ and $\frac{\partial g}{\partial v} = f_2(u, v)$. Next, we will prove only the former because a symmetric argument shows the latter.

Choose C_{pq} to be a path consisting of a vertical segment C_1, followed by a horizontal segment C_2. Specifically, C_1 is from p to $m(u_0, v)$, and C_2 is from m to q.



Hence:

$$
\begin{aligned}
g(u, v) &= \int_C f_1\,dx + \int_C f_2\,dy \\
&= \left(\int_{C_1} f_1\,dx + \int_{C_1} f_2\,dy\right) + \left(\int_{C_2} f_1\,dx + \int_{C_2} f_2\,dy\right) \\
&= \int_{C_1} f_2\,dy + \int_{C_2} f_1\,dx \\
&= \int_{v_0}^{v} f_2(u_0, y)\,dy + \int_{u_0}^{u} f_1(x, v)\,dx.
\end{aligned}
$$

Note that $\int_{v_0}^{v} f_2(u_0, y)\, dy$ has nothing to do with u (namely, the term is always the same no matter what is the first coordinate of q). Hence:

$$\frac{\partial g}{\partial u}(u, v) \quad = \quad \frac{\partial \left(\int_{u_0}^{u} f_1(x, v)\, dx \right)}{\partial u} = f(u, v).$$

□

Corollary 1. *Suppose that we can find a function $g(x, y)$ satisfying (2). Then, for any points $p = (x_p, y_p)$, $q = (x_q, y_q)$, and any piecewise-smooth curve C from p to q, it holds that*

$$\int_C f_1\, dx + \int_C f_2\, dy \quad = \quad g(x_q, y_q) - g(x_p, y_p).$$

Proof. Follows directly from the proof of the if-direction of Theorem 1. □

Example 1. Let C be a piecewise smooth curve from point $p = (2, 0)$ to $q = (1, 1)$ in $\mathbb{R}^2$. Calculate:

$$\int_C y\, dx + \int_C x\, dy. \tag{3}$$

Solution. Let $g(x, y) = xy$. We have that $\frac{\partial g}{\partial x} = y$ and $\frac{\partial g}{\partial y} = x$. Hence, (3) $= g(1, 1) - g(2, 0) = 1$. □

Example 2. Let C be a piecewise smooth curve from point $p = (2, 0)$ to $q = (1, 1)$ in $\mathbb{R}^2$. Calculate:

$$\int_C y^2 (\sin(x) + x \cdot \cos(x))\, dx + \int_C 2xy \sin(x)\, dy. \tag{4}$$

Solution. Let $g(x, y) = x \sin(x) \cdot y^2$. We have that $\frac{\partial g}{\partial x} = y^2(\sin(x) + x\cos(x))$ and $\frac{\partial g}{\partial y} = 2xy\sin(x)$. Hence, (3) $= g(1, 1) - g(2, 0) = \sin(1)$. □

2 Path Independence in $\mathbb{R}^d$

The discussion in the previous section can be readily generalized to $\mathbb{R}^d$ of an arbitrary d. Let $f_1(x_1, x_2, ..., x_d)$, $f_2(x_1, x_2, ..., x_d)$, ..., $f_d(x_1, x_2, ..., x_d)$ be d scalar functions. Define S to be the set of all possible line integrals of the form

$$\int_C f_1\, dx_1 + \int_C f_2\, dx_2 + ... + \int_C f_d\, dx_d.$$

Definition 2. *We say that S is* **path independent** *in $\mathbb{R}^d$ if, for any two piecewise-smooth curves C_1, C_2 with the same starting and ending points, it holds that*

$$\int_{C_1} f_1\, dx_1 + \int_{C_1} f_2\, dx_2 + ... + \int_{C_1} f_d\, dx_d \quad = \quad \int_{C_2} f_1\, dx_1 + \int_{C_2} f_2\, dx_2 + ... + \int_{C_2} f_d\, dx_d.$$

The next theorem generalizes Theorem 1:

Theorem 2. *S is path independent if and only if we can find a function $g(x_1, x_2, ..., x_d)$ such that*

$$\begin{aligned} f_1(x_1, .., x_d) &= \frac{\partial g}{\partial x_1}(x_1, .., x_d) \\ f_2(x_1, .., x_d) &= \frac{\partial g}{\partial x_2}(x_1, .., x_d) \\ &\cdots \\ f_d(x_1, .., x_d) &= \frac{\partial g}{\partial x_d}(x_1, .., x_d). \end{aligned}$$

When S is path independent, for any points $p = (x_{p1}, x_{p2}, ..., x_{pd})$, $q = (x_{q1}, x_{q2}, ..., x_{qd})$, and any piecewise-smooth curve C from p to q, it holds that

$$\int_C f_1\, dx_1 + \int_C f_2\, dx_2 + ... + \int_C f_d\, dx_d. \quad = \quad g(x_{q1}, x_{q2}, ..., x_{qd}) - g(x_{p1}, x_{p2}, ..., x_{pd}).$$

Proof. Direct extension of the proof of Theorem 1. □

Example 3. Let C be a piecewise smooth curve from point $p = (2, 3, 4)$ to $q = (1, 1, 1)$ in $\mathbb{R}^3$. Calculate:

$$\int_C 2xy^2z\, dx + \int_C 2x^2yz\, dy + \int_C x^2y^2\, dz. \tag{5}$$

Solution. Let $g(x, y, z) = x^2y^2z$. We have that $\frac{\partial g}{\partial x} = 2xy^2z$, $\frac{\partial g}{\partial y} = 2x^2yz$, and $\frac{\partial g}{\partial z} = 2x^2y^2$. Hence, $(3) = g(2, 3, 4) - g(1, 1, 1) = 143$. □