# Lecture Note 4

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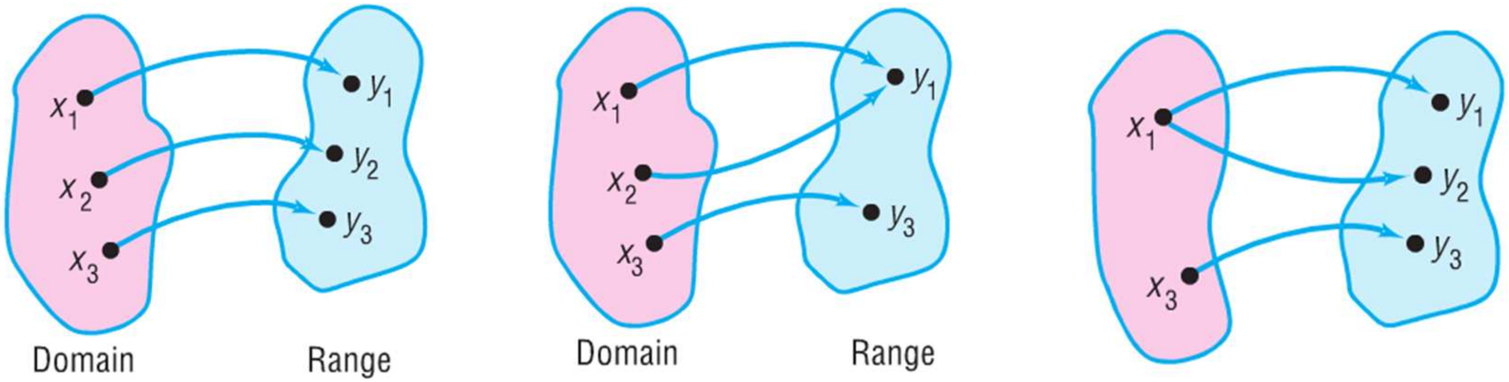
MATH1020
General Mathematics

# One–to–one Functions and Inverse Functions

## What will you learn?

- One—to—one Functions
- Inverse Functions

**Definition 1** A function is one–to–one if any two different inputs in the domain correspond to two different outputs in the range. That is, if $x_1$ and $x_2$ are two different inputs of a function $f$, then $f$ is one–to–one if $f(x_1) \neq f(x_2)$.




(a) One—to—one function: Each $x$ in the domain has one and only one image in the range. No $y$ in the range is the image of more than one $x$.

(b) Not a one—to—one function $y_1$ is the image of both $x_1$ and $x_2$.

(c) Not a function $x_1$ has two images, $y_1$ and $y_2$.

Figure 1:

**Theorem 1** **Horizontal–line Test** If every horizontal line intersects the graph of a function $f$ in at most one point, then $f$ is one–to–one.

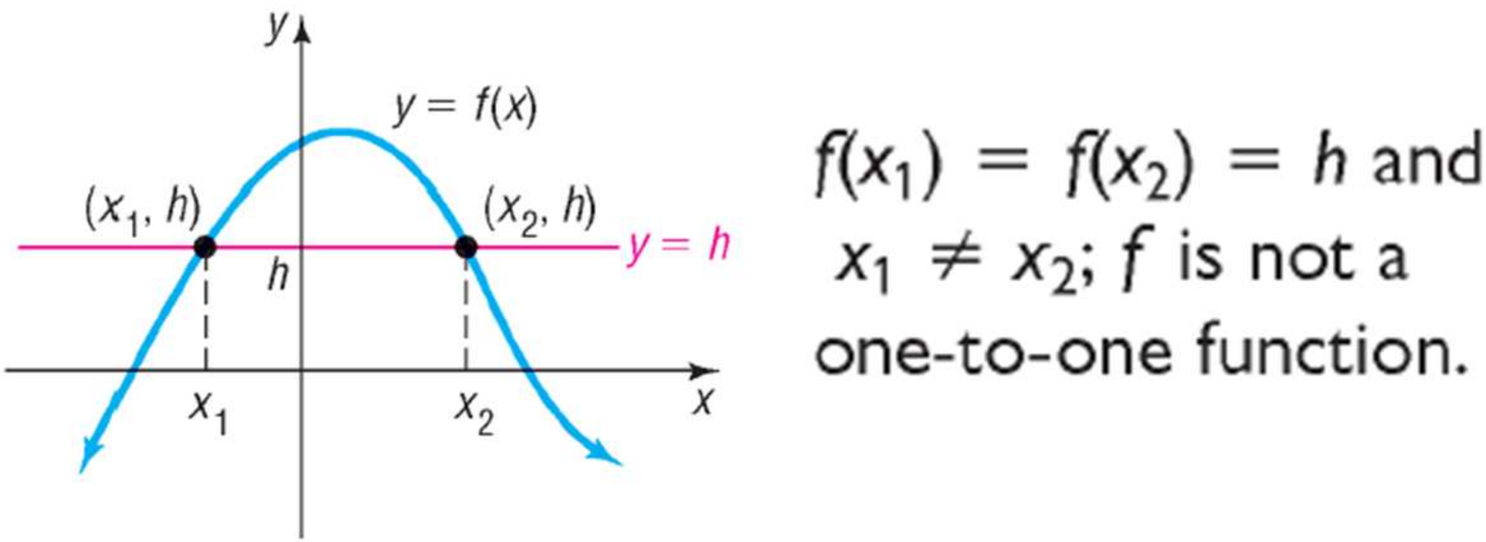
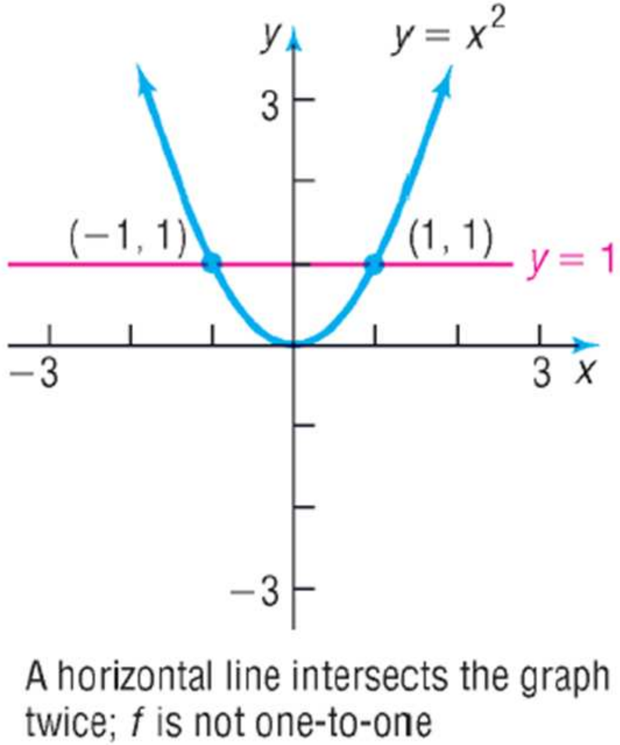


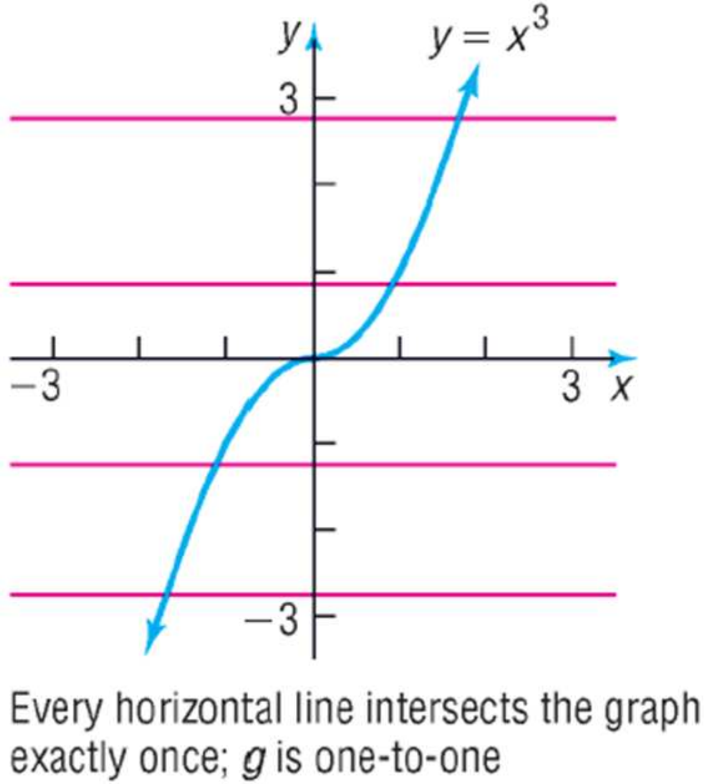

Figure 2: Horizontal–line Test.

**Exercises 1** For each function, use its graph to determine whether the function is one–to–one.

(a) $f(x) = x^2$ (b) $g(x) = x^3$.




(a) $f(x) = x^2$




(b) $g(x) = x^3$

Figure 3:

**Theorem 2**

1. A function that increasing $\nearrow$ on an interval $I$ is one$-$to$-$one function on $I$.
2. A function that is decreasing $\searrow$ on an interval $I$ is a one$-$to$-$one function on $I$.

**Definition 2** Suppose $f$ one$-$to$-$one function. Then, to each $x$ in the domain of $f$, there is exactly one $y$ in the range $($because $f$ is one$-$to$-$one$)$. The correspondence from the range of $f$ back to the domain of $f$ is called the **inverse function of** $f$. We use the symbol $f^{-1}$ to denote the inverse of $f$.

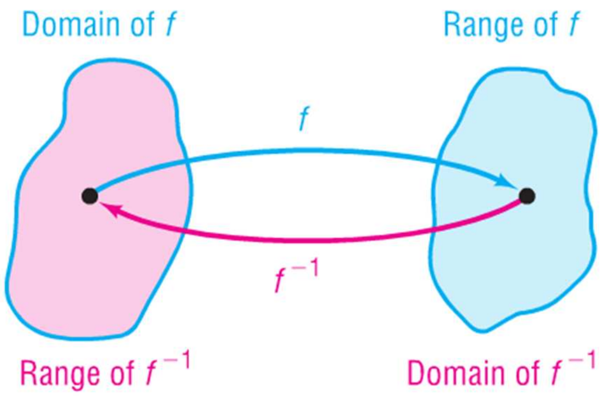



Figure 4:

Two facts are now apparent about a one-to-one function $f$ and its inverse $f^{-1}$:

$$\boxed{\text{Domain of } f = \text{Range of } f^{-1} \qquad \text{Range of } f = \text{Domain of } f^{-1}}$$

Figure 4 illustrates the relationship between $f$ and $f^{-1}$.

- If we start with $x$, apply $f$ and then apply $f^{-1}$, we get $x$ back again.
- If we start with $x$, apply $f^{-1}$, and then apply $f$, we get the number $x$ back again.
- To put it simply, what $f$ does, $f^{-1}$ undoes, and vice versa.

**Exercises 2 Verify Inverse Function**

(a) Verify the inverse of $g(x) = x^5$ is $g^{-1}(x) = \sqrt[5]{x}$.

(b) Verify the inverse of $f(x) = 3x + 5$ is $f^{-1} = \frac{1}{3}(x - 5)$.

**Exercises 3 Verify Inverse Function**

Verify that the inverse of $f(x) = \dfrac{1}{x-1}$ is $f^{-1}(x) = \dfrac{1}{x} + 1$.

For what values of $x$ is $f^{-1}(f(x)) = x$?

For what values of $f$ is $f(f^{-1}(x)) = x$?

Suppose that $(a, b)$ is a point on the graph of a one–to–one function $f$ defined by $y = f(x)$. Then $b = f(a)$. This means that $a = f^{-1}(b)$, so $(b, a)$ is a point on the graph of the inverse function $f^{-1}$. The relationship between on the point $(a, b)$ on $f$ and the point $(b, a)$ on $f$ is shown in Figure 5.

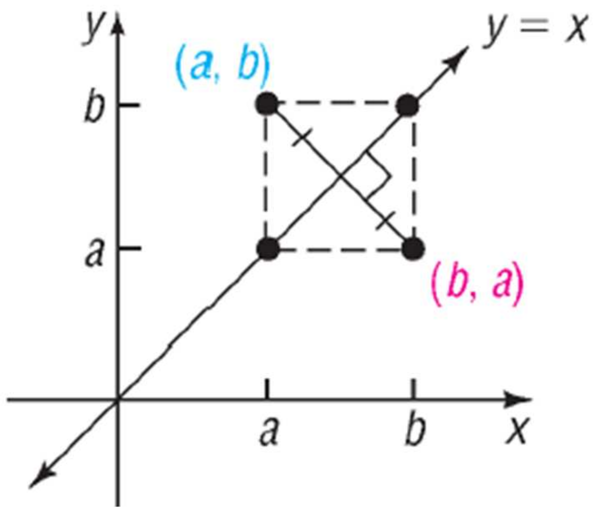
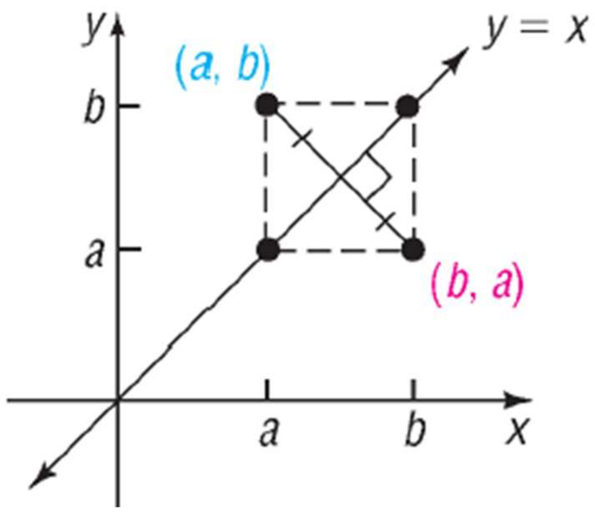



Figure 5:

The line segment containing $(a, b)$ and $(b, a)$ is perpendicular to the line $y = x$ and is bisected by the line $y = x$. (Do you see why?) It follows that the point $(b, a)$ on $f^{-1}$ is the reflection about the line $y = x$ of the point $(b, a)$ on $f$.



**Theorem 3** The graph of a function $f$ and the graph of its inverse $f^{-1}$ are symmetric with respect to the line $y = x$.

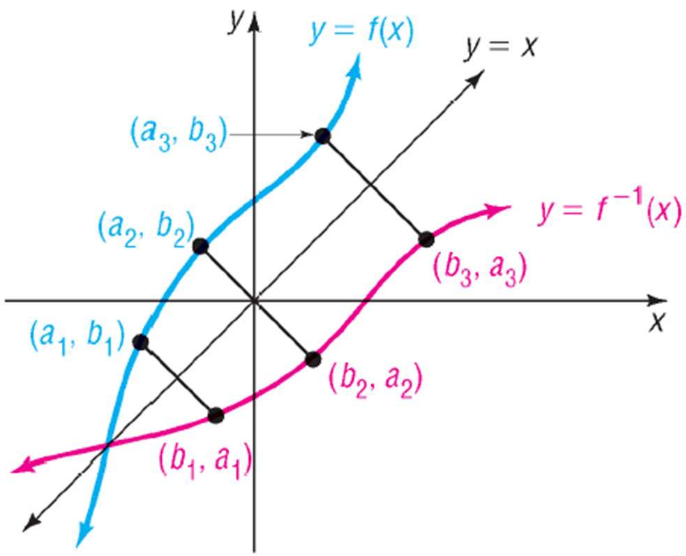



Figure 6:

Figure 6 illustrates the application of Theorem 3. Notice that, once the graph of $f$ is known, the graph of $f^{-1}$ may be obtained by reflecting the graph of $f$ about the line $y = x$.

**Procedure for Finding the Inverse of a One−to−One Function**

**STEP** 1:In $y = f(x)$, interchange the variables $x$ and $y$ to obtain

$$x = f(y).$$

This equation defines the inverse function $f^{-1}$ implicitly.

**STEP** 2**:** If possible, solve the implicit equation for $y$ in terms of $x$ to obtain the explicit form of $f^{-1}$:

$$y = f^{-1}(x).$$

**STEP** 3**:** Check the result by showing that

$$f^{-1}(f(x)) = x \qquad \text{and} \qquad f(f^{-1}(x)) = x.$$

**Exercises 4 Finding the Inverse Function**

The function

$$f(x) = \frac{2x+1}{x-1} \qquad x \neq 1$$

is one–to–one. Find its inverse and check the result.

**Exercises 5 Find the Range of a Function**

Find the domain and the range of

$$f(x) = \frac{2x+1}{x-1}.$$

**Exercises 6 Finding the Inverse of a Domain$-$restricted Function**

Find the inverse of $y = f(x) = x^2$ if $x \geq 0$.