Exercises: Dot Product and Cross Product

Problem 1. Give the result of $\boldsymbol{a} \cdot \boldsymbol{b}$ for each of the following:

1. $\boldsymbol{a} = [1, 2], \boldsymbol{b} = [2, 5]$
2. $\boldsymbol{a} = [1, 2, 3], \boldsymbol{b} = [2, 5, -7]$

Problem 2. Give the result of $\boldsymbol{a} \times \boldsymbol{b}$ for each of the following:

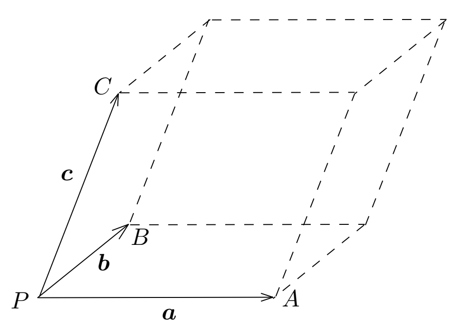
1. $\boldsymbol{a} = [1, 2, 3], \boldsymbol{b} = [3, 2, 1]$
2. $\boldsymbol{a} = \boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}, \boldsymbol{b} = [3, 2, 1]$

Problem 3. In each of the following, you are given two vectors $\boldsymbol{a} \cdot \boldsymbol{b}$. Let γ be the angle between the two vectors' directions. Give the value of $\cos\gamma$.

1. $\boldsymbol{a} = [1, 2], \boldsymbol{b} = [2, 5]$
2. $\boldsymbol{a} = [1, 2, 3], \boldsymbol{b} = [3, 2, 1]$

Problem 4. This exercise explores the usage of dot product for calculation of projection lengths. Consider points $P(1, 2, 3), A(2, -1, 4), B(3, 2, 5)$. Let ℓ be the line passing P and A. Now, let us project point B onto ℓ; denote by C the projection. Calculate the distance between P and C.

Problem 5. Let $\overrightarrow{P, A}$, $\overrightarrow{P, B}$, and $\overrightarrow{P, C}$ be directed segments that are not in the same plane. They determine a parallelepiped as shown below:



Suppose that $\overrightarrow{P, A}$, $\overrightarrow{P, B}$, and $\overrightarrow{P, C}$ are instantiations of vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$, respectively. Prove that the volume of the parallelepiped equals $|(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}|$.

Problem 6. Given a point $p(x, y, z)$ in $\mathbb{R}^3$, we use $\boldsymbol{p}$ to denote the corresponding vector $[x, y, z]$. Let q be a point in $\mathbb{R}^3$, and $\boldsymbol{v}$ be a non-zero 3d vector. Denote by ρ the plane passing q that is perpendicular to the direction of $\boldsymbol{v}$. Prove that for any p on ρ, it holds that $(\boldsymbol{p} - \boldsymbol{q}) \cdot \boldsymbol{v} = 0$.

Problem 7. Given a point $p(x, y, z)$ in $\mathbb{R}^3$, we use $\boldsymbol{p}$ to denote the corresponding vector $[x, y, z]$. Let q be a point in $\mathbb{R}^3$, and $\boldsymbol{u}$ be a unit 3d vector (i.e., $|\boldsymbol{u}| = 1$). Denote by ρ the plane passing q that is perpendicular to the direction of $\boldsymbol{u}$. Prove that for any p in $\mathbb{R}^3$, its distance to ρ equals $|(\boldsymbol{p} - \boldsymbol{q}) \cdot \boldsymbol{u}|$.

Problem 8. Consider the plane $x + 2y + 3z = 4$ in $\mathbb{R}^3$. Calculate the distance from point $(0, 0, 0)$ to the plane.

Problem 9. Consider the line $x + 2y = 4$ in $\mathbb{R}^2$. Calculate the distance from point $(0, 0)$ to the line.

Problem 10. Given a point $p(x, y, z)$ in $\mathbb{R}^3$, we use $\boldsymbol{p}$ to denote the corresponding vector $[x, y, z]$. Let q be a fixed point in $\mathbb{R}^3$, and $\boldsymbol{v}$ a non-zero 3d vector. Given a real value s, $f(s)$ gives a point p in $\mathbb{R}^3$ such that $\boldsymbol{p} = \boldsymbol{q} + s \cdot \boldsymbol{v}$. As s goes from $-\infty$ to ∞, what is the locus of $f(s)$?