THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1020
Exercise 9
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1. Figure 1 shows the graph of $y = \sin\, x$. Because every horizontal line $y = h$, where b is between -1 and 1, intersects the graph of $y = \sin\, x$ infinitely many times, it follows from the horizontal-line test that the function $y = \sin\, x$ is not one-to-one.

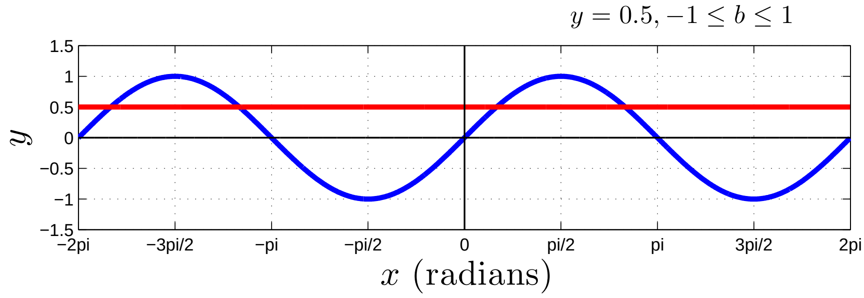


Figure 1: $y = \sin x$, where $x \in [-2\pi, 2\pi]$.

However, if we restrict the domain of $y = \sin\, x$ to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the restricted function

$$y = \sin x, \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

is one-to-one and so will have an inverse function, as illustrated in Figure 2.

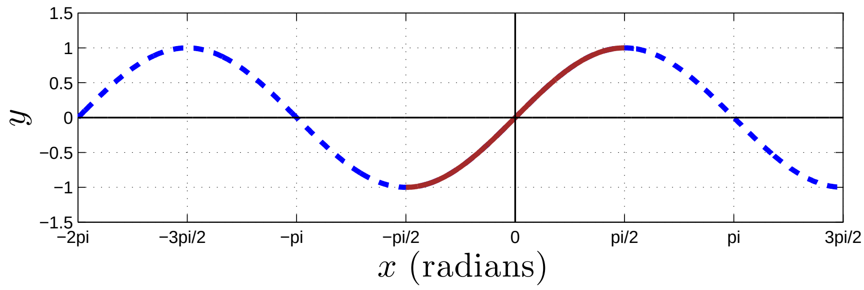


Figure 2: $y = \sin x$ (brown), where $x \in [-\pi/2, \pi/2]$.

Remark 1 Although there are many other ways to restrict the domain and obtain a one-to-one function,mathematicians have agreed to use the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ to define the inverse of $y = \sin x$.

2. An equation for the inverse of $y = f(x) = \sin x$ is obtained by interchanging x and y. The implicit form of the inverse function is $x = \sin y,\ -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. The explicit form is called the **inverse sine** of x and is symbolized by $y = f^{-1}(x) = \sin^{-1} x$.

3. **Definition 1**

$$y = \sin^{-1} x \quad \text{means} \quad x = \sin y,$$
$$\text{where} \quad -1 \le x \le 1 \quad \text{and} \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}. \tag{1}$$

4. Because $y = \sin^{-1} x$ means $x = \sin y$, we read $y = \sin^{-1} x$ as "y is the angle or real number whose sine equals x." Alternatively, we can say that "y is the inverse sine of x."

   Note that the superscript -1 that appears in $y = \sin^{-1} x$ is not an exponent, but is reminiscent of the symbolism f^{-1} used to denote the inverse function of f. (To avoid this notation, some books use the notation $y = \arcsin x$ instead of $y = \sin^{-1} x$.)

   The graph of the inverse sine function can be obtained by reflecting the restricted portion of the graph of $y = f(x) = \sin x$ about the line $y = x$, as shown in Figure 3.

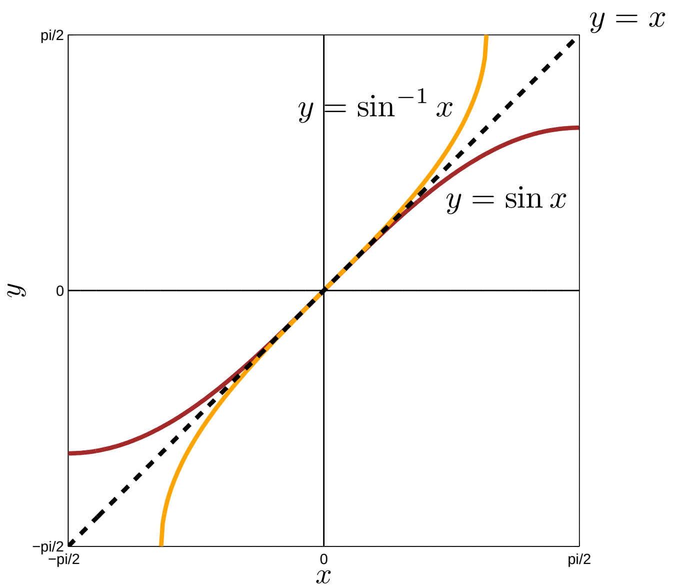


Figure 3: Graphs of $\sin x$ and $\sin^{-1} x$.

5. **Exercise 1 Finding the Exact Value of an Inverse Sine Function**

Find the exact value of: $\sin^{-1} 1$.

Solution

Let $\theta = \sin^{-1} 1$.

We seek the angle θ, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals 1.

$$\begin{array}{lll} \theta = \sin^{-1} 1, & -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} & \\ \sin\theta = 1 & -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} & \text{By definition of } y = \sin^{-1} x. \end{array}$$

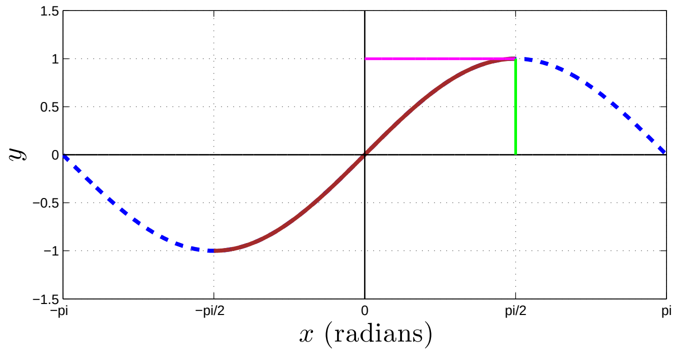


Figure 4: $y = \sin x$ (brown), where $x \in [-\pi/2, \pi/2]$.

Now look at Table 1 and Figure 4.

Table 1

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

We see that the only angle θ within the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is 1 is $\frac{\pi}{2}$, (Note that $\sin\frac{5\pi}{2}$ also equals 1, but $\frac{5\pi}{2}$ lies outside the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and hence is not admissible.)

So, since $\frac{\pi}{2} = 1$ and $\frac{\pi}{2}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we conclude that

$$\sin^{-1} 1 = \frac{\pi}{2}.$$

6. **Exercise 2 Finding the Exact Value of an Inverse Sine Function**

Find the exact value of: $\sin^{-1}\left(-\frac{1}{2}\right)$.

Solution:

Let $\theta = \sin^{-1}\left(-\frac{1}{2}\right)$.

We seek the angle θ, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, whose sine equals $-\frac{1}{2}$.

$$\begin{array}{ll} \theta = \sin^{-1}\left(-\frac{1}{2}\right) & \qquad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \\ \sin\theta = -\frac{1}{2} & \qquad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}. \end{array}$$

(Refer to Table 1 and Figure 2, if necessary.)

The only angle within the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is $-\frac{1}{2}$ is $-\frac{\pi}{6}$.

So, since $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$ and $-\frac{\pi}{6}$ is in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we conclude that

$$\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}.$$

Table 1

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

7. **Use Properties of Inverse Functions to Find Exact Values of Certain Composite Functions**

   We know that $f^{-1}(f(x)) = x$ for all x in the domain of f and $f^{-1}(f(x)) = x$ for all x in the domain of f^{-1}. In terms of the sine function and its inverse, these properties are of the form:

$$f^{-1}(f(x)) = \sin^{-1}(\sin x) = x \quad \text{where} \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2} \tag{2}$$

$$f(f^{-1}(x)) = \sin(\sin^{-1} x) = x \quad \text{where} \quad -1 \le x \le 1 \tag{3}$$

8. **Exercise 3 Finding the Exact Value of Certain Composite Functions**

Find the exact value of each of the following composite functions:

(a) $\sin^{-1}\left(\sin\frac{\pi}{8}\right)$

(b) $\sin^{-1}\left(\sin\frac{5\pi}{8}\right)$

Solution

(a) The composite function $\sin^{-1}\left(\sin\frac{\pi}{8}\right)$ follows the form of equation (2). Because $\frac{\pi}{8}$ is in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we can use (2). Then

$$\sin^{-1}\left(\sin\frac{\pi}{8}\right) = \frac{\pi}{8}.$$

(b) The composite function $\sin^{-1}\left(\sin\frac{5\pi}{8}\right)$ follows the form of equation (2) but $\frac{5\pi}{8}$ is not in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

To use (2) we need to find an angle θ in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ for which $\sin\theta = \sin\frac{5\pi}{8}$.

Then, using (2), $\sin^{-1}\left(\sin\frac{5\pi}{8}\right) = \sin^{-1}(\sin\theta) = \theta$, and we are finished.

Using a graphing calculator, we verify that

$$\sin^{-1}\left(\sin\frac{5\pi}{8}\right) \neq \frac{5\pi}{8}$$

and

$$\sin^{-1}\left(\sin\frac{5\pi}{8}\right) = \sin^{-1}\left(\sin\frac{3\pi}{8}\right) = \frac{3\pi}{8}.$$

9. **Exercise 4 Finding the Exact Value of Certain Composite Functions**

   Find the exact value,if any,of each composite function.

   (a) $\sin(\sin^{-1} 0.8)$
   (b) $\sin(\sin^{-1} 1.8)$

   Solution

   (a) The composite function $\sin(\sin^{-1} 0.8)$ follows the form of equation (3) and 0.8 is in the interval $[-1, 1]$. So we use (3):

   $$\sin(\sin^{-1} 0.8) = 0.8$$

   (b) The composite function $\sin(\sin^{-1} 1.8)$ follows the form of equation (3), but 1.8 is not in the domain of the inverse sine function. This composite function is not defined. Can you explain why the error occurs?

10. **The Inverse Cosine Function**

In Figure 5 we show the graph of $y = \cos x$. Because every horizontal line $y = b$, where b is between -1 and 1, intersects the graph of $y = \cos x$ infinitely many times, it follows that the cosine function is not one-to-one.

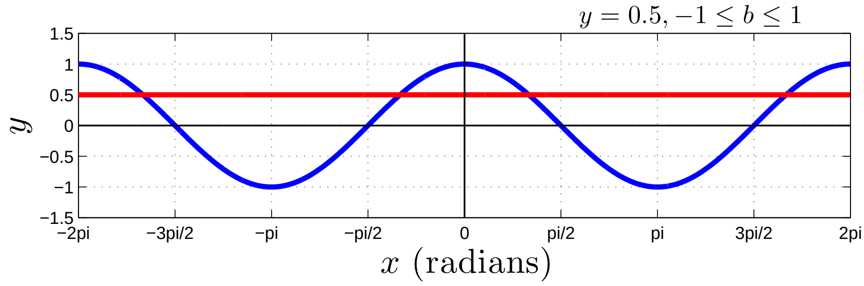


Figure 5: $y = \cos x$, where $x \in [-2\pi, 2\pi]$.

However, if we restrict the domain of $y = \cos x$ to the interval $[0, \pi]$, the restricted function

$$y = \cos x \quad 0 \le x \le \pi$$

is one-to-one and hence will have an inverse function. Figure 6 and 7.

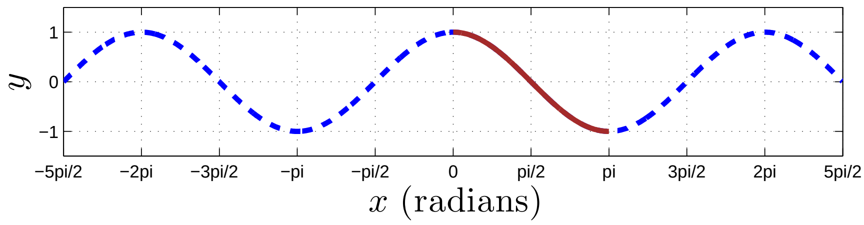


Figure 6: $y = \cos x$, where $x \in \left[\frac{-5\pi}{2}, \frac{5\pi}{2}\right]$.

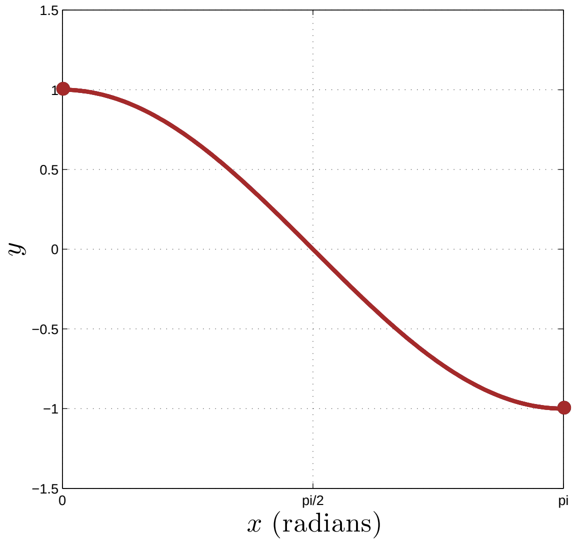


Figure 7: $y = \cos x$ (brown), where $x \in [0, \pi]$ and $y \in [-1, 1]$.

An equation for the inverse of $y = f(x) = \cos y$ is obtained by interchanging x and y. The implicit form of the inverse function is $x = \cos y, 0 \leq y \leq \pi$. The explicit form is called the **inverse cosine** of x and is symbolized by $y = f^{-1}(x) = \cos^{-1} x$ (or by $y = \arccos x$).

11. **Definition 2**

$$\begin{aligned} &y = \cos^{-1} x \quad \text{means} \quad x = \cos y, \\ &\text{where} \quad -1 \leq x \leq 1 \quad \text{and} \quad 0 \leq y \leq \pi. \end{aligned} \tag{4}$$

Here y is the angle whose cosine is x. Because the range of the cosine function,$y = \cos x$, is $-1 \leq y \leq 1$, the domain of the inverse function $y = \cos^{-1} x$ is $-1 \leq x \leq 1$. Because the restricted domain of the cosine function,$y = \cos x$, is $0 \leq x \leq \pi$, the range of the inverse function $y = \cos^{-1} x$ is $0 \leq y \leq \pi$.

The graph of $y = \cos^{-1} x$ can be obtained by reflecting the restricted portion of the graph of $y = \cos x$ about the line $y = x$, as shown in Figure 8.

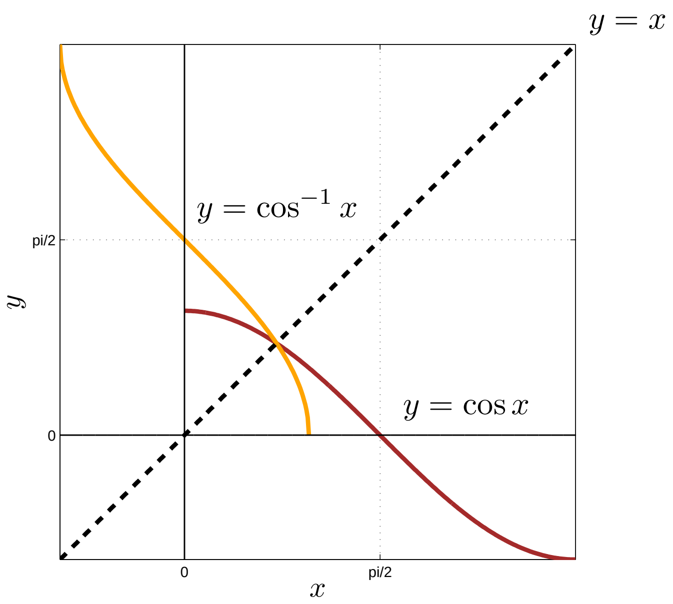


Figure 8: Graphs of $\cos x$ and $\cos^{-1} x$.

12. **Exercise 5 Finding the Exact Value of an Inverse Cosine Function**

Find the exact value of: $\cos^{-1} 0$.

Solution:

Let $\theta = \cos^{-1} 0$.

We seek the angle θ, $0 \leq \theta \leq \pi$, whose cosine equals 0.

$$\theta = \cos^{-1} 0, \quad 0 \leq \theta \leq \pi;$$

$$\cos\theta = 0, \quad 0 \leq \theta \leq \pi.$$

Look at Table 2. See Figure 6.

Table 2

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1

We see that the only angle θ within the interval $[0, \pi]$ whose cosine is 0 is $\frac{\pi}{2}$.

[Note that $\cos\frac{3\pi}{2}$ and $\cos\left(-\frac{\pi}{2}\right)$ also equal 0, but they lie outside the interval $[0, \pi]$ and hence are not admissible.]

Since $\cos\frac{\pi}{2} = 0$ and $\frac{\pi}{2}$ is in the interval $[0, \pi]$, we conclude that

$$\cos^{-1} 0 = \frac{\pi}{2}.$$

13. **Exercise 6 Finding the Exact Value of an Inverse Cosine Function**

Find the exact value of: $\cos^{-1}\Big(-\frac{\sqrt{2}}{2}\Big)$.

Solution:

Let $\theta = \cos^{-1}\Big(-\frac{\sqrt{2}}{2}\Big)$.

We seek the angle $\theta, 0 \le \theta \le \pi$, whose cosine equals $-\frac{\sqrt{2}}{2}$.

$$\theta = \cos^{-1}\Big(-\frac{\sqrt{2}}{2}\Big) \quad 0 \le \theta \le \pi;$$

$$\cos\theta = -\frac{\sqrt{2}}{2} \quad 0 \le \theta \le \pi.$$

Look at Table 2.

We see that only angle θ within the interval $[0, \pi]$ whose cosine is $-\frac{\sqrt{2}}{2}$ is $\frac{3\pi}{4}$.

So, since $\cos\frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$ and $\frac{3\pi}{4}$ is in the interval $[0, \pi]$, we conclude that

$$\cos^{-1}\Big(-\frac{\sqrt{2}}{2}\Big) = \frac{3\pi}{4}.$$

Table 2

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1

14. **Definition 3**

$$f^{-1}(f(x)) = \cos^{-1}(\cos x) = x \quad \text{where } 0 \leq x \leq \pi; \tag{5}$$
$$f(f^{-1}(x)) = \cos(\cos^{-1} x) = x \quad \text{where } -1 \leq x \leq 1. \tag{6}$$

15. **Exercise 7 Using Properties of Inverse Functions to Find the Exact Value of Certain Composite Functions**

Find the exact value of:

(a) $\cos^{-1}\left(\cos \frac{\pi}{12}\right)$

(b) $\cos[\cos^{-1}(-0.4)]$

(c) $\cos^{-1}\left[\cos\left(-\frac{2\pi}{3}\right)\right]$

(d) $\cos(\cos^{-1} \pi)$

Solution:

(a) $\cos^{-1}\left(\cos \frac{\pi}{12}\right) = \frac{\pi}{12}$.
$\frac{\pi}{12}$ **is in the interval** $[0, \pi]$; **use Property (5).**

(b) $\cos[\cos^{-1}(-0.4)] = -0.4$.
-0.4 **is in the interval** $[-1, 1]$; **use Property (6).**

(c) The angle $-\frac{2\pi}{3}$ is not in the interval $[0, \pi]$, so we cannot use (5). However, because the cosine function is even, $\cos\left(-\frac{2\pi}{3}\right) = \cos \frac{2\pi}{3}$. Since $\frac{2\pi}{3}$ is in the interval $[0, \pi]$, we have

$$\cos^{-1}\left[\cos\left(-\frac{2\pi}{3}\right)\right] = \cos^{-1}\left(\cos \frac{2\pi}{3}\right) = \frac{2\pi}{3}.$$

$\frac{2\pi}{3}$ **is in the interval** $[0, \pi]$; **apply (5).**

(d) Because π is not in the interval $[-1, 1]$, the domain of the inverse cosine function, $\cos^{-1} \pi$ is not defined. This means the composite function $\cos(\cos^{-1} \pi)$ is also not defined.

16. **The Inverse Tangent Function**

In Figure 9, we show the graph of $y = \tan x$. Because every horizontal line intersects the graph infinitely many times, it follows that the tangent function is not one-to-one.

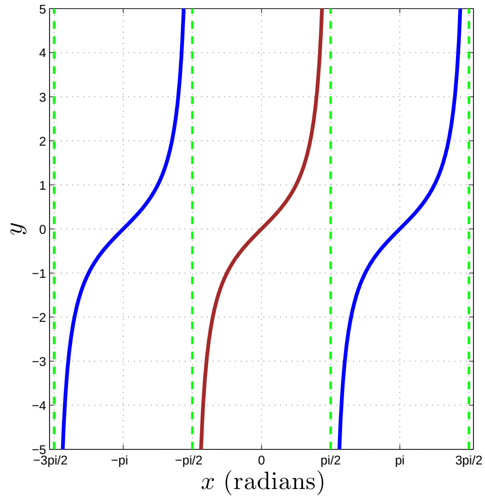


Figure 9: $y = \tan x$, where $x \in (-3\pi/2, -\pi/2) \cup (-\pi/2, \pi/2) \cup (\pi/2, 3\pi/2)$.

However, if we restrict the domain of $y = \tan x$ to the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the restricted function

$$y = \tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

is one-to-one and hence has an inverse function. See Figure 10.

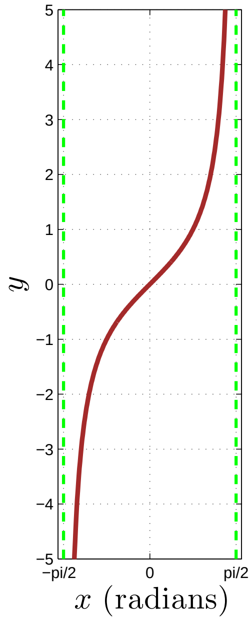


Figure 10: $y = f_3(x)$ (brown), where $x \in \left(-\frac{1}{2}\pi, \frac{1}{2}\pi\right)$.

An equation for the inverse of $y = f(x) = \tan x$ is obtained by interchanging x and y. The implicit form of the inverse function is $x = \tan y$, $-\frac{\pi}{2} < y < \frac{\pi}{2}$. The explicit form is called the **inverse tangent** of x and is symbolized by $y = f^{-1}(x) = \tan^{-1} x$ (or by $y = \arctan x$).

17. **Definition 4** $y = \tan^{-1} x$ means $x = \tan y$
where $-\infty < x < \infty$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$ (5)

Here y is the angle whose tangent is x. The domain of the function $y = \tan^{-1} x$ is $-\infty < x < \infty$, and its range is $-\frac{\pi}{2} < y < \frac{\pi}{2}$. The graph of $y = \tan^{-1} x$ can be obtained by reflecting the restricted portion of the graph of $y = \tan x$ about the line $y = x$, as shown in Figure 11.

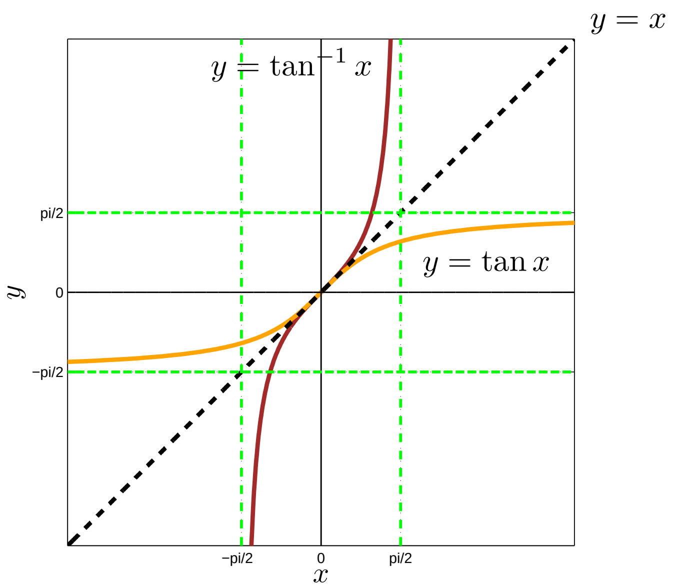


Figure 11: Graphs of $\tan x$ and $\tan^{-1} x$.

18. **Exercise 8 Finding the Exact Value of an Inverse Tangent Function**

Find the exact value of:

(a)$\tan^{-1} 1$ (b)$\tan^{-1}(-\sqrt{3})$

Solution:

(a) Let $\theta = \tan^{-1} 1$.

We seek the angle θ, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, whose tangent equals 1.

$$\theta = \tan^{-1} 1 \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$\tan\theta = 1 \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

Look at Table 3.

The only angle θ within the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent is 1 is $\frac{\pi}{4}$. Since $\tan\frac{\pi}{4} = 1$ and $\frac{\pi}{4}$ is in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we conclude that

$$\tan^{-1} 1 = \frac{\pi}{4}.$$

Table 3

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\tan\theta$	Undefined	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undefined

(b)

Let $\theta = \tan^{-1}(-\sqrt{3})$.

We seek the angle θ, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, whose tangent equals $-\sqrt{3}$.

$$\theta = \tan^{-1}(-\sqrt{3}) \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$\tan\theta = -\sqrt{3} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

Look at Table 3.

The only angle θ within the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent is $-\sqrt{3}$ is $-\frac{\pi}{3}$. Since $\tan\left(-\frac{\pi}{3}\right) = -\sqrt{3}$ and $-\frac{\pi}{3}$ is in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we conclude that

$$\tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}.$$

Table 3

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\tan\theta$	Undefined	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undefined

19. **Exercise 9 Finding the Inverse Function of a Trigonometric Function**

Find the inverse function f^{-1} of $f(x) = 2\sin x - 1$. State the domain and the range of f and f^{-1}.

Solution:

For finding the inverse function, we have

$$\begin{array}{rcll} y & = & 2\sin x - 1 & \\ x & = & 2\sin y - 1 & \textbf{Interchange } x \textbf{ and } y. \\ x+1 & = & 2\sin y & \textbf{Proceed to solve for } y. \\ \sin y & = & \frac{x+1}{2} & \\ y & = & \sin^{-1}\frac{x+1}{2} & \textbf{Apply Definition } (1). \end{array}$$

The inverse function is $f^{-1}(x) = \sin^{-1}\frac{x+1}{2}$.

[Properties of f and f^{-1}]: The domain of $f(x) = 2\sin x - 1$ is the set of all real numbers. Therefore, the range of f^{-1} is the set of all real numbers.

[Graph]: The graph of $f(x) = 2\sin x - 1$ can be obtained by vertically stretching $y = \sin x$ by a factor of 2 and then shifting the resulting graph down 1 unit. So the range of $f(x) = 2\sin x - 1$ is $[-3, 1]$. Therefore, the domain of f^{-1} is $[-3, 1]$.

We conclude also find the domain of f^{-1} by noting that the argument of the inverse sine function is $\frac{x+1}{2}$ and that it must lie in the interval $[-1, 1]$. That is,

$$\begin{array}{cl} -1 \le \frac{x+1}{2} \le 1 & \\ -2 \le x+1 \le 2 & \textbf{Multiply each part by 2.} \\ -3 \le x \le 1 & \textbf{Add } -1 \textbf{ to each part.} \end{array}$$

The domain of f^{-1} is $\{x \mid -3 \le x \le 1\}$.

20. **Exercise 10 Solving an Equation Involving an Inverse Trigonometric Function.** Solve the equation:

$$3\sin^{-1} x = \pi.$$

Solution:

To solve an equation involving a single inverse trigonometric function, first isolate the inverse trigonometric function.

$$\begin{array}{rcll} 3\sin^{-1} x & = & \pi & \\ \sin^{-1} x & = & \frac{\pi}{3} & \text{Divide both sides by 3.} \\ x & = & \sin\frac{\pi}{3} & y = \sin^{-1} x \text{ means } x = \sin y. \\ x & = & \frac{\sqrt{3}}{2} & \end{array}$$

The solution set is $\left\{\frac{\sqrt{3}}{2}\right\}$.

21. Know the Definitions of the Inverse Secant, Cosecant, and Cotangent Functions

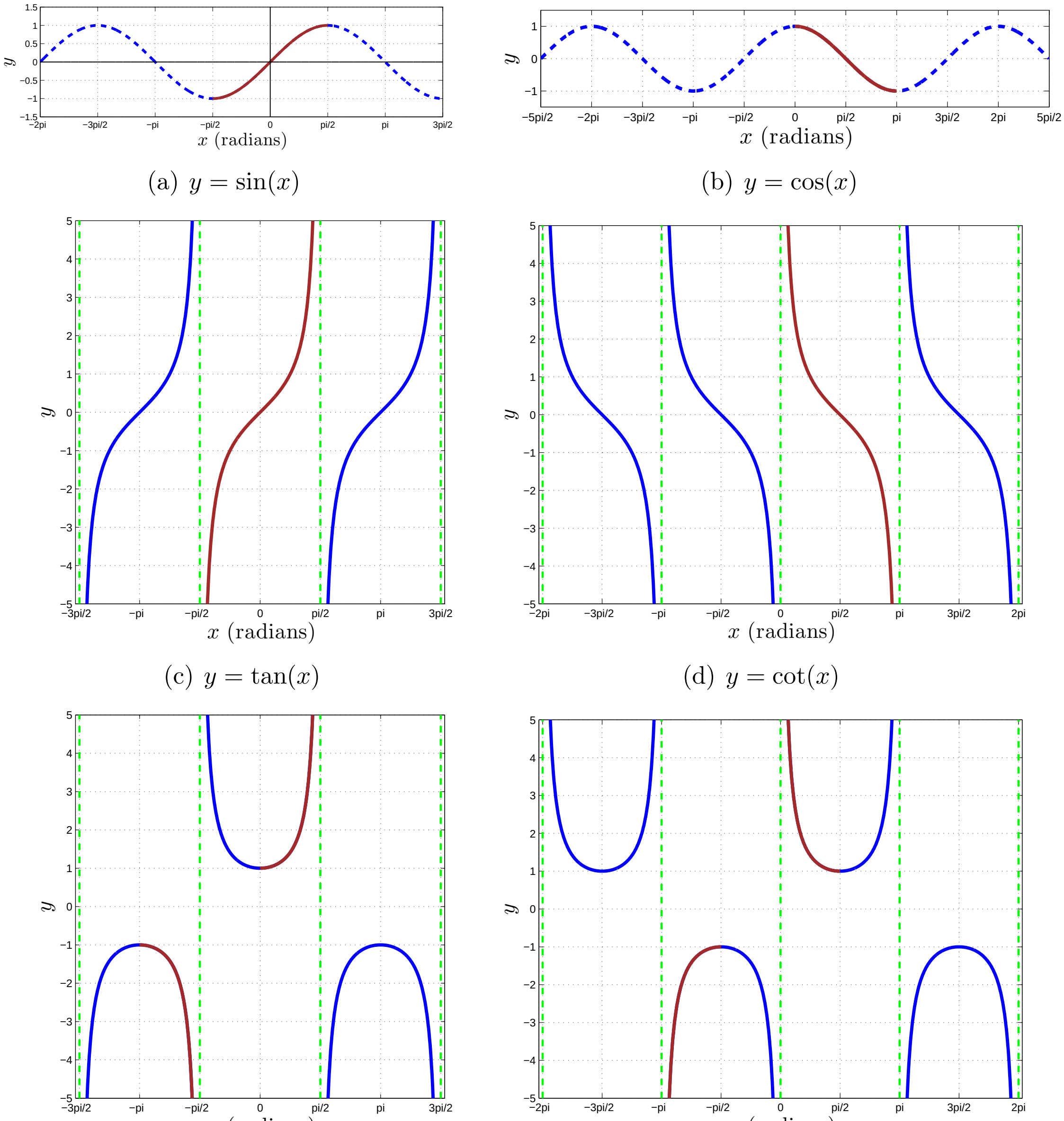
Definition 5 The inverse secant, inverse cosecant, and inverse cotangent functions are defined as follows:

$$y = \sec^{-1} x \quad \text{means} \quad x = \sec y \text{ where } \mid x \mid \geq 1 \text{ and } 0 \leq y \leq \pi,\ y \neq \frac{\pi}{2}. \tag{7}$$

$$y = \csc^{-1} x \quad \text{means} \quad x = \csc y \text{ where } \mid x \mid \geq 1 \text{ and } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0. \tag{8}$$

$$y = \cot^{-1} x \quad \text{means} \quad x = \cot y \text{ where } -\infty < x < \infty \text{ and } 0 < y < \pi. \tag{9}$$

See Figures 12, 13 and 14.



(a) $y = \sin(x)$

(b) $y = \cos(x)$

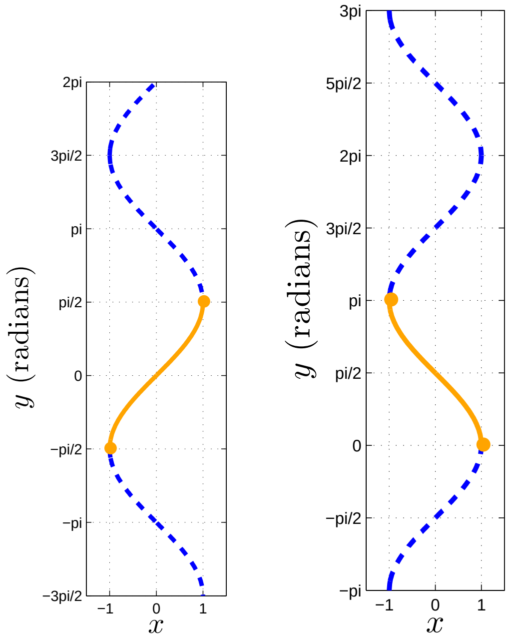
(c) $y = \tan(x)$

(d) $y = \cot(x)$

(e) $y = \sec(x)$

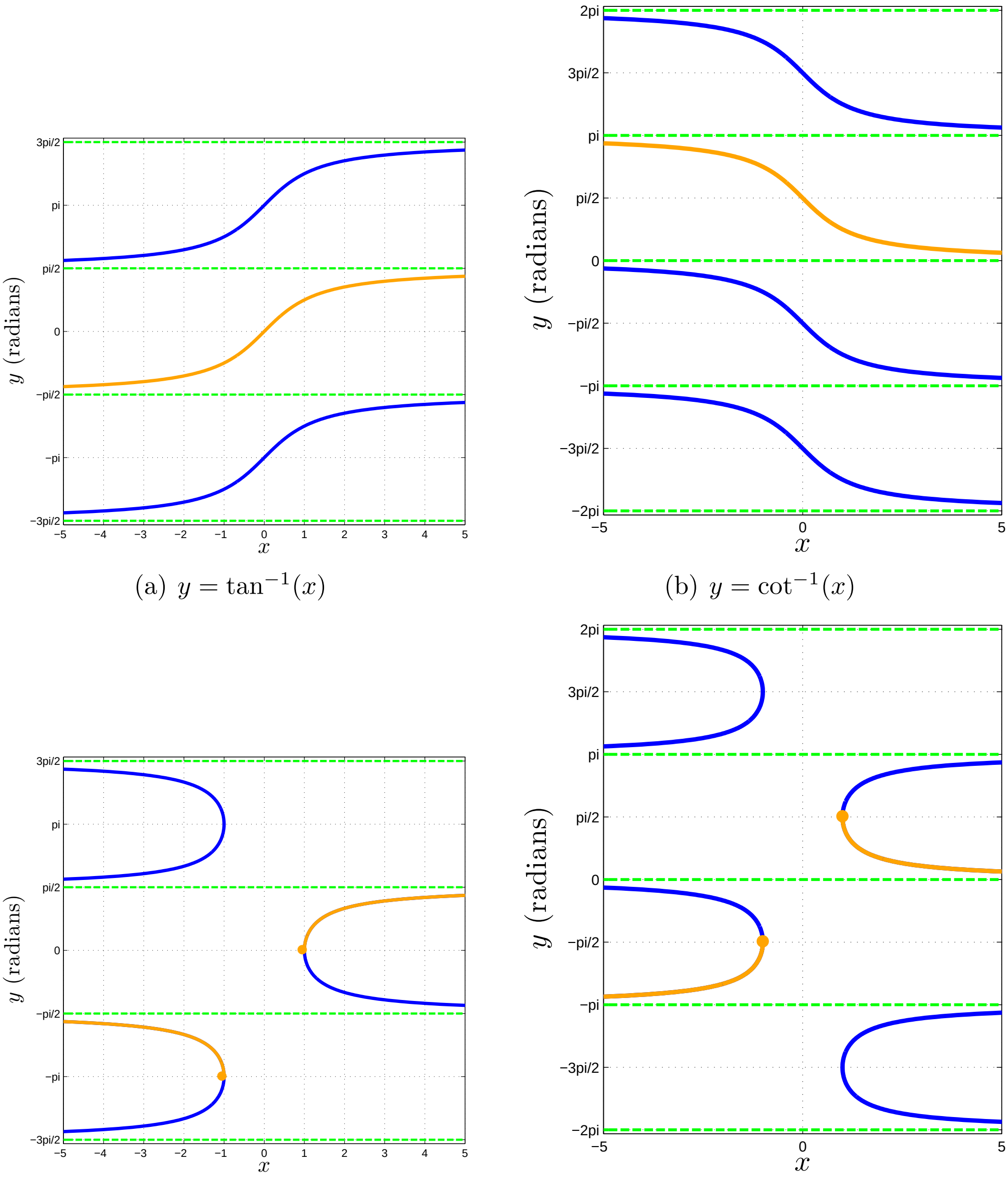
(f) $y = \mathrm{cosec}(x)$

Figure 12: Six trigonometric functions with restricted domains.



(a) $y = \sin^{-1}(x)$ (b) $y = \cos^{-1}(x)$

Figure 13:



(a) $y = \tan^{-1}(x)$

(b) $y = \cot^{-1}(x)$

(c) $y = \sec^{-1}(x)$

(d) $y = \mathrm{cosec}^{-1}(x)$

Figure 14:

Exercise 11 Finding the Exact Value of an Inverse Cosecant Function

Find the exact value of: $\csc^{-1} 2$.

Solution:

Let $\theta = \csc^{-1} 2$. We seek the angle $\theta, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \theta \neq 0$, whose cosecant equals 2 (or, equivalently,whose sine equals $\frac{1}{2}$).

$$\begin{array}{rcll} \theta & = & \csc^{-1} 2 & -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \theta \neq 0; \\ \csc\theta & = & 2 & -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \theta \neq 0 \quad \sin\theta = \frac{1}{2}. \end{array}$$

The only angle θ in the interval $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \theta \neq 0$, whose cosecant is 2 $\left[\sin\theta = \frac{1}{2}\right]$ is $\frac{\pi}{6}$, so $\csc^{-1} 2 = \frac{\pi}{6}$.

22. **Exercise 12 Writing a Trigonometric Expression as an Algebraic Expression**

Write $\sin(\tan^{-1} u)$ as an algebraic expression containing u.

Solution:

Let $\theta = \tan^{-1} u$ so that $\tan\theta = u, \;\; -\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}, \;\; -\infty < u < \infty.$

As a result, we know that $\sec\theta > 0$.

Then

$$\begin{aligned}\sin(\tan^{-1} u) &= \sin\theta = \sin\theta\cdot\frac{\cos\theta}{\cos\theta}\\ &= \tan\theta\cos\theta = \frac{\tan\theta}{\sec\theta}\\ &= \frac{\tan\theta}{\sqrt{1+\tan^2\theta}}\\ &= \frac{u}{\sqrt{1+u^2}}.\end{aligned}$$