CSCI2100B Data Structures Analysis

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Algorithm

- An **algorithm** is a clearly specified set of simple instructions to be followed to solve a problem.
  - How to estimate the time required for a program.
  - How to reduce the running time of a program from days or years to fractions of a second.
  - What is the storage complexity of the program.
  - How to deal with trade-offs.

Running Time

- There are two contradictory goals:
  - We would like an algorithm that is **easy** to understand, code, and debug.
  - We would like an algorithm that makes **efficient** use of the computer's resources, especially, one that runs as fast as possible.

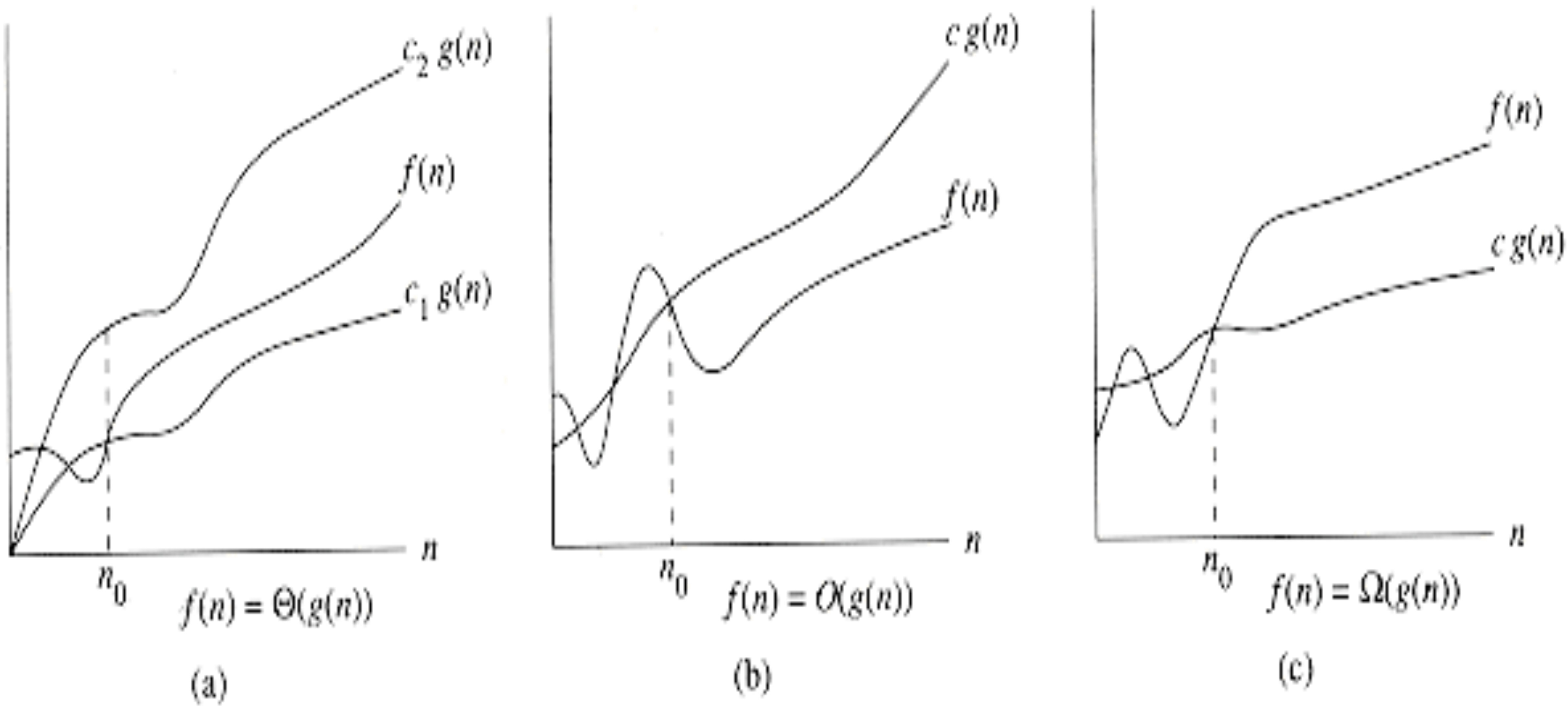
Function Comparison

- Given two functions, $f(N)$ and $g(N)$, what does it mean when we say that

  $$f(N) < g(N)?$$

  - Should this hold for all N?
  - We need to compare their relative rates of growth.

Example



http://science.slc.edu/~jmarshall/courses/2002/spring/cs50/BigO/index.html

Why Use Bounds

- The idea is to establish a **relative order** among functions.
- We are more concerned about the **relative rates of growth** of functions.
- For example, which function is greater, $1{,}000N$ or N^2?
- The turning point is $N = 1{,}000$ where N^2 will be greater for larger N.

First Definition

- It says that there is some point n_0 past which $c\,f(N)$ is always at least as large as $T(N)$.
- In our case, $T(N) = 1000N$, $f(N) = N^2$, $n_0 = 1{,}000$, and $c = 1$.
- We could also use $n_0 = 10$, and $c = 100$.
- So we can say that $1000N = O(N^2)$.
- It is an **upper bound** on $T(N)$.

Other Definitions

- The second definition says that the growth rate of $T(N)$ is greater than or equal to that of $g(N)$.
- The third definition says that the growth rate of $T(N)$ equals the growth rate of $h(N)$.
- The fourth definition says that the growth rate of $T(N)$ is less than the growth rate of $p(N)$.

Big-O Notation

- If $f(n)$ and $g(n)$ are functions defined for positive integers, then to write $f(n)$ is $O(g(n))$.
- $f(n)$ is big-O of $g(n)$ means that there exists a constant c such that $|f(x)| \leq c\,|g(n)|$ for all sufficiently large positive integers n.
- Under these conditions we also say that "$f(n)$ has **order** at most $g(n)$" or "$f(n)$ grows no more rapidly than $g(n)$".

Examples

- $f(n) = 100n$ then $f(n) = O(n)$.

- $f(n) = 4n + 200$ then $f(n) = O(n)$.

- $f(n) = n^2$ then $f(n) = O(n^2)$.

- $f(n) = 3\ n^2 - 100$ then $f(n) = O(n^2)$.

Rules

- If $T_1(N) = O(f(N))$ and $T_2(N) = O(g(N))$, then
  - $T_1(N) + T_2(N) = \max(O(f(N)), O(g(N)))$,
  - $T_1(N) * T_2(N) = O(\mathrm{f}(N)) * \mathrm{g}(N))$,
- If $T(\mathrm{N})$ is a polynomial of degree k, then $T(N) = (N^k)$.
- $\log^k N = O(N)$ for any constant k.
  - This tells us that logarithms grow very slowly.

Watch Out!

- It is bad to include constants or low-order terms inside a Big-Oh notation.
- Do not say $T(N) = O(2N^2)$ or $T(N) = O(N^2 + N)$.
- In both cases, $T(N) = O(N^2)$.

Observations

- If $f(n)$ is a polynomial in n with degree r, then $f(n)$ is $O(n^r)$, but $f(n)$ is not $O(n^s)$ for any power s less than r.

- Any logarithm of n grows more slowly (as n increases) than any positive power of n.
  - Hence log n is $O(n^k)$ for any $k > 0$, but n^k is never $O(\log n)$ for any power $k > 0$.

Common Orders

- $O(1)$ means computing time that is bounded by a constant (not dependent on n)
- $O(n)$ means that the time is directly proportional to n, and is called linear time.
- $O(n^2)$ means quadratic time.
- $O(n^3)$ means cubic time.
- $O(2^n)$ means exponential time.
- $O(\log n)$ means logarithmic time.
- $O(\log^2 n)$ means log-squared time.

Algorithm Analyses

- On a list of length n, sequential search has running time $O(n)$.
- On a ordered list of length n, binary search has running time $O(\log n)$.
- The sum of the sum of integer index of a loop from 1 to n is $O(n^2)$, i.e., $1 + 2 + 3 + \ldots + n$.
    - For $i = 1$ to n
        - For $j = i$ to n

Recurrence Relations

- Recurrence relations are useful in certain counting problems.
- A recurrence relation relates the *n*-th element of a sequence to its predecessors.
- Recurrence relations arise naturally in the analysis of recursive algorithms.

Sequences and Recurrence Relations

- A (numerical) sequence is an ordered list of number.
  - 2, 4, 6, 8, … (positive even numbers)
  - 0, 1, 1, 2, 3, 5, 8, … (the Fibonacci numbers)
  - 0, 1, 3, 6, 10, 15, … (numbers of key comparisons in selection sort)

Definitions

- A recurrence relation for the sequence $a_0, a_1, \ldots$ is an equation that relates a_n to certain of its predecessors $a_0, a_1, \ldots, a_{n-1}$.

- Initial conditions for the sequence $a_0, a_1, \ldots$ are explicitly given values for a finite number of the terms of the sequence.

Example

- A person invests \$1,000 at 12% compounded annually. If A_n represents the amount at the end of n years, find a recurrence relation and initial conditions that define the sequence A_n.

- At the end of n-1 years, the amount is A_{n-1}. After one more year, we will have the amount A_{n-1} plus the interest. Thus $A_n = A_{n-1} + (0.12)\,A_{n-1} = (1.12)\,A_{n-1}, n \geq 1$.

- To apply this recurrence relation for $n = 1$, we need to know the value of A_0 which is 1,000.

Solving Recurrence Relations

- Iteration - we use the recurrence relation to write the n-th term an in terms of certain of its predecessors a_{n-1}, …, a_0.
- We then successively use the recurrence relation to replace each of a_{n-1}, …by certain of their predecessors.
- We continue until an explicit formula is obtained.

Some Definitions of Linear Second-order recurrences with constant coefficients

- *k*-th-order
  - Elements $x(n)$ and $x(n\text{-}k)$ are k positions apart in the unknown sequence.
- Linear
  - It is a linear combination of the unknown terms of the sequence.
- Constant coefficients
  - The assumption that a, b, and c are some fixed numbers.
- Homogeneous
  - If $f(x) = 0$ for every n.

Solving Recurrence Relations

- Linear homogeneous recurrence relations with constant coefficients - a linear homogeneous recurrence relation of order k with constant coefficients is a recurrence relation of the form

$$a_0 = c_0, a_1 = c_1, \ldots, a_{k-1} = c_{k-1}$$

- Notice that a linear homogeneous recurrence relation of order K with constant coefficients, together with the k initial conditions

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}, c_k \geq 0$$

uniquely defines a sequence $a_0, a_1, \ldots$

Example

- Nonlinear

$$a_n = 3a_{n-1}a_{n-2}.$$

- Inhomogeneous

$$a_n - a_{n-1} = 2n.$$

- Homogeneous recurrence relation with nonconstant coefficients

$$a_n = 3n \cdot a_{n-1}.$$

Iteration Example

- We can solve the recurrence relation $a_n = a_{n-1} + 3$ subject to the initial condition $a_1 = 2$ by iteration.
- $a_{n-1} = a_{n-2} + 3$
- $a_n = a_{n-1} + 3 = a_{n-2} + 3 + 3 = a_{n-2} + 2 \times 3.$
- $a_{n-2} = a_{n-3} + 3.$
- $a_n = a_{n-2} + 2 \times 3 = a_{n-3} + 3 + 2 \times 3 = a_{n-3} + 3 \times 3.$
- $a_n = a_{n-k} + k \times 3 = 2 + 3(n-1).$

Iteration Example

- **In general, to solve $a_n = a_{n-1} + k, a_1 = c$, one obtains $a_n = c + k(n-1)$.**
- **We can solve the recurrence relation**
  - $a_n = ka_{n-1}, a_0 = c$.
  - $a_n = ka_{n-1} = k(ka_{n-2}) = \ldots = k^n a_0 = ck^n$.

Linear Homogeneous Recurrence Example

$$a_n = 5a_{n-1} - 6a_{n-2}, a_0 = 7, a_1 = 16$$

- Since the solution was of the form $a_n = t^n$, thus for our first attempt at finding a solution of the second-order recurrence relation, we will search for a solution of the form $a_n = t^n$.
  - $t^n = 5t^{n-1} - 6t^{n-2}$
  - $t^2 - 5t + 6 = 0$

Example

- **Solving the above we obtain,** $t = 2, t = 3$**.**
- **At this point, we have two solutions** S **and** T **given by**
  - $S_n = 2^n, T_n = 3^n$.
- **We can verify that is** S **and** T **are solutions of the above, then** $bS + dT$**, where** b **and** d **are any numbers whatever, is also a solution of the above.**

Example

- **In our case, if we define the sequence U by the equation**
  - $U_n = bS_n + dT_n$
  - $= b2^n + d3^n$
- **To satisfy the initial conditions, we must have**
  - $7 = U_0 = b2^0 + d3^0 = b + d.$
  - $16 = U_1 = b2^1 + d3^1 = 2b + 3d.$

Example

- **Solving these equations for b and d, we obtain**
  - $b = 5, d = 2$
- **Therefore, the sequence U defined by**
  - $U_n = 5 \times 2^n + 2 \times 3^n$

  satisfies the recurrence relation and the initial conditions.

Fibonacci Sequence

- **The Fibonacci sequence is defined by the recurrence relation**
  - $f_n = f_{n-1} + f_{n-2}, n \geq 3$ **and initial conditions**
  - $f_1 = 1, f_2 = 2$**.**
- **We begin by using the quadratic formula to solve**
  - $t^2 - t - 1 = 0$
- **The solutions are**
  - $t = \frac{1 \pm \sqrt{5}}{2}$

Example

- Thus the solution is of the form

$$f_n = b(\tfrac{1+\sqrt{5}}{2})^n + d(\tfrac{1-\sqrt{5}}{2})^n.$$

- To satisfy the initial conditions, we must have

$$b(\tfrac{1+\sqrt{5}}{2}) + d(\tfrac{1-\sqrt{5}}{2}) = 1,$$

$$b(\tfrac{1+\sqrt{5}}{2})^2 + d(\tfrac{1-\sqrt{5}}{2})^2 = 2.$$

Example

- Solving these equations for b and d, we obtain

$$b = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2}), d = -\frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2}).$$

- Therefore, an explicit formula for the Fibonacci sequence is

$$f_n = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^{n+1} - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^{n+1}.$$

Tower of Hanoi

- **Find an explicit formula for a_n, the minimum number of moves in which the n-disk Tower of Hanoi puzzle can be solved.**

- $a_n = 2a_{n-1} + 1, a_1 = 1.$

- **Applying the iterative method, we obtain**

$$\begin{aligned}
a_n &= 2a_{n-1} + 1 \\
&= 2(2a_{n-2} + 1) + 1 \\
&= 2^2 a_{n-2} + 2 + 1 \\
&= 2^2(2a_{n-3} + 1) + 2 + 1 \\
&= 2^3 a_{n-3} + 2^2 + 2 + 1 \\
&\ldots \\
&= 2^{n-1} a_1 + 2^{n-2} + 2^{n-3} + \ldots + 2 + 1 \\
&= 2^{n-1} + 2^{n-2} + 2^{n-3} + \ldots + 2 + 1 \\
&= 2^n - 1
\end{aligned}$$

Common Recurrence Types

- **Decrease-by-one**
  - $T(n) = T(n-1) + f(n)$
- **Decrease-by-a-constant-factor**
  - $T(n) = T(n/b) + f(n)$
- **Divide-and-conquer**
  - $T(n) = aT(n/b) + f(n)$