Exercises: Matrix Rank

Problem 1. Calculate the rank of the following matrix:

$$\begin{bmatrix} 0 & 16 & 8 & 4 \\ 2 & 4 & 8 & 16 \\ 16 & 8 & 4 & 2 \\ 4 & 8 & 16 & 2 \end{bmatrix}$$

Solution. To compute the rank of a matrix, remember two key points: (i) the rank does not change under elementary row operations; (ii) the rank of a row-echelon matrix is easy to acquire. Motivated by this, we convert the given matrix into row echelon form using elementary row operations:

$$\begin{aligned} \begin{bmatrix} 0 & 16 & 8 & 4 \\ 2 & 4 & 8 & 16 \\ 16 & 8 & 4 & 2 \\ 4 & 8 & 16 & 2 \end{bmatrix} &\Rightarrow \begin{bmatrix} 2 & 4 & 8 & 16 \\ 16 & 8 & 4 & 2 \\ 4 & 8 & 16 & 2 \\ 0 & 16 & 8 & 4 \end{bmatrix} \\ &\Rightarrow \begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & -24 & -60 & -126 \\ 0 & 0 & 0 & -30 \\ 0 & 4 & 2 & 1 \end{bmatrix} \\ &\Rightarrow \begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 4 & 2 & 1 \\ 0 & -24 & -60 & -126 \\ 0 & 0 & 0 & -30 \end{bmatrix} \\ &\Rightarrow \begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 4 & 2 & 1 \\ 0 & 0 & -48 & -120 \\ 0 & 0 & 0 & -30 \end{bmatrix} \end{aligned}$$

As this matrix has 4 non-zero rows, we conclude that the original matrix has rank 4.

Problem 2. Calculate the rank of the following matrix:

$$\begin{bmatrix} 4 & -6 & 0 \\ -6 & 0 & 1 \\ 0 & 9 & -1 \\ 0 & 1 & 4 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} 4 & -6 & 0 \\ -6 & 0 & 1 \\ 0 & 9 & -1 \\ 0 & 1 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -3 & 0 \\ -6 & 0 & 1 \\ 0 & 9 & -1 \\ 0 & 1 & 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & -3 & 0 \\ 0 & -9 & 1 \\ 0 & 9 & -1 \\ 0 & 1 & 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & -3 & 0 \\ 0 & -9 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 37/9 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & -3 & 0 \\ 0 & -9 & 1 \\ 0 & 0 & 37/9 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, the rank of the original matrix is 3.

Problem 3. Judge whether the following vectors are linearly independent.

$$[3, 0, 1, 2]$$
$$[6, 1, 0, 0]$$
$$[12, 1, 2, 4]$$
$$[6, 0, 2, 4]$$
$$[9, 0, 1, 2]$$

If they are not, find the largest number of linearly independent vectors among them.

Solution. This question is essentially asking for the rank of matrix:

$$\begin{bmatrix} 3 & 0 & 1 & 2 \\ 6 & 1 & 0 & 0 \\ 12 & 1 & 2 & 4 \\ 6 & 0 & 2 & 4 \\ 9 & 0 & 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 0 & 1 & 2 \\ 0 & 1 & -2 & -4 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 0 & 1 & 2 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & -2 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank of the matrix is 3. This means that the maximum number of linearly independent vectors is 3. They are the ones that correspond to the non-zero rows of the final matrix:

$$[3, 0, 1, 2]$$
$$[6, 1, 0, 0]$$
$$[9, 0, 1, 2]$$

Problem 4. Prove: if $\boldsymbol{A}$ is not square, then either the row vectors or the column vectors are linearly dependent.

Proof. The maximum number of linearly independent row vectors is the rank of $\boldsymbol{A}$, while the maximum number of linearly independent column vectors is the rank of $\boldsymbol{A}^T$. Suppose that $\boldsymbol{A}$ is an $m \times n$ matrix. If $m < n$, then $rank\, \boldsymbol{A}^T = rank\, \boldsymbol{A} \leq m < n$. Therefore, the column vectors are linear dependent. Similarly, if $n < m$, then the row vectors are linearly dependent. □

Problem 5. Let S be an arbitrary set of vectors in $\mathbb{R}^3$. Prove that there are at most 3 linearly independent vectors in S.

Proof. Let n be the number of vectors in S. For an $n \times 3$ matrix $\boldsymbol{A}$ where the i-th ($1 \leq i \leq n$) row is the i-th vector in S. Clearly, $rank\, \boldsymbol{A} = rank\, \boldsymbol{A}^T \leq 3$. Hence, S can have at most 3 linearly independent vectors. □

Problem 6 (Hard). Prove: $rank(\boldsymbol{AB}) \leq rank \boldsymbol{A}$.

Proof. Suppose that $\boldsymbol{A}$ is an $m \times n$ matrix, and $\boldsymbol{B}$ an $n \times p$ matrix. Let $d = rank \boldsymbol{A}$. Without loss of generality, assume that the first d rows of $\boldsymbol{A}$ are linearly independent. Denote the row vectors of $\boldsymbol{A}$ as $\boldsymbol{r}_1$, ..., $\boldsymbol{r}_m$ in top down order, and the column vectors of $\boldsymbol{B}$ as $\boldsymbol{c}_1, ..., \boldsymbol{c}_p$ in left-to-right order.

We will prove that for any $i \in [d+1, m]$, the i-th row of $\boldsymbol{AB}$ is a linear combination of the first d rows of $\boldsymbol{AB}$. This, in effect, shows that $rank(\boldsymbol{AB}) \leq d$.

We know that the first d rows of $\boldsymbol{AB}$ are:

$$\begin{aligned}
\boldsymbol{v}_1 &= [\boldsymbol{r}_1 \cdot \boldsymbol{c}_1, \boldsymbol{r}_1 \cdot \boldsymbol{c}_2, ..., \boldsymbol{r}_1 \cdot \boldsymbol{c}_p] \\
\boldsymbol{v}_2 &= [\boldsymbol{r}_2 \cdot \boldsymbol{c}_1, \boldsymbol{r}_2 \cdot \boldsymbol{c}_2, ..., \boldsymbol{r}_2 \cdot \boldsymbol{c}_p] \\
&\quad ... \\
\boldsymbol{v}_d &= [\boldsymbol{r}_d \cdot \boldsymbol{c}_1, \boldsymbol{r}_d \cdot \boldsymbol{c}_2, ..., \boldsymbol{r}_d \cdot \boldsymbol{c}_p]
\end{aligned}$$

while the i-th ($i \in [d+1, m]$) row of $\boldsymbol{AB}$ is:

$$\boldsymbol{v}_i = [\boldsymbol{r}_i \cdot \boldsymbol{c}_1, \boldsymbol{r}_i \cdot \boldsymbol{c}_2, ..., \boldsymbol{r}_i \cdot \boldsymbol{c}_p]$$

Since $\boldsymbol{r}_i$ is a linear combination of $\boldsymbol{r}_1, \boldsymbol{r}_2, ..., \boldsymbol{r}_d$, there exist real values $\alpha_1, ..., \alpha_d$ that (i) are not all zero, and (ii) satisfy:

$$\boldsymbol{r}_i = \sum_{z=1}^{d} \alpha_z \boldsymbol{r}_z$$

This means that for any $j \in [1, p]$, we have

$$\boldsymbol{r}_i \cdot \boldsymbol{c}_j = \sum_{z=1}^{d} \alpha_z (\boldsymbol{r}_z \cdot \boldsymbol{c}_j)$$

This, in turn, indicates that

$$\boldsymbol{v}_i = \sum_{z=1}^{d} \alpha_z \boldsymbol{v}_z$$

namely, $\boldsymbol{v}_i$ is a linear combination of $\boldsymbol{v}_1, ..., \boldsymbol{v}_d$. □

Problem 7 (Very Hard). Prove: $rank(\boldsymbol{A}+\boldsymbol{B}) \le rank\,\boldsymbol{A} + rank\,\boldsymbol{B}$.

Proof. Let $\boldsymbol{A}, \boldsymbol{B}$ be $m \times n$ matrices. Construct an $(2m) \times (2n)$ matrix:

$$\boldsymbol{Q} \quad = \quad \left[\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{0} \\ \hline \boldsymbol{0} & \boldsymbol{B} \end{array}\right]$$

$rank\,\boldsymbol{Q} = rank\,\boldsymbol{A} + rank\,\boldsymbol{B}$ (you can see this by converting $\boldsymbol{Q}$ into row-echelon form).

Also observe that $\boldsymbol{Q}$ has the same rank as

$$\left[\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{0} \\ \hline \boldsymbol{A} & \boldsymbol{B} \end{array}\right]$$

which has the same rank as

$$\left[\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{A} \\ \hline \boldsymbol{A} & \boldsymbol{A}+\boldsymbol{B} \end{array}\right]$$

Since the rank of a submatrix cannot exceed the rank of the whole matrix, we know that $rank\,(\boldsymbol{A}+\boldsymbol{B})$ is at most the rank of $\boldsymbol{Q}$, which as mentioned earlier is $rank\,\boldsymbol{A} + rank\,\boldsymbol{B}$. □