Exercises: Vector Basics

Problem 1. For each of the following directed segments, give the vector of which the directed segment is an instantiation:

1. $\overrightarrow{(1,2),(2,3)}$
2. $\overrightarrow{(10,20),(11,21)}$
3. $\overrightarrow{(1,-2),(2,3)}$
4. $\overrightarrow{(1,-2,0),(2,3,10)}$

Solution:

1. By definition of instantiation, we know that the vector is $[2-1, 3-2] = [1,1]$.
2. $[1,1]$
3. $[1,5]$
4. $[1,5,10]$

Problem 2. Give the default instantiations and the norms of the following vectors:

1. $[1,2]$
2. $[1,2,3]$
3. $[1,-2,3]$

Solution:

1. By definition, the default instantiation of $[1,2]$ is $\overrightarrow{(0,0),(1,2)}$; and the vector's norm is $\sqrt{1^2+2^2} = \sqrt{5}$.
2. Default instantiation $\overrightarrow{(0,0,0),(1,2,3)}$; norm:$\sqrt{14}$.
3. Default instantiation $\overrightarrow{(0,0,0),(1,-2,3)}$; norm:$\sqrt{14}$.

Problem 3. Give the results of $\boldsymbol{a}+\boldsymbol{b}$ and $\boldsymbol{a}-\boldsymbol{b}$ for each of the following:

1. $\boldsymbol{a} = [1,2], \boldsymbol{b} = [2,5]$
2. $\boldsymbol{a} = [1,2,3], \boldsymbol{b} = [2,5,-7]$
3. $\boldsymbol{a} = 10\boldsymbol{i} - 209\boldsymbol{j} + 32\boldsymbol{k}, \boldsymbol{b} = [2,5,-7]$

Solution:

1. By definition of the operators $+$ and $-$ on vectors, we have $\boldsymbol{a}+\boldsymbol{b} = [1+2, 2+5] = [3,7]$ and $\boldsymbol{a}-\boldsymbol{b} = [1-2, 2-5] = [-1,-3]$.

2. $\boldsymbol{a}+\boldsymbol{b}=[3,7,-4]$ and $\boldsymbol{a}-\boldsymbol{b}=[-1,-3,10]$.

3. $\boldsymbol{a}+\boldsymbol{b}=[12,-204,25]$ and $\boldsymbol{a}-\boldsymbol{b}=[8,-214,39]$. Note that $10\boldsymbol{i}-209\boldsymbol{j}+32\boldsymbol{k}=[10,-209,32]$ are equivalent (recall the definitions of $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$).

Problem 4. Give the results of $c\boldsymbol{a}$ for each of the following:

1. $\boldsymbol{a}=[1,2], c=5$

2. $\boldsymbol{a}=[1,2,3], c=-5$

3. $\boldsymbol{a}=10\boldsymbol{i}-209\boldsymbol{j}+32\boldsymbol{k}, c=10$

Solution:

1. By definition of the scalar-multiplication operator, $c\boldsymbol{a}=[5\cdot 1, 5\cdot 2]=[5,10]$.

2. $[-5,-10,-15]$.

3. $[100,-2090,320]$.

Problem 5. Indicate whether $\boldsymbol{a}$ and $\boldsymbol{b}$ have the same directions in each of the following cases:

1. $\boldsymbol{a}=[1,1], \boldsymbol{b}=[2,2]$

2. $\boldsymbol{a}=[1,2,3], \boldsymbol{b}=[20,40,60]$

3. $\boldsymbol{a}=[1,2,3], \boldsymbol{b}=[2,-4,6]$

Solution:

1. The direction of $\boldsymbol{a}$ is defined to be the ray emanating from the origin $(0,0)$ and passing the point $(1,1)$. Similarly, The direction of $\boldsymbol{b}$ is defined to be the ray emanating from the origin $(0,0)$ and passing the point $(2,2)$. These two rays are the same; namely, $\boldsymbol{a}$ and $\boldsymbol{b}$ have the same direction.

2. Same direction.

3. Different directions.

Problem 6. Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be 2d vectors such that $\boldsymbol{a}+\boldsymbol{b}=[3,5]$, and $\boldsymbol{a}-\boldsymbol{b}=[4,6]$. What are $\boldsymbol{a}$ and $\boldsymbol{b}$?

Solution 1: Suppose that $\boldsymbol{a}=[a_1,a_2]$ and $\boldsymbol{b}=[b_1,b_2]$. From $\boldsymbol{a}+\boldsymbol{b}=[3,5]$, we have $a_1+b_1=3$ and $a_2+b_2=5$, whereas from $\boldsymbol{a}-\boldsymbol{b}=[4,6]$, we have $a_1-b_1=4$ and $a_2-b_2=6$. Solving these equations gives $\boldsymbol{a}=[3.5,5.5]$ and $\boldsymbol{b}=[-0.5,-0.5]$.

Solution 2: From $\boldsymbol{a}+\boldsymbol{b}=[3,5]$ and $\boldsymbol{a}-\boldsymbol{b}=[4,6]$, we can directly obtain $(\boldsymbol{a}+\boldsymbol{b})+(\boldsymbol{a}-\boldsymbol{b})=2\boldsymbol{a}=[3,5]+[4,6]=[7,11]$. Hence, $\boldsymbol{a}=[3.5,5.5]$. Then, $\boldsymbol{b}=[3,5]-\boldsymbol{a}=[-0.5,-0.5]$.

Problem 7. Let $\boldsymbol{a}$ be a vector and c a scalar. Prove: $|c\boldsymbol{a}|=|c||\boldsymbol{a}|$.

Proof. Let $\boldsymbol{a} = [a_1, a_2, \ldots, a_d]$. Hence, $c\boldsymbol{a} = [ca_1, ca_2, \ldots, ca_d]$. Thus, $|c\boldsymbol{a}| = \sqrt{\sum_{i=1}^{d}(ca_i)^2} = \sqrt{c^2 \sum_{i=1}^{d}(a_i)^2} = \sqrt{c^2}\sqrt{\sum_{i=1}^{d}(a_i)^2} = |c||\boldsymbol{a}|$. ☐

Problem 8. Let A, B, C, D be 4 points in $\mathbb{R}^d$. Suppose that $\overrightarrow{A,B}$, $\overrightarrow{B,C}$, and $\overrightarrow{C,D}$ are instantiations of $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$, respectively; see Figure 1. Prove that $\overrightarrow{A,D}$ is an instantiation of $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}$.

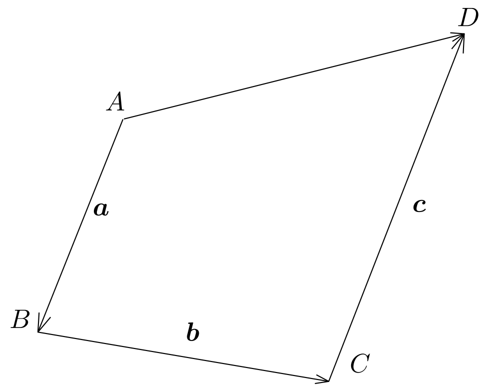

Figure 1: Problem 8

Proof. Consider the directed segment $\overrightarrow{A,C}$ as shown in Figure 2. By Lemma 2 in the notes of Lecture 1, $\overrightarrow{A,C}$ is an instantiation of $\boldsymbol{a}+\boldsymbol{b}$ (i.e., $\boldsymbol{d} = \boldsymbol{a}+\boldsymbol{b}$). By applying the lemma again, we obtain that $\overrightarrow{A,D}$ is an instantiation of $\boldsymbol{d}+\boldsymbol{c} = \boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}$. ☐

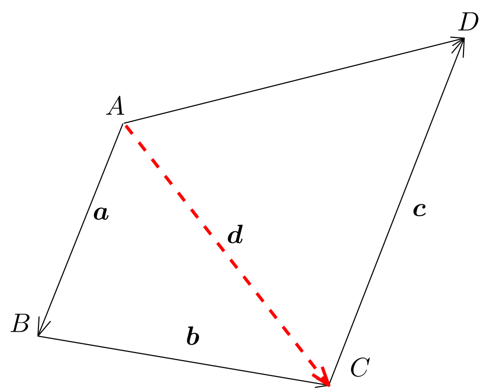

Figure 2: Proof of Problem 8

Problem 9. Let A, B, C, D be 4 points in $\mathbb{R}^d$. Suppose that $\overrightarrow{A,B}$, $\overrightarrow{C,B}$, and $\overrightarrow{C,D}$ are instantiations of $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$, respectively; see Figure 3. Give the vector of which $\overrightarrow{A,D}$ is an instantiation.

Solution: Consider the directed segment $\overrightarrow{A,C}$ as shown in Figure 4. By Lemma 2 in the notes of Lecture 1, $\overrightarrow{A,B}$ is an instantiation of $\boldsymbol{b}+\boldsymbol{c}$. This means that $\boldsymbol{a} = \boldsymbol{b}+\boldsymbol{d}$, leading to $\boldsymbol{d} = \boldsymbol{a}-\boldsymbol{b}$. By applying the lemma again, we obtain that $\overrightarrow{A,D}$ is an instantiation of $\boldsymbol{d}+\boldsymbol{c} = \boldsymbol{a}-\boldsymbol{b}+\boldsymbol{c}$.

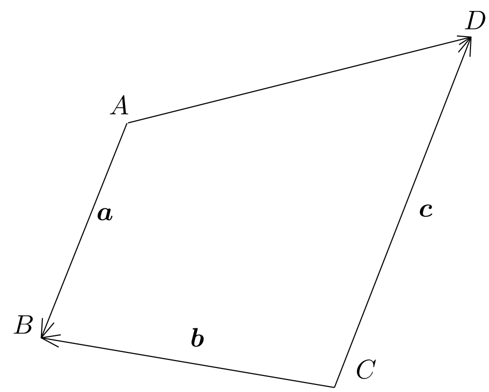

Figure 3: Problem 9

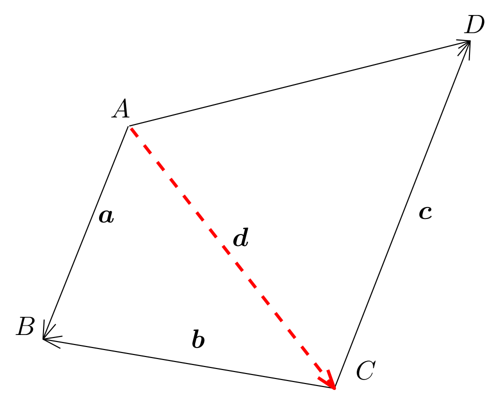

Figure 4: Solution to Problem 9