Lecture Notes: Dot Product and Cross Product

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1 Dot Product

Let us start by defining dot product:

Definition 1. *Given two vectors $\boldsymbol{a} = [a_1, ..., a_d]$ and $\boldsymbol{b} = [b_1, ..., b_d]$, we define $\boldsymbol{a} \cdot \boldsymbol{b}$, which is called the* **dot product** *of $\boldsymbol{a}$ and $\boldsymbol{b}$, as the real value $\sum_{i=1}^{d} a_i b_i$.*

$\boldsymbol{a} \cdot \boldsymbol{b}$ is also called the *inner product* of $\boldsymbol{a}$ and $\boldsymbol{b}$. Note that its value is a scalar, as opposed to a vector. It is easy to prove by definition the following properties:

- (Commutativity) $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$.
- (Distributivity) $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$, and $\boldsymbol{a} \cdot (\boldsymbol{b} - \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} - \boldsymbol{a} \cdot \boldsymbol{c}$.
- $|\boldsymbol{a}| = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}}$.

Next, we will prove another important but less trivial property of dot product. Let $\gamma \in [0°, 180°]$ be the angle between the directions of two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. Recall that the direction of a vector is a ray emanating from the origin of $\mathbb{R}^d$. Hence, γ is well defined.

Lemma 1. *If $\boldsymbol{a} \neq \boldsymbol{0}$ and $\boldsymbol{b} \neq \boldsymbol{0}$, then $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos \gamma$.*

Proof. Let $\overrightarrow{O, A}$ and $\overrightarrow{O, B}$ be the default instantiations of $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively; see Figure 1. We know that $\overrightarrow{A, B}$ is an instantiation of $\boldsymbol{b} - \boldsymbol{a}$. By the law of cosine, we have:

$$\begin{aligned}
|\overrightarrow{A, B}|^2 &= |\overrightarrow{O, A}|^2 + |\overrightarrow{O, B}|^2 - 2|\overrightarrow{O, A}||\overrightarrow{O, B}| \cos \gamma \quad \Rightarrow \\
\cos \gamma &= \frac{|\overrightarrow{O, A}|^2 + |\overrightarrow{O, B}|^2 - |\overrightarrow{A, B}|^2}{2|\overrightarrow{O, A}||\overrightarrow{O, B}|} \qquad (1)
\end{aligned}$$

From

$$\begin{aligned}
|\overrightarrow{O, A}|^2 &= |\boldsymbol{a}|^2 = \boldsymbol{a} \cdot \boldsymbol{a} \\
|\overrightarrow{O, B}|^2 &= |\boldsymbol{b}|^2 = \boldsymbol{b} \cdot \boldsymbol{b} \\
|\overrightarrow{A, B}|^2 &= |\boldsymbol{b} - \boldsymbol{a}|^2 = (\boldsymbol{b} - \boldsymbol{a}) \cdot (\boldsymbol{b} - \boldsymbol{a}) \\
\text{(by distributivity)} &= (\boldsymbol{b} - \boldsymbol{a}) \cdot \boldsymbol{b} - (\boldsymbol{b} - \boldsymbol{a}) \cdot \boldsymbol{a} \\
\text{(by distributivity)} &= \boldsymbol{b} \cdot \boldsymbol{b} - \boldsymbol{a} \cdot \boldsymbol{b} - \boldsymbol{b} \cdot \boldsymbol{a} + \boldsymbol{a} \cdot \boldsymbol{a} \\
\text{(by commutativity and def. of scalar multiplication)} &= \boldsymbol{b} \cdot \boldsymbol{b} - 2\boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{a}
\end{aligned}$$

we can derive from (1)

$$\cos \gamma = \frac{\boldsymbol{a} \cdot \boldsymbol{a} + \boldsymbol{b} \cdot \boldsymbol{b} - (\boldsymbol{b} \cdot \boldsymbol{b} - 2\boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{a})}{2|\boldsymbol{a}||\boldsymbol{b}|} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$$

thus completing the proof. $\square$

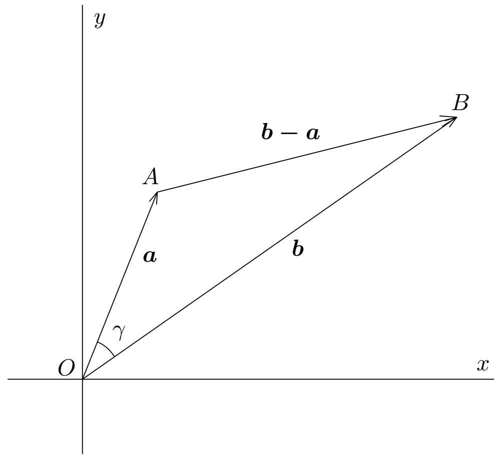
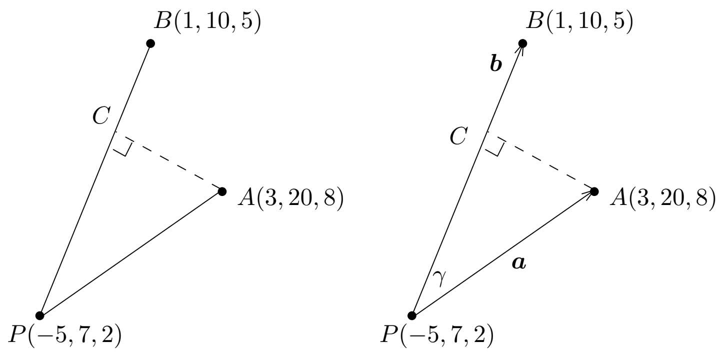


Figure 1: Proof of Lemma 1



(a) (b)

Figure 2: Using dot product to calculate projection lengths

We say that vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are *orthogonal* if the angle γ is $90°$. Lemma 1 leads to:

Corollary 1. *When $\boldsymbol{a} \neq \boldsymbol{0}$ and $\boldsymbol{b} \neq \boldsymbol{0}$, then $\boldsymbol{a} \cdot \boldsymbol{b} = 0$ if and only if $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal.*

It may come as a bit of surprise that we managed to prove such a non-trivial geometric fact without doing any calculation on the coordinates.

Dot Product and Projection Length. Let us now see an important use of dot product: computing the projection length of a line segment on a line. Figure 2a shows 3 points $P(-5,7,2)$ (where the numbers in the parentheses are coordinates of P), $A(3,20,8)$, and $B(1,10,5)$. Let $\overline{PA}$ denote the segment connecting P and A, and $|\overline{PA}|$ be the length of the segment. Project point A onto the line passing through segment $\overline{PC}$. We want to know how long is $\overline{PC}$, namely, the value of $|\overline{PC}|$.

Next we will give an easy solution to this problem. Let $\boldsymbol{a}$ be the vector of which the directed segment $\overrightarrow{P,A}$ is an instantiation, and $\vec{b}$ the vector of which the directed segment $\overrightarrow{P,B}$ is an instantiation; see Figure 2b. Apparently, $\boldsymbol{a} = [8,13,6]$ and $\boldsymbol{b} = [6,3,3]$. It thus follows that $\boldsymbol{a} \cdot \boldsymbol{b} = [8 \cdot 6 + 13 \cdot 3 + 6 \cdot 3] = 105$. On the other hand, from Lemma 1, we know that $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos \gamma$, where γ is the angle as shown in Figure 2b. As $|\boldsymbol{b}| = \sqrt{54}$, we know that

$$|\boldsymbol{a}|\sqrt{54}\cos\gamma = 105 \quad \Rightarrow$$
$$|\boldsymbol{a}|\cos\gamma = 105/\sqrt{54}.$$

Observe from Figure 2b $|\boldsymbol{a}|\cos\gamma$ is exactly the length of segment $\overline{PC}$.

2 Cross Product

Unlike dot product which is defined on vectors of arbitrary dimensionality d, cross product is defined only on 3d vectors:

Definition 2. *Given two 3d vectors $\boldsymbol{a} = [a_1, a_2, a_3]$ and $\boldsymbol{b} = [b_1, b_2, b_3]$, we define $\boldsymbol{a} \times \boldsymbol{b}$, which is called the* **cross product** *of $\boldsymbol{a}$ and $\boldsymbol{b}$, as the vector $\boldsymbol{c} = [c_1, c_2, c_3]$ where*

$$\begin{aligned} c_1 &= a_2 b_3 - a_3 b_2 \\ c_2 &= a_3 b_1 - a_1 b_3 \\ c_3 &= a_1 b_2 - a_2 b_1. \end{aligned}$$

The definition probably looks a bit complicated, but the next three lemmas will help us gain a geometric sense about what the cross product really is.

Lemma 2. *Let $\gamma \in [0°, 180°]$ be the angle between the directions of two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, and $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$. Then, $|\boldsymbol{c}| = |\boldsymbol{a}||\boldsymbol{b}|\sin\gamma$.*

Proof. See appendix. □

As an immediate corollary, we know that $\boldsymbol{c} = \boldsymbol{0}$ in each of the following scenarios:

- $\boldsymbol{a} = \boldsymbol{0}$ or $\boldsymbol{b} = \boldsymbol{0}$.
- The support lines of the directions of $\boldsymbol{a}$ and $\boldsymbol{b}$ are the same, namely, $\boldsymbol{a}$ and $\boldsymbol{b}$ have the same direction ($\gamma = 0°$) or opposite directions ($\gamma = 180°$).

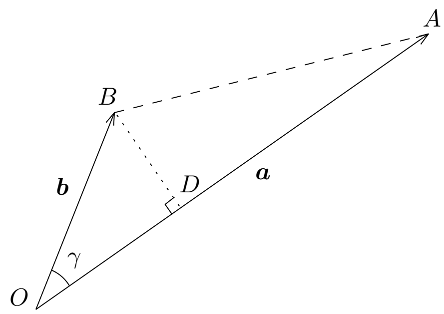
If $\boldsymbol{c} \neq \boldsymbol{0}$, its length $|\boldsymbol{c}|$ has a beautiful explanation. Let O be the origin; and let $\overrightarrow{O,A}$ and $\overrightarrow{O,B}$ be the default instantiations of $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively. Then, $|\boldsymbol{c}|$ is twice the area of the triangle OAB, as can be verified with simple geometry; see Figure 3a (note that the length of segment BD equals $|\boldsymbol{b}|\sin\gamma$).

Lemma 3. *Let $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$. Then, $\boldsymbol{a} \cdot \boldsymbol{c} = 0$ and $\boldsymbol{b} \cdot \boldsymbol{c} = 0$.*

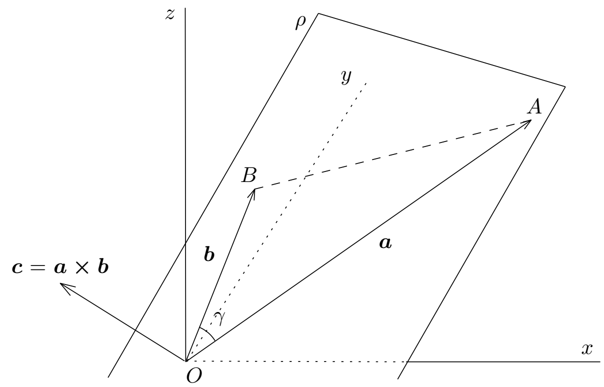
Proof. Let $\boldsymbol{a} = [a_1, a_2, a_3]$, $\boldsymbol{b} = [b_1, b_2, b_3]$, and $\boldsymbol{c} = [c_1, c_2, c_3]$. We will prove only $\boldsymbol{a} \cdot \boldsymbol{c} = 0$ because an analogous argument shows $\boldsymbol{b} \cdot \boldsymbol{c} = 0$.

$$\begin{aligned} \boldsymbol{a} \cdot \boldsymbol{c} &= a_1 c_1 + a_2 c_2 + a_3 c_3 \\ &= a_1(a_2 b_3 - a_3 b_2) + a_2(a_3 b_1 - a_1 b_3) + a_3(a_1 b_2 - a_2 b_1) \\ &= 0. \end{aligned}$$

□



(a) Length of $\boldsymbol{a} \times \boldsymbol{b}$



(b) Direction of $\boldsymbol{a} \times \boldsymbol{b}$

Figure 3: Illustration of cross product

The lemma leads to the following important corollary:

Corollary 2. *Let $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$. If $\boldsymbol{c} \neq \boldsymbol{0}$, then the direction of $\boldsymbol{c}$ is perpendicular to the plane determined by the default instantiations of $\boldsymbol{a}$ and $\boldsymbol{b}$ (see Figure 3b, where the plane is ρ).*

Proof. Since $\boldsymbol{c} \neq \boldsymbol{0}$, we know that (i) neither $\boldsymbol{a}$ nor $\boldsymbol{b}$ is $\boldsymbol{0}$, and (ii) the angle γ between the directions of $\boldsymbol{a}$ and $\boldsymbol{b}$ is larger than $0°$ but smaller than $180°$. Hence, the default instantiations of $\boldsymbol{a}$ and $\boldsymbol{b}$ uniquely determine a plane ρ.

Since $\boldsymbol{a} \cdot \boldsymbol{c} = 0$ and $\boldsymbol{b} \cdot \boldsymbol{c} = 0$, we know that the direction of $\boldsymbol{c}$ is orthogonal to the directions of both $\boldsymbol{a}$ and $\boldsymbol{b}$. Hence, the direction of $\boldsymbol{c}$ is perpendicular to ρ. □

We are almost ready to explain $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$ in a way much more intuitive than Definition 2. Recall that to unambiguously pinpoint a vector, we need to specify (i) its length, and (ii) its direction. Lemma 2 has given the length, and Corollary 1 has *almost* given its direction. Why did we say "almost"? Because there are two vectors that are perpendicular to the plane ρ in Figure 3b: besides the $\boldsymbol{c}$ shown, $-\boldsymbol{c}$ is also perpendicular to ρ.

We can remove this last piece of ambiguity as follows. Let us see the plane ρ from the side such that $\boldsymbol{c}$ shoots into our eyes. The direction of $\boldsymbol{a}$ should turn *counter-clockwise* to the direction of $\boldsymbol{b}$ by an angle less than $180°$ (i.e., γ in Figure 3b). Notice that if we see the plane ρ from the wrong side, then $\boldsymbol{a}$ needs to do so *clockwise* to reach $\boldsymbol{b}$. We will state the next lemma without proof.

Lemma 4. *The above approach of determining the direction of $\boldsymbol{c}$ is correct.*

At this point, we have obtained a complete geometric description about $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$. Interestingly, often times it suffices to hold only this view of $\boldsymbol{c}$, and turn to Definition 2 only when necessary.

It is easy to verify by definition the following properties of cross product:

- (Anti-Commutativity) $\boldsymbol{a} \times \boldsymbol{b} = -(\boldsymbol{b} \times \boldsymbol{a})$.
- (Distributivity) $\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = (\boldsymbol{a} \times \boldsymbol{b}) + (\boldsymbol{a} \times \boldsymbol{c})$, and $(\boldsymbol{b} + \boldsymbol{c}) \times \boldsymbol{a} = (\boldsymbol{b} \times \boldsymbol{a}) + (\boldsymbol{c} \times \boldsymbol{a})$.

Note that in general cross product does <u>*not*</u> necessarily obey associativity. Here is a counter example: $\boldsymbol{i} \times \boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{0} \times \boldsymbol{j} = \boldsymbol{0}$, but $\boldsymbol{i} \times (\boldsymbol{i} \times \boldsymbol{j}) = \boldsymbol{i} \times \boldsymbol{k} = -\boldsymbol{j}$.

Appendix

Proof of Lemma 2

Let $\boldsymbol{a} = [a_1, a_2, a_3]$, $\boldsymbol{b} = [b_1, b_2, b_3]$, and $\boldsymbol{c} = [c_1, c_2, c_3]$ (remember $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$). We will first establish another lemma which is interesting in its own right:

Lemma 5. $(|\boldsymbol{a}||\boldsymbol{b}|)^2 = |\boldsymbol{c}|^2 + (\boldsymbol{a} \cdot \boldsymbol{b})^2$.

Proof. We will take a bruteforce approach to prove the lemma, by representing all the quantities in the target equation with coordinates.

$$\begin{aligned}
(|\boldsymbol{a}||\boldsymbol{b}|)^2 &= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) \\
&= a_1^2 b_1^2 + a_1^2 b_2^2 + a_1^2 b_3^2 + a_2^2 b_1^2 + a_2^2 b_2^2 + a_2^2 b_3^2 + a_3^2 b_1^2 + a_3^2 b_2^2 + a_3^2 b_3^2 \\
|\boldsymbol{a} \times \boldsymbol{b}|^2 &= c_1^2 + c_2^2 + c_3^2 \\
&= (a_2 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2 + (a_1 b_2 - a_2 b_1)^2 \\
&= a_2^2 b_3^2 + a_3^2 b_2^2 + a_3^2 b_1^2 + a_1^2 b_3^2 + a_1^2 b_2^2 + a_2^2 b_1^2 - 2a_2 b_2 a_3 b_3 - 2a_1 b_1 a_3 b_3 - 2a_1 b_1 a_2 b_2 \\
(\boldsymbol{a} \cdot \boldsymbol{b})^2 &= (a_1 b_1 + a_2 b_2 + a_3 b_3)^2 \\
&= a_1^2 b_1^2 + a_2^2 b_2^2 + a_3^2 b_3^2 + 2a_1 b_1 a_2 b_2 + 2a_1 b_1 a_3 b_3 + 2a_2 b_2 a_3 b_3
\end{aligned}$$

The lemma thus follows. $\square$

Now we proceed to prove Lemma 2. From Lemma 1, we know that $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| cos\gamma$. Hence:

$$\begin{aligned}
(|\boldsymbol{a}||\boldsymbol{b}|)^2 - (\boldsymbol{a} \cdot \boldsymbol{b})^2 &= (|\boldsymbol{a}||\boldsymbol{b}|)^2 - (|\boldsymbol{a}||\boldsymbol{b}|)^2 \cos^2 \gamma \\
&= (|\boldsymbol{a}||\boldsymbol{b}|)^2 (1 - \cos^2 \gamma) \\
&= (|\boldsymbol{a}||\boldsymbol{b}|)^2 \sin^2 \gamma.
\end{aligned}$$

By combining the above with Lemma 5, we obtain:

$$|\boldsymbol{c}|^2 = (|\boldsymbol{a}||\boldsymbol{b}|)^2 \sin^2 \gamma.$$

Since $\sin \gamma \geq 0$ (recall that $\gamma \in [0^\circ, 180^\circ]$), it follows that $|\boldsymbol{c}| = |\boldsymbol{a}||\boldsymbol{b}| \sin \gamma$.