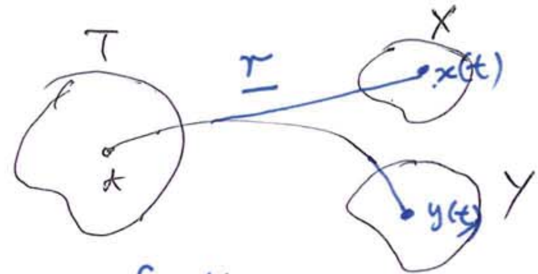
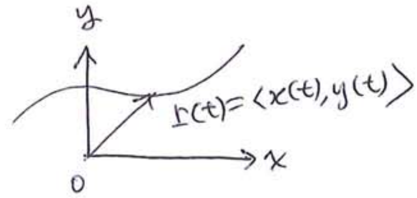
Chapter 11 Vector-valued functions

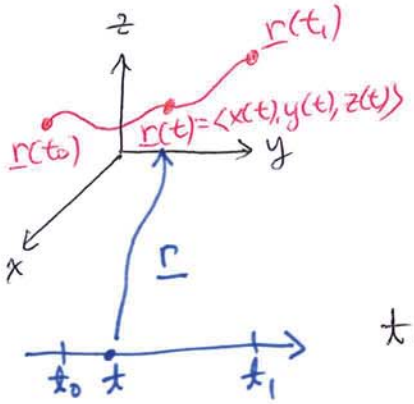
$y = f(x)$ curve in 2D

$z = f(\underline{r}) = f(x,y)$ surface in 3D

$\underline{r}(t) = \langle x(t), y(t), z(t) \rangle$ curve in 3D







$\underline{r}$: function

t: independent variable

$t \in I = [t_0, t_1]$

$\underline{r} : I \to \mathbb{R}^2$ or $\mathbb{R}^3$

$$\underline{r}(t) = \langle x(t), y(t) \rangle = x(t)\hat{\underline{i}} + y(t)\hat{\underline{j}} \quad (\mathbb{R}^2)$$

$$\underline{r}(t) = \langle x(t), y(t), z(t) \rangle = x(t)\hat{\underline{i}} + y(t)\hat{\underline{j}} + z(t)\hat{\underline{k}}$$

$x(t), y(t), z(t) \in \mathbb{R}$ $\quad (\mathbb{R}^3)$

Describe the following curves

① $\underline{r}(t) = t\hat{\underline{i}} + 2t\hat{\underline{j}}$

② $\underline{r}(t) = t\hat{\underline{i}} + t^2\hat{\underline{j}}$

③ $\underline{r}_1(t) = \sin t\,\hat{\underline{i}} + \cos t\,\hat{\underline{j}}$, $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

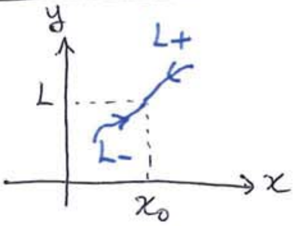
$\underline{r}_2(t) = t\hat{\underline{i}} + \sqrt{1-t^2}\,\hat{\underline{j}}$, $t \in [-1, 1]$

Are there any other parametrizations that represent a half circle?

NOTE: $\underline{r}(t)$ is called a closed curve if $\underline{r}(a) = \underline{r}(b)$ $\quad (a \le t \le b)$

"$\underline{r}(t_1) = \underline{r}(t_2) \Rightarrow t_1 = t_2$" → $\underline{r}(t)$ is a non-self intersecting curve, and $a \le t_1 < t_2 \le b$

Calculus of vector-valued functions



$y = f(x)$

Left Limit $\lim_{x \to x_0^-} f(x) = L_-$

Right Limit $\lim_{x \to x_+} f(x) = L_+$

$\lim_{x \to x_0} f(x)$ exists if $L_- = L_+$

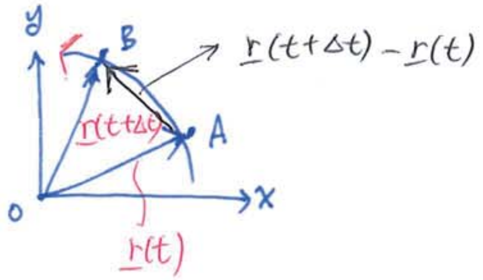
$f(x)$ is continuous at $x = x_0$ if $\lim_{x \to x_0} f(x) = f(x_0)$.

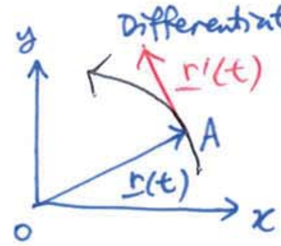
Let $\underline{r}(t) = x(t)\hat{\underline{i}} + y(t)\hat{\underline{j}} + z(t)\hat{\underline{k}}$

① $\lim_{t \to a} \underline{r}(t) = \lim_{t\to a} x(t)\hat{\underline{i}} + \lim_{t\to a} y(t)\hat{\underline{j}} + \lim_{t\to a} z(t)\hat{\underline{k}}$

② $\underline{r}(t)$ is continuous at $a \in I$ if $\lim_{t \to a} \underline{r}(t) = \underline{r}(a)$

③ $\underline{r}'(t) = \lim_{h \to 0} \dfrac{\underline{r}(t+h) - \underline{r}(t)}{h} = \underbrace{x'(t)\hat{\underline{i}} + y'(t)\hat{\underline{j}} + z'(t)\hat{\underline{k}}}_{\text{Differentiate its component by component}}$





If $\underline{r}'(t_0)$ exists, $\neq \underline{0}$, then $\underline{r}'(t_0)$ is tangent to the curve $\underline{r}(t)$ at t_0, and points in the direction of increasing t.

Rules $k \in \mathbb{R}$, $f(t)$: real-valued function, $\underline{c}$: constant vector

$\underline{r}_i(t)$: vector-valued functions, $g(\underline{r}(t))$: multivariable function.

① $(\underline{c})' = \underline{0}$

② $(k\underline{r}(t))' = k\underline{r}'(t)$

③ $(\underline{r}_1(t) \pm \underline{r}_2(t))' = \underline{r}_1'(t) \pm \underline{r}_2'(t)$

④ $(f(t)\underline{r}(t))' = f'(t)\underline{r}(t) + f(t)\underline{r}'(t)$

⑤ $(\underline{r}_1(t) \cdot \underline{r}_2(t))' = \underline{r}_1'(t) \cdot \underline{r}_2(t) + \underline{r}_1(t) \cdot \underline{r}_2'(t)$ — Product Rules

⑥ $(\underline{r}_1(t) \times \underline{r}_2(t))' = \underline{r}_1'(t) \times \underline{r}_2(t) + \underline{r}_1(t) \times \underline{r}_2'(t)$

⑦ $(\underline{r}(f(t)))' = \underline{r}'(f(t)) \cdot f'(t)$ → Chain Rule

⑧ $\int_a^b k\underline{r}(t)\,dt = k\int_a^b \underline{r}(t)\,dt$

⑨ $\int_a^b (\underline{r}_1(t) \pm \underline{r}_2(t))\,dt = \int_a^b \underline{r}_1(t)\,dt \pm \int_a^b \underline{r}_2(t)\,dt$

⑩ $\int_a^b \underline{r}(t)\,dt = \underline{R}(b) - \underline{R}(a)$, where $\dfrac{d\underline{R}}{dt} = \underline{r}(t)$ $\left(\dfrac{d}{dt}\int_a^t \underline{r}(s)\,ds = \underline{r}(t)\right)$

Fundamental Thm. of Calculus

<u>Higher order derivatives</u> $\quad \underline{r}''(t) = \frac{d}{dt}\underline{r}'(t)$

<u>Geometrical interpretation</u> $\quad$ Position $\underline{r}(t)$

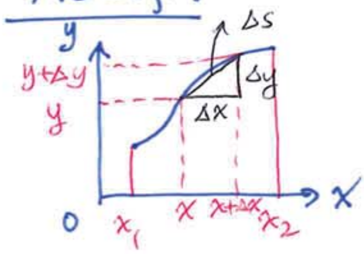
Velocity $\underline{r}'(t)$

Acceleration $\underline{r}''(t)$

<u>Example</u>: A particle has position vector $\underline{r}(t) = \cos t \sin t\, \hat{\underline{i}} + \sin^2 t\, \hat{\underline{j}} + \cos t\, \hat{\underline{k}}$ at time t.
Describe its path.

<u>Newton's Second Law of Motion</u> $\quad \underline{F}(t) = m\underline{a}(t)$

<u>Arc length</u>

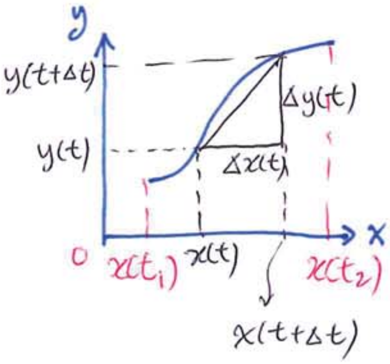


$$\Delta s = \sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x$$

$$s = \int_{x_1}^{x_2}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

<u>How?</u>

s: Total arc length of the curve from x_1 to x_2



$$\Delta s = \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2+\left(\frac{\Delta y}{\Delta t}\right)^2}\,\Delta t$$

$$s = \int_{t_1}^{t_2}\|\underline{r}'(t)\|\,dt$$

<u>How?</u>

s: Total arc length of the curve from $x(t_1)$ to $x(t_2)$

$$\frac{ds}{dt} = \|\underline{r}'(t)\|$$

If $\underline{r}(t)$ is a smooth vector-valued function and if s is an arc-length parameter, then $\left\|\frac{d\underline{r}}{ds}\right\| = 1 \quad \forall s$.

<u>Def</u>: If $\underline{r}(s)$ is a parametric curve such that $\|\underline{r}'(s)\| = 1\ \forall s$, we say the curve is parametrized by arc-length.

- Given $\underline{r}(t): [a,b] \to \mathbb{R}^3$, compute $s = \int_a^t \|\underline{r}'(\tau)\|\,d\tau$.
- Express t in terms of s, i.e. $t = t(s)$.
- Replace all t's by this function of s in the curve $\underline{r}(t)$.

The new parametrization $\underline{r}(s)$ will be arc-length parametrized.

Why $\|\underline{r}'(s)\| = 1$?

<u>Can you prove it?</u>

<u>Hint</u>: You need Chain rule!