Lecture Notes: Line (Curve) Integral by Coordinate

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We will introduce a convention. Suppose that $f(x_1, x_2, ..., x_d)$ is a scalar function with real-valued parameters. Given a point $p = (x_1, x_2, ..., x_d)$ in $\mathbb{R}^d$, we use $f(p)$ as a shortform for $f(x_1, x_2, ..., x_d)$.

1 Line Integral by One Coordinate

We have already discussed a form of line integrals—with respect to length s. Next, we will introduce a new type of line integrals, with respect to a single dimension:

Definition 1. *Let C be a smooth curve in $\mathbb{R}^d$ with a finite length from a starting point to an ending point. Break C into a sequence of n curves $C_1, C_2, ..., C_n$. Define ℓ to be the maximum length of $C_1, C_2, ..., C_n$. For each $i \in [1, n]$:*

- *choose an arbitrary point p_i on C_i*
- *denote by $\Delta_i = \beta_i - \alpha_i$ where α_i is the x_1-coordinate of the starting point of C_i, and β_i is the x_1-coordinate of the ending point of C_i.*

For a scalar function $f(x_1, x_2, ..., x_d)$, if the following limit exists:

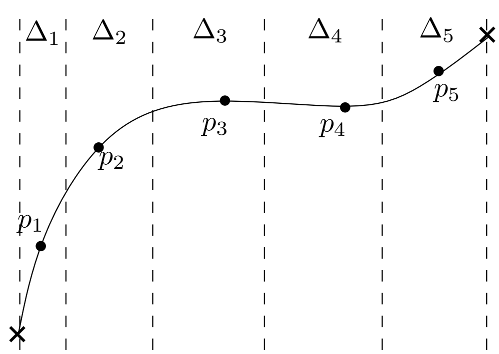
$$\lim_{\ell \to 0} \sum_{i=1}^{n} f(p_i) \cdot \Delta_i$$

then we define

$$\int_C f(x_1, ..., x_d)\, dx_1$$

to be the above limit.

The figure below illustrates the curve partitioning in the above definition for $n = 5$ where x_1 refers to the horizontal dimension:



Note that as ℓ tends to 0, n tends to ∞. We state the next intuitive lemma without proof:

Lemma 1. *Suppose that the curve C in Definition 1 is defined by $\boldsymbol{r}(t) = [x_1(t), x_2(t), ..., x_d(t)]$ with $t \in [t_1, t_2]$. When $f(x_1(t), x_2(t), ..., x_d(t))$ is continuous in $[t_1, t_2]$, it holds that*

$$\int_C f(x_1, ..., x_d)\, dx_1 = \int_{t_1}^{t_2} f(x_1(t), ..., x_d(t))\, \frac{dx_1}{dt} dt.$$

Example 1. Consider the circle $x^2 + y^2 = 1$. Let C be the arc from point $q_1 = (\sqrt{3}/2, 1/2)$ counterclockwise to point $q_2 = (1/2, \sqrt{3}/2)$. Calculate $\int_C \frac{1}{y}\, dx$.

Solution. The circle can be represented with $\boldsymbol{r}(t) = [x(t), y(t)]$ where $x(t) = \cos(t)$ and $y(t) = \sin(t)$. q_1 and q_2 correspond to $\boldsymbol{r}(\pi/6)$ and $\boldsymbol{r}(\pi/3)$, respectively. Hence, we have:

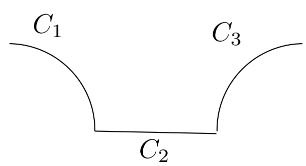
$$\begin{aligned}
\int_C \frac{1}{y}\, dx &= \int_{\pi/6}^{\pi/3} \frac{1}{y} \frac{dx}{dt} dt \\
&= \int_{\pi/6}^{\pi/3} \frac{1}{\sin(t)} \cdot (-\sin(t))\, dt \\
&= \int_{\pi/6}^{\pi/3} -1\, dt = -\pi/6.
\end{aligned}$$

□

Definition 1 requires that C should be smooth. Suppose that C is not a smooth curve, but can be broken into a *finite* number of smooth curves $C_1, C_2, ..., C_k$ (for some k). We say that C is *piecewise smooth.* For such a curve C, we define

$$\int_C f(x_1, ..., x_d)\, dx_1 = \sum_{i=1}^{k} \int_{C_i} f(x_1(t), ..., x_d(t))\, dx_1.$$

For example, in the figure below, let curve C be the concatenation of C_1, C_2 and C_3. C is not smooth, but is piecewise smooth.



2 Line Integral by All Coordinates

Suppose that we are given d scalar functions $f_1(x_1, ..., x_d), f_2(x_1, ..., x_d), ..., f_d(x_1, ..., x_d)$. Let C be a smooth curve in $\mathbb{R}^d$ from point p to point q. Also, let $\boldsymbol{r}(t) = [x_1(t), x_2(t), ..., x_d(t)]$ be a parametric form of C, such that p and q are given by $t = t_p$ and $t = t_q$, respectively.

From our earlier discussion, when all of $f_1(x_1(t), ..., x_d(t)), f_2(x_1(t), ..., x_d(t)), ..., f_d(x_1(t), ..., x_d(t))$ are continuous in $[t_p, t_q]$, it holds that

$$\begin{aligned}
&\int_C f_1(x_1, ..., x_d)\, dx_1 + \int_C f_2(x_2, , ..., x_d)\, dx_2 + ... + \int_C f_d(x_d, , ..., x_d)\, dx_d \\
= &\int_{t_p}^{t_q} \left(f_1(x_1(t), ..., x_d(t)) \frac{dx_1}{dt} + f_2(x_1(t), ..., x_d(t)) \frac{dx_2}{dt} + ... + f_d(x_1(t), ..., x_d(t)) \frac{dx_d}{dt} \right) dt.
\end{aligned} \tag{1}$$

Example 2. Consider the circle $x^2 + y^2 = 1$. Let C be the arc from $q_1 = (\sqrt{3}/2, 1/2)$ counter-clockwise to point $q_2 = (1/2, \sqrt{3}/2)$. Calculate

$$\int_C \frac{1}{y}\,dx + \int_C \frac{y}{x}\,dy.$$

Solution. The circle can be represented with $\boldsymbol{r}(t) = [x(t), y(t)]$ where $x(t) = \cos(t)$ and $y(t) = \sin(t)$. Points p and q correspond to $\boldsymbol{r}(\pi/6)$ and $\boldsymbol{r}(\pi/3)$, respectively. Hence, we have:

$$\begin{aligned}
\int_C \frac{1}{y}\,dx + \int_C \frac{y}{x}\,dy &= \int_{\pi/6}^{\pi/3} \frac{1}{y}\frac{dx}{dt}dt + \int_{\pi/6}^{\pi/3} \frac{y}{x}\frac{dy}{dt}dt \\
&= \int_{\pi/6}^{\pi/3} \frac{1}{\sin(t)} \cdot (-\sin(t))\,dt + \int_{\pi/6}^{\pi/3} \frac{\sin(t)}{\cos(t)} \cdot \cos(t)\,dt \\
&= \int_{\pi/6}^{\pi/3} -1\,dt + \int_{\pi/6}^{\pi/3} \sin(t)dt \\
&= -\frac{\pi}{6} + \frac{\sqrt{3}-1}{2}.
\end{aligned}$$

□

3 Vector Representation of Line Integrals

(1) shows the general form of line integral (by coordinate). In this section, we will introduce an alternative representation of (1).

First, recall that the curve is $\boldsymbol{r}(t) = [x_1(t), x_2(t), ..., x_d(t)]$. Let us define:

$$\begin{aligned}
d\boldsymbol{r} &= [dx_1, dx_2, ..., dx_d] \\
\frac{d\boldsymbol{r}}{dt} &= \boldsymbol{r}'(t) = [x_1'(t), x_2'(t), ..., x_d'(t)] \\
\boldsymbol{f}(\boldsymbol{r}) &= [f_1(x_1, ..., x_d), f_2(x_1, ..., x_d), ..., f_d(x_1, ..., x_d)].
\end{aligned}$$

We now define

$$\int_C \boldsymbol{f}(\boldsymbol{r}) \cdot d\boldsymbol{r} \tag{2}$$

as a shortform for the left hand side of (1) (note that the dot in between means *dot product*). It should be reiterated that we are *not* defining a new form of integrals. Instead, we are merely giving another way to express the same thing as the left hand side (1).

One benefit of the representation in (2) is that we can change the integral variable to t in a concise way. Specifically, let C be a curve from $t = t_1$ to $t = t_2$. Then, we have:

$$\begin{aligned}
\int_C \boldsymbol{f}(\boldsymbol{r}) \cdot d\boldsymbol{r} &= \int_{t_1}^{t_2} \boldsymbol{f}(\boldsymbol{r}(t)) \cdot \frac{d\boldsymbol{r}}{dt}dt \\
&= \int_{t_1}^{t_2} \boldsymbol{f}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)\,dt
\end{aligned}$$

which says precisely the same as (1).

Example 3. Consider the circle $x^2 + y^2 = 1$. Let C be the arc from $q_1 = (\sqrt{3}/2, 1/2)$ counter-clockwise to point $q_2 = (1/2, \sqrt{3}/2)$. Calculate

$$\int_C \frac{1}{y}\,dx + \int_C \frac{y}{x}\,dy.$$

Solution. The circle can be represented with $\boldsymbol{r}(t) = [x(t), y(t)]$ where $x(t) = \cos(t)$ and $y(t) = \sin(t)$. q_1 and q_2 correspond to $\boldsymbol{r}(\pi/6)$ and $\boldsymbol{r}(\pi/3)$, respectively. Clearly, $\boldsymbol{r}'(t) = [-\sin(t), \cos(t)]$. Define:

$$\boldsymbol{f}(\boldsymbol{r}) = \left[\frac{1}{y}, \frac{y}{x}\right].$$

We have:

$$\begin{aligned}
\int_C \frac{1}{y}\,dx + \int_C \frac{y}{x}\,dy &= \int_C \boldsymbol{f}(\boldsymbol{r}) \cdot d\boldsymbol{r} \\
&= \int_{\pi/6}^{\pi/3} \boldsymbol{f}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)\,dt. \\
&= \int_{\pi/6}^{\pi/3} \left[\frac{1}{\sin(t)}, \frac{\sin(t)}{\cos(t)}\right] \cdot [-\sin(t), \cos(t)]\ dt. \\
&= \int_{\pi/6}^{\pi/3} -1 + \sin(t)\,dt. \\
&= -\frac{\pi}{6} + \frac{\sqrt{3}-1}{2}.
\end{aligned}$$

□