THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1020
Exercise 15
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Let (h, k) be any point in the plane.

1. If a and b are real numbers with $a > b > 0$, then the graph of each of the following equations is an ellipse with center (h, k).

   (a)
   $$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$
   - major axis on the horizontal line $y = k$.
   - minor axis on the vertical line $x = h$.
   - vertices: $(h \pm a, k)$.
   - foci: $(h - c, k)$ and $(h + c, k)$, where $c = \sqrt{a^2 - b^2}$.

   (b)
   $$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$
   - major axis on the horizontal line $x = h$.
   - minor axis on the vertical line $y = k$.
   - vertices: $(h, k \pm a)$.
   - foci: $(h, k - c)$ and $(h, k + c)$, where $c = \sqrt{a^2 - b^2}$.

2. If a and b are positive real numbers, then the graph of each of the following equations is a hyperbola with center (h, k).

   (a)
   $$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$
   - focal axis on the horizontal line $y = k$.
   - vertices: $(h - a, k)$ and $(h + a, k)$.
   - foci: $(h - c, k)$ and $(h + c, k)$, where $c = \sqrt{a^2 + b^2}$.
   - asymptotes: $y = \pm\frac{b}{a}(x - h) + k$.

(b)

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

- focal axis on the vertical line $x = h$.
- vertices: $(h, k-c)$ and $(h, k+c)$.
- foci: $(h, k-c)$ and $(h, k+c)$, where $c = \sqrt{a^2+b^2}$.
- asymptotes: $y = \pm\frac{a}{b}(x-h)+k$.

3. If p are non-zero real numbers, then the graph of each of the following equations is a parabola with center (h, k).

(a)

$$(x-h)^2 = 4p(y-k)$$

- focus: $(h, k+p)$.
- directrix: horizontal line $y = k-p$.
- axis: the vertical line $x = h$.
- opens upward if $p > 0$, downward if $p < 0$.

(b)

$$(y-k)^2 = 4p(x-h)$$

- focus: $(h+p, k)$.
- directrix: vertical line $x = h-p$.
- axis: the horizontal line $y = k$.
- opens to right if $p > 0$, to left if $p < 0$.

Exercise 1 Graph the parabola

$$y^2 + 2y + 8x + 17 = 0$$

and specify its vertex, focus, directrix, and axis of symmetry.

Solution: We have

$$\begin{array}{ll}
y^2 + 2y + 8x + 17 = 0 & \\
y^2 + 2y = -8x - 17 & \text{(separate } x- \text{ and } y- \text{terms)} \\
y^2 + 2y + 1 = -8x - 17 + 1 = -8x - 16 & \text{(complete the squared term)} \\
(y+1)^2 = -8(x+2) & \text{(factor both sides)}
\end{array}$$

Therefore, the vertex is $(-2, -1)$. Since the $y-$term is squared and p is negative, the graph of the parabola opens left.

The focus is a distance p from the vertex. Since

$$\begin{aligned} 4p &= 8, \\ p &= 2 \end{aligned}$$

and the focus is $(-4, -1)$. The directrix is a distance p from the vertex on the opposite side from the focus, i.e., $x = 0$. The axis of symmetry is the horizontal line passing through the vertex

$$y = -1.$$

We leave it to the reader to graph the equation $(y + 1)^2 = -8(x + 2)$.

Exercise 2 Given the focus $(3, 1)$ and the directrix $y = -3$, find the equation of the parabola.

Solution:

To find the standard form, we need to find h, k, and p; and we need to determine which of the six standard forms to use.

The vertex lies halfway between the vertex and the focus or at the point $(3, -1)$. So $h = 3$ and $k = -1$.

To find p, we find the distance between the focus and the vertex, here 2.

A quick sketch of the directrix and the vertex will determine which standard form to use. Since the directrix is horizontal and lies below the vertex, the parabola curves up; and the correct standard form is

$$(x - h)^2 = 4p(y - k).$$

Therefore, we have

$$(x - 3)^2 = 4(2)(y + 1) = 8(y + 1).$$

We leave it to the reader to graph the equation $(x - 3)^2 = 8(y + 1)$.