THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1020
Exercise 13
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(Properties of the Cross Product)

(i) $\mathbf{a}\times\mathbf{b} = \mathbf{0}$ if $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$

(ii) $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$

(iii) $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = (\mathbf{a}\times\mathbf{b}) + (\mathbf{a}\times\mathbf{c})$ left distributive law

(iv) $(\mathbf{a}+\mathbf{b})\times\mathbf{c} = (\mathbf{a}\times\mathbf{c}) + (\mathbf{b}\times\mathbf{c})$ right distributive law

(v) $\mathbf{a}\times(k\mathbf{b}) = (k\mathbf{a})\times\mathbf{b} = k(\mathbf{a}\times\mathbf{b}), k$ a scalar

(vi) $\mathbf{a}\times\mathbf{a} = \mathbf{0}$

(vii) $\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{0}$

(viii) $\mathbf{b}\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{0}$

Exercise 1 Show that $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = (\mathbf{a}\times\mathbf{b}) + (\mathbf{a}\times\mathbf{c})$.

Solution: Let $\mathbf{a} = a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$ and $\mathbf{c} = c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}$. Then

$$\begin{aligned}
\mathbf{a}\times(\mathbf{b}+\mathbf{c}) &= \begin{vmatrix} a_2 & a_3 \\ b_2+c_2 & b_3+c_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1+c_1 & b_3+c_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1+c_1 & b_2+c_2 \end{vmatrix}\mathbf{k} \\
&= [(a_2b_3+a_2c_3)-(a_3b_2+a_3c_2)]\mathbf{i} - [(a_1b_3+a_1c_3)-(a_3b_1+a_3c_1)]\mathbf{j} \\
&\quad + [(a_1b_2+a_1c_2)-(a_2b_1+a_2c_1)]\mathbf{k} \\
&= [(a_2b_3-a_3b_2)\mathbf{i} - (a_1b_3-a_3b_1)\mathbf{j} + (a_1b_2-a_2b_1)\mathbf{k}] \\
&\quad + [(a_2c_3-a_3c_2)]\mathbf{i} - (a_1c_3-a_3c_1)\mathbf{j} + (a_1c_2-a_2c_1)\mathbf{k}] \\
&= (\mathbf{a}\times\mathbf{b}) + (\mathbf{a}\times\mathbf{c}).
\end{aligned}$$

Exercise 2 Show that $(\mathbf{a}+\mathbf{b})\times\mathbf{c} = (\mathbf{a}\times\mathbf{c}) + (\mathbf{b}\times\mathbf{c})$.

Solution: We have

$$\begin{aligned}
(\mathbf{a}+\mathbf{b})\times\mathbf{c} &= -\mathbf{c}\times(\mathbf{a}+\mathbf{b}) && \text{by property (ii)} \\
&= (-\mathbf{c}\times\mathbf{c}) + (-\mathbf{c}\times\mathbf{b}) && \text{by property (iii)} \\
&= (\mathbf{a}\times\mathbf{c}) + (\mathbf{b}\times\mathbf{c}) && \text{by property (ii)} \\
&= (\mathbf{a}\times\mathbf{b}) + (\mathbf{a}\times\mathbf{c}).
\end{aligned}$$

Exercise 3 Find the area of the triangle determined by the points $P_1(1,1,1)$, $P_2(2,3,4)$, and $P_3(3,0,-1)$.

Solution: The vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_2P_3}$ can be taken as two sides of the triangle. Since

$$\overrightarrow{P_1P_2} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \text{ and } \overrightarrow{P_2P_3} = \mathbf{i} - 3\mathbf{j} - 5\mathbf{k}$$

we have

$$\begin{aligned}
\overrightarrow{P_1P_2} \times \overrightarrow{P_2P_3} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 1 & -3 & -5 \end{vmatrix} \\
&= \begin{vmatrix} 2 & 3 \\ -3 & -5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 3 \\ 1 & -5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ 1 & -3 \end{vmatrix} \mathbf{k} \\
&= -\mathbf{i} + 8\mathbf{j} - 5\mathbf{k}.
\end{aligned}$$

We see that the area is

$$A = \frac{1}{2}|-\mathbf{i} + 8\mathbf{j} - 5\mathbf{k}| = \frac{3}{2}\sqrt{10}.$$