THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1020
Exercise 8
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1. **Trigonometric Identities:**

   Before establishing some additional identities, let's review the definition of an identity.

   Definition 1 Two functions f and g are said to be **identically equal** if

   $$f(x) = g(x)$$

   for every value of x for which both functions are defined.Such an equation is referred to as an **identity.** An equation that is not an identity is called a **conditional equation.**

2. **Example 1** The following are identities:

$$(x+1)^2 = x^2 + 2x + 1$$

$$\sin^2 x + \cos^2 x = 1$$

$$\csc x = \frac{1}{\sin x}$$

3. **Example 2** The following are conditional equations:

$$2x + 5 = 0 \quad \text{True} \quad \text{only} \quad \text{if} \quad x = -\frac{5}{2}$$

$$\sin x = 0 \quad \text{True} \quad \text{only} \quad \text{if} \quad x = k\pi, \ k \quad \text{an} \quad \text{integer}$$

$$\sin x = \cos x \quad \text{True} \quad \text{only} \quad \text{if} \quad x = \frac{\pi}{4} + 2k\pi \quad \text{or} \quad x = \frac{5\pi}{4} + 2k\pi, \ \text{k} \quad \text{an} \quad \text{integer}$$

4. A summary of the **basic trigonometric identities**

- **Quotient Identities**

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad \cot\theta = \frac{\cos\theta}{\sin\theta}.$$

- **Reciprocal Identities**

$$\csc\theta = \frac{1}{\sin\theta} \quad \sec\theta = \frac{1}{\cos\theta} \quad \cot\theta = \frac{1}{\tan\theta}.$$

- **Pythagorean Identities**

$$\sin^2\theta + \cos^2\theta = 1 \quad \tan^2\theta + 1 = \sec^2\theta;$$
$$\cot^2\theta + 1 = \csc^2\theta.$$

- **Even-Odd Identities**

$$\sin(-\theta) = -\sin\theta \quad \cos(-\theta) = \cos\theta \quad \tan(-\theta) = -\tan\theta;$$
$$\csc(-\theta) = -\csc\theta \quad \sec(-\theta) = \sec\theta \quad \cot(-\theta) = -\cot\theta.$$

5. **Use Algebra to Simplify Trigonometric Expressions**

Exercise 1 Using Algebraic Techniques to Simplify Trigonometric Expressions

(a) Simplify $\frac{\cot\theta}{\csc\theta}$ by rewriting each trigonometric function in terms of sine and cosine functions.

(b) Show that $\frac{\cos\theta}{1+\sin\theta} = \frac{1-\sin\theta}{\cos\theta}$ by multiplying the numerator and denominator by $1-\sin\theta$.

(c) Simplify $\frac{1+\sin u}{\sin u} + \frac{\cot u - \cos u}{\cos u}$ by rewriting the expression over a common denominator.

(d) Simplify $\frac{\sin^2 v - 1}{\tan v \sin v - \tan v}$ by factoring.

Solution:

(a)

$$\frac{\cot\theta}{\csc\theta} = \frac{\frac{\cos\theta}{\sin\theta}}{\frac{1}{\sin\theta}} = \frac{\cos\theta}{\sin\theta} \cdot \frac{\sin\theta}{1} = \cos\theta.$$

(b)

$$\frac{\cos\theta}{1+\sin\theta} = \frac{\cos\theta}{1+\sin\theta} \cdot \frac{1-\sin\theta}{1-\sin\theta} = \frac{\cos\theta(1-\sin\theta)}{1-\sin^2\theta} = \frac{\cos\theta(1-\sin\theta)}{\cos^2\theta} = \frac{1-\sin\theta}{\cos\theta}.$$

(c)

$$\begin{aligned}\frac{1+\sin u}{\sin u}+\frac{\cot u-\cos u}{\cos u} &= \frac{1+\sin u}{\sin u}\frac{\cos u}{\cos u}+\frac{\cot u-\cos u}{\cos u}\frac{\sin u}{\sin u}\\ &= \frac{\cos u+\sin u\cos u+\cot u\sin u-\cos u\sin u}{\sin u\cos u}=\frac{\cos u+\frac{\cos u}{\sin u}\cdot\sin u}{\sin u\cos u}\\ &= \frac{\cos u+\sin u}{\sin u\cos u}=\frac{2\cos u}{\sin u\cos u}=\frac{2}{\sin u}.\end{aligned}$$

(d)

$$\frac{\sin^2 v-1}{\tan v\sin v-\tan v}=\frac{(\sin v+1)(\sin v-1)}{\tan v(\sin v-1)}=\frac{\sin v+1}{\tan v}.$$

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6. **Establish Identities**

Exercise 2 Establishing the identity: $\csc\theta\cdot\tan\theta=\sec\theta$.

Solution:

We start with the left side, because it contains the more complicated expression, and apply a reciprocal identity and a quotient identity.

$$\csc\theta\cdot\tan\theta=\frac{1}{\sin\theta}\cdot\frac{\sin\theta}{\cos\theta}=\frac{1}{\cos\theta}=\sec\theta.$$

Having arrived at the right side, the identity is established.

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Exercise 3 Establishing the identity: $\sin^2(-\theta)+\cos^2(-\theta)=1$.

Solution:

We begin with the left side and, because the arguments are $-\theta$, apply Even-Odd Identities.

$$\begin{aligned}\sin^2(-\theta)+\cos^2(-\theta) &= [\sin(-\theta)]^2+[\cos(-\theta)]^2 & \\ &= (-\sin\theta)^2+(\cos\theta)^2 & \text{Even-Odd Identities}\\ &= (\sin\theta)^2+(\cos\theta)^2 & \\ &= 1 & \text{Pythagorean Identity}\end{aligned}$$

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Exercise 4 Establish the identity: $\dfrac{\sin^2(-\theta)-\cos^2(-\theta)}{\sin(-\theta)-\cos(-\theta)}=\cos\theta-\sin\theta$.

Solution:

Two observations are: The left side contains the more complicated expression. Also, the left side contains expressions with the argument $-\theta$, whereas the right side contains expressions with the argument θ. Starting with the left side and applying Even-Odd Identities, we have

$$\begin{aligned}
\frac{\sin^2(-\theta)-\cos^2(-\theta)}{\sin(-\theta)-cos(-\theta)} &= \frac{[\sin(-\theta)]^2-[\cos(-\theta)]^2}{\sin(-\theta)-\cos(-\theta)} \\
&= \frac{(-\sin\theta)^2-(\cos\theta)^2}{-\sin\theta-\cos\theta} && \text{Even-Odd Identites} \\
&= \frac{(\sin\theta-\cos\theta)(\sin\theta+\cos\theta)}{-(\sin\theta+\cos\theta)} && \text{Simplfy} \\
&= \frac{(-\sin\theta)^2-(\cos\theta)^2}{-\sin\theta-\cos\theta} && \text{Factor} \\
&= \frac{\sin\theta-\cos\theta}{-1} = -(\sin\theta-\cos) = -\sin\theta+\cos\theta \\
&= \cos\theta-\sin\theta && \text{Cancel simplify.}
\end{aligned}$$

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Exercise 5 Establish the identity: $\dfrac{1+\tan u}{1+\cot u}=\tan u$.

Solution:

$$\begin{aligned}
\frac{1+\tan u}{1+\cot u} &= \frac{1+\tan u}{1+\frac{1}{\tan u}} = \frac{1+\tan u}{\frac{\tan u+1}{\tan u}} \\
&= \frac{\tan u(1+\tan u)}{\tan u+1} = \tan u
\end{aligned}$$

When sums or differences of quotients appear, it is usually best to rewrite them as a single quotient, especially if the other side of the identity consists of only one term.

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Exercise 6 Establish the identity: $\dfrac{\sin\theta}{1+\cos\theta}+\dfrac{1+\cos\theta}{\sin\theta}=2\csc\theta$.

Solution:

The left side is more complicated, so we begin with it and proceed to add.

$$\begin{aligned}
\frac{\sin\theta}{1+\cos\theta}+\frac{1+\cos\theta}{\sin\theta} &= \frac{\sin^2\theta+(1+\cos\theta)^2}{(1+\cos\theta)(\sin\theta)} && \text{Add the quotients.} \\
&= \frac{\sin^2\theta+1+2\cos\theta+\cos^2\theta}{(1+\cos\theta)(\sin\theta)} && \text{Remove parentheses in the numerator.} \\
&= \frac{(\sin^2\theta+\cos^2\theta)+1+2\cos\theta}{(1+\cos\theta)(\sin\theta)} && \text{Regroup} \\
&= \frac{2+2\cos\theta}{(1+\cos\theta)(\sin\theta)} && \text{Pythagorean Identity.} \\
&= \frac{2(1+\cos\theta)}{(1+\cos\theta)(\sin\theta)} && \text{Factor and cancel.} \\
&= \frac{2}{(\sin\theta} \\
&= 2\csc\theta && \text{Reciprocal Identity.}
\end{aligned}$$

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Exercise 7 Establish the identity: $\dfrac{\tan v + \cot v}{\sec v \csc v} = 1$.

Solution:

$$
\begin{aligned}
\frac{\tan v + \cot v}{\sec v \csc v} &= \frac{\frac{\sin v}{\cos v} + \frac{\cos v}{\sin v \csc v}}{\frac{1}{\cos v} \cdot \frac{1}{\sin v}} && \text{Change to sines and cosines.} \\
&= \frac{\frac{\sin^2 v + \cos^2 v}{\cos v \sin v}}{\frac{1}{\cos v \sin v}} && \text{Add the quotients in the numerator.} \\
&= \frac{1}{\cos v \sin v} \cdot \frac{\cos v \sin v}{1} && \text{Divide the quotients; } \sin^2 v + \cos^2 v = 1. \\
&= 1
\end{aligned}
$$

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Exercise 8 Establish the identity: $\dfrac{1 - \sin\theta}{\cos\theta} = \dfrac{\cos\theta}{1 + \sin\theta}$.

Solution:

Starting with the left side and multiplying the numerator and the denominator by $1 + \sin\theta$, (Alternatively, we could multiply the numerator and denominator of the right side by $1 - \sin\theta$.), we have

$$
\begin{aligned}
\frac{1 - \sin\theta}{\cos\theta} &= \frac{1 - \sin\theta}{\cos\theta} \cdot \frac{1+\sin\theta}{1+\sin\theta} && \text{Multiply the numerator and denominator by } 1 + \sin\theta. \\
&= \frac{1 - \sin^2\theta}{\cos\theta(1 + \sin\theta)} \\
&= \frac{\cos^2\theta}{\cos\theta(1 + \sin\theta)} && 1 - \sin^2\theta = \cos^2\theta. \\
&= \frac{\cos\theta}{1 + \sin\theta} && \text{Cancel.}
\end{aligned}
$$

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7. Guidelines for Establishing Identities

Step 1: It is almost always preferable to start with the side containing the more complicated expression.

Step 2: Rewrite sums or differences of quotients as a single quotient.

Step 3: Sometimes rewriting one side in terms of sine and cosine functions only will help.

Step 4: Always keep your goal in mind. As you manipulate one side of the expression, you must keep in mind the form of the expression on the other side.

8. <u>**Sum and Difference Formulas**</u>

Theorem 1 Sum and Difference Formulas for the Cosine Function

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta; \tag{1}$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta. \tag{2}$$

9. **Use Sum and Difference Formulas to Find Exact Values**

One use of formulas (1) and (2) is to obtain the exact value of the cosine of an angle that can be expressed as the sum or difference of angles whose sine and cosine are known exactly.

Exercise 9 Using the Sum Formula to Find Exact Values

Find the exact value of $\cos 75°$.

Solution: Since $75° = 45° + 30°$, we use formulas (1) to obtain

$$\begin{aligned} \cos 75° &= \cos(45°+30°) = \cos 45°\cos 30° - \sin 45°\sin 30° \quad \text{Use formulas (1)} \\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{1}{4}(\sqrt{6}-\sqrt{2}). \end{aligned}$$

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Exercise 10 Using the Difference Formula to Find Exact Values

Find the exact value of $\cos\frac{\pi}{12}$. **Solution:**

Since $\frac{\pi}{12} = \frac{3\pi}{12} - \frac{2\pi}{12}$, we use formulas (2) to obtain

$$\begin{aligned} \cos\frac{\pi}{12} &= \cos\left(\frac{3\pi}{12} - \frac{2\pi}{12}\right) = \cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right) \\ &= \cos\frac{\pi}{4}\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\sin\frac{\pi}{6} \qquad \text{Use formula (2).} \\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{1}{4}(\sqrt{6}+\sqrt{2}). \end{aligned}$$

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10. **Use Sum and Difference Formulas to Establish Identities**

Another use of formulas (1) and (2) is to establish other identities. Two important identities are given next:

$$\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta; \tag{3}$$
$$\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta. \tag{4}$$

Proof of formula (3): Using the formula for $\cos(\alpha-\beta)$ with $\alpha=\dfrac{\pi}{2}$ and $\beta=\theta$, we have

$$\begin{aligned}\cos\left(\frac{\pi}{2}-\theta\right)&=\cos\frac{\pi}{2}\cos\theta+\sin\frac{\pi}{2}\sin\theta\\&=0\cdot\cos\theta+1\cdot\sin\theta\\&=\sin\theta\end{aligned}$$

Proof of formula (4): By making use of the identity (3) just established, we have

$$\sin\left(\frac{\pi}{2}-\theta\right)=\cos\left[\frac{\pi}{2}-\left(\frac{\pi}{2}-\theta\right)\right]=\cos\theta \quad \text{Use (3).}$$

Remark 1 Formulas (3) and (4) should look familiar. Using Even property of cosine, we have

$$\begin{aligned}\cos\left(\frac{\pi}{2}-\theta\right)&=\cos\left[-\left(\theta-\frac{\pi}{2}\right)\right]\\&=\cos\left(\theta-\frac{\pi}{2}\right)\end{aligned}$$

and since

$$\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta \qquad (3)$$

it follows that $\cos\left(\theta-\frac{\pi}{2}\right)=\sin\theta$. The graphs of $y=\cos\left(\theta-\dfrac{\pi}{2}\right)$ and $y=\sin\theta$ are identical.

11. **Theorem 2 Sum and Difference Formulas for the Sine Function**

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta; \tag{5}$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta. \tag{6}$$

Having established the identities in formulas (3) and (4), we now can derive the sum and difference formulas for $\sin(\alpha+\beta)$ and $\sin(\alpha-\beta)$.

Proof of formula (5):

$$\begin{aligned}
\sin(\alpha+\beta) &= \cos\left[\tfrac{\pi}{2}-(\alpha+\beta)\right] && \text{Formula (3)}\\
&= \cos\left[\left(\tfrac{\pi}{2}-\alpha\right)-\beta\right] \\
&= \cos\left(\tfrac{\pi}{2}-\alpha\right)\cos\beta + \sin\left(\tfrac{\pi}{2}-\alpha\right)\sin\beta && \text{Formula (2)}\\
&= \sin\alpha\cos\beta + \cos\alpha\sin\beta. && \text{Formulas (3) and (4)}
\end{aligned}$$

Proof of formula (6):

$$\begin{aligned}
\sin(\alpha-\beta) &= \sin[\alpha+(-\beta)] \\
&= \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta) && \text{Use the sum formula for the sine just obtained.}\\
&= \sin\alpha\cos\beta + \cos\alpha(-\sin\beta) && \text{Even-Odd Identities}\\
&= \sin\alpha\cos\beta - \cos\alpha\sin\beta.
\end{aligned}$$

Exercise 11 Using the Sum Formula to Find Exact Values

Find the exact value of $\sin\dfrac{7\pi}{12}$.

Solution:

Since $\dfrac{7\pi}{12} = \dfrac{3\pi}{12} + \dfrac{4\pi}{12}$, we use formulas (5) to obtain

$$\begin{aligned}
\sin\frac{7\pi}{12} &= \sin\left(\frac{3\pi}{12}+\frac{4\pi}{12}\right) = \sin\left(\frac{\pi}{4}+\frac{\pi}{3}\right)\\
&= \sin\frac{\pi}{4}\cos\frac{\pi}{3} + \cos\frac{\pi}{4}\sin\frac{\pi}{3}\\
&= \frac{\sqrt{2}}{2}\cdot\frac{1}{2} + \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} = \frac{1}{4}(\sqrt{2}+\sqrt{6}).
\end{aligned}$$

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Exercise 12 Using the Difference Formula to Find Exact Values

Find the exact value of $\sin 80^\circ\cos 20^\circ - \cos 80^\circ\sin 20^\circ$.

Solution: The form of the expression $\sin 80^\circ\cos 20^\circ - \cos 80^\circ\sin 20^\circ$ is that of the right side of formula(6) for $\sin(\alpha-\beta)$ with $\alpha = 80^\circ$ and $\beta = 20^\circ$. That is,

$$\sin 80^\circ\cos 20^\circ - \cos 80^\circ\sin 20^\circ = \sin(80^\circ - 20^\circ) = \sin 60^\circ = \frac{\sqrt{3}}{2}.$$

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Exercise 13 Finding Exact Values

If it is known that $\sin\alpha = \frac{4}{5}, \frac{\pi}{2} < \alpha < \pi$, and that $\sin\beta = -\frac{2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}, \pi < \beta < \frac{3\pi}{2}$, find the exact value of

(a) $\cos\alpha$ (b) $\cos\beta$ (c) $\cos(\alpha+\beta)$ (d) $\sin(\alpha+\beta)$

Solution:

(a) Since $\sin\alpha = \dfrac{4}{5} = \dfrac{y}{r}$ and $\dfrac{\pi}{2} < \alpha < \pi$, we let $y = 4$ and $r = 5$ and place α in quadrant II. The point $P = (x, y) = (x, 4), x < 0$, is on a circle of radius 5, that is, $x^2 + y^2 = 25$. Then

$$\begin{array}{lll} x^2+y^2 & = & 25 \\ x^2+16 & = & 25 \qquad\qquad y=4 \\ x^2 & = & 25-16=9 \\ x & = & -3 \qquad\qquad x<0 \end{array}$$

Then

$$\cos\alpha = \frac{x}{r} = -\frac{3}{5}.$$

Alternatively, we can find $\cos\alpha$ using identities, as follows:

$$\begin{array}{lll} \cos\alpha & = & -\sqrt{1-\sin^2\alpha} \quad \alpha \text{in quadrant II}, \ \cos\alpha < 0 \\ & = & -\sqrt{1-\frac{16}{25}} \\ & = & -\sqrt{\frac{9}{25}} = -\frac{3}{5} \end{array}$$

(b) Since $\sin\beta = \dfrac{-2}{\sqrt{5}} = \dfrac{y}{r}$ and $\pi < \beta < \dfrac{3\pi}{2}$, we let $y = -2$ and $r = \sqrt{5}$ and place β in quadrant III. The point $P = (x, y) = (x, -2),\ x < 0$, is on a circle of radius $\sqrt{5}$, that is,$x^2 + y^2 = 5$. Then

$$\begin{array}{lll} x^2+y^2 & = & 5 \\ x^2+4 & = & 5 \quad y=-2 \\ x^2 & = & 1 \\ x & = & -1 \quad x<0 \end{array}$$

Then

$$\cos\beta = \frac{x}{r} = \frac{-1}{\sqrt{5}} = -\frac{\sqrt{5}}{5}.$$

Alternatively, we can find $\cos\beta$ using identities, as follows:

$$\begin{aligned} \cos\beta &= -\sqrt{1-\sin^2\beta} \\ &= -\sqrt{1-\frac{4}{5}} \\ &= -\sqrt{\frac{1}{5}} = -\frac{\sqrt{5}}{5}. \end{aligned}$$

(c) Using the results found in parts (a) and (b) and formula (1), we have

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ &= -\frac{3}{5}\Big(-\frac{\sqrt{5}}{5}\Big) - \frac{4}{5}\Big(-\frac{2\sqrt{5}}{5}\Big) \\ &= \frac{11\sqrt{5}}{25}.\end{aligned}$$

(d)

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ &= \frac{4}{5}\Big(-\frac{\sqrt{5}}{5}\Big) + \Big(-\frac{3}{5}\Big)\Big(-\frac{2\sqrt{5}}{5}\Big) \\ &= \frac{2\sqrt{5}}{25}.\end{aligned}$$

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Exercise 14 Establishing an Identity

Establish the identity: $\frac{\cos(\alpha-\beta)}{\sin\alpha\sin\beta} = \cot\alpha\cot\beta + 1$.

Solution:

$$\begin{aligned}\frac{\cos(\alpha-\beta)}{\sin\alpha\sin\beta} &= \frac{\cos\alpha\cos\beta + \sin\alpha\sin\beta}{\sin\alpha\sin\beta} \\ &= \frac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta} + \frac{\sin\alpha\sin\beta}{\sin\alpha\sin\beta} \\ &= \frac{\cos\alpha}{\sin\alpha}\cdot\frac{\cos\beta}{\sin\beta} + 1 \\ &= \cot\alpha\cot\beta + 1\end{aligned}$$

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Theorem 3 Sum and Difference Formulas for the Tangent Function

$$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}; \tag{7}$$

$$\tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}. \tag{8}$$

We use the identity $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and the sum formulas for $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$ to derive a formula for $\tan(\alpha+\beta)$.

Proof of formula (7):

$$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}$$

Now we divide the numerator and denominator by $\cos\alpha\cos\beta$.

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}\\ &= \frac{\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}-\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}\\ &= \frac{\frac{\sin\alpha}{\cos\alpha}+\frac{\sin\beta}{\cos\beta}}{1-\frac{\sin\alpha}{\cos\alpha}\cdot\frac{\sin\beta}{\cos\beta}}\\ &= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}.\end{aligned}$$

Proof of formula (8): Using the sum formula for $\tan(\alpha+\beta)$ and Even-Odd Properties to get the difference formula, we have

$$\tan(\alpha-\beta) = \tan[\alpha+(-\beta)] = \frac{\tan\alpha+\tan(-\beta)}{1-\tan\alpha\tan(-\beta)} = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}.$$

Exercise 15 Establishing an Identity

Prove the identity: $\tan(\theta+\pi) = \tan\theta$.

Solution: Using formula (7), we have

$$\tan(\theta+\pi) = \frac{\tan\theta+\tan\pi}{1-\tan\theta\tan\pi} = \frac{\tan\theta+0}{1-\tan\theta\cdot 0} = \tan\theta.$$

The result obtained in Example 15 verifies that the tangent function is periodic with period π, a fact that we discussed earlier.

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Exercise 16 Establishing an Identity

Prove the identity: $\tan\left(\theta + \frac{\pi}{2}\right) = -\cot\theta$.

Solution: We cannot use formula (7), since $\tan\frac{\pi}{2}$ is not defined. Instead, we proceed as follows.

$$\begin{aligned}
\tan\left(\theta + \frac{\pi}{2}\right) = \frac{\sin\left(\theta + \frac{\pi}{2}\right)}{\cos\left(\theta + \frac{\pi}{2}\right)} &= \frac{\sin\theta\cos\frac{\pi}{2} + \cos\theta\sin\frac{\pi}{2}}{\cos\theta\cos\frac{\pi}{2} - \sin\theta\sin\frac{\pi}{2}} \\
&= \frac{(\sin\theta)(0) + (\cos\theta)(1)}{(\cos\theta)(0) - (\sin\theta)(1)} \\
&= \frac{\cos\theta}{-\sin\theta} \\
&= -\cot\theta
\end{aligned}$$

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12. **Double-angle and Half-angle Formulas**

In what follows, we derive formulas for $\sin(2\theta), \cos(2\theta), \sin\left(\frac{1}{2}\theta\right)$, and $\cos(\frac{1}{2}\theta)$ in terms of $\sin\theta$ and $\cos\theta$. They are derived using the sum formulas.

In the sum formulas for $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$, let $\alpha=\beta=\theta$. Then

$$\begin{aligned}
\sin(\alpha+\beta) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta;\\
\sin(\theta+\theta) &= \sin\theta\cos\theta+\cos\theta\sin\theta;\\
\sin(2\theta) &= 2\sin\theta\cos\theta.
\end{aligned}$$

and

$$\begin{aligned}
\cos(\alpha+\beta) &= \cos\alpha\cos\beta-\sin\alpha\sin\beta;\\
\cos(\theta+\theta) &= \cos\theta\cos\theta-\sin\theta\sin\theta;\\
\cos(2\theta) &= \cos^2\theta-\sin^2\theta.
\end{aligned}$$

An application of the Pythagorean Identity $\sin^2\theta+\cos^2\theta=1$ results in two other ways to express $\cos(2\theta)$.

$$\cos(2\theta)=\cos^2\theta-\sin^2\theta=(1-\sin^2\theta)-\sin^2\theta=1-2\sin^2\theta$$

and

$$\cos(2\theta)=\cos^2\theta-\sin^2\theta=\cos^2\theta-(1-\cos^2\theta)=2\cos^2-1$$

We have established the following **Double-angle Formulas:**

Theorem 4 Double-angle Formulas

$$\sin(2\theta)=2\sin\theta\cos\theta; \tag{9}$$
$$\cos(2\theta)=\cos^2-\sin^2\theta; \tag{10}$$
$$\cos(2\theta)=1-2\sin^2\theta; \tag{11}$$
$$\cos(2\theta)=2\cos^2\theta-1. \tag{12}$$

Exercise 17 quad Finding Exact Values Using the Double-angle Formula If $\sin\theta = \frac{3}{5}, \frac{\pi}{2} < \theta < \pi$, find the exact value of:

(a) $\sin(2\theta)$ (b) $\cos(2\theta)$.

Solution:

(a) Because $\sin(2\theta) = 2\sin\theta\cos\theta$ and we already know that $\sin\theta = \frac{3}{5}$, we only need to find $\cos\theta$. Since $\sin\theta = \frac{3}{5} = \frac{y}{r}, \frac{\pi}{2} < \theta < \pi$, we let $y = 3$ and $r = 5$ and place θ in quadrant II. The point $P = (x, y) = (x, 3), x < 0$, is on a circle of radius 5, that is,$x^2 + y^2 = 25$. Then

$$\begin{aligned}
x^2 + y^2 &= 25 \\
x^2 + 9 &= 25 && y = 3 \\
x^2 &= 25 - 9 = 16 \\
x &= -4 && x < 0
\end{aligned}$$

We find that $\cos\theta = \dfrac{x}{r} = \dfrac{-4}{5}$. Now we use formula (9) to obtain

$$\sin(2\theta) = 2\sin\theta\cos\theta = 2\Big(\frac{3}{5}\Big)\Big(-\frac{4}{5}\Big) = -\frac{24}{25}.$$

(b) Because we are given $\sin\theta = \dfrac{3}{5}$, it is easiest to use formula (11) to get $\cos(2\theta)$. As a result,

$$\cos(2\theta) = 1 - 2\sin^2\theta = 1 - 2\Big(\frac{9}{25}\Big) = 1 - \frac{18}{25} = \frac{7}{25}.$$

■

Remark 2 In finding $\cos(2\theta)$ in Example 17 (b), we chose to use a version of the Double-angle Formula, formula (11). Note that we are unable to use the Pythagorean Identity $\cos(2\theta) = \pm\sqrt{1 - \sin^2(2\theta)}$, with $\sin(2\theta) = -\dfrac{24}{25}$, because we have no way of knowing which sign to choose.

13. **Use Double-angle Formulas to Establish Identities**

Exercise 18 Establishing Identities

(a) Develop a formula for $\tan(2\theta)$ in terms of $\tan\theta$.

(b) Develop a formula for $\sin(3\theta)$ in terms of $\sin\theta$ and $\cos\theta$.

Solution:

(a) In the sum formula for $\tan(\alpha+\beta)$, let $\alpha=\beta=\theta$. Then

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\\ \tan(\theta+\theta) &= \frac{\tan\theta+\tan\theta}{1-\tan\theta\tan\theta}\end{aligned}$$

Hence,

$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta} \tag{13}$$

(b) To get a formula for $\sin(3\theta)$, we use the sum formula and write 3θ as $2\theta+\theta$.

$$\sin(3\theta) = \sin(2\theta+\theta) = \sin(2\theta)\cos\theta + \cos(2\theta)\sin\theta$$

Now use the Double-angle Formulas to get

$$\begin{aligned}\sin(3\theta) &= (2\sin\theta\cos\theta)(\cos\theta) + (\cos^2\theta - \sin^2\theta)(\sin\theta)\\ &= 2\sin\theta\cos^2\theta + \sin\theta\cos^2\theta - \sin^3\theta\\ &= 3\sin\theta\cos^2\theta - \sin^3\theta\end{aligned}$$

■

Remark 3 The formula obtained in Example 18(b) can also be written as

$$\begin{aligned}\sin(3\theta) = 3\sin\theta\cos^2\theta - \sin^3\theta &= 3\sin\theta(1-\sin^2\theta) - \sin^3\theta\\ &= 3\sin\theta - 4\sin^3\theta\end{aligned}$$

That is, $\sin(3\theta)$ is a third-degree polynomial in the variable $\sin\theta$. In fact,$\sin(n\theta), n$ a positive odd integer,can always be written as a polynomial of degree n in the variable $\sin\theta$.

By rearranging the Double-angle Formulas (11) and(12), we obtain other formulas that we will use later in this course.

We begin with formula (11) and proceed to solve for $\sin^2\theta$.

$$\begin{aligned} \cos(2\theta) &= 1 - 2\sin^2\theta; \\ 2\sin^2\theta &= 1 - \cos(2\theta). \end{aligned}$$

Hence,

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2} \tag{14}$$

Similarly,using formula (4), we proceed to solve for $\cos^2\theta$.

$$\begin{aligned} \cos(2\theta) &= 2\cos^2\theta - 1; \\ 2\cos^2\theta &= 1 + \cos(2\theta). \end{aligned}$$

Hence,

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2} \tag{15}$$

Formulas (14) and (15) can be used to develop a formula for $\tan^2\theta$.

$$\tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta} = \frac{\frac{1-\cos(2\theta)}{2}}{\frac{1+\cos(2\theta)}{2}}$$

Hence,

$$\tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)} \tag{16}$$

Formulas (14) through (16) do not have to be memorized since their derivations are so straightforward.

Formulas (14) and (15)are important in calculus. The next example illustrates a problem that arises in calculus requiring the use of formula (15).

Exercise 19 Establishing an Identity

Write an equivalent expression for $\cos^4\theta$ that does not involve any powers of sine or cosine greater than 1.

Solution: The idea here is to apply formula (15) twice.

$$\begin{array}{rcll}
\cos^4\theta & = & (\cos^2\theta)^2 & \\
 & = & \left(\frac{1+\cos(2\theta)}{2}\right)^2 & \text{Formula (15)} \\
 & = & \frac{1}{4}[1+2\cos(2\theta)+\cos^2(2\theta)] & \\
 & = & \frac{1}{4}+\frac{1}{2}\cos(2\theta)+\frac{1}{4}\cos^2(2\theta) & \\
 & = & \frac{1}{4}+\frac{1}{2}\cos(2\theta)+\frac{1}{4}\left[\frac{1+\cos[2(2\theta)]}{2}\right] & \text{Formula (15)} \\
 & = & \frac{1}{4}+\frac{1}{2}\cos(2\theta)+\frac{1}{8}[1+\cos(4\theta)] & \\
 & = & \frac{3}{8}+\frac{1}{2}\cos(2\theta)+\frac{1}{8}\cos(4\theta). &
\end{array}$$

■

14. **Using Half-angle Formulas to Find Exact Values**

Another important use of formulas (14) through (16) is prove the Half-angle Formulas.In formulas (14) through (16), let $\theta = \frac{\alpha}{2}$. Then

$$\begin{cases} \sin^2 \frac{\alpha}{2} & = \dfrac{1-\cos\alpha}{2}; \\ \cos^2 \frac{\alpha}{2} & = \dfrac{1+\cos\alpha}{2}; \\ \tan^2 \frac{\alpha}{2} & = \dfrac{1-\cos\alpha}{1+\cos\alpha}. \end{cases} \tag{17}$$

If we solve for the trigonometric functions on the left sides of equations (17) we obtain the Half-angle Formulas.

Theorem 5 Half-angle Formulas

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}; \tag{18}$$

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}; \tag{19}$$

$$\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}. \tag{20}$$

where the $+$ or $-$ sign is determined by the quadrant of the angle $\dfrac{\alpha}{2}$.

Exercise 20 Finding Exact Values Using Half-angle Formulas

Use a Half-angle Formula to find the exact value of:

(a) $\cos 15^\circ$ (b) $\sin(-15^\circ)$

Solution:

(a) Because $15^\circ = \frac{30^\circ}{2}$, we can use the Half-angle Formula for $\cos\frac{\alpha}{2}$ with $\alpha = 30^\circ$. Also, because 15° is in quadrant I, $\cos 15^\circ > 0$, we choose the $+$ sign in using formula (19):

$$\begin{aligned} \cos 15^\circ = \cos\frac{30^\circ}{2} &= \sqrt{\frac{1+\cos 30^\circ}{2}} \\ &= \sqrt{\frac{1+\sqrt{3}/2}{2}} = \sqrt{\frac{2+\sqrt{3}}{4}} = \frac{\sqrt{2+\sqrt{3}}}{2}. \end{aligned}$$

(b) We use the fact that $\sin(-15^\circ) = -\sin 15^\circ$ and then apply formula (19).

$$\begin{aligned} \sin(-15^\circ) = -\sin\frac{30^\circ}{2} &= -\sqrt{\frac{1-\cos 30^\circ}{2}} \\ &= -\sqrt{\frac{1-\sqrt{3}/2}{2}} = -\sqrt{\frac{2-\sqrt{3}}{4}} = -\frac{\sqrt{2-\sqrt{3}}}{2} \end{aligned}$$

■

Remark 4 Let us compare the answer found in Example (20)(a) with the answer to Example 10. There we calculated

$$\cos\frac{\pi}{12} = \cos 15^\circ = \frac{1}{4}(\sqrt{6}+\sqrt{2})$$

Based on this and the result of Example (20)(a), we conclude that
$\frac{1}{4}(\sqrt{6}+\sqrt{2})$ and $\frac{\sqrt{2+\sqrt{3}}}{2}$
are equal. (Since each expression is positive,you can verify this equality by squaring each expression.) Two very different looking, yet correct, answers can be obtained, depending on the approach taken to solve a problem.

Exercise 21 Finding Exact Values Using Half-angle Formulas
If $\cos\alpha = -\frac{3}{5}, \pi < \alpha < \frac{3\pi}{2}$, find the exact value of:
(a) $\sin\frac{\alpha}{2}$ (b) $\cos\frac{\alpha}{2}$ (c) $\tan\frac{\alpha}{2}$
Solution
First, we observe that if $\pi < \alpha < \dfrac{3\pi}{2}$ then $\frac{\pi}{2} < \frac{\alpha}{2} < \frac{3\pi}{4}$. As a result, $\dfrac{\alpha}{2}$ lies in quadrant II.
(a) Because $\dfrac{\alpha}{2}$ lies in quadrant II, $\sin\dfrac{\alpha}{2} > 0$, so we use the + sign in formula (18) to get

$$\begin{aligned}\sin\frac{\alpha}{2} &= \sqrt{\frac{1-\cos\alpha}{2}} = \sqrt{\frac{1-\left(-\frac{3}{5}\right)}{2}}\\ &= \sqrt{\frac{\frac{8}{5}}{2}} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}.\end{aligned}$$

(b) Because $\dfrac{\alpha}{2}$ lies in quadrant II, $\cos\dfrac{\alpha}{2} < 0$, so we use the − sign in formula (**??**) to get

$$\begin{aligned}\cos\frac{\alpha}{2} &= -\sqrt{\frac{1+\cos\alpha}{2}} = -\sqrt{\frac{1+\left(-\frac{3}{5}\right)}{2}}\\ &= -\sqrt{\frac{\frac{2}{5}}{2}} = -\frac{1}{\sqrt{5}} = -\frac{\sqrt{5}}{5}.\end{aligned}$$

(c) Because $\dfrac{\alpha}{2}$ lies in quadrant II, $\tan\dfrac{\alpha}{2} < 0$, so we use the − sign in formula (20) to get

$$\tan\frac{\alpha}{2} = -\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = -\sqrt{\frac{1-\left(-\frac{3}{5}\right)}{1+\left(-\frac{3}{5}\right)}} = -\sqrt{\frac{\frac{8}{5}}{\frac{2}{5}}} = -2$$

■

Remark 5 Another way to solve Example 21(c) is to use the solutions found in parts (a) and (b).

$$\tan\frac{\alpha}{2} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \frac{\frac{2\sqrt{5}}{5}}{-\frac{\sqrt{5}}{5}} = -2.$$

Half-angle Formulas for $\tan\frac{\alpha}{2}$

$$\tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1+\cos\alpha}. \tag{21}$$

Proof of formula (21): There is a formula for $\tan\frac{\alpha}{2}$ that does not contain $+$ and $-$ signs, making it more useful than Formula 20. To derive it,we use the formulas

$$1-\cos\alpha = 2\sin^2\frac{\alpha}{2}17$$

and

$$\sin\alpha = \sin\left[2\left(\frac{\alpha}{2}\right)\right] = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} \quad \text{Double-angle Formula}$$

Then

$$\frac{1-\cos\alpha}{\sin\alpha} = \frac{2\sin^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}} = \tan\frac{\alpha}{2}.$$

Since it also can be shown that

$$\frac{1-\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1+\cos\alpha}$$

we have the following two Half-angle Formulas:

Example 3 With this formula, the solution to Example 21 can be obtained as follows:

$$\cos\alpha = -\frac{3}{5} \quad \pi < \alpha < \frac{3\pi}{2}$$

$$\sin\alpha = -\sqrt{1-\cos^2\alpha} = -\sqrt{1-\frac{9}{25}} = -\sqrt{\frac{16}{25}} = -\frac{4}{5}$$

Then,by equation (11),

$$\tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha} = \frac{1-\left(-\frac{3}{5}\right)}{-\frac{4}{5}} = -2.$$

15. **Product-to-Sum and Sum-to-Product Formulas**

Express Products as Sums

Sum and difference formulas can be used to derive formulas for writing the products of sines and/or cosines as sums or differences. These identities are usually called the **Product-to-Sum Formulas.**

Theorem 6 Product-to-Sum Formulas

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]; \tag{22}$$
$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]; \tag{23}$$
$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha-\beta)+\sin(\alpha+\beta)]. \tag{24}$$

These formulas do not have to be memorized. Instead,you should remember how they are derived. Then,when you want to use them, either look them up or derive them, as needed.

Proof of formulas (22) and (23): Writing down the sum and difference formulas for the cosine, we have

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta; \tag{25}$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta. \tag{26}$$

Subtract equation (26) from equation (25) to get

$$\cos(\alpha-\beta)-\cos(\alpha+\beta) = 2\sin\alpha\sin\beta$$

from which

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$$

Now add equations (4) and (5) to get

$$\cos(\alpha-\beta)+\cos(\alpha+\beta) = 2\cos\alpha\cos\beta$$

from which

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$$

Proof of formula (24): To derive Product-to-Sum Formula (24), use the sum and difference formulas for sine in a similar way.

Exercise 22 Expressing Products as Sums

Express each of the following products as a sum containing only sines or only cosines.

(a) $\sin(6\theta)\sin(4\theta)$ (b) $\cos(3\theta)\cos\theta$ (c) $\sin(3\theta)\cos(5\theta)$

Solution

(a) We use formula (22) to get

$$\begin{aligned}\sin(6\theta)\sin(4\theta) &= \frac{1}{2}[\cos(6\theta-4\theta)-\cos(6\theta+4\theta)]\\ &= \frac{1}{2}[\cos(2\theta)-cos(10\theta)].\end{aligned}$$

(b) We use formula (23) to get

$$\begin{aligned}\cos(3\theta)\cos\theta &= \frac{1}{2}[\cos(3\theta-\theta)+\cos(3\theta+\theta)]\\ &= \frac{1}{2}[\cos(2\theta)+\cos(4\theta)].\end{aligned}$$

(b) We use formula (24) to get

$$\begin{aligned}\sin(3\theta)\cos(5\theta) &= \frac{1}{2}[\sin(3\theta+5\theta)+\sin(3\theta-5\theta)]\\ &= \frac{1}{2}[\sin(8\theta)+\sin(-2\theta)] = \frac{1}{2}[\sin(8\theta)-\sin(2\theta)].\end{aligned}$$

■

16. The **Sum-to-Product Formulas** are given next.

Theorem 7 Sum-to-Product Formulas

$$\sin\alpha+\sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}; \tag{27}$$

$$\sin\alpha-\sin\beta = 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}; \tag{28}$$

$$\cos\alpha+\cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}; \tag{29}$$

$$\cos\alpha+\cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}. \tag{30}$$

We will derive formula (27) and leave the derivations of formulas (28) through (30) as exercises.

Proof of formula (27): Using Product-to-Sum Formula (24), we have

$$2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = 2\cdot\frac{1}{2}\left[\sin\left(\frac{\alpha+\beta}{2}+\frac{\alpha-\beta}{2}\right)+\sin\left(\frac{\alpha+\beta}{2}-\frac{\alpha-\beta}{2}\right)\right]$$

$$= \sin\frac{2\alpha}{2}+\sin\frac{2\beta}{2} = \sin\alpha+\sin\beta.$$

Exercise 23 Expressing Sums (or Differences)as a Product

Express each sum or difference as a product of sines and/or cosines.

(a) $\sin(5\theta)-\sin(3\theta)$ (b) $\cos(3\theta)+\cos(2\theta)$

Solution:

(a) We use formula (28) to get

$$\begin{aligned}\sin(5\theta)-\sin(3\theta) &= 2\sin\frac{5\theta-3\theta}{2}\cos\frac{5\theta+3\theta}{2}\\ &= 2\sin\theta\cos(4\theta).\end{aligned}$$

(b) We use formula (29) to get

$$\begin{aligned}\cos(3\theta)+\cos(2\theta) &= 2\cos\frac{3\theta+2\theta}{2}\cos\frac{3\theta-2\theta}{2}\\ &= 2\cos\frac{5\theta}{2}\cos\frac{\theta}{2}.\end{aligned}$$

■

17. **Trigonometric Equations(I)**

Solve Equations Involving a Single Trigonometric Function

In what follows, we discuss **trigonometric equations,** that is, equations involving trigonometric functions that are satisfied only by some values of the variable (or,possibly,are not satisfied by any values of the variable). The values that satisfy the equation are called **solutions** of the equation.

Exercise 24 Checking Whether a Given Number Is a Solution of a Trigonometric Equation Determine whether $\theta = \dfrac{\pi}{4}$ is a solution of the equation $2\sin\theta - 1$ a solution?

Solution:

First, we replace θ by $\dfrac{\pi}{4}$ in the given equation. The result is

$$2\sin\frac{\pi}{4} - 1 = 2\cdot\frac{\sqrt{2}}{2} - 1 = \sqrt{2} - 1 \neq 0.$$

We conclude that $\dfrac{\pi}{4}$ is not a solution.

Next we replace θ by $\frac{\pi}{6}$ in the equation. The result is

$$2\sin\frac{\pi}{6} - 1 = 2\cdot\frac{1}{2} - 1 = 1 - 1 = 0.$$

We conclude that $\dfrac{\pi}{6}$ is a solution of the given equation.

■

Remark 6 The equation given in Example 24 has other solutions besides $\theta = \dfrac{\pi}{6}$. For example, $\theta = \dfrac{5\pi}{6}$ is also a solution, as is $\theta = \dfrac{13\pi}{6}$. (You should check this for yourself.)

Remark 7 Unless the domain of the variable is restricted, we need to find *all* the solutions of a trigonometric equation. As the next example illustrates, finding all the solutions can be accomplished by first finding solutions over an interval whose length equals the period of the function and then adding multiples of that period to the solutions found.

Exercise 25 Finding All the Solutions of a Trigonometric Equation Solve the equation: $\cos\theta = \frac{1}{2}$

Given a general formula for all the solutions.List eight of the solutions.

Solution:

The period of the cosine function is 2π. In the interval $[0, 2\pi)$, there are two angles θ for which $\cos\theta = \dfrac{1}{2} : \theta = \dfrac{\pi}{3}$ and $\theta = \dfrac{5\pi}{3}$. Because the cosine function has period 2π, all the solutions of $\cos\theta = \frac{1}{2}$ may be given by the general formula

$$\theta = \frac{\pi}{3} + 2k\pi \quad or \quad \theta = \frac{5\pi}{3} + 2k\pi \quad k \text{ any } \text{ integer.}$$

Eight of the solutions are

$$\underbrace{-\frac{5\pi}{3}, -\frac{\pi}{3}}_{k=-1}, \underbrace{\frac{\pi}{3}, \frac{5\pi}{3}}_{k=0}, \underbrace{\frac{7\pi}{3}, \frac{11\pi}{3}}_{k=1}, \underbrace{\frac{13\pi}{3}, \frac{17\pi}{3}}_{k=2}.$$

■

Remark 8 We can verify the solutions by graphing $Y_1 = \cos x$ and $Y_2 = \frac{1}{2}$ to determine where the graphs intersect. Can you plot the graph of $Y_1 = \cos x$ and $Y_2 = \frac{1}{2}$? The graph of Y_1 intersects the graph of Y_2 at $x = 1.05\Big(\approx \frac{\pi}{3}\Big), 5.24\Big(\approx \frac{5\pi}{3}\Big), 7.33\Big(\approx \frac{7\pi}{3}\Big)$, and $11.52\Big(\approx \frac{11\pi}{3}\Big)$, rounded to two decimal places.

In most of our work, we shall be interested only in finding solutions of trigonometric equations for $0 \le \theta < 2\pi$.

Exercise 26 Solving a Linear Trigonometric Equation Solve the equation:

$$2\sin\theta + \sqrt{3} = 0, 0 \le \theta < 2\pi.$$

Solution:

We solve the equation for $\sin\theta$.

$$\begin{array}{rcll} 2\sin\theta + \sqrt{3} & = & 0 & \\ 2\sin\theta & = & -\sqrt{3} & \text{Subtract } \sqrt{3} \text{ from both sides.} \\ \sin\theta & = & -\frac{\sqrt{3}}{2} & \text{Divide both sides by 2} \end{array}$$

In the interval $[0, 2\pi)$, there are two angles θ for which $\sin\theta = -\dfrac{\sqrt{3}}{2} : \theta = \dfrac{4\pi}{3}$ and $\theta = \dfrac{5\pi}{3}$. The solution set is $\left\{\frac{4\pi}{3}, \frac{5\pi}{3}\right\}$.

■

Exercise 27 Solving a Trigonometric Equation

Solve the equation:

$$\sin(2\theta) = \frac{1}{2}, \ 0 \le \theta < 2\pi.$$

Solution:

In the interval $[0, 2\pi)$, the sine function has a value $\dfrac{1}{2}$ at $\dfrac{\pi}{6}$ and $\dfrac{5\pi}{6}$. Do you know why? Since the period of the sine function is 2π and the argument is 2θ in the equation $\sin(2\theta) = \dfrac{1}{2}$, we write the general formula that gives all the solutions.

$$\begin{array}{rlrll} 2\theta = \dfrac{\pi}{6} + 2k\pi & \text{or} & 2\theta = \dfrac{5\pi}{6} + 2k\pi & k \quad \text{any} \quad \text{integer} \\ \theta = \dfrac{\pi}{12} + k\pi & \text{or} & \theta = \dfrac{5\pi}{12} + k\pi & \text{Divide by 2.} \end{array}$$

Then

$$\begin{array}{ccc} \theta = \dfrac{\pi}{12} + (-1)\pi = \dfrac{-11\pi}{12} & k = -1 & \theta = \dfrac{5\pi}{12} + (-1)\pi = \dfrac{-7\pi}{12} \\ \theta = \dfrac{\pi}{12} + (0)\pi = \dfrac{\pi}{12} & k = 0 & \theta = \dfrac{5\pi}{12} + (0)\pi = \dfrac{5\pi}{12} \\ \theta = \dfrac{\pi}{12} + (1)\pi = \dfrac{13\pi}{12} & k = 1 & \theta = \dfrac{5\pi}{12} + (1)\pi = \dfrac{17\pi}{12} \\ \theta = \dfrac{\pi}{12} + (2)\pi = \dfrac{25\pi}{12} & k = 2 & \theta = \dfrac{5\pi}{12} + (2)\pi = \dfrac{29\pi}{12} \end{array}$$

In the interval $[0, 2\pi)$, the solutions of $\sin(2\theta) = \frac{1}{2}$ are $\theta = \frac{\pi}{12}, \theta = \frac{5\pi}{12}, \theta = \frac{13\pi}{12}$, and $\theta = \frac{17\pi}{12}$. The solution set is $\left\{\frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}\right\}$.

Remark 9 Check of Example 27: Verify these solutions by graphing $Y_1 = \sin(2x)$ and $Y_2 = \frac{1}{2}$ for $0 \le x \le 2\pi$.

Remark 10 In solving a trigonometric equation for θ, $0 \le x < 2\pi$, in which the argument is not θ (as in Example 27), you must write down all the solutions first and then list those that are in the interval $[0, 2\pi)$. Otherwise, solutions may be lost. For example, in solving $\sin(2\theta) = \dfrac{1}{2}$, if you write the solutions $2\theta = \dfrac{\pi}{6}$ and $2\theta = \dfrac{5\pi}{6}$, you will find only $\theta = \dfrac{\pi}{12}$ and $\theta = \dfrac{5\pi}{12}$ and miss the other solutions.

■

Exercise 28 Solving a Trigonometric Equation

Solve the equation:

$$\tan\left(\theta - \frac{\pi}{2}\right) = 1,\ \ \theta, 0 \le x < 2\pi.$$

Solution:

The period of the tangent function is π. In the interval $[0, \pi)$, the tangent function has the value 1 when the argument is $\frac{\pi}{4}$. Because the argument is $\theta - \frac{\pi}{2}$ in the given equation, we write the general formula that gives all the solutions.

$$\begin{aligned} \theta - \frac{\pi}{2} &= \frac{\pi}{4} + k\pi \qquad k \text{ any integer} \\ \theta &= \frac{3\pi}{4} + k\pi. \end{aligned}$$

In the interval $[0, 2\pi)$, $\theta = \frac{3\pi}{4}$ and $\theta = \frac{3\pi}{4} + \pi = \frac{7\pi}{4}$ are the only solutions. The solution set is $\left\{\frac{3\pi}{4}, \frac{7\pi}{4}\right\}$.

■

18. **Trigonometric Equations (II)**

Solve Trigonometric Equations Quadratic in Form

In this section we continue our study of trigonometric equations.Many trigonometric equations can be solved by applying techniques that we already know,such as applying the quadratic formula (if the equation is a second-degree polynomial) or factoring.

Exercise 29 Solving a Trigonometric Equation Quadratic in Form

Solve the equation:

$$2\sin^2\theta - 3\sin\theta + 1 = 0,\ 0 \le \theta < 2\pi.$$

Solution:

This equation is a quadratic equation (in $\sin\theta$) that can be factored.

$$\begin{array}{rclcl} 2\sin^2\theta - 3\sin\theta + 1 & = & 0 & \quad & 2x^2 - 3x + 1 = 0, x = \sin\theta \\ (2\sin\theta - 1)(\sin\theta - 1) & = & 0 & \quad & (2x-1)(x-1) = 0 \end{array}$$

Now, using the Zero-Product Property, we have

$$\begin{aligned} 2\sin\theta - 1 = 0 \quad &\text{or} \quad \sin\theta - 1 = 0 \\ \sin\theta = \frac{1}{2} \quad &\text{or} \quad \sin\theta = 1 \end{aligned}$$

Solving each equation in the interval $[0, 2\pi)$, we obtain

$$\theta = \frac{\pi}{6} \quad \theta = \frac{5\pi}{6} \quad \theta = \frac{\pi}{2}$$

The solution set is $\left\{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{2}\right\}$.

■

Solve Trigonometric Equations Using Identities

When a trigonometric equation contains more than trigonometric function,identities sometimes can be used to obtain an equation that contains only one trigonometric function.

Exercise 30 Solving a Trigonometric Equation Using Identities

Solve the equation:

$$3\cos\theta + 3 = 2\sin^2\theta,\ 0 \le \theta < 2\pi.$$

Solution:

The equation in its present form contains a sine and a cosine. However, a form of the Pythagorean Identity can be used to transform the equation into an equivalent expression containing only cosines.

$$\begin{array}{rcll} 3\cos\theta + 3 & = & 2\sin^2\theta & \\ 3\cos\theta + 3 & = & 2(1-\cos^2\theta) & \sin^2\theta = 1-\cos^2\theta \\ 3\cos\theta + 3 & = & 2 - 2\cos^2\theta & \\ 2\cos^2\theta + 3\cos\theta + 1 & = & 0 & \text{Quadratic in } \cos\theta \\ (2\cos\theta + 1)(\cos\theta + 1) & = & 0 & \text{Factor} \end{array}$$

Now, using the Zero-Product Property, we have

$$\begin{array}{rcl} 2\cos\theta + 1 = 0 & \text{or} & \cos\theta + 1 = 0 \\ \cos\theta = -\dfrac{1}{2} & \text{or} & \cos\theta = -1 \end{array}$$

Solving each equation in the interval $[0, 2\pi)$, we obtain

$$\theta = \frac{2\pi}{3} \quad \theta = \frac{4\pi}{3} \quad \theta = \pi$$

The solution set is $\left\{\frac{2\pi}{3}, \pi, \frac{4\pi}{3}\right\}$.

■

Exercise 31 Check of Example 30: Graph $Y_1 = 3\cos x + 3$ and $Y_2 = 2\sin^2 x, 0 \le x \le 2\pi$, and approximate the points if intersection.

When a trigonometric equation contains trigonometric functions with different arguments, identities can sometimes be used to obtain an equivalent equation with the same argument.

Exercise 32 Solving a Trigonometric Equation Using Identities

Solve the equation:

$$\cos(2\theta) + 3 = 5\cos\theta,\ \ 0 \le \theta < 2\pi$$

Solution:

First, we observe that the given equation contains two cosine function,but with different arguments, θ and 2θ. We use the Double-angle Formula $\cos(2\theta) = 2\cos^2 -1$ to obtain an equivalent equation containing only $\cos\theta$.

$$\begin{array}{rcll}
\cos(2\theta) + 3 & = & 5\cos\theta & \\
(2\cos^2 -1) + 3 & = & 5\cos\theta & \cos(2\theta) = 2\cos^2 -1 \\
2\cos^2\theta - 5\cos\theta + 2 & = & 0 & \text{Place in standard form.} \\
(\cos\theta - 2)(2\cos\theta - 1) & = & 0 & \text{Factor.} \\
\cos\theta = 2 \text{ or } \cos\theta & = & \frac{1}{2} & \text{Solve by using the Zero-Product property.}
\end{array}$$

For any angle θ, $\ -1 \le \cos\theta \le 1$; therefore,the equation $\cos\theta = 2$ has no solution. The solutions of $\cos\theta = \dfrac{1}{2}, 0 \le \theta < 2\pi$, are

$$\theta = \frac{\pi}{3}\ \theta = \frac{5\pi}{3}.$$

The solution set is $\left\{\dfrac{\pi}{3}, \dfrac{5\pi}{3}\right\}$.

■

Exercise 33 Check of Example 32: Graph $Y_1 = \cos(2x){+}3$ and $Y_2 = 5\cos x, 0 \le x \le 2\pi$, and approximate the points of intersection.

Exercise 34 Solving a Trigonometric Equation Using Identities Solve the equation:

$$\cos^2\theta + \sin\theta = 2,\ 0 \leq \theta < 2\pi.$$

Solution: This equation involves two trigonometric functions,sine and cosine. We use a form of the Pythagorean Identity, $\sin^2\theta + \cos^2\theta = 1$, to rewrite the equation in terms of $\sin\theta$.

$$\begin{array}{rcll} \cos^2\theta + \sin\theta & = & 2 & \\ (1 - \sin^2\theta) + \sin\theta & = & 2 & \cos^2\theta = 1 - \sin^2\theta \\ \sin^2\theta - \sin\theta + 1 & = & 0 & \end{array}$$

This is a quadratic equation in $\sin\theta$. The discriminant is

$$b^2 - 4ac = (-1)^2 - 4 \cdot 1 \cdot 1 = -3 < 0.$$

Therefore, the equation has no real solution. The solution set is the empty set, $\varnothing$.

■

Exercise 35 Check of Example 34: Graph $Y_1 = \cos^2 x + \sin x$ and $Y_2 = 2$. The two graphs never intersect, so the equation $Y_1 = Y_2$ has no real solution.

Exercise 36 Solving a Trigonometric Equation Using Identities

Solve the equation:

$$\sin\theta\cos\theta = -\frac{1}{2},\ 0 \le \theta < 2\pi.$$

Solution:

The left side of the given equation is in the form of the Double-angle Formula $2\sin\theta\cos\theta = \sin(2\theta)$, except for a factor of 2. We multiply each side by 2.

$$\begin{array}{rcll} \sin\theta\cos\theta & = & -\frac{1}{2} & \\ 2\sin\theta\cos\theta & = & -1 & \text{Multiply each side by 2.} \\ \sin(2\theta) & = & -1 & \text{Double-angle Formula} \end{array}$$

The argument here is 2θ. So we need to write all the solutions of this equation and then list those that are in the interval $[0, 2\pi)$. Because $\sin\left(\frac{3\pi}{2} + 2\pi k\right) = -1$ for any integer k, we have

$$\begin{array}{rcll} 2\theta & = & \frac{3\pi}{2} + 2k\pi & k \text{ any integer} \\ \theta & = & \frac{3\pi}{4} + k\pi & \\ \theta & = & \frac{3\pi}{4} + (-1)\pi = -\frac{\pi}{4} & k = -1 \\ \theta & = & \frac{3\pi}{4} + (0)\pi = \frac{3\pi}{4} & k = 0 \\ \theta & = & \frac{3\pi}{4} + (1)\pi = \frac{7\pi}{4} & k = 1 \\ \theta & = & \frac{3\pi}{4} + (2)\pi = \frac{11\pi}{4} & k = 2 \end{array}$$

The solutions in the interval $[0, 2\pi)$ are

$$\theta = \frac{3\pi}{4} \quad \theta = \frac{7\pi}{4}$$

The solution set is $\left\{\frac{3\pi}{4}, \frac{7\pi}{4}\right\}$.

■

Solve Trigonometric Equations Linear in Sine and Cosine

Sometimes it is necessary to square both sides of an equation to obtain expressions that allow the use of identities. Remember, squaring both sides of an equation may introduce extraneous solution. As a result, apparent solutions must be checked.

Exercise 37 Solving a Trigonometric Equation Linear in Sine and Cosine

Solve the equation:

$$\sin\theta + \cos\theta = 1,\ 0 \le \theta < 2\pi.$$

Solution A:

Attempts to use available identities do not lead to equations that are easy to solve. (Try it yourself.) Given the form of this equation,we decide to square each side.

$$\begin{aligned}
\sin\theta + \cos\theta &= 1 & & \\
(\sin\theta + \cos\theta)^2 &= 1 & &\text{Square each side.} \\
\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta &= 1 & &\text{Remove parentheses.} \\
2\sin\theta\cos\theta &= 0 & &\sin^2\theta + \cos^2\theta = 1 \\
\sin\theta\cos\theta &= 0 & &
\end{aligned}$$

Setting each factor equal to zero, we obtain

$$\sin\theta = 0 \quad \text{or} \quad \cos\theta = 0.$$

The apparent solutions are

$$\theta = 0\ \theta = \pi\ \theta = \frac{\pi}{2}\ \theta = \frac{3\pi}{2}.$$

Because we squared both sides of the original equation, we must check these apparent solutions to see if any are extraneous.

$$\begin{array}{rll}
\theta = 0: & \sin 0 + \cos 0 = 0 + 1 = 1 & \text{A solution} \\
\theta = \pi: & \sin\pi + \cos\pi = 0 + (-1) = -1 & \text{Not a solution} \\
\theta = \frac{\pi}{2}: & \sin\frac{\pi}{2} + \cos\frac{\pi}{2} = 1 + 0 = 1 & \text{A solution} \\
\theta = \frac{3\pi}{2}: & \sin\frac{3\pi}{2} + \cos\frac{3\pi}{2} = -1 + 0 = -1 & \text{Not a solution}
\end{array}$$

The values $\theta = \pi$ and $\theta = \dfrac{3\pi}{2}$ are extraneous. The solution set is $\left\{0, \dfrac{\pi}{2}\right\}$.

Solution B:

We start with the equation

$$\sin\theta + \cos\theta = 1$$

and divide each side by $\sqrt{2}$. (The reason for this choice will become apparent shortly.) Then

$$\frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta = \frac{1}{\sqrt{2}}.$$

The left side now resembles the formula for the sine of the sum of two angles, one of which is θ. The other angle is unknown (call it ψ.) Then

$$\sin(\theta+\psi) = \sin\theta\cos\psi + \cos\theta\sin\psi = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \tag{31}$$

where

$$\cos\psi = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}, \sin\psi = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2},\ 0 \le \psi < 2\pi.$$

The angle ψ is therefore $\frac{\pi}{4}$. As a result, equation (31) becomes

$$\sin\left(\theta + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

In the interval $[0, 2\pi)$, there are two angles whose sine is $\frac{\sqrt{2}}{2}$: $\frac{\pi}{4}$ and $\frac{3\pi}{4}$. As a result,

$$\begin{aligned} \theta + \frac{\pi}{4} = \frac{\pi}{4} \ &\mathit{or}\ \theta + \frac{\pi}{4} = \frac{3\pi}{4} \\ \theta = 0 \text{ or } \theta &= \frac{\pi}{2}. \end{aligned}$$

■