Questions 1 to 6 are worth 10 points each. If you are in ENGG 2440A, please turn in solutions to *four* questions of your choice. If you are in ESTR 2004, please turn in solutions to *three* questions of your choice and the Mini-project (worth 30 points).

Write your solutions clearly and concisely. If you do not explain your answer you will be given no credit. You are encouraged to collaborate on the homework, but you must write your own solutions, give credit to your collaborators on your solution sheet, and follow the faculty guidelines and university policies on academic honesty regarding the use of external references.

Questions

1. Find exact closed-form solutions to the following recurrences.

   (a) $f(n) = 11f(n-1) - 2f(n-2)$, $f(0) = 1$, $f(1) = 3$.

   (b) $f(n) = 4f(n/3) + \log(n)$, $f(1) = 3$, where n is a power of 3.

   (c) $f(n) = f(n-1) + f(n-2) + 2$, $f(0) = 0$, $f(1) = 1$.

   (d) $f(n) = \frac{1}{2}f(n-2) + n/2$, $f(0) = 1$, where n is even.

2. A password consists of the digits `0` to `9` and the special symbols `*` and `#`. How many 7 to 9-symbol passwords are there if

   (a) there are the same number of each type of special symbol?

   (b) the password must have at least one even and one odd digit?

   (c) digits and special symbols alternate?

   (d) no special symbol ever immediately follows an odd number? (**Hint:** Write a recurrence.)

3. Use the pigeonhole principle to answer the following questions.

   (a) The population of India is 1.36 billion. Show that there must be a set of people of size least 128400, all of whom have the same birthday and live in the same state. (There are 29 states in India.)

   (b) As humans colonize the galaxy, the number of habitated planets grows as the recurrence: $P(t+1) = 1.1P(t) + 7$, where t denotes years after the start of colonization. The human population growth asymptotically behaves like $O(2^t)$. Show that some planet must have a human population of at least $\Omega(1.8^t)$.

   (c) Dice can have the following properties: number of sides (at least 6 and at most 24), color (red, blue, green, or yellow), and country of manufacture (China, USA, or Japan). What is the largest set of die that you could possibly have such that no 50 of them share all three attributes in common?

4. DNA (Deoxyribonucleic acid) is a molecule that carries the genetic instructions for all known organisms and many viruses. It consists of a chain of bases. In DNA chain, there are four types of bases: **A**, **C**, **G**, **T**. For example, a DNA chain of length 10 can be **ACGTACGTAT**.

   (a) Let $g(n)$ be the number of configurations of a DNA chain of length n, in which the sequences **TT** and **TG** never appear. Write a recurrence for $g(n)$.

   (b) Solve the recurrence from part (a).

   (c) Determine the asymptotic behavior of the recurrence from (a). Also, prove whether this rate of growth is $o(n^{\sqrt{n}})$, $\Theta(n^{\sqrt{n}})$, or $\Omega(n^{\sqrt{n}})$.

5. Let T be the set of all positive integers that do not contain a 4 anywhere in their base-10 representation. For example, 7 and 335 are in T, but 5642 isn't. This question concerns the value of the sum

$$S = \sum_{t \in T} \frac{1}{t} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{13} + \frac{1}{15} + \cdots$$

   (a) Show that the number of d-digit numbers in T is at most $8 \cdot 9^{d-1}$.

   (b) Use part (a) to show that the d-digit numbers contribute at most $8 \cdot 0.9^{d-1}$ to S.

   (c) Prove that $S < 100$. Can you prove that $S < 10$?

   (d) (**Extra credit**) Now let T_d be the set of positive integers that do not contain d *consecutive* 4s. For example, 1454 is in T_2 but 1445 is not. Let $f(d) = \sum_{t \in T_d} 1/t$. Prove that $f(d)$ is finite and find the asymptotic growth of f (e.g., $f(d)$ is $\Theta(2^d)$ or $\Theta(d^2)$ or $\Theta(1)$.)

6. A pair of permutations of $\{1, \dots, n\}$ is a *special pair* if there is some position in which they differ by exactly one. For example, $\{(3, 1, 2, 4), (1, 4, 3, 2)\}$ (when $n = 4$) is a special pair because they differ by exactly one in the third position, but $\{(1, 2, 3, 4), (1, 4, 3, 2)\}$ is not a special pair. A set S_n of permutations of $\{1, \dots, n\}$ is a *special set* if every two permutations within S_n are a special pair.

   (a) Show that when $n = 3$, there exists a special set of size 3 but no special set of size 4.

   (b) Show that if S_n is a special set, the function $f\colon S_n \to \{0, 1\}^n$ given by $f((x_1, x_2, \dots, x_n)) = (x_1 \bmod 2, x_2 \bmod 2, \dots, x_n \bmod 2)$ is injective.

   (c) Use part (b) to show that if S_n is a special set then $|S_n| \le 2^n$.

   (d) Define the sets $S_1, S_2, \dots$ recursively by the formula $S_n = A_n \cup B_n \cup C_n$ where

$$\begin{aligned}
A_n &= \{(n, n-1, x_1, x_2, \dots, x_{n-2})\colon (x_1, x_2, \dots, x_{n-2}) \in S_{n-2}\}, \\
B_n &= \{(x_1, n, n-1, x_2, \dots, x_{n-2})\colon (x_1, x_2, \dots, x_{n-2}) \in S_{n-2}\}, \\
C_n &= \{(n-1, x_1, n, x_2, \dots, x_{n-2})\colon (x_1, x_2, \dots, x_{n-2}) \in S_{n-2}\}.
\end{aligned}$$

   with $S_1 = \{(1)\}$ and $S_2 = \{(1, 2), (2, 1)\}$. Show that S_n is a special set for all n.

   (e) Give a formula for the size of the sets S_n from part (d).

   (f) (**Extra credit**) For $n = 8$, can you find a special set larger than S_8 from part (d)?

ESTR 2004 mini-project You start with a cyclic sequence of n numbers. A particle sits on top of one of the numbers. At each step, the particle moves left or right according to the rule described below, and the number under it is incremented by 1.

(a) The particle moves left if the number under it is even, and right if the number under it is odd. For example:

$$(10, \dot{4}, 3) \to (\dot{10}, 5, 3) \to (11, 5, \dot{3}) \to (\dot{11}, 5, 4) \to (12, \dot{5}, 4)$$

Show that the particle returns to its initial position after finitely many steps.

(b) Let $t(n)$ be the number of steps after which the particle returns to the origin regardless of the initial configuration. Find the best asymptotic upper and lower bounds for $t(n)$ that you can.

(c) The particle now moves left if the number under it is 0 or 1 modulo 4, and right if it is 2 or 3 modulo 4. Repeat parts (a) and (b) for this rule of motion.