Exercises: Divergence Theorem and Stokes' Theorem

Problem 1. This exercise allows you to see the main idea behind the proof of the Divergence Theorem. Suppose that T is a closed region in $\mathbb{R}^3$ whose boundary surface S can be divided into xy-monotone surfaces: S_1 and S_2, whose projections onto the xy-plane are the same region D. (For example, the ball $x^2 + y^2 + z^2 \leq 1$ is such a region because we can divide its boundary into two xy-monotone surfaces (i) S_1: $x^2 + y^2 + z^2 = 1$ with $z \geq 0$, and (ii) S_2: $x^2 + y^2 + z^2 = 1$ with $z \leq 0$.) Let $f(x, y, z)$ be a function that is continuous on S. Orient S by taking its outer side. Prove that

$$\iiint_T \frac{\partial f}{\partial z}\, dxdydz \quad = \quad \iint_S f\, dxdy.$$

Proof: Let S_1 be described by $z = \phi_1(x, y)$ and S_2 by $z = \phi_2(x, y)$, with $(x, y) \in D$.

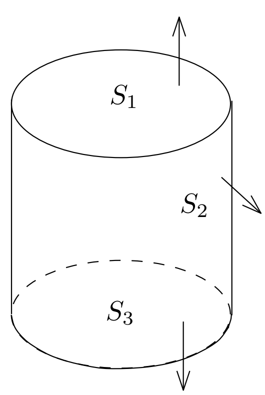
$$\begin{aligned} \iint_S f\, dxdy &= \iint_{S_1} f\, dxdy + \iint_{S_2} f\, dxdy \\ \text{(by the orientation of } S_1,\ S_2) &= \iint_D f(x, y, \phi_1(x, y))\, dxdy - \iint_D f(x, y, \phi_2(x, y))\, dxdy. \quad (1) \end{aligned}$$

On the other hand:

$$\begin{aligned} \iiint_T \frac{\partial f}{\partial z}\, dxdydz &= \iint_D \left(\int_{\phi_2(x,y)}^{\phi_1(x,y)} \frac{\partial f}{\partial z}\, dz \right) dxdy \\ &= \iint_D f(x, y, \phi_1(x, y)) - f(x, y, \phi_2(x, y))\, dxdy \\ &= \text{right hand side of (1).} \end{aligned}$$

□

Problem 2. Consider the cylinder $x^2 + y^2 \leq 1$ and $0 \leq z \leq 1$. Let S be the boundary of the cylinder; see below. Use the Divergence Theorem to calculate $\iint_S xy\, dydz + y^2\, dxdz + z\, dxdy$.



Solution. Introduce $f_1 = xy$, $f_2 = y^2$, and $f_3 = z$. Let T be the cylinder. By Divergence Theorem, we have:

$$
\begin{aligned}
\iint_S xy\,dydz + y^2\,dxdz + z\,dxdy &= \iiint_T \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\,dxdydz \\
&= \iiint_T y + 2y + 1\,dxdydz \\
&= \iiint_T 3y + 1\,dxdydz. \qquad (2)
\end{aligned}
$$

Clearly, $\iiint_T 1\,dxdydz = \pi$ (volume of the cylinder). On the other hand, letting D be the disc $x^2 + y^2 \leq 1$ in the xy-plane, we have:

$$
\begin{aligned}
\iiint_T y\,dxdydz &= \int_0^1 \left(\iint_D y\,dxdy \right) dz \\
&= \iint_D y\,dxdy.
\end{aligned}
$$

To evaluate the above, we represent D using polar coordinates:

$$
\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta
\end{aligned}
$$

where $r \in [0, 1]$ and $\theta \in [0, 2\pi]$. Let R be the set of all such (r, θ). Then:

$$
\begin{aligned}
\iint_D y\,dxdy &= \iint_R r\sin\theta \cdot r\,drd\theta \\
&= \int_0^1 \left(\int_0^{2\pi} r^2\,\sin\theta d\theta \right) dr = 0.
\end{aligned}
$$

Therefore, we know that (2) equals π.

Problem 3. This exercise allows you to derive another popular form of the Divergence theorem. Let T be a closed region in $\mathbb{R}^3$ that is bounded by a surface S, which is the union of a finite number of smooth surfaces $S_1, ..., S_k$. Define $\boldsymbol{f}(x, y, z)$ to be a vector function that is continuous on each S_i $(1 \leq i \leq k)$. For each point $p = (x, y, z)$, define $\boldsymbol{n}(x, y, z)$ to be the unit vector of S at p pointing towards the outside of S. Prove:

$$
\iiint_T \text{div } \boldsymbol{f}\,dxdydz = \iint_S \boldsymbol{f} \cdot \boldsymbol{n}\,dA.
$$

Proof. We can write $\boldsymbol{n}$ as $[\cos\alpha, \cos\beta, \cos\gamma]$, where

- α is the angle between $\boldsymbol{n}$ and $\boldsymbol{i} = [1, 0, 0]$;
- β is the angle between $\boldsymbol{n}$ and $\boldsymbol{j} = [0, 1, 0]$;
- γ is the angle between $\boldsymbol{n}$ and $\boldsymbol{k} = [0, 0, 1]$.

Hence:

$$\begin{aligned}
\iint_S \boldsymbol{f}\cdot\boldsymbol{n}\,dA &= \iint_S (f_1\cdot\cos\alpha + f_2\cdot\cos\beta + f_3\cdot\cos\gamma)\,dA \\
\text{(taking the outer side of } S\text{)} &= \iint_S f_1\,dydz + f_2\,dxdz + f_3\,dxdy \\
&= \iiint_T \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\,dxdydz \\
&= \iiint_T \operatorname{div}\,\boldsymbol{f}\,dxdydz.
\end{aligned}$$

□

Problem 4. Let $\boldsymbol{f} = [e^x, e^y, e^z]$. Let S be the boundary of the cube with $|x| \le 1$, $|y| \le 1$, and $|z| \le 1$. Let $\boldsymbol{n}$ be defined as in the previous problem. Calculate $\iint_S \boldsymbol{f}\cdot\boldsymbol{n}\,dA$.

Solution. Let T be the region enclosed by the cube. We have:

$$\iint_S \boldsymbol{f}\cdot\boldsymbol{n}\,dA = \iiint_T e^x + e^y + e^z\,dxdydz. \tag{3}$$

Focusing on the e^z term, we have:

$$\begin{aligned}
\iiint_T e^z\,dxdydz &= \int_{-1}^{1}\left(\int_{-1}^{1}\left(\int_{-1}^{1} e^z\,dz\right)dy\right)dx \\
&= (e - 1/e)\int_{-1}^{1}\left(\int_{-1}^{1} dy\right)dx \\
&= 4(e - 1/e).
\end{aligned}$$

Similarly, $\iiint_T e^x\,dxdydz = \iiint_T e^y\,dxdydz = 4(e - 1/e)$. Hence, (3) equals $12(e - 1/e)$.

Problem 5. Let C be the curve that is the intersection of

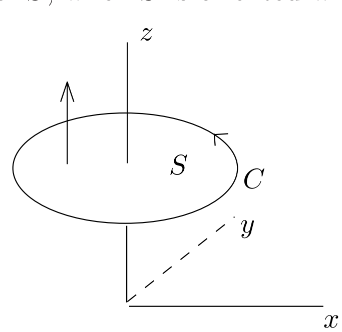
$$\begin{aligned}
x^2 + y^2 &= 2z \\
z &= 2
\end{aligned}$$

Designate the direction of C as passing points $(2, 0, 2)$, $(0, 2, 2)$, and $(-2, 0, 2)$ in this sequence. Use the Stokes' theorem to calculate $\int_C y\,dx - xz\,dy + yz^2\,dz$.

Solution. Let S be the surface satisfying the following two equations simultaneously:

$$\begin{aligned}
x^2 + y^2 &\le 4 \\
z &= 2.
\end{aligned}$$

We can regard C as the boundary of S, when S is oriented with its upper side taken.



Introduce functions $f_1 = y$, $f_2 = -xz$, and $f_3 = yz^2$. Then,

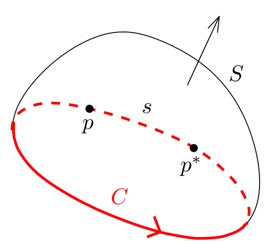
$$\begin{aligned}
\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} &= z^2 + x = 4 + x \\
\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} &= 0 \\
\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} &= -z - 1 = -3
\end{aligned}$$

By Stokes' Theorem, we know:

$$\int_C y\,dx - xz\,dy + yz^2\,dz = \iint_S (4+x)dydz - 3\,dxdy. \tag{4}$$

Note that S is perpendicular to the yz-plane. Hence, $\iint_S (4+x)dydz = 0$. On the other hand, $3\iint_S dxdy$ is clearly $3 \cdot (4\pi) = 12\pi$. Therefore, (4) equals -12π.

Problem 6. This exercise allows you to see an alternative form of the Stokes' theorem. Let S be a piecewise surface and C its boundary curve, both oriented in the way described in the Stokes theorem (see lecture notes). Also, let f_1, f_2, f_3 be functions that have continuous partial derivatives on each smooth surface that constitutes S.



Define $\boldsymbol{f}(x,y,z) = [f_1, f_2, f_3]$, and $\boldsymbol{n}(x,y,z)$ be the unit normal vector of S at point (x,y,z), emanating from the side of S chosen. Fix any point p^* on C. Given any point p on C, denote by s the length of the curve from p^* to p, following the direction of C. Let $\boldsymbol{r}(s) = [x(s), y(s), z(s)]$ be a parametric form of C. Prove:

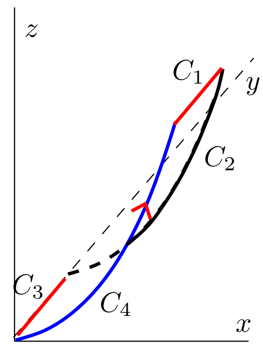
$$\iint_S \mathrm{curl}\boldsymbol{f} \cdot \boldsymbol{n}\,dA = \int_C \boldsymbol{f} \cdot \boldsymbol{r}'(s)\,ds.$$

Proof. By Stokes' theorem, we have

$$\begin{aligned}
\iint_S \mathrm{curl}\boldsymbol{f} \cdot \boldsymbol{n}\,dA &= \int_C f_1\,dx + f_2\,dy + f_3\,dz \\
&= \int_C \boldsymbol{f} \cdot d\boldsymbol{r} \\
&= \int_C \boldsymbol{f} \cdot \frac{d\boldsymbol{r}}{ds}\,ds \\
&= \int_C \boldsymbol{f} \cdot \boldsymbol{r}'(s)\,ds.
\end{aligned}$$

□

Problem 7. Let S be the surface $z = x^2$ with $0 \le x \le 2$ and $0 \le y \le 1$. Orient S by taking its upper side. Define $\boldsymbol{f} = [e^y, e^z, e^x]$. Calculate $\iint_S \mathrm{curl}\boldsymbol{f} \cdot \boldsymbol{n}\,dA$, where $\boldsymbol{n}$ is as defined in the previous problem. Calculate $\iint_S \mathrm{curl}\boldsymbol{f} \cdot \boldsymbol{n}\,dA$.



Solution. Let C be the boundary curve of S, directed as shown above. C consists of 4 curves: $C_1, C_2, ..., C_4$. By Stokes' theorem, we have:

$$\iint_S \mathrm{curl}\boldsymbol{f} \cdot \boldsymbol{n}\, dA \;=\; \int_C e^y\, dx + e^z\, dy + e^x\, dz. \tag{5}$$

Next, we focus on each of the above terms separately.

$$\begin{aligned}
\int_C e^y\, dx &= \sum_{i=1}^{4} \int_{C_i} e^y\, dx \\
&= \int_{C_2} e^y\, dx + \int_{C_4} e^y\, dx \\
&= e^1 \int_{C_2} dx + e^0 \int_{C_4} dx \\
\text{(note the directions of } C_2,\, C_4) &= -2e + 2.
\end{aligned}$$

$$\begin{aligned}
\int_C e^z\, dy &= \sum_{i=1}^{4} \int_{C_i} e^z\, dy \\
&= \int_{C_1} e^z\, dy + \int_{C_3} e^z\, dy \\
&= e^4 \int_{C_1} dy + e^0 \int_{C_3} dy \\
&= e^4 - 1.
\end{aligned}$$

$$\begin{aligned}
\int_C e^x\, dz &= \sum_{i=1}^{4} \int_{C_i} e^x\, dz \\
&= \int_{C_2} e^x\, dz + \int_{C_4} e^x\, dz \\
&= \int_{C_2} e^x \frac{dz}{dx}\, dx + \int_{C_4} e^x \frac{dz}{dx}\, dx \\
&= \int_2^0 2x \cdot e^x\, dx + \int_0^2 2x \cdot e^x\, dx \\
&= 0.
\end{aligned} \tag{6}$$

Therefore, (5) equals $e^4 - 2e + 1$.

Problem 8. Fix a vector function $\boldsymbol{f}(x, y, z) = [f_1, f_2, f_3]$. Prove that if $\mathrm{curl}\boldsymbol{f} = \mathbf{0}$, then the class of line integrals $\int_C f_1\,dx + f_2\,dy + f_3\,dz$ is path independent.

Proof. It suffices to prove that for any closed piecewise smooth curve C, it holds that $\int_C f_1\,dx + f_2\,dy + f_3\,dz = 0$. Find an arbitrary piecewise smooth surface S whose boundary curve is C. Orient S according to the direction of C. By the Stokes' theorem, it holds that

$$\int_C f_1\,dx + f_2\,dy + f_3\,dz \quad = \quad \iint_S \mathrm{curl}\boldsymbol{f} \cdot \boldsymbol{n}\,dA = 0.$$

We thus complete the proof. □