Battle of the Sexes Revisited

		Wife	
		Boxing	*Opera*
Husband	*Boxing*	2, 1	0, 0
	Opera	0, 0	1, 2

There are two Nash equilibria: (B, B) and (O, O).

Battle of the Sexes Revisited

		Wife	
		Boxing	*Opera*
Husband	*Boxing*	2, 1	0, 0
	Opera	0, 0	1, 2

- What if the wife knows that there is a probability of $\frac{1}{3}$ that the husband goes to Boxing, and $\frac{2}{3}$ that the husband goes to Opera?
- What if the chances are $\frac{1}{2}$ and $\frac{1}{2}$?

Mixed Strategies

Player 1's available actions $A_1 = \{a_1, a_2, a_3, a_4, a_5\}$.

A **mixed strategy** of player 1:

$$\alpha_1 = (a_1(0), a_2(\tfrac{1}{2}), a_3(0), a_4(\tfrac{1}{4}), a_5(\tfrac{1}{4})).$$

Notations:

$$\alpha_1(a_1) = 0,\ \alpha_1(a_2) = \frac{1}{2},\ \alpha_1(a_3) = 0,$$

$$\alpha_1(a_4) = \frac{1}{4},\ \alpha_1(a_5) = \frac{1}{4}.$$

Support of α_1 is $\{a_2, a_4, a_5\}$.

- Another **mixed strategy** of player 1:

$$\alpha_1' = (a_1(\tfrac{1}{2}), a_2(\tfrac{1}{6}), a_3(0), a_4(\tfrac{1}{6}), a_5(\tfrac{1}{6})).$$

$$\alpha_1'(a_1) = \frac{1}{2},\ \alpha_1'(a_2) = \frac{1}{6},\ \alpha_1'(a_3) = 0,$$
$$\alpha_1'(a_4) = \frac{1}{6},\ \alpha_1'(a_5) = \frac{1}{6}.$$

Support of α_1' is $\{a_1, a_2, a_4, a_5\}$.

Denote by $\Delta(A_1) = \{\alpha_1, \alpha_1', \alpha_1'', \alpha_1''', \ldots\}$ the set of mixed strategies over the set A_1 of player 1's available actions.

Denote by $\Delta(A_1) = \{\alpha_1, \alpha_1', \alpha_1'', \alpha_1''', \ldots\}$ the set of mixed strategies over the set A_1 of player 1's available actions.

<u>Example.</u> If $A_1 = \{a_1, a_2, a_3\}$ **then**

$\Delta(A_1) =$

$\{(a_1(\frac{1}{2}), a_2(\frac{1}{4}), a_3(\frac{1}{4})),\ (a_1(\frac{1}{4}), a_2(\frac{1}{12}), a_3(\frac{2}{3})),$

$(a_1(0), a_2(\frac{1}{2}), a_3(\frac{1}{2})),\ (a_1(\frac{1}{9}), a_2(\frac{5}{9}), a_3(\frac{1}{3})),$

$\ldots\}$

Mixed Strategies

		Wife	
		Boxing	*Opera*
Husband	*Boxing*	2, 1	0, 0
	Opera	0, 0	1, 2

$$A_h = \{B, O\} A_w = \{B, O\}$$
$$\Delta(A_h) = \{(B(p), O(q)) : 1 \geq p \geq 0, 1 \geq q \geq 0, p + q = 1\}$$
$$\Delta(A_w) = \{(B(p), O(q)) : 1 \geq p \geq 0, 1 \geq q \geq 0, p + q = 1\}$$

		Wife	
		Boxing	*Opera*
Husband	*Boxing*	2, 1	0, 0
	Opera	0, 0	1, 2

$$\Delta(A_h) = \{(B(p), O(q)) : 1 \geq p \geq 0, 1 \geq q \geq 0, p + q = 1\}$$
$$\Delta(A_w) = \{(B(p), O(q)) : 1 \geq p \geq 0, 1 \geq q \geq 0, p + q = 1\}$$

A mixed strategy profile:

$$(\underbrace{((B(\tfrac{1}{2}), O(\tfrac{1}{2}))}_{\alpha_h}, \underbrace{((B(\tfrac{1}{3}), O(\tfrac{2}{3}))}_{\alpha_w}) \in \Delta(A_h) \times \Delta(A_w)$$

If all n players are playing mixed strategies,

Player 1 plays α_1;
Player 2 plays α_2;
…
Player n plays α_n.

Then we have a mixed strategy profile

$$(\alpha_1, \alpha_2, \dots \alpha_n) = (\alpha_i)_{i \in N}, \text{ or simply } (\alpha_i).$$

Note

$$(\alpha_1, \alpha_2, \dots \alpha_n) \in \Delta(A_1) \times \Delta(A_2) \times \dots \times \Delta(A_n)$$

$$(\alpha_1, \alpha_2, \dots \alpha_n) \in \times_{j \in N} \Delta\left(A_j\right)$$

$$((B(\tfrac{1}{2}), O(\tfrac{1}{2})), (B(\tfrac{1}{3}), O(\tfrac{2}{3})))$$

Wife

Husband		*Boxing*	*Opera*
	Boxing	prob.=1/6	prob.=1/3
	Opera	prob.=1/6	prob.=1/3

$$((B(\tfrac{1}{3}), O(\tfrac{2}{3})), (B(\tfrac{2}{3}), O(\tfrac{1}{3})))$$

Wife

Husband		*Boxing*	*Opera*
	Boxing	prob.=2/9	prob.=1/9
	Opera	prob.=4/9	prob.=2/9

Let there be three players $N = \{1,2,3\}$.

Consider a profile $(\alpha_1, \alpha_2, \alpha_3)$ of mixed strategies.

$$\alpha_1 = (a_1(\frac{1}{2}), a_2(\frac{1}{12}), a_3(\frac{1}{4}), a_4(\frac{1}{12}), a_5(\frac{1}{12}))$$
$$\alpha_2 = (b_1(\frac{1}{4}), b_2(\frac{1}{4}), b_3(0), b_4(\frac{1}{2}))$$
$$\alpha_3 = (c_1(\frac{1}{3}), c_2(\frac{2}{3}), c_3(0))$$

With this $(\alpha_j)_{j\in N} \in \times_{j\in N} \Delta(A_j)$,

- The probability of $(a_1, b_2, c_2) \in A$ is $\frac{1}{2} \times \frac{1}{4} \times \frac{2}{3} = \frac{1}{12}$.
- The probability of $(a_3, b_3, c_1) \in A$ is $\frac{1}{4} \times 0 \times \frac{1}{3} = 0$.
- and so on.

$$\alpha_1 = (a_1(\tfrac{1}{2}), a_2(\tfrac{1}{12}), a_3(\tfrac{1}{4}), a_4(\tfrac{1}{12}), a_5(\tfrac{1}{12}))$$
$$\alpha_2 = (b_1(\tfrac{1}{4}), b_2(\tfrac{1}{4}), b_3(0), b_4(\tfrac{1}{2}))$$
$$\alpha_3 = (c_1(\tfrac{1}{3}), c_2(\tfrac{2}{3}), c_3(0))$$

- The probability of $(a_1, b_2, c_2) \in A$ is $\frac{1}{2} \times \frac{1}{4} \times \frac{2}{3} = \frac{1}{12}$.
- The probability of $(a_3, b_3, c_1) \in A$ is $\frac{1}{4} \times 0 \times \frac{1}{3} = 0$.
- The probability of $(a_3, b_4, c_2) \in A$ is $\frac{1}{4} \times \frac{1}{2} \times \frac{2}{3} = \frac{1}{12}$.
- and so on.

This particular profile $(\alpha_1, \alpha_2, \alpha_3) \in \times_{j \in N} \Delta(A_j)$ induces a probability for each member profile $a \in A = \times_{j \in N} A_j$.

- The probability of $(a_1, b_2, c_2) \in A$ is $\frac{1}{2} \times \frac{1}{4} \times \frac{2}{3} = \frac{1}{12}$.
- The probability of $(a_3, b_3, c_1) \in A$ is $\frac{1}{4} \times 0 \times \frac{1}{3} = 0$.
- The probability of $(a_3, b_4, c_2) \in A$ is $\frac{1}{4} \times \frac{1}{2} \times \frac{2}{3} = \frac{1}{12}$.
- and so on.

Q: How does player 1 evaluate this profile $\left(\alpha_j\right)_{j \in N}$?

A: Player 1′s evaluation of the profile $\left(\alpha_j\right)_{j \in N}$ is

$$\underbrace{\left(\frac{1}{2} \cdot \frac{1}{4} \cdot \frac{2}{3}\right) \cdot u_1\big((a_1, b_8, c_2)\big) + \left(\frac{1}{4} \cdot 0 \cdot \frac{1}{3}\right) \cdot u_1\big((a_3, b_9, c_1)\big) + \cdots}_{\text{one term for each } a \in A}$$

In general,

Q: How does player i evaluate a profile $\left(\alpha_j\right)_{j\in N}$?

A: Player i's evaluation of the profile $\left(\alpha_j\right)_{j\in N}$ is

$$U_i(\alpha) = \sum_{a\in A}\left(\prod_{j\in N}\alpha_j\left(a_j\right)\right)\cdot u_i(a)$$

if A is finite.

Hence $U_i : \times_{j\in N}\Delta\left(A_j\right) \to \mathbb{R}$ is the **utility function** for player i to evaluate the profiles of mixed strategies.

Note that $\langle N, (\Delta(A_i)), (U_i) \rangle$ can be seen as a strategic game.

- The set of players is N.
- The set of **'actions'** available to player i is a set of mixed strategies.
- Each player uses $U_i : \times_{j \in N} \Delta(A_j) \to \mathbb{R}$ to evaluate a mixed strategy profile.

The game $\langle N, (\Delta(A_i)), (U_i) \rangle$ is called the **mixed extension** of $\langle N, (A_i), (u_i) \rangle$.

Mixed Extensions of Strategic Games

DEFINITION. The **mixed extension** of the strategic game $\langle N, (A_i), (u_i) \rangle$ is the strategic game $\left\langle N, \left(\Delta(A_i)\right), (U_i) \right\rangle$ in which $\Delta(A_i)$ is the set of probability distributions over A_i, and $U_i : \times_{j \in N} \Delta\left(A_j\right) \to \mathbb{R}$ assigns to each $\alpha \in \times_{j \in N} \Delta\left(A_j\right)$ the expected value, under u_i, of the lottery over A that is induced by α (so that $U_i(\alpha) = \sum_{a \in A} \left(\prod_{j \in N} \alpha_j\left(a_j\right) \right) u_i(a)$ if A is finite).

Notation: Lottery

$$A = \{a, b, c, d, e\}$$

A sample lottery over A:

$\left(a\left({}^{1}/_{5}\right), b\left({}^{2}/_{5}\right), c\left({}^{1}/_{10}\right), d\left({}^{1}/_{10}\right), e\left({}^{1}/_{5}\right)\right).$

Other lotteries over A:

$$(a\left(\frac{1}{5}\right), b\left(\frac{1}{5}\right), c\left(\frac{1}{5}\right), d\left(\frac{1}{5}\right), e\left(\frac{1}{5}\right))$$

$$(a\left(\frac{1}{3}\right), b\left(\frac{1}{9}\right), c\left(\frac{1}{9}\right), d(0), e\left(\frac{4}{9}\right))$$

$$(a\left(\frac{1}{7}\right), b\left(\frac{3}{7}\right), c\left(\frac{1}{14}\right), d\left(\frac{2}{7}\right), e\left(\frac{1}{14}\right))$$

$$(a(1), b(0), c(0), d(0), e(0))$$

$$\ldots$$

Mixed Strategy Nash Equilibrium

DEFINITION. A **mixed strategy Nash equilibrium of a strategic game** is a Nash equilibrium of its mixed extension.

That is, a **mixed strategy Nash equilibrium** of $\langle N, (A_i), (u_i) \rangle$ is defined to be a Nash equilibrium of $\left\langle N, \left(\Delta(A_i)\right), (U_i) \right\rangle$.

Class Discussion

Let there be three players $N = \{1,2,3\}$. Let $\alpha^* = (\alpha_1^*, \alpha_2^*, \alpha_3^*) \in \times_{j \in N} \Delta\left(A_j\right)$ be a **mixed strategy Nash equilibrium** of $G = \langle N, (A_i), (u_i) \rangle$, and

$$\alpha_1 = \left(a_1(1), a_2(0), a_3(0), a_4(0), a_5(0)\right)$$
$$\alpha_2 = \left(b_1(0), b_2(1), b_3(0), b_4(0)\right)$$
$$\alpha_3 = \left(c_1(0), c_2(1), c_3(0)\right)$$

Q: Is $(a_1, b_2, c_2) \in A$ a Nash equilibrium of G?

Notations:

$$e(a_1) = \left(a_1(1), a_2(0), a_3(0), a_4(0), a_5(0)\right)$$
$$e(b_2) = \left(b_1(0), b_2(1), b_3(0), b_4(0)\right)$$
$$e(c_2) = \left(c_1(0), c_2(1), c_3(0)\right)$$

That is, $e(a_i)$ denotes the degenerate mixed strategy of player i that attaches probability one to $a_i \in A_i$.

Notations:

If

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n),$$

then

$$\alpha_{-i} = (\alpha_1, \alpha_2, \dots \alpha_{i-1}, \quad \alpha_{i+1}, \dots, \alpha_n),$$

$$(\alpha_{-i}, \beta) = (\alpha_1, \alpha_2, \dots \alpha_{i-1}, \beta, \alpha_{i+1}, \dots, \alpha_n)$$

Notations:

If $\alpha_1 = \big(a_1(p_1), a_2(p_2), \dots, a_n(p_n)\big)$, then

$$\lambda\alpha_1 = \big(a_1(\lambda p_1), a_2(\lambda p_2), \dots, a_n(\lambda p_n)\big).$$

For example, let

$$\alpha_1 = \left(a_1(0), a_2\left(\frac{1}{2}\right), a_3(0), a_4\left(\frac{1}{4}\right), a_5\left(\frac{1}{4}\right)\right),$$

$$\alpha_2 = \left(a_1\left(\frac{1}{2}\right), a_2\left(\frac{1}{3}\right), a_3(0), a_4\left(\frac{1}{12}\right), a_5\left(\frac{1}{12}\right)\right),$$

then

$$\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 = \left(a_1\left(\frac{1}{4}\right), a_2\left(\frac{5}{12}\right), a_3(0), a_4\left(\frac{1}{6}\right), a_5\left(\frac{1}{6}\right)\right).$$

Multilinearity of Function $\boldsymbol{U_i}$

Let α be a mixed strategy profile; β_i and γ_i two mixed strategies of player i; $\lambda \in [0,1]$ a real number.

We have

$$U_i(\alpha_{-i}, \lambda\beta_i + (1-\lambda)\gamma_i) \\ = \lambda U_i(\alpha_{-i}, \beta_i) + (1-\lambda)U_i(\alpha_{-i}, \gamma_i)$$

EXAMPLE. Suppose player 1 has only three available actions r, s and t. In a profile α, player 1 plays $\alpha_1 = \left(r\left(\frac{1}{7}\right), s\left(\frac{5}{6}\right), t\left(\frac{1}{42}\right)\right)$. Then by the multilinearity of U_i in α, we have

$$U_1(\alpha) = \frac{1}{7} \cdot U_1\big(\alpha_{-1}, e(r)\big) + \frac{5}{6} \cdot U_1\big(\alpha_{-1}, e(s)\big) + \frac{1}{42} \cdot U_1\big(\alpha_{-1}, e(t)\big)$$

In general, we have

$$U_i(\alpha) = \sum_{a \in A_i} \alpha_i(a) U_i\big(\alpha_{-i}, e(a)\big)$$

Class Discussion

Suppose player 2 has four available actions: $A_2 = \{u, v, w, x\}$. Let α be a profile such that

$$U_2((\alpha_{-2}, e(u))) = 5$$
$$U_2((\alpha_{-2}, e(v))) = 2$$
$$U_2((\alpha_{-2}, e(w))) = 0$$
$$U_2((\alpha_{-2}, e(x))) = 5$$

Q: What **mixed strategy** should player 2 play?

LEMMA. Let $G = \langle N, (A_i), (u_i) \rangle$ be a finite strategic game. Then $\alpha^* \in \times_{i \in N} \Delta(A_i)$ is a mixed strategy Nash equilibrium of G if and only if for every player $i \in N$, every pure strategy in the support of α_i^* is a best response to α_{-i}^*.

Every action in the support of any player's equilibrium mixed strategy yields that player the same payoff.

$$U_2((\alpha_{-2}, (u(\tfrac{1}{2}), v(0), w(0), x(\tfrac{1}{2})))) =$$
$$U_2((\alpha_{-2}, (u(\tfrac{1}{9}), v(0), w(0), x(\tfrac{8}{9})))) =$$
$$U_2((\alpha_{-2}, (u(0), v(0), w(0), x(1)))) =$$
$$U_2((\alpha_{-2}, (u(0), v(0), w(0), x(1)))) = \cdots = 5$$

Class Discussion

Let α be an arbitrary profile of mixed strategies. Suppose

$$U_2((\alpha_{-2}, e(u))) = 5$$
$$U_2((\alpha_{-2}, e(v))) = 2$$
$$U_2((\alpha_{-2}, e(w))) = 0$$
$$U_2((\alpha_{-2}, e(x))) = 5$$

Q: Does it make any difference if player 2 plays $(u(\frac{1}{2}), v(0), w(0), x(\frac{1}{2}))$, or $(u(0), v(0), w(0), x(1))$, or $(u(\frac{998}{1000}), v(0), w(0), x(\frac{2}{1000}))$ in response to α_{-2}?

Class Discussion

Let α^* be a mixed strategy Nash equilibrium. Suppose

$$U_2((\alpha_{-2}, e(u))) = 5$$
$$U_2((\alpha_{-2}, e(v))) = 2$$
$$U_2((\alpha_{-2}, e(w))) = 0$$
$$U_2((\alpha_{-2}, e(x))) = 5$$

Q: Does it make any difference if player 2 plays $(u(\frac{1}{2}), v(0), w(0), x(\frac{1}{2}))$, or $(u(0), v(0), w(0), x(1))$, or $(u(\frac{998}{1000}), v(0), w(0), x(\frac{2}{1000}))$ in response to α_{-2}?

That is, if

$$\left(\alpha^*_{-2}, (u(\tfrac{1}{2}), v(0), w(0), x(\tfrac{1}{2}))\right)$$

is a mixed strategy Nash equilibrium, then

$$\left(\alpha^*_{-2}, e(x)\right) \text{ or}$$

$$\left(\alpha^*_{-2}, (u(\tfrac{998}{1000}), v(0), w(0), x(\tfrac{2}{1000}))\right)$$

is not necessarily a mixed strategy Nash equilibrium.

Q:

$U_2((\alpha^*_{-2}, (u\left(\frac{1}{2}\right), v(0), w(0), x(\frac{1}{2})))) = U_2((\alpha^*_{-2}, e(x)))$?

Class Discussion

$$U_i(\alpha_{-i}, \lambda\beta_i + (1-\lambda)\gamma_i)$$
$$= \lambda U_i(\alpha_{-i}, \beta_i) + (1-\lambda)U_i(\alpha_{-i}, \gamma_i)$$

Q1: If $\big(e(a_1), e(b_8), e(c_2)\big) \in \Delta(A_1) \times \Delta(A_2) \times \Delta(A_3)$ is a mixed strategy Nash equilibrium of G, then is $(a_1, b_8, c_2) \in A$ a Nash equilibrium of G?

Q2: And vice versa? (*Is it true that, if* a^* *is a Nash equilibrium of* G, *then* $\big(e(a_i^*)\big)$ *is a mixed strategy Nash equilibrium of* G?)

Existence of Mixed Strategy Nash Equilibrium

PROPOSITION. Every finite strategic game has a mixed strategy Nash equilibrium.

		Wife	
		Boxing	*Opera*
Husband	*Boxing*	2, 1	0, 0
	Opera	0, 0	1, 2

Let (α_h, α_w) be a mixed strategy Nash equilibrium.

CASE 1. $\alpha_h(B) = 0$ or 1.

There are two Nash equilibria (B, B) and (O, O), as we already knew.

		Wife	
		Boxing	*Opera*
Husband	*Boxing*	2, 1	0, 0
	Opera	0, 0	1, 2

Let (α_h, α_w) be a mixed strategy Nash equilibrium.

CASE 2. $0 < \alpha_h(B) < 1$.

So we must have $U_h(e(B), \alpha_w) = U_h(e(O), \alpha_w)$, or $2 \cdot \alpha_w(B) + 0 \cdot \alpha_w(O) = 0 \cdot \alpha_w(B) + 1 \cdot \alpha_w(O)$. Since $\alpha_w(B) + \alpha_w(O) = 1$, we have $\alpha_w(B) = \frac{1}{3}$ and $\alpha_w(O) = \frac{2}{3}$.

Moreover, since $0 < \alpha_w(B) < 1$, we must also have $U_w\big(\alpha_h, e(B)\big) = U_w\big(\alpha_h, e(O)\big)$, or $\alpha_h(B) = 2\alpha_h(O)$. Thus $\alpha_h(B) = \frac{2}{3}$.

Thus, the only nondegenerate mixed strategy Nash equilibrium of the game is $((B\left(\frac{2}{3}\right), O\left(\frac{1}{3}\right)), (B\left(\frac{1}{3}\right), O\left(\frac{2}{3}\right)))$, or more simply denoted $((\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3}))$, .

Q: What are the husband's and the wife's payoffs in the mixed strategy Nash equilibrium $((\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3}))$?

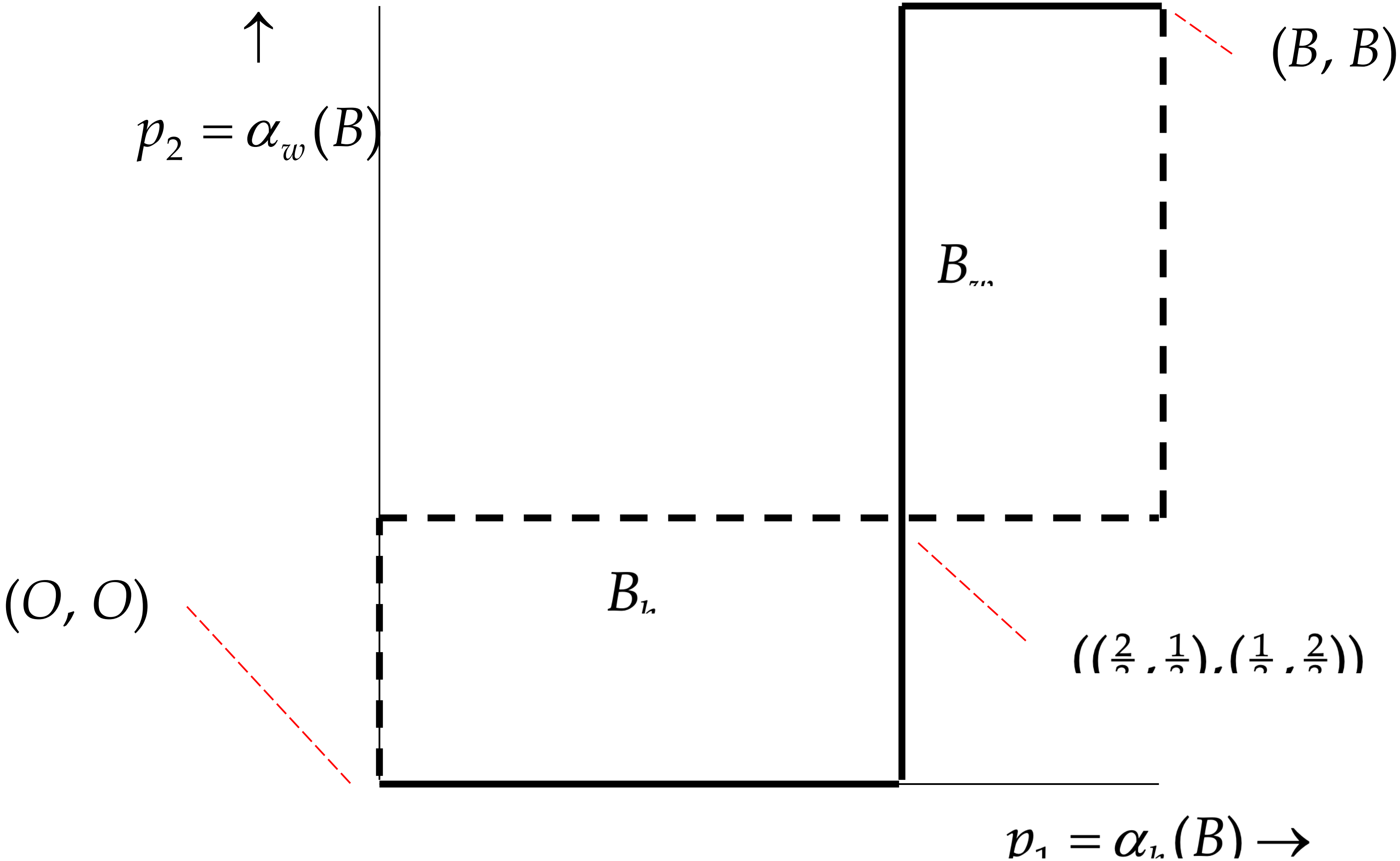
A: Both are $\frac{2}{3}$

		Wife	
		Boxing	*Opera*
Husband	*Boxing*	2, 1	0, 0
	Opera	0, 0	1, 2

<u>An In-depth Investigation</u>

Suppose $\alpha_h = \big(B(p_1), O(1-p_1)\big)$ and $\alpha_w = \big(B(p_2), O(1-p_2)\big)$. The expected utility of the wife is $U_w = p_1 p_2 + 2(1-p_1)(1-p_2)$, or $(3p_1 - 2)p_2 - 2p_1 + 2$. To maximise it, the wife should take $p_2 = 0$ if $p_1 < \frac{2}{3}$; $p_2 = 1$ if $p_1 > \frac{2}{3}$; and $p_2 =$ any value $\in [0,1]$ if $p_1 = \frac{2}{3}$.



The Players' Best Response Functions in the Mixed Extension of Battle of the Sexes

Class Discussion

		Player 2	
		Head	*Tail*
Player 1	*Head*	1, –1	–1, 1
	Tail	–1, 1	1, –1

Find a mixed strategy Nash equilibrium (α_1, α_2).

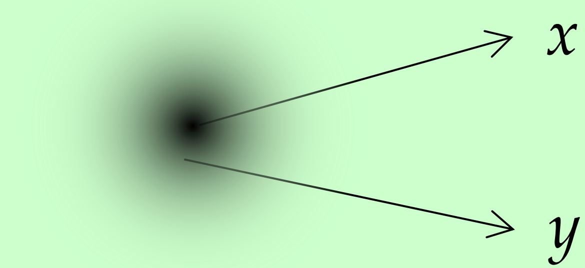
Assume $0 < \alpha_1(H) < 1$.

- $U_1(e(H), \alpha_2) = U_1(e(T), \alpha_2)$.
- $\alpha_2(H) + (-1) \cdot \alpha_2(T) = (-1) \cdot \alpha_2(H) + \alpha_2(T)$.
- Therefore, $\alpha_2(H) = \alpha_2(T) = \frac{1}{2}$.

Battle of the Sexes Revisited, Again

		Wife: *Boxing*	Wife: *Opera*
Husband	*Boxing*	2, 1	0, 0
	Opera	0, 0	1, 2

(random variable)



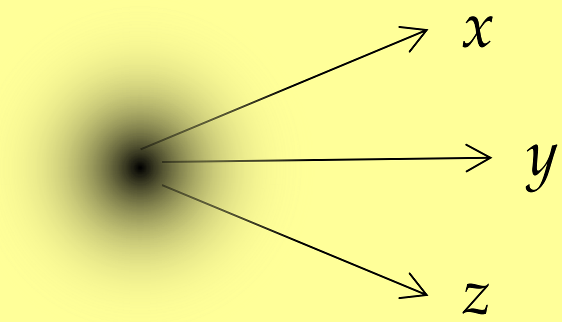
	Husband	**Wife**
x	B	B
y	O	O

Battle of the Sexes Revisited, Again

Wife

		Boxing	*Opera*
Husband	*Boxing*	2, 1	0, 0
	Opera	0, 0	1, 2

(random variable)



	Husband	**Wife**
x	$a_{h,1}$	$a_{w,1}$
y	$a_{h,2}$	$a_{w,1}$
z	$a_{h,2}$	$a_{w,2}$

Battle of the Sexes Revisited, Again

<table>
<tr><th></th><th>Husband</th><th>Wife</th></tr>
<tr><td>x</td><td>$a_{h,1}$</td><td rowspan="2">$a_{w,1}$</td></tr>
<tr><td>y (prob. $= p$)</td><td rowspan="2">$a_{h,2}$</td></tr>
<tr><td>z (prob. $= q$)</td><td>$a_{w,2}$</td></tr>
</table>

If the husband is informed that either y or z has occurred, then he chooses an action $a_{h,2}$ that is optimal given that the wife chooses $a_{w,1}$ with probability $\frac{p}{p+q}$ and $a_{w,2}$ with probability $\frac{q}{p+q}$.

Battle of the Sexes Revisited, Again

Ω	π	**Husband**	**Wife**
x	(prob. $= r$)	$a_{h,1}$	$a_{w,1}$
y	(prob. $= p$)	$a_{h,2}$	
z	(prob. $= q$)		$a_{w,2}$

Set of states: $\Omega = \{x, y, z\}$.

Probability measures: $\pi(x) = r,\ \pi(y) = p,\ \pi(z) = q$.

Husband's Information Partition: $\mathcal{P}_h = \big\{\{x\}, \{y, z\}\big\}$.

Husband's Strategy: $\sigma_h = \big\{x \mapsto a_{h,1}, y \mapsto a_{h,2}, z \mapsto a_{h,2}\big\}$.

Wife's Information Partition: $\mathcal{P}_w = \{\{x, y\}, \{z\}\}$.

Wife's Strategy: $\sigma_w = \big\{x \mapsto a_{w,1}, y \mapsto a_{w,1}, z \mapsto a_{w,2}\big\}$.

<table>
<tr><th>Ω</th><th>π</th><th>Husband</th><th>Wife</th></tr>
<tr><td>x</td><td>(prob. = r)</td><td>$a_{h,1}$</td><td rowspan="2">$a_{w,1}$</td></tr>
<tr><td>y</td><td>(prob. = p)</td><td rowspan="2">$a_{h,2}$</td></tr>
<tr><td>z</td><td>(prob. = q)</td><td>$a_{w,2}$</td></tr>
</table>

Husband's Strategy: $\sigma_h = \{\underbrace{x \mapsto a_{h,1}}_{x}, \underbrace{y \mapsto a_{h,2}, z \mapsto a_{h,2}}_{y,z}\}$

Consider: $\tau_h = \{\underbrace{x \mapsto a'_{h,1}}_{x}, \underbrace{y \mapsto a'_{h,2}, z \mapsto a'_{h,2}}_{y,z}\}$.

We say σ_h is better than τ_h if and only if

$$ru_h(a_{h,1}, a_{w,1}) + pu_h(a_{h,2}, a_{w,1}) + qu_h(a_{h,2}, a_{w,2}) >$$
$$ru_h(a'_{h,1}, a_{w,1}) + pu_h(a'_{h,2}, a_{w,1}) + qu_h(a'_{h,2}, a_{w,2})$$

Correlated Equilibrium

If both the husband and the wife are choosing their best strategies, that is, for any τ_h and τ_w we have

$$ru_h(a_{h,1}, a_{w,1}) + pu_h(a_{h,2}, a_{w,1}) + qu_h(a_{h,2}, a_{w,2}) \geq$$
$$ru_h(a'_{h,1}, a_{w,1}) + pu_h(a'_{h,2}, a_{w,1}) + qu_h(a'_{h,2}, a_{w,2})$$

and

$$ru_w(a_{h,1}, a_{w,1}) + pu_w(a_{h,2}, a_{w,1}) + qu_w(a_{h,2}, a_{w,2}) \geq$$
$$ru_w(a_{h,1}, a'_{w,1}) + pu_w(a_{h,2}, a'_{w,1}) + qu_w(a_{h,2}, a'_{w,2})$$

then the situation is said to be in **correlated equilibrium**.

Correlated Equilibrium

DEFINITION. A **correlated equilibrium of a strategic game** $\langle N, (A_i), (u_i) \rangle$ consists of

- a finite probability space (Ω, π) (Ω is a set of **states** and π is a probability measure on Ω)

- for each player $i \in N$ a partition $\mathcal{P}_i$ of Ω (player i's **information partition**)

- for each player $i \in N$ a function $\sigma_i \colon \Omega \to A_i$ with $\sigma_i(\omega) = \sigma_i(\omega')$ whenever $\omega \in P_i$ and $\omega' \in P_i$ for some $P_i \in \mathcal{P}_i$ (σ_i is player i's **strategy**)

such that for every $i \in N$ and every function $\tau_i \colon \Omega \to A_i$ for which $\tau_i(\omega) = \tau_i(\omega')$ whenever $\omega \in P_i$ and $\omega' \in P_i$ for some $P_i \in \mathcal{P}_i$ (*i.e.* for every strategy of player i) we have

$$\sum_{\omega\in\Omega} \pi(\omega) u_i\big(\sigma_{-i}(\omega), \sigma_i(\omega)\big) \geq \sum_{\omega\in\Omega} \pi(\omega) u_i\big(\sigma_{-i}(\omega), \tau_i(\omega)\big).$$

Correlated Equilibrium and Mixed Strategy Nash Equilibrium

PROPOSITION. For every mixed strategy Nash equilibrium α of a finite strategic game $\langle N, (A_i), (u_i) \rangle$ there is a correlated equilibrium $\langle (\Omega, \pi), (\mathcal{P}_i), (\sigma_i) \rangle$, in which for each player $i \in N$ the distribution on A_i induced by σ_i is α_i.

		Wife	
		Boxing	*Opera*
Husband	*Boxing*	2, 1	0, 0
	Opera	0, 0	1, 2

The mixed strategy Nash equilibrium: $((\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3}))$.

The corresponding correlated equilibrium is:

- $\Omega = \{x_{BB}, x_{BO}, x_{OB}, x_{OO}\}$.
- $\pi(x_{BB}) = \pi(x_{OO}) = \frac{2}{9}$, $\pi(x_{BO}) = \frac{4}{9}$, $\pi(x_{OB}) = \frac{1}{9}$.
- $\mathcal{P}_h = \{\{x_{BB}, x_{BO}\}, \{x_{OB}, x_{OO}\}\}$.
- $\mathcal{P}_w = \{\{x_{BB}, x_{OB}\}, \{x_{BO}, x_{OO}\}\}$.
- $\sigma_h = \{x_{BB} \mapsto B, x_{BO} \mapsto B, x_{OB} \mapsto O, x_{OO} \mapsto O\}$.
- $\sigma_w = \{x_{BB} \mapsto B, x_{BO} \mapsto O, x_{OB} \mapsto B, x_{OO} \mapsto O\}$.

Class Discussion

There is yet another correlated equilibrium.

- $\Omega = \{x, y\}$.
- $\pi(x) = \pi(y) = \frac{1}{2}$.
- $\mathcal{P}_h = \mathcal{P}_w = \big\{\{x\}, \{y\}\big\}$.
- $\sigma_h = \sigma_w = \{x \mapsto B, y \mapsto O\}$.

Q: Is this a correlated equilibrium?

Q: What are the husband's and the wife's payoffs?

A: Both are $\frac{3}{2}$.

Class Discussion

Consider the following game.

	L	R
T	6, 6	2, 7
B	7, 2	0, 0

Nash equilibrium payoff profiles:

- (2,7) and (7,2) (*pure*)
- $(4\frac{2}{3}, 4\frac{2}{3})$ (*mixed*)

Class Discussion

	L	R
T	6, 6	2, 7

- $\Omega = \{x_{TL}, x_{TR}, x_{BL}\}$.
- $\pi(x_{TL}) = \pi(x_{TR}) = \pi(x_{BL}) = \frac{1}{3}$.
- $\mathcal{P}_1 = \{\{x_{BL}\}, \{x_{TL}, x_{TR}\}\}$.
- $\mathcal{P}_2 = \{\{x_{TL}, x_{BL}\}, \{x_{TR}\}\}$.
- $\sigma_1 = \{x_{BL} \mapsto B, x_{TL} \mapsto T, x_{TR} \mapsto T\}$.
- $\sigma_2 = \{x_{BL} \mapsto L, x_{TL} \mapsto L, x_{TR} \mapsto R\}$.

Q: Is this a correlated equilibrium?

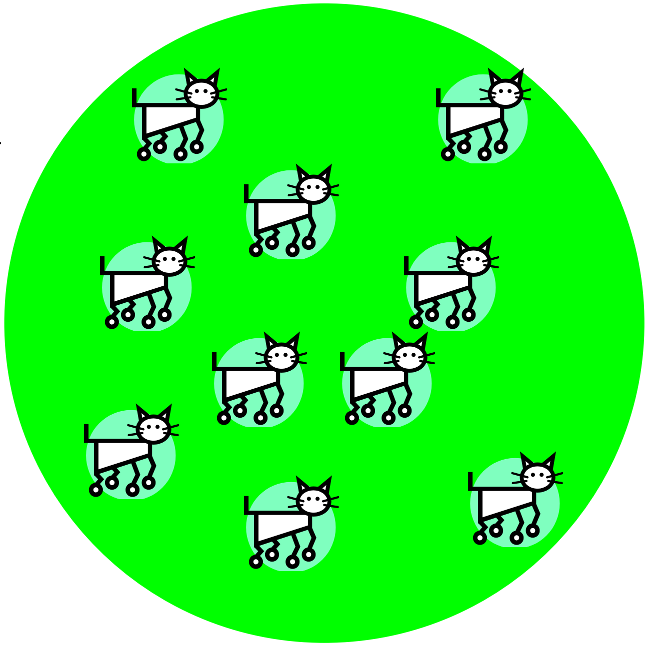
Q: What is the payoff profile?

A: (5,5).

Evolutionary Equilibrium

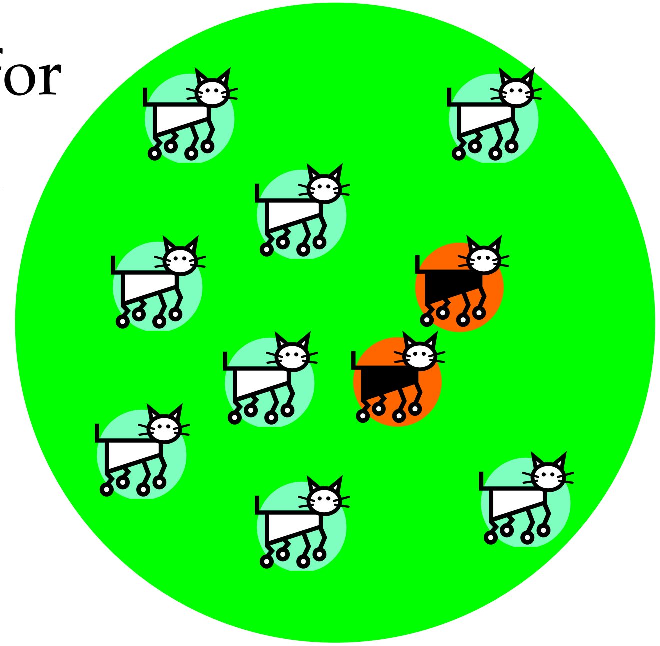
Consider a population of animals, the set of available actions of each animal is the same B.

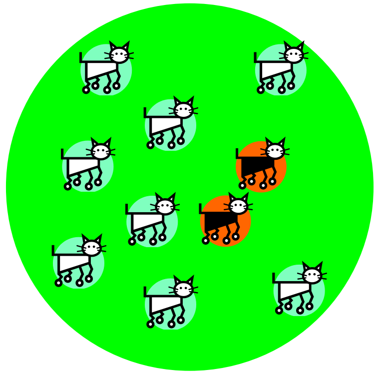
An animal does not consciously choose actions. It plays by its instincts.



Evolutionary Equilibrium

From time to time, mutations occur: for every possible $b \in B$ some mutants will follow b.



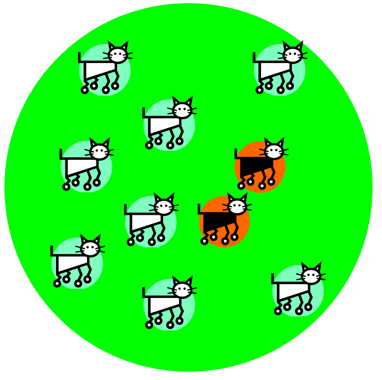


The animals interact with each other pairwise. Each match is a two player symmetric strategic game

$$\langle\{1,2\},(B,B),(u_i)\rangle,$$

where $u_1(a,b) = u_2(b,a) = u(a,b)$.

Suppose in an equilibrium all animals take the action b^*. Now a fraction $\varepsilon > 0$ of the population mutates and take the action b.



For b^* to be an evolutionary equilibrium ('**evolutionarily stable solution (ESS)**'), we require

$$(1-\varepsilon)u(b,b^*) + \varepsilon u(b,b) < (1-\varepsilon)u(b^*,b^*) + \varepsilon u(b^*,b)$$

for any value of ε sufficiently small.

$$\boxed{(1-\varepsilon)u(b,b^*)+\varepsilon u(b,b)<(1-\varepsilon)u(b^*,b^*)+\varepsilon u(b^*,b)}$$

This inequality is satisfied if and only if for every $b \neq b^*$, either

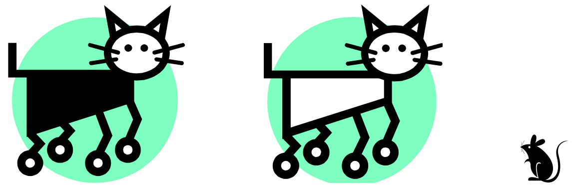
- $u(b,b^*) < u(b^*,b^*)$, or
- $u(b,b^*) = u(b^*,b^*)$ and $u(b,b) < u(b^*,b)$, or
- $u(b,b^*) > u(b^*,b^*)$ (but this is impossible…)

Therefore, this inequality is satisfied if for every best response $b \in B$ to b^* with $b \neq b^*$, $u(b,b) < u(b^*,b)$.

Evolutionary Equilibrium

DEFINITION. Let $G = \langle \{1,2\}, (B, B), (u_i) \rangle$ be a symmetric strategic game, where $u_1(a, b) = u_2(b, a) = u(a, b)$ for some function u. An **evolutionarily stable solution** (ESS) of G is an action $b^* \in B$ for which (b^*, b^*) is a Nash equilibrium of G and $u(b, b) < u(b^*, b)$ for every best response $b \in B$ to b^* with $b \neq b^*$.

Class Discussion



EXAMPLE. Two cats in a population always fight over a rat. Cats can behave like a dove (D) or like a hawk (H).

	D	H
D	$\frac{1}{2}, \frac{1}{2}$	$0, 1$
H	$1, 0$	$\frac{1}{2}(1-c), \frac{1}{2}(1-c)$

Q: Show a mixed strategy Nash equilibrium if $c > 1$.

Q: What if $c < 1$?

	D	H
D	$\frac{1}{2}, \frac{1}{2}$	$0, 1$
H	$1, 0$	$\frac{1}{2}(1-c), \frac{1}{2}(1-c)$

A: If $c > 1$, the unique mixed strategy Nash equilibrium is $\left(\left(1-\frac{1}{c}, \frac{1}{c}\right), \left(1-\frac{1}{c}, \frac{1}{c}\right)\right)$. This equilibrium mixed strategy $\left(1-\frac{1}{c}, \frac{1}{c}\right)$ is the only ESS.

A: If $c < 1$, the unique mixed strategy Nash equilibrium is $\left(e(H), e(H)\right)$. This equilibrium strategy H is the only ESS.

Class Discussion

$\frac{1}{2}, \frac{1}{2}$	$1, -1$	$-1, 1$
$-1, 1$	$\frac{1}{2}, \frac{1}{2}$	$1, -1$
$1, -1$	$-1, 1$	$\frac{1}{2}, \frac{1}{2}$

Q: Show a mixed strategy Nash equilibrium.

Q: What about a mutant that uses a pure strategy?

Q: Is the equilibrium mixed strategy an ESS?