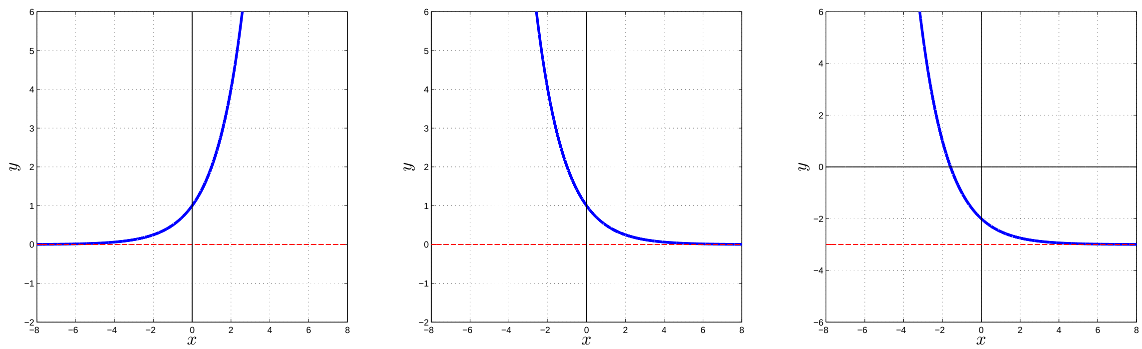
THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1020
Exercise 5
Produced by Jeff Chak-Fu WONG

Exercise 1 Graphing an exponential Function Using Transformations

Graph $f(x) = 2^{-x} - 3$ and determine the domain, range, and horizontal asymptote of f.



(a) $y = 2^x$

(b) $y = 2^{-x}$: Replace x by reflect about $y-$axis

(c) $y = 2^{-x} - 3$: Subtract 3; shift down 3 units

Figure 1:

Solution:

We begin with the graph of $y = 2^x$. Figure 1 shows the three stages.

As Figure 1(c) illustrates, the domain of $f(x) = 2^{-x} - 3$ is the interval $\mathbb{R} = (-\infty, \infty)$, and the range is the interval $(-3, \infty)$.

The limit of $f(x)$ is -3 as x increases without bound, i.e.,

$$\lim_{x\to+\infty} \left(2^{-x} - 3\right) = \lim_{x\to+\infty} 2^{-x} - \lim_{x\to+\infty} 3 = -3$$

Hence, the horizontal asymptote of f is the line $y = -3$ (in red).

The limit of $f(x)$ does not exist as x decreases without bound, i.e.,

$$\lim_{x\to-\infty} \left(2^{-x} - 3\right) = +\infty,$$

as shown in Figure 2.

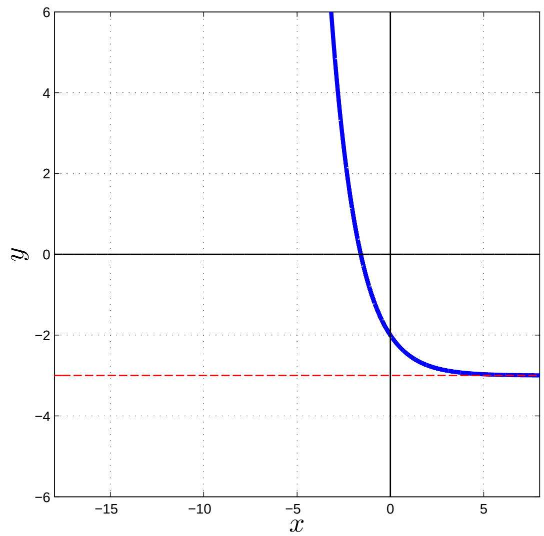


Figure 2: Graph of $y = 2^{-x} - 3$, $x \in [-18, 6]$.

Exercise 2 Graphing exponential Functions Using Transformations

Graph $f(x) = -e^{x-3}$ and determine the domain, range, and horizontal asymptote of f.

Solution:

As Figure 3(c) illustrates,the domain of $f(x) = -e^{x-3}$ is the interval $\mathbb{R} = (-\infty, \infty)$, and the range is the interval $(-\infty, 0)$.

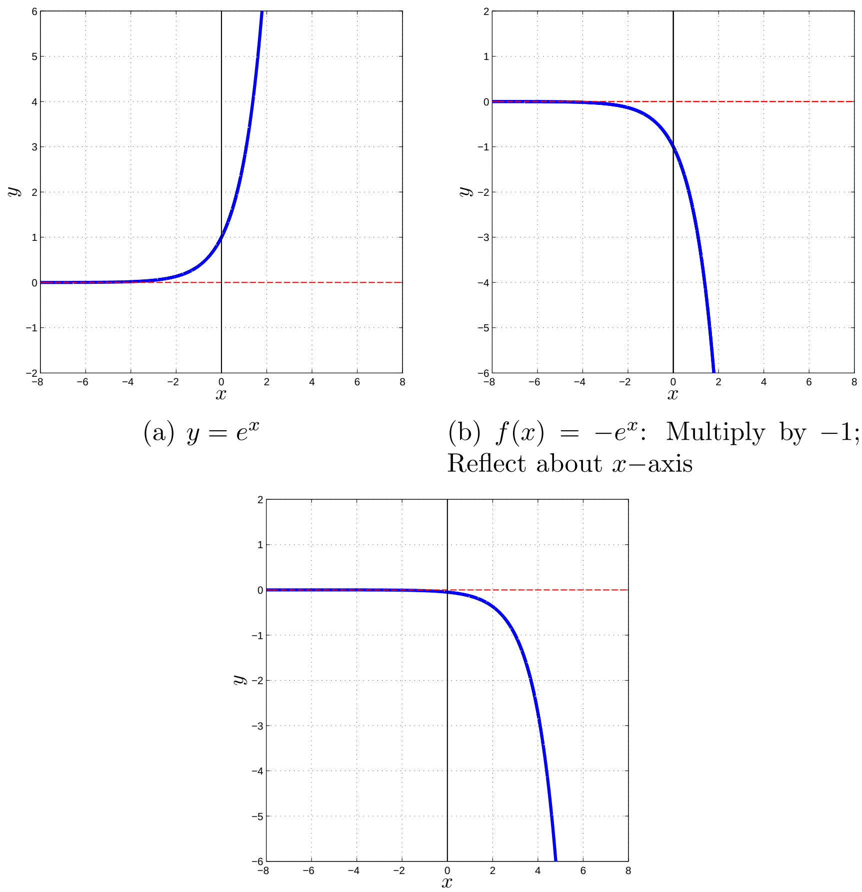
The limit of $f(x)$ does not exist as x increases without bound, i.e.,

$$\lim_{x \to +\infty} \left(-e^{x-3}\right) = -\infty$$

while the limit of $f(x)$ is 0 as x decreases without bound, i.e.,

$$\lim_{x \to -\infty} \left(-e^{x-3}\right) = 0.$$

Hence, the horizontal asymptote is the line $y = 0$.



(a) $y = e^x$

(b) $f(x) = -e^x$: Multiply by -1; Reflect about $x-$axis

(c) $f(x) = -e^{x-3}$: Replace x by $x-3$; Shift right 3 units

Figure 3:

Exercise 3 Solving an Exponential Equation

Solve: $3^{x+1} = 81$

Solution:

Let us simplify the given equation:

$$3^{x+1} = 81$$

$$3^{x+1} = 3^4$$

Now we have the same base, 3 on each side of the equation, so we can set the exponents equal to each other to obtain

$$x + 1 = 4$$

$$x = 3$$

The solution set is $\{3\}$.

Exercise 4 Solving an Exponential Equation

Solve: $e^{-x^2} = (e^x)^2 \cdot \dfrac{1}{e^3}$.

Solution:

Using the Laws of Exponents first to get the base e on the right side, we have

$$(e^x)^2 \cdot \frac{1}{e^3} = e^{2x} \cdot e^{-3} = e^{2x-3}.$$

Then, we have

$$\begin{aligned}
e^{-x^2} &= e^{2x-3} && \\
-x^2 &= 2x - 3 && \text{Apply Property (3).} \\
x^2 + 2x - 3 &= 0 && \text{Place the quadratic equation in standard form.} \\
(x+3)(x-1) &= 0 && \text{Factor.} \\
x = -3 \text{ or } x &= 1 && \text{Use the Zero-Product Property.}
\end{aligned}$$

The solution set is $\{-3, 1\}$.