Probability 2017 Homework 2

Problem 1 (20 points) Fischer and Spassky play a chess match in which the first player to win a game wins the match. After 10 successive draws, the match is declared drawn. Each game is won by Fischer with probability 0.4, is won by Spassky with probability 0.3, and is a draw with probability 0.3, independent of previous games.

- What is the probability that Fischer wins the match?
- What is the PMF of the duration of the match?

Solution The probability p_i that Fischer wins at round i is $0.3^{i-1} \cdot 0.4$. Hence the probability that Fischer wins is

$$\sum_{i=1}^{10} p_i \approx 0.57. \tag{1}$$

Similarly, let $q_i = 0.3^{i-1} \cdot 0.3$ be the probability that Spassky wins at round i. Let X be the duration of the match, we have $\mathbb{P}(X = i) = p_i + q_i = 0.3^{i-1} * 0.7$ for $i = 1, \ldots, 9$, as the match terminates at round i iff either of the player wins at that round. We have $\mathbb{P}(X = 10) = 1 - \sum_{i=1}^{9} \mathbb{P}(X = i)$.

Problem 2 (10 points) A family has 5 natural children and has adopted 2 girls. Each natural child has equal probability of being a girl or a boy, independent of the other children. Find the PMF of the number of girls out of the 7 children.

Solution Let Z be the number of girls. $Z = X + 2$, where $X \sim B(5, \frac{1}{2})$. Hence,

$$\mathbb{P}(Z = i) = \binom{5}{i-2} (\frac{1}{2})^5 \quad i = 2, 3, 4, 5, 6, 7. \tag{2}$$

Problem 3 (10 points) You toss independently a fair coin and you count the number of tosses until the first tail appears. If this number is n, you receive 2^n dollars. What is the expected amount that you will receive?

Solution Let X be the number of tosses until the first tail. Then $\mathbb{P}(X = i) = (\frac{1}{2})^i$. Hence

$$\mathbb{E}[2^X] = \sum_{i=1}^{\infty} 2^i \mathbb{P}(X = i) = \infty \tag{3}$$

Problem 4 (20 points) On a given day, your golf score takes integer values ranging from 101 to 110 with probability 0.1 for each integer, independent of other days. Determined to improve your score, you decide to play on three different days and declare as your score the minimum X of the scores X_1, X_2, and X_3 on the different days.

- Calculate the PMF of X.
- By how much has your expected score improved as a result of playing on three days?

Solution The event $X = 100 + i$ can be split into 3 disjoint cases

- X_1, X_2, and X_3 are all X, with probability $\frac{1}{1000}$
- Two and only two among X_1, X_2, and X_3 are X, and the other one is greater than X. That is with probability $\frac{1}{100} \cdot \binom{3}{2} \cdot \frac{10-i}{10}$
- One and only one among X_1, X_2, and X_3 is X, and the other two are greater than X. That is with probability $\frac{1}{10} \cdot \binom{3}{1} \cdot (\frac{10-i}{10})^2$

Hence,

$$\mathbb{P}(X = i) = \frac{1}{1000} + \frac{1}{100} \cdot \binom{3}{2} \cdot \frac{110-i}{10} + \frac{1}{10} \cdot \binom{3}{1} \cdot (\frac{110-i}{10})^2 \quad i = 101, \ldots 110 \tag{4}$$

The expected improvement is

$$105.5 - \mathbb{E}[X] = 2.475 \tag{5}$$

X	101	102	103	104	105	106	107	108	109	110
P_X	0.271	0.217	0.169	0.127	0.091	0.061	0.037	0.019	0.007	0.001

Problem 5 (10 points) Each morning, Hungry Harry eats some eggs. On any given morning, the number of eggs he eats is equally likely to be 1, 2, 3, 4, 5, or 6, independent of what he has done in the past. Let X be the number of eggs that Harry eats in 10 days. Find the mean and variance of X.

Solution Let Z be the number of eggs Hungry eats in one day. We have $\mathbb{E}[Z] = \frac{7}{2}$ and $\mathrm{var}[Z] = \frac{35}{12}$. Hence

$$\mathbb{E}[X] = 10 \cdot \mathbb{E}[Z] = 35, \mathrm{var}[X] = 100 \cdot \mathrm{var}[z] = \frac{875}{3}. \tag{6}$$