Lecture Note 8

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MATH1020
General Mathematics

Trigonometric Identities

Before establishing some additional identities, let us review the definition of an identity.

Definition 1 Two functions f and g are said to be **identically equal** if

$$f(x) = g(x)$$

for every value of x for which both functions are defined. Such an equation is referred to as an **identity.** An equation that is not an identity is called a **conditional equation.**

Example 1 The following are identities:

$$(x + 1)^2 = x^2 + 2x + 1$$

$$\sin^2 x + \cos^2 x = 1$$

$$\csc x = \frac{1}{\sin x}$$

Example 2 The following are conditional equations:

$$2x + 5 = 0 \quad \text{True} \quad \text{only} \quad \text{if} \quad x = -\frac{5}{2}$$

$$\sin x = 0 \quad \text{True} \quad \text{only} \quad \text{if} \quad x = k\pi,\ k \text{ an integer}$$

$$\sin x = \cos x \quad \text{True} \quad \text{only} \quad \text{if} \quad x = \frac{\pi}{4} + 2k\pi \quad \text{or} \quad x = \frac{5\pi}{4} + 2k\pi,\ k \text{ an integer}$$

A summary of the **basic trigonometric identities**

- **Quotient Identities**

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}.$$

- **Reciprocal Identities**

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}.$$

- **Pythagorean Identities**

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta;$$
$$\cot^2\theta + 1 = \csc^2\theta.$$

- **Even-Odd Identities**

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = -\tan\theta;$$
$$\csc(-\theta) = -\csc\theta \qquad \sec(-\theta) = \sec\theta \qquad \cot(-\theta) = -\cot\theta.$$

Use Algebra to Simplify Trigonometric Expressions

Exercises 1 Using Algebraic Techniques to Simplify Trigonometric Expressions

(a) Simplify $\dfrac{\cot\theta}{\csc\theta}$ by rewriting each trigonometric function in terms of sine and cosine functions.

(b) Show that $\dfrac{\cos\theta}{1+\sin\theta} = \dfrac{1-\sin\theta}{\cos\theta}$ by multiplying the numerator and denominator by $1-\sin\theta$.

(c) Simplify $\dfrac{1+\sin u}{\sin u} + \dfrac{\cot u - \cos u}{\cos u}$ by rewriting the expression over a common denominator.

(d) Simplify $\dfrac{\sin^2 v - 1}{\tan v \sin v - \tan v}$ by factoring.

Establish Identities

Exercises 2 Establishing the identity:

$$\csc\theta \cdot \tan\theta = \sec\theta.$$

Exercises 3 Establishing the identity:

$$\sin^2(-\theta) + \cos^2(-\theta) = 1.$$

Exercises 4 Establish the identity:

$$\frac{\sin^2(-\theta) - \cos^2(-\theta)}{\sin(-\theta) - \cos(-\theta)} = \cos\theta - \sin\theta.$$

Exercises 5 Establish the identity:

$$\frac{1+\tan u}{1+\cot u} = \tan u.$$

Exercises 6 Establish the identity:

$$\frac{\sin\theta}{1+\cos\theta}+\frac{1+\cos\theta}{\sin\theta}=2\csc\theta.$$

Exercises 7 Establish the identity:

$$\frac{\tan v + \cot v}{\sec v \csc v} = 1.$$

Exercises 8 Establish the identity:

$$\frac{1 - \sin\theta}{\cos\theta} = \frac{\cos\theta}{1 + \sin\theta}.$$

Guidelines for Establishing Identities

Step 1: It is almost always preferable to start with the side containing the more complicated expression.

Step 2: Rewrite sums or differences of quotients as a single quotient.

Step 3: Sometimes rewriting one side in terms of sine and cosine functions only will help.

Step 4: Always keep your goal in mind. As you manipulate one side of the expression, you must keep in mind the form of the expression on the other side.

Sum and Difference Formulas

Theorem 1 **Sum and Difference Formulas for the Cosine Function**

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta; \tag{1}$$

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta. \tag{2}$$

Use Sum and Difference Formulas to Find Exact Values

One use of formulas (1) and (2) is to obtain the exact value of the cosine of an angle that can be expressed as the sum or difference of angles whose sine and cosine are known exactly.

Exercises 9 Using the Sum Formula to Find Exact Values

Find the exact value of $\cos 75°$.

Exercises 10 Using the Difference Formula to Find Exact Values

Find the exact value of $\cos\dfrac{\pi}{12}$.

Use Sum and Difference Formulas to Establish Identities

Theorem 2 Another use of formulas (1) and (2) is to establish other identities. Two important identities are given next:

$$\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta; \tag{3}$$

$$\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta. \tag{4}$$

Proof of formula (3): Using the formula for $\cos(\alpha-\beta)$ with $\alpha=\dfrac{\pi}{2}$ and $\beta=\theta$, we have

$$\begin{aligned}\cos\left(\frac{\pi}{2}-\theta\right)&=\cos\frac{\pi}{2}\cos\theta+\sin\frac{\pi}{2}\sin\theta\\&=0\cdot\cos\theta+1\cdot\sin\theta\\&=\sin\theta.\end{aligned}$$

Proof of formula (4): By making use of the identity (3) just established, we have

$$\sin\left(\frac{\pi}{2}-\theta\right) = \cos\left[\frac{\pi}{2}-\left(\frac{\pi}{2}-\theta\right)\right] = \cos\theta \quad \text{Use } (3).$$

Remark 1 Formulas (3) and (4) should look familiar. Using Even property of cosine, we have

$$\cos\left(\frac{\pi}{2}-\theta\right) = \cos\left[-\left(\theta-\frac{\pi}{2}\right)\right]$$
$$= \cos\left(\theta-\frac{\pi}{2}\right)$$

and since

$$\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta \qquad (3)$$

it follows that $\cos\left(\theta-\frac{\pi}{2}\right) = \sin\theta$. The graphs of $y = \cos\left(\theta-\dfrac{\pi}{2}\right)$ and $y = \sin\theta$ are identical.

Theorem 3 Sum and Difference Formulas for the Sine Function

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta; \tag{5}$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta. \tag{6}$$

Having established the identities in formulas (3) and (4), we now can derive the sum and difference formulas for $\sin(\alpha+\beta)$ and $\sin(\alpha-\beta)$.

Proof of formula (5):

$$\begin{aligned}
\sin(\alpha+\beta) &= \cos\left[\tfrac{\pi}{2}-(\alpha+\beta)\right] && \text{Formula (3)}\\
&= \cos\left[\left(\tfrac{\pi}{2}-\alpha\right)-\beta\right] && \\
&= \cos\left(\tfrac{\pi}{2}-\alpha\right)\cos\beta+\sin\left(\tfrac{\pi}{2}-\alpha\right)\sin\beta && \text{Formula (2)}\\
&= \sin\alpha\cos\beta+\cos\alpha\sin\beta. && \text{Formulas (3) and (4)}
\end{aligned}$$

Proof of formula (6):

$$\begin{aligned}
\sin(\alpha-\beta) &= \sin[\alpha+(-\beta)] \\
&= \sin\alpha\cos(-\beta)+\cos\alpha\sin(-\beta) && \text{Use the sum formula for the sine just obtained.} \\
&= \sin\alpha\cos\beta+\cos\alpha(-\sin\beta) && \text{Even}-\text{Odd Identities} \\
&= \sin\alpha\cos\beta-\cos\alpha\sin\beta.
\end{aligned}$$

Exercises 11 Using the Sum Formula to Find Exact Values

Find the exact value of $\sin \dfrac{7\pi}{12}$.

Exercises 12 Using the Difference Formula to Find Exact Values

Find the exact value of $\sin 80^\circ \cos 20^\circ - \cos 80^\circ \sin 20^\circ$.

Exercises 13 Finding Exact Values

If it is known that $\sin\alpha = \frac{4}{5}$, $\frac{\pi}{2} < \alpha < \pi$, and that $\sin\beta = -\frac{2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}$, $\pi < \beta < \frac{3\pi}{2}$, find the exact value of

(a) $\cos\alpha$ (b) $\cos\beta$ (c) $\cos(\alpha+\beta)$ (d) $\sin(\alpha+\beta)$

Exercises 14 Establishing an Identity

Establish the identity:

$$\frac{\cos(\alpha - \beta)}{\sin\alpha \sin\beta} = \cot\alpha \cot\beta + 1$$

Theorem 4 Sum and Difference Formulas for the Tangent Function

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}; \tag{7}$$

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}. \tag{8}$$

We use the identity $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and the sum formulas for $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$ to derive a formula for $\tan(\alpha+\beta)$.

Proof of formula (7):

$$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}.$$

Now we divide the numerator and denominator by $\cos\alpha\cos\beta$.

$$\begin{aligned}
\tan(\alpha+\beta) &= \frac{\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} \\
&= \frac{\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}-\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} \\
&= \frac{\frac{\sin\alpha}{\cos\alpha}+\frac{\sin\beta}{\cos\beta}}{1-\frac{\sin\alpha}{\cos\alpha}\cdot\frac{\sin\beta}{\cos\beta}} \\
&= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}.
\end{aligned}$$

Proof of formula (8): Using the sum formula for $\tan(\alpha + \beta)$ and Even–Odd Properties to get the difference formula, we have

$$\tan(\alpha - \beta) = \tan[\alpha + (-\beta)] = \frac{\tan\alpha + \tan(-\beta)}{1 - \tan\alpha\tan(-\beta)} = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}.$$

Exercises 15 Establishing an Identity

Prove the identity:

$$\tan(\theta + \pi) = \tan\theta.$$

Exercises 16 Establishing an Identity

Prove the identity:

$$\tan\left(\theta + \frac{\pi}{2}\right) = -\cot\theta.$$

Double-angle and Half-angle Formulas

In what follows, we derive formulas for $\sin(2\theta), \cos(2\theta), \sin\left(\frac{1}{2}\theta\right)$, and $\cos(\frac{1}{2}\theta)$ in terms of $\sin\theta$ and $\cos\theta$. They are derived using the sum formulas.

In the sum formulas for $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$, let $\alpha=\beta=\theta$. Then

$$\begin{aligned}
\sin(\alpha+\beta) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta; \\
\sin(\theta+\theta) &= \sin\theta\cos\theta+\cos\theta\sin\theta; \\
\sin(2\theta) &= 2\sin\theta\cos\theta.
\end{aligned}$$

and

$$\begin{aligned}
\cos(\alpha+\beta) &= \cos\alpha\cos\beta-\sin\alpha\sin\beta; \\
\cos(\theta+\theta) &= \cos\theta\cos\theta-\sin\theta\sin\theta; \\
\cos(2\theta) &= \cos^2\theta-\sin^2\theta.
\end{aligned}$$

An application of the Pythagorean Identity $\sin^2\theta + \cos^2\theta = 1$ results in two other ways to express $\cos(2\theta)$.

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = (1 - \sin^2\theta) - \sin^2\theta = 1 - 2\sin^2\theta.$$

and

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = \cos^2\theta - (1 - \cos^2\theta) = 2\cos^2 - 1.$$

We have established the following **Double-angle Formulas:**

Theorem 5 Double-angle Formulas

$$\sin(2\theta) = 2\sin\theta\cos\theta; \tag{9}$$

$$\cos(2\theta) = \cos^2 - \sin^2\theta; \tag{10}$$

$$\cos(2\theta) = 1 - 2\sin^2\theta; \tag{11}$$

$$\cos(2\theta) = 2\cos^2\theta - 1. \tag{12}$$

Exercises 17 Finding Exact Values Using the Double-angle Formula

If $\sin\theta = \frac{3}{5}$, $\frac{\pi}{2} < \theta < \pi$, find the exact value of:

(a) $\sin(2\theta)$ (b) $\cos(2\theta)$

Remark 2 In finding $\cos(2\theta)$ in Example 17 (b), we chose to use a version of the Double-angle Formula, formula (11). Note that we are unable to use the Pythagorean Identity $\cos(2\theta) = \pm\sqrt{1-\sin^2(2\theta)}$, with $\sin(2\theta) = -\dfrac{24}{25}$, because we have no way of knowing which sign to choose.

Use Double-angle Formulas to Establish Identities

Exercises 18 Establishing Identities

(a) Develop a formula for $\tan(2\theta)$ in terms of $\tan\theta$.

(b) Develop a formula for $\sin(3\theta)$ in terms of $\sin\theta$ and $\cos\theta$.

Remark 3 The formula obtained in Example 18(b) can also be written as

$$\begin{aligned}\sin(3\theta) = 3\sin\theta\cos^2\theta - \sin^3\theta &= 3\sin\theta(1-\sin^2\theta) - \sin^3\theta \\ &= 3\sin\theta - 4\sin^3\theta.\end{aligned}$$

That is, $\sin(3\theta)$ is a third-degree polynomial in the variable $\sin\theta$. In fact, $\sin(n\theta)$, n a positive odd integer, can always be written as a polynomial of degree n in the variable $\sin\theta$.

By rearranging the Double-angle Formulas (11) and(12), we obtain other formulas that we will use later in this course.

We begin with formula (11) and proceed to solve for $\sin^2\theta$:

$$\cos(2\theta) = 1 - 2\sin^2\theta;$$

$$2\sin^2\theta = 1 - \cos(2\theta).$$

Hence,

$$\sin^2\theta = \frac{1 - \cos(2\theta)}{2}. \tag{13}$$

Similarly, using formula (4), we proceed to solve for $\cos^2\theta$.

$$\cos(2\theta) = 2\cos^2\theta - 1;$$
$$2\cos^2\theta = 1 + \cos(2\theta).$$

Hence,

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}. \tag{14}$$

Formulas (13) and (14) can be used to develop a formula for $\tan^2\theta$.

$$\tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta} = \frac{\frac{1-\cos(2\theta)}{2}}{\frac{1+\cos(2\theta)}{2}}.$$

Hence,

$$\tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)}. \tag{15}$$

Formulas (13) through (15) do not have to be memorized since their derivations are so straightforward.

Formulas (13) and (14)are important in calculus. The next example illustrates a problem that arises in calculus requiring the use of formula (14).

Exercises 19 Establishing an Identity

Write an equivalent expression for $\cos^4\theta$ that does not involve any powers of sine or cosine greater than 1.

Using Half-angle Formulas to Find Exact Values

Another important use of formulas (13) through (15) is prove the Half-angle Formulas.In formulas (13) through (15), let $\theta = \dfrac{\alpha}{2}$. Then

$$\begin{cases} \sin^2 \dfrac{\alpha}{2} & = \dfrac{1-\cos\alpha}{2}; \\ \cos^2 \dfrac{\alpha}{2} & = \dfrac{1+\cos\alpha}{2}; \\ \tan^2 \dfrac{\alpha}{2} & = \dfrac{1-\cos\alpha}{1+\cos\alpha}. \end{cases} \tag{16}$$

If we solve for the trigonometric functions on the left sides of equations (16) we obtain the Half-angle Formulas.

Theorem 6 Half-angle Formulas

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}; \tag{17}$$

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}; \tag{18}$$

$$\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}. \tag{19}$$

where the $+$ or $-$ sign is determined by the quadrant of the angle $\dfrac{\alpha}{2}$.

Exercises 20 Finding Exact Values Using Half-angle Formulas

Use a Half$-$angle Formula to find the exact value of:

(a) $\cos 15^\circ$ (b) $\sin(-15^\circ)$

Remark 4 Let us compare the answer found in Example (20)(a) with the answer to Example 10. There we calculated

$$\cos\frac{\pi}{12} = \cos 15^\circ = \frac{1}{4}(\sqrt{6}+\sqrt{2})$$

Based on this and the result of Example (20)(a), we conclude that

$\frac{1}{4}(\sqrt{6}+\sqrt{2})$ and $\frac{\sqrt{2+\sqrt{3}}}{2}$

are equal. (Since each expression is positive,you can verify this equality by squaring each expression.) Two very different looking, yet correct, answers can be obtained, depending on the approach taken to solve a problem.

Exercises 21 Finding Exact Values Using Half-angle Formulas

If $\cos\alpha = -\frac{3}{5},\ \pi < \alpha < \frac{3\pi}{2}$, find the exact value of:

(a) $\sin\frac{\alpha}{2}$ (b) $\cos\frac{\alpha}{2}$ (c) $\tan\frac{\alpha}{2}$

Remark 5 Another way to solve Example 21(c) is to use the solutions found in parts (a) and (b).

$$\tan\frac{\alpha}{2} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \frac{\frac{2\sqrt{5}}{5}}{-\frac{\sqrt{5}}{5}} = -2.$$

Half-angle Formulas for $\tan\frac{\alpha}{2}$

$$\tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1+\cos\alpha}. \qquad (20)$$

Proof of formula (20): There is a formula for $\tan\frac{\alpha}{2}$ that does not contain $+$ and $-$ signs, making it more useful than Formula 19. To derive it, we use the formulas

$$1-\cos\alpha = 2\sin^2\frac{\alpha}{2}16$$

and

$$\sin\alpha = \sin\left[2\left(\frac{\alpha}{2}\right)\right] = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} \quad \text{Double}-\text{angle Formula.}$$

Then

$$\frac{1-\cos\alpha}{\sin\alpha} = \frac{2\sin^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}} = \tan\frac{\alpha}{2}.$$

Since it also can be shown that

$$\frac{1-\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1+\cos\alpha}$$

we have the following two Half−angle Formulas:

Example 3 With this formula, the solution to Example 21 can be obtained as follows:

$$\cos\alpha = -\frac{3}{5} \quad \pi < \alpha < \frac{3\pi}{2}$$

$$\sin\alpha = -\sqrt{1-\cos^2\alpha} = -\sqrt{1-\frac{9}{25}} = -\sqrt{\frac{16}{25}} = -\frac{4}{5}$$

Then, by equation (11),

$$\tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha} = \frac{1-\left(-\frac{3}{5}\right)}{-\frac{4}{5}} = -2.$$

Product-to-Sum and Sum-to-Product Formulas

Express Products as Sums

Sum and difference formulas can be used to derive formulas for writing the products of sines and/or cosines as sums or differences. These identities are usually called the **Product-to-Sum Formulas.**

Theorem 7 **Product-to-Sum Formulas**

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]; \tag{21}$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]; \tag{22}$$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha-\beta) + \sin(\alpha+\beta)]. \tag{23}$$

These formulas do not have to be memorized. Instead, you should remember how they are derived. Then, when you want to use them, either look them up or derive them, as needed.

Proof of formulas (21) and (22): Writing down the sum and difference formulas for the cosine, we have

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta; \tag{24}$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta. \tag{25}$$

Subtract equation (25) from equation (24) to get

$$\cos(\alpha - \beta) - \cos(\alpha + \beta) = 2\sin\alpha\sin\beta$$

from which

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

Now add equations (4) and (5) to get

$$\cos(\alpha - \beta) + \cos(\alpha + \beta) = 2\cos\alpha\cos\beta$$

from which

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$$

Proof of formula (23): To derive Product-to-Sum Formula (23), use the sum and difference formulas for sine in a similar way.

Exercises 22 Expressing Products as Sums

Express each of the following products as a sum containing only sines or only cosines.

(a) $\sin(6\theta)\sin(4\theta)$ (b) $\cos(3\theta)\cos\theta$ (c) $\sin(3\theta)\cos(5\theta)$

The **Sum−to−Product Formulas** are given next.

Theorem 8 Sum−to−Product Formulas

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}; \tag{26}$$

$$\sin\alpha - \sin\beta = 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}; \tag{27}$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}; \tag{28}$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}. \tag{29}$$

We will derive formula (26) and leave the derivations of formulas (27) through (29) as exercises.

Proof of formula (26): Using Product−to−Sum Formula (23), we have

$$2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = 2\cdot\frac{1}{2}\left[\sin\left(\frac{\alpha+\beta}{2}+\frac{\alpha-\beta}{2}\right)+\sin\left(\frac{\alpha+\beta}{2}-\frac{\alpha-\beta}{2}\right)\right]$$

$$= \sin\frac{2\alpha}{2}+\sin\frac{2\beta}{2} = \sin\alpha+\sin\beta.$$

Exercises 23 Expressing Sums (or Differences)as a Product

Express each sum or difference as a product of sines and/or cosines.

(a) $\sin(5\theta) - \sin(3\theta)$ (b) $\cos(3\theta) + \cos(2\theta)$

Trigonometric Equations(I)

Solve Equations Involving a Single Trigonometric Function

In what follows, we discuss **trigonometric equations,** that is, equations involving trigonometric functions that are satisfied only by some values of the variable (or,possibly,are not satisfied by any values of the variable). The values that satisfy the equation are called **solutions** of the equation.

Exercises 24 Checking Whether a Given Number Is a Solution of a Trigonometric Equation Determine whether $\theta = \dfrac{\pi}{4}$ is a solution of the equation $2\sin\theta - 1$ a solution?

Remark 6 The equation given in Example 24 has other solutions besides $\theta = \dfrac{\pi}{6}$. For example, $\theta = \dfrac{5\pi}{6}$ is also a solution, as is $\theta = \dfrac{13\pi}{6}$. (You should check this for yourself.)

Remark 7 Unless the domain of the variable is restricted, we need to find *all* the solutions of a trigonometric equation. As the next example illustrates, finding all the solutions can be accomplished by first finding solutions over an interval whose length equals the period of the function and then adding multiples of that period to the solutions found.

Exercises 25 Finding All the Solutions of a Trigonometric Equation

Solve the equation: $\cos\theta = \frac{1}{2}$

Given a general formula for all the solutions.List eight of the solutions.

Remark 8 We can verify the solutions by graphing $Y_1 = \cos x$ and $Y_2 = \frac{1}{2}$ to determine where the graphs intersect. Can you plot the graph of $Y_1 = \cos x$ and $Y_2 = \frac{1}{2}$? The graph of Y_1 intersects the graph of Y_2 at $x = 1.05\left(\approx \frac{\pi}{3}\right), 5.24\left(\approx \frac{5\pi}{3}\right), 7.33\left(\approx \frac{7\pi}{3}\right)$, and $11.52\left(\approx \frac{11\pi}{3}\right)$, rounded to two decimal places.

In most of our work, we shall be interested only in finding solutions of trigonometric equations for $0 \leq \theta < 2\pi$.

Exercises 26 Solving a Linear Trigonometric Equation

Solve the equation:

$$2\sin\theta + \sqrt{3} = 0, 0 \leq \theta < 2\pi.$$

Exercises 27 Solving a Trigonometric Equation

Solve the equation:

$$\sin(2\theta) = \frac{1}{2},\ 0 \leq \theta < 2\pi.$$

Exercises 28 Solving a Trigonometric Equation

Solve the equation:

$$\tan\left(\theta - \frac{\pi}{2}\right) = 1,\ \theta, 0 \le x < 2\pi.$$

Trigonometric Equations (II)

Solve Trigonometric Equations Quadratic in Form

In this section we continue our study of trigonometric equations.Many trigonometric equations can be solved by applying techniques that we already know,such as applying the quadratic formula (if the equation is a second-degree polynomial) or factoring.

Exercises 29 Solving a Trigonometric Equation Quadratic in Form

Solve the equation:

$$2\sin^2\theta - 3\sin\theta + 1 = 0,\ 0 \le \theta < 2\pi.$$

Solve Trigonometric Equations Using Identities

When a trigonometric equation contains more than trigonometric function,identities sometimes can be used to obtain an equation that contains only one trigonometric function.

Exercises 30 Solving a Trigonometric Equation Using Identities

Solve the equation:

$$3\cos\theta + 3 = 2\sin^2\theta,\ 0 \leq \theta < 2\pi.$$

Exercises 31 Check of Example 30: Graph $Y_1 = 3\cos x + 3$ and $Y_2 = 2\sin^2 x, 0 \leq x \leq 2\pi$, and approximate the points if intersection.

When a trigonometric equation contains trigonometric functions with different arguments, identities can sometimes be used to obtain an equivalent equation with the same argument.

Exercises 32 Solving a Trigonometric Equation Using Identities

Solve the equation:

$$\cos(2\theta) + 3 = 5\cos\theta,\ 0 \le \theta < 2\pi$$

Exercises 33 Check of Example 32: Graph $Y_1 = \cos(2x) + 3$ and $Y_2 = 5\cos x, 0 \le x \le 2\pi$, and approximate the points of intersection.

Exercises 34 Solving a Trigonometric Equation Using Identities

Solve the equation:

$$\cos^2\theta + \sin\theta = 2,\ 0 \leq \theta < 2\pi.$$

Exercises 35 Check of Example 34: Graph $Y_1 = \cos^2 x + \sin x$ and $Y_2 = 2$. The two graphs never intersect, so the equation $Y_1 = Y_2$ has no real solution.

Exercises 36 Solving a Trigonometric Equation Using Identities

Solve the equation:

$$\sin\theta\cos\theta = -\frac{1}{2},\ 0 \le \theta < 2\pi.$$

Solve Trigonometric Equations Linear in Sine and Cosine

Sometimes it is necessary to square both sides of an equation to obtain expressions that allow the use of identities. Remember, squaring both sides of an equation may introduce extraneous solution. As a result, apparent solutions must be checked.

Exercises 37 Solving a Trigonometric Equation Linear in Sine and Cosine

Solve the equation:

$$\sin\theta + \cos\theta = 1,\ 0 \leq \theta < 2\pi.$$