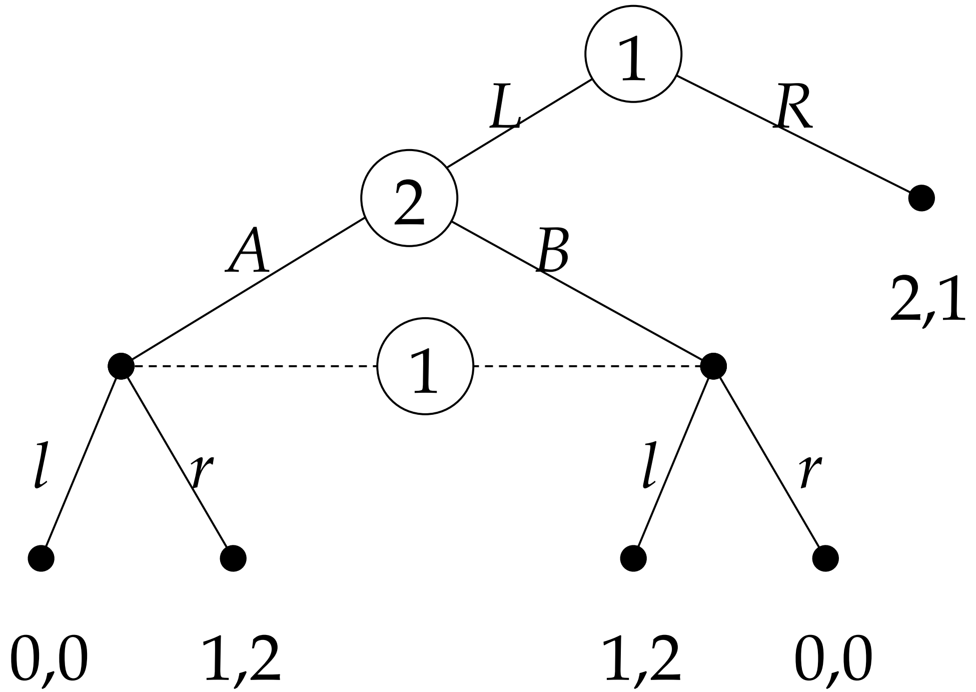
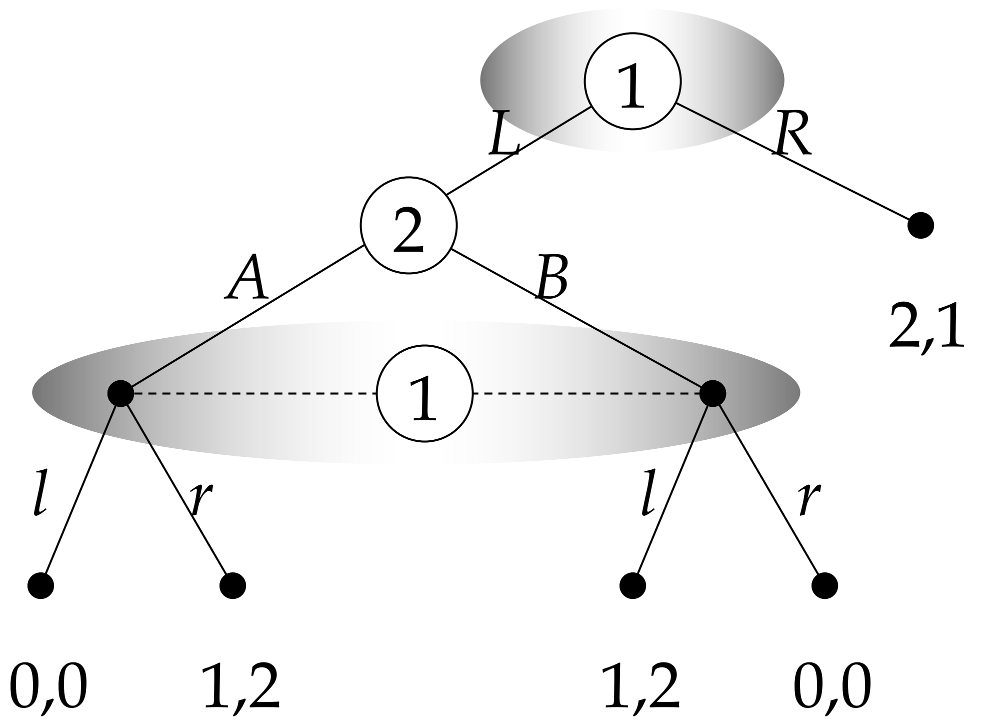
Extensive Games with Imperfect Information

Extensive games with imperfect information are extensive games in which the players are imperfectly informed about some or all of the choice that have *already* been made.

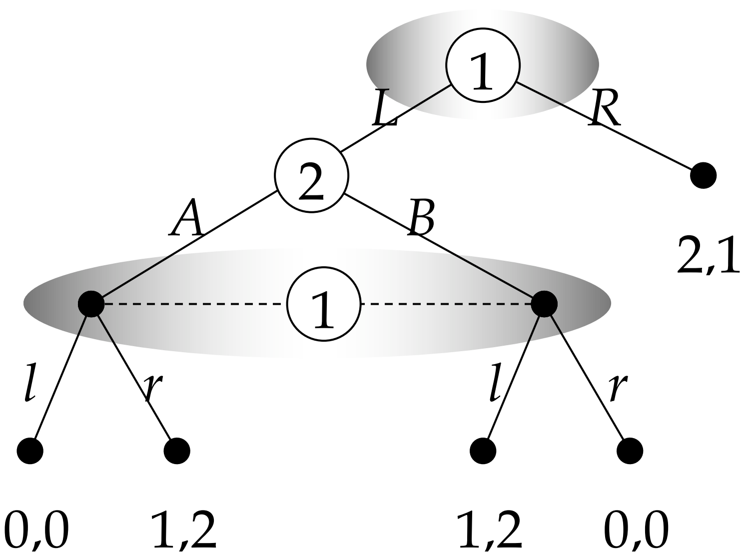
EXAMPLE.





Player 1's *information sets*: $\{\emptyset\}$ and $\{(L, A), (L, B)\}$.

Player 2's *information set*: $\{L\}$.



- $N = \{1,2\}$
- $H = \{\emptyset, L, R, (L, A), (L, B), (L, A, l), (L, A, r), (L, B, l), (L, B, r)\}$
- $Z = \{R, (L, A, l), (L, A, r), (L, B, l), (L, B, r)\}$
- $P(\emptyset) = 1 \quad P(L) = 2 \quad P(L, A) = 1 \quad P(L, B) = 1$
- $\mathcal{I}_1 = \{\{\emptyset\}, \{(L, A), (L, B)\}\}$
- $\mathcal{I}_2 = \{\{L\}\}$

DEFINITION. An **extensive game** has the following components.

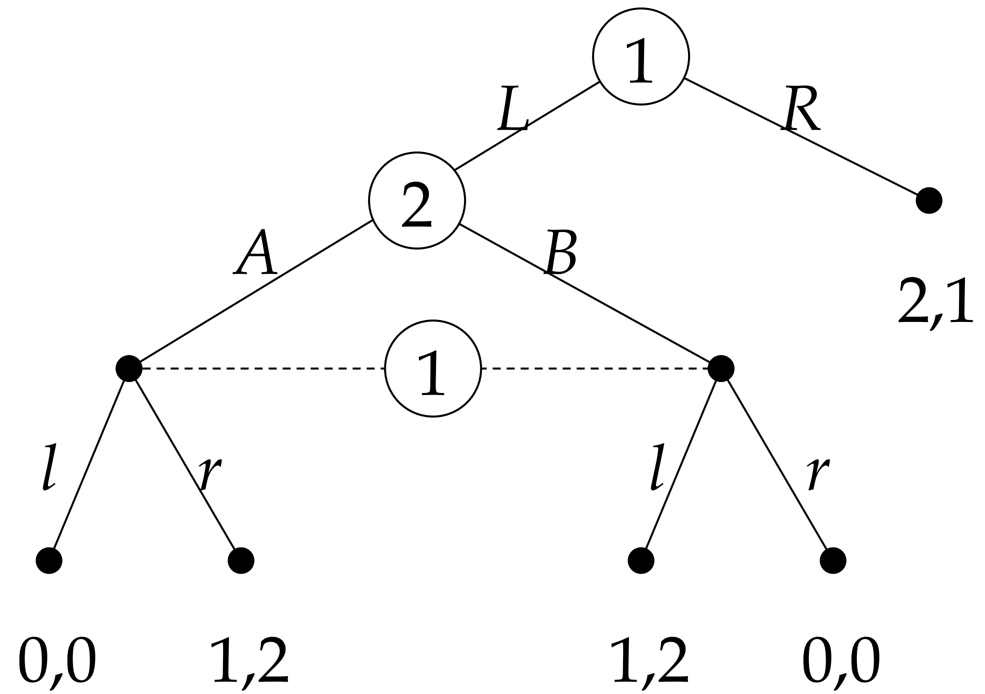
- A set N (the set of **players**).
- A set H of sequences (finite or infinite) that satisfies the following three properties.
  - The empty sequence $\emptyset$ is a member of H.
  - If $(a^k)_{k=1,\ldots,K} \in H$ (where K may be infinite) and $L < K$, then $(a^k)_{k=1,\ldots,L} \in H$.
  - If an infinite sequence $(a^k)_{k=1}^{\infty}$ satisfies $(a^k)_{k=1,\ldots,L} \in H$ for every positive integer L then $(a^k)_{k=1}^{\infty} \in H$.

(H is the set of **histories**. A history $(a^k)_{k=1,\ldots,K} \in H$ is **terminal** if it is infinite, or if there is no a^{K+1} such that $(a^k)_{k=1,\ldots,K+1} \in H$. $Z \subseteq H$ is the set of terminal histories.)

- A function P that assigns to each nonterminal sequence (each member of $H \backslash Z$) a member of $N \cup \{c\}$. (P is the **player function**, $P(h)$ being the player who takes an action after the history h. If $P(h) = c$ then chance determines the action taken after the history h.)

- A function f_c that associates with every history h for which $P(h) = c$ a probability measure $f_c(\cdot|h)$ on $A(h)$, where each such probability measure is independent of every other such measure. ($f_c(a|h)$ is the probability that a occurs after the history h.)

EXAMPLE.



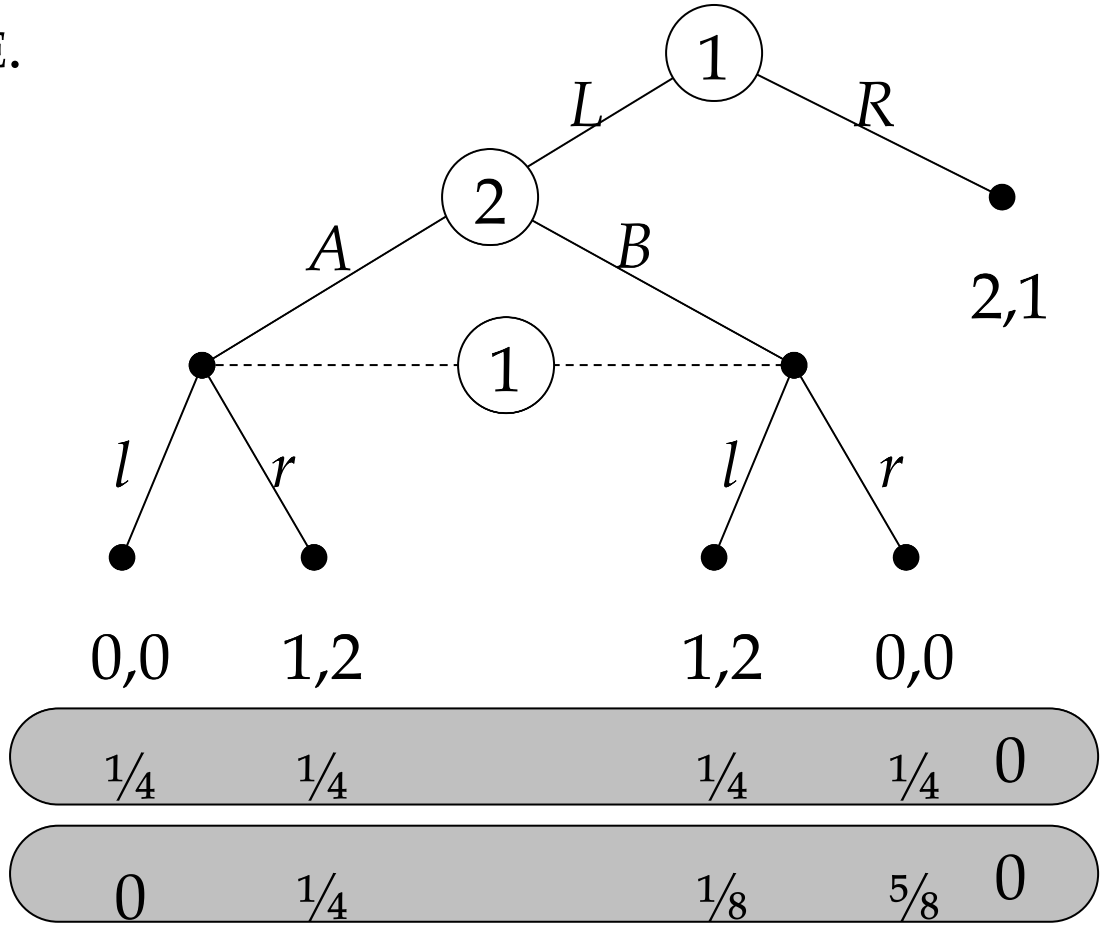
Players' *information partitions*:

$$\mathcal{I}_1 = \{\underbrace{\{\emptyset\}}_{\text{information set}}, \underbrace{\{(L,A),(L,B)\}}_{\text{information set}}\}. \quad \mathcal{I}_2 = \{\underbrace{\{L\}}_{\text{information set}}\}.$$

(Something new…)

- For each player $i \in N$ a partition $\mathcal{I}_i$ of $\{h \in H: P(h) = i\}$ with the property that $A(h) = A(h')$ whenever h and h' are in the same member of the partition. For $I_i \in \mathcal{I}_i$ we denote by $A(I_i)$ the set $A(h)$ and by $P(I_i)$ the player $P(h)$ for any $h \in I_i$. ($\mathcal{I}_i$ is the **information partition** of player i; a set $I_i \in \mathcal{I}_i$ is an **information set** of player i.)

EXAMPLE.



Question: Which lottery does player 1 prefer?

- For each player $i \in N$ a preference relation $\succsim_i$ on lotteries over Z (the **preference relation** of player i) that can be represented as the expected value of a payoff function defined on Z.

(Even if the players' actions are deterministic, the chance moves induce lotteries.)

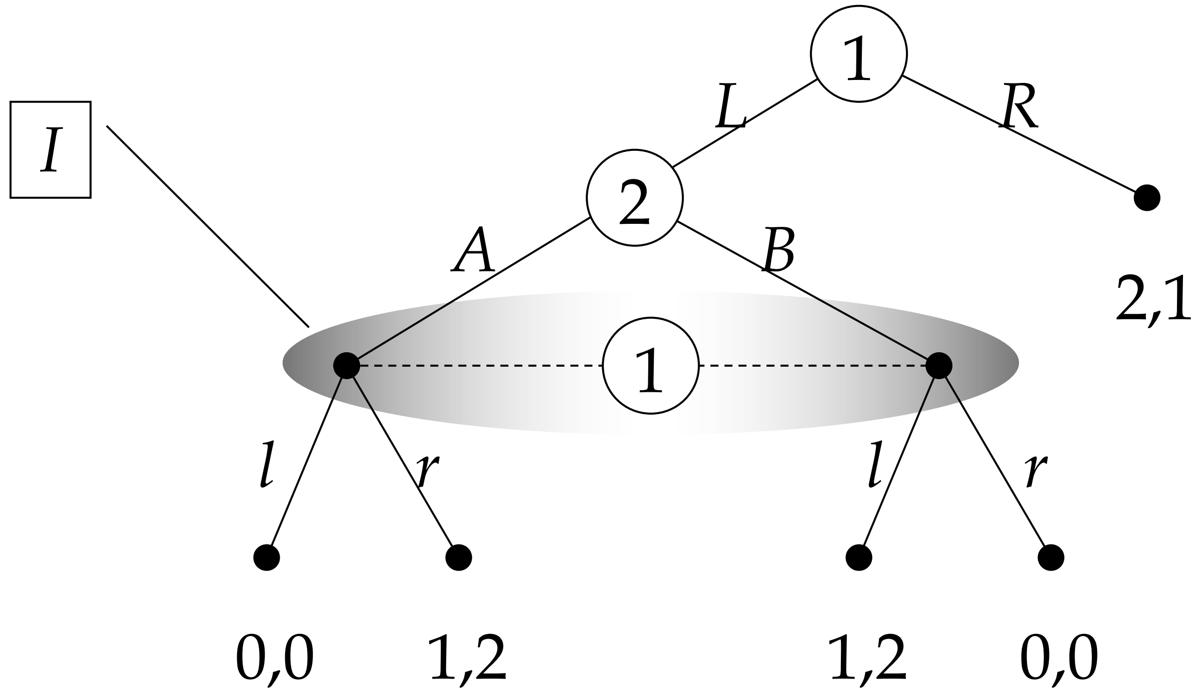
Extensive Game with Imperfect Information:

$$\langle N, H, P, f_c, (\mathcal{I}_i), (\succsim_i) \rangle.$$

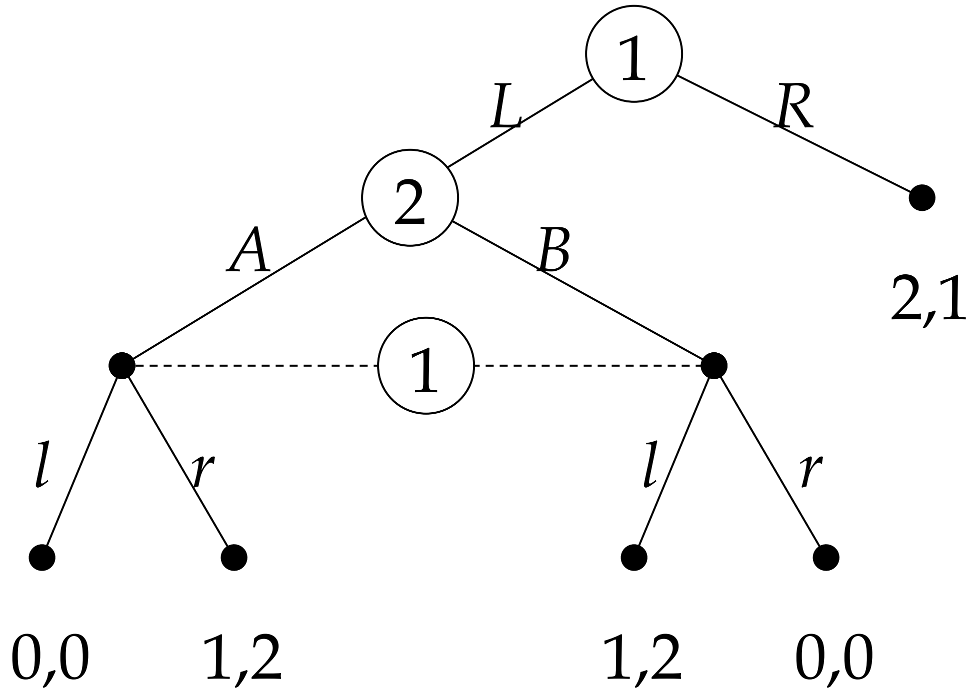
<u>COMPARE</u>

Extensive Game with Perfect Information:

$$\langle N, H, P, f_c, (\succsim_i) \rangle.$$



Player 1 cannot distinguish between (L, A) and (L, B) as these two histories are in the same information set I: $(L, A) \in I \in \mathcal{I}_1$ and $(L, B) \in I \in \mathcal{I}_1$. He only knows that some history in I has occurred.



Generally, player i cannot distinguish between h and h' <u>if</u> these two histories are in the same information set: $h \in I_i \in \mathcal{I}_i$ and $h' \in I_i \in \mathcal{I}_i$. He only knows that some history in I_i has occurred.

Therefore,

for available actions,

instead of $A(h)$**, we have** $A(I_i)$**;**

for the player function,

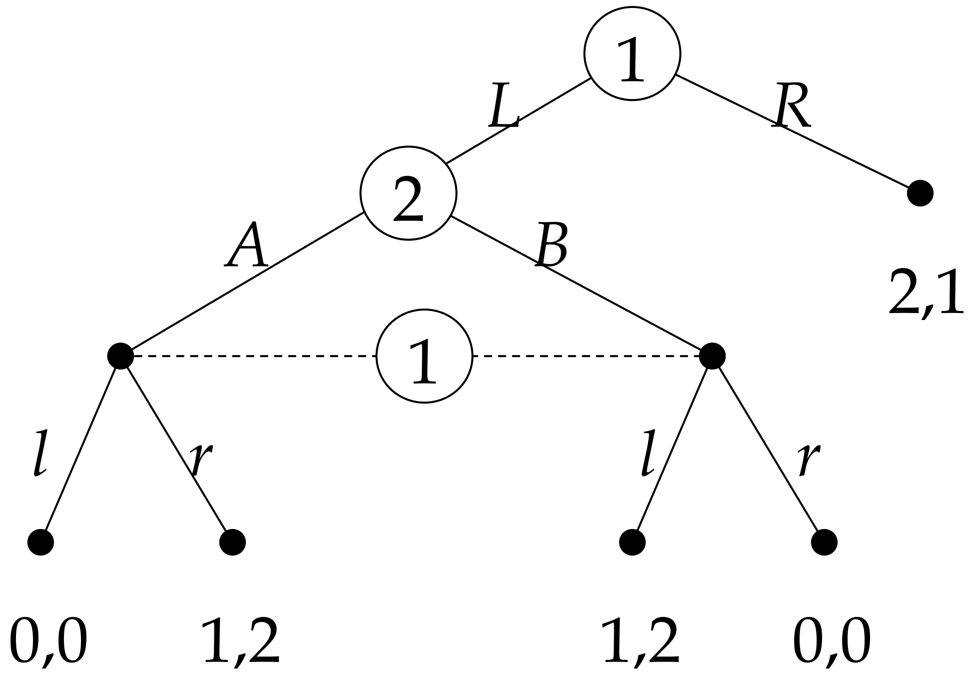
instead of $P(h)$**, we have** $P(I_i)$**.**

In general, we do not talk about h anymore. Whenever we want to talk about h, we talk about I_i instead (of course, we mean $h \in I_i$).

Class Discussion

Q: Are extensive games with perfect information special cases of extensive games with imperfect information?

Class Discussion



Q: What are the possible strategies of player 1?

Q: What are the possible strategies of player 2?

Strategies in Extensive Games

DEFINITION. A **pure strategy of player** $i \in N$ in an extensive game $\langle N, H, P, f_c, (\mathcal{I}_i), (\succsim_i)\rangle$ is a function that assigns an action in $A(I_i)$ to each information set $I_i \in \mathcal{I}_i$.

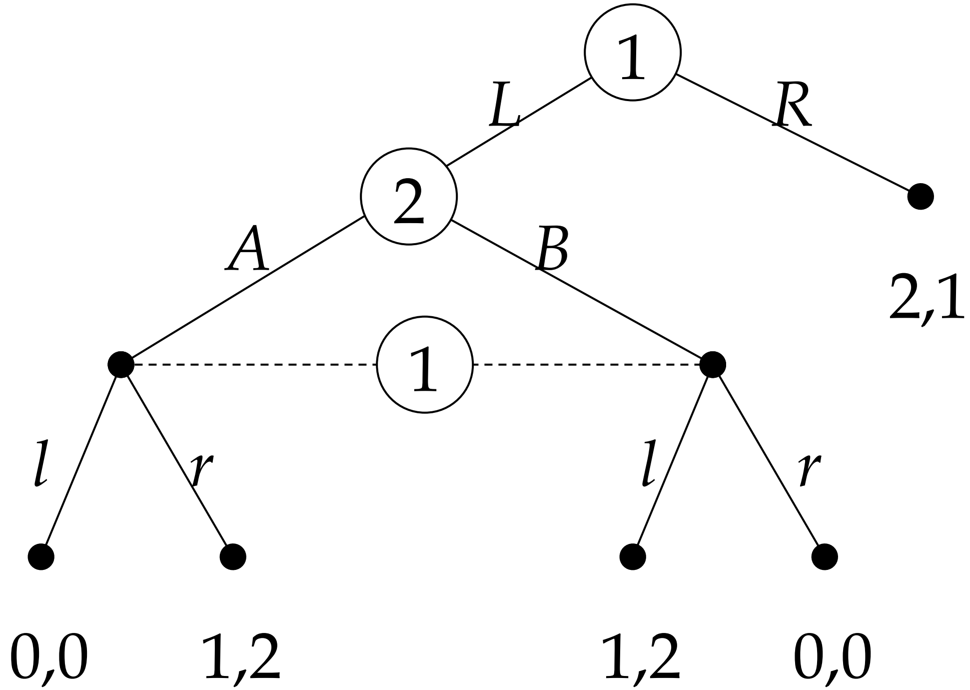
Remember?

In general, we do not talk about h anymore. Whenever we want to talk about h, we talk about I_i instead (of course, we mean $h \in I_i$).

Games with Perfect Recall

If at every point, every player remembers whatever he knew in the past, then the game is known as a **game with perfect recall**.

First, let $X_i(h)$ be the record of player i's experience along the history h.



$$X_1((L, A)) = (\emptyset, L)$$

Class Discussion

$X_i(h)$ is the sequence consisting of the information sets that the player encounters in the history h and the actions that he takes at them, in order.

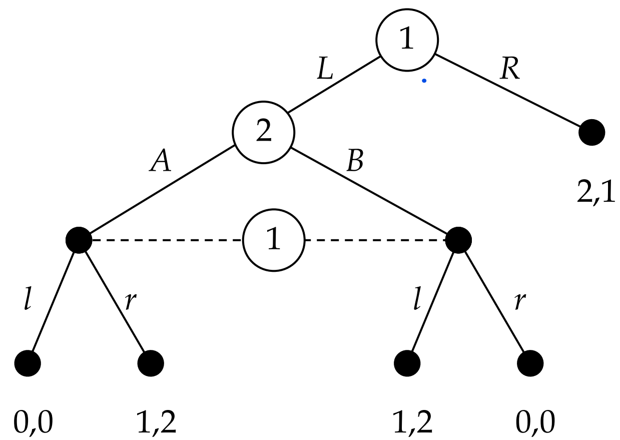
$X_1\big((L,A)\big) = (\emptyset, L)$

$X_1\big((L,B)\big) = ?$

$X_1(\emptyset) = ?$

$X_1\big((L,A,r)\big) = ?$

$X_1(R) = ?$



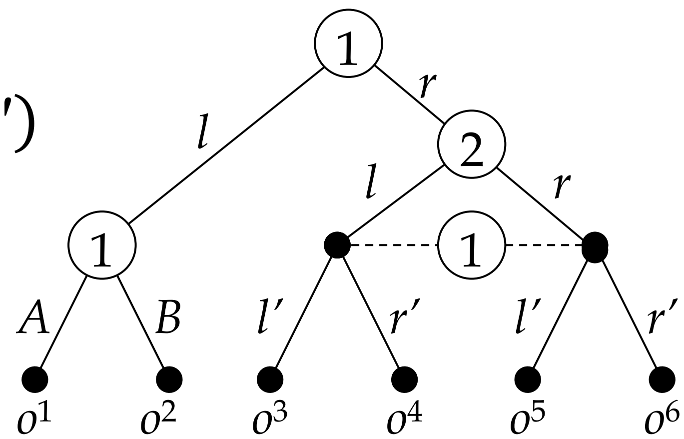
Class Discussion

$X_i(h)$ is the sequence consisting of the information sets that the player encounters in the history h and the actions that he takes at them, in order.

$X_1\big((r, l, r')\big) = (\emptyset, r, \{(r, l), (r, r)\}, r')$

$X_1\big((r, r, r')\big) = ?$

$X_1\big((l, B)\big) = ?$

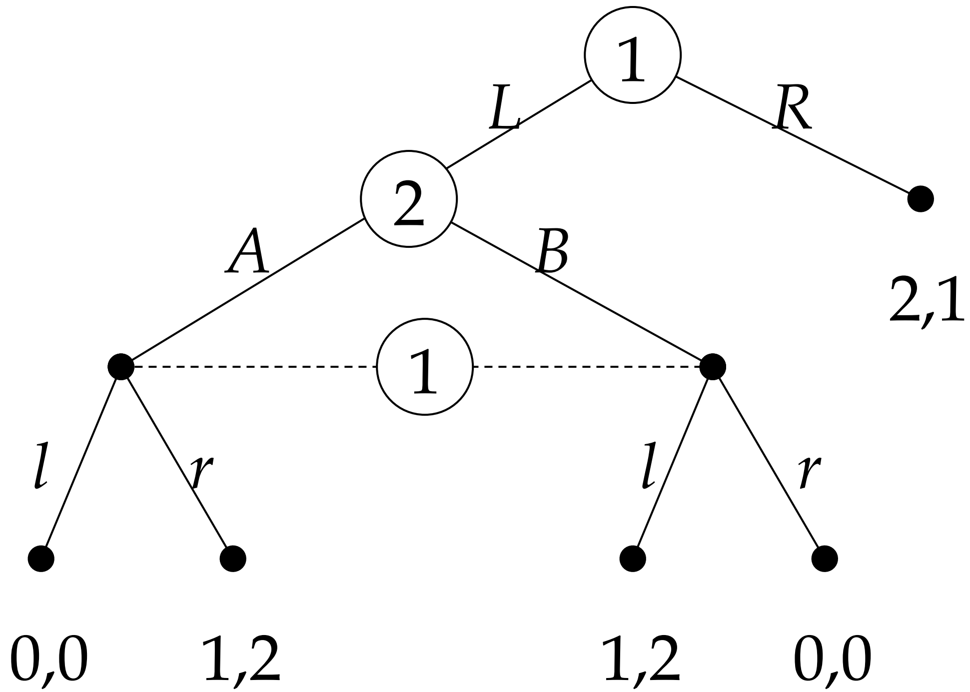


Games with Perfect Recall

An extensive game is a **game with perfect recall** if for each player i, we have $X_i(h) = X_i(h')$ whenever h and h' are in the same information set.

Q: Is an extensive game with perfect information a game with perfect recall?

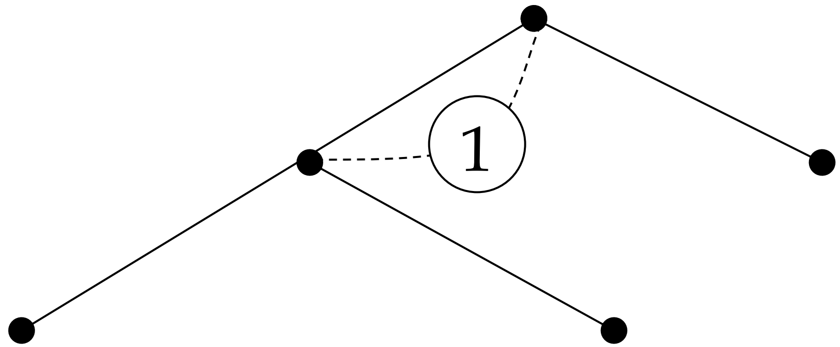
Class Discussion



Q: What are the information sets of player 1?

Q: Does this game have perfect recall?

Class Discussion



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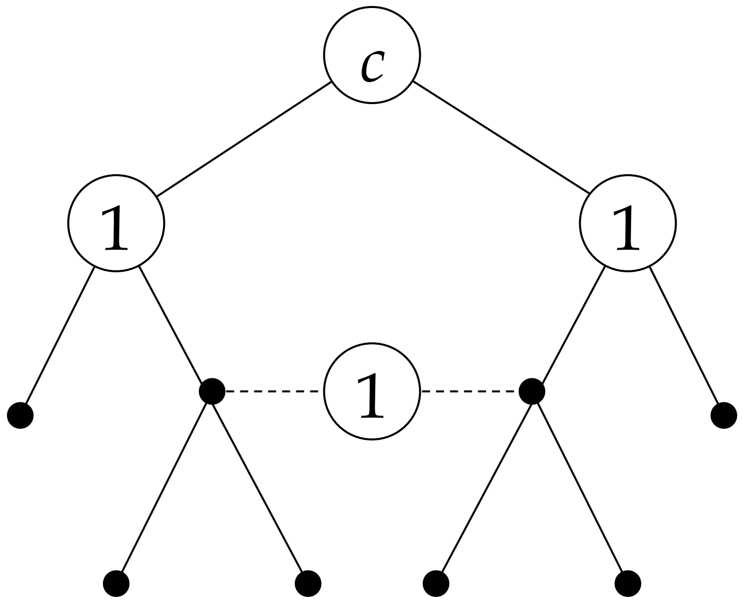
Class Discussion



Q: What are the information sets of player 1?

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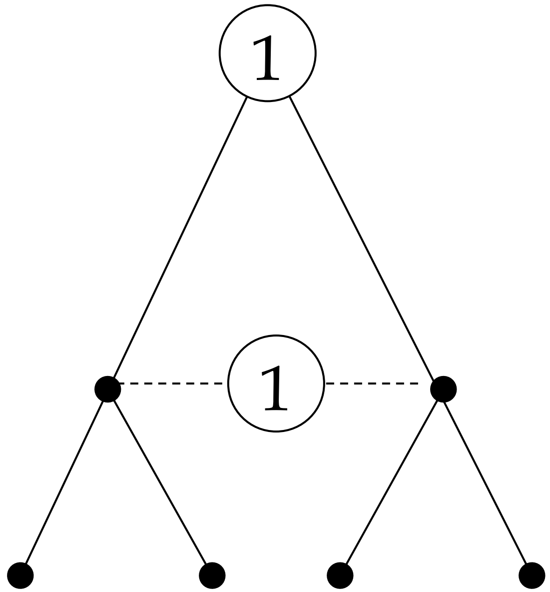
Class Discussion



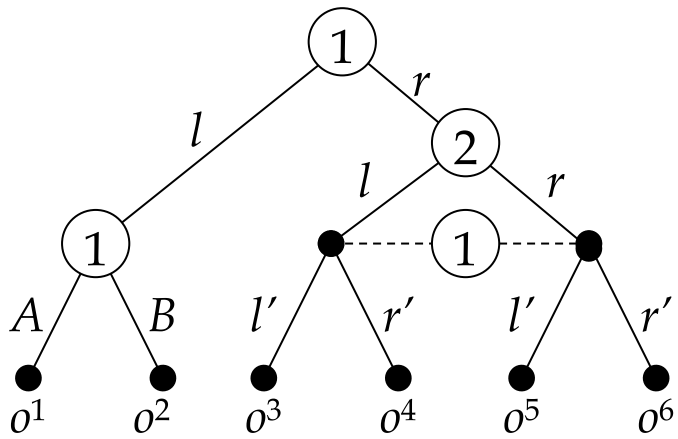
Q: What are the information sets of player 1?

Q: Does this game have perfect recall?

Mixed Strategies in Extensive Games

DEFINITION. A **mixed strategy of player** i in an extensive game $\langle N, H, P, f_c, (\mathcal{I}_i), (\succsim_i)\rangle$ is a probability measure over the set of player i's pure strategies.

Class Discussion

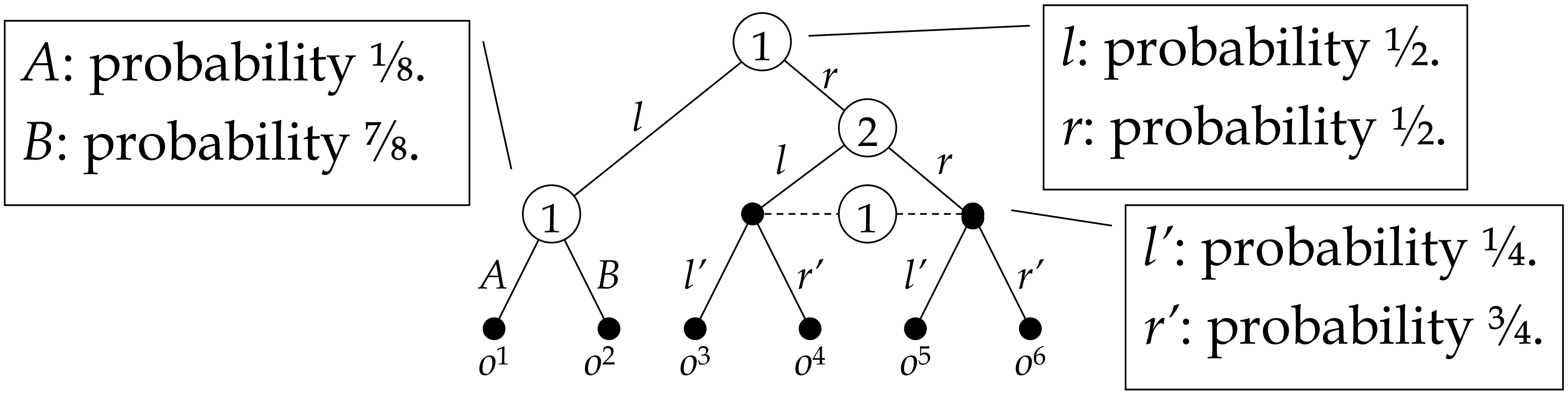


Q: What are the pure strategies for player 1?

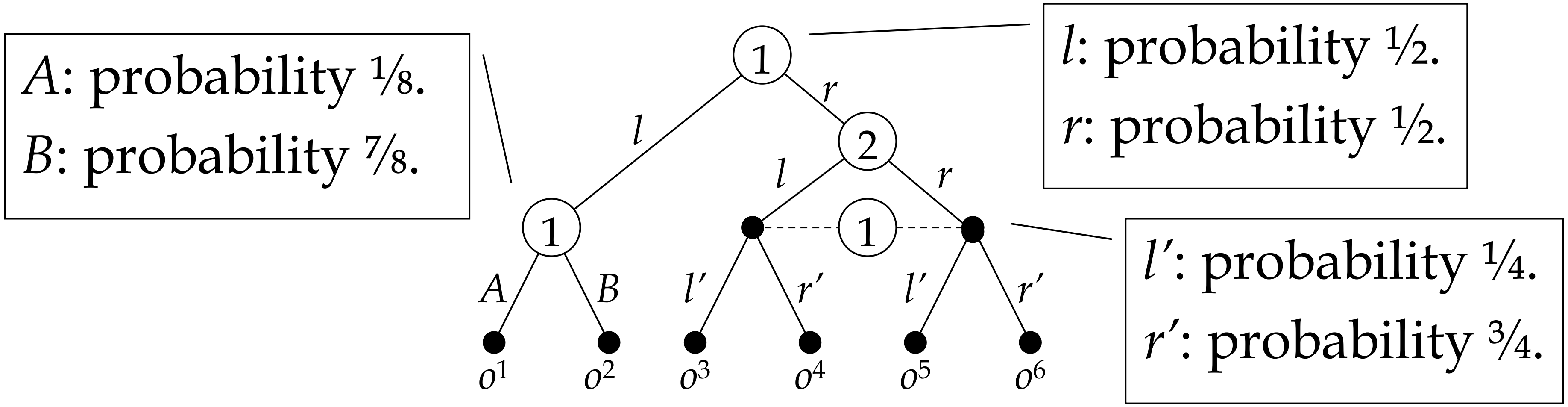
Q: Give one example of mixed strategy for player 1.

Behavioural Strategies in Extensive Games

Some players randomise their actions in a ***different*** way.

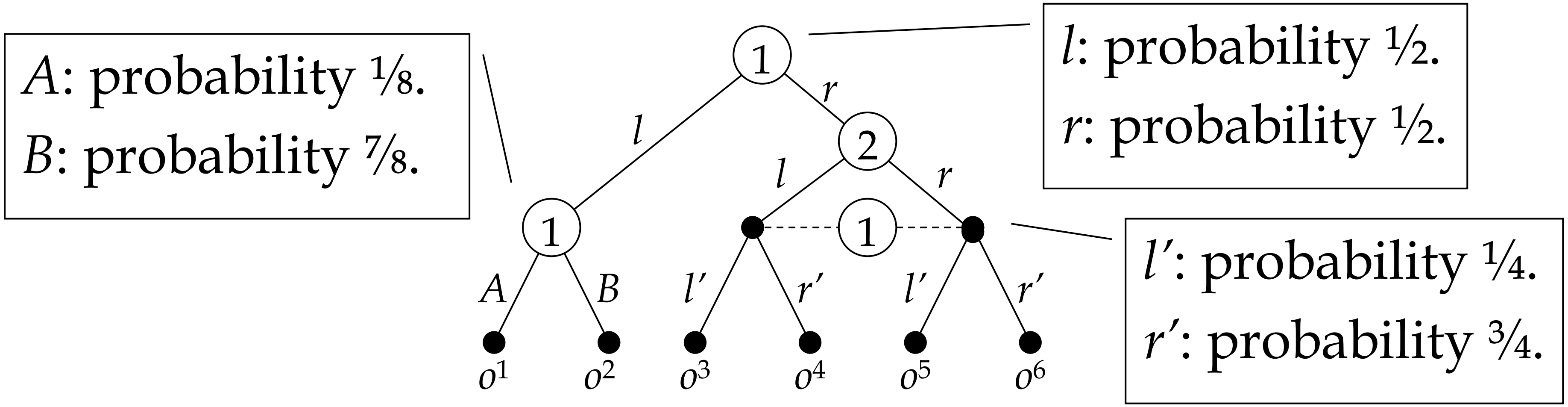


These are called **behavioural strategies**.



Player 1′s **behavioural strategies**:

$$\begin{pmatrix} \beta_1(\emptyset), \\ \beta_1(l), \\ \beta_1(\{(r,l),(r,r)\}) \end{pmatrix} = \begin{pmatrix} \left(l\left(\frac{1}{2}\right), r\left(\frac{1}{2}\right)\right), \\ \left(A\left(\frac{1}{8}\right), B\left(\frac{7}{8}\right)\right), \\ \left(l'\left(\frac{1}{4}\right), r'\left(\frac{3}{4}\right)\right) \end{pmatrix}$$

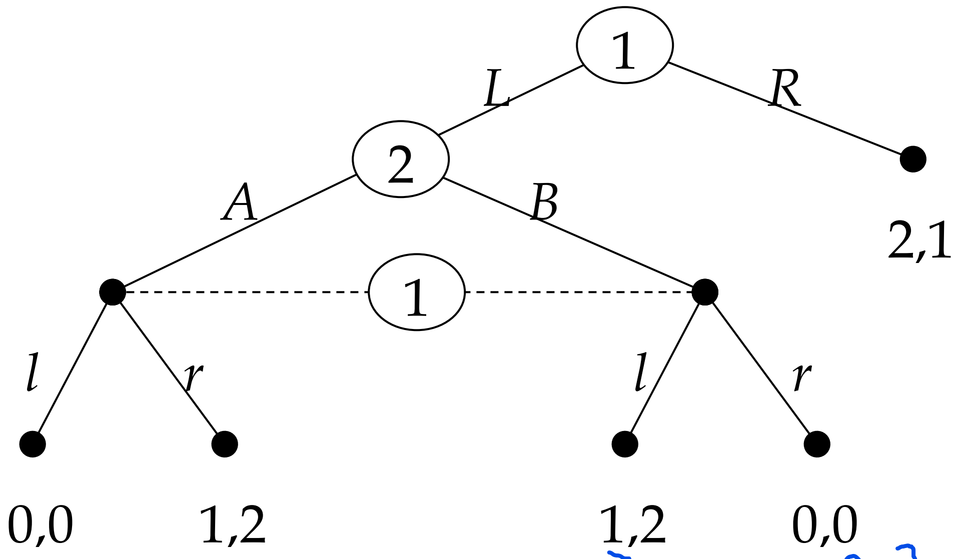


Player 1′s **behavioural strategies**:

$$\beta_1 = ((l(\tfrac{1}{2}), r(\tfrac{1}{2})), (A(\tfrac{1}{8}), B(\tfrac{7}{8})), (l'(\tfrac{1}{4}), r'(\tfrac{3}{4})))$$

$$\beta_1(\emptyset) = (l(\tfrac{1}{2}), r(\tfrac{1}{2})) \quad \beta_1(l) = (A(\tfrac{1}{8}), B(\tfrac{7}{8}))$$

$$\beta_1(\{(r,l),(r,r)\}) = (l'(\tfrac{1}{4}), r'(\tfrac{3}{4}))$$



Player 1′s information sets: {∅} and {L}, {R}, {{L,A}, {L,B}}.

{l}, {r}

Player 1′s pure strategies: Ll, Lr, Rl, and Rr.

An example of player 1′s mixed strategy:

An example of player 1′s behavioural strategy:

$$(\beta_1(\_\_), \beta_1(\_\_\_\_\_\_)) =$$

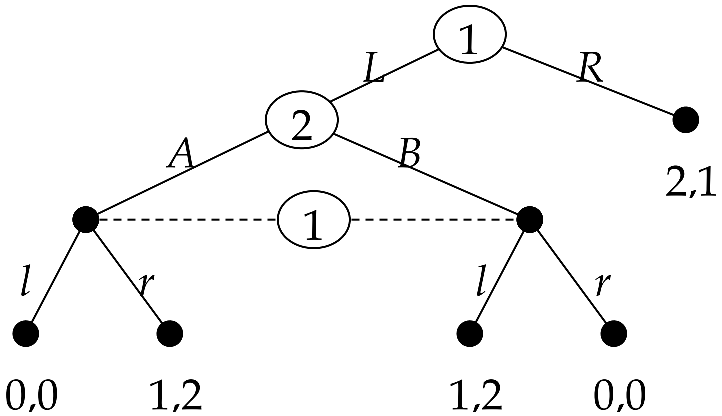
Behavioural Strategies in Extensive Games

DEFINITION. A **behavioural strategy of player** i is a collection $(\beta_i(I_i))_{I_i \in \mathcal{I}_i}$ of independent probability measures, where $\beta_i(I_i)$ is a probability measure over $A(I_i)$.

Notations:

For any $h \in I_i \in \mathcal{I}_i$ and action $a \in A(h)$ we denote by $\beta_i(h)(a)$ the probability $\beta_i(I_i)(a)$ assigned by $\beta_i(I_i)$ to the action a.

EXAMPLE. Let $I = \{(L, A), (L, B)\}$ and $\beta_1(I) = (l(\frac{1}{4}), r(\frac{3}{4}))$, then



$$\beta_1(I)(l) = \underline{\frac{1}{4}}.$$

$$\beta_1(\{(L,A),(L,B)\})(l) = \underline{\frac{1}{4}}.$$

$$\beta_1(\{(L,A),(L,B)\})(r) = \underline{\frac{3}{4}}.$$

$$\beta_1(L,A)(l) = \underline{\frac{1}{4}}.$$

$$\beta_1(L,B)(r) = \underline{\frac{3}{4}}.$$

A *mixed strategy* is a probability measure over the set of pure strategies (**the player randomly selects a pure strategy**),

whereas

a *behavioural strategy* specifies a probability measure over the actions available at each of the information sets (**the player plans a collection of randomisations, one for each of the point at which he has to take an action**).

Outcomes

For any profile $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ of either mixed or behavioural strategies, the outcome $O(\sigma)$ of σ is the **probability distribution over the terminal histories** that results when each player i follows the precepts of σ_i.

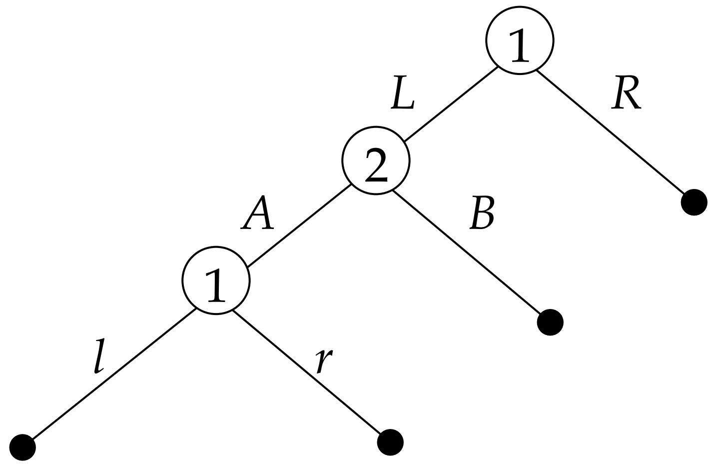
Class Discussion

Player 1's behavioural strategy:

$$((L(\frac{1}{4}), R(\frac{3}{4})), (l(\frac{1}{3}), r(\frac{2}{3}))).$$

Player 2's behavioural strategy:

$$(A(\frac{1}{2}), B(\frac{1}{2})).$$



Probability of Outcomes

(L, A, l): ________.

(L, A, r): ________.

(L, B): __________.

(R): ____________.

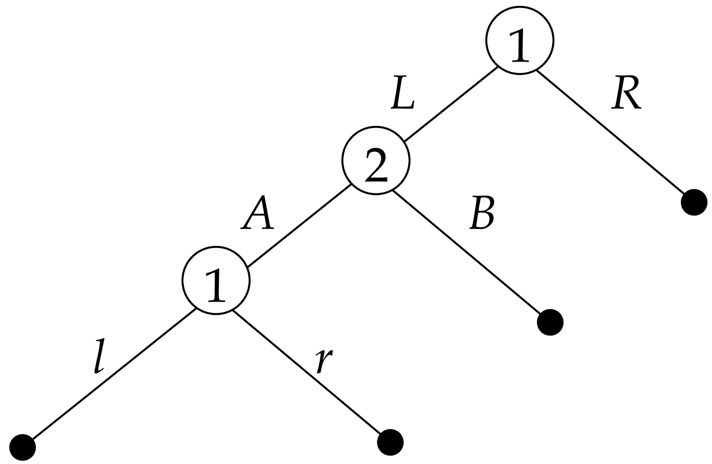
Class Discussion

Player 1's behavioural strategy:

$$((L(\frac{1}{4}), R(\frac{3}{4})), (l(\frac{1}{3}), r(\frac{2}{3}))).$$

Player 2's behavioural strategy:

$$(A(\frac{1}{2}), B(\frac{1}{2})).$$



Probability of Outcomes

(L, A, l): ________.

(L, A, r): ________.

(L, B): __________.

(R): _____________.

Outcome-Equivalence of Strategies

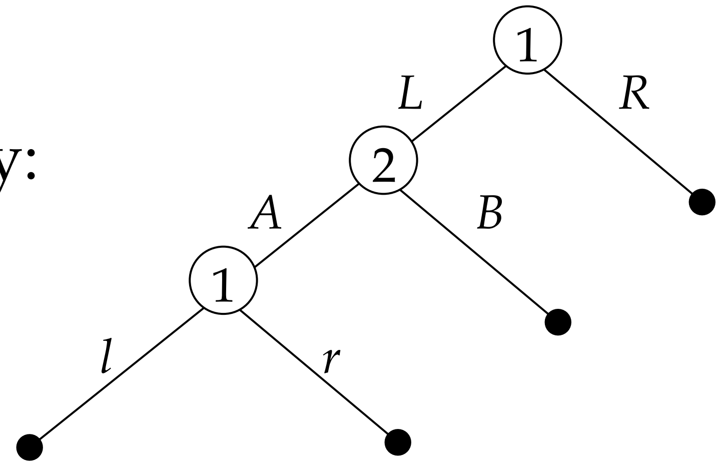
Two (mixed or behavioural) strategies of any players are outcome-equivalent if, for *every collection of **pure** strategies* of the other players, the two strategies induce the same outcome.

Under certain conditions (to be discussed soon), for any mixed strategy there is an outcome-equivalent behavioural strategy, and *vice versa*.

Class Discussion

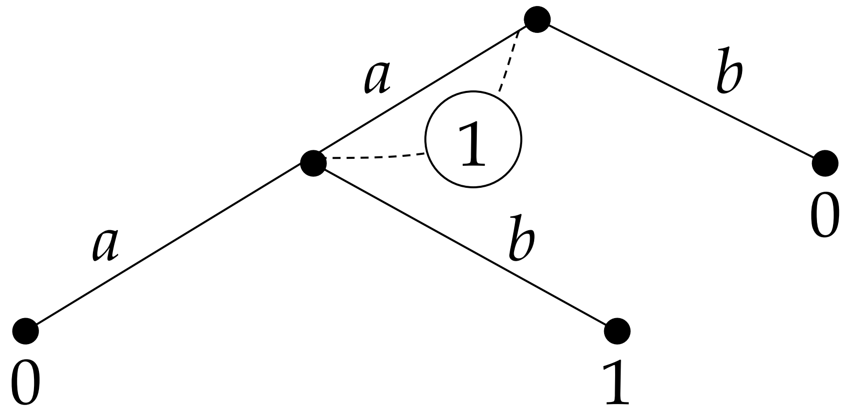
Player 1's behavioural strategy:

$$\left(\left(L\left(\frac{1}{4}\right), R\left(\frac{3}{4}\right)\right), \left(l\left(\frac{1}{3}\right), r\left(\frac{2}{3}\right)\right)\right).$$



Q: Find a *mixed* strategy for player 1 that is outcome-equivalent to the above behavioural strategy.

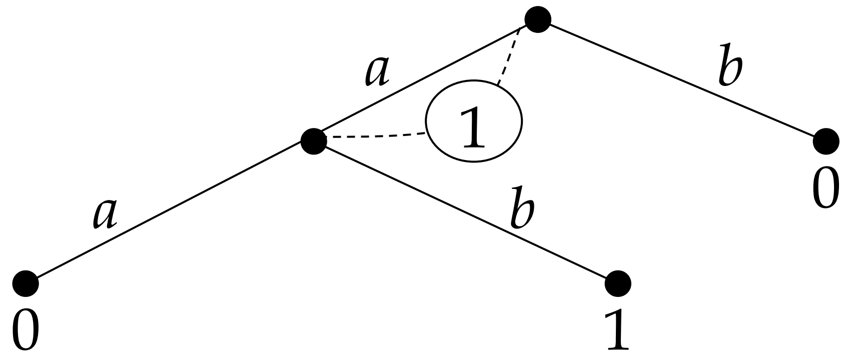
Class Discussion



Suppose a behavioural strategy assigns probability p to a (and hence $1 - p$ to b).

- Probability of outcome (a, a) is: p^2.
- Probability of outcome (a, b) is: $p(1-p)$.
- Probability of outcome b is: $(1-p)$.

Class Discussion

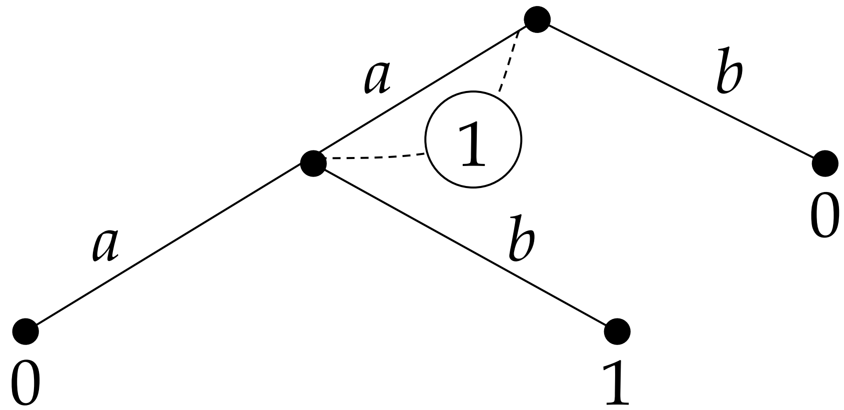


Q: Is there any mixed strategy that assigns probabilities to outcomes as follows?

$$(a, a): p^2,\ (a, b): p(1-p),\ b: 1-p.$$

A: Consider the mixed strategy $s_1 = (a(p'), b(q'))$, the probabilities of outcomes are (a, a): $\underline{p^2}$, (a, b): $\underline{p(1-p)}$, b: $\underline{(1-p)}$.

Class Discussion



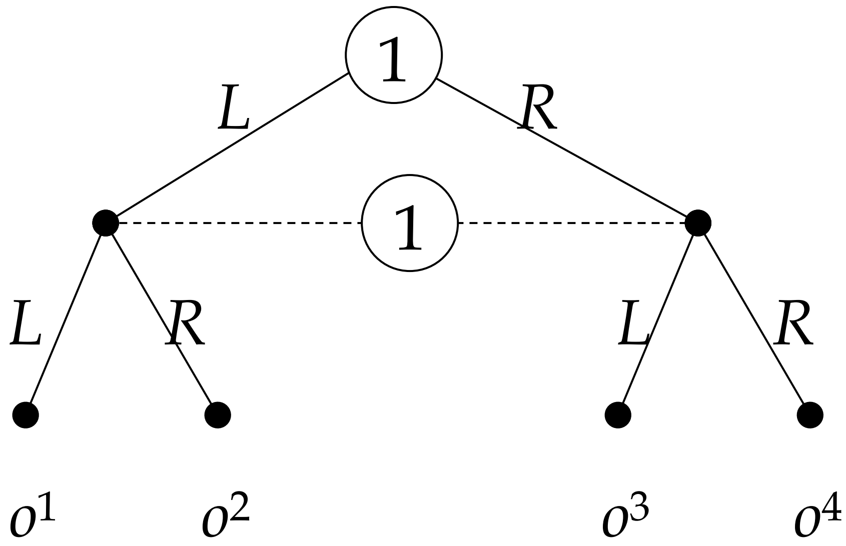
Q: What is the best behavioural strategy?

Q: What is the best mixed strategy?

Outcome-Equivalence of Mixed/Behavioural Strategies in Finite Extensive Games with Perfect Recall

PROPOSITION. For any mixed strategy of a player in a finite extensive game with perfect recall, there is an outcome-equivalent behavioural strategy.

Class Discussion



Consider this game with imperfect recall and $s_1 = (LL(\frac{1}{2}), LR(0), RL(0), RR(\frac{1}{2}))$. The outcome is $(\frac{1}{2}, 0, 0, \frac{1}{2})$. This outcome cannot be achieved by any behavioural strategy (***why?***).

Nash Equilibrium in Mixed Strategies

A **Nash equilibrium in mixed strategies** of an extensive game is a profile σ^* of mixed strategies with the property that for every player $i \in N$ we have

$$O(\sigma^*_{-i}, \sigma^*_i) \succsim_i O(\sigma^*_{-i}, \sigma_i)$$

for every mixed strategy σ_i of player i.

Nash Equilibrium in Behavioural Strategies

A **Nash equilibrium in behavioural strategies** of an extensive game is a profile σ^* of behavioural strategies with the property that for every player $i \in N$ we have

$$O(\sigma^*_{-i}, \sigma^*_i) \succsim_i O(\sigma^*_{-i}, \sigma_i)$$

for every behavioural strategy σ_i of player i.