# Arrangement of Course Examination

The course examination will be held on

**Wednesday 16 December 2020**

**Time: 15:00 – 17:00**

I shall send the examination paper to your CUHK email addresses (xxxxxxxxxx@link.cuhk.edu.hk) at 15:00 on 16 December. You can work on it for 120 minutes. Please submit your answer to CUHK Blackboard by **17:30** on 16 December.

# A Majority Game

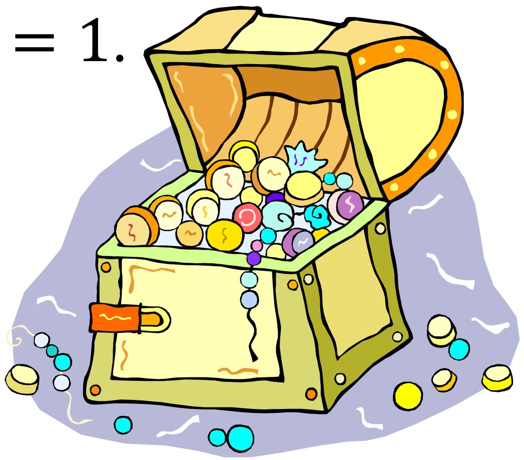
There are three people.

- $N = \{1,2,3\}$.

The *worth* of the teams is:

- $v(\{1,2,3\}) = 1$.
- $v(\{1,2\}) = 1,\ v(\{2,3\}) = 1,\ v(\{1,3\}) = 1$.
- $v(\{1\}) = 0,\ v(\{2\}) = 0,\ v(\{3\}) = 0$.

**Q**: What should be the payoff profile?



- $v(\{1,2,3\}) = 1.$
- $v(\{1,2\}) = 1,\ v(\{2,3\}) = 1,\ v(\{1,3\}) = 1.$
- $v(\{1\}) = 0,\ v(\{2\}) = 0,\ v(\{3\}) = 0.$

How about this payoff profile: $x = (\frac{1}{2}, \frac{1}{2}, 0)$.

1. It is a feasible (*i.e.*, $v(N) = \sum_{i \in N} x_i$).
2. Everyone feels 'OK' (*i.e.*, $x_i \geq v(\{i\})$).

A payoff profile that satisfies these two conditions is called an **imputation**.

**Question**: is $x = (\frac{1}{2}, \frac{1}{2}, 0)$ in the core?

# Imputations

An **imputation** is a feasible payoff profile $x$ for which $x_i \geq v(\{i\})$ for all $i \in N$.

The set of imputations is denoted $X$.

$$X = \{(x_i)_{i \in N} : v(N) = \sum_{i \in N} x_i \wedge \forall i \in N [x_i \geq v(\{i\})]\}$$

**Question**: are imputations always in the core?

- $v(\{1,2,3\}) = 1.$
- $v(\{1,2\}) = 1,\ v(\{2,3\}) = 1,\ v(\{1,3\}) = 1.$
- $v(\{1\}) = 0,\ v(\{2\}) = 0,\ v(\{3\}) = 0.$

How about this payoff profile: $x = (\frac{1}{2}, \frac{1}{2}, 0)$.

1. It is efficient (*i.e.*, $v(N) = \sum_{i \in N} x_i$).
2. It is individually rational (*i.e.*, $x_i \geq v(\{i\})$).

**Q**: Can you find a better imputation for <u>some</u> coalition $S$ (*that is, an imputation* $y$, *such that for some coalition* $S$, $y_i > x_i$ *for* <u>*all*</u> $i \in S$ *and* $y(S) \leq v(S)$)?

- $v(\{1,2,3\}) = 1.$
- $v(\{1,2\}) = 1,\ v(\{2,3\}) = 1,\ v(\{1,3\}) = 1.$
- $v(\{1\}) = 0,\ v(\{2\}) = 0,\ v(\{3\}) = 0.$

How about this payoff profile: $x = (\frac{1}{2}, \frac{1}{2}, 0)$.

1. It is efficient (*i.e.,* $v(N) = \sum_{i \in N} x_i$).
2. It is individually rational (*i.e.,* $x_i \geq v(\{i\})$).

But there is an imputation $(\frac{1}{4}, \frac{5}{8}, \frac{1}{8})$ that is better for the coalition $\{2,3\}$. *(And other imputations as well, of course.)*

# Objections

- $v(\{1,2,3\}) = 1.$
- $v(\{1,2\}) = 1,\ v(\{2,3\}) = 1,\ v(\{1,3\}) = 1.$
- $v(\{1\}) = 0,\ v(\{2\}) = 0,\ v(\{3\}) = 0.$

*The coalition $\{2,3\}$ is unsatisfied with $x = (\frac{1}{2}, \frac{1}{2}, 0)$, and it can* **object** *by suggesting $(\frac{1}{4}, \frac{5}{8}, \frac{1}{8})$ that is better for all the members of $\{2,3\}$. This is backed up by a threat to implement $(\frac{1}{4}, \frac{5}{8}, \frac{1}{8})$ on its own by dividing the worth among its members.* ***(How?)***

# Objections

An imputation $y$ is an **objection of the coalition** $S$ **to the imputation** $x$ if $y_i > x_i$ for all $i \in S$ and $y(S) \leq v(S)$, in which case we write $y \succ_S x$.

It is sometimes said that '$y$ dominates $x$ via $S$.'

*(That is, the coalition $S$ can object to the imputation $x$ by proposing the imputation $y$, because with the imputation $y$, all members in $S$ will be better off.)*

The imputation $y = (\frac{1}{4}, \frac{5}{8}, \frac{1}{8})$ is an **objection of the coalition** $\{2,3\}$ **to the imputation** $x = (\frac{1}{2}, \frac{1}{2}, 0)$ because $y_i > x_i$ for all $i \in \{2,3\}$ and $y(\{2,3\}) = \frac{3}{4} \leq v(\{2,3\}) = 1$.

In this case we write $(\frac{1}{4}, \frac{5}{8}, \frac{1}{8}) \succ_{\{2,3\}} (\frac{1}{2}, \frac{1}{2}, 0)$.

**Question**: if $x$ is in the core, then is there an imputation $y$ that is an **objection of some coalition** $S$ **to the imputation** $x$?

**Question**: is there any objection to $y = (\frac{1}{4}, \frac{5}{8}, \frac{1}{8})$?

- $v(\{1,2,3\}) = 1$.
- $v(\{1,2\}) = 1,\ v(\{2,3\}) = 1,\ v(\{1,3\}) = 1$.
- $v(\{1\}) = 0,\ v(\{2\}) = 0,\ v(\{3\}) = 0$.

How about this payoff profile: $x = (\frac{1}{2}, \frac{1}{2}, 0)$.

**Q**: Can you find any possible objections to the imputation $x$ proposed by any coalition?

**Q**: Can you find any imputations that have no objections?

# The Stable Sets

Since $\langle N, v\rangle$ is cohesive (*i.e.*, $v(N) \geq \sum_{k=1}^{K} v(S_k)$), we have $y \succ_S x$ if and only if there is an $S$-feasible payoff vector $(y_i)_{i\in S}$ for which $y_i > x_i$ for all $i \in S$.

**Compare**: the core = $\{x \in X$: there is no coalition $S$ and imputation $y$ for which $y \succ_S x\}$.

- $v(\{1,2,3\}) = 1.$
- $v(\{1,2\}) = 1,\ v(\{2,3\}) = 1,\ v(\{1,3\}) = 1.$
- $v(\{1\}) = 0,\ v(\{2\}) = 0,\ v(\{3\}) = 0.$

$$x = (\frac{1}{2}, \frac{1}{2}, 0).$$

Consider $\{(\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, 0, \frac{1}{2}), (0, \frac{1}{2}, \frac{1}{2})\}.$

**Q**: Do they dominate one another?

**Q**: Do they have objections from outside of the set?

# The Stable Sets: Internal Stability

An subset $Y$ of imputations is *internally stable*, if for any imputation $y \in Y$ there is no $z \in Y$ such that $z \succ_S y$ for some coalition $S$.

# The Stable Sets: External Stability

An subset $Y$ of imputations is *externally stable*, if for any imputation $u \notin Y$ there exists $w \in Y$ such that $w \succ_S u$ for some coalition $S$.

# The Stable Sets

DEFINITION. A subset $Y$ of the set $X$ of imputations of a coalitional game with transferable payoff $\langle N, v \rangle$ is a **stable set** if it satisfies the following two conditions:

- *Internal stability* If $y \in Y$ then for no $z \in Y$ does there exist a coalition $S$ for which $z \succ_S y$.
- *External stability* If $u \in X \backslash Y$ then there exists $w \in Y$ such that $w \succ_S u$ for some coalition $S$.

- $v(\{1,2,3\}) = 1$
- $v(\{1,2\}) = 1,\ v(\{2,3\}) = 1,\ v(\{1,3\}) = 1.$
- $v(\{1\}) = 0,\ v(\{2\}) = 0,\ v(\{3\}) = 0.$

$$Y = \{(\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, 0, \frac{1}{2}), (0, \frac{1}{2}, \frac{1}{2})\}.$$

**Q**: Is it internally stable?

**Q**: Is it externally stable?

This stable set corresponds to the 'standard of behaviour' in which some pair of players shares equally the single unit of payoff that is available.

- $v(\{1,2,3\}) = 1.$
- $v(\{1,2\}) = 1,\ v(\{2,3\}) = 1,\ v(\{1,3\}) = 1.$
- $v(\{1\}) = 0,\ v(\{2\}) = 0,\ v(\{3\}) = 0.$

$$Y = \{(\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, 0, \frac{1}{2}), (0, \frac{1}{2}, \frac{1}{2})\}.$$

Let $\mathcal{D}(Y)$ be the set of imputations objected to by one or more imputations in $Y$.
Then $\mathcal{D}(Y)^c = X \backslash \mathcal{D}(Y)$ is the set of imputations objected to by **none** of the imputations in $Y$.

# The Stable Sets

Consider a **stable set** $Y$ of imputations. It is *internally stable*: If $y \in Y$ then for no $z \in Y$ does there exist a coalition $S$ for which $z \succ_S y$.

$\mathcal{D}(Y)^c = X \backslash \mathcal{D}(Y)$ is the set of imputations objected to by **none** of the imputations in $Y$.

*Internal stability of* $Y$: Any imputation in $Y$ is objected to by **none** of the imputations in $Y$.

$$Y \subseteq \mathcal{D}(Y)^c = X \backslash \mathcal{D}(Y).$$

# The Stable Sets

Consider a **stable set** $Y$ of imputations. It is *externally stable*: If $u \in X \backslash Y$ then there exists $w \in Y$ such that $w \succ_S u$ for some coalition $S$.

$\mathcal{D}(Y)^c = X \backslash \mathcal{D}(Y)$ is the set of imputations objected to by **none** of the imputations in $Y$.

*External stability of* $Y$: Any imputation not objected to by any imputation in $Y$ must not be outside $Y$.

$$Y \supseteq \mathcal{D}(Y)^c = X \backslash \mathcal{D}(Y).$$

# The Stable Sets in Other Words

Let $Y$ be a stable set. Let $\mathcal{D}(Y)$ be the set of imputations objected to by one or more imputations in $Y$.

- *Internal stability of* $Y$: $Y \subseteq \mathcal{D}(Y)^c = X \backslash \mathcal{D}(Y)$.
- *External stability of* $Y$: $Y \supseteq \mathcal{D}(Y)^c = X \backslash \mathcal{D}(Y)$.

So a set $Y$ of imputations is a stable set if and only if $Y = X \backslash \mathcal{D}(Y)$.

- $v(\{1,2,3\}) = 1.$
- $v(\{1,2\}) = 1,\ v(\{2,3\}) = 1,\ v(\{1,3\}) = 1.$
- $v(\{1\}) = 0,\ v(\{2\}) = 0,\ v(\{3\}) = 0.$

Consider $Y' = \{(\frac{1}{6}, x, y) : x + y = \frac{5}{6}\}$

**Q**: Is $Y'$ a stable set?

**Q**: What is the '*standard of behaviour*?'

**Q**: Are there any other stable sets?

# Stable Sets as Standards of Behaviour

Each stable set can be interpreted as corresponding to a *standard of behaviour* (all the imputations in a stable set correspond to some particular mode of behaviour).

# Class Discussion

DEFINITION. The **core of the coalitional game with transferable payoff** $\langle N, v \rangle$ is the set of feasible payoff profiles $(x_i)_{i \in N}$ for which there is no coalition $S$ and $S$-feasible payoff vector $(y_i)_{i \in S}$ for which $y_i > x_i$ for all $i \in S$.

**Q**: What is the difference between the Core and Stable Sets?

# The Stable Sets

PROPOSITION.

*a.* The core is a subset of <u>every</u> stable set.

*Every member of the <u>core</u> is an imputation and no member is dominated by an imputation. So the result follows from external stability.*

*b.* No stable set is a proper subset of any other.

*This follows from external stability.*

*c.* If the core is a stable set then it is the only stable set.

*This follows from (a) and (b).*

# Convex Coalitional Games

A game is **convex** if $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ for all $S$ and $T$.

THEOREM. The core of a convex game is not empty.

THEOREM. The core of a convex game is stable.

*Reference:*

Shapley, L. S., 1971. Cores of convex games. *Int. J. Game Theory*, 1(1). Physica-Verlag, 11-26. [ doi:10.1007/BF01753431 ]

# The Shapley Value

Another important concept in coalitional games is the Shapley Value.

# Subgames of Coalitional Games with Transferable Payoff

There are three people.

- $N = \{1,2,3\}$.

The *worth* of the teams is:

- $v(\{1,2,3\}) = 1.$
- $v(\{1,2\}) = 1,\ v(\{2,3\}) = 1,\ v(\{1,3\}) = 1.$
- $v(\{1\}) = 0,\ v(\{2\}) = 0,\ v(\{3\}) = 0.$

# Subgames of Coalitional Games with Transferable Payoff

$\langle S, v^S \rangle$ is a subgame of $\langle N, v \rangle$

There are **two** people.

- $S = \{1,2\}$.

The *worth* of the teams is:

- $v^S(\{1,2\}) = v(\{1,2\}) = 1$.
- $v^S(\{1\}) = v(\{1\}) = 0,\ v^S(\{2\}) = v(\{2\}) = 0$.

# Subgames of Coalitional Games with Transferable Payoff

Let $\langle N, v\rangle$ be a coalitional game with transferable payoff. For each coalition S the **subgame**

$$\langle S, v^S\rangle$$

**of** $\langle N, v\rangle$ is the coalitional game with transferable payoff in which

$$v^S(T) = v(T)$$

for any $T \subseteq S$.

**Value** of a subgame: $\psi(S, v^S) = (\psi_i(S, v^S))_{i\in S}$

# Shapley Value

DEFINITION. The **Shapley value** $\varphi$ is defined by the condition that, for each $i \in N$,

$$\varphi_i(N, v) = \frac{1}{|N|!} \sum_{R \in \mathcal{R}} \underbrace{v(S_i(R) \cup \{i\}) - v(S_i(R))}_{\Delta_i(S_i(R))}$$

where $\mathcal{R}$ is the set of all $|N|!$ orderings of $N$, and $S_i(R)$ is the set of players preceding $i$ in the ordering $R$.

**Q**: What is the meaning of $S_i(R)$?

**Q**: What is the meaning of $\Delta_i(S_i(R))$?

EXAMPLE.

$$\langle \{1,2,3\}, v \rangle.$$

- $v(\{1,2,3\}) = 1.$
- $v(\{1,2\}) = 1,\ v(\{2,3\}) = 0,\ v(\{1,3\}) = 1.$
- $v(\{1\}) = 0,\ v(\{2\}) = 0,\ v(\{3\}) = 0.$

(Denote $\Delta_i(S_i(R)) = v(S_i(R) \cup \{i\}) - v(S_i(R))$)

$$\mathcal{R} = \{(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1)\}$$

| $R$ | | (1,2,3) | | (1,3,2) | | (2,1,3) | | (2,3,1) | | (3,1,2) | | (3,2,1) | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $S_1(R)$ | $\Delta_1(S_1(R))$ | ∅ | 0 | ∅ | 0 | {2} | 1 | {2,3} | 1 | {3} | 1 | {2,3} | 1 |
| $S_2(R)$ | $\Delta_2(S_2(R))$ | {1} | 1 | {1,3} | 0 | ∅ | 0 | ∅ | 0 | {1,3} | 0 | {3} | 0 |
| $S_3(R)$ | $\Delta_3(S_3(R))$ | {1,2} | 0 | {1} | 1 | {1,2} | 0 | {2} | 0 | ∅ | 0 | ∅ | 0 |

$$\varphi(N, v) = \left(\frac{1}{|N|!} \sum_{R \in \mathcal{R}} v(S_i(R) \cup \{i\}) - v(S_i(R))\right)_{i \in N} = \left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right)$$

# EXAMPLE.

$$\left\langle \{1,3\}, v^{\{1,3\}} \right\rangle.$$

- $v^{\{1,3\}}(\{1,3\}) = 1.$
- $v^{\{1,3\}}(\{1\}) = 0,\ v^{\{1,3\}}(\{3\}) = 0.$

$$\mathcal{R} = \{(1,3), (3,1)\}$$

| $R$ | | (1,3) | | (3,1) | |
|---|---|---|---|---|---|
| $S_1(R)$ | $\Delta_1(S_1(R))$ | $\emptyset$ | 0 | {3} | 1 |
| $S_3(R)$ | $\Delta_3(S_3(R))$ | {1} | 1 | $\emptyset$ | 0 |

Therefore, the Shapley value $\varphi(\{1,3\}, v^{\{1,3\}}) = (\frac{1}{2}, \frac{1}{2})$.

EXAMPLE.

$$\left\langle \{2,3\}, v^{\{2,3\}} \right\rangle.$$

- $v^{\{2,3\}}(\{2,3\}) = 0.$
- $v^{\{2,3\}}(\{2\}) = 0,\ \ v^{\{2,3\}}(\{3\}) = 0.$

$$\mathcal{R} = \{(2,3), (3,2)\}$$

| $R$ | | (2,3) | | (3,2) | |
|---|---|---|---|---|---|
| $S_2(R)$ | $\Delta_2(S_2(R))$ | $\emptyset$ | 0 | {3} | 0 |
| $S_3(R)$ | $\Delta_3(S_3(R))$ | {2} | 0 | $\emptyset$ | 0 |

Therefore, the Shapley value $\varphi(\{2,3\}, v^{\{2,3\}}) = (0,0)$.

# Shapley Value

$$\varphi_i(N, v) = \frac{1}{|N|!} \sum_{R \in \mathcal{R}} \Delta_i(S_i(R)) \quad \text{for each} \quad i \in N$$

If all the players are arranged in some arbitrary order, then $\varphi_i(N, v)$ is the expected marginal contribution over all orders of player $i$ to the set of players who precede him.

- $v(\{1,2,3\}) = 1.$
- $v(\{1,2\}) = 1,\ v(\{2,3\}) = 0,\ v(\{1,3\}) = 1.$
- $v(\{1\}) = 0,\ v(\{2\}) = 0,\ v(\{3\}) = 0.$

Shapley value: $\varphi(N, v) = (\frac{2}{3}, \frac{1}{6}, \frac{1}{6}).$

**Player 1** to **Player 2**: Give me more since otherwise I will leave the game, causing you to obtain only $\varphi_2(\{2,3\}, v^{\{2,3\}}) = 0$ rather than the larger payoff $\varphi_2(\{1,2,3\}, v) = \frac{1}{6}$.

**Player 2** to **Player 1**: It is true that if you leave then I will lose, but if *I* leave then *you* will lose at least as much $\varphi_1(\{1,2,3\}, v) - \varphi_1(\{1,3\}, v^{\{1,3\}}) = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$, which is not better than what I will lose $\varphi_2(\{1,2,3\}, v) - \varphi_2(\{2,3\}, v^{\{2,3\}}) = \frac{1}{6} - 0 = \frac{1}{6}$!!

**Note that the roles of players 1 and 2 can be swapped. This is known as the <u>Balanced Contributions Property</u> of Shapley Value.**

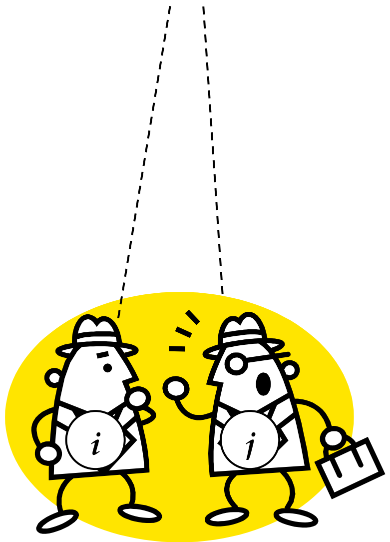
Shapley value: $\varphi(N, v) = (\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$.

**Player 2** to **Player 1**: Give me more since otherwise I will leave the game, causing you to obtain only $\varphi_1(\{1,3\}, v^{\{1,3\}}) = \frac{1}{2}$ rather than the larger payoff $\varphi_1(\{1,2,3\}, v) = \frac{2}{3}$.

**Player 1** to **Player 2**: It is true that if you leave then I will lose, but if *I* leave then *you* will lose at least as much $\varphi_2(\{1,2,3\}, v) - \varphi_2(\{2,3\}, v^{\{1,3\}}) = \frac{1}{6} - 0 = \frac{1}{6}$, which is not better than what I will lose $\varphi_1(\{1,2,3\}, v) - \varphi_1(\{1,3\}, v^{\{1,3\}}) = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$!!

# Objection and Counterobjection

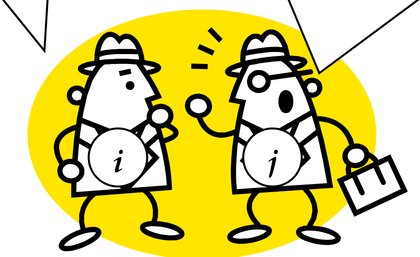
imputation: $x$



# Objection and Counterobjection

Give me more since otherwise I will leave the game, causing you to obtain only $\psi_j(N \setminus \{i\}, v^{N \setminus \{i\}})$ rather than the larger payoff $x_j$.

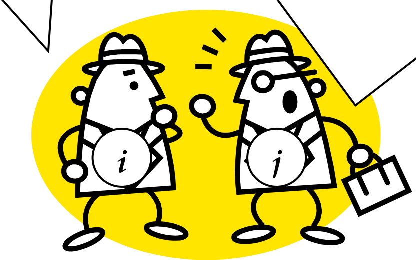
It is true that if you leave then I will lose, but if *I* leave then *you* will lose at least as much: $x_i - \psi_i(N \setminus \{j\}, v^{N \setminus \{j\}}) \geq x_j - \psi_j(N \setminus \{i\}, v^{N \setminus \{i\}})$!



# Objection and Counterobjection

Give me more since otherwise I will persuade the other players to exclude you from the game, causing me to obtain $\psi_i(N\setminus\{j\}, v^{N\setminus\{j\}})$ rather than the smaller payoff $x_i$.

It is true that if you exclude me then you will gain, but if *I* exclude *you* then I will gain at least as much $\psi_j(N\setminus\{i\}, v^{N\setminus\{i\}}) - x_j \geq \psi_i(N\setminus\{j\}, v^{N\setminus\{j\}}) - x_i$!



# Shapley Value

The Shapley value of a game $\langle N, v\rangle$ is the feasible payoff profile that for every objection of any player $i$ against any player $j$ there is a counterobjection of player $j$.

# Balanced Contributions Property

DEFINITION. A value $\psi$ satisfies the **balanced contributions property** if for every coalitional game with transferable payoff $\langle N, v\rangle$ we have

$$\psi_i(N,v) - \psi_i(N\backslash\{j\}, v^{N\backslash\{j\}}) \\ = \psi_j(N,v) - \psi_j(N\backslash\{i\}, v^{N\backslash\{i\}})$$

where $i \in N$ and $j \in N$.

The *unique* value that satisfies this property is the **Shapley value**.

# Shapley Value

PROPOSITION. The unique value that satisfies the balance contributions property is the Shapley value.