PART B

Linear Algebra. Vector Calculus

Advanced Engineering Mathematics, 10/e by Edwin Kreyszig

CHAPTER 7

Linear Algebra: Matrices, Vectors, Determinants. Linear Systems

- **Linear algebra**
  - Matrices (vector is special case of matrix) and their manipulations (e.g., determinants and rank)
  - Generalized concepts such as vector space and transformation
  - Applications in wide ranging fields (including social science, science, engineering, …)

7.1 Matrices, Vectors: Addition and Scalar Multiplication

General Concepts and Notations

- A **matrix** is a rectangular array of numbers or functions enclosed in brackets
- Example of matrices:

$$\begin{bmatrix} 0.3 & 1 & -5 \\ 0 & -0.2 & 16 \end{bmatrix}, \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \tag{1}$$

$$\begin{bmatrix} e^{-x} & 2x^2 \\ e^{6x} & 4x \end{bmatrix}, \quad \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ \frac{1}{2} \end{bmatrix}$$

- The numbers (or functions) are ***entries*** (or ***elements***) of the matrix

- **Standard index notation:** An entry of a matrix is uniquely identified by 2 indices
  - first index showing its row number
  - the second showing its column number
- Example: a_{23} is the entry in Row 2 and Column 3 of the matrix

$$\begin{bmatrix} 4 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 5 & 8 & 2 \\ 1 & 3 & 2 & 3 \end{bmatrix}$$

a_{23}

Column index comes second

Row index comes first

- Matrices are denoted by capital boldface letters **A**, **B**,…, or by writing the entry in brackets, e.g., $\mathbf{A} = [a_{jk}]$.

- Generally, an $\boldsymbol{m} \times \boldsymbol{n}$ **matrix** is a matrix with m rows and n columns:

$$\mathbf{A} = \left[a_{jk}\right] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

- $m \times n$ is called the **size** of the matrix.
- The matrices in (1) are of sizes 2 × 3, 3 × 3, 2 × 2, 1 × 3, and 2 × 1, respectively.

Square Matrix

- A matrix having as many rows as columns, $m = n$, is a **square matrix**
- For square matrix **A** of size $n \times n$, the entries a_{11}, a_{22}, … , a_{nn} is called the **main diagonal** of **A**
- Example,

$$\mathbf{A} = \begin{bmatrix} 4 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 5 & 8 & 2 \\ 1 & 3 & 2 & 3 \end{bmatrix}$$

Main diagonal of **A**:
4, 1, 8, 3

- Square matrices are particularly important
- A matrix of any size $m \times n$ is called a **rectangular matrix**; with square matrices as a special case.

Vectors

- A **vector** is a matrix with just a single row or column
- Its entries are called the **components** of the vector
- Vectors are denoted by ***lowercase*** boldface letters **a**, **b**, … or by its general component in brackets, e.g., $\mathbf{a} = [a_j]$.
- A general **row vector** is of the form:

$$\mathbf{a} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}, \text{ e.g., } \mathbf{a} = \begin{bmatrix} -2 & 5 & 0.8 & 0 & 1 \end{bmatrix}.$$

- A general **column vector** is of the form

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \text{ e.g., } \mathbf{b} = \begin{bmatrix} 4 \\ 0 \\ -7 \end{bmatrix}.$$

Components are denoted by one index only

b_3

Definition

Equality of Matrices

- Matrices $\mathbf{A} = [a_{jk}]$ and $\mathbf{B} = [b_{jk}]$ are **equal**, written $\mathbf{A} = \mathbf{B}$, if and only if that
  1. they have the same size, and
  2. their corresponding entries are equal, that is, $a_{11} = b_{11}$, $a_{12} = b_{12}$, etc.
- Matrices that are not equal are called **different**.
- Matrices of different sizes are always different.

Definition

Addition of Matrices

- The **sum** of two matrices $\mathbf{A} = [a_{jk}]$ and $\mathbf{B} = [b_{jk}]$ of the *same* size is written $\mathbf{A} + \mathbf{B}$, and is the matrix having the entries $a_{jk} + b_{jk}$ obtained by adding the corresponding entries of **A** and **B**.
- Matrices of different sizes cannot be added.

Scalar Multiplication (Multiplication by a Number)

- The **product** of any $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ and any **scalar** c (number c) is written $c\mathbf{A}$ and is the $m \times n$ matrix $c\mathbf{A} = [ca_{jk}]$ obtained by multiplying each entry of **A** by c.

Rules for Matrix Addition and Scalar Multiplication.

- From the familiar laws for the addition of numbers we obtain similar laws for the addition of matrices of the same size $m \times n$, namely,

$$
\text{(3)} \quad
\begin{aligned}
&\text{(a)} && \mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A} \\
&\text{(b)} && (\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C}) \qquad (\text{written } \mathbf{A}+\mathbf{B}+\mathbf{C}) \\
&\text{(c)} && \mathbf{A}+\mathbf{0}=\mathbf{A} \\
&\text{(d)} && \mathbf{A}+(-\mathbf{A})=\mathbf{0}.
\end{aligned}
$$

  Here, $\mathbf{0}$ denotes the **zero matrix** (of size $m \times n$), i.e., the $m \times n$ matrix with all entries zero. If $m = 1$ or $n = 1$, this is a vector, called a **zero vector**.

- Matrix addition is *commutative* and *associative* [by (3a) and (3b)].

Rules for Matrix Addition and Scalar Multiplication.

- Similarly, for scalar multiplication we obtain the rules

(4)

$$
\begin{aligned}
&\text{(a)}\quad c(\mathbf{A}+\mathbf{B}) = c\mathbf{A} + c\mathbf{B} \\
&\text{(b)}\quad (c+k)\mathbf{A} = c\mathbf{A} + k\mathbf{A} \\
&\text{(c)}\quad c(k\mathbf{A}) = (ck)\mathbf{A} \qquad \text{(written } ck\mathbf{A}\text{)} \\
&\text{(d)}\quad 1\mathbf{A} = \mathbf{A}.
\end{aligned}
$$

7.2 Matrix Multiplication

Definition

Multiplication of a Matrix by a Matrix

The **product C** = **AB** (in this order) of an $m \times n$ matrix **A** = $[a_{jk}]$ times an $r \times p$ matrix **B** = $[b_{jk}]$ is defined if and only if $r = n$, and that the product is the $m \times p$ matrix **C** = $[c_{jk}]$ with entries

(1)
$$c_{jk} = \sum_{l=1}^{n} a_{jl} b_{lk} = a_{j1} b_{1k} + a_{j2} b_{2k} + \cdots + a_{jn} b_{nk} \qquad \begin{array}{l} j = 1, \cdots, m \\ k = 1, \cdots, p. \end{array}$$

The condition $r = n$ means that the second factor, **B**, must have as many rows as the first factor has columns, namely n. A diagram of sizes that shows when matrix multiplication is as follows:

$$\begin{array}{cccc} \mathbf{A} & \mathbf{B} & = & \mathbf{C} \\ [m \times n] & [n \times p] & = & [m \times p]. \end{array}$$

- The entry c_{jk} in (1) is obtained by multiplying each entry in the jth row of **A** by the corresponding entry in the kth column of **B** and then adding these n products. For instance, $c_{21} = a_{21}b_{11} + a_{22}b_{21} + \dots + a_{2n}b_{n1}$, and so on.
- One calls this briefly a ***multiplication of rows into columns***.
- For $n = 3$, this is illustrated by

$$m = 4\left\{\overbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}}^{n=3}\right. \overbrace{\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}}^{p=2} = \left.\overbrace{\begin{bmatrix} c_{11} & c_{12} \\ \boxed{c_{21}} & c_{22} \\ c_{31} & c_{32} \\ c_{41} & c_{42} \end{bmatrix}}^{p=2}\right\} m = 4$$

where the shaded entries are those that contribute to the calculation of entry c_{21} just discussed.

EXAMPLE 1
Matrix Multiplication

$$\mathbf{AB} = \begin{bmatrix} 3 & 5 & -1 \\ 4 & 0 & 2 \\ -6 & -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 & 1 \\ 5 & 0 & 7 & 8 \\ 9 & -4 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 22 & -2 & 43 & 42 \\ 26 & -16 & \boxed{14} & 6 \\ -9 & 4 & -37 & -28 \end{bmatrix}$$

Here, $c_{11} = 3 \cdot 2 + 5 \cdot 5 + (-1) \cdot 9 = 22$, and so on. The entry in the box is $c_{23} = 4 \cdot 3 + 0 \cdot 7 + 2 \cdot 1 = 14$. The product **BA** is not defined.

EXAMPLE 2: Matrix Multiplication

$$\begin{bmatrix} 4 & 2 \\ 1 & 8 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 22 \\ 43 \end{bmatrix}, \text{ and } \begin{bmatrix} 3 \\ 5 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 1 & 8 \end{bmatrix} = ? \text{ Not defined!}$$

EXAMPLE 3: Matrix Multiplication

$$\begin{bmatrix} 3 & 6 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 19 \end{bmatrix} = 19$$

$$\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}\begin{bmatrix} 3 & 6 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 1 \\ 6 & 12 & 2 \\ 12 & 24 & 4 \end{bmatrix}$$

EXAMPLE 4 CAUTION!

Matrix Multiplication Is Not Commutative, AB ≠ BA in General

- This is illustrated by Examples 1 and 2, where one of the two products is not even defined, and by Example 3, where the two products have different sizes.
- This also holds for square matrices. For instance,

$$\begin{bmatrix} 1 & 1 \\ 100 & 100 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{but} \quad \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 100 & 100 \end{bmatrix} = \begin{bmatrix} 99 & 99 \\ -99 & -99 \end{bmatrix}.$$

(It is interesting that $\mathbf{AB} = \mathbf{0}$ this also shows that does *not* necessarily imply $\mathbf{BA} = \mathbf{0}$ or $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$.)

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Our examples show that in matrix products ***the order of factors must always be observed very carefully***. Otherwise matrix multiplication satisfies rules similar to those for numbers, namely.

$$
(2)\quad
\begin{aligned}
&\text{(a)}\quad (k\mathbf{A})\mathbf{B} = k(\mathbf{AB}) = \mathbf{A}(k\mathbf{B}) && \textit{written } k\mathbf{AB} \textit{ or } \mathbf{A}k\mathbf{B}\\
&\text{(b)}\quad \mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C} && \textit{written } \mathbf{ABC}\\
&\text{(c)}\quad (\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC}+\mathbf{BC}\\
&\text{(d)}\quad \mathbf{C}(\mathbf{A}+\mathbf{B}) = \mathbf{CA}+\mathbf{CB}
\end{aligned}
$$

provided **A**, **B**, and **C** are such that the expressions on the left are defined; here, k is any scalar. (2b) is called the **associative law**. (2c) and (2d) are called the **distributive laws**.

Expressing Matrix Multiplication more compactly

The **product** **C** = **AB** of an $m \times n$ matrix **A** times an $n \times p$ matrix **B** is the $m \times p$ matrix **C** = $[c_{jk}]$ with entries

(3)

$$c_{jk} = \mathbf{a}_j \mathbf{b}_k = \begin{bmatrix} a_{j1} & a_{j2} & \cdots & a_{jn} \end{bmatrix} \begin{bmatrix} b_{1k} \\ b_{2k} \\ \vdots \\ b_{nk} \end{bmatrix}$$

$$= a_{j1}b_{1k} + a_{j2}b_{2k} + \cdots + a_{jn}b_{nk} \qquad i = 1, 2, \ldots, m; \; j = 1, 2, \cdots, p$$

where $\mathbf{a}_j$ is the jth row vector of **A** and $\mathbf{b}_k$ is the kth column vector of **B**.

Parallel processing of products $\mathbf{C} = \mathbf{AB}$ on the computer

- Facilitated by a variant of (3), which is used by standard algorithms (such as in Lapack).
- With $\mathbf{A}$ as given, decompose $\mathbf{B}$ in terms of its column vectors, and compute the product column by column,

$$\textbf{(5)} \qquad \mathbf{AB} = \mathbf{A}[\mathbf{b}_1 \;\; \mathbf{b}_2 \;\; \dots \;\; \mathbf{b}_p] = [\mathbf{Ab}_1 \;\; \mathbf{Ab}_2 \;\; \dots \;\; \mathbf{Ab}_p].$$

- Assign columns of $\mathbf{B}$ to different processors (individually or several to each processor) to simultaneously compute the columns of the product matrix $\mathbf{Ab}_1$, $\mathbf{Ab}_2$, etc.

Applications of Matrices

Example 10.5. In CUHK's ENGG1410, the grading scheme is given as: Homework assignment (HW) 20%, midterm examination (MID) 30%, and Final Examination (FIN) 50%.

Let the marks of the HW, MID and FIN of the students be tabulated as matrix **A**:

$$\begin{array}{rc} & \begin{array}{ccc}\text{HW} & \text{MID} & \text{FIN}\end{array} \\ \begin{array}{r}\text{Student \#1}\\ \text{Student \#2}\\ \text{Student \#3}\\ \vdots \\ \text{Student \#99}\\ \text{Student\#100}\end{array} & \begin{bmatrix} 89 & 75 & 82 \\ 50 & 55 & 85 \\ 87 & 91 & 97 \\ \vdots & \vdots & \vdots \\ 97 & 88 & 67 \\ 65 & 78 & 75 \end{bmatrix} \end{array} = \mathbf{A}$$

The overall marks of the students are given by $\mathbf{M} = \mathbf{A}\begin{bmatrix} 0.2 \\ 0.3 \\ 0.5 \end{bmatrix}$

EXAMPLE 11

11 Computer Production. Matrix Times Matrix

Supercomp Ltd produces two computer models PC1086 and PC1186. The matrix **A** shows the cost per computer (in thousands of dollars) and **B** the production figures for the year 2010 (in multiples of 10,000 units.) Find a matrix **C** that shows the shareholders the cost per quarter (in millions of dollars) for raw material, labor, and miscellaneous.

$$\begin{array}{cc} & \begin{array}{cc} \text{PC1086} & \text{PC1186} \end{array} & \\ \mathbf{A} = & \begin{bmatrix} 1.2 & 1.6 \\ 0.3 & 0.4 \\ 0.5 & 0.6 \end{bmatrix} & \begin{array}{l} \text{Raw Components} \\ \text{Labor} \\ \text{Miscellaneous} \end{array} \end{array} \qquad \begin{array}{ccc} & \begin{array}{c} \text{Quarter} \\ \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \end{array} & \\ \mathbf{B} = & \begin{bmatrix} 3 & 8 & 6 & 9 \\ 6 & 2 & 4 & 3 \end{bmatrix} & \begin{array}{l} \text{PC1086} \\ \text{PC1186} \end{array} \end{array}$$

Solution.

$$\begin{array}{ccc} & \begin{array}{c} \text{Quarter} \\ \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \end{array} & \\ \mathbf{C} = \mathbf{AB} = & \begin{bmatrix} 13.2 & 12.8 & 13.6 & 15.6 \\ 3.3 & 3.2 & 3.4 & 3.9 \\ 5.1 & 5.2 & 5.4 & 6.3 \end{bmatrix} & \begin{array}{l} \text{Raw Components} \\ \text{Labor} \\ \text{Miscellaneous} \end{array} \end{array}$$

Since cost is given in multiples of \$1000 and production in multiples of 10,000 units, the entries of **C** are multiples of \$10 millions; thus $c_{11} = 13.2$ means \$132 million, etc. ■

EXAMPLE 12

Weight Watching. Matrix Times Vector

Suppose that in a weight-watching program, a person of 185 lb burns 350 cal/hr in walking (3 mph), 500 in bicycling (13 mph), and 950 in jogging (5.5 mph). Bill, weighing 185 lb, plans to exercise according to the matrix shown. Verify the calculations (W = Walking, B = Bicycling, J = Jogging).

$$
\begin{array}{c} \\ \text{MON} \\ \text{WED} \\ \text{FRI} \\ \text{SAT} \end{array}
\begin{array}{c} \begin{array}{ccc} \text{W} & \text{B} & \text{J} \end{array} \\ \begin{bmatrix} 1.0 & 0 & 0.5 \\ 1.0 & 1.0 & 0.5 \\ 1.5 & 0 & 0.5 \\ 2.0 & 1.5 & 1.0 \end{bmatrix} \end{array}
\begin{bmatrix} 350 \\ 500 \\ 950 \end{bmatrix} = \begin{bmatrix} 825 \\ 1325 \\ 1000 \\ 2400 \end{bmatrix}
\begin{array}{c} \text{MON} \\ \text{WED} \\ \text{FRI} \\ \text{SAT} \end{array}
$$

EXAMPLE 13

- A city of size 60 square miles has the following data for change of land use (as Commercial (C), Industrial (I) and Residential (R) in every 5 years:
  - For C-district: 70% remains C, 20% turns I and 10% turns R
  - For I-district: 10% turns C, 90% remains I and 0% turns R
  - For R-district: 0% turns C, 20% turns I and 80% remains R
- Land use percentage in 2004: 25% C, 20% I, and 55% R.
- Land us percentage in 2009?
- Form Land use Transition Matrix **A** and 2004 Land Use Percentage vector

$$\begin{array}{cc} \text{From} & \begin{array}{ccc} \text{C} & \text{I} & \text{R} \end{array} & \text{To} \\ \mathbf{A} = & \begin{bmatrix} 0.7 & 0.1 & 0 \\ 0.2 & 0.9 & 0.2 \\ 0.1 & 0 & 0.8 \end{bmatrix} & \begin{array}{c} \text{C} \\ \text{I} \\ \text{R} \end{array} \end{array} \quad \text{and} \quad \mathbf{L}_{2004} = \begin{bmatrix} 25 \\ 20 \\ 55 \end{bmatrix}$$

- Land Use Percentage vector in 2009

$$\mathbf{L}_{2009} = \mathbf{A}\mathbf{L}_{2004} = \begin{bmatrix} 0.7 & 0.1 & 0 \\ 0.2 & 0.9 & 0.2 \\ 0.1 & 0 & 0.8 \end{bmatrix} \begin{bmatrix} 25 \\ 20 \\ 55 \end{bmatrix} = \begin{bmatrix} 19.5 \\ 34 \\ 46.5 \end{bmatrix}$$

- Land Use Percentage vector in 2014

$$\mathbf{L}_{2014} = \mathbf{A}\mathbf{L}_{2009} = \mathbf{A}^2\mathbf{L}_{2004}$$

- Generally, Land Use Percentage vector in Year 2004+5*n

$$\mathbf{L}_{2014+n*5} = \mathbf{A}^n\mathbf{L}_{2004}$$

- This assumes constant Land Transition Matrix A (which is not true).
- Question: Is there a steady state Land Use Percentage Vector $\mathbf{L}_{ss}$ such that $\mathbf{L}_{ss} = \mathbf{A}\mathbf{L}_{ss}$? (Later!)

Definition: Transposition

- The transpose of a matrix **A**, denoted by $\mathbf{A}^\mathsf{T}$, is obtained by writing its rows as columns (or equivalently its columns as rows).
- For vectors – transpose of a row vector becomes a column vector and vice versa.
- For square matrices, we can "reflect" the elements along the main diagonal, i.e., a_{12} becomes a_{21}, a_{31} becomes a_{13}, etc.
- The transpose of an $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ is the $n \times m$ matrix $\mathbf{A}^\mathsf{T} = [a_{kj}]$ as

$$(9)\quad \mathbf{A} = [a_{jk}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}; \quad \mathbf{A}^\mathrm{T} = [a_{kj}] = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

- Rules for transposition are

$$
\begin{aligned}
&\text{(a)} \quad (\mathbf{A}^{\mathsf{T}})^{\mathsf{T}} = \mathbf{A} \\
&\text{(b)} \quad (\mathbf{A}+\mathbf{B})^{\mathsf{T}} = \mathbf{A}^{\mathsf{T}} + \mathbf{B}^{\mathsf{T}} \\
&\text{(c)} \quad c\mathbf{A})^{\mathsf{T}} = c\mathbf{A}^{\mathsf{T}} \\
&\text{(d)} \quad (\mathbf{A}\mathbf{B})^{\mathsf{T}} = \mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}.
\end{aligned}
\tag{10}
$$

CAUTION! Note that in (10d) the transposed matrices are ***in reversed order.***

Example:

$$\mathbf{A}=\begin{bmatrix}1&5\\2&1\\2&6\end{bmatrix};\ \mathbf{A}^{\mathrm{T}}=\begin{bmatrix}1&2&2\\5&1&6\end{bmatrix};\ \left(\mathbf{A}^{\mathrm{T}}\right)^{\mathrm{T}}=\begin{bmatrix}1&5\\2&1\\2&6\end{bmatrix}=\mathbf{A}$$

Also,

$$\mathbf{A}=\begin{bmatrix}1&5\\2&1\\2&6\end{bmatrix};\ \mathbf{B}=\begin{bmatrix}1&2\\1&1\end{bmatrix};\ \mathbf{AB}=\begin{bmatrix}6&7\\3&5\\8&10\end{bmatrix}$$

Note: multiplying **B** to left of **A** not possible

$$(\mathbf{A})^{\mathrm{T}}=\begin{bmatrix}1&2&2\\5&1&6\end{bmatrix};\ (\mathbf{B})^{\mathrm{T}}=\begin{bmatrix}1&1\\2&1\end{bmatrix};\ (\mathbf{B})^{\mathrm{T}}(\mathbf{A})^{\mathrm{T}}=\begin{bmatrix}6&3&8\\7&5&10\end{bmatrix}=(\mathbf{AB})^{\mathrm{T}}$$

Special Matrices

Symmetric and Skew-Symmetric Matrices.
Transposition gives rise to two useful classes of matrices:

- **Symmetric** matrices are <u>square</u> matrices whose transpose equals the matrix itself.

(11a) $\mathbf{A}^{\mathsf{T}} = \mathbf{A}$ (thus $a_{kj} = a_{jk}$). Symmetric Matrix

- **Skew-symmetric** matrices are <u>square</u> matrices whose transpose equals ***minus*** the matrix.

(11b) $\mathbf{A}^{\mathsf{T}} = -\mathbf{A}$ (thus $a_{kj} = -a_{jk}$, hence $a_{jj} = 0$). Skew-Symmetric Matrix

- **Note:** Equations (11a) and (11b) imply that **Symmetric** matrices and **Skew-symmetric** matrices are <u>square</u> matrices

EXAMPLE 8

Symmetric and Skew-Symmetric Matrices

$$\mathbf{A} = \begin{bmatrix} 20 & 120 & 200 \\ 120 & 10 & 150 \\ 200 & 150 & 30 \end{bmatrix} \quad \text{is symmetric, and}$$

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix} \quad \text{is skew-symmetric.}$$

Note: non-square matrix can never be symmetric or skew-symmetric. For instance,

$$\mathbf{A} = \begin{bmatrix} 1 & 5 \\ 2 & 1 \\ 2 & 6 \end{bmatrix}; \mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 1 & 2 & 2 \\ 5 & 1 & 6 \end{bmatrix};$$

There can never be a relation such as $\mathbf{A}^{\mathrm{T}} = \mathbf{A}$ or $\mathbf{A}^{\mathrm{T}} = -\mathbf{A}$.

EXAMPLE MTR Fare is a Symmetric Matrix

終點站 Destination station / 起點站 Origin station		港島綫 Island Line															
		上環 Sheung Wan	中環 Central	金鐘 Admiralty	灣仔 Wan Chai	銅鑼灣 Causeway Bay	天后 Tin Hau	炮台山 Fortress Hill	北角 North Point	鰂魚涌 Quarry Bay	太古 Tai Koo	西灣河 Sai Wan Ho	筲箕灣 Shau Kei Wan	杏花邨 Heng Fa Chuen	柴灣 Chai Wan	尖沙咀 Tsim Sha Tsui	佐敦 Jordan
港島綫 Island Line	上環 Sheung Wan		4.0	4.0	4.9	4.9	4.9	5.9	5.9	5.9	7.3	7.3	7.3	7.3	7.3	8.6	8.6
	中環 Central	4.0		4.0	4.0	4.9	4.9	4.9	5.9	5.9	5.9	7.3	7.3	7.3	7.3	8.6	8.6
	金鐘 Admiralty	4.0	4.0		4.0	4.0	4.9	4.9	5.9	5.9	5.9	7.3	7.3	7.3	7.3	8.6	8.6
	灣仔 Wan Chai	4.9	4.0	4.0		4.0	4.0	4.9	4.9	5.9	5.9	7.3	7.3	7.3	7.3	8.6	8.6
	銅鑼灣 Causeway Bay	4.9	4.9	4.0	4.0		4.0	4.0	4.9	4.9	5.9	5.9	7.3	7.3	7.3	10.5	10.5
	天后 Tin Hau	4.9	4.9	4.9	4.0	4.0		4.0	4.0	4.9	4.9	5.9	5.9	7.3	7.3	10.5	10.5
	炮台山 Fortress Hill	5.9	4.9	4.9	4.9	4.0	4.0		4.0	4.0	4.9	4.9	5.9	5.9	5.9	10.5	10.5
	北角 North Point	5.9	5.9	5.9	4.9	4.9	4.0	4.0		4.0	4.0	4.9	4.9	5.9	5.9	10.5	10.5
	鰂魚涌 Quarry Bay	5.9	5.9	5.9	5.9	4.9	4.9	4.0	4.0		4.0	4.0	4.9	4.9	5.9	10.5	10.5
	太古 Tai Koo	7.3	5.9	5.9	5.9	5.9	4.9	4.9	4.0	4.0		4.0	4.0	4.9	4.9	12.5	12.5
	西灣河 Sai Wan Ho	7.3	7.3	7.3	7.3	5.9	5.9	4.9	4.9	4.0	4.0		4.0	4.0	4.0	12.5	12.5
	筲箕灣 Shau Kei Wan	7.3	7.3	7.3	7.3	7.3	5.9	5.9	4.9	4.9	4.0	4.0		4.0	4.0	12.5	12.5
	杏花邨 Heng Fa Chuen	7.3	7.3	7.3	7.3	7.3	7.3	5.9	5.9	4.9	4.9	4.0	4.0		4.0	12.5	12.5
	柴灣 Chai Wan	7.3	7.3	7.3	7.3	7.3	7.3	5.9	5.9	5.9	4.9	4.0	4.0	4.0		12.5	12.5
	尖沙咀 Tsim Sha Tsui	8.6	8.6	8.6	8.6	10.5	10.5	10.5	10.5	10.5	12.5	12.5	12.5	12.5	12.5		4.0
	佐敦 Jordan	8.6	8.6	8.6	8.6	10.5	10.5	10.5	10.5	10.5	12.5	12.5	12.5	12.5	12.5	4.0	

- Any <u>square</u> matrix can be decomposed as sum of a Symmetric and Skew-Symmetric matrices.

$$\mathbf{M} = \underbrace{\frac{(\mathbf{M} + \mathbf{M}^T)}{2}}_{\mathbf{M}_{sym}} + \underbrace{\frac{(\mathbf{M} - \mathbf{M}^T)}{2}}_{\mathbf{M}_{skew}}$$

Symmetric matrix **Skew-Symmetric matrix**

- Example:

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 2 & 2 \\ 1 & 4 & 1 \end{bmatrix}; \mathbf{A} = \mathbf{A}_{sym} + \mathbf{A}_{skew};$$

$$\mathbf{A}_{sym} = \frac{\left(\mathbf{A} + \mathbf{A}^{\mathrm{T}}\right)}{2} = \begin{bmatrix} 1 & 3.5 & 1.5 \\ 3.5 & 2 & 3 \\ 1.5 & 3 & 1 \end{bmatrix}; \mathbf{A}_{skew} = \frac{\left(\mathbf{A} - \mathbf{A}^{\mathrm{T}}\right)}{2} = \begin{bmatrix} 0 & 0.5 & 0.5 \\ -0.5 & 0 & -1 \\ -0.5 & 1 & 0 \end{bmatrix}$$

Triangular Matrices

- **Upper triangular matrices** are square matrices that can have nonzero entries only on and ***above*** the main diagonal (any entry below the diagonal must be zero).
- Similarly, **lower triangular matrices** are square matrices that can have nonzero entries only on and ***below*** the main diagonal (any entry below the diagonal must be zero).
- For both upper and lower triangular matrix, entry on the main diagonal may or may not be zero.

EXAMPLE 9

Upper and Lower Triangular Matrices

$$\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 6 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 & 0 \\ 8 & -1 & 0 \\ 7 & 6 & 8 \end{bmatrix}, \quad \begin{bmatrix} 3 & 0 & 0 & 0 \\ 9 & -3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 9 & 3 & 6 \end{bmatrix}$$

Upper triangular　　　　Lower triangular

Diagonal Matrices.

Diagonal matrices are square matrices that can have nonzero entries only on the main diagonal. Any entry above or below the main diagonal must be zero.

$$\begin{bmatrix} 1 & 0 \\ 0 & -8 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1.2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

- Given **S** a diagonal matrix **S** with all its diagonal entries are equal, say, c, then **S** a **scalar matrix** because multiplying **S** to any square matrix **A** of the same size is equivalent to multiplying **A** by a scalar, that is,

$$\mathbf{AS} = \mathbf{SA} = c\mathbf{A}. \tag{12}$$

- Particularly, a scalar matrix, whose entries on the main diagonal are all 1, is called a **unit matrix** (or **identity matrix**) and is denoted by $\mathbf{I}_n$ or simply by **I**. we have

$$\mathbf{AI} = \mathbf{IA} = \mathbf{A}. \tag{13}$$

7.3 Linear Systems of Equations. Gauss Elimination

Linear System of Equations

- A **linear system of *m* equations in *n* unknowns** $x_1, \ldots, x_n$ is a set of equations of the form

$$
\text{(1)} \qquad
\begin{aligned}
a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\
\cdots\cdots&\cdots\cdots\cdots\cdots \\
a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m.
\end{aligned}
$$

- System is *linear* – each variable x_j appears in the first power only, as in the equation of a straight line.
- $a_{11}, \ldots, a_{mn}$ are the given **coefficients** of the system.
- $b_1, \ldots, b_m$ on the right are also given numbers:
  -- If all the $b_j = 0$, then (1) is called a **homogeneous system**.
  -- If at least one b_j not zero, (1) is called a **nonhomogeneous system**.

Linear System of Equations

- A **solution** of (1) is a set of numbers $x_1, \dots, x_n$ that satisfies all the m equations.
- A **solution vector** of (1) is a vector $\mathbf{x}$ whose components form a solution of (1), i.e.,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- If the system (1) is homogeneous, it always has at least the **trivial solution** $x_1 = 0, \dots, x_n = 0$.

Linear System of Equations

- Applications -- Linear system of equations appear in many problems in sciences, engineering, networking, urban planning, etc.

- **Wassily Wassilyevich Leontief** of Harvard divide US economy into 500 sectors and formulated it as 500 equations with 500 unknowns in 1949. He received Nobel Prize in Economic science in 1973.

- In China,雞鹿同籠 from《孫子算經》下卷

Introducing Gauss Elimination and Back Substitution

EXAMPLE 2 Solve the linear system

$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ -x_1 + x_2 - x_3 &= 0 \\ 10x_2 + 25x_3 &= 90 \\ 20x_1 + 10x_2 &= 80. \end{aligned}$$

Conduct Gauss Elimination.

Step 1. Elimination of x_1

Use Pivot equation

to eliminate

$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ -x_1 + x_2 - x_3 &= 0 \\ 10x_2 + 25x_3 &= 90 \\ 20x_1 + 10x_2 &= 80. \end{aligned}$$

The result is

$$\begin{array}{lr} & x_1 - \quad x_2 + \quad x_3 = \ 0 \\ \text{eq2+eq1} & 0 = \ 0 \\ & 10x_2 + 25x_3 = 90 \\ \text{eq4-20*eq1} & 30x_2 - 20x_3 = 80. \end{array}$$

Step 2. Change equation order (if needed) and elimination of x_2

$$\begin{array}{lr} & x_1 - \quad x_2 + \quad x_3 = \ 0 \\ \text{Pivot 10} \longrightarrow & 10x_2 + 25x_3 = 90 \\ & \boxed{30x_2} - 20x_3 = 80 \\ \text{Eliminate } 30x_2 & 0 = \ 0. \end{array}$$

The result is

$$
\begin{aligned}
x_1 - \quad x_2 + \quad x_3 &= \quad 0 \\
10x_2 + 25x_3 &= \quad 90 \\
\text{eq3} - 3^*\text{eq2} \qquad -95x_3 &= -190 \\
0 &= \quad 0.
\end{aligned}
$$

Conduct Back Substitution.

Determination of $x_3 \rightarrow x_2 \rightarrow x_1$ (in this order)

$$
\begin{aligned}
-95x_3 &= -190 && \Rightarrow x_3 = 2 \\
10x_2 + 25x_3 &= \quad 90 && \Rightarrow x_2 = 4 \\
x_1 - \quad x_2 + \quad x_3 &= \quad 0. && \Rightarrow x_1 = 2
\end{aligned}
$$

For simplicity, prefer conducting Gauss elimination/Back substitution using matrices and not equations.

Example: 雞鹿同籠

•第一次在《孫子算經》的下卷中的一道算題：今有雉、兔同籠，上有三十五頭，下九十四足。問雉、兔各幾何？

•在同一本書中也記載了解法：上置三十五頭，下置九十四足。半其足，得四十七。以少減多。(也就是說，將腳的總數九十四除以二得到四十七，然後減去頭數三十五就得到兔子的數目，然後自然可以得到雞的數目。)

- Solve the linear system

$$\begin{array}{ccccc} x_1 & + & x_2 & = & 35 \\ 2x_1 & + & 4x_2 & = & 94 \end{array}$$

Conduct Gauss Elimination.

Step 1. Elimination of x_1

Use Pivot equation

to eliminate

$$\begin{array}{ccccc} x_1 & + & x_2 & = & 35 \\ \boxed{2x_1} & + & 4x_2 & = & 94 \end{array}$$

The result is

$$\begin{array}{ccccc} x_1 & + & x_2 & = & 35 \\ & & x_2 & = & 12 \end{array}$$

0.5*eq2 – eq1

半其足，得四十七。

以少減多。

Conduct Back Substitution.

Determination of x_2 → x_1 , or

$$x_2 = 12$$

$$x_1 = 23$$

Coefficient Matrix and Augmented Matrix

Matrix Form of the Linear System (1).

• The m equations of (1) may be written as a single vector equation

(2) $$\mathbf{A}\mathbf{x} = \mathbf{b}$$

where the **coefficient matrix** $\mathbf{A} = [a_{jk}]$ is the $m \times n$ matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

are column vectors.

- Assume some coefficients a_{jk} not zero, so **A** is not a zero matrix.
- With **x** having n components and **b** having m components, the m x (n+1) matrix

$$\tilde{\mathbf{A}} = \left[\begin{array}{ccc:c} a_{11} & \cdots & a_{1n} & b_1 \\ \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$$

is called the **augmented matrix** of the system (1).
- The dashed vertical line indicates that last column of $\tilde{\mathbf{A}}$ comes from vector **b**. (The line can be omitted after one has "mastered" the skill.

- **The augmented matrix Ã determines the system (1) completely** because it contains all the given numbers appearing in (1).
- **Gauss elimination and back substitution** can be performed based on augmented matrix Ã.

EXAMPLE 2 (Matrix formulation) Gauss Elimination.

- Solve the linear system

$$\begin{aligned} x_1 - \quad x_2 + \quad x_3 &= 0 \\ -x_1 + \quad x_2 - \quad x_3 &= 0 \\ 10x_2 + 25x_3 &= 90 \\ 20x_1 + 10x_2 \qquad &= 80. \end{aligned}$$

Solution by Gauss Elimination.

- Manipulating the augmented matrix on the left vs manipulating the equations on the right:

Augmented Matrix $\tilde{\mathbf{A}}$

Pivot 1

Eliminate

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array}\right]$$

Equations

Pivot 1

Eliminate

$$\begin{aligned} x_1 - \quad x_2 + \quad x_3 &= 0 \\ -x_1 + \quad x_2 - \quad x_3 &= 0 \\ 10x_2 + 25x_3 &= 90 \\ 20x_1 + 10x_2 \qquad &= 80. \end{aligned}$$

EXAMPLE 2 (continued) Gauss Elimination.

Step 1. Elimination of x_1

- Call the first row of **A** the **pivot row** (as the first equation is the **pivot equation)**.
- Call the coefficient 1 of its x_1-term the **pivot** in this step.
- Previously, we use the pivot equation to eliminate x_1 (get rid of x_1) in the remaining equations by:
  - Add 1 times the pivot equation to the second equation.
  - Add −20 times the pivot equation to the fourth equation.
- This corresponds to the **row operations** on the augmented matrix as indicated in BLUE behind the ***new matrix*** in (3).
- Operations are performed on the ***preceding matrix***.

EXAMPLE 2 (continued) Gauss Elimination.

Step 1. Elimination of x_1 **(continued)**

The result is

$$(3)\quad \left[\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \end{array}\right] \begin{array}{l} \\ \text{Row 2 + Row 1} \\ \\ \text{Row 4} - \text{20 Row 1} \end{array} \qquad \begin{aligned} x_1 - x_2 + x_3 &= 0 \\ 0 &= 0 \\ 10x_2 + 25x_3 &= 90 \\ 30x_2 - 20x_3 &= 80. \end{aligned}$$

EXAMPLE 2 (continued) Gauss Elimination.

Step 2. Elimination of x_2

- The first equation remains as it is.
- The new second equation ($0 = 0$) cannot serve as next pivot equation – it has no x_2-term. Hence, need to change the order of equations by putting $0 = 0$ to the end and move the third equation and the fourth equation one place up.
- Do the same to the corresponding rows of the new matrix.
- This is called **partial pivoting** (as opposed to the rarely used *total pivoting*, in which the order of the unknowns is also changed).

EXAMPLE 2 (continued) Gauss Elimination.

Step 2. Elimination of x_2 **(continued)**

It gives

Pivot 10

Eliminate 30

$$\left[\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & \boxed{30} & -20 & 80 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Pivot 10

Eliminate $30x_2$

$$\begin{aligned} x_1 - \quad x_2 + \quad x_3 &= \ 0 \\ 10x_2 + 25x_3 &= 90 \\ \boxed{30x_2} - 20x_3 &= 80 \\ 0 &= \ 0. \end{aligned}$$

EXAMPLE 2 (continued) Gauss Elimination.

Step 2. Elimination of x_2 **(continued)**

- To eliminate x_2, add −3 times the pivot equation to the third equation. Corresponding operation to row in new matrix in BLUE.
- The result is

$$
(4)\quad \left[\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{Row 3} - 3\ \text{Row 2} \qquad \begin{aligned} x_1 - x_2 + x_3 &= 0 \\ 10x_2 + 25x_3 &= 90 \\ -95x_3 &= -190 \\ 0 &= 0. \end{aligned}
$$

EXAMPLE 2 (continued) Gauss Elimination.

Back Substitution. Determination of x_3, x_2, x_1 **(in such order)**

- Working backward from last to first equation of the "triangular" system (4) to readily determine x_3, then x_2, and then x_1:

$$\begin{aligned} -95x_3 &= -190 \quad &\Rightarrow x_3 = 2 \\ 10x_2 + 25x_3 &= \quad 90 \quad &\Rightarrow x_2 = 4 \\ x_1 - \quad x_2 + \quad x_3 &= \quad 0. \quad &\Rightarrow x_1 = 2 \end{aligned}$$

In this case, the solution is unique. (More later)

- **Advantages of Gauss elimination**:
  - General, systematic, and
  - Applicable to large systems.

Elementary Row Operations in Gauss Elimination.

- **Elementary Row Operations for Matrices (only three):**
  - *Interchange of two rows*
  - *Addition of a constant multiple of one row to another row*
  - *Multiplication of a row by a* ***nonzero*** *constant c*

- **Corresponding Elementary Operations for Equations:**
  - *Interchange of two equations*
  - *Addition of a constant multiple of one equation to another equation*
  - *Multiplication of an equation by a* ***nonzero*** *constant c*

- **Elementary Operations** does not alter the solution set.

Row-Equivalent Systems

- A linear system S_1 is **row-equivalent** to another linear system S_2 if S_1 can be obtained from S_2 by (finitely many!) row operations.
- Gauss elimination produces row equivalent systems.

Theorem 1

Row-Equivalent Systems

Row-equivalent linear systems have the same set of solutions.

- Gauss elimination preserves the solution set.
- Note: we are dealing with ***row operations here***. Column operations on the augmented matrix will generally alter the solution set and are NOT permitted in this context.

Row Echelon Form

- After Gauss elimination, the form of the coefficient matrix, the augmented matrix, and the system itself are called the **row echelon form**.
- **Definition of row echelon form**:
  - *r*ows of zeros, if present, are the last rows,
  - in each nonzero row, the leftmost nonzero entry is farther to the right than in the previous row.
- Example of row echelon form:

$$\left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 0 & 0 \end{array}\right], \quad \left[\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ 0 & 0.5 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right],$$

Coefficient matrix

$$\underbrace{\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 2 \\ 0 & 0 & 2 & -2 & 1 \\ 0 & 0 & 0 & 1/4 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]}_{\text{Augmented matrix}}$$

- Note: the leftmost nonzero entries do not have to be 1 in row echelon form.

Row Echelon Form (continued)

- The original system of m equations in n unknowns has augmented matrix $[\mathbf{A} \mid \mathbf{b}]$.
- After Gauss elimination, we convert matrix $[\mathbf{A} \mid \mathbf{b}]$ to become matrix $[\mathbf{R} \mid \mathbf{f}]$, where both $\mathbf{R}$ and $[\mathbf{R} \mid \mathbf{f}]$ are in row echelon form.
- The two systems $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{Rx} = \mathbf{f}$ are equivalent: if either one has a solution, so does the other, and the solutions are identical.

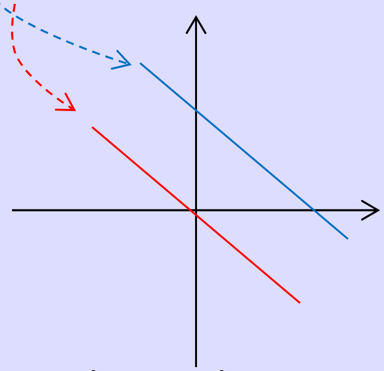
Types of Solutions For Linear System

System (1) may be **inconsistent** (no solutions at all) or **consistent** (it has one solution or infinitely many solutions).

Example X

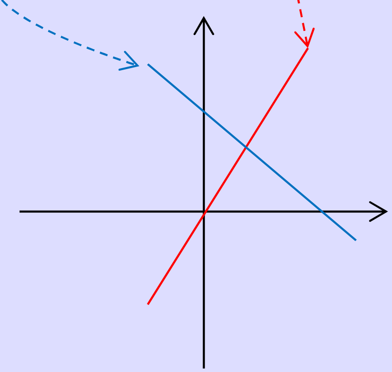
- 2 equations and 2 unknown case

$$x_1 + x_2 = 1$$
$$x_1 + x_2 = 0$$



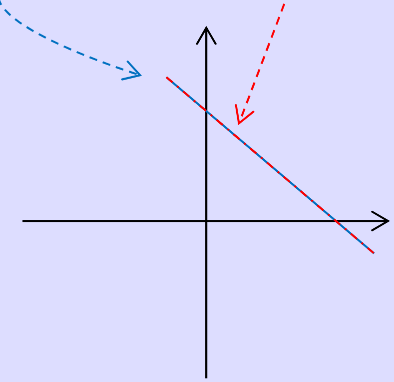
inconsistent
(No solution)

$$x_1 + x_2 = 1$$
$$2x_1 - x_2 = 0$$



consistent
(unique solution)

$$x_1 + x_2 = 1$$
$$2x_1 + 2x_2 = 2$$



consistent
(infinitely many solution)

Example 1 (continued)

- 2 equations and 2 unknown case

$$x_1 + x_2 = 1$$
$$x_1 + x_2 = 0$$

$$x_1 + x_2 = 1$$
$$2x_1 - x_2 = 0$$

$$x_1 + x_2 = 1$$
$$2x_1 + 2x_2 = 2$$

- Conduct Gauss elimination

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & -1 \end{array}\right]$$

Row echelon form

inconsistent
(No solution)

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -3 & -2 \end{array}\right]$$

Row echelon form

consistent
(unique solution)

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 2 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

Row echelon form

consistent
(infinitely many solution)

Row Echelon Form and Information about Solution

Summarizing: After Gauss elimination (but before back substitution), the row echelon form of the augmented matrix will be

# of nonzero rows = r

Total m rows

$$\text{(9)} \qquad \left[\begin{array}{cccccc|c} r_{11} & r_{12} & \cdots & \cdots & \cdots & r_{1n} & f_1 \\ & r_{22} & \cdots & \cdots & \cdots & r_{2n} & f_2 \\ & & \ddots & \cdots & \cdots & \vdots & \vdots \\ & & & r_{rr} & \cdots & r_{rn} & f_r \\ & & & & & & f_{r+1} \\ & & & & & & \vdots \\ & & & & & & f_m \end{array}\right]$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{\mathbf{R}} \qquad \underbrace{\quad}_{\mathbf{f}}$$

All entries in the blue triangle and blue rectangle are zero.

Section 7.3 p64

- The number of nonzero rows, r, in the row-reduced coefficient matrix **R** is called the **rank of R** and also the **rank of A** (later).
- r may not be the same as m nor n. Note that $r \le m$, the number of given equations
- The types of solution of $\mathbf{Ax} = \mathbf{b}$ are determined as:

  (a) Inconsistent Case (No solution).
  If $r < m$ (**R** has at least one row of all 0s) and at least one of the numbers $f_{r+1}, f_{r+2}, \dots, f_m$ is not zero,
  → inconsistency in the equations,
  → $\mathbf{Rx} = \mathbf{f}$ inconsistent,
  → $\mathbf{Ax} = \mathbf{b}$ inconsistent as well.
  → No solution!
  (May find approximate solution -- LATER)

EXAMPLE 4 Inconsistent (No solution) case

$n = 3$

$m = 3$

What will happen if we apply the Gauss elimination to a linear system that has no solution? The answer is that in this case the method will show this fact by producing a contradiction. For instance, consider

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{array}\right] \qquad \begin{aligned} 3x_1 + 2x_2 + x_3 &= 3 \\ \boxed{2x_1} + x_2 + x_3 &= 0 \\ \boxed{6x_1} + 2x_2 + 4x_3 &= 6. \end{aligned}$$

***Step 1. Elimination of** x_1* from the second and third equations by adding

$-\frac{2}{3}$ times the first equation to the second equation,

$-\frac{6}{3} = -2$ times the first equation to the third equation.

This gives

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & -2 & 2 & 0 \end{array}\right] \quad \begin{array}{l} \\ \text{Row 2} - \frac{2}{3}\text{ Row 1} \\ \text{Row 3} - 2\text{ Row 1} \end{array} \qquad \begin{aligned} 3x_1 + 2x_2 + x_3 &= 3 \\ -\tfrac{1}{3}x_2 + \tfrac{1}{3}x_3 &= -2 \\ \boxed{-2x_2} + 2x_3 &= 0. \end{aligned}$$

***Step 2. Elimination of** x_2* from the third equation gives

$r = 2 < m$,
R has one zero row
and yet f_3 nonzero

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & 0 & 0 & 12 \end{array}\right] \quad \begin{array}{l} \\ \\ \text{Row 3} - 6\text{ Row 2} \end{array} \qquad \begin{aligned} 3x_1 + 2x_2 + x_3 &= 3 \\ -\tfrac{1}{3}x_2 + \tfrac{1}{3}x_3 &= -2 \\ 0 &= 12. \end{aligned}$$

The false statement $0 = 12$ shows that the system has no solution. ■

(b) Consistent Case (Solution(s) possible) occurs

when **R** has no zero row, i.e., $r = m$,

$$
r = m \left\{ \left[\begin{array}{ccccccc|c}
r_{11} & r_{12} & \cdots & \cdots & \cdots & \cdots & r_{1n} & f_1 \\
 & r_{22} & \cdots & \cdots & \cdots & \cdots & r_{2n} & f_2 \\
 & & \ddots & \cdots & \cdots & \cdots & \vdots & \vdots \\
 & & & \ddots & \cdots & \cdots & \vdots & \vdots \\
 & & & & r_{mm} & \cdots & r_{mn} & f_m
\end{array}\right] \right.
$$

$\mathbf{R}$ $\qquad\qquad$ $\mathbf{f}$

or

when **R** has some zero rows, i.e., $r < m$ and all the numbers $f_{r+1}, f_{r+2}, \ldots, f_m$ are zero

$$
\begin{array}{l}
r < m \left\{ \begin{array}{l} \\ \\ \\ \\ \end{array} \right. \\
\\
\\
\end{array}
\left[\begin{array}{cccccc|c}
r_{11} & r_{12} & \cdots & \cdots & \cdots & r_{1n} & f_1 \\
 & r_{22} & \cdots & \cdots & \cdots & r_{2n} & f_2 \\
 & & \ddots & \cdots & \cdots & \vdots & \vdots \\
 & & & r_{rr} & \cdots & r_{rn} & f_r \\
 & & & & & & f_{r+1} \\
 & & & & & & \vdots \\
 & & & & & & f_m
\end{array}\right]
$$

All zero

$\mathbf{R}$ $\qquad\qquad$ $\mathbf{f}$

Hence, for **consistent** system, after keeping the r nonzero rows, we have

$$
r\left\{\left[\underbrace{\overbrace{\begin{matrix}
r_{11} & r_{12} & \cdots & \cdots & \cdots & \cdots & r_{1n} \\
 & r_{22} & \cdots & \cdots & \cdots & \cdots & r_{2n} \\
 & & \ddots & \cdots & \cdots & \cdots & \vdots \\
 & & & \ddots & \cdots & \cdots & \vdots \\
 & & & & r_{rr} & \cdots & r_{rn}
\end{matrix}}^{n}}_{\mathbf{R}}
\;\middle|\;
\underbrace{\begin{matrix} f_1 \\ f_2 \\ \vdots \\ \vdots \\ f_r \end{matrix}}_{\mathbf{f}}\right]\right.
$$

with $r = m$ when **R** has no zero row, or

$r < m$ when **R** has some zero rows.

Now we can compare r with n to see if system has unique or multiple solutions.

(i) Unique solution case. If $r = n$, then $\mathbf{Ax} = \mathbf{b}$ has exactly one solution, which can be found by back substitution using $\mathbf{Rx} = \mathbf{f}$. Example 2 is one such case.

(ii) Infinitely many solutions case. If $r < n$, then one can choose $(n\text{-}r)$ values of the $x_1, \ldots, x_n$ freely. Then Back substitution to solve for remaining ones in terms of these free values.

EXAMPLE 2 revisited Consistent case: unique solution

Solve the linear system

$$
\begin{aligned}
x_1 - x_2 + x_3 &= 0 \\
-x_1 + x_2 - x_3 &= 0 \\
10x_2 + 25x_3 &= 90 \\
20x_1 + 10x_2 \qquad &= 80.
\end{aligned}
$$

Solution: Augmented matrix $\tilde{\mathbf{A}}$

$$
\tilde{\mathbf{A}} = \left[\begin{array}{ccc:c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array}\right]
$$

$\underbrace{\qquad\qquad}_{\mathbf{A}}$ $\underbrace{\quad}_{\mathbf{b}}$

$m = 4$

$n = 3$

Elementary row operations

Row echelon form

$$
r = 3 \left\{ \left[\begin{array}{ccc:c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{array}\right] \right.
$$

$\underbrace{\qquad\qquad}_{\mathbf{R}}$ $\underbrace{\quad}_{\mathbf{F}}$

Keeping all nonzero rows

$$
\left[\begin{array}{ccc:c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \end{array}\right]
$$

$r = n = 3$

Unique solution as obtained previously using Back Substitution

EXAMPLE 3 Consistent (Multiple Solution) Case

$n = 4, m = 3$

$$\tilde{\mathbf{A}} = \left[\begin{array}{cccc:c} 3 & 2 & 2 & -5 & 8 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{array}\right]$$

Elementary row operations

$$\left[\begin{array}{cccc:c} 3 & 2 & 2 & -5 & 8 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Keeping nonzero rows

$$\left[\begin{array}{cccc:c} 3 & 2 & 2 & -5 & 8 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \end{array}\right]$$

$f_3 = 0$ → Consistent case

$r = 2 < n = 4$ → Multiple solutions

Solve the following linear system of three equations in four unknowns whose augmented matrix is

$$(5)\quad \left[\begin{array}{cccc:c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{array}\right]. \qquad \text{Thus,} \qquad \begin{aligned} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 &= 8.0 \\ \boxed{0.6x_1} + 1.5x_2 + 1.5x_3 - 5.4x_4 &= 2.7 \\ \boxed{1.2x_1} - 0.3x_2 - 0.3x_3 + 2.4x_4 &= 2.1. \end{aligned}$$

Solution. As in the previous example, we circle pivots and box terms of equations and corresponding entries to be eliminated. We indicate the operations in terms of equations and operate on both equations and matrices.

Step 1. Elimination of x_1 from the second and third equations by adding

$-0.6/3.0 = -0.2$ times the first equation to the second equation,

$-1.2/3.0 = -0.4$ times the first equation to the third equation.

This gives the following, in which the pivot of the next step is circled.

$$(6)\quad \left[\begin{array}{cccc:c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & -1.1 & -1.1 & 4.4 & -1.1 \end{array}\right] \begin{array}{l} \\ \text{Row 2} - 0.2\text{ Row 1} \\ \text{Row 3} - 0.4\text{ Row 1} \end{array} \qquad \begin{aligned} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 &= 8.0 \\ 1.1x_2 + 1.1x_3 - 4.4x_4 &= 1.1 \\ \boxed{-1.1x_2} - 1.1x_3 + 4.4x_4 &= -1.1. \end{aligned}$$

Step 2. Elimination of x_2 from the third equation of (6) by adding

$1.1/1.1 = 1$ times the second equation to the third equation.

This gives

$$(7)\quad \left[\begin{array}{cccc:c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ \text{Row 3} + \text{Row 2} \end{array} \qquad \begin{aligned} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 &= 8.0 \\ 1.1x_2 + 1.1x_3 - 4.4x_4 &= 1.1 \\ 0 &= 0. \end{aligned}$$

Back Substitution. From the second equation, $x_2 = 1 - x_3 + 4x_4$. From this and the first equation, $x_1 = 2 - x_4$. Since x_3 and x_4 remain arbitrary, we have infinitely many solutions. If we choose a value of x_3 and a value of x_4, then the corresponding values of x_1 and x_2 are uniquely determined.

On Notation. If unknowns remain arbitrary, it is also customary to denote them by other letters $t_1, t_2, \cdots$. In this example we may thus write $x_1 = 2 - x_4 = 2 - t_2$, $x_2 = 1 - x_3 + 4x_4 = 1 - t_1 + 4t_2$, $x_3 = t_1$ (first arbitrary unknown), $x_4 = t_2$ (second arbitrary unknown). ■

- Characterization of solutions:

$$3x_1 + 2x_2 + 2x_3 - 5x_4 = 8$$
$$1.1x_2 + 1.1x_3 - 4.4x_4 = 1.1$$

- Pick $(n\text{-}r)$=(4-2)=2 variables as free variables. In this case, most conveniently pick $\boldsymbol{x}_3$ and $\boldsymbol{x}_4$ as free variables, then by Back substitution:

$$x_2 = 1 - x_3 + 4x_4$$
$$x_1 = \frac{1}{3}\left(8 - 2x_2 - 2x_3 + 5x_4\right) = \left(2 - x_4\right)$$

Hence, infinite number of solutions.

- Furthermore, denote free parameters by $\boldsymbol{x}_3 = \boldsymbol{t}_1$ and $\boldsymbol{x}_4 = \boldsymbol{t}_2$, we can express the solution as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 - t_2 \\ 1 - t_1 + 4t_2 \\ t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_1 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -1 \\ 4 \\ 0 \\ 1 \end{bmatrix}$$