Lecture Note 7

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MATH1020
General Mathematics

Properties of Logarithms

Work with the properties of Logarithms

Logarithms have some very useful properties that can be derived directly from its definition and the laws of exponents.

Exercises 1 Establishing Properties of Logarithms

(a) Show that $\log_a 1 = 0$.

(b) Show that $\log_a a = 1$.

Theorem 1 Properties of Logarithms

In the properties given next, M and a are positive real numbers, $a \neq 1$, and r is any real number.

The number $\log_a M$ is the exponent to which a must be raised to obtain M. That is,

$$\boxed{a^{\log_a M} = M \qquad (1)}$$

The logarithm to the base a of a raised to a power equals that power. That is,

$$\boxed{\log_a a^r = r \qquad (2)}$$

The proof uses the fact that $y = a^x$ and $y = \log_a x$ are inverses.

Proof of Property (1)

For inverse function,

$$f(f^{-1}(x)) = x \qquad \text{for all } x \text{ in the domain of } f^{-1}$$

Using $f(x) = a^x$ and $f^{-1} = \log_a x$, we find

$$f(f^{-1}(x)) = a^{\log_a x} = x \qquad \text{for } x > 0$$

Now let $x = M$ to obtain $a^{\log_a M} = M$, where $M > 0$.

Proof of Property (2)

For inverse function,

$$f(f^{-1}(x)) = x \qquad \text{for all } x \text{ in the domain of } f$$

Using $f(x) = a^x$ and $f^{-1} = \log_a x$, we find

$$f(f^{-1}(x)) = \log_a a^x = x \qquad \text{for all real number } x$$

Now let $x = r$ to obtain $\log_a a^r = r$, where r is any real number.

Exercises 2 **Using Properties** (1) **and** (2)

(a) $2^{\log_2 \pi} =$

(b) $\log_{0.2} 0.2^{-\sqrt{3}} =$

(c) $\ln e^{3kt} =$

Other useful properties of logarithms are given next.

Theorem 2 **Properties of Logarithms**

In the following properties, $M,\ N$, and a are positive real numbers, $a > 0,\ a \neq 1$, and r is any real number.

The Log of a Product Equals the Sum of the Logs

$$\boxed{\log_a(MN) = \log_a M + \log_a N \qquad (3)}$$

The Log of a Quotient Equals the Difference of the Logs

$$\boxed{\log_a \frac{M}{N} = \log_a M - \log_a N \qquad (4)}$$

The Log of a Power Equals the Product of the Power and the Log

$$\boxed{\log_a M^r = r \log_a M \qquad (5)}$$

Proof of Property (3) Let $A = \log_a M$ and let $B = \log_a N$. These expressions are equivalent to the exponential expressions

$$a^A = M \qquad \text{and} \qquad a^B = N.$$

Now

$$\begin{aligned} \log_a(MN) = \log_a(a^A a^B) &= \log_a a^{A+B} && \text{Law of Exponents} \\ &= A + B && \text{Property } (2) \text{ of logarithms} \\ &= \log_a M + \log_a N. \end{aligned}$$

Proof of Property (4) Exercise!

Proof of Property (5) Let $A = \log_a M$. This expression is equivalent to

$$a^A = M.$$

Now

$$\begin{aligned} \log_a M^r = \log_a (a^A)^r &= \log_a a^{rA} && \text{Law of Exponents} \\ &= rA && \text{Property } (2) \text{ of logarithms} \\ &= r \log_a M. \end{aligned}$$

Write a Logarithmic Expression as a Sum or Difference of Logarithms

Logarithms can be used to transform products into sums, quotients into differences,and powers into factors. Such transformations prove useful in certain types of calculus problems.

Exercises 3 **Writing a Logarithmic Expression as a Sum of Logarithms**

Write $\log_a(x\sqrt{x^2+1}),\ x > 0$, as a sum of logarithms. Express all powers as factors.

Exercises 4 Writing a Logarithmic Expression as a Difference of Logarithms

Write

$$\ln \frac{x^2}{(x-1)^3} \qquad x > 1.$$

as a difference of logarithms. Express all powers as factors.

Exercises 5 Writing a Logarithmic Expression as a Sum and Difference of Logarithms

Write

$$\log_a \frac{\sqrt{x^2+1}}{x^3(x+1)^4} \qquad x > 0.$$

as a sum and difference of logarithms. Express all powers as factors.

Write a Logarithmic Expression as a Single Logarithm

Another use of properties (3) through (5) is to write sums and/or differences of logarithms with the same base as a single logarithm. This skill will be needed to solve certain logarithmic equation discussed in the next section.

Exercises 6 **Writing Expressions as a Single Logarithm**

Write each of the following as a single logarithm.

(a) $\log_a 5 + 4\log_a 3$;

(b) $\frac{2}{3}\ln 8 - \ln(3^4 - 5)$;

(c) $\log_a x + \log_a 7 + \log_a(x^2 + 1) - \log_a 3$.

WARNING

A common error made by some students is to express the logarithm of a sum as the sum of logarithms.

$$\log_a(M+N) \qquad \text{is not equal to} \qquad \log_a M + \log_a N$$

Correct statement: $\log_a MN = \log_a M + \log_a N$ Property (3)

Another common error is to express the difference of logarithms as the quotient of logarithms.

$$\log_a M - \log_a N \qquad \text{is not equal to} \qquad \frac{\log_a M}{\log_a N}$$

Correct statement: $\log_a M - \log_a N = \log_a \left(\dfrac{M}{N}\right)$ Property (4)

A third common error is to express a logarithm raised to a power as the product of the power times the logarithm.

$$(\log_a M)^r \qquad \text{is not equal to} \qquad r \log_a M$$

Correct statement: $\log_a M^r = r \log_a M$ Property (5)

Two other properties of logarithms that we need to know are consequences of the fact that the logarithmic function $y = \log_a x$ is a one–to–one function.

Theorem 3 Properties of Logarithms

In the following properties, $M,\ N$, and a are positive real numbers, $a > 0,\ a \neq 1$.

$$\text{If } M = N, \text{ then } \qquad \log_a M = \log_a N. \tag{6}$$

$$\text{If } \log_a M = \log_a N, \text{ then } \qquad M = N. \tag{7}$$

When property (6) is used, we start with the equation $M = N$ and say "take the logarithm of both sides" to obtain $\log_a M = \log_a N$.

Properties (6) and (7) are useful for solving exponential and logarithmic equations, a topic discussed in the next section.

Evaluate Logarithms Whose Base is Neither 10 Nor e

Logarithms to the base 10, common logarithms, were used to facilitate arithmetic computations before the widespread use of any computing softwares. Natural logarithms, that is, logarithms whose base is the number e, remain very important because they arise frequently in the study of natural phenomena.

Common logarithms are usually abbreviated by writing $\log$, with the base understood to be 10, just as natural logarithms are abbreviated by $\ln$, with the base understood to be e.

Exercises 7 Approximating a Logarithm Whose Base Is Neither 10 Nor e

Approximate $\log_2 7$. Round the answer to four decimal places.

Theorem 4 Change-of-Base Formula

If $a > 0,\ a \neq 1,\ b > 0,\ b \neq 1$, and M are positive real numbers, then

$$\boxed{\log_a M = \frac{\log_b M}{\log_b a} \qquad (8)}$$

Proof:

We derive this formula as follows: Let $y = \log_a M$. Then

$$
\begin{aligned}
a^y &= M \\
\log_b a^y &= \log_b M && \text{Property } (6) \\
y \log_b a &= \log_b M && \text{Property } (5) \\
y &= \frac{\log_b M}{\log_b a} && \text{Solve for } y \\
\log_a M &= \frac{\log_b M}{\log_b a} && y = \log_a M
\end{aligned}
$$

Since calculators have keys only for $\boxed{\log}$ and $\boxed{\ln}$, in practice, the Change–of–Base Formula uses either $b = 10$ or $b = e$. That is,

$$\boxed{\log_a M = \frac{\log M}{\log a} \qquad \text{and} \qquad \log_a M = \frac{\ln M}{\ln a} \qquad (9)}$$

Exercises 8 Using the Change-of-Base Formula

Approximate: (a) $\log_5 89$ (b) $\log_{\sqrt{2}} \sqrt{5}$.

Round answers to four decimal places.

Graph Logarithmic Functions Whose Base is Neither 10 Nor e

We also use the Change-of-Base Formula to graph logarithmic functions whose base is neither 10 nor e.

Exercises 9 **Graphing a Logarithmic Function Whose Base Is Neither 10 Nor e**

Use Property (9) and use MATLAB to graph $y = \log_2 x$ and $y = \log_{1/3} x$.