ENGG1410 Midterm (105 minutes)

No calculators allowed. Double-sided HANDWRITTEN A4 sized cheat-sheet allowed. Cheating will be dealt with severely.

Name: Student ID:

1. Let $f(x, y) = x^2 + y^2 - 2x + 4y + 10$.
   (a) *(4 points)* Find the point where ∇f, the gradient of f, equals $\mathbf{0}$, the zero vector. Sketch the vector field ∇f around this point. (Draw at least 8 arrows.)
   (b) *(3 points)* For any positive c, find the curve C such that the length of ∇f equals c.
   (c) *(3 points)* For any point (x_0, y_0), find a vector $\mathbf{v}$ such that the directional derivative of f at (x_0, y_0) in the direction of $\mathbf{v}$ equals zero.

2. Given a vector field:
$$\mathbf{v}(x, y, z) = \begin{bmatrix} 2xy, & x^2 + 2yz, & x + y^2 \end{bmatrix}.$$
   (a) *(5 points)* Let $g(x, y, z) = x$. Show that $\operatorname{div}(g\mathbf{v}) = g \operatorname{div}(\mathbf{v}) + \mathbf{v} \cdot \nabla g$.
   (b) *(5 points)* Prove that there does NOT exist a function $f(x, y, z)$ such that $\nabla f(x, y, z) = \mathbf{v}(x, y, z)$.

3. Consider the curve C: $y^2 = x^3$, where $0 \leq x \leq 4$ and $y \leq 0$.
   (a) *(3 points)* Find a parametric representation for the curve C.
   (b) *(3 points)* Find its tangent line at the point $P : (1, -1)$.
   (c) *(4 points)* Calculate the length of the curve C.

4. (a) *(6 points)* Let $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + 2\mathbf{j}$, and θ be the angle between these two vectors. Calculate $\mathbf{a} \cdot \mathbf{b}$, $\mathbf{a} \times \mathbf{b}$, $\cos\theta$ and $\sin\theta$.
   (b) *(4 points)* Find the distance from the point $q = (4, 3, 2)$ to the line l which goes through the point $p = (1, 4, 1)$ and is parallel to the vector $\mathbf{b}$.

5. Let C be the triangle constructed from the three points $(0, 0)$, (a, a), $(-a, a)$ with some $a > 0$, and traversed counterclockwise. Denote $\boldsymbol{F} = [y - \frac{2}{3}y^3 + 6x^2 y, 0]$.
   (a) *(5 points)* Represent $\oint_C \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r}$ as a double integral.
   (b) *(5 points)* For what value of $a > 0$ do we have $\oint_C \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = -1$?

6. (a) *(5 points)* Let C be the curve $[2t(\cos t)^5, 2t^2(\sin t)^7, 3t]$ with t from 0 to 2π. Calculate $\int_C (y\,dx + x\,dy + dz)$.
   (b) *(5 points)* Consider the following two surfaces:
$$\begin{aligned} x^2 + y^2 - z &= 1 \\ x &= 2y. \end{aligned}$$
   Both points $p = (0, 0, -1)$ and $q = (2, 1, 4)$ are on the above surfaces. Let C be the curve that goes from p to q along the intersection of the above surfaces. Calculate $\int_C (y\,dx - x\,dy + dz)$.

Solutions

1. (a) By direct computation, $\nabla f = (2x-2)\mathbf{i} + (2y+4)\mathbf{j}$. Therefore $\nabla f = \mathbf{0}$ at the point $(1, -2)$. The vector field of the gradient function points outward from this point, in a radially symmetric manner.

   (b) Since the length of ∇f equals $\sqrt{4(x-1)^2 + 4(y+2)^2}$, hence the curve is $4(x-1)^2 + 4(y+2)^2 = c^2$.

   (c) One can directly observe that the vector $-(2y_0+4)\mathbf{i} + (2x_0-2)\mathbf{j}$ is perpendicular to the vector $(2x_0-2)\mathbf{i} + (2y_0+4)\mathbf{j}$. (Alternatively, one can assume the vector as the form $[a, b]$, and then solve the equation $[a, b] \cdot [2x-2, 2y+4] = 0$, and choose a as an arbitrary non-zero value.)

2. (a)

$$
\begin{aligned}
LHS &= \operatorname{div}(g\mathbf{v}) \\
&= \frac{\partial x v_1}{\partial x} + \frac{\partial x v_2}{\partial y} + \frac{\partial x v_3}{\partial z} \\
&= 4xy + 2xz + 0 \\
&= x(4y + 2z) \\
RHS &= g \operatorname{div}(\mathbf{v}) + \mathbf{v} \bullet \nabla g \\
&= x\left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right) + \mathbf{v} \bullet \begin{bmatrix} 1, & 0, & 0 \end{bmatrix} \\
&= x(2y + 2z + 0) + 2xy \\
&= x(4y + 2z)
\end{aligned}
$$

   (b) Consider

$$
\begin{aligned}
\operatorname{curl}(\mathbf{v}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & x^2 + 2yz & x + y^2 \end{vmatrix} \\
&= \left[\frac{\partial(x+y^2)}{\partial y} - \frac{\partial(x^2+2y-z)}{\partial z}\right]\mathbf{i} - \left[\frac{\partial(x+2y)}{\partial x} - \frac{\partial(2xy)}{\partial z}\right]\mathbf{j} \\
&\quad + \left[\frac{\partial(x^2+2yz)}{\partial x} - \frac{\partial(2xy)}{\partial y}\right]\mathbf{k} \\
&= (2y-2y)\mathbf{i} - (1-0)\mathbf{j} + (2x-2x)\mathbf{k} \\
&= -\mathbf{j} \\
&\neq \mathbf{0}
\end{aligned}
$$

   Thus, the desired function does not exist for $\mathbf{v}$.

3. (a) Let $x = t$. Then, $y = -t^{3/2}$, and a parametric representation of the curve is given by $\mathbf{r}(t) = [t, -t^{3/2}]$, with $0 \le t \le 4$. *An alternative solution is to let* $x = t^2$. *Then,* $y = -t^3$, *and the parametric representation is given by* $\mathbf{r}(t) = [t^2, -t^3]$, *with* $0 \le t \le 2$.

   (b) Note that the point P corresponds to the position vector $\mathbf{r}(1) = [1, -1]$. Since $\mathbf{r}'(t) = [1, -3\sqrt{t}/2]$, we have $\mathbf{r}'(1) = [1, -3/2]$. Therefore, the tangent line at point P can be expressed as

$$\mathbf{q}(w) = \mathbf{r}(1) + w\mathbf{r}'(1) = [1 + w, -1 - 3w/2], \quad -\infty < w < +\infty.$$

   If the curve C is expressed in the alternative way, then the point P corresponds to the position vector $\mathbf{r}(1) = [1, -1]$. Since $\mathbf{r}'(t) = [2t, -3t^2]$, we have $\mathbf{r}'(1) = [2, -3]$. Therefore, the tangent line at point P can be expressed as

$$\mathbf{q}(w) = \mathbf{r}(1) + w\mathbf{r}'(1) = [1 + 2w, -1 - 3w], \quad -\infty < w < +\infty.$$

   (c) According to the formula for curve length, we have

$$\begin{aligned} s &= \int_0^4 \sqrt{\mathbf{r}'(t) \cdot \mathbf{r}'(t)} dt = \int_0^4 \sqrt{1 + \tfrac{9}{4}t} dt \\ &= \frac{4}{9} \int_0^9 \sqrt{u + 1} du = \frac{8}{27}(u + 1)^{\frac{3}{2}}|_{u=0}^{9} = \frac{8}{27}(10\sqrt{10} - 1). \end{aligned}$$

   The alternative representation will give the same result after change of variables.

4. (a) $\mathbf{a} \cdot \mathbf{b} = 3 \times 2 - 1 \times 2 + 1 \times 0 = 4$.

$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 1 \\ 2 & 2 & 0 \end{vmatrix} = -2\mathbf{i} + 2\mathbf{j} + 8\mathbf{k}$.

$\cos\theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \dfrac{4}{\sqrt{11 \times 8}} = \dfrac{2}{\sqrt{22}}$.

$\sin\theta = \dfrac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}||\mathbf{b}|} = \dfrac{\sqrt{72}}{\sqrt{11 \times 8}} = \dfrac{3}{\sqrt{11}}$.

(b) $\overrightarrow{pq} = [3, -1, 1] = \mathbf{a}$, the distance from the point q to the line l is $|\overrightarrow{pq}| \sin\theta = 3$.

5. (a) Note that $\frac{\partial F_1}{\partial y} = 1 - 2y^2 + 6x^2$, $\frac{\partial F_2}{\partial x} = 0$. By Green's Theorem, we have

$$\oint_C \boldsymbol{F}(\boldsymbol{r})\cdot d\boldsymbol{r} = \oint_C F_1 dx + F_2 dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy = \iint_D (-1 + 2y^2 - 6x^2) dxdy,$$

where D is the area in the triangle.

(b)

$$\begin{aligned}\oint_C \boldsymbol{F}(\boldsymbol{r})\cdot d\boldsymbol{r} &= \iint_D (-1 + 2y^2 - 6x^2) dxdy \\ &= \int_0^a \int_{-y}^{y} (-1 + 2y^2 - 6x^2) dxdy \\ &= \int_0^a \left(-x + 2y^2 x - 2x^3\right)\Big|_{x=-y}^{y} dy \\ &= \int_0^a -2y dy \\ &= -a^2.\end{aligned}$$

Hence, by $-a^2 = -1$ and $a > 0$ we can solve $a = 1$.

6. (a) Let $f_1(x, y, z) = y$, $f_2(x, y, z) = x$, and $f_3(x, y_z) = 1$. We can find $g(x, y, z) = xy + z$ satisfying $\frac{\partial g}{\partial x} = f_1$, $\frac{\partial g}{\partial y} = f_2$, and $\frac{\partial g}{\partial z} = f_3$. Hence, the set of integrals of the form $\int_C (y\, dx + x\, dy + dz)$ is path independent.

Returning to the curve C given in the problem, we know that it starts from point $(0, 0, 0)$ and ends at $(4\pi, 0, 6\pi)$. Therefore, $\int_C (y\, dx + x\, dy + dz) = g(4\pi, 0, 6\pi) - g(0, 0, 0) = 6\pi$.

(b) We can represent the intersection of the two surfaces in a parametric form $\boldsymbol{r}(t) = [x(t), y(t), z(t)]$ where

$$\begin{aligned}x(t) &= 2t \\ y(t) &= t \\ z(t) &= 5t^2 - 1.\end{aligned}$$

C is the part of the above curve from $t = 0$ to $t = 1$. Hence:

$$\begin{aligned}\int_C (y\, dx - x\, dy + dz) &= \int_0^1 (t\frac{dx}{dt} - 2t\frac{dy}{dt} + \frac{dz}{dt}) dt \\ &= \int_0^1 (2t - 2t + 10t) dt = 5.\end{aligned}$$