Lecture Note 1

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MATH1020
General Mathematics

Example 1 **Using Algebra to solve Geometry problems**

Consider the three point $A = (-2, 1)$, $B = (2, 3)$, and $C = (3, 1)$.

1. Plot each point and form the triangle ABC.
2. Find the length of each side of the triangle.
3. Verify that the triangle is a right triangle.
4. Find the area of the triangle.

Example 2 Finding the midpoint of a line segment

Find the midpoint of a line segment from $P_1 = (-5, 5)$ to $P_2 = (3, 1)$. Plot the points P_1 and P_2 and their midpoint.

Example 3 **Finding intercepts from an equation**

Find the $x-$intercept(s) and the $y-$intercept(s) of the graph of $y = x^2 - 4$. Then graph $y = x^2 - 4$ by plotting points.

Solution:

To find the $x-$intercept(s), we let $y = 0$ and obtain the equation:

$$\begin{aligned} x^2 - 4 &= 0 && y = x^2 - 4 \text{ with } y = 0 \\ (x+2)(x-2) &= 0 && \text{Factor} \\ (x+2) = 0 \text{ or } (x-2) &= 0 && \text{Zero} - \text{Product Property} \\ x = -2 \text{ or } x &= 2 && \text{Solve} \end{aligned}$$

The equation has two solutions, -2 and 2. The $x-$intercepts are -2 and 2.

To find the $y-$intercept(s), we let $x = 0$ and obtain the equation:

$$\begin{aligned} y &= x^2 - 4 \\ &= (0)^2 - 4 \\ &= -4 \end{aligned}$$

The $y-$intercept is -4.

Since $x^2 \geq 0$ for all x, we deduce from $y = x^2 - 4$ that $y \geq -4$ for all x. This useful information, the intercepts, and the points from Table enables us to graph $y = x^2 - 4$ by hand.

x	$y = x^2 - 4$	(x, y)
-3	$y = (-3)^2 - 4 = 5$	$(-3, 5)$
-1	$y = (-1)^2 - 4 = -3$	$(-1, -3)$
1	$y = (1)^2 - 4 = -3$	$(1, -3)$
3	$y = (3)^2 - 4 = 5$	$(3, 5)$

■

Test for symmetry:

To test the graph of an equation for symmetry with respect to

$x-$**axis:** Replace y by $-y$ in the equation. If an equivalent equation results, the graph of the equation is symmetric with respect to the $x-$axis.

$y-$**axis:** Replace x by $-x$ in the equation. If an equivalent equation results, the graph of the equation is symmetric with respect to the $y-$axis.

Origin: Replace x by $-x$ and y by $-y$ in the equation. If an equivalent equation results, the graph of the equation is symmetric with respect to the origin.

Example 4 **Find the intercepts and testing an equation for symmetry**

For the equation $y = \dfrac{x^2 - 4}{x^2 + 1}$.

1. Find the intercepts.

2. Test for symmetry.

Solution:

1. To obtain the $x-$intercept(s), let $y = 0$ in the equation and solve for x :

$$\frac{x^2-4}{x^2+1} = 0 \qquad \text{Let } y = 0$$

$$x^2 - 4 = 0 \qquad \text{Multiply both sides by } x^2+1$$

$$x = -2 \text{ or } x = 2 \qquad \text{Factor and use the Zero} - \text{Product Property}$$

To obtain the $y-$intercept(s), let $x = 0$ in the equation and solve for y :

$$\begin{aligned} y &= \frac{x^2-4}{x^2+1} \\ &= \frac{(0)^2-4}{(0)^2+1} \\ &= \frac{-4}{1} \\ &= -4 \end{aligned}$$

The $x-$intercepts are -2 and 2, while the The $y-$intercept is 4.

Remark 1

$$
\begin{aligned}
\frac{x^2-4}{x^2+1}\cdot(x^2+1) &= 0\cdot(x^2+1) \\
x^2-4\cdot\left(\frac{x^2+1}{x^2+1}\right) &= 0 \\
(x^2-4)\cdot 1 &= 0 \\
(x^2-4) &= 0.
\end{aligned}
$$

2. We now test the equation from symmetry with respect to the $x-$axis, the $y-$axis and the origin.

$x-$**axis:** To test for symmetry with respect to the $x-$axis, replace y by $-y$. Since

$$-y = \frac{x^2 - 4}{x^2 + 1}$$

is not equivalent to

$$y = \frac{x^2 - 4}{x^2 + 1}.$$

$y-$**axis:** To test for symmetry with respect to the $y-$axis, replace x by $-x$. Since

$$y = \frac{(-x)^2 - 4}{(-x)^2 + 1}$$

is equivalent to

$$y = \frac{x^2 - 4}{x^2 + 1}.$$

The graph of the equation is symmetry with respect to the $y-$axis.

Origin: To test for symmetry with respect to the origin, replace x by $-x$,, replace y by $-y$. Since

$$\begin{aligned} -y &= \frac{(-x)^2-4}{(-x)^2+1}) && \text{replace } x \text{ by } -x, \text{ and replace } y \text{ by } -y \\ y &= \frac{x^2-4}{x^2+1} && \text{Simplify} \end{aligned}$$

Since the result is not equivalent to the origin equation, the graph of the equation

$$y = \frac{x^2-4}{x^2+1})$$

is not symmetry with respect to the origin.

■

Example 5 Consider a linear equation $f(x) = mx + b$.

Draw the following graphs:

1. slope $m > 0$ with $y-$intercept b.
2. slope $m < 0$ with $y-$intercept b.
3. slope $m = 0$ with $y-$intercept b.
4. slope $m = 1$ with $y-$intercept $b = 0$.

Example 6 Finding values of a function

For the function f defined by $f(x) = 2x^2 - 3x$, evaluate

(a)	$f(3)$	(b)	$f(x) + f(3)$	(c)	$3f(x)$
(d)	$f(-x)$	(e)	$-f(x)$	(f)	$f(3x)$
(g)	$f(x+3)$	(e)	$\dfrac{f(x+h) - f(x)}{h}, \ h \neq 0$		

Solution:

(a) We substitute 3 for x in the equation for f $f(x) = 2x^2 - 3x$ to get $f(3) = 2(3)^2 - 3(3) = 18 - 9 = 9$. The image of 3 is 9.

(b) $f(x) + f(3) = 2x^2 - 3x + 9$.

(c) We multiply the equation for f by 3

$$3f(x) = 3\left(2x^2 - 3x\right) = 6x^2 - 9x.$$

(d) We substitute $-x$ for x in the equation for f and simplify

$$f(-x) = 2(-x)^2 - 3(-x) = 2x^2 + 3x.$$

(e)

$$-f(x) = -\left(2x^2 - 3x\right) = -2x^2 + 3x.$$

(f) We substitute $3x$ for x in the equation for f and simplify

$$f(3x) = 2(3x)^2 - 3(3x) = 18x^2 - 9x.$$

(g) Exercise!

(h) Exercise!

■

Finding the domain of a function defined by an equation:

1. Start with the domain as the set of real numbers.
2. If the equation has a denominator, exclude any numbers that give a zero denominator.
3. If the equation has a radical of even index, exclude any numbers that cause the expression inside the radical to be negative.

Example 7 **Finding the domain of a function**

Find the domain of each of the following functions:

1. $f(x) = x^2 + 5x$;
2. $f(x) = \dfrac{3x}{x^2 - 4}$;
3. $f(x) = \sqrt{4 - 3t}$.

Solution:

1. The function tells us to square a number and then add five times the number. Since these operations can be performed on any real number, we conclude that the domain of f is the set of all real numbers.

2. The function f tell us to divide $3x$ by $x^2 - 4$. Since division by 0 is not defined, the denominator $x^2 - 4$ can never be 0, so$x$ can be never equal to -2 or 2.

   The domain of the function f is

$$\begin{aligned} &\{x| \ x \neq -2, \ x = 2\} \\ &= (-\infty, 2) \cup (-2, 2) \cup (2, +\infty) \\ &= \mathbb{R} \setminus \{-2, 2\}. \end{aligned}$$

3. The function f tells us to take the square root of $4 - 3x$. But only non-negative numbers have real square roots, so the

expression under the square root (the radicand) must be non-negative (greater than or equal to zero). This requires that

$$\begin{aligned} 4 - 3x &\geq 0 \\ -3x &\geq -4 \\ x &\leq \frac{4}{3}. \end{aligned}$$

The domain of the function f is $\left\{ x |\ x \leq \dfrac{4}{3} \right\}$ or $(-\infty, 4/3]$.

■

The domains of $f+g$, $f-g$, $f \cdot g$ and $\dfrac{f}{g}$

The domain of $f+g$ consists of the numbers x that are in the domains of both f and g. That is,

$$\text{domain of } f+g = \text{ domain of } f \cap \text{ domain of } g.$$

The domain of $f - g$ consists of the numbers x that are in the domains of both f and g. That is,

$$\text{domain of } f - g = \text{ domain of } f \cap \text{ domain of } g.$$

The domain of $f \cdot g$ consists of the numbers x that are in the domains of both f and g. That is,

$$\text{domain of } f \cdot g = \text{ domain of } f \cap \text{ domain of } g.$$

The domain of $\dfrac{f}{g}$ consists of the numbers x for which $g(x) \neq 0$ that are in the domains of both f and g. That is,

$$\text{domain of } \frac{f}{g} = \{x |\ g(x) \neq 0\} \text{ domain of } f \cap \text{ domain of } g.$$

Example 8 Let f and g be two functions defined as

$$f(x) = \frac{1}{x+2} \qquad \text{and} \qquad g(x) = \frac{x}{x-1}.$$

Find the following, and determine the domain in each case:

1. $(f+g)(x)$;
2. $(f-g)(x)$;
3. $(f \cdot g)(x)$;
4. $\left(\dfrac{f}{g}\right)(x)$.

Solution:

The domain of f is $\{x|\ x \neq -2\}$ and The domain of g is $\{x|\ x \neq 1\}$.

1.

$$
\begin{aligned}
(f+g)(x) &= f(x)+g(x) \\
&= \frac{1}{x+2}+\frac{x}{x-1} \\
&= \frac{1}{x+2}\cdot\frac{x-1}{x-1}+\frac{x}{x-1}\cdot\frac{x+2}{x+2} \\
&= \frac{x^2+3x-1}{(x+2)(x-1)}.
\end{aligned}
$$

The domain of $f+g$ consists of those numbers that are in the domains of f and g. Therefore, the domain of $f+g$ is

$$\{x|\ x \neq -2,\ x \neq 1\}$$

or the interval

$$(-\infty,-2)\cup(-2,1)\cup(1,+\infty)$$

or the interval

$$\mathbb{R} \setminus \{-2, 1\}.$$

2.

$$\begin{aligned}(f-g)(x) &= f(x) - g(x) \\ &= \frac{1}{x+2} - \frac{x}{x-1} \\ &= \frac{1}{x+2} \cdot \frac{x-1}{x-1} - \frac{x}{x-1} \cdot \frac{x+2}{x+2} \\ &= \frac{-(x^2+x+1)}{(x+2)(x-1)}.\end{aligned}$$

The domain of $f-g$ consists of those numbers that are in the domains of f and g. Therefore, the domain of $f-g$ is

$$\{x|\ x \neq -2,\ x \neq 1\}$$

or the interval

$$(-\infty,-2) \cup (-2,1) \cup (1,+\infty)$$

or the interval

$$\mathbb{R} \setminus \{-2, 1\}.$$

3.

$$
\begin{aligned}
(f \cdot g)(x) &= f(x) \cdot g(x) \\
&= \frac{1}{x+2} \cdot \frac{x}{x-1} \\
&= \frac{x}{(x+2)(x-1)}.
\end{aligned}
$$

The domain of $f \cdot g$ consists of those numbers that are in the domains of f and g. Therefore, the domain of $f \cdot g$ is

$$\{x|\ x \neq -2,\ x \neq 1\}$$

or the interval

$$(-\infty, -2) \cup (-2, 1) \cup (1, +\infty)$$

or the interval

$$\mathbb{R} \setminus \{-2, -1\}.$$

4.

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$
$$= \frac{\dfrac{1}{x+2}}{\dfrac{x}{x-1}}$$
$$= \frac{x-1}{x(x+2)}.$$

The domain of $\dfrac{f}{g}$ consists of those numbers x for which $g(x) \neq 0$ that are in the domains of both f and g. Since $g(x) = 0$ when $x = 0$, we exclude 0 as well as -2 and 1 from the domain. Therefore, the domain of $\dfrac{f}{g}$ is

$$\{\, x|\ x \neq -2,\ x \neq 0,\ x \neq 1\}$$

or the interval

$$(-\infty, -2) \cup (-2, 0) \cup (0, 1) \cup (1, +\infty)$$

or the interval

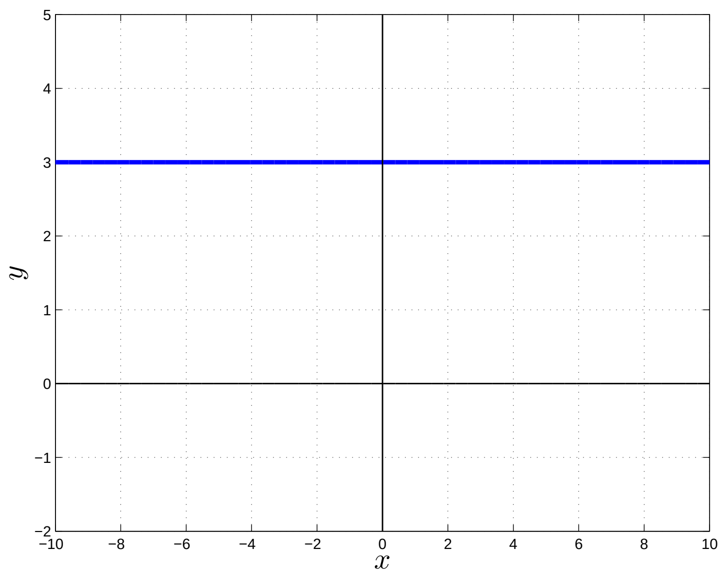
$$\mathbb{R} \setminus \{-2, 0, 1\}.$$

$\blacksquare$

Classification Of Functions

Constant function

$$f(x) = b, \qquad b \text{ is a real number}.$$



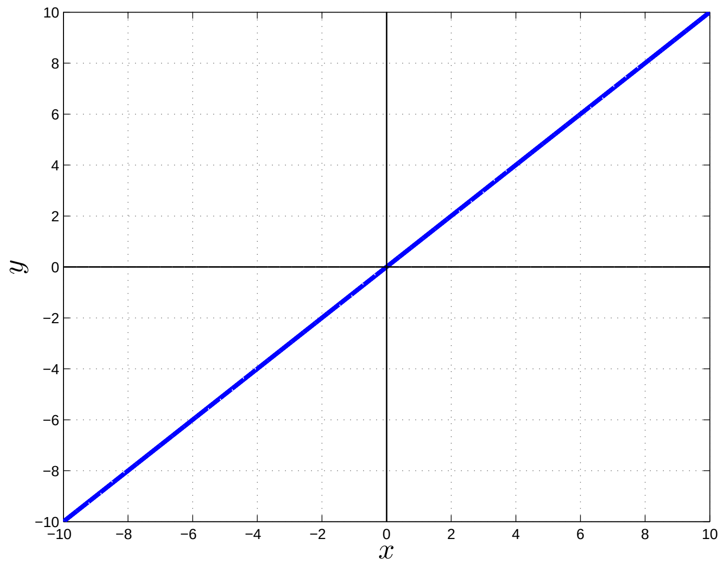
The domain of a constant function is the set of all real numbers; its range is the set consisting of a single number b.

Its graph os a horizontal line whose $y-$intercept is b.

The constant function is an even function whose graph is a horizontal line over its domain.

Identity function

$$f(x) = x.$$



The domain and the range of the identity function are the set of all real numbers.

Its graph is a line whose slope $m = 1$ and whose $y-$intercept is 0.

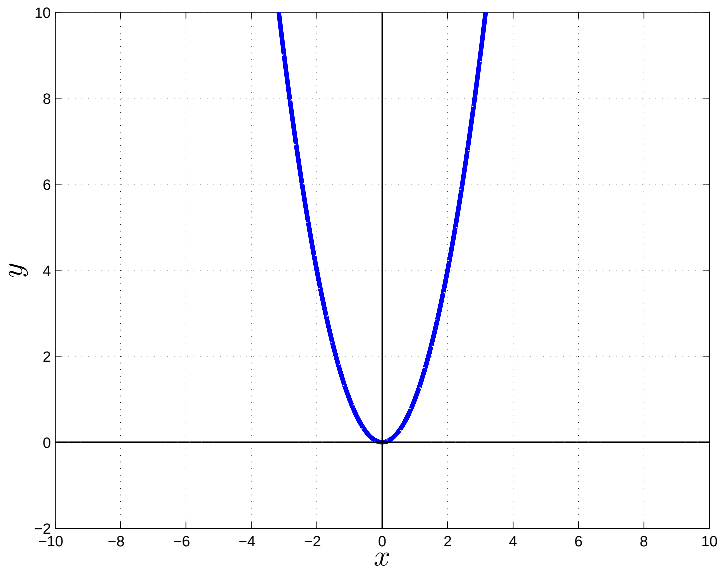
The line consists of all points for which the $x-$coordinate equals the $y-$coordinate.

The identity function is an odd function that is increasing over its domain.

Note that the graph bisects quadrants I and III.

Square function

$$f(x) = x^2.$$



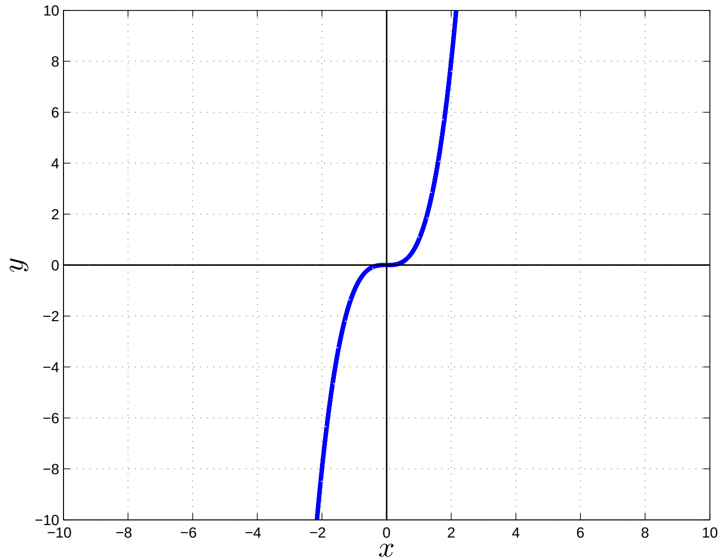
The domain of the square function f is the set of all real numbers; its range is the set of nonnegative real numbers.

The graph of this function is parabola whose intercept is at $(0, 0)$.

The square function is an even function that is decreasing on the interval $(-\infty, 0)$ and increasing on the interval $(0, \infty)$.

Cubic function

$$f(x) = x^3.$$



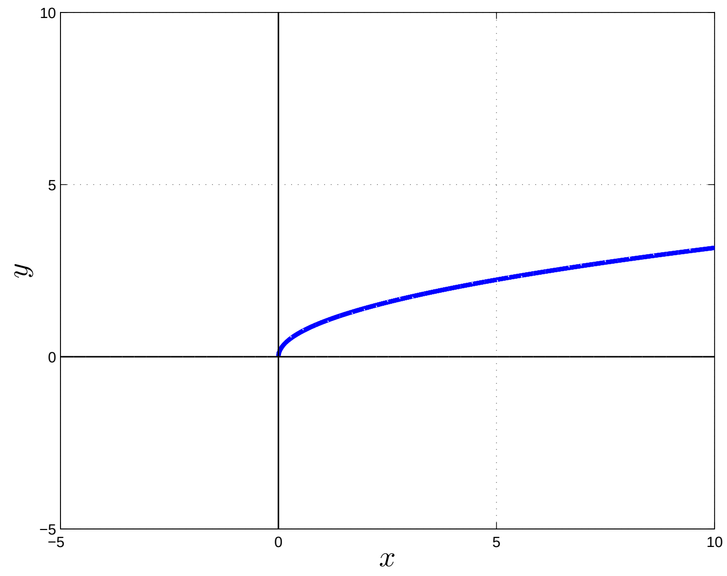
The domain and the range of the cubic function are the set of all real numbers.

The intercept of the graph is at $(0, 0)$.

The cubic function is odd and is increasing on the interval $(-\infty, \infty)$.

Square root function

$$f(x) = \sqrt{x}.$$



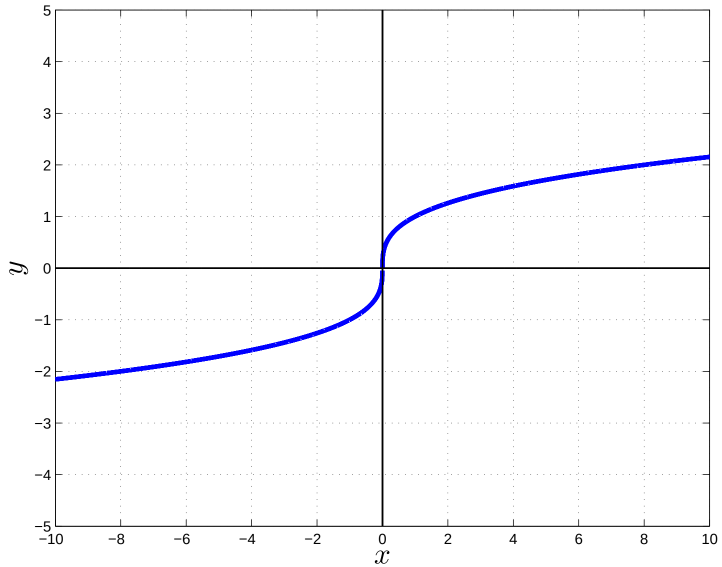
The domain and the range of the square root function are the set of nonnegative real numbers.

The intercept of the graph is at $(0, 0)$.

The square root function is neither even nor odd and is increasing on the interval $(0, \infty)$.

Cubic root function

$$f(x) = \sqrt[3]{x}.$$



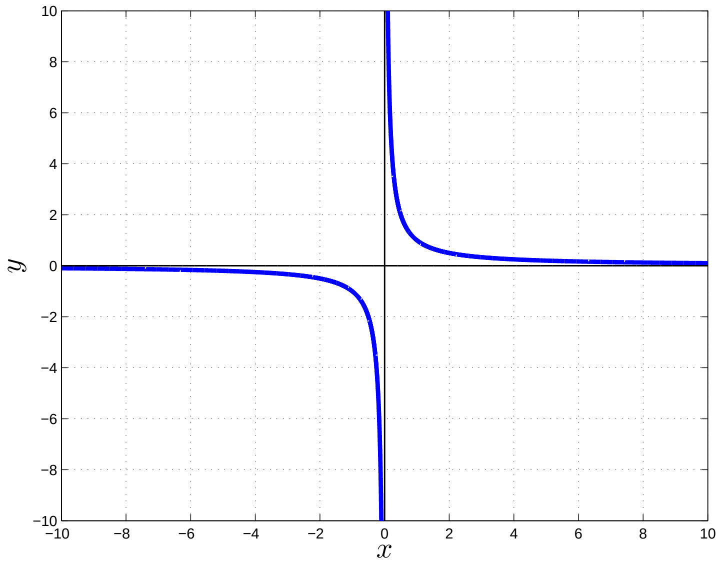
The domain and the range of the cubic root function are the set of nonnegative real numbers.

The intercept of the graph is at $(0, 0)$.

The square root function is odd and is increasing on the interval $(-\infty, \infty)$.

Reciprocal function

$$f(x) = \frac{1}{x}.$$



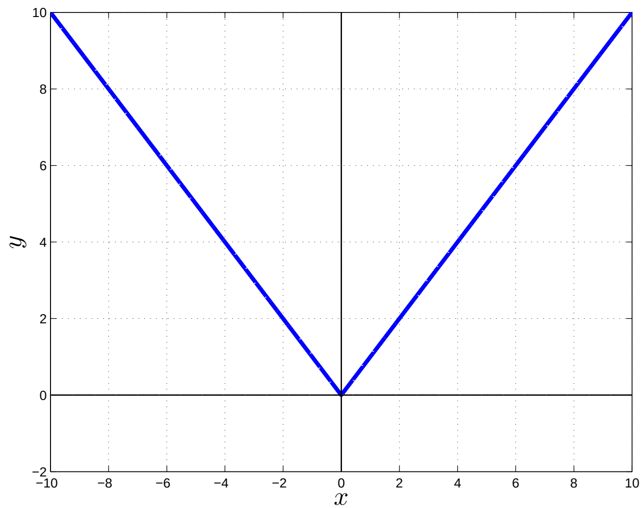
The domain and the range of the reciprocal function are the set of all nonzero real numbers.

The graph has no intercepts.

The reciprocal function is decreasing on the intervals $(-\infty, 0)$ and $(0, \infty)$ and is an odd function.

Absolute value function

$$f(x) = |x|.$$



The domain of the absolute value function is the set of all real numbers; its range is the set of nonnegative real numbers.

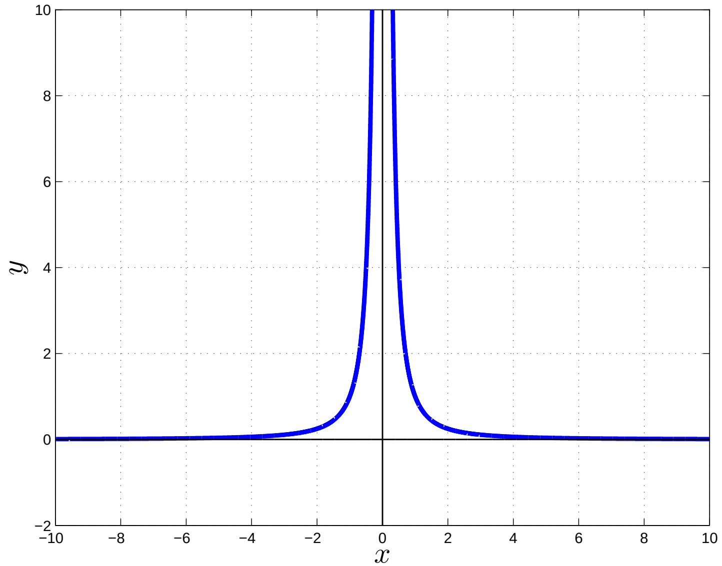
The intercept of the graph is at $(0, 0)$.

If $x \geq 0$, then $f(x) = x$, and the graph of f is part of the line $y = x$; if $x < 0$, then $f(x) = -x$, and the graph of f is part of the line $y = -x$.

Then absolute value function is an even function; it is decreasing on the interval $(-\infty, 0)$ and increasing on the interval $(0, \infty)$.

Reciprocal Square function

$$f(x) = \frac{1}{x^2}.$$



Domain: ________________

Range: ________________

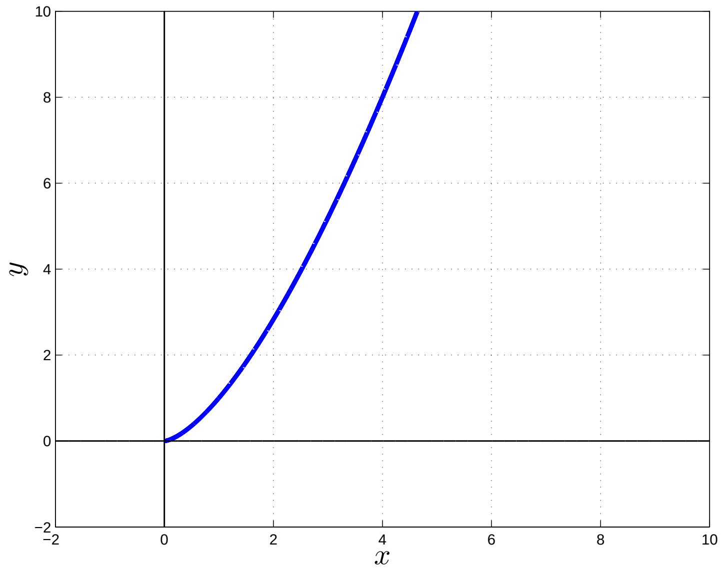
Increasing on: ________________

Decreasing on: ________________

Either even or odd, Neither even nor odd: ________________

Rational Power function

$$f(x) = x^{3/2}.$$



Domain: ________________

Range: ________________

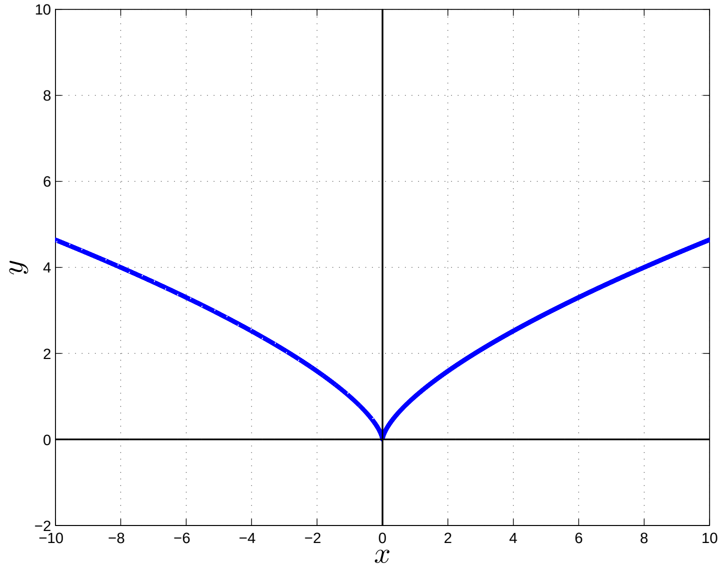
Increasing on: ________________

Decreasing on: ________________

Either even or odd, Neither even nor odd: ________________

Example 9 Consider the function:

$$f(x) = x^{2/3}.$$



Domain: ________________

Range: ________________

Increasing on: ________________

Decreasing on: ________________

Either even or odd, Neither even nor odd: ________________

Example 10 Analyzing a piecewise-defined function

The function f is defined as

$$f(x) = \begin{cases} -x+1 & \text{if} \quad -3 \le x < 1; \\ 2 & \text{if} \quad x = 1; \\ x^2 & \text{if} \quad x > 1. \end{cases}$$

Answer the following questions:

1. Find $f(0)$, $f(1)$, and $f(2)$.
2. Determine the domain of f.
3. Graph f.
4. Use the graph to find the range of f.
5. Is f continuous on its domain?

Solution:

1. To find $f(0)$, we observe that when $x = 0$, the equation for f is given by
$$f(x) = -x + 1.$$
So we have
$$f(0) = -(0) + 1 = 1.$$
When $x = 1$, the equation for f is $f(x) = 2$. Thus
$$f(1) = 2.$$
When $x = 2$, the equation for f is $f(x) = x^2$. So
$$f(2) = 2^2 = 4.$$

2. To find the domain of f, we look at its definition. Since f is

defined for all x greater than or equal to -2, the domain f is

$$\{x|\ x \geq -3\}$$

or the interval

$$[-3, +\infty).$$

3. To graph f, we graph each piece. First we graph the line

$$y = -x + 1$$

and keep only the part for which $-3 \leq x < 1$. Then we plot the point $(1, 2)$ because when $x = 1$, $f(x) = 2$. Finally, we graph the parabola $y = x^2$ and keep only the past for which $x > 1$.

4. From the graph, we conclude that the range of f is

$$\{y|\ y > 0\}$$

or the interval

$$(0, +\infty).$$

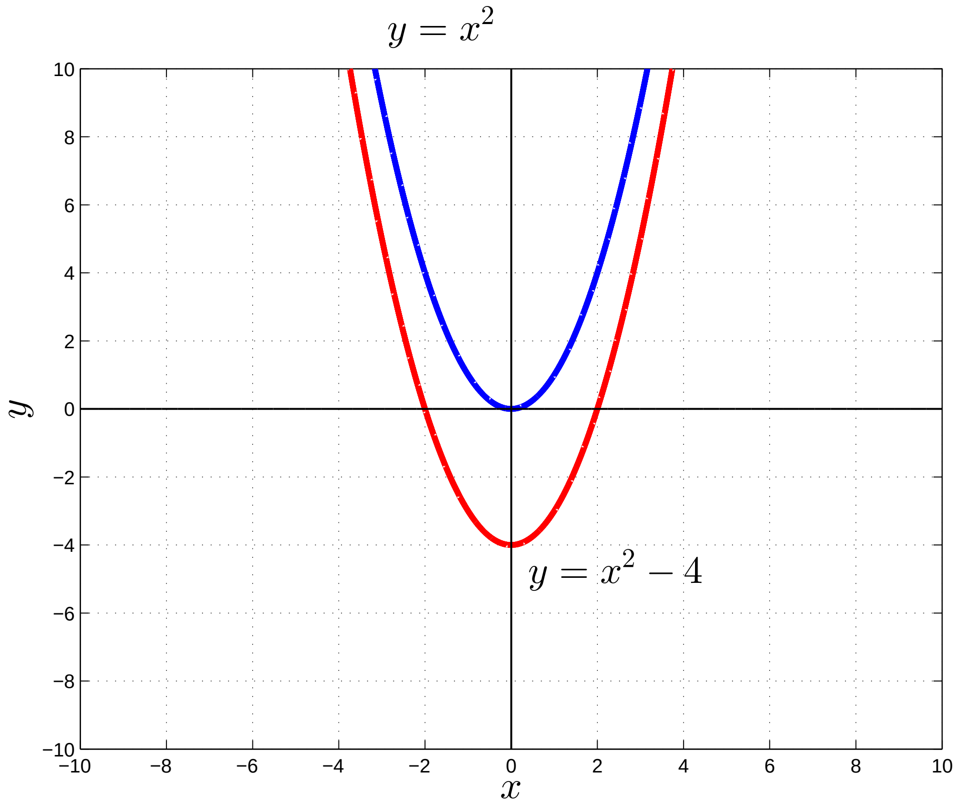
5. The function f is not continuous because there is a jump in the graph at $x = 1$.

$\blacksquare$

Example 11 **Vertical shift down**

Use the graph of $f(x) = x^2$ to obtain the graph of $h(x) = x^2 - 4$.

Solution: We notice that each $y-$coordinate of h is 4 units less than the corresponding $y-$coordinate of f. To obtain the graph of h from the graph of f, subtract 4 from each $y-$coordinate on the graph of f. The graph of h is identical to that of f, except that is shifted down 4 units, as shown the figure below.


$y = x^2$
$y = x^2 - 4$
10
8
6
4
2
0
−2
−4
−6
−8
−10
−10
−8
−6
−4
−2
0
2
4
6
8
10
x
y

Example 12 Combining vertical and horizontal shifts

Graph the function $f(x) = (x + 3)^2 - 5$.

Solution:

We graph f in steps.

First, we note that the rule for f is basically a square function, so we begin with the graph of $y = x^2$ as shown in Figure 1.

To get the graph of $y = (x + 3)^2$, we shift the graph of $y = x^2$ horizontally 3 units to the left. See Figure 1.

Finally, to get the graph of $f(x) = (x + 3)^2 - 5$ vertically down 5 units. See Figure 1

Note the minimum points plotted on each graph.

minimum

local	global
relative	absolute

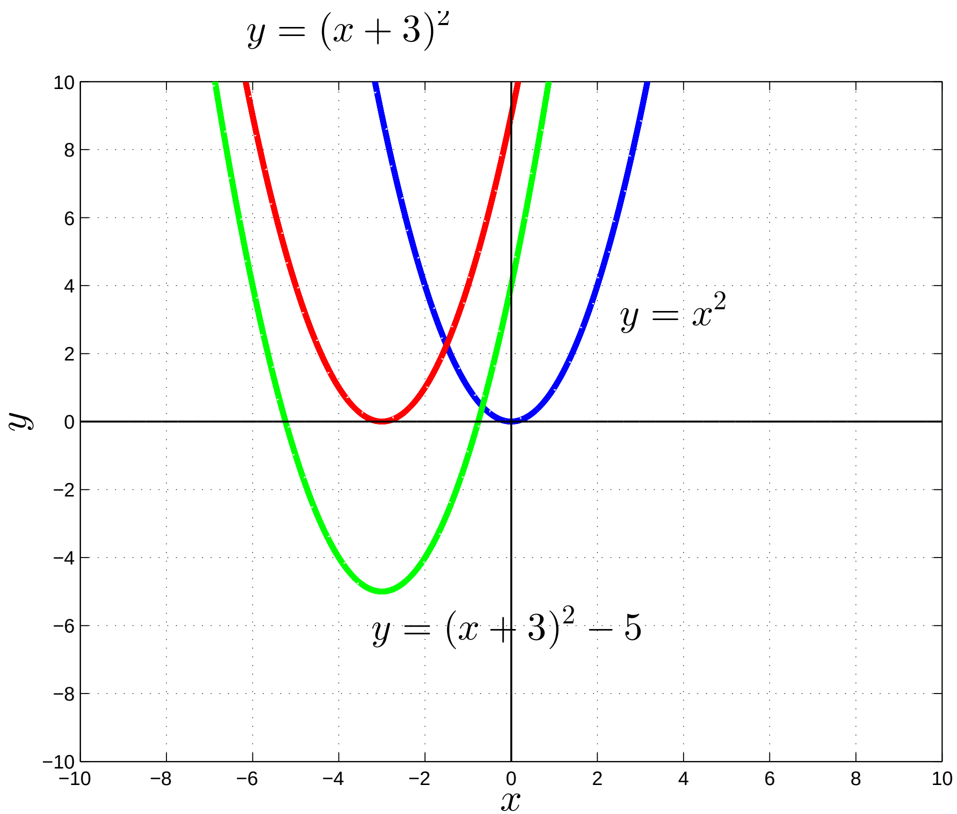


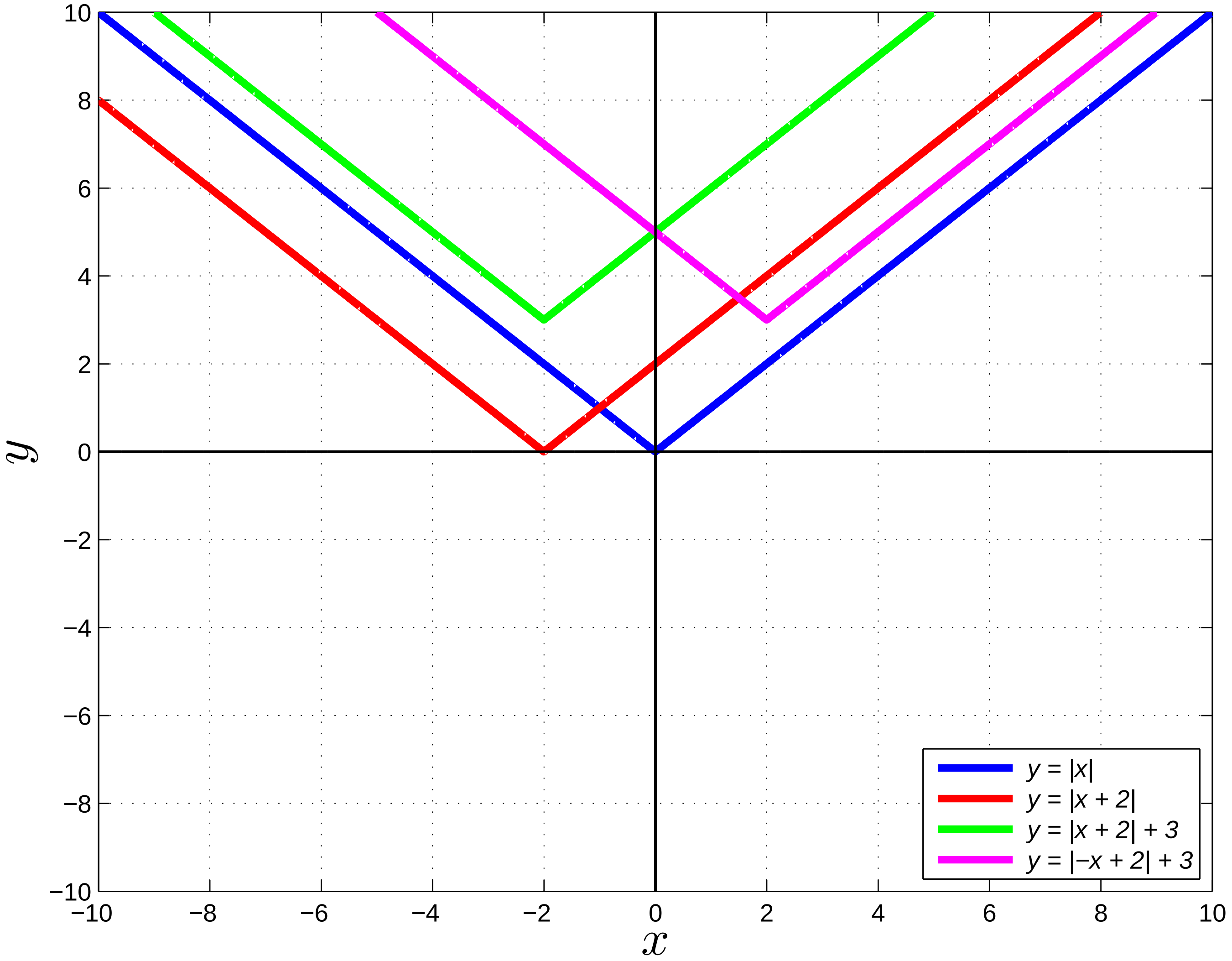
Figure 1:

Example 13 **Determining the function obtained from a series of transformations**

Find the function that is finally graphed after the following three transformations are applied to the graph of $y = |x|$.

1. Shift left 2 units.
2. Shift up 3 units.
3. Reflect about the $y-$axis.

1. Shift left 2 units.	Replace x by $x + 2$.	$y = \|x + 2\|$
2. Shift up 3 units.	Add 3.	$y = \|x + 2\| + 3$
3. Reflect about the $y-$axis.	Replace x by $-x$.	$y = \|-x + 2\| + 3$



y = |x|
y = |x + 2|
y = |x + 2| + 3
y = |−x + 2| + 3
x
y

Exercises A **Combining graphing procedures**

Graph the function $f(x) = \dfrac{3}{x-2} + 1$. Find the domain and the range of f.

Solution:

It is helpful to write f as $f(x) = \dfrac{3}{x-2} + 1$. Now we use the following steps to obtain the graph of f.

Step 1	$y = \dfrac{1}{x}$	Reciprocal function
Step 2	$y = 3\left(\dfrac{1}{x}\right) = \frac{3}{x}$	Multiply by 3.
		Vertical stretch of the graph of $y = \dfrac{1}{x}$ by a factor of 3.
Step 3	$y = \dfrac{3}{x-2}$	Replace x by $x-2$.
		Horizontal shift to the right 2 units.
Step 4	$y = \dfrac{3}{x-2} + 1$	Add 1.
		Vertical shift up 1 unit.

The domain of $y = \dfrac{1}{x}$ is $\{x \mid x \neq 0\}$ and its range $\{y \mid y \neq 0\}$.

Because we shifted right 2 units and up 1 unit to obtain f, the domain of f is $\{x \mid x \neq 2\}$ and its range $\{y \mid y \neq 1\}$.

See Figure 2.

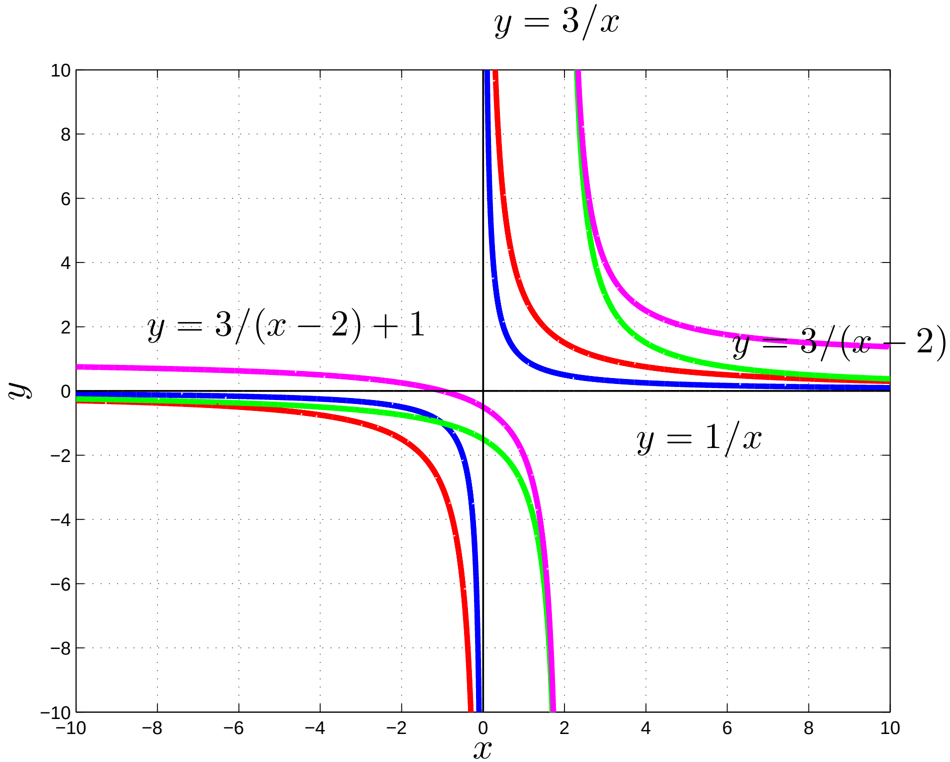


Figure 2:

Exercises B Combining graphing procedures

Graph the function $f(x) = \sqrt{1-x} + 2$. Find the domain and the range of f.

Solution:

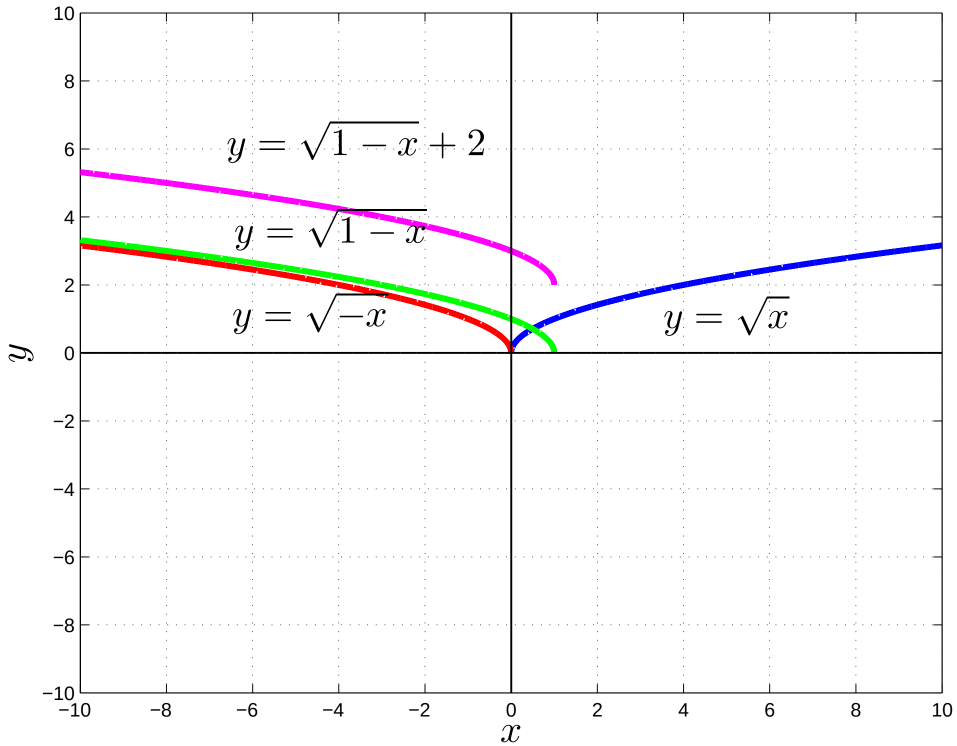
It is because horizontal shifts require the form x - h, we begin by rewriting f as

$$\begin{aligned} f(x) &= \sqrt{1-x} + 2 \\ &= \sqrt{-(x-1)} + 2. \end{aligned}$$

Now use the following steps:

Step 1	$y = \sqrt{x}$	Square root function.
Step 2	$y = \sqrt{-x}$	Replace x by $-x$.
		Reflect about the $y-$axis.
Step 3	$y = \sqrt{-(x-1)}$	Replace x by $x-1$.
		Horizontal shift to the right 1 unit.
Step 4	$y = \sqrt{1-x} + 2$	Add 2.
		Vertical shift up 2 units.

The domain of $y = \dfrac{1}{x}$ is $(-\infty, 1]$ and its range $[2, +\infty)$.


$y = \sqrt{1-x} + 2$
$y = \sqrt{1-x}$
$y = \sqrt{-x}$
$y = \sqrt{x}$
x
y

			Change the graph point (x, y) to
Vertical shifts			
$y = f(x) + k,\ k > 0$	Raise the graph of f by k times.	Add k to $f(x)$.	$(x, y + k)$
$y = f(x) - k,\ k > 0$	Lower the graph of f by k times.	Subtract k from $f(x)$.	$(x, y - k)$
Horizontal shifts			
$y = f(x + h),\ h > 0$	Shift the graph of f to the left h times.	Replace x by $x + h$.	$(x + h, y)$
$y = f(x - h),\ h > 0$	Shift the graph of f to the right h times.	Replace x by $x - h$.	$(x - h, y)$
Compressing or Stretching			
$y = af(x),\ a > 0$	Multiply each $y-$coordinate of $y = f(x)$ by a. Stretch the graph of f vertically if $a > 1$. Compress the graph of f vertically if $0 < a < 1$.	Multiply $f(x)$ by a.	(x, ay)
$y = f(ax),\ a > 0$	Multiply each $x-$coordinate of $y = f(x)$ by $1/a$. Stretch the graph of f horizontally if $0 < a < 1$. Compress the graph of f horizontally if $a > 1$.	Replace x by $\frac{x}{a}$.	$\left(\frac{x}{a}, y\right)$
Reflection about the $x-$axis			
$y = -f(x)$	Reflect the graph of f about the $x-$axis.	Multiply $f(x)$ by -1.	$(x, -y)$
Reflection about the $y-$axis			
$y = f(-x)$	Reflect the graph of f about the $y-$axis.	Replace x by $-x$.	$(-x, y)$