Lecture Note 5

Dr. Jeff Chak-Fu WONG

Department of Mathematics
Chinese University of Hong Kong
jwong@math.cuhk.edu.hk

MATH1020
General Mathematics

Exponential Functions

Theorem 1 **Laws of Exponents**

If s, t, a,and b are real numbers with $a > 0$ and $b > 0$, then

$$\begin{aligned} a^s \cdot a^t &= a^{s+t} \\ (a^s)^t &= a^{st} \\ (ab)^s &= a^s \cdot b^s \\ 1^s &= 1 \\ a^{-s} &= \frac{1}{a^s} = \left(\frac{1}{a}\right)^s \\ a^0 &= 1. \end{aligned}$$

Definition 1 An **exponential function** is a function of the form

$$\boxed{f(x) = Ca^x}$$

where a is a positive real number $(a > 0)$ and $a \neq 1$ and $C \neq 0$ is a real number. The domain of f is the set of all real numbers. The base a is the **growth factor**, and because $f(0) = Ca^0 = C$, we call C the **initial value**.

Theorem 2 For an exponential function $f(x) = C \cdot a^x,\ a > 0,\ a \neq 1$, if x is any real number, then

$$\boxed{\frac{f(x+1)}{f(x)} = a \qquad \text{or} \qquad f(x+1) = af(x).}$$

Proof:

$$\frac{f(x+1)}{f(x)} = \frac{C \cdot a^{x+1}}{C \cdot a^x} = a^{x+1-x} = a^1 = a.$$

The following display summarizes the information that we have about the function $f(x) = a^x, \ a > 1$.

Properties of the Exponential Function $f(x) = a^x, \ a > 1$

1. The domain is the set of all real numbers: the range is the set of positive real numbers.

2. There are no $x-$intercepts; the $y-$intercept is 1.

3. The $x-$axis $(y = 0)$ is a horizontal asymptote as $x \to -\infty$.

4. $f(x) = a^x, \ a > 1$, is a decreasing function and is one$-$to$-$one.

5. The graph of f contains the points $(0, 1), \ (1, a)$, and $\left(-1, \dfrac{1}{a}\right)$

6. The graph of f is smooth and continuous, with no corners or gaps, See Figure 1.

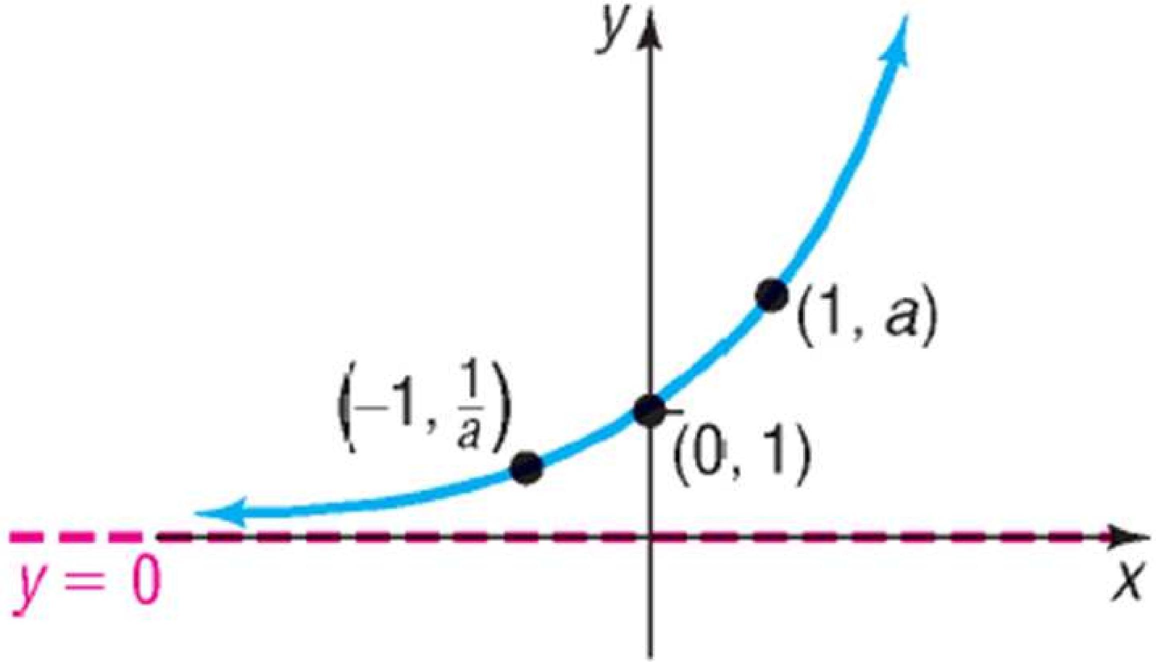

Figure 1:

The following display summarizes the information that we have about the function $f(x) = a^x,\ 0 < a < 1$.

Properties of the Exponential Function $f(x) = a^x,\ 0 < a < 1$

1. The domain is the set of all real numbers: the range is the set of positive real numbers.

2. There are no $x-$intercepts; the $y-$intercept is 1.

3. The $x-$axis $(y = 0)$ is a horizontal asymptote as $x \to \infty$.

4. $f(x) = a^x,\ 0 < a < 1$, is a decreasing function and is one$-$to$-$one.

5. The graph of f contains the points $(0, 1),\ (1, a)$, and $\left(-1, \dfrac{1}{a}\right)$.

6. The graph of f is smooth and continuous, with no corners or gaps. See Figure 2.

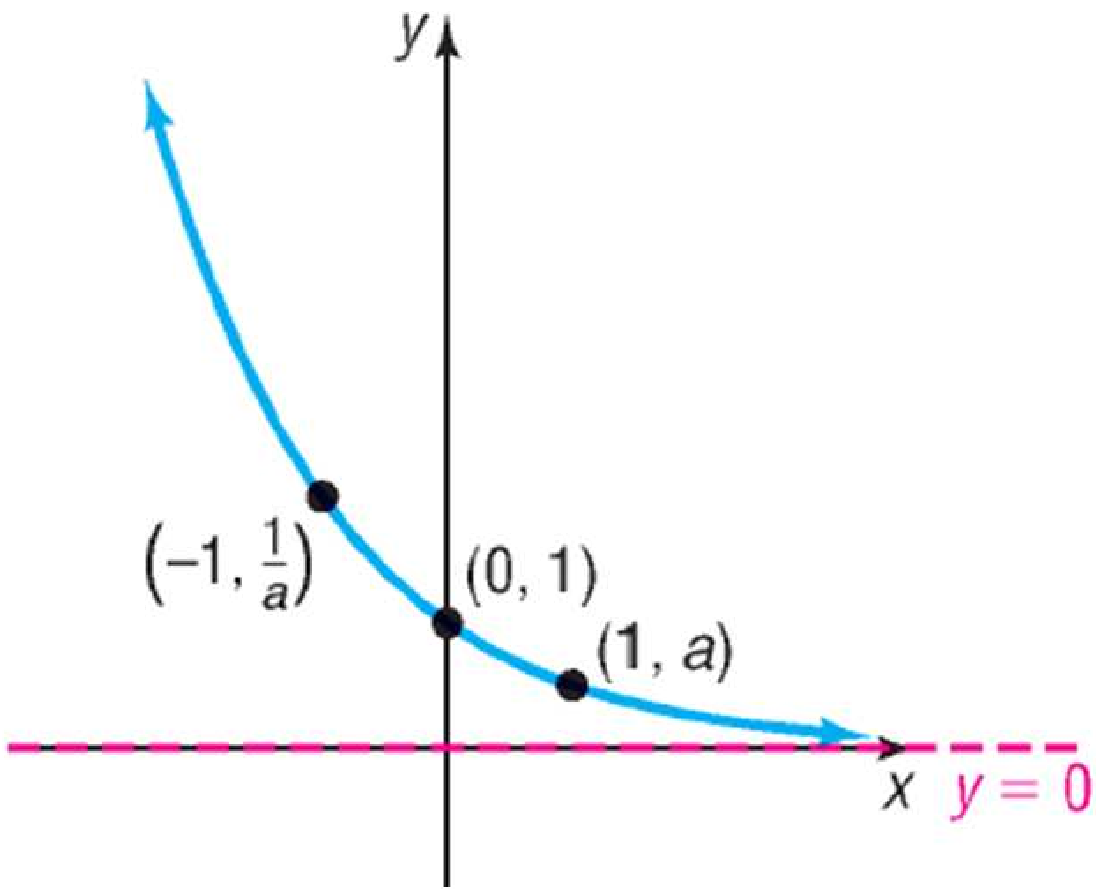

Figure 2:

Exercises 1 Graphing an exponential Function Using Transformations

Graph $f(x) = 2^{-x} - 3$ and determine the domain, range, and horizontal asymptote of f.

Let's look at one way of arriving at this important number e.

Definition 2 The number e is defined as the number that the expression

$$\boxed{\left(1+\frac{1}{n}\right)^n \qquad (2)}$$

approaches as $n \to \infty$. In calculus, this is expressed using limit notation as

$$\boxed{e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.}$$

Table 1 illustrates what happens to the defining expression (2) as n takes on increasingly large values. The last number in the right column in the table is correct to nine decimal places and id the same as the entry given for e on your calculator (if expressed correctly to nine decimal places).

Table 1

n	$\frac{1}{n}$	$1+\frac{1}{n}$	$\left(1+\frac{1}{n}\right)^n$
1	2	2	2
2	0.5	1.5	2.25
5	0.2	1.2	2.48832
10	0.1	1.1	2.59374246
100	0.01	1.01	2.704813829
$1,000$	0.001	1.001	2.716923932
$10,000$	0.0001	1.0001	2.718145927
$100,000$	0.00001	1.00001	2.718268237
$1,000,000$	0.000001	1.000001	2.718280469
$1,000,000,000$	10^{-9}	$1+10^{-9}$	2.718281827

Exercises 2 Graphing exponential Functions Using Transformations

Graph $f(x) = -e^{x-3}$ and determine the domain, range, and horizontal asymptote of f.

Solve Exponential Equations

Equations that involve terms of the form $a^x, a > 0, a \neq 1$, are referred to as exponential equations. Such equations can sometimes be solved by appropriately applying the Laws of Exponents and property (3)

$$\boxed{\text{If } a^u = a^v, \text{ then } u = v. \qquad (3)}$$

Property (3) is a consequence of the fact that exponential functions are one$-$to$-$one. To use property (3), each side of the equality must be written with the same base.

Exercises 3 Solving an Exponential Equation

Solve: $3^{x+1} = 81$

Exercises 4 Solving an Exponential Equation

Solve: $e^{-x^2} = (e^x)^2 \cdot \dfrac{1}{e^3}$.