Lecture Notes: Vector Space

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1 Vector Space Definition

Let V be a set of elements (what the elements actually are does not matter). Suppose that two operations are defined on V:

- **Addition:** for any elements e_1 and e_2 in V, $e_1 + e_2$ is defined to be an element in V.
- **Scalar Multiplication:** for any real value c and any element $e \in V$, both ce and ec are defined to be an identical element in V.

Also, suppose that the addition operation satisfies the following conditions:

- *Addition Commutativity:* for any elements e_1, e_2 in V, it holds that $e_1 + e_2 = e_2 + e_1$.
- *Addition Associativity:* for any elements e_1, e_2, e_3 in V, it holds that $(e_1 + e_2) + e_3 = e_1 + (e_2 + e_3)$.
- *Zero Element:* there is a unique element $z \in V$ such that $e + z = e$ for any element $e \in V$. z is called the *zero element*.
- *Existence of Opposites:* for any element $e \in V$, there exists a unique element $e' \in V$ such that $e + e' = z$. Elements e, e' are *opposites* of each other.

Furthermore, suppose that the scalar multiplication operation satisfies the following conditions:

- *Distributivity on Elements:* for any real value c, and any elements $e_1, e_2 \in V$, it holds that $c(e_1 + e_2) = ce_1 + ce_2$.
- *Distributivity on Scalars:* for any real values c_1, c_2, and any element $e \in V$, it holds that $(c_1 + c_2)e = c_1e + c_2e$.
- *Scalar Associativity:* for any real values c_1, c_2, and any element $e \in V$, it holds that $c_1(c_2e) = (c_1c_2)e$.
- *Product with 1:* for any element $e \in V$, it holds that $1e = e$.
- *Product with 0:* for any element $e \in V$, it holds that $0e = z$.

Then, we say that V is a *vector space*.

Example 1. Let V be the set of all vectors in $\mathbb{R}^3$. Define addition and scalar multiplication in the same way as on vectors. Then, V is a vector space. □

Example 2. Let V be the set of all vectors in $\mathbb{R}^3$, except $[0,0,0]$. Define addition and scalar multiplication in the same way as on vectors. Then, V is not a vector space because it does not have a zero element. $\square$

Example 3. Let V be the set of all vectors in $\mathbb{R}^3$, except $[1,1,1]$. Define addition and scalar multiplication in the same way as on vectors. Then, V is not a vector space because $[0.1,0.5,0.3]+[0.9,0.5,0.7]$ is not in V. $\square$

Example 4. Let V be the set of all possible matrices. Define addition and scalar multiplication in the same way as on matrices. Then, V is not a vector space because addition is undefined on a 2×2 matrix and a 3×3 matrix. $\square$

Example 5. Let V be the set of all 2×2 matrices. Define addition and scalar multiplication in the same way as on matrices. Then, V is a vector space. Think: what is the zero element? $\square$

2 Dimension and Basis

Now that both addition and scalar multiplication are properly defined, we can extend the notions of linear combination and linear independence to any vector space V. Denote by z its zero element. Specifically, let $e_1, e_2, ..., e_k$ be distinct elements of V for any $k \geq 1$. If there exist real values $c_1, ..., c_k$ such that

$$e_{k+1} = \sum_{i=1}^{k} c_i e_i$$

then we say that e_{k+1} is a *linear combination* of $e_1, e_2, ..., e_k$. If there exist real values $c_1, ..., c_k$ that are not all zero, and satisfy:

$$\sum_{i=1}^{k} c_i e_i = z$$

then we say that $e_1, e_2, ..., e_k$ are *linearly dependent*; otherwise, they are *linearly independent*.

Lemma 1. *For $k \geq 2$, distinct elements $e_1, e_2, ..., e_k$ are linearly dependent if and only if at least one of $e_1, e_2, ..., e_k$ is a linear combination of the other elements.*

Proof. *If Direction.* Suppose that $e_k = \sum_{i=1}^{k-1} e_i$. Then, we have

$$\begin{aligned} (-1)e_k + e_k &= (-1)e_k + \sum_{i=1}^{k-1} e_i \Rightarrow \\ z &= (-1)e_k + \sum_{i=1}^{k-1} e_i. \end{aligned}$$

Thus, $e_1, e_2, ..., e_k$ are linearly dependent.

Only-If Direction. Suppose that $e_1, e_2, ..., e_k$ are linearly dependent, namely, there exist real values $c_1, ..., c_k$ that are not all zero, and satisfy:

$$\sum_{i=1}^{k} c_i e_i = z$$

Without loss of generality, assume that $c_k \neq 0$. Then we have:

$$\begin{aligned}
(-c_k)e_k + \sum_{i=1}^{k} c_i e_i &= (-c_k)e_k \Rightarrow \\
\sum_{i=1}^{k-1} c_i e_i &= (-c_k)e_k \Rightarrow \\
\sum_{i=1}^{k-1} -\frac{c_i}{c_k} e_i &= e_k
\end{aligned}$$

Thus, e_k is a linear combination of $e_1, ..., e_k$. $\square$

We say that the *dimension* of V is an integer d if both the following conditions are satisfied:

1. there are d distinct elements $e_1, e_2, ..., e_d$ that are linearly independent.
2. any element of V is a linear combination of $e_1, e_2, ..., e_d$.

Furthermore, we refer to the set of $\{e_1, ..., e_d\}$ in the first condition as a *basis* of V. Also, V is said to be the *span* of $e_1, ..., e_d$.

Example 7.

- The vector space V in Example 1 has dimension 3. A basis is $\{[1, 0, 0],\ [0, 1, 0],\ [0, 0, 1]\}$. In other words, V is the span of $[1, 0, 0],\ [0, 1, 0],\ [0, 0, 1]$.
- The vector space V in Example 5 has dimension 4. A basis is the set of the following matrices:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

  V is the span of these 4 matrices. $\square$

Example 8. Consider the set V of vectors given by the vector function $\boldsymbol{r}(t) = [x(t), y(t), z(t)]$ (where $t \in \mathbb{R}$) satisfying:

$$\begin{aligned}
x(t) &= 6t \\
y(t) &= 2t \\
z(t) &= 10t
\end{aligned}$$

Define addition and scalar multiplication in the same way as on vectors. What is the dimension of V? Find a basis of V.

Solution. Intuitively, since V is controlled by only 1 parameter, its dimension ought to be 1. The following analysis confirms the intuition. Note that the zero element of V is $\boldsymbol{z} = [0, 0, 0]$.

Let $\boldsymbol{u} = \boldsymbol{r}(1) = [6, 2, 10]$. We prove that any vector $\boldsymbol{v}$ in V is a linear combination of $\boldsymbol{u}$. In fact, assume that $\boldsymbol{v} = \boldsymbol{r}(t) = [6t, 2t, 10t]$. Then, $\boldsymbol{v} = t\boldsymbol{u}$. Furthermore, $\boldsymbol{u}$ itself is clearly linearly independent. Hence, the dimension of V is 1, and a basis is $\{[6, 2, 10]\}$. $\square$

Example 9. Consider the set V of vectors given by the vector function $\boldsymbol{r}(x, y) = [2x+3y, x+y, y]$ (where $x, y \in \mathbb{R}$). Define addition and scalar multiplication in the same way as on vectors. What is the dimension of V? Find a basis of V.

Solution. Intuitively, since V is controlled by 2 parameters, its dimension ought to be 2. The following analysis confirms the intuition. Note that the zero element of V is $\boldsymbol{z} = [0, 0, 0]$.

Let $\boldsymbol{u_1} = \boldsymbol{r}(1, 0) = [2, 1, 0]$ and $\boldsymbol{u_2} = \boldsymbol{r}(0, 1) = [3, 1, 1]$. Since the rank of matrix

$$\begin{bmatrix} 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}$$

is 2, we know that $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$ are linearly independent. Next, we prove that any vector $\boldsymbol{v} = \boldsymbol{r}(x, y)$ is a linear combination of $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$. This is true because, as you can verify easily, $\boldsymbol{v} = x\,\boldsymbol{u_1} + y\,\boldsymbol{u_2}$. $\square$

3 Linear Transformation

Let V_1 be the set of all m-dimensional vectors in $\mathbb{R}^m$, and V_2 be the set of all n-dimensional vectors in $\mathbb{R}^n$. Let $\boldsymbol{A}$ be an $m \times n$ matrix. Then, given a vector $\boldsymbol{x} \in V_2$, define function

$$\boldsymbol{y}(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}.$$

Note that $\boldsymbol{y}(\boldsymbol{x})$ is an m-dimensional vector, namely, a vector in V_1. We say that function $\boldsymbol{y}$ is a *linear transformation* from V_2 to V_1. Also, we refer to $\boldsymbol{y}(\boldsymbol{x})$ as the *image* of $\boldsymbol{x}$.

Example 10. Consider the following mapping from each point $(x, y, z) \in \mathbb{R}^3$ to a point $(u, v) \in \mathbb{R}^2$:

$$\begin{aligned} u &= 2x + y + 3z \\ v &= -x - y + 2z. \end{aligned}$$

This is a linear transformation given by

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$\square$

Now consider the situation where $n = m$, namely, V_1 and V_2 contain vectors of the same dimensionality. We say that a linear transformation $\boldsymbol{y}$ from V_2 to V_1 is a *linear one-one mapping* if the following condition holds: $\boldsymbol{y}(\boldsymbol{x_1}) = \boldsymbol{y}(\boldsymbol{x_2})$ if and only if $\boldsymbol{x_1} = \boldsymbol{x_2}$.

Example 11. Let $\boldsymbol{x} = [a, b, c]$ be a vector in $\mathbb{R}^3$. Consider the linear transformation

$$\boldsymbol{y}(\boldsymbol{x}) = \begin{bmatrix} 2 & 1 & 3 \\ -1 & -1 & 2 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Function $\boldsymbol{y}(\boldsymbol{x})$ is not a linear one-one mapping because, as you can easily verify, $\boldsymbol{y}([0, 0, 0]) = \boldsymbol{y}([5, -7, -1])$. $\square$

Lemma 2. *Let $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$ be a linear transformation with $\boldsymbol{A}$ being an $n \times n$ matrix. $\boldsymbol{y}$ is a linear one-one mapping if and only if $\boldsymbol{A}$ has rank n.*

Proof. *If-Direction.* Assuming that $\boldsymbol{A}$ has rank n, next we prove that $\boldsymbol{y}$ is a linear one-one mapping. That $\boldsymbol{A}$ has rank n indicates that $\boldsymbol{A}^{-1}$ exists. Suppose that there were $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ satisfying $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x_1} = \boldsymbol{A}\boldsymbol{x_2}$. We thus know that $\boldsymbol{x_1} = \boldsymbol{x_2} = \boldsymbol{A}^{-1}\boldsymbol{y}$.

Only-If Direction. Assuming that $\boldsymbol{y}$ is a linear one-one mapping, next we prove that $\boldsymbol{A}$ has rank n. It suffices to prove that equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ has only a unique solution $\boldsymbol{x} = \boldsymbol{0}$. Assume on the contrary that $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ had a non-zero solution $\boldsymbol{x'}$; then, any $\boldsymbol{x_1}, \boldsymbol{x_2}$ obeying $\boldsymbol{x_1} - \boldsymbol{x_2} = \boldsymbol{x'}$ would satisfy $A\boldsymbol{x_1} = A\boldsymbol{x_2}$, contradicting the fact that $\boldsymbol{y}$ is a linear one-one mapping. □

Suppose that $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$ is a linear one-one mapping. The above lemma indicates that $\boldsymbol{A}^{-1}$ exists. Hence:

$$\boldsymbol{x} \;=\; \boldsymbol{A}^{-1}\boldsymbol{y}.$$

We refer to the above function as the *inverse transformation* of $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$. Note that, according to Lemma 2, if $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$ is not a linear one-one mapping, then it does not have any inverse transformation. For example, the linear transformation in Example 11 has no inverse transformation.

Example 12. Let $\boldsymbol{x} = [a, b, c]$, $\boldsymbol{y} = [u, v, w]$ be vectors in $\mathbb{R}^3$. Consider the linear transformation

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 4 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{1}$$

Since matrix $\boldsymbol{A} = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 4 \\ 0 & -2 & 1 \end{bmatrix}$ has inverse $\boldsymbol{A}^{-1} = \begin{bmatrix} -4 & 1 & -4 \\ 1/2 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, the linear transformation in (1) has the inverse transformation

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -4 & 1 & -4 \\ 1/2 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

□