THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1020
Exercise 2
Produced by Jeff Chak-Fu WONG

Exercise 1 Analyze the graph of the rational function:

$$R(x) = \frac{x-1}{x^2-4}.$$

Solution:

Step 1:

$$R(x) = \frac{x-1}{x^2-4} = \frac{x-1}{(x+2)(x-2)}.$$

The domain of R is

$$\{x| \ x \neq -2, x \neq 2\}$$

or

$$(-\infty, -2) \cup (-2, 2) \cup (2, +\infty)$$

or

$$\mathbb{R} \setminus \{-2, 2\}.$$

Step 2: R is in lowest terms because there are no common factors between the numerator and denominator.

Step 3: The $x-$intercepts are found by determining the real zeros of the numerator of R written in lowest terms. By solving $x - 1 = 0$, we find the only real zero of the numerator is 1, so the only $x-$intercept of the graph R is 1. Since 0 is in the domain of R, the $y-$intercept is $R(0) = \frac{1}{4}$.

Step 4: Because

$$R(-x) = \frac{(-x)-1}{(-x)^2-4} = \frac{-x-1}{(-x)^2-4} = -\frac{x+1}{x^2-4} \neq R(x) = \frac{x-1}{x^2-4}$$

we conclude that $R(-x) \neq R(x)$, so R is not even. Because $R(-x) \neq -R(x)$, R is not odd. So there is no symmetry with respect to the $y-$axis or the origin.

Step 5: We locate the vertical asymptotes by finding the zeros of the denominator with the rational function in lowest terms. With R in lowest terms, we find that the graph of R has two vertical asymptotes: the lines $x = -2$ and $x = 2$.

Step 6: Because the degree of the numerator is less than the degree of the denominator, R is proper and the line $y = 0$(the $x-$axis) is a horizontal asymptote of the graph.

To determine if the graph of R intersects the horizontal asymptote, we solve the equation $R(x) = 0$:

$$\begin{aligned} \frac{x-1}{x^2-4} &= 0 \\ \frac{x-1}{x^2-4} \cdot (x^2-4) &= 0 \cdot (x^2-4) \\ x-1 &= 0 \\ x &= 1. \end{aligned}$$

Step 7: Figure 1 shows the graph of $R(x) = \dfrac{x-1}{x^2-4}$. We clearly observe that the graph of R does not cross the lines $x = -2$ and $x = 2$, since R is not defined at $x = -2$ or $x = 2$.

Step 8: Exercise!

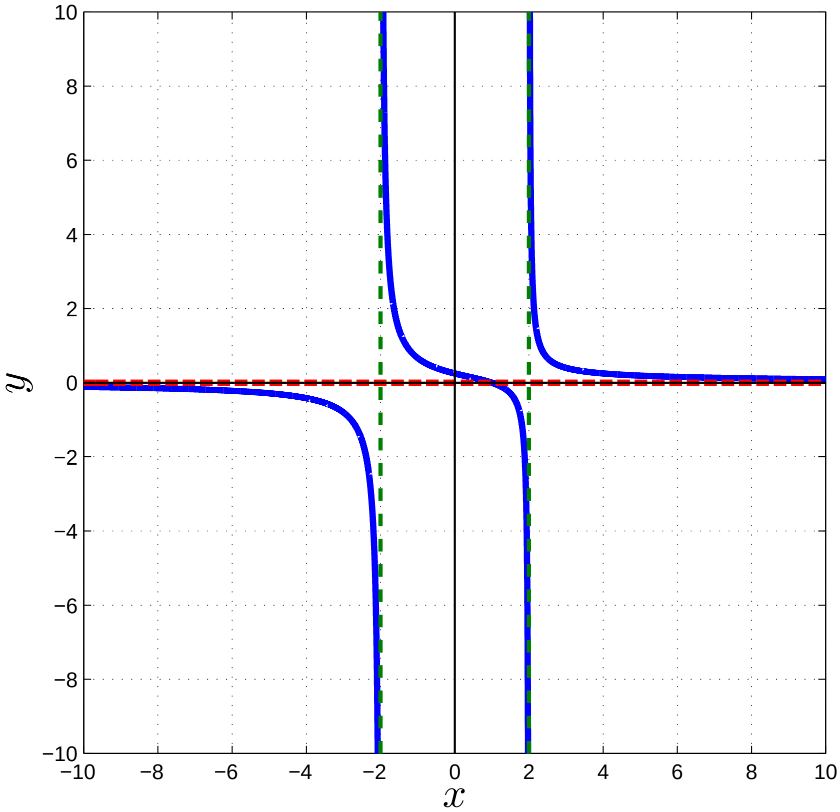

Figure 1:

Exercise 2 Analyze the graph of the rational function:

$$R(x) = \frac{x^2 - 1}{x}.$$

Solution:

Step 1:

$$R(x) = \frac{x^2 - 1}{x} = \frac{(x+1)(x-1)}{x}.$$

The domain of R is

$$\{x| \ x \neq 0\}$$

or

$$(-\infty, 0) \cup (0, +\infty)$$

or

$$\mathbb{R} \setminus \{0\}.$$

Step 2: R is in lowest terms because there are no common factors between the numerator and denominator.

Step 3: The graph has two $x-$intercepts; -1 and 1. There is no $y-$intercept, since x cannot equal 0.

Step 4: Since

$$R(-x) = \frac{(-x)^2 - 1}{(-x)} = -\frac{x^2 - 1}{x} = -R(x),$$

the function is odd and the graph is symmetric with respect to the origin.

Step 5: The real zero of the denominator with R in lowest terms is 0, so the graph of $R(x)$ has the line $x = 0$ (the $y-$axis) as a vertical asymptote.

Step 6: Since the degree of the numerator, 2, is more than the degree of the denominator, 1, the rational function R is improper and will have a slant or an oblique asymptote. To find the oblique asymptote we use the long division:

$$\begin{array}{ll}
 & \phantom{x^2 - {}} x \\
x + 0 & \sqrt{x^2 - 1} \\
 & \underline{\ x^2 \quad 0\ } \\
 & - 1
\end{array}$$

The quotient is x, so the line $y = x$ is an oblique asymptote of the graph. To determine whether the graph of R intersects the asymptote $y = x$, we solve

$$\begin{aligned} R(x) = \frac{x^2 - 1}{x} &= x \\ \frac{x^2 - 1}{x} \cdot x &= x \cdot x \\ x^2 - 1 &= x^2 \\ -1 &\neq 0. \end{aligned}$$

We conclude that the equation $\dfrac{x^2 - 1}{x} = x$ has no solution, so the graph of $R(x)$ does not intersect the line $y = x$.

Step 7: Figure 2 shows the graph of $R(x) = \dfrac{x^2 - 1}{x}$. We clearly observe that

- There is no $y-$intercept and there are two $x-$intercepts, -1 and 1.
- The symmetry with respect to the origin is also evident.
- There is a vertical asymptote at $x = 0$.

Step 8: Exercise!

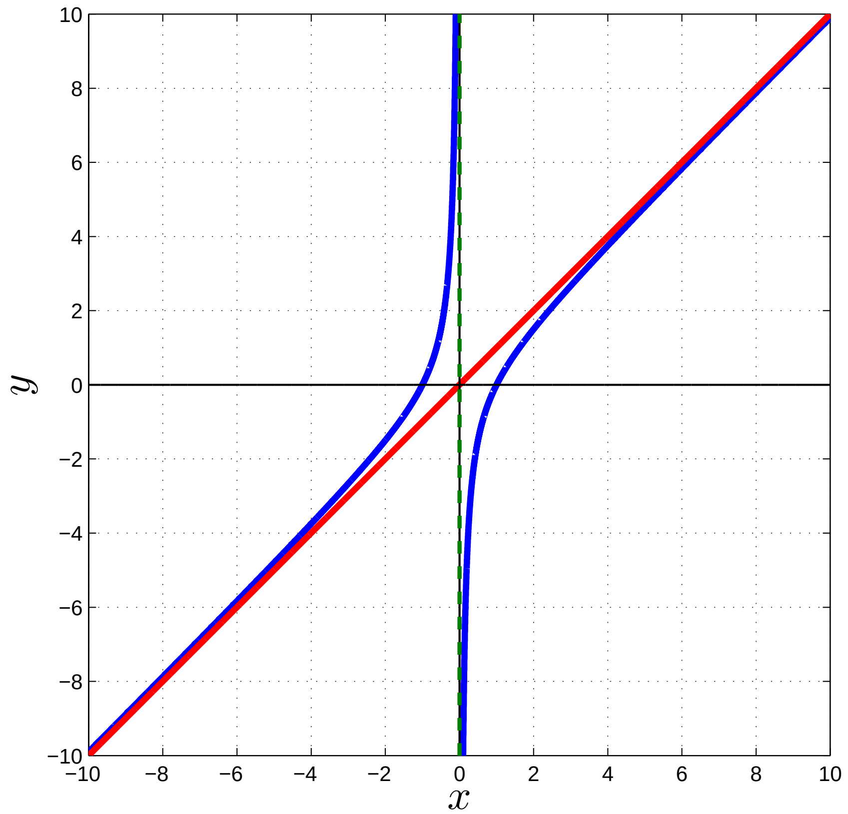

Figure 2:

Exercise 3 Analyze the graph of the rational function:

$$R(x) = \frac{3x^2 - 3x}{x^2 + x - 12}.$$

Solution:

Step 1:

$$R(x) = \frac{3x^2 - 3x}{x^2 + x - 12} = \frac{3x(x-1)}{(x+4)(x-3)}.$$

The domain of R is

$$\{x|\ x \neq -4, x \neq 3\}$$

or

$$(-\infty, -4) \cup (-4, 3) \cup (3, +\infty)$$

or

$$\mathbb{R} \setminus \{-4, 3\}.$$

Step 2: R is in lowest terms because there are no common factors between the numerator and denominator.

Step 3: The graph has two x-intercepts; 0 and 1. Since 0 is in the domain of R, the $y-$intercept is $R(0) = 0$.

Step 4: Because

$$R(-x) = \frac{3(-x)^2 - 3(-x)}{(-x)^2 + (-x) - 12} = \frac{3x^2 + 3x}{x^2 - x - 12}$$

we conclude that R is neither even nor odd. There is no symmetry with respect to the $y-$axis or the origin.

Step 5: We locate the vertical asymptotes by finding the zeros of the denominator with the rational function in lowest terms. With R in lowest terms, we find that the graph of R has two vertical asymptotes: the lines $x = -4$ and $x = 3$.

Step 6: Since the degree of the numerator equals the degree of the denominator the graph has a horizontal asymptote. To find it, we form the quotient of the leading coefficient of the numerator, 3, and the leading coefficient of the denominator, 1. The graph of R has the horizontal asymptote $y = 3$. To find out whether the graph of R intersects the asymptote, we solve the equation $R(x) = 3$.

$$\begin{aligned} R(x) = \frac{3x^2 - 3x}{x^2 + x - 12} &= 3 \\ 3x^2 - 3x &= 3 \cdot (x^2 + x - 12) \\ 3x^2 - 3x &= 3x^2 + 3x - 36 \\ -6x &= -36 \\ x &= 6 \end{aligned}$$

Step 7: Figure 3 shows the graph of $R(x) = \dfrac{3x^2 - 3x}{x^2 + x - 12}$. We clearly observe that the graph of R does not cross the lines $x = -4$ and $x = 3$, since R is not defined at $x = -3$ or $x = 4$.

Step 8: Exercise!

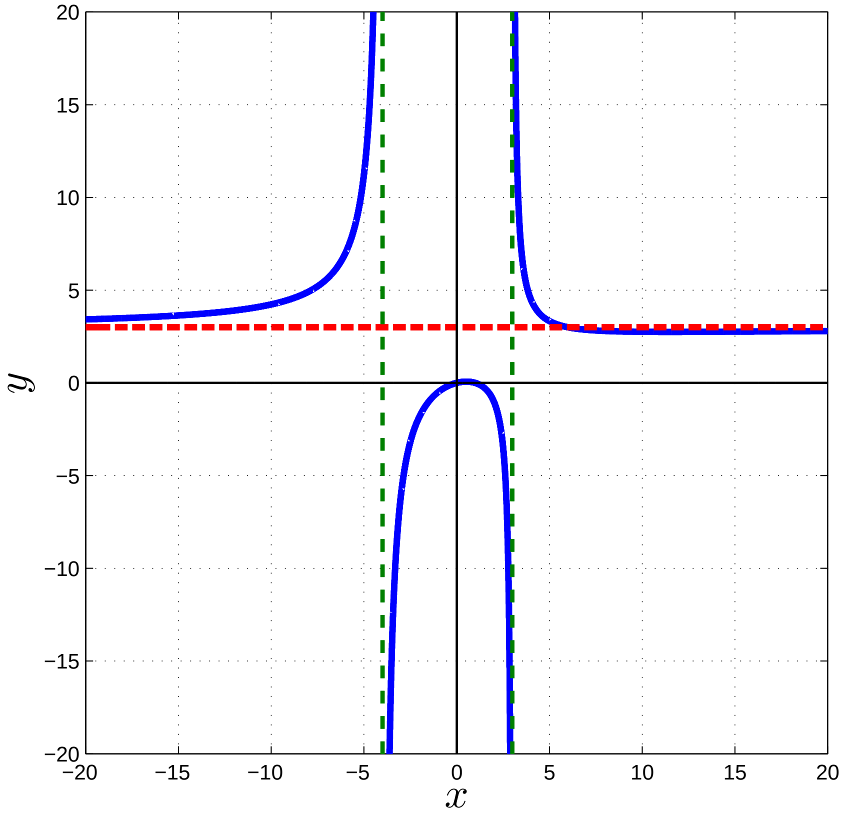

Figure 3:

Exercise 4 Analyze the graph of the rational function:

$$R(x) = \frac{2x^2 - 5x + 2}{x^2 - 4}.$$

Solution:

Step 1:

$$R(x) = \frac{2x^2 - 5x + 2}{x^2 - 4} = \frac{(2x-1)(x-2)}{(x+2)(x-2)}.$$

The domain of R is

$$\{x| \ x \neq -2, x \neq 2\}$$

or

$$(-\infty, -2) \cup (-2, 2) \cup (2, +\infty)$$

or

$$\mathbb{R} \setminus \{-2, 2\}.$$

Step 2: In lowest terms,

$$R(x) = \frac{2x-1}{x+2}, \quad x \neq 2.$$

Step 3: The graph has one x-intercepts; $1/2$. Since 0 is in the domain of R, the $y-$intercept is $R(0) = -1/2$.

Step 4: Because

$$R(-x) = \frac{2(-x)^2 - 5(-x) + 2}{(-x)^2 - 4} = \frac{2x^2 + 5x + 2}{x^2 - 4}.$$

we conclude that R is neither even nor odd. There is no symmetry with respect to the $y-$axis or the origin.

Step 5: Since $x+2$ is the only factor of the denominator of R(x) in lowest terms, the graph has one vertical asymptote, $x = -2$. However, the rational function is undefined at both $x = 2$ and $x = -2$.

Step 6: Since the degree of the numerator equals the degree of the denominator the graph has a horizontal asymptote. To find it, we form the quotient of the leading coefficient of the numerator, 2, and the leading coefficient of the denominator, 1. The graph of R has the horizontal asymptote $y = 2$. To find out whether the graph of R intersects the asymptote, we solve the equation $R(x) = 2$.

$$\begin{aligned} R(x) = \frac{2x-1}{x+2} &= 2 \\ 2x - 1 &= 2 \cdot (x+2) \\ 2x - 1 &= 2x + 4 \\ -1 &\neq 4 \end{aligned}$$

Step 7: Figure4 shows the graph of $R(x) = \dfrac{2x^2 - 5x + 2}{x^2 - 4}$. We clearly observe that the graph has on vertical asymptote at $x = -2$. Also, the function appears to be continuous at $x = 2$.

Step 8: Exercise!

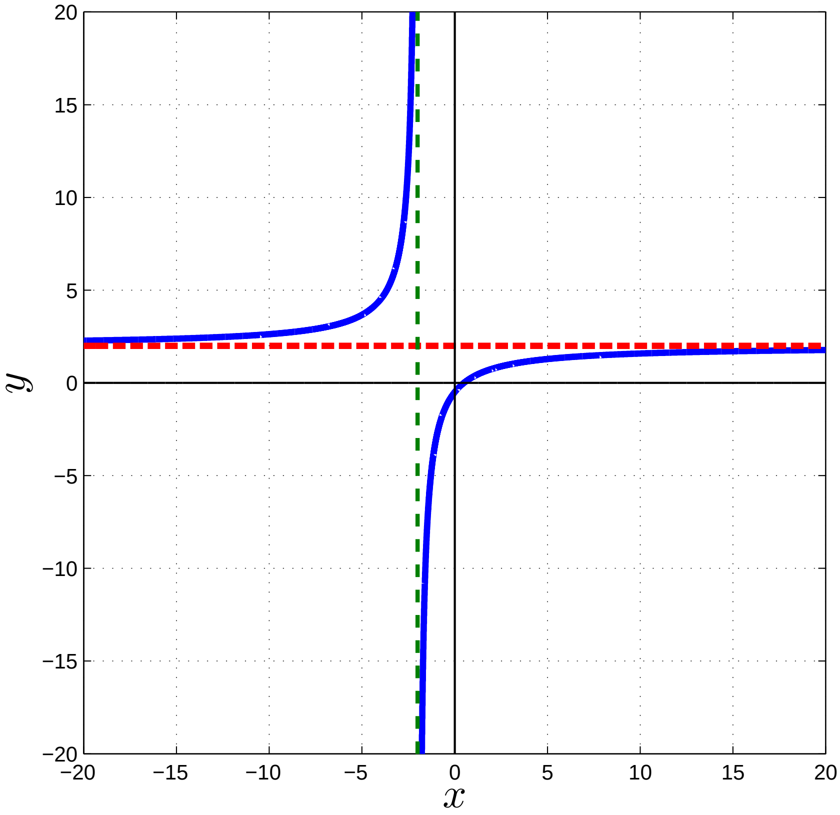

Figure 4: