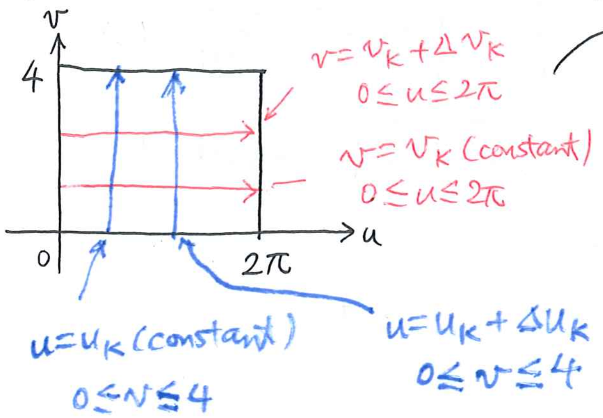
In Example ②, we may represent it as cylindrical coordinates and rectangular coordinates as follows.

Cylindrical: $\text{mass} = \int_0^{2\pi}\int_0^a\int_0^{\sqrt{a^2-r^2}} k\left(2a-\sqrt{r^2+z^2}\right) r\,dz\,dr\,d\theta$

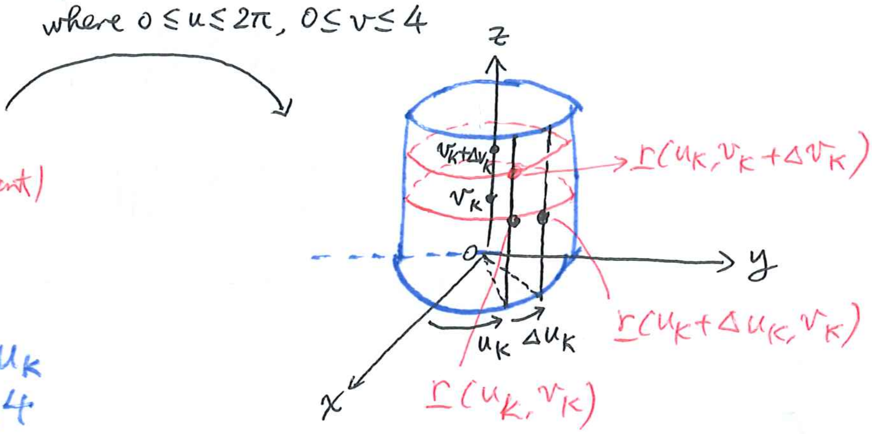
Rectangular: $\text{mass} = 4\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} k\left(2a-\sqrt{x^2+y^2+z^2}\right) dz\,dy\,dx$

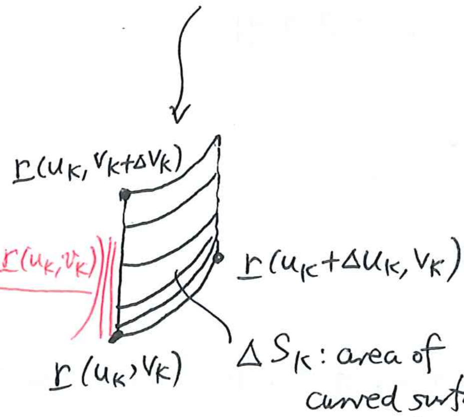
However, these two integrals are very hard to be evaluated by hand.

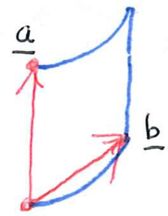
Surface Areas



$\underline{r}(u,v) = \langle \cos u, \sin u, v\rangle$
where $0 \le u \le 2\pi$, $0 \le v \le 4$







ΔS_K: area of curved surface $\approx$ area of parallelogram with sides $\|\underline{a}\|$ and $\|\underline{b}\|$.

$$\Delta S_K \approx \left\| \left(\underline{r}(u_K+\Delta u_K, v_K) - \underline{r}(u_K, v_K)\right) \times \left(\underline{r}(u_K, v_K+\Delta v_K) - \underline{r}(u_K, v_K)\right) \right\|$$

$$\approx \left\| \left(\frac{\underline{r}(u_K+\Delta u_K, v_K) - \underline{r}(u_K, v_K)}{\Delta u_K}\right) \times \left(\frac{\underline{r}(u_K, v_K+\Delta v_K) - \underline{r}(u_K, v_K)}{\Delta v_K}\right) \right\| \cdot \Delta u_K \cdot \Delta v_K$$

$$\approx \left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\| \cdot \Delta u_K \cdot \Delta v_K$$

$$\frac{\partial r}{\partial u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle, \quad \frac{\partial r}{\partial v} = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$$

Total surface area of cylinder $= S = \sum_{K=1}^{n} \Delta S_K \approx \sum_{K=1}^{n} \left\| \left(\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right) \right\| \cdot \Delta u_K \cdot \Delta v_K$

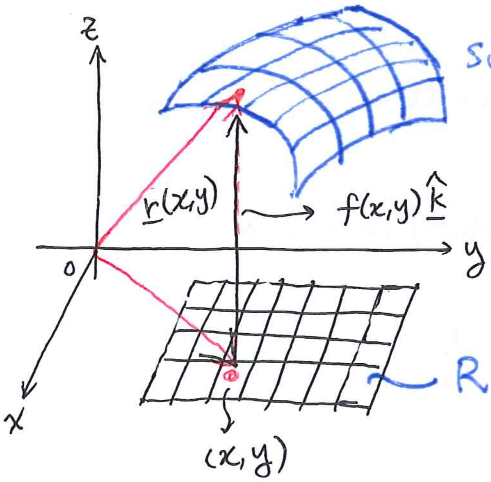
Take $n \to \infty$, $S = \iint_R \left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\| du\,dv$

Consider the example of vertical cylinder above. Desired surface area

$$= \int_0^{2\pi}\int_0^4 \left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\| dv\,du = \int_0^{2\pi}\int_0^4 \left\| \begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin u & \cos u & 0 \\ 0 & 0 & 1 \end{matrix} \right\| dv\,du$$

$$= \int_0^{2\pi}\int_0^4 \left\| \cos u\,\hat{\underline{i}} + \sin u\,\hat{\underline{j}} \right\| dv\,du = \underline{\underline{8\pi}}$$

Finding surface area of a surface (in rectangular coordinates)



Surface $z=f(x,y)$: continuously differentiable function on R.
S

R: planar region lying on the xy-plane.

$$\text{Surface area of } S = \iint_R \sqrt{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+1}\ dA$$

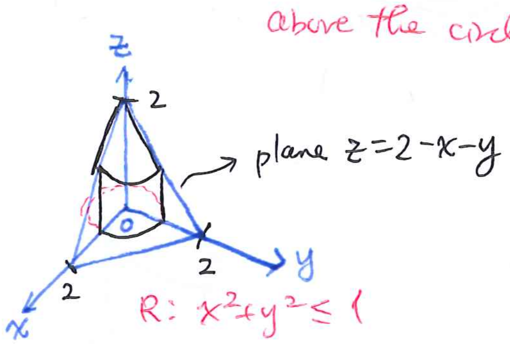
Proof: Let $\underline{r}(x,y)=\langle x,y,f(x,y)\rangle$

$$\frac{\partial \underline{r}}{\partial x}=\left\langle 1,0,\frac{\partial f}{\partial x}\right\rangle \ ; \quad \frac{\partial \underline{r}}{\partial y}=\left\langle 0,1,\frac{\partial f}{\partial y}\right\rangle$$

$$\frac{\partial \underline{r}}{\partial x}\times\frac{\partial \underline{r}}{\partial y}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 1 & 0 & f_x\\ 0 & 1 & f_y\end{vmatrix}=\left\langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right\rangle$$

$$\left\|\frac{\partial \underline{r}}{\partial x}\times\frac{\partial \underline{r}}{\partial y}\right\|=\sqrt{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+1}\ ,$$ result follows.

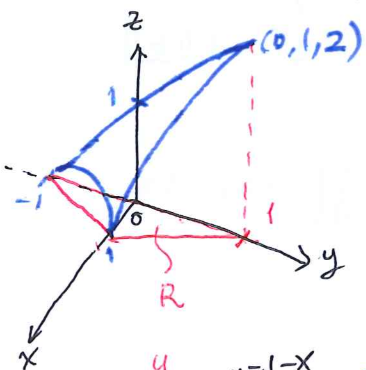
Examples ① Find the surface area of the portion of the plane $z=2-x-y$ that lies above the circle $x^2+y^2\le 1$ in the first quadrant.
↳ i.e. $x\ge 0, y\ge 0$.



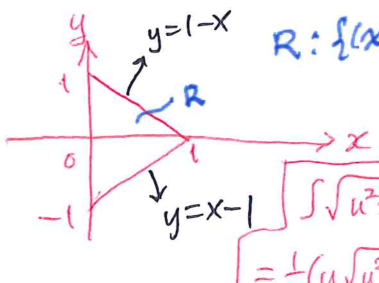
Let $z=f(x,y)=2-x-y$, $f_x(x,y)=-1$, $f_y(x,y)=-1$ are continuous within R.

$$\text{Surface area}=\iint_R\sqrt{1+(-1)^2+(-1)^2}\,dA=\sqrt{3}\iint_R dA$$
$$=\sqrt{3}\left(\frac{1}{4}\pi(1)^2\right)$$
$$=\frac{\sqrt{3}\pi}{4}$$

② Find the area of the portion of the surface $f(x,y)=1-x^2+y$ that lies above the triangular region with vertizes $(1,0,0)$, $(0,-1,0)$ and $(0,1,0)$.



$f(x,y)=1-x^2+y$
$f_x(x,y)=-2x$
$f_y(x,y)=1$
are continuous within R



$R:\{(x,y)\in\mathbb{R}^2 \mid 0\le x\le 1$ and $x-1\le y\le 1-x\}$

Surface area

$$=\int_0^1\int_{x-1}^{1-x}\sqrt{1+1^2+(-2x)^2}\,dy\,dx$$
$$=\int_0^1 y\sqrt{2+4x^2}\,\Big|_{x-1}^{1-x}dx$$

let $x=\frac{1}{\sqrt{2}}\tan\theta$

$$=\int_0^1\left[(1-x)\sqrt{2+4x^2}-(x-1)\sqrt{2+4x^2}\right]dx$$
$$=\int_0^1\left(2\sqrt{2+4x^2}-2x\sqrt{2+4x^2}\right)dx$$
$$=\left[x\sqrt{2+4x^2}+\ln\left(2x+\sqrt{2+4x^2}\right)-\frac{(2+4x^2)^{\frac{3}{2}}}{6}\right]_0^1$$
$$=\sqrt{6}+\ln(2+\sqrt{6})-\sqrt{6}-\ln\sqrt{2}+\frac{1}{3}\sqrt{2}$$
$$=\ln(2+\sqrt{6})-\ln\sqrt{2}+\frac{1}{3}\sqrt{2}$$

$$\boxed{\int\sqrt{u^2\pm a^2}\,du\ (a>0)=\frac{1}{2}\left(u\sqrt{u^2\pm a^2}\pm a^2\ln\left|u+\sqrt{u^2\pm a^2}\right|\right)+C}$$