ENGG 2440A / ESTR 2004: Discrete Mathematics for Engineers — Homework 1

The Chinese University of Hong Kong, Fall 2018 — due Thursday 20 September

Questions 1 to 6 are worth 10 points each. If you are in ENGG 2440A, please turn in solutions to *four* questions of your choice. If you are in ESTR 2004, please turn in solutions to *three* questions of your choice and the Mini-project (worth 30 points).

Write your solutions clearly and concisely. If you do not explain your answer you will be given no credit. You are encouraged to collaborate on the homework, but you must write your own solutions, give credit to your collaborators on your solution sheet, and follow the faculty guidelines and university policies on academic honesty regarding the use of external references.

Questions

1. Which of the following pairs of statements are logically equivalent?

   (a) $P \longrightarrow (\text{NOT } Q)$
   $Q \longrightarrow (\text{NOT } P)$

   (b) If it is raining then the ground is wet.
   If the ground is not wet then it isn't raining.

   (c) If it is raining then the ground is wet.
   It is not raining or the ground is wet.

   (d) Not all three balls are of the same colour. (Colours are black or white.)
   Among the three balls, there is exactly one white ball, or there is exactly one black ball.

2. The following propositions are about a group of tennis players, some of whom win the grand slam title. $W(x)$ means that person x is a tennis player who won a Grand Slam title. $K(x, y)$ means that person x knows that person y is a Gland Slam winner. Translate the following propositions into plain English.

   (a) $\exists x : \text{NOT } K(x, x)$

   (b) $\forall x \exists y : K(x, y) \text{ AND } W(y)$

   (c) $\forall x : (W(x) \text{ AND } x \neq \text{Naomi}) \longrightarrow \forall y : K(x, y)$

   (d) $\exists x \forall y, z : W(x) \text{ AND } K(y, x) \text{ AND } (z \neq x \longrightarrow \text{NOT } K(x, z))$

3. Express the following propositions about people and their relative heights using quantifiers and logical operators. Use x, y, z as variables and $T(x, y)$ for "x is taller than y". Make sure that all your variables are quantified. Explain your answer.

   (a) Bob is neither the tallest person nor the shortest person.

   (b) A person cannot be both taller and shorter than another person.

   (c) There is a shortest person, but there is no tallest person.

   (d) There are at least two people that are taller than Bob.
   (You can use $x \neq y$ for "x and y are different people".)

4. The following propositions are about students and the exams they pass: $E(s, c)$ means "Student s passed the exam in course c", and $P(s)$ means "Student s won a prize".

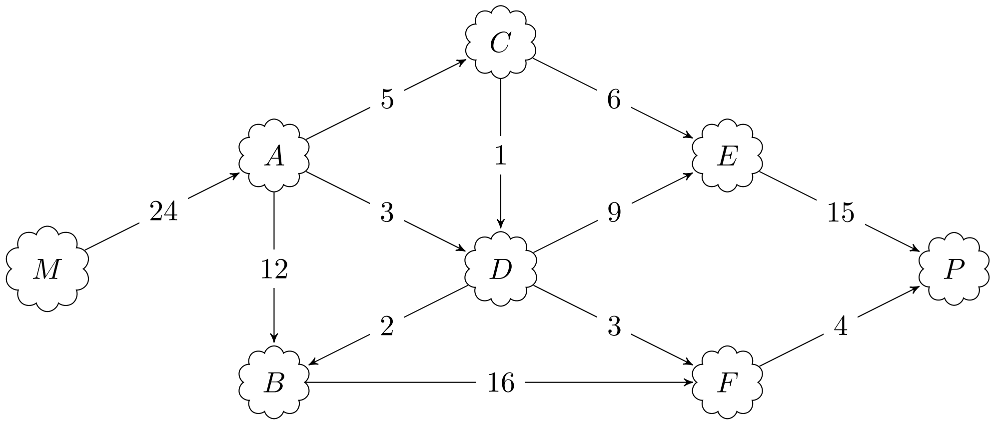
   (a) Explain the meaning of these two propositions in plain English:
   $A\colon \forall c\ \exists s\colon E(s, c)$ AND (NOT $P(s)$)
   $B\colon \exists c\ \forall s\colon E(s, c) \longrightarrow P(s)$

   (b) Can A and B both be true? Justify your answer.

   (c) Explain the meaning of these two propositions in plain English:
   $C\colon \forall c\ \exists s\ \forall d\colon E(s, c)$ AND $((c \neq d) \longrightarrow$ (NOT $E(s, d)$))
   $D\colon \forall c, d\ \exists s\colon E(s, c)$ AND $((c \neq d) \longrightarrow$ (NOT $E(s, d)$))

   (d) Are C and D logically equivalent? Justify your answer.

5. You are organizing the distribution of durians around CUHK. You will hire a porter to get as many as possible from the University MTR station (M) to the Science Park (P). Here is the network of trails that can be used to carry the durians.



   The number along each arrow is the largest number of durians that the porter is willing to move along that trail. The porter may make multiple trips, but is not allowed to move more than this many durians overall. For example, the porter can move a total of 6 but not 7 durians from C to E.

   (a) How should you instruct the porter to move the durians so that as many as possible make it to the Science Park?

   (b) Explain convincingly why no additional durians can reach the Science Park.

6. Express the following predicates about numbers (non-negative integers) using quantified formulas and the symbols 0, 1, =, +, ×, and E. The first five symbols have their usual meaning and $m\mathrm{E}n$ stands for m^n. For example, "$m = \sqrt{n} + 1$" can be expressed as $\exists r\colon m = r + 1$ AND $n = r \times r$.

   (a) i. $m \leq n$ and ii. $m < n$.

   (b) $m \bmod q = r$ meaning "r is the remainder when m is divided by q."

   A (binary) string is a finite sequence of `0`s and `1`s. We will represent the string x by the number whose binary expansion is `1`x. For instance, string `0` is represented by number 2, string `110` is represented by number 14, and the empty string is represented by number 1.

   With this convention, predicates about strings can be expressed as predicated about numbers. For example, the predicate "the last bit of (string) x is `1`" is expressed by $x \bmod 2 = 1$. Express the following predicates:

(c) $\mathrm{bit}(x, m)$ meaning "the m-th bit of string x is `1`".

(d) $\mathrm{len}(x, m)$ meaning "string x has length m".

(e) $z = x \circ y$ meaning "string z is the concatenation of strings x and y".
(Concatenation means z is obtained by typing first x then y, e.g., $\texttt{10011} = \texttt{10} \circ \texttt{011}$.)

(f) **(Extra credit)** $m = 1 \times 2 \times \cdots \times n$.
(**Hint:** Although this question is about *numbers*, strings may come in handy.)

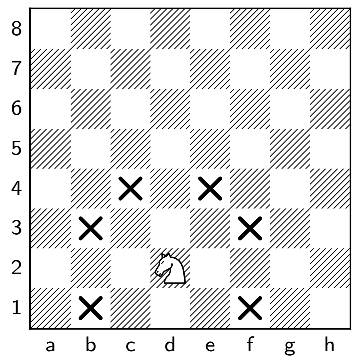
ESTR 2004 mini-project Call a chessboard configuration *good* if it satisfies the following constraints:

i. Each row and each column contains exactly one white knight and exactly one black knight.

ii. No two knights of the same color attack one another.

iii. Each knight attacks at least one knight of the opposite color.

A knight attacks all pieces that are two squares away in one direction and one square away in the other direction, as indicated by the crossed out squares in the following diagram:



(a) Write a propositional formula in conjunctive normal form representing the good configurations. Explain clearly the meaning of your formula.

(b) Use the computer to find good configurations for boards of size 6×6, 8×8, and 12×12 (if any).

(c) For which values of n do good $n \times n$ chessboard configurations exist?

(d) Now replace the constraints ii. and iii. by

ii'. No two knights of opposite colors attack one another.

iii'. Each knight attacks at least one knight of the same color.

and repeat parts (a), (b), and (c).