# Lecture Note 2

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MATH1020
General Mathematics

# Polynomial and Rational Functions

## What will you learn?

- Polynomial functions
- Properties of Rational Functions

**Definition 1** A polynomial function is a function of the form:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0$$

where $a_n, a_{n-1}, \cdots, a_1, a_0$ are real numbers and $n$ is a nonnegative integer. The domain of a polynomial function is the set of all real numbers.

**Definition 2** A power function of degree $n$ is a monomial of the form

$$f(x) = ax^n$$

where $a$ is a real number, $a \neq 0$, and $n > 0$ is an integer.

Properties of power functions, $f(x) = x^n$, $n$ is an even integer.

1. The graph is symmetric with respect to the $y$-axis, so $f$ is even.
2. The domain is the set of all real numbers. The range is the set of nonnegative real numbers.
3. The graph always contains the points $(0,0)$, $(1,1)$ and $(-1,1)$.
4. An exponent $n$ increases in magnitude, the graph becomes more vertical when $x < -1$ or $x > 1$; but for $x$ near the origin, the graph tends to flatten out and lie closer to the $x-$axis.

**Example 1** Graph: $f(x) = \frac{1}{2}(x-1)^4$. Figures 1 - 3 show the required stages.

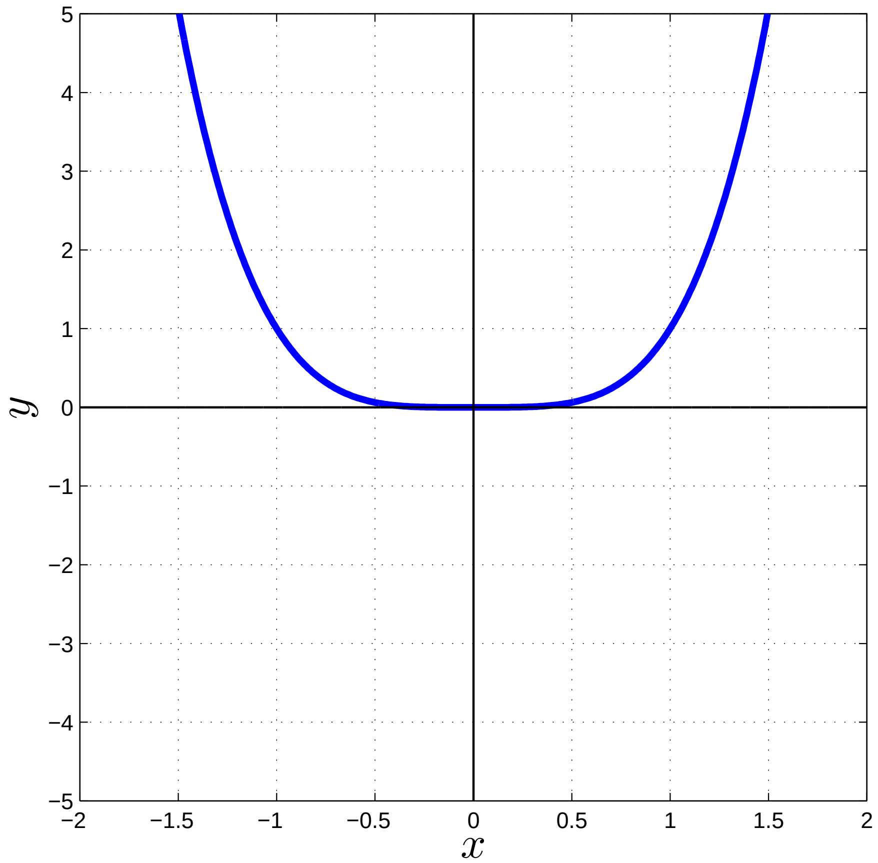



Figure 1: $y = x^4$.

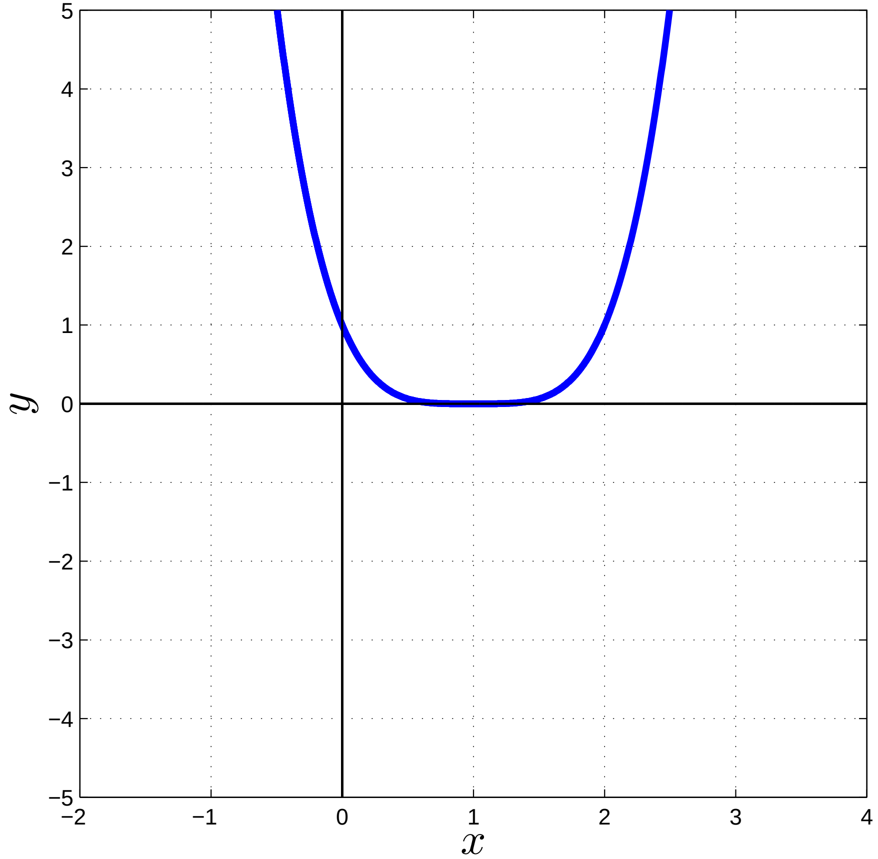



Figure 2: $y = (x-1)^4$.

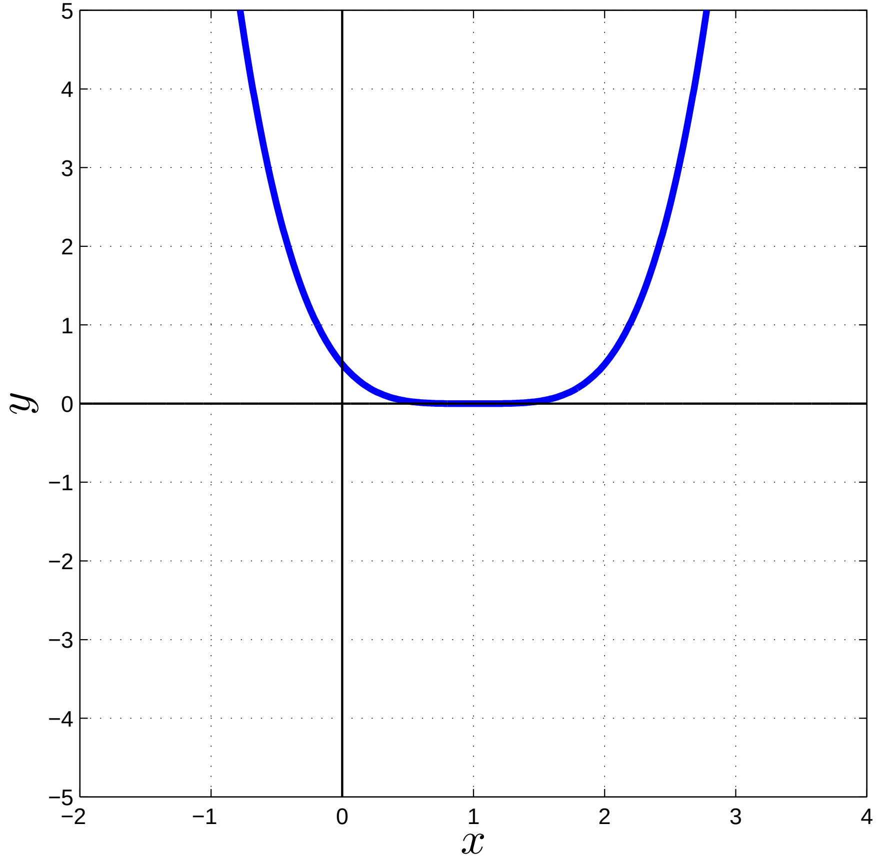



Figure 3: $y = \frac{1}{2}(x-1)^4$.

Properties of power functions, $f(x) = x^n$, $n$ is an odd integer.

1. The graph is symmetric with respect to the origin, so $f$ is odd.
2. The domain and the range are the set of all real numbers. numbers.
3. The graph always contains the points $(0, 0)$, $(1, 1)$ and $(-1, 1)$.
4. An exponent $n$ increases in magnitude, the graph becomes more vertical when $x < -1$ or $x > 1$; but for $x$ near the origin, the graph tends to flatten out and lie closer to the $x-$axis.

**Example 2** Graph: $f(x) = 1 - x^5$. Figures 4 - 6 show the required stages.

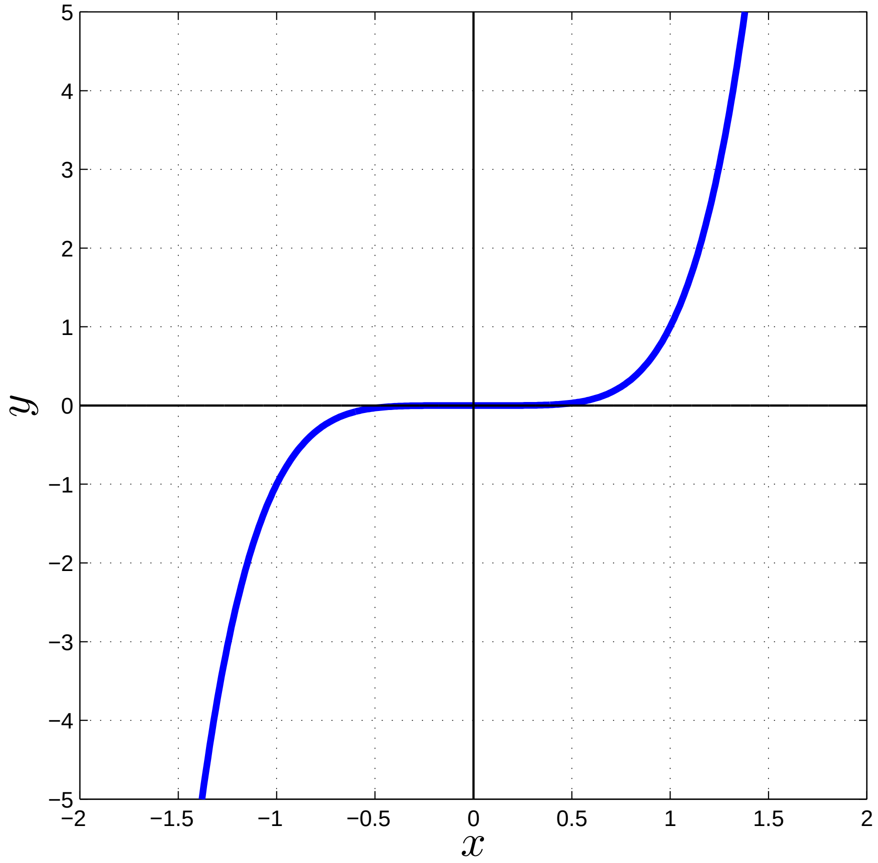



Figure 4: $y = x^5$.

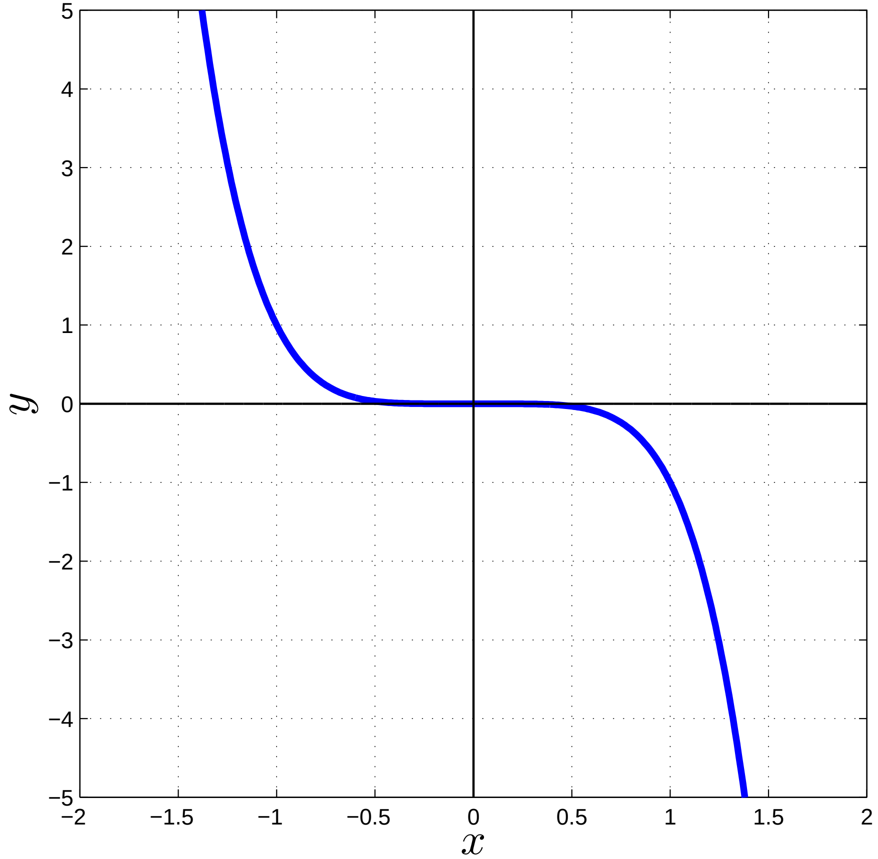



Figure 5: $y = -x^5$.

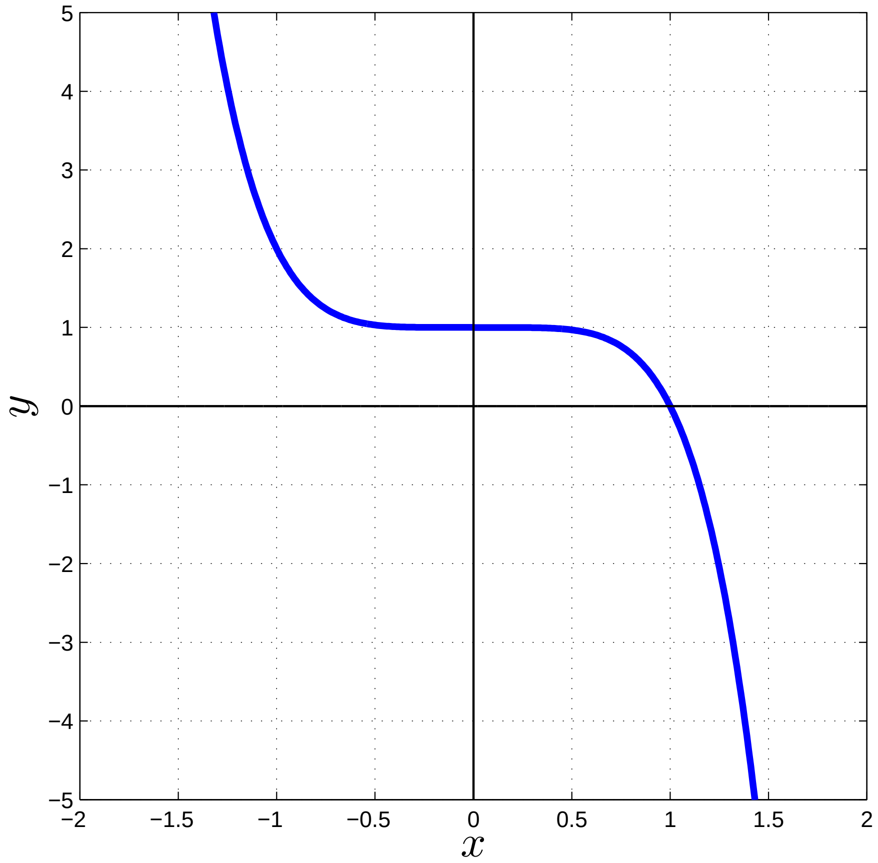



Figure 6: $y = 1 - x^5$.

**Definition 3** If a function and $r$ is a real number for which $f(r) = 0$, then $r$ is called a real zero of $f$.

As a consequence of this definition, the following statements are equivalent:

1. $r$ is a real zero of a polynomial function $f$.
2. $r$ is the $x-$intercept of the graph of $f$.
3. $x - r$ is a factor of $f$.
4. $r$ is a solution to the equation $f(x) = 0$.

So the real zeros of a polynomial functions are the $x-$intercepts of its graph, and they are found by solving the equation $f(x) = 0$.

**Example 3** Find a polynomial from its zeros.

1. Find a polynomial of degree $3$ whose zeros are $-3,\ 2$ and $5$.
2. Using a graphing utility to graph the polynomial found in part (a) to verify your result (see Figure 7).

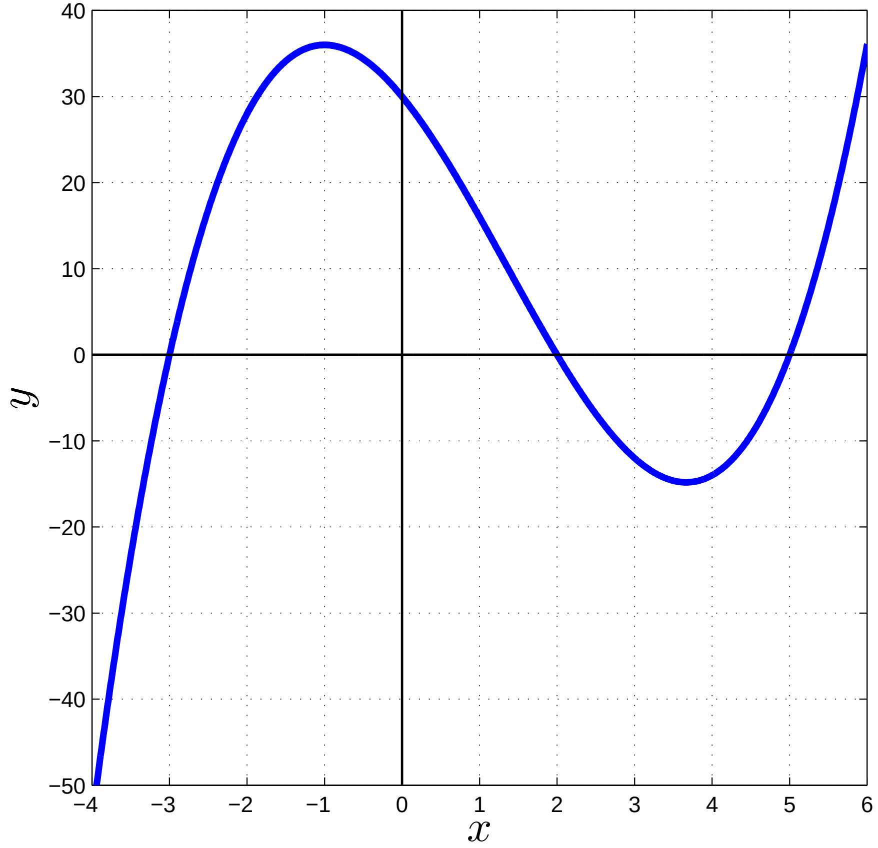



Figure 7: Graph of $y = x^3 - 4x^2 - 11x + 30$.

If the same factor $x - r$ occurs more than once, $r$ is called a repeated, or multiple, zero of $f$. More precisely, we have the following definition.

**Definition 4** If $(x - r)^m$ is a factor of a polynomial $f$ and $(x - r)^{m+1}$ is not a factor of $f$, then $r$ is called **a zero of multiplicity** $m$ of $f$.

Note that $m \geq 1$ is an integer.

**Example 4** Identifying zeros and their multiplicities:

$$f(x) = 5(x-2)(x+3)^2\left(x-\frac{1}{2}\right)^4.$$

## Investigating the role of multiplicity

For the polynomial

$$f(x) = x^2(x - 2).$$

1. Find the $x-$ and $y-$intercepts of the graph of $f$.
2. Using a graphing utility, graph the polynomial. (see Figure 8)
3. For each $x-$intercept, determine whether it is of odd or even multiplicity.

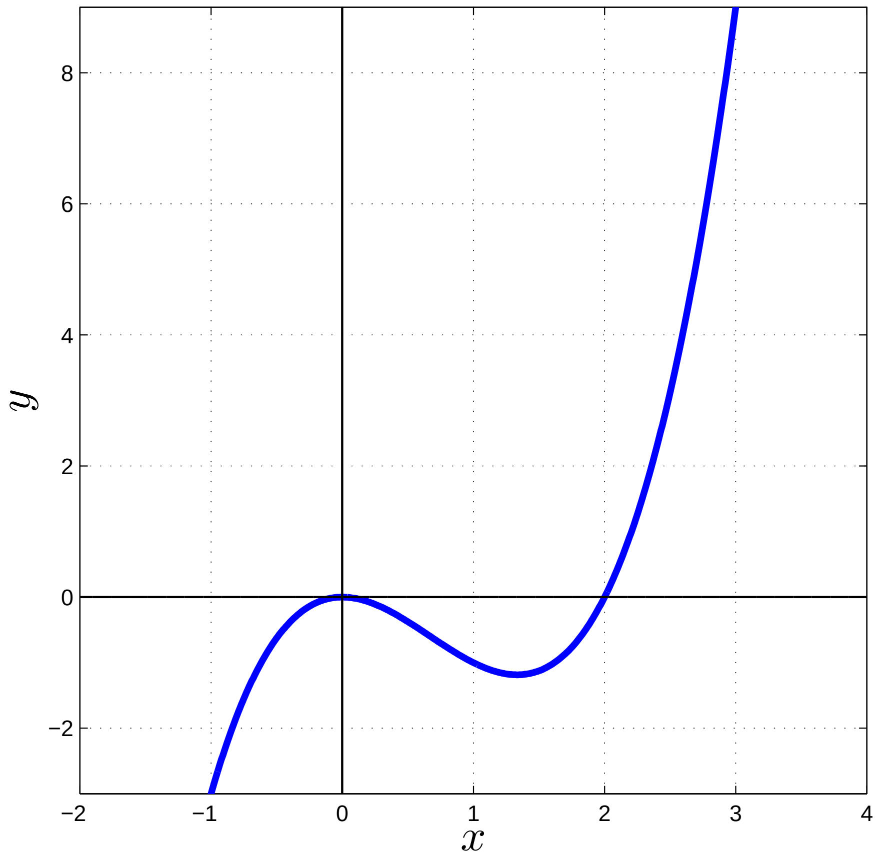



Figure 8: Graph of $y = x^2(x-2)$.

Points on the graph where the graph changes from an increasing function to a decreasing function, vice versa, are called **a turning point.**

**Theorem 1 Turning Points**

If $f$ is a polynomial of degree $n$, the $f$ has at most $n - 1$ turning points.

If the graph of a polynomial function $f$ has $n - 1$ turning points, the degree of $f$ is at least $n$.

For very large values of $x$, either positive or negative, the graph of $f(x) = x^2(x-2)$ looks like the graph of $y = x^3$. To see why, we write $f$ in the form

$$f(x) = x^2(x-2) = x^3 - 2x^2 = x^3\left(1 - \frac{2}{x}\right).$$

Now, for large values of $x$, either positive or negative, the term $\dfrac{2}{x}$ is close to zero (or approaches zero), so for large values of $x$ :

$$f(x) = x^3 - 2x^2 = x^3\left(1 - \frac{2}{x}\right) \approx x^3.$$

The behaviour of the graph of a function for large values of $x$, either positive or negative, is referred to as its end behaviour. The end behaviour of $f(x) = x^2(x-2)$ is $y = x^3$.

Note that

$$\lim_{x \to +\infty} \frac{2}{x} = +\infty.$$

**Theorem 2 End Behaviour**

For large values of $x$, either positive or negative, the graph of the polynomial

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,\ a_n \neq 0$$

resembles the graph of the power function.

$$y = a_n x^n.$$

Graph of a polynomial function

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,\ a_n \neq 0.$$

1. Degree of the polynomial $f$: $n$
2. Maximum number of turning points: $n-1$
3. At a zero of even multiplicity: The graph of $f$ touches the $x-$axis.
4. At a zero of odd multiplicity: The graph of $f$ crosses the $x-$axis.
5. Between zeros, the graph of $f$ is either above or below the $x-$axis.
6. End behaviour: For large $|x|$, the graph of f behaves like the graph of $y = a_n x^n$.

Analyzing the graph of a polynomial function:

**Step 1:** Determine the end behavior of the graph of the function.

**Step 2:** Find the $x-$ and $y-$intercepts of the graph of the function.

**Step 3:** Determine the zeros of the function and their multiplicity. Use this information to determine whether the graph crosses or touches the $x-$axis at each $x-$intercept.

**Step 4:** Using a graphing utility to graph the graph.

**Step 5:** Approximate the turning points of the graph.

**Step 6:** Use the information in Steps 1 through 5 to draw a complete graph of the function by hand.

**Step 7:** Find the domain and the range of the function.

**Step 8:** Use the graph to determine where the function is increasing and where it is decreasing.

**Example 5** Analyze the graph of the polynomial function:

$$f(x) = (2x+1)(x-3)^2.$$

See Figure 11.

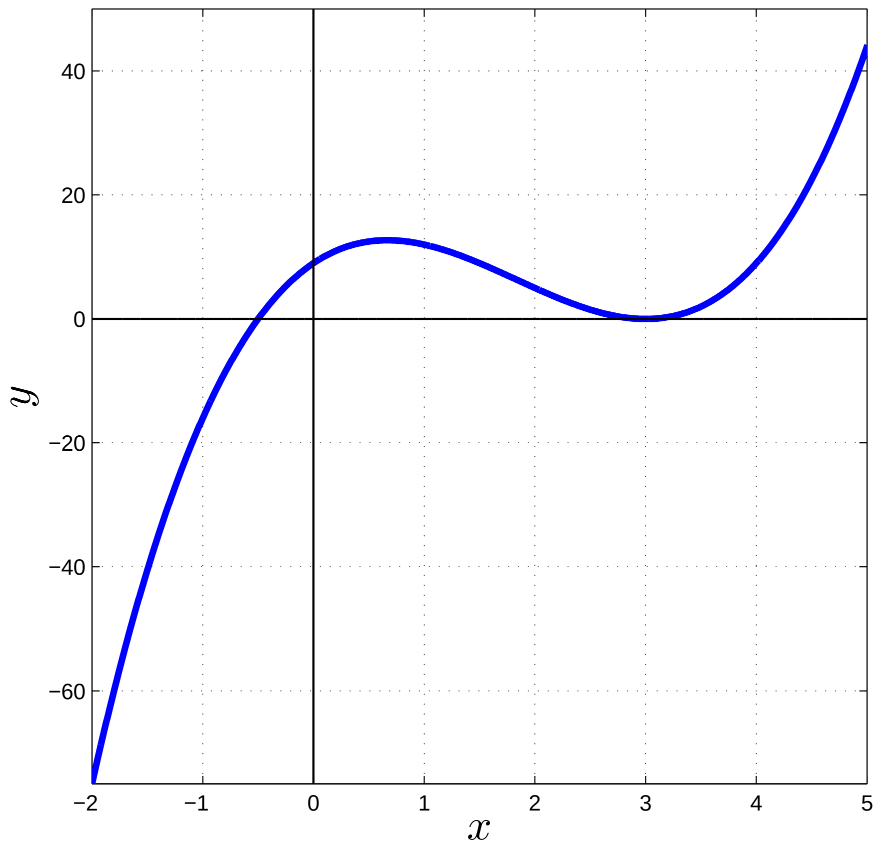



Figure 9: Graph of $y = (2x+1)(x-3)^2$.

**Remark 1**

| | |
|---|---|
| If $r$ is a zero of even multiplicity<br>Sign of $f(x)$ does not change from one side<br>of $r$ to the other side of $r$. | Graph touches<br>$x-$axis at $r$ |
| If $r$ is a zero of odd multiplicity<br>Sign of $f(x)$ changes from one side<br>of $r$ to the other side of $r$. | Graph crosses<br>$x-$axis at $r$ |

Ratios of integers are called **rational numbers**.

Similarly, ratios of polynomial functions are called **rational functions.**

**Definition 5** A rational function is a function of the form:

$$R(x) = \frac{p(x)}{q(x)}$$

where $p$ and $q$ are polynomial functions and $q$ is not the zero polynomial. The domain of a rational function is the set of all real numbers except those for which the denominator $q$ is $0$.

**Example 6** Find the domain of the following rational functions:

1. $R(x) = \dfrac{2x^2 - 4}{x + 5}$;
2. $R(x) = \dfrac{1}{x^2 - 4}$;
3. $R(x) = \dfrac{x^3}{x^2 + 1}$;
4. $R(x) = \dfrac{-x^2 + 2}{3}$;
5. $R(x) = \dfrac{x^2 - 1}{x - 1}$.

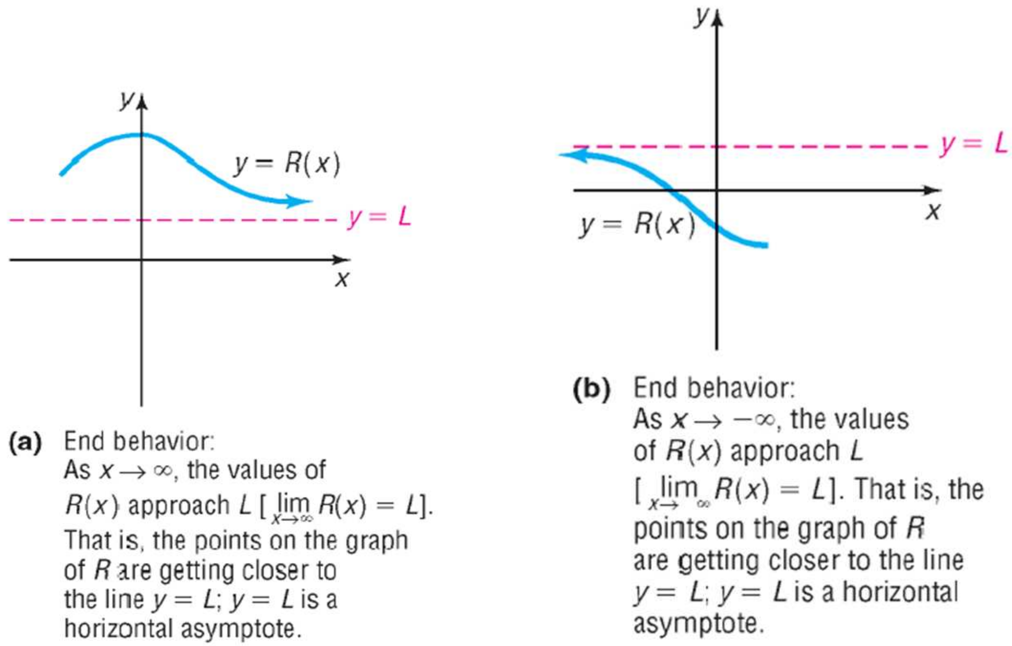
## Asymptotes

Let $R$ denote a function:

1. If, as $x \to -\infty$ or as $x \to +\infty$, the values of $R(x)$ approach some fixed number $L$, then the line $y = L$ is a horizontal asymptote of the graph of $R$.

2. If, as $x$ approaches some number $c$, the values $|R(x)| \to +\infty$, then the line $x = c$ is a vertical asymptote of the graph $R$. The graph of $R$ never intersects a vertical asymptote.

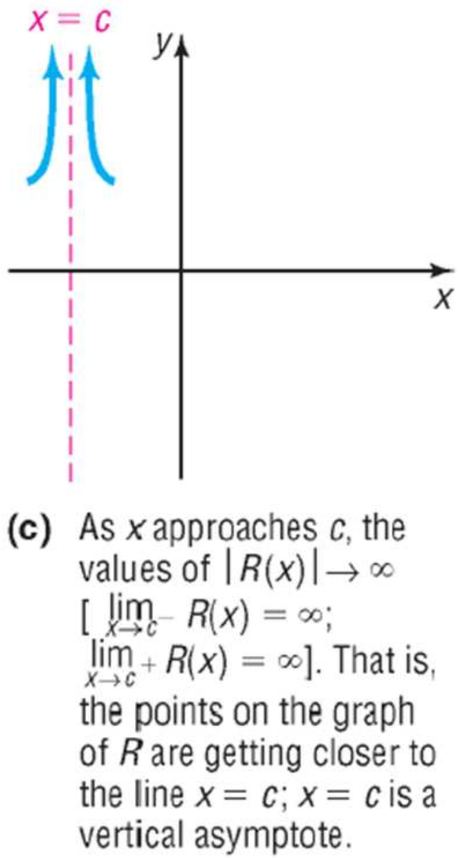
A horizontal asymptote, when it occurs, describes the end behaviour of the graph as $x \to +\infty$ or as $x \to -\infty$. The graph of a function may intersect a horizontal asymptote. See Figure 10.



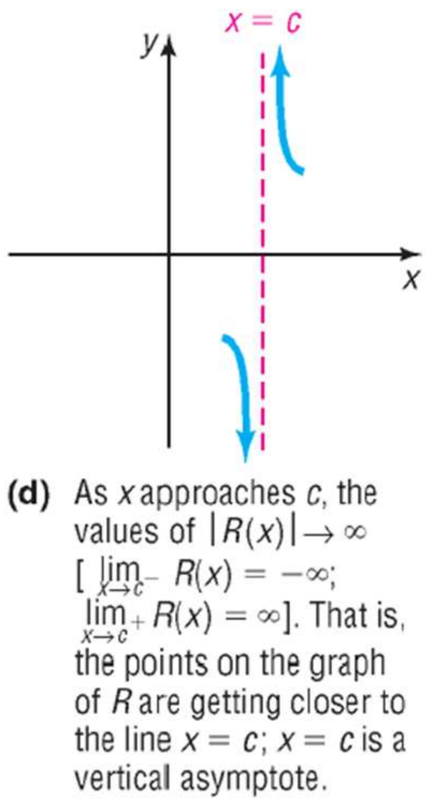

(a) (b)

Figure 10: Horizontal asymptote

A vertical asymptote, when it occurs, describes the end behavior of the graph when $x$ is closes to some number $c$. The graph of a function will never intersect a vertical asymptote. See Figure 11.




(a)




(b)

Figure 11: Vertical asymptote

There is a third possibility. If, as $x \to -\infty$ or as $x \to +\infty$, the value of a function $R(x)$ approaches a linear expression $ax + b,\ a \neq 0$, then the line $y = ax + b,\ a \neq 0$, is an oblique asymptote (or a slant asymptote) of $R$. Figure 12 shows an oblique asymptote. An oblique asymptote, when it occurs, describes the end behaviour of the graph. The graph of a function will never intersect a oblique asymptote.

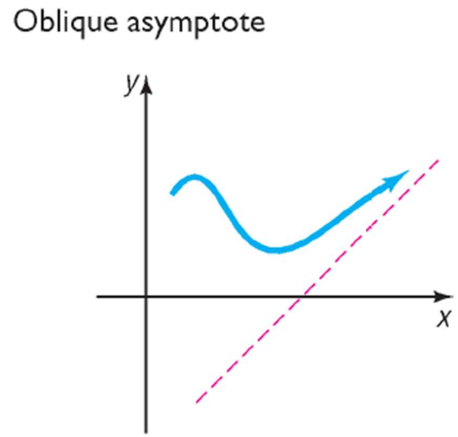



Figure 12: Oblique asymptote

## Locating Vertical Asymptotes

**Theorem 3** A rational function $R(x) = \dfrac{p(x)}{q(x)}$, in lowest terms, will have a vertical asymptote $x = r$ if $r$ is a real zero of the denominator $q$.

That is, if $x - r$ is a factor of the denominator $q$ of a rational function $R(x) = \dfrac{p(x)}{q(x)}$, in lowest terms, $R$ will have the vertical asymptote $x - r$.

**Example 7** Find vertical asymptotes:

Find the vertical asymptotes, if any, of the graph of each rational function.

1. $R(x) = \dfrac{x}{x^2 - 4}$;
2. $R(x) = \dfrac{x+3}{x-1}$;
3. $R(x) = \dfrac{x^2}{x^2+1}$;
4. $R(x) = \dfrac{x^2 - 9}{x^2 + 4x - 21}$.

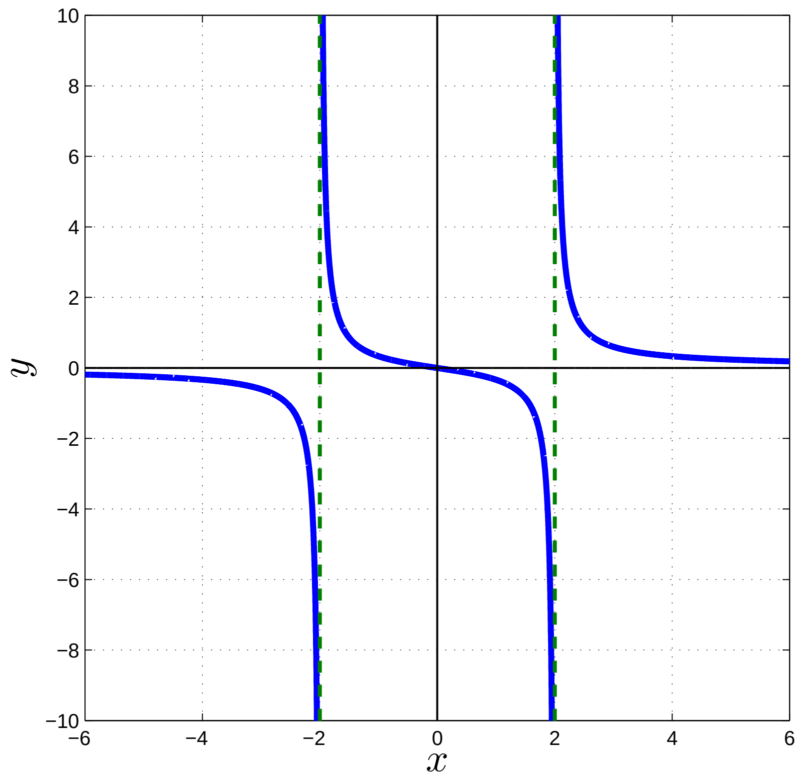



Figure 13: Example 7: $R(x) = \dfrac{x}{x^2 - 4}$.

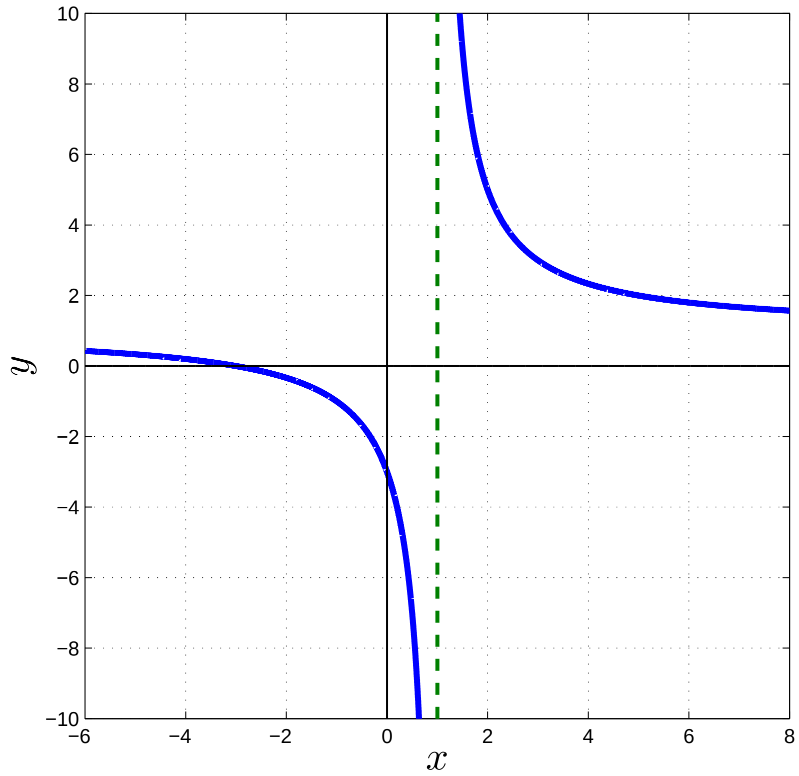



Figure 14: Example 7: $R(x) = \dfrac{x+3}{x-1}$.

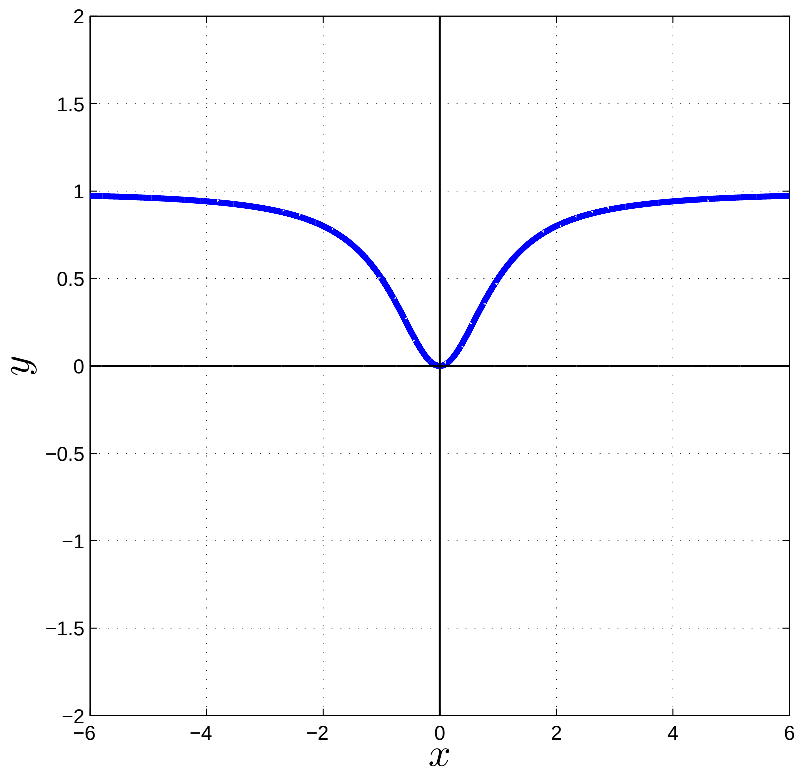



Figure 15: Example 7: $R(x) = \dfrac{x^2}{x^2+1}$.

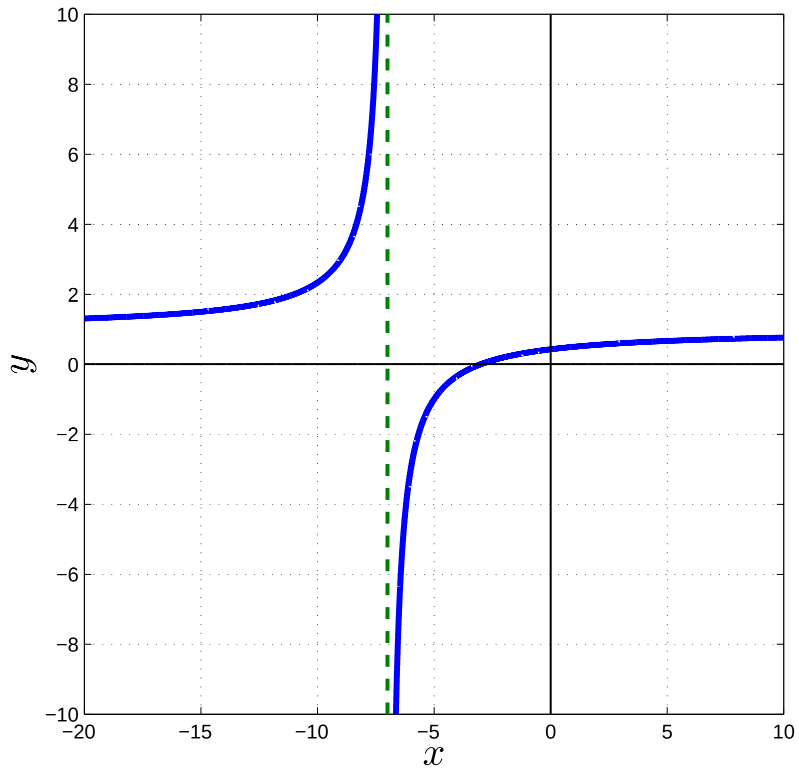



Figure 16: Example 7: $R(x) = \dfrac{x^2 - 9}{x^2 + 4x - 21}$.

## Find the Horizontal Asymptotes of a Rational Function

**Theorem 4** If a rational function is proper, the line $y = 0$ is a horizonal asymptote of its graph.

**Example 8** Find the horizontal asymptotes, if any, of the graph of

$$R(x) = \frac{x - 12}{4x^2 + x + 1}.$$

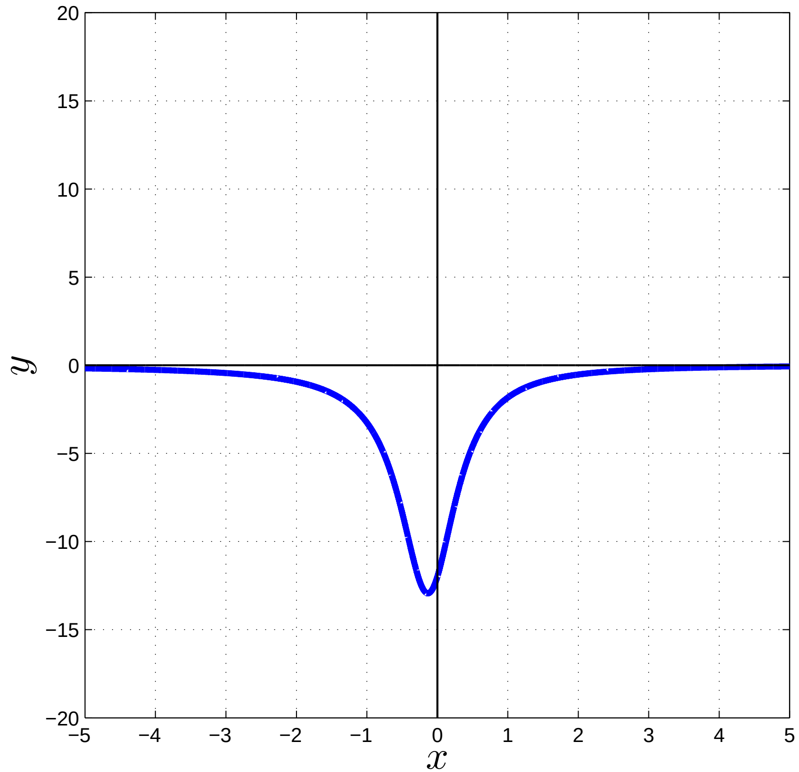



Figure 17: Example 8.

**Example 9** Find the horizontal or oblique asymptotes, if any, of the graph of

$$R(x) = \frac{8x^2 - x + 2}{4x^2 - 1}.$$

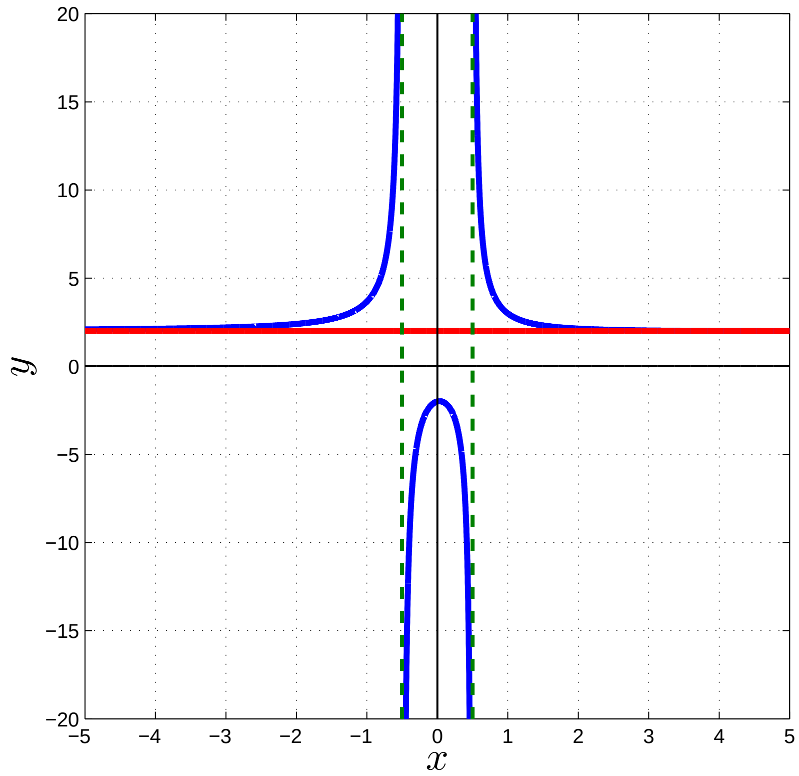



Figure 18: Example 9.

**Example 10** Find the horizontal or oblique asymptotes, if any, of the graph of

$$R(x) = \frac{3x^4 - x^2}{x^3 - x^2 + 1}.$$

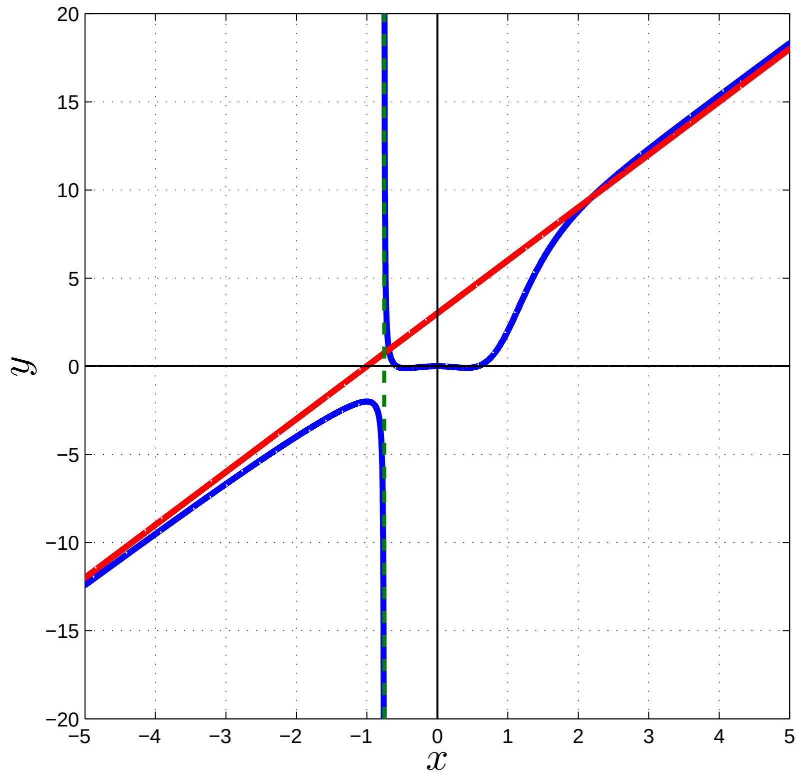



Figure 19: Example 10.

**Example 11** Find the horizontal or oblique asymptotes, if any, of the graph of

$$R(x) = \frac{2x^5 - x^3 + 2}{x^3 - 1}.$$

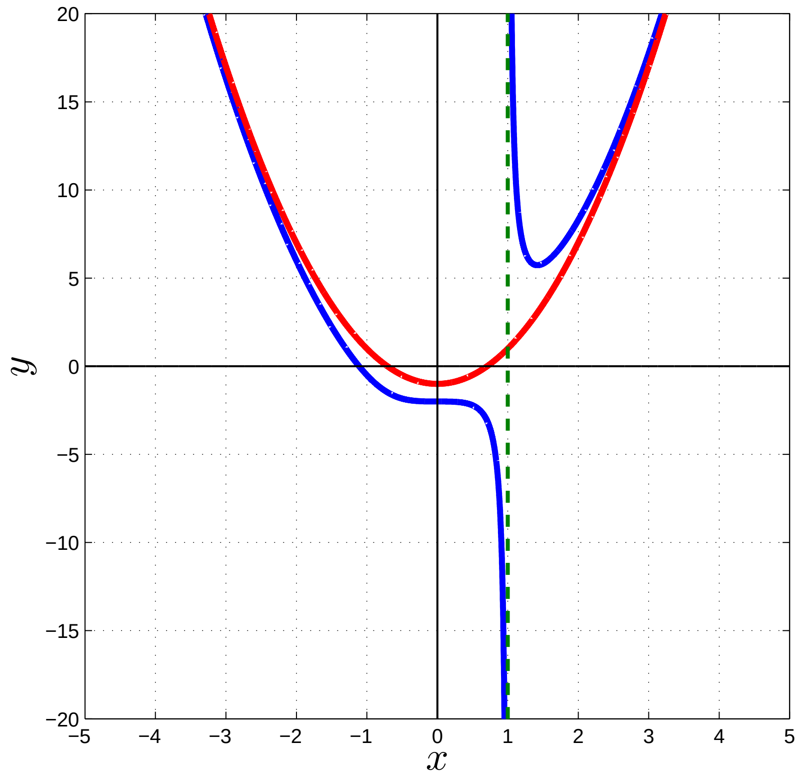



Figure 20: Example 11.

## Find Horizontal and Oblique Asymptotes of a rational function $R$:

Consider the rational function:

$$R(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x^1 + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x^1 + b_0}$$

in which the degree of the numerator is $n$ and the degree of the denominator is $m$.

1. If $n < m$ (the degree of the numerator is less than the degree of the denominator), then $R$ is a proper rational function, and the graph of $R$ will have the horizontal asymptote $y = 0$ (the $x-$axis).

2. If $n \geq m$ (the degree of the numerator is greater than the degree of the denominator), then $R$ is an improper rational function. Here long division is used.

(a) If $n = m$ (the degree of the numerator equals the degree of the denominator), the quotient obtained will be the number $\dfrac{a_n}{a_m}$, and the line $y = \dfrac{a_n}{a_m}$ is a horizontal asymptote.

(b) If $n = m + 1$ (the degree of the numerator is one more than the degree of the denominator), the quotient obtained is of the form $ax + b$ (a polynomial of degree $1$), and the line $y = ax + b$ is an oblique asymptote.

(c) If $n \geq m$ (the degree of the numerator is two or more greater than the degree of the denominator), the quotient obtained is a polynomial of degree $2$ or higher, and $R$ has neither a horizontal nor an oblique asymptote. In this case, for $|x|$ unbounded, the graph of $R$ will behavior like the graph of the quotient.

## Analyzing the graph of a rational function:

**Step 1:** Factor the numerator and denominator of a rational function $R$. Find the domain of the rational function.

**Step 2:** Write $R$ in lower terms.

**Step 3:** Locate the intercepts of the graph. Then $x-$intercepts, if any, of

$$R(x) = \frac{p(x)}{q(x)}$$

in lowest terms satisfy the equation $p(x) = 0$. The $y-$intercept, if there is one, is $R(0)$.

**Step 4:** Test for symmetry. Replace $x$ by $-x$ in $R(x)$. If $R(-x) = R(x)$, there is symmetry with respect to the $y-$axis; if $R(-x) = -R(x)$, there is symmetry with respect to the origin.

**Step 5:** Locate the vertical asymptotes. The vertical asymptotes, if any, of

$$R(x) = \frac{p(x)}{q(x)}$$

in lowest terms are found by identifying the real zeros of $q(x)$. Each zero of the denominator gives rise to a vertical asymptote.

**Step 6:** Locate the horizontal or oblique asymptotes, if any, using the procedure given in Section. Determine points, if any, at which the graph of $R$ intersects these asymptotes.

**Step 7:** Graph $R$ using a graphing utility.

**Step 8:** Using the results obtained in Steps 1 through 7 to graph $R$ by hand.

**Exercises 1** Analyze the graph of the rational function:

$$R(x) = \frac{x-1}{x^2-4}.$$

**Exercises 2** Analyze the graph of the rational function:

$$R(x) = \frac{x^2 - 1}{x}.$$

**Exercises 3** Analyze the graph of the rational function:

$$R(x) = \frac{3x^2 - 3x}{x^2 + x - 12}.$$

**Exercises 4** Analyze the graph of the rational function:

$$R(x) = \frac{2x^2 - 5x + 2}{x^2 - 4}.$$