THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1020
Exercise 6
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Exercise 1 Relating Logarithms to Exponents

(a) If $y = \log_3 x$, then $x = 3^y$. For example, $2 = \log_3 9$ is equivalent to $9 = 3^2$.

(b) If $y = \log_7 x$, then $x = 7^y$. For example, $-1 = \log_7 \left(\frac{1}{7}\right)$ is equivalent to $\frac{1}{7} = 7^{-1}$.

Exercise 2 Changing Exponential Statements to Logarithmic Statements

Change each exponential statement to an equivalent statement involving a logarithm.

(a) $1.4^3 = k$ (b) $e^m = 9$ (c) $a^4 = 25$

Solution: We use the fact that $y = \log_a x$ and $x = a^y$, $a > 0, a \neq 1$, are equivalent.

(a) If $1.4^3 = k$, then $3 = \log_{1.4} k$.

(b) If $e^m = 9$, then $m = \log_e 9$.

(c) If $a^4 = 25$, then $4 = \log_a 25$.

Exercise 3 Changing Logarithmic Statements to Exponential Statements

Change each logarithmic statement to an equivalent statement involving an exponent.

(a) $\log_a 4 = 5$ (b) $\log_e b = -3$ (c) $\log_3 5 = c$

Solution:

(a) If $\log_a 4 = 5$, then $a^5 = 4$.

(b) If $\log_e b = -3$, then $e^{-3} = b$.

(c) If $\log_3 5 = c$, then $3^c = 5$.

Exercise 4 Finding the Exact Value of a Logarithmic Expression

Find the exact value of:

(a) $\log_2 16$ (b) $\log_3 \left(\dfrac{1}{27}\right)$

Solution:

(a) $y = \log_2 16$

$$\begin{aligned} 2^y &= 16 && \text{Change to exponential form} \\ 2^y &= 2^4 && 16 = 2^4 \\ y &= 4 && \text{Equate exponents} \end{aligned}$$

Therefore, $\log_2 16 = 4$.

(b) $y = \log_3 \dfrac{1}{27}$

$$\begin{aligned} 3^y &= \frac{1}{27} && \text{Change to exponential form} \\ 3^y &= 3^{-3} && \frac{1}{27} = \frac{1}{3^3} = 3^{-3} \\ y &= -3 && \text{Equate exponents} \end{aligned}$$

Therefore, $\log_3 \left(\dfrac{1}{27}\right) = -3$.

Exercise 5 Finding the Domain of a Logarithmic Function

Find the domain of each logarithmic function.

(a) $F(x) = \log_2(x+3)$ (b) $g(x) = \log_5\left(\dfrac{1+x}{1-x}\right)$ (c) $h(x) = \log_{1/2}|x|$

Solution:

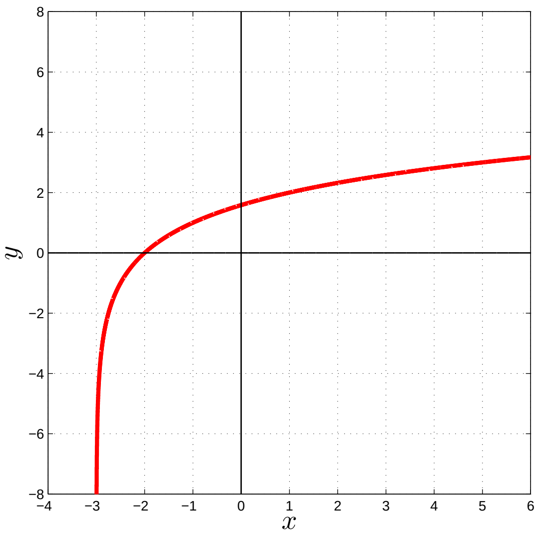
(a) The domain of F consists of all x for which $x+3>0$, that is, $x>-3$. Using interval notion, the domain of f is $(-3,\infty)$, as shown in Figure 3(a).

(b) The domain of g is restricted to

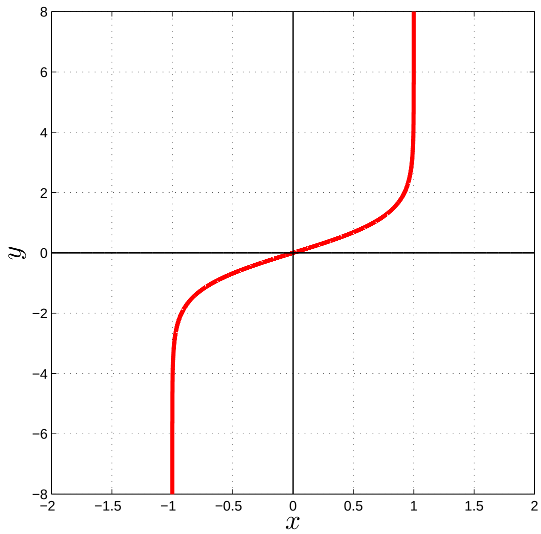
$$\frac{1+x}{1-x} > 0.$$

Solving this inequality, we find that the domain of g consists of all x between -1 and 1, that is, $-1<x<1$ or, using interval notation, $(-1,1)$, as shown in Figure 3(b).

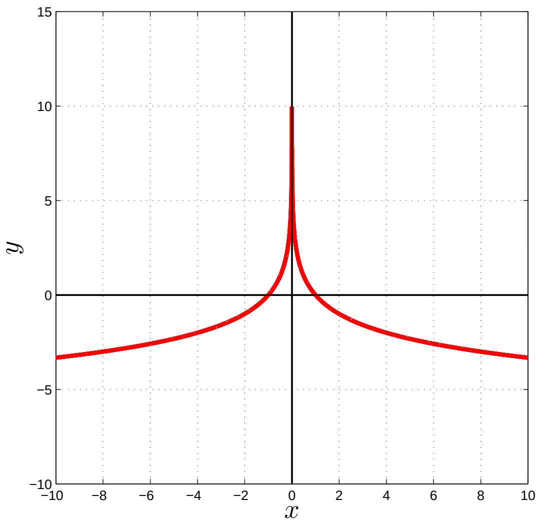
(c) Since $|x|>0$, provided that $x \neq 0$, the domain of h consists of all real numbers except zero or, using interval notation, $(-\infty,0)\cup(0,\infty)$, as shown in Figure 3(c).



(a) $y = \log_2(x+3)$



(b) $y = \log_5\left(\dfrac{1+x}{1-x}\right)$



(c) $y = \log_{1/2}|x|$

Figure 1:

Exercise 6 Graphing a Logarithmic Function and Its Inverse

(a) Find the domain of the logarithmic function $f(x) = -\ln(x-2)$.

(b) Graph f.

(c) From the graph, determine the range and vertical asymptote of f.

(d) Find f^{-1}, the inverse of f.

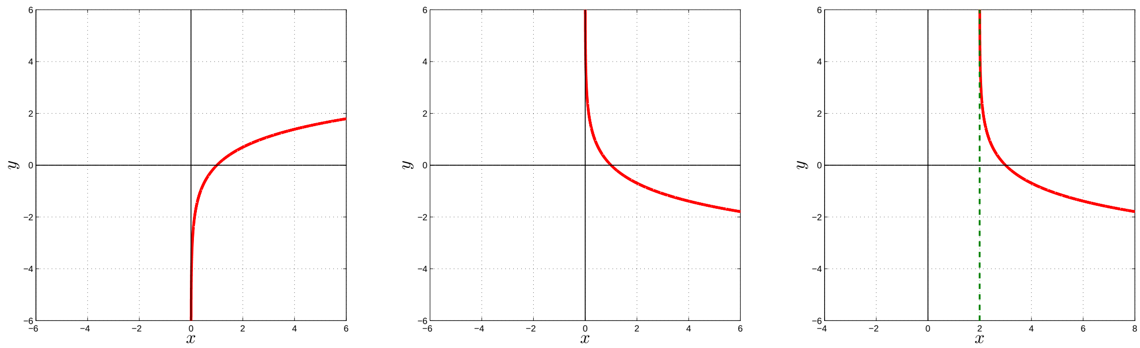
(e) Use f^{-1} to confirm the range of f found in part (c). From the domain of f, find the range of f^{-1}.

(f) Graph f^{-1}.

Solution:

(a) The domain of f consists of all x for which $x - 2 > 0$ or, equivalently,$x > 2$. The domain of f is $\{x|\ x > 2\}$ or $(2, \infty)$.

(b) To obtain the graph of $y = -\ln(x-2)$, we begin with the graph of $y = \ln x$ and use transformations. See Figure 2.



(a) $y = \ln x$ (b) $y = -\ln x$ (c) $y = -\ln(x-2)$

Figure 2:

(c) The range of $f(x) = -\ln(x-2)$ is the set of all real numbers, $\mathbb{R}$. The vertical asymptote is $x = 2$. [Do you see why? The original asymptote $(x = 0)$ is shifted to the right 2 units.]

(d) We begin with $y = -\ln(x-2)$. The inverse function is defined (implicitly) by the equation

$$x = -\ln(y-2).$$

We proceed to solve for y.

$-x = \ln(y-2)$	Isolate the logarithm.
$e^{-x} = y - 2$	Change to an exponential expression.
$y = e^{-x} + 2$	Solve for y

The inverse of f is $f^{-1}(x) = e^{-x} + 2$.

(e) The range of f is the domain of f^{-1}, which is the set of all real numbers, $\mathbb{R}$, confirming what we found from the graph of f. The range of f^{-1} is the domain of f, which is $(2, \infty)$.

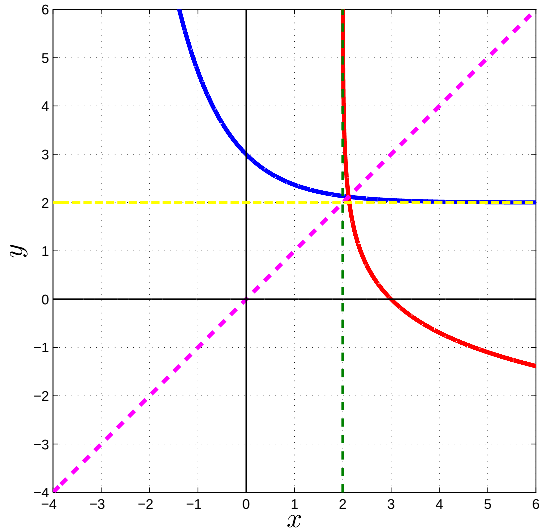


Figure 3:

(f) To graph f^{-1},we use the graph of f in Figure 2(c) and reflect it about the line $y = x$. See Figure 3. We could also graph $f^{-1}(x) = e^{-x} + 2$ using transformations.

Exercise 7 Graphing a Logarithmic Function and Its Inverse

(a) Find the domain of the logarithmic function $f(x) = 3\log(x-1)$.

(b) Graph f.

(c) From the graph, determine the range and vertical asymptote of f.

(d) Find f^{-1}, the inverse of f.

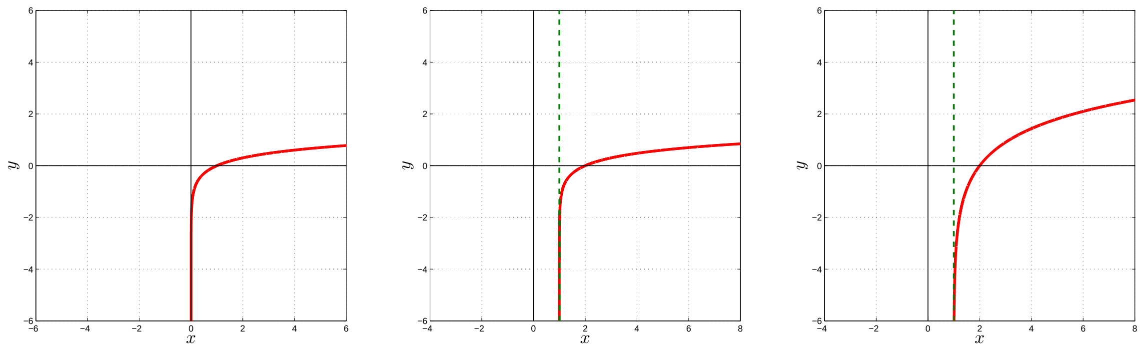
(e) Use f^{-1} to confirm the range of f found in part (c). From the domain of f, find the range of f^{-1}.

(f) Graph f^{-1}.

Solution:

(a) The domain of f consists of all x for which $x - 1 > 0$ or, equivalently, $x > 1$. The domain of f is $\{x |\ x > 1\}$ or $(1, \infty)$.

(b) To obtain the graph of $y = 3\log(x-1)$, we begin with the graph of $y = \log x$ and use transformations. See Figure 4.



(a) $y = \log x$ (b) $y = \log(x-1)$ (c) $y = 3\log(x-1)$

Figure 4:

(c) The range of $f(x) = 3\log(x-1)$ is the set of all real numbers. The vertical asymptote is $x = 1$.

(d) We begin with $y = 3\log(x-1)$. The inverse function is defined (implicitly) by the equation

$$x = 3\log(y-1).$$

We proceed to solve for y.

$\frac{x}{3} = \log(y-1)$	Isolate the logarithm
$10^{x/3} = y - 1$	Change to an exponential expression
$y = 10^{x/3} + 1$	Solve for y

The inverse of f is $f^{-1}(x) = 10^{x/3} + 1$.

(e) The range of f is the domain of f^{-1}, which is the set of all real numbers, confirming what we found from the graph of f. The range of f^{-1} is the domain of f, which is $(1, \infty)$.

(f) To graph f^{-1}, we use the graph of f in Figure 4(c) and reflect it about the line $y = x$. See Figure 5. We could also graph $f^{-1}(x) = 10^{x/3} + 1$ using transformations.

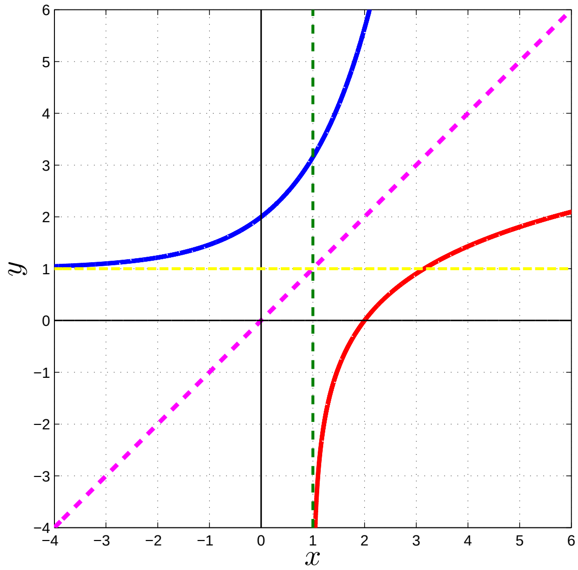


Figure 5:

Exercise 8 Solving a Logarithmic Equation

Solve: (a) $\log_3(4x-7)=2$ (b) $\log_x 64 = 2$

Solution

(a) We can obtain an exact solution by changing the logarithmic equation to exponential form.

$$\begin{aligned} \log_3(4x-7) &= 2 \\ 4x-7 &= 3^2 \qquad \text{Change to an exponential equation.} \\ 4x-7 &= 9 \\ 4x &= 16 \\ x &= 4. \end{aligned}$$

Check:

$$\begin{aligned} \log_3(4x-7) &= \log_3(4\cdot 4-7) \\ &= \log_3 9 = 2 \qquad 3^2 = 9. \end{aligned}$$

The solution set is $\{4\}$.

(b) We can obtain an exact solution by changing the logarithmic equation to exponential form.

$$\begin{array}{rcll} \log_x 64 &=& 2 & \\ x^2 &=& 64 & \text{Change to an exponential equation.} \\ x &=& \pm\sqrt{64} = \pm 8 & \text{Take square root} \end{array}$$

The base of a logarithm is always positive. Therefore, we discard/neglect -8. We check the solution 8.

Check:

$$\log_8 64 = 2 \qquad 8^2 = 64.$$

The solution set is $\{8\}$.

Exercise 9 Using Logarithms to Solve Exponential Equations

Solve: $e^{2x} = 5$

Solution:

We can obtain an exact solution by changing the exponential equation to logarithmic form.

$$\begin{array}{rcll} e^{2x} & = & 5 & \\ \ln 5 & = & 2x & \text{Change to a logarithmic equation} \\ & & & \text{using the fact that if } e^y = x \text{ then } y = \ln x. \\ x & = & \dfrac{\ln 5}{2} & \text{Exact solution} \\ & \approx & 0.805 & \text{Approximate solution} \end{array}$$

The solution set is $\left\{\dfrac{\ln 5}{2}\right\}$.