# Tutorial on CHAPTER 7.1-7.2

LINEAR ALGEBRA: MATRICES, VECTORS, DETERMINANTS.

LINEAR SYSTEMS

HE, CHANGRAN

OFFICE: ERB411

# Matrix

Def: A **matrix** is a rectangular array of numbers or functions enclosed in brackets.

The numbers (or functions) are ***entries*** **(**or ***elements)*** of the matrix.

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Comment:

1. The matrix is a chart of elements. (it is only a way of representation)

2. The entries of a matrix can be numbers or functions(e.g. a11=x^2)

3. We identify entries by their coordinates in the matrix. (e.g. a23 is the entry in Row 2 and Column 3 of the matrix)

# Matrix

Matrices are denoted by capital boldface letters **A, B**

Generally, an ***m*** × ***n*** **matrix** is a matrix with *m* **rows** and *n* **columns**:

$$\mathbf{A}_{mn} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

If ***m=n,*** it is a ***square matrix.***

# Matrix

For square matrix **A** of size *n × n,* the entries *a*11, *a*22, … , *ann* is called the **main diagonal** of **A.**

A matrix with only diagonal entries is call **diagonal matrix**.

$$\mathbf{A} = \begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix}$$

Properties of **diagonal matrix**:

$$\mathbf{A}^n = \begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix}^n = \begin{pmatrix} {a_{11}}^n & & 0 \\ & \ddots & \\ 0 & & {a_{nn}}^n \end{pmatrix}$$

$$\mathbf{AB} = \begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix}\begin{pmatrix} b_{11} & & 0 \\ & \ddots & \\ 0 & & b_{nn} \end{pmatrix} = \begin{pmatrix} b_{11} & & 0 \\ & \ddots & \\ 0 & & b_{nn} \end{pmatrix}\begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix} = \mathbf{BA}$$

# Vector

Def: A **vector** is a matrix with just a single row or column.

Its entries are called the **components** of the vector.

Vectors are denoted by ***lowercase*** boldface letters **a**, **b**.

**row vector**

$$size: 1\times n \quad \mathbf{A}_{13} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$$

**column vector**

$$size: n\times 1 \quad \mathbf{A}_{31} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Comment: A vector can be viewed as a special case of matrix where the size is $1\times n$ or $n\times 1$

# Vector products

Dot product: $$u \bullet v = u^T v = \begin{pmatrix} u_1 & u_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = u_1 v_1 + u_2 v_2$$

Outer product: $$uv^T = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \begin{pmatrix} v_1 & v_2 \end{pmatrix} = \begin{pmatrix} u_1 v_1 & u_1 v_2 \\ u_2 v_1 & u_2 v_2 \end{pmatrix}$$

# Equality of Matrices

1. Check whether their sizes are same.

2. Check whether each correspondent entry are the same.

# Addition of Matrices

Rule: Add each correspondent entries together.

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$AB = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

Note: Two matrices have to be same size.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = ?$$

# Scalar Multiplication (Multiplication by a Number)

$$c\mathbf{A} = c\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} ca_{11} & \dots & ca_{1n} \\ \vdots & \ddots & \vdots \\ ca_{m1} & \cdots & ca_{mn} \end{pmatrix}$$

Note: c is a number.

# Rules for Matrix Addition and Scalar Multiplication

Note: here **0** is not a number, it is called **zero matrix**, whose entries are all zeros.

These 8 rules holds because of **linearity(Linear Algebra)**.

**Commutative, associative**

$$\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$$

$$\mathbf{A}+(\mathbf{B}+\mathbf{C})=(\mathbf{A}+\mathbf{B})+\mathbf{C}$$

$$\mathbf{A}+\mathbf{0}=\mathbf{A}$$

$$\mathbf{A}+(\mathbf{-A})=\mathbf{0}$$

$$\mathrm{c}(\mathbf{A}+\mathbf{B})=c\mathbf{A}+c\mathbf{B}$$

$$(c+k)\mathbf{A}=c\mathbf{A}+k\mathbf{A}$$

$$c(k\mathbf{A})=(ck)\mathbf{A}$$

$$1\mathbf{A}=\mathbf{A}$$

# Matrix Multiplication

This is because of definition, people define the result of matrix multiplication in this way.

Note: Before multiplying **A** and **B**, first check their sizes. The number of columns of **A** has to be the same with the number of rows of **B.**

You can write the size of **A** and **B** and check whether the two number in the middle are the same.

For instance, **A** is 2x3 matrix, **B** is 3x4 matrix, implies $\underset{2\times 3 \vdots 3\times 4}{\mathbf{AB}}$ $\underset{2\times 2 \vdots 3\times 4}{\mathbf{AB}}$

$$\mathbf{C}_{nq} = \mathbf{A}_{mn}\mathbf{B}_{nq}$$

$$c_{jk} = \sum_{l=1}^{n} a_{jl}b_{lk} = a_{j1}b_{1k} + a_{j2}b_{2k} + \cdots + a_{jn}b_{nk}$$

# Exercises

Calculate **AB**.

$$\mathbf{A} = \begin{pmatrix} a & b & c \\ c & b & a \\ 1 & 1 & 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & a & c \\ 1 & b & b \\ 1 & c & a \end{pmatrix}$$

# Matrix Multiplication

Rule: Multiplication of rows into columns.

Matrix Multiplication is **not commutative, AB ≠ BA** in general.

However, If **A** or **B** are matrices in form of

$$\mathbf{A}\ or\ \mathbf{B} = \begin{pmatrix} c & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & c \end{pmatrix} = c\begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} = c\mathbf{I}_n \text{ ,where c is a number}$$

Here, **A** is called **scalar matrix** and **I** is called **identity matrix**, then for any **B** of the same size(e.g. nxn in this case), **AB = BA** holds**.**

Note that there are other situations that **AB = BA** holds.

# Identity Matrix

$$\mathbf{I}_n = \begin{pmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}_{\mathrm{n}\times\mathrm{n}}$$

$$\mathbf{I}_{\mathrm{n}}\mathbf{A}_{np} = \mathbf{A}_{np}\mathbf{I}_n = \mathbf{A}_{np}$$

Just think about the role of 1 in number multiplication.

# Matrix Multiplication

**Note:** **AB** = **0** does ***not*** necessarily imply **BA** = **0** or **A** = **0** or **B** = **0**.

$$(k\mathbf{A})\mathbf{B} = k(\mathbf{AB}) = \mathbf{A}(k\mathbf{B})$$ , where k here is a number(scalar).

$$\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$$

$$(\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC}+\mathbf{BC}$$

$$\mathbf{C}(\mathbf{A}+\mathbf{B}) = \mathbf{CA}+\mathbf{CB}$$

# Exercises

Find the matrix that is commutative with **A**.

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 3 & 1 & 2 \end{pmatrix}$$

# Compact form of multiplication

$$\mathbf{C} = \mathbf{AB} = \mathbf{A}\begin{bmatrix} \mathbf{b_1} & \mathbf{b_2} & \cdots & \mathbf{b_p} \end{bmatrix} = \begin{bmatrix} \mathbf{Ab_1} & \mathbf{Ab_2} & \cdots & \mathbf{Ab_p} \end{bmatrix}, \text{where } \mathrm{b_i} \text{ are columns, } 1 \le \mathrm{i} \le p$$

$$\mathbf{C} = \mathbf{AB} = \begin{bmatrix} \mathbf{a_1} \\ \mathbf{a_2} \\ \vdots \\ \mathbf{a_n} \end{bmatrix} \mathbf{B} \text{ ,where } \mathrm{a_i} \text{ are rows, } 1 \le \mathrm{i} \le n$$

Just think about the definition of matrix multiplication. <u>Multiplication of rows into columns.</u>

# Matrix exponential

Def:

$$\mathbf{A}^r = \mathbf{A}\mathbf{A}\cdots\mathbf{A} \text{ ,with r } \mathbf{A} \text{ matrices}$$

$$\mathbf{A}^0 = \mathbf{I}$$

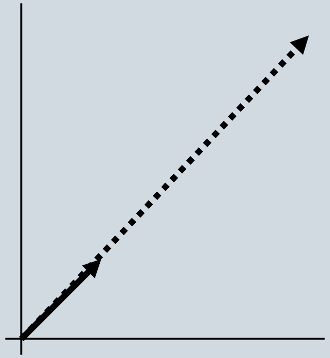
# Exercises

Calculate

$$\begin{pmatrix} 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}^2$$

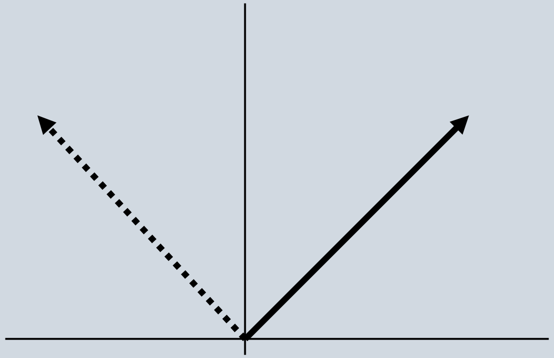
# Matrices as linear transformations

$$\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$$



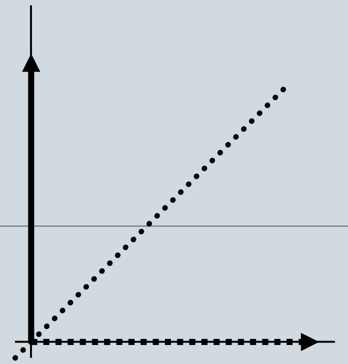
(stretching)

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$



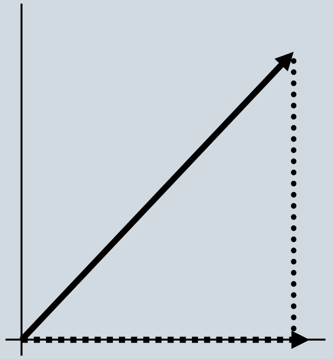
(rotation)

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



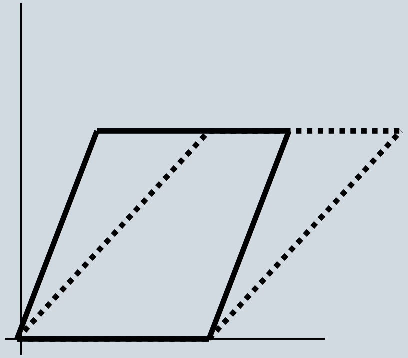
(reflection)

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



(projection)

$$\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + cy \\ y \end{pmatrix}$$



(shearing)

# Transpose

A “tall” matrix after transpose will be a “fat” one.

A “fat” one after transpose will be a “tall” one.

$$\mathbf{A}^T = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}^T = \begin{pmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{nm} \end{pmatrix}$$

# Transpose

$$(\mathbf{A}^T)^T = \mathbf{A}$$

$$(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$$

$$(c\mathbf{A})^T = c\mathbf{A}^T$$ , where c is a scalar(number)

$$(\mathbf{A}\mathbf{B})^T = \mathbf{B}^T\mathbf{A}^T$$

$$\rightarrow (\mathbf{A}_1 + \cdots + \mathbf{A}_n)^T = \mathbf{A}_1{}^T + \cdots + \mathbf{A}_n{}^T$$

$$(\mathbf{A}_1 \cdots \mathbf{A}_n)^T = \mathbf{A}_n{}^T \cdots \mathbf{A}_1{}^T$$

# Symmetric and Skew-Symmetric Matrices (square)

For square matrices:

$$\mathbf{A}^T = \mathbf{A} \rightarrow (a_{ij} = a_{ji}) \quad symmetric$$

$$\mathbf{A}^T = -\mathbf{A} \rightarrow (a_{ij} = -a_{ji} \; and \; \mathrm{a}_{ii} = 0) \quad skew-symmetric$$

$$\mathbf{A} = \frac{(\mathbf{A}+\mathbf{A}^T)}{2} + \frac{(\mathbf{A}-\mathbf{A}^T)}{2}$$

$$symmetric \qquad skew-symmetric$$

# Triangular Matrices(square)

**Upper triangular matrices**

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{pmatrix} \qquad \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$$

**Lower triangular matrices**

$$\begin{pmatrix} a_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \qquad \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix}$$

**Comment:**

**A**,**B** are triangular matrices then their product **AB** is also triangular matrix.

# Blocking of Matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \left(\begin{array}{c|cc} a_{11} & a_{12} & a_{13} \\ \hline a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}\right) = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}$$

Divide a big matrix to smaller matrices with suitable sizes.

You have many ways to divide a matrix.

# Blocking of Matrix

$$\mathbf{A} = \left(\begin{array}{cc|c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \hline a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{array}\right) = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}$$

$$\mathbf{B} = \left(\begin{array}{cc} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \hline b_{31} & b_{32} \end{array}\right) = \begin{pmatrix} \mathbf{B}_{11} \\ \mathbf{B}_{21} \end{pmatrix}$$

$$\rightarrow \mathbf{AB} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{B}_{11} \\ \mathbf{B}_{21} \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{11}\mathbf{B}_{11} + \mathbf{A}_{12}\mathbf{B}_{21} \\ \mathbf{A}_{21}\mathbf{B}_{11} + \mathbf{A}_{22}\mathbf{B}_{21} \end{pmatrix} = \begin{pmatrix} \mathbf{C}_{11} \\ \mathbf{C}_{21} \end{pmatrix}$$

# Blocking of Matrix

$$\mathbf{A}+\mathbf{B}=\begin{pmatrix}\mathbf{A}_1 & \dots & \mathbf{0}\\ \vdots & \ddots & \vdots\\ \mathbf{0} & \cdots & \mathbf{A}_n\end{pmatrix}+\begin{pmatrix}\mathbf{B}_1 & \dots & \mathbf{0}\\ \vdots & \ddots & \vdots\\ \mathbf{0} & \cdots & \mathbf{B}_n\end{pmatrix}=\begin{pmatrix}\mathbf{A}_1+\mathbf{B}_1 & \dots & \mathbf{0}\\ \vdots & \ddots & \vdots\\ \mathbf{0} & \cdots & \mathbf{A}_n+\mathbf{B}_n\end{pmatrix}$$

$$\mathbf{AB}=\begin{pmatrix}\mathbf{A}_1 & \dots & \mathbf{0}\\ \vdots & \ddots & \vdots\\ \mathbf{0} & \cdots & \mathbf{A}_n\end{pmatrix}\begin{pmatrix}\mathbf{B}_1 & \dots & \mathbf{0}\\ \vdots & \ddots & \vdots\\ \mathbf{0} & \cdots & \mathbf{B}_n\end{pmatrix}=\begin{pmatrix}\mathbf{A}_1\mathbf{B}_1 & \dots & \mathbf{0}\\ \vdots & \ddots & \vdots\\ \mathbf{0} & \cdots & \mathbf{A}_n\mathbf{B}_n\end{pmatrix}$$

# Exercise

Calculate **AB**

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} 1 & 0 & 3 & 2 \\ -1 & 2 & 0 & 1 \\ 1 & 0 & 4 & 1 \\ -1 & -1 & 2 & 0 \end{pmatrix}$$

$$\mathbf{A} = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline -1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{A}_1 & \mathbf{I} \end{pmatrix}$$

$$\mathbf{B} = \left(\begin{array}{cc|cc} 1 & 0 & 3 & 2 \\ -1 & 2 & 0 & 1 \\ \hline 1 & 0 & 4 & 1 \\ -1 & -1 & 2 & 0 \end{array}\right) = \begin{pmatrix} \mathbf{B_1} & \mathbf{B_2} \\ \mathbf{B_3} & \mathbf{B_4} \end{pmatrix}$$

# Exercise

$$\mathbf{A} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{A}_1 & \mathbf{I} \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} \mathbf{B}_1 & \mathbf{B}_2 \\ \mathbf{B}_3 & \mathbf{B}_4 \end{pmatrix}$$

$$\rightarrow \mathbf{AB} = \begin{pmatrix} \mathbf{B}_1 & \mathbf{B}_2 \\ \mathbf{A}_1\mathbf{B}_1 + \mathbf{B}_3 & \mathbf{A}_1\mathbf{B}_2 + \mathbf{B}_4 \end{pmatrix}$$

Next, just need to calculate

$\mathbf{A}_1\mathbf{B}_1 + \mathbf{B}_3$

$\mathbf{A}_1\mathbf{B}_2 + \mathbf{B}_4$

# Exercise

$answer:$

$$\mathbf{AB} = \left(\begin{array}{cc|cc} 1 & 0 & 3 & 2 \\ -1 & 2 & 0 & 1 \\ \hline -2 & 4 & 1 & 1 \\ -1 & 1 & 5 & 3 \end{array}\right)$$