Lecture Notes: Vectors and Their Basic Operators

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1 Vectors

We will use $\mathbb{R}$ to denote the set of all the real values. Given an integer $d \geq 1$, we use $\mathbb{R}^d$ to denote the d-dimensional space where each dimension has a domain of $\mathbb{R}$.

Definition 1. *A d-dimensional* **vector** *is a sequence of d real values* v_1, v_2, ..., v_d, *and is represented as* $[v_1, v_2, ..., v_d]$.

Each v_i ($1 \leq i \leq d$) in the above definition is called a *component* of the vector. Henceforth, we will use boldfaces to denote vectors, e.g., $\boldsymbol{v} = [v_1, v_2, ..., v_d]$. We use $\mathbf{0}$ to represent the specific vector $[0, 0, ..., 0]$ called the *zero vector*. We will be concerned primarily with $d = 2$ and $d = 3$.

Let $p_1 = (a_1, a_2, ..., a_d)$ and $p_2 = (b_1, b_2, ..., b_d)$ be two points in $\mathbb{R}^d$. They define a *directed segment* $\overrightarrow{p_1, p_2}$ which is the segment connecting p_1 and p_2, but also carrying a direction from p_1 to p_2.

Definition 2. *Let* $\boldsymbol{v} = [v_1, v_2, ..., v_d]$ *be a vector. An* **instantiation** *of* $\boldsymbol{v}$ *is a directed segment* $\overrightarrow{p_1, p_2}$ *where the points* $p_1 = (a_1, a_2, ..., a_d), p_2 = (b_1, b_2, ..., b_d)$ *satisfy:*

$$v_i = b_i - a_i, \forall i \in [1, d].$$

We say that $\overrightarrow{p_1, p_2}$ *is the* **default instantiation** *of* $\boldsymbol{v}$ *if* $p_1 = (0, 0, ..., 0)$ *and* $p_2 = (v_1, v_2, ..., v_d)$.

Note that a vector has an infinite number of instantiations. Consider, for example, $\boldsymbol{v} = [1, 2, -3]$ (namely, $d = 3$). Its default instantiation is $\overrightarrow{(0, 0, 0), (1, 2, -3)}$, but $\overrightarrow{(10, 20, 30), (11, 22, 27)}$ is also an instantiation, and so is $\overrightarrow{(-5, 8, -10), (-4, 10, -13)}$.

Definition 3. *The* **length**, *also called the* **norm**, *of a vector* $\boldsymbol{v} = [v_1, v_2, ..., v_d]$ *is defined to be*

$$|\boldsymbol{v}| = \sqrt{\sum_{i=1}^{d} v_i^2}.$$

We refer to $\boldsymbol{v}$ as a *unit vector* if $|\boldsymbol{v}| = 1$.

Definition 4. *Let* $\boldsymbol{v} = [v_1, v_2, ..., v_d]$ *be a vector that is not the zero vector. The* **direction** *of a vector* $\boldsymbol{v} = [v_1, v_2, ..., v_d]$ *is the ray*[1] *that emanates from the origin* $(0, 0, ...0)$ *and passes the point* $(v_1, v_2, ..., v_d)$.

[1] A *ray* emanating from a point p is a line segment that has p as an end point, and extends infinitely on the other end.

For example, [1, 2, 3] has the same direction as [2, 4, 6] and [5, 10, 15]. However, [1, 2, 3] does *not* have the same direction as $[-1, -2, -3]$. Note that the direction definition does not apply to the zero vector $\mathbf{0}$, which does not have a direction.

Henceforth, we say that a directed segment $\overrightarrow{p_1, p_2}$ is *parallel* to a ray, if the support line[2] of $\overrightarrow{p_1, p_2}$ is parallel to the support line of the ray. We have:

Lemma 1. *All instantiations of a vector $\boldsymbol{v}$ are parallel to the direction of $\boldsymbol{v}$.*

Proof. We first show that any instantiation of $\boldsymbol{v} = [v_1, ..., v_d]$ can be obtained from another by translation[3]. Let $\overrightarrow{p_1, p_2}$ and $\overrightarrow{q_1, q_2}$ be two instantiations of $\boldsymbol{v}$. Suppose that

$$p_1 = (a_1, ..., a_d),\ p_2 = (b_1, ..., b_d)$$
$$q_1 = (x_1, ..., x_d),\ q_2 = (y_1, ..., y_d).$$

By definition of instantiation, we know:

$$b_i - a_i = y_i - x_i = v_i,\ \forall i \in [1, d].$$

Hence:

$$x_i - a_i = y_i - b_i,\ \forall i \in [1, d].$$

This means that q_1 can be obtained from p_1 by applying the same translation as obtaining q_2 from p_2. Hence, $\overrightarrow{q_1, q_2}$ can be obtained from $\overrightarrow{p_1, p_2}$ by translation. It thus follows that the support lines of $\overrightarrow{q_1, q_2}$ and $\overrightarrow{p_1, p_2}$ are parallel.

The lemma then follows from the obvious fact that the default instantiation of $\boldsymbol{v}$ is parallel to the direction of $\boldsymbol{v}$. $\square$

Remark. It is clear from Definitions 3 and 4 that a vector $\boldsymbol{v}$ has both a length and a direction. Henceforth, we will use the term *scalar* as a synonym for "real value" (e.g., $15.235, 0$, and -3 are all scalars). The introduction of this term is to emphasize that real values do not have directions, unlike vectors.

2 Equality, Addition, Subtraction, and Scalar Multiplication

Definition 5. *Two vectors $\boldsymbol{a} = [a_1, ..., a_d]$ and $\boldsymbol{b} = [b_1, ..., b_d]$ are* **equivalent** *if $a_i = b_i$ for all $i \in [1, d]$.*

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are equivalent, we write $\boldsymbol{a} = \boldsymbol{b}$; otherwise, we write $\boldsymbol{a} \neq \boldsymbol{b}$. Next, we define the operators $+$ and $-$ on vectors.

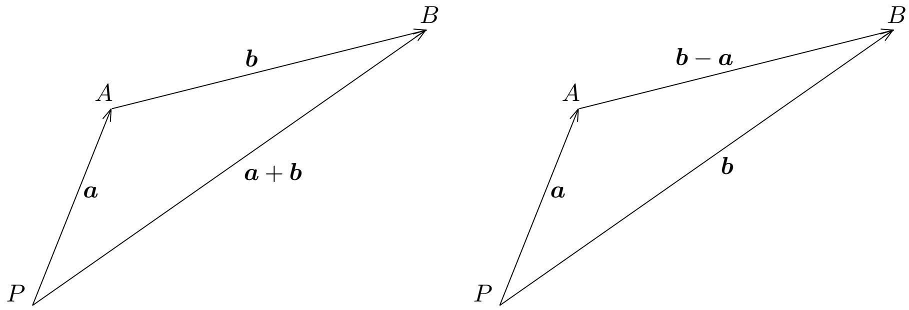
Definition 6. *Given two vectors $\boldsymbol{a} = [a_1, ..., a_d]$ and $\boldsymbol{b} = [b_1, ..., b_d]$, we define $\boldsymbol{a} + \boldsymbol{b}$ as the vector $[a_1 + b_1, a_2 + b_2, ..., a_d + b_d]$, and $\boldsymbol{a} - \boldsymbol{b}$ as the vector $[a_1 - b_1, a_2 - b_2, ..., a_d - b_d]$*

It is easy to prove by definition that the $+$ and $-$ operators have the following properties:

- (Commutativity) $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$.

[2]The support line of a segment is the line that passes the segment.

[3]Translating a geometric object (such as a directed segment, line, circle, etc.) is to move the object in $\mathbb{R}^d$ without applying any rotation.



(a) (b)

Figure 1: Geometric view of vector addition and subtraction

- (Associativity) $(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c})$.
- $\boldsymbol{a} - \boldsymbol{b} - \boldsymbol{c} = \boldsymbol{a} - (\boldsymbol{b} + \boldsymbol{c})$.
- If $\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}$, then $\boldsymbol{b} = \boldsymbol{c} - \boldsymbol{a}$ and $\boldsymbol{a} = \boldsymbol{c} - \boldsymbol{b}$.

The addition operator has an important geometric property:

Lemma 2. *Suppose that $\overrightarrow{PA}$ and $\overrightarrow{AB}$ are instantiations of $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively. Then, $\overrightarrow{PB}$ is an instantiation of $\boldsymbol{a} + \boldsymbol{b}$; see Figure 1a.*

Proof. Suppose that $\boldsymbol{a} = [a_1, a_2, ..., a_d]$ and $\boldsymbol{b} = [b_1, b_2, ..., b_d]$. Also, assume that $P = (p_1, p_2, ..., p_d)$, $A = (x_1, x_2, ..., x_d)$, and $B = (y_1, y_2, ..., y_d)$.

Because $\overrightarrow{PA}$ and $\overrightarrow{AB}$ are instantiations of $\boldsymbol{a}$ and $\boldsymbol{b}$, we know by definition that

$$\begin{aligned} a_i &= x_i - p_i, \forall i \in [1, d] \\ b_i &= y_i - x_i, \forall i \in [1, d]. \end{aligned}$$

It thus follows that

$$a_i + b_i = y_i - p_i, \forall i \in [1, d].$$

Therefore, $\overrightarrow{PB}$ is an instantiation of $\boldsymbol{a} + \boldsymbol{b}$. □

Corollary 1. *Suppose that $\overrightarrow{PA}$ and $\overrightarrow{PB}$ are instantiations of $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively. Then, $\overrightarrow{AB}$ is an instantiation of $\boldsymbol{b} - \boldsymbol{a}$; see Figure 1b.*

Next we define a multiplication operator between a vector and a scalar:

Definition 7. *Given a vector $\boldsymbol{v} = [v_1, ..., v_d]$ and a scalar (a.k.a., a real value) c, we define the product of $\boldsymbol{v}$ and c—denoted as $c\boldsymbol{v}$ or $\boldsymbol{v}c$—as the vector $[cv_1, cv_2, ..., cv_d]$.*

Specifically, we will denote by $-\boldsymbol{v}$ as the product of $\boldsymbol{v}$ and -1. It is easy to prove by definition the following properties:

- $c(\boldsymbol{a} + \boldsymbol{b}) = c\boldsymbol{a} + c\boldsymbol{b}$.
- $\boldsymbol{a} + (-\boldsymbol{a}) = \boldsymbol{0}$.

When $d = 3$, we define 3 special unit vectors:

$$\boldsymbol{i} = [1, 0, 0],\ \boldsymbol{j} = [0, 1, 0],\ \boldsymbol{k} = [0, 0, 1].$$

This allows us to represent a 3d vector $\boldsymbol{v} = [v_1, v_2, v_3]$ as $\boldsymbol{v} = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}$ (note that all the operators in this equation are now well defined). Similarly, when $d = 2$, we define 2 special unit vectors:

$$\boldsymbol{i} = [1, 0],\ \boldsymbol{j} = [0, 1].$$

A 2d vector $\boldsymbol{v} = [v_1, v_2]$ can therefore be represented as $\boldsymbol{v} = v_1\boldsymbol{i} + v_2\boldsymbol{j}$.