Multivariable Calculus

Week 7 NOTES – Double Integration

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Ref.: Ch. 14.1 – 14.3 of Adams and Essex

Ch. 14.1 – 14.2 of Larson and Edwards

Recall: Integration is the reverse process of differentiation.

(1) e^{ax} → (2) Integration → (3) ? → (4) Differentiation → (1)

$\frac{d}{dx}[\,?\,] = e^{ax}$, $a \neq 0$.

Note that one of the anti-derivatives of e^{ax} is $\frac{1}{a}e^{ax}$

Thus, we have $\frac{d}{dx}\left[\frac{1}{a}e^{ax}\right] = e^{ax}$

The general form of anti-derivatives of e^{ax} is denoted by

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C \rightsquigarrow C: \text{constant of integration.}$$

Example 1: Let $f(x,y) = e^{ax} + y$. Integrate f with respect to y.
Integrate f with respect to x.

(1) Original function: $f(x,y) = e^{ax} + y$

$\xrightarrow{\text{(2) Treat } x \text{ as a constant (say } x=\pi\text{)} \text{ Find } \int f(x,y)\,dy}$

(3) We obtain $f(\pi, y) = e^{a\pi} + y$

↓

(4) Integrate w.r.t. y

$$\int (e^{a\pi} + y)\,dy = e^{a\pi}y + \frac{y^2}{2} + C$$

$\xleftarrow{\text{(5) Replace } \pi \text{ by } x \text{ and } C \text{ by } C(x)}$

(6) Integral of f w.r.t. y is

$$\boxed{\int (e^{ax} + y)\,dy = e^{ax}y + \frac{y^2}{2} + C(x)}$$

Similarly, we can treat y as a constant (say $y = \pi$)

Find $\int f(x,y)\,dx$. We obtain $f(x,\pi) = e^{ax} + \pi$.

Integrate w.r.t. x: $\int (e^{ax} + \pi)\,dx = \frac{1}{a}e^{ax} + \pi x + C$

Replace π by y and C by $C(y)$

$$\Rightarrow \boxed{\int (e^{ax} + y)\,dx = \frac{1}{a}e^{ax} + yx + C(y)}$$

Single integral (Definite)

$$\int_a^b f(x,y)\,dx = \int_{x=a}^{x=b} f(x,y)\,dx$$

$x=b$ → upper limit, $x=a$ → lower limit

: Definite integral of $f(x,y)$ w.r.t. x over interval $[a,b]$.

(Note: y is treated as constant).

$$\int_c^d f(x,y)\,dy = \int_{y=c}^{y=d} f(x,y)\,dy$$

$y=d$ → upper limit, $y=c$ → lower limit

: Definite integral of $f(x,y)$ w.r.t. y over interval $[c,d]$

(Note: x is treated as constant)

Eg. ① Since $\int x^3\,dx = \frac{x^4}{4} + C$, by Fund. Thm of Calculus,

$$\int_3^4 x^3\,dx = \left[\frac{x^4}{4}+C\right]_3^4 = \left(\frac{4^4}{4}+C\right) - \left(\frac{3^4}{4}+C\right) = 64 - \frac{81}{4} = \frac{175}{4}$$

$\| \left[\frac{x^4}{4}\right]_3^4$ (no need care the constant C).

② Since $\int \sec^2(x^5 y)\,dy = \frac{1}{x^5}\tan(x^5 y) + C(x)$, by Fund. Thm of Calculus,

$$\int_2^3 \sec^2(x^5 y)\,dy = \left[\frac{1}{x^5}\tan(x^5 y) + C(x)\right]_{y=2}^{y=3} = \left(\frac{1}{x^5}\tan(3x^5) + C(x)\right) - \left(\frac{1}{x^5}\tan(2x^5) + C(x)\right)$$

$\| \left[\frac{1}{x^5}\tan(x^5 y)\right]_{y=2}^{y=3}$

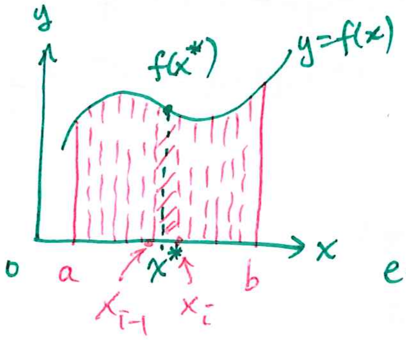
$$= \frac{\tan(3x^5) - \tan(2x^5)}{x^5}, \quad x \neq 0$$

Geometrically, if a function of one variable is defined and bounded,

then $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$ (Riemann sum).

$[a,b]$ is divided into points $a = x_0 < x_1 < x_2 < \dots < x_n = b$.

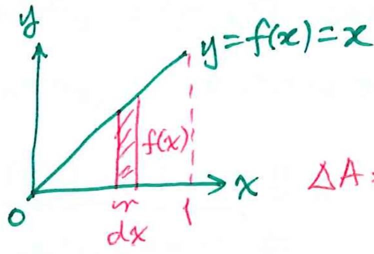
$x_i^* \in [x_{i-1}, x_i]$, $\Delta x_i = x_i - x_{i-1}$



NOTE: $n \to \infty$, $\Delta x_i \to 0$ $\left(\because \Delta x_i = \frac{b-a}{n}\right)$

$\int_a^b f(x)\,dx$ = signed area of $f(x)$ on $[a,b]$.

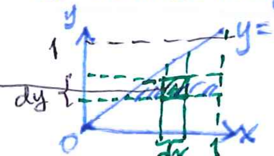
e.g. Let $f(x) = x$.



$$\int_0^1 f(x)\,dx = \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

$\Delta A = f(x)\cdot dx$

as double integral.



$dA = dx\,dy$
$= dy\,dx$

$$\int_0^1 \int_y^1 dx\,dy = \int_0^1 x\Big|_y^1\,dy = \int_0^1 (1-y)\,dy = y - \frac{y^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$\int_0^1 \int_0^x dy\,dx$$

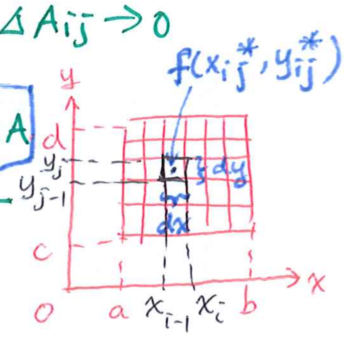
Definition: Suppose that $z=f(x,y)$ is a real-valued function of 2 real variables. Let $f(x,y)$ be defined and bounded in some region $R=[a,b]\times[c,d]$ of the xy-plane with a finite area.

We divide R into $m\times n$ rectangles of area ΔA_{ij} $(1\le i\le m,\ 1\le j\le n)$ and $(x_{ij}^*, y_{ij}^*)\in[x_{i-1},x_i]\times[y_{j-1},y_j]$.

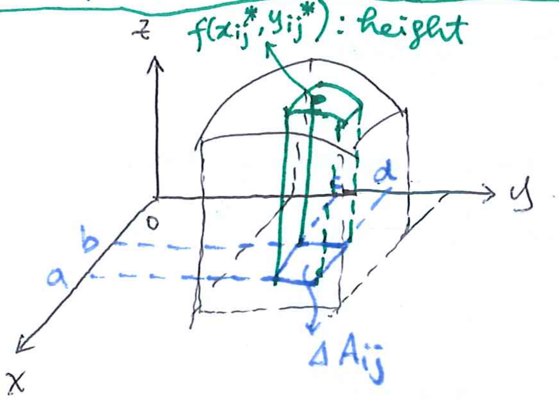
The product $\Delta V_{ij} = f(x_{ij}^*, y_{ij}^*)\cdot\Delta A_{ij}$ is the signed volume of a rectangular box.

$$\lim_{\substack{n\to\infty\\ m\to\infty}} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A_{ij} \text{ exists as } m,n\to\infty,\ \Delta A_{ij}\to 0$$

Then, the double integral of f over R is $\boxed{V=\iint_R f(x,y)\,dA}$



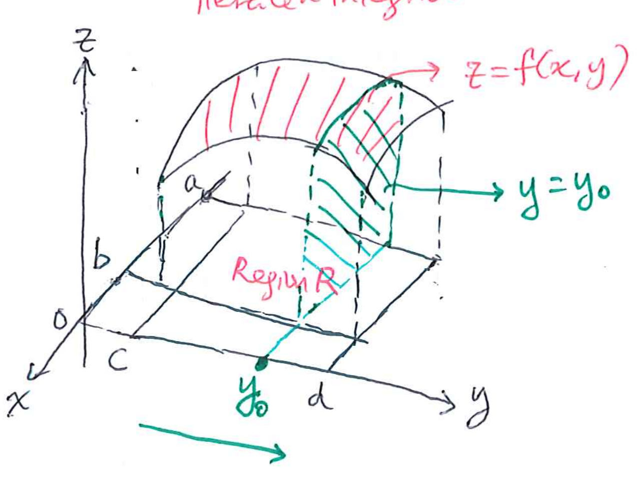
volume under the surface $z=f(x,y)$ over region R.

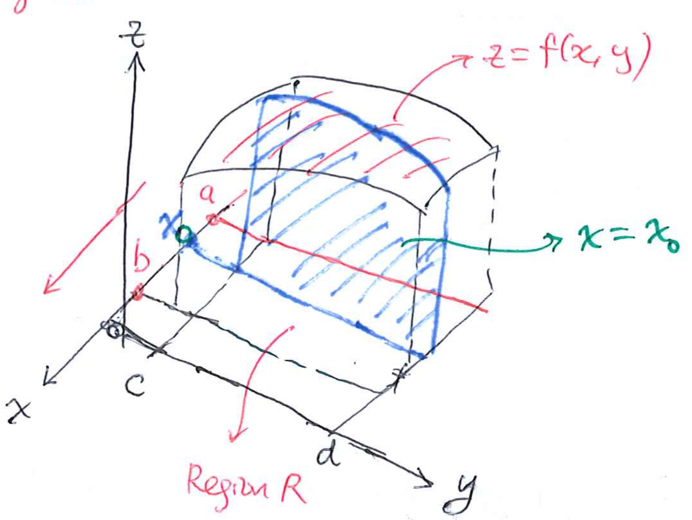


Fubini's Theorem (First Form)

Suppose that $z=f(x,y)$ is continuous throughout the rectangular region $R:[a,b]\times[c,d]$.

Then $\underbrace{\int_a^b\int_c^d f(x,y)\,dy\,dx}_{\text{iterated integral}} = \underbrace{\iint_R f(x,y)\,dA}_{\text{double integral}} = \underbrace{\int_c^d\int_a^b f(x,y)\,dx\,dy}_{\text{iterated integral}}$





If $z=f(x,y)\ge 0$ over region R, then $\int_a^b f(x,y)\,dx$ is the area under the curve $z=f(x,y)$ in the plane of cross-section at y, while $\int_a^b f(x,y)\,dy$ is the area under the curve $z=f(x,y)$ in the plane of cross-section at x.

Properties of double integral

① $\iint_R f(x,y)\,dA = 0$ if R has a zero area.

② $\iint_R 1\,dA$ = area of R ($\because$ volume of a cylinder with base R and height 1)

③ $\iint_R f(x,y)\,dA \geq 0$ if $f(x,y) \geq 0$

$\iint_R f(x,y)\,dA \leq 0$ if $f(x,y) \leq 0$

④ (Linearity of integrals) $\iint_R (Lf(x,y) + Mg(x,y))\,dA = L\iint_R f(x,y)\,dA + M\iint_R g(x,y)\,dA$

⑤ (Preserving inequalities) If $f(x,y) \leq g(x,y)$ on R, then

$$\iint_R f(x,y)\,dA \leq \iint_R g(x,y)\,dA.$$

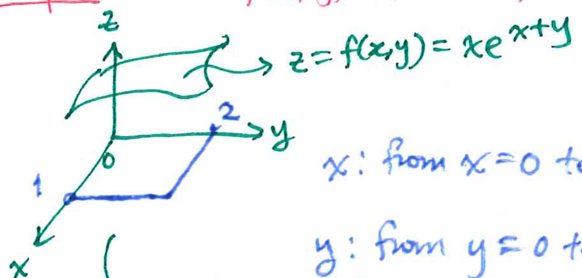
⑥ (Triangle Inequality) $\left|\iint_R f(x,y)\,dA\right| \leq \iint_R |f(x,y)|\,dA$

NOT interior pts
can share boundary pts

⑦ (Additivity of domains). If $D_1, D_2, \ldots, D_K$ are non-overlapping domains on each of which f is integrable, then f is integrable over the union $R = D_1 \cup D_2 \cup \cdots \cup D_K$ and

$$\iint_R f(x,y)\,dA = \sum_{i=1}^{K} \iint_{D_i} f(x,y)\,dA$$

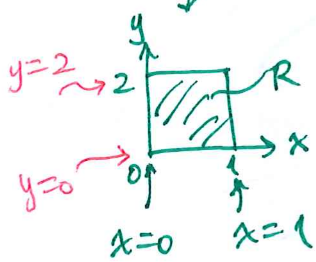
Example ① $f(x,y) = xe^{x+y}$, $R = [0,1] \times [0,2]$



$$\int_0^2\int_0^1 xe^{x+y}\,dx\,dy$$

x: from $x=0$ to $x=1$ (curves on the xy-plane)

y: from $y=0$ to $y=2$ (points on the y-axis)



$$= \int_0^2 e^y \left[\int_0^1 xe^x\,dx\right] dy$$

$$= \int_0^2 e^y \left[xe^x - e^x\right]\Big|_0^1 dy$$

$$= \int_0^2 e^y\,dy = e^2 - 1$$

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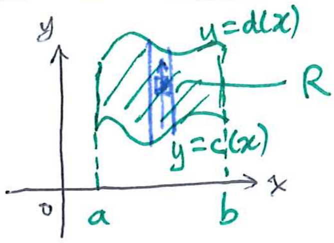
② $\iint_R f(x,y)\,dA$ where $z = f(x,y) = y\sin x$ and the region $R = [0,\frac{\pi}{2}] \times [-1,2]$

$$\iint_R f(x,y)\,dA = \int_{y=-1}^{y=2}\int_{x=0}^{x=\frac{\pi}{2}} y\sin x\,dx\,dy = \left(\int_{y=-1}^{y=2} y\,dy\right)\left(\int_{x=0}^{x=\frac{\pi}{2}} \sin x\,dx\right)$$

$$= \left[\frac{y^2}{2}\right]_{y=-1}^{y=2} \cdot \left[-\cos x\right]_{x=0}^{x=\frac{\pi}{2}} = \frac{3}{2}$$

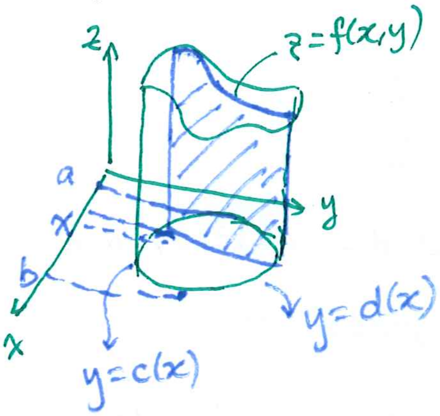
Double Integrals over General Regions

Type I regions: $R = \{(x,y) \mid a \le x \le b,\ c(x) \le y \le d(x)\}$.



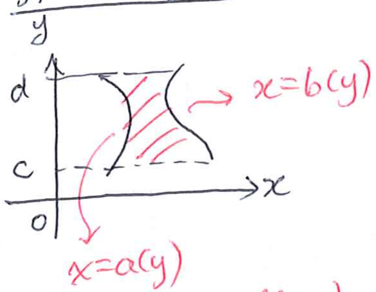
$$\iint_R f(x,y)\,dA = \int_a^b \int_{c(x)}^{d(x)} f(x,y)\,dy\,dx$$

points curves



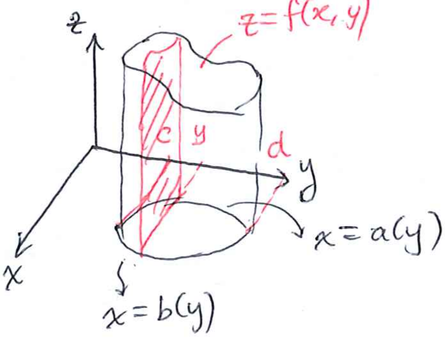
slices perpendicular to the x-axis in Type I regions

Type II regions: $R = \{(x,y) \mid c \le y \le d,\ a(y) \le x \le b(y)\}$



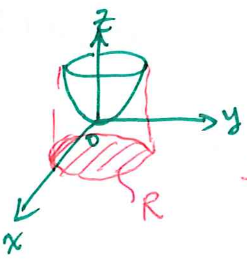
$$\iint_R f(x,y)\,dA = \int_c^d \int_{a(y)}^{b(y)} f(x,y)\,dx\,dy$$

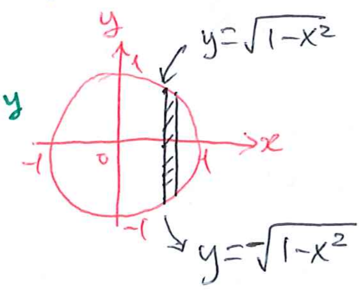
points curves

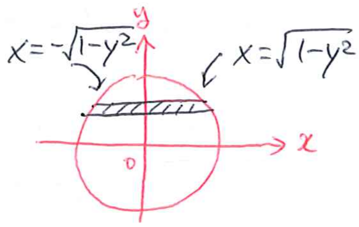


slices perpendicular to the y-axis in Type II regions.

Examples ① $\iint_R (x^2+y^2)\,dA$, where R is the region enclosed by the circle $x^2+y^2=1$.





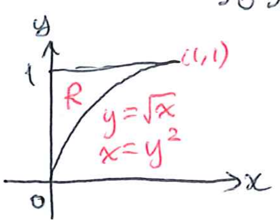


$$\iint_R (x^2+y^2)\,dA = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx \quad \text{(Type I)}$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (x^2+y^2)\,dx\,dy \quad \text{(Type II)}$$

$$= \frac{\pi}{2}$$

② Evaluate $\int_0^1 \int_{\sqrt{x}}^1 e^{y^3}\,dy\,dx$

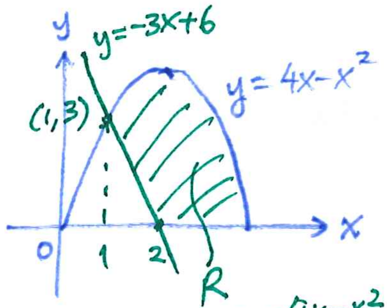


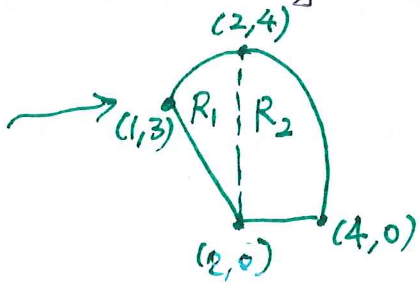
$$= \int_0^1 \int_0^{y^2} e^{y^3}\,dx\,dy$$

$$= \int_0^1 y^2 e^{y^3}\,dy = \frac{e^{y^3}}{3}\Big|_0^1$$

$$= \frac{e-1}{3}$$

③ Find the area of the region R that lies below the parabola $y=4x-x^2$ above the x-axis, and above the line $y=-3x+6$.





$R = R_1 \cup R_2$

$$\text{Area} = \int_1^2 \int_{-3x+6}^{4x-x^2} dy\,dx + \int_2^4 \int_0^{4x-x^2} dy\,dx$$

$$= \int_1^2 (4x-x^2+3x-6)\,dx + \int_2^4 (4x-x^2)\,dx = \underline{\underline{\frac{15}{2}}}$$

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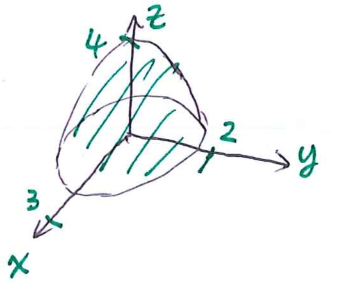
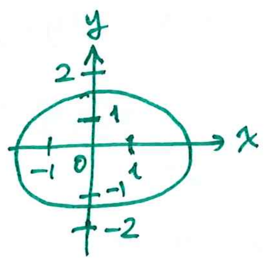
④ Find the volume of the solid region bounded by the paraboloid $z=4-x^2-2y^2$ and the xy-plane.

Let $z=0$, base of the region: ellipse $x^2+2y^2=4$





Base: $-2 \le x \le 2$

$-\sqrt{\frac{4-x^2}{2}} \le y \le \sqrt{\frac{4-x^2}{2}}$

$$\text{Volume} = \int_{-2}^{2} \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} (4-x^2-2y^2)\,dy\,dx$$

$$= \int_{-2}^{2} \left[(4-x^2)y - \frac{2y^3}{3}\right]_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} dx$$

$$= \frac{4}{3\sqrt{2}} \int_{-2}^{2} (4-x^2)^{\frac{3}{2}}\,dx$$

Let $x = 2\sin\theta \Rightarrow \frac{4}{3\sqrt{2}} \int_{-\pi/2}^{\pi/2} 16\cos^4\theta\,d\theta$

$$= \frac{64}{3\sqrt{2}}\, 2\int_0^{\pi/2} \cos^4\theta\,d\theta$$

$$= \frac{128}{3\sqrt{2}} \left(\frac{3\pi}{16}\right) = 4\sqrt{2}\pi$$

↳ Wallis's Formula. (need steps!)

⑤ Definition of the Average Value of a Function over a region.

If f is integrable over the plane region R, then the average value of f over R $= \frac{1}{\text{Area of } R} \iint_R f(x,y)\,dA$.

Question: Find the average value of $f(x,y) = x\cos xy$ over the rectangle $R=[0,\pi]\times[0,1]$.

Sol^n Value of the integral of f over $R = \int_0^\pi \int_0^1 x\cos xy\,dy\,dx = \int_0^\pi [\sin xy]_{y=0}^{y=1} dx$

$$= \int_0^\pi \sin x\,dx = -\cos x\Big|_0^\pi = 2$$

Area of $R = \pi \times 1 = \pi \Rightarrow$ Average value of f over $R = \frac{2}{\pi}$.