Lecture Notes: Surfaces, Tangent Planes, and Surface Normals

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1 Surfaces

We will focus on $\mathbb{R}^3$ with dimensions x, y, and z. Consider a plane $x + 2y + 3z - 4 = 0$, or a sphere $x^2 + y^2 + z^2 = 1$. What is in common in these two geometric objects is that they seem to have defined a 2d region, perhaps warped (as in the sphere case). In mathematics, we call them *surfaces*.

Formally, there are two standard ways to describe a surface. The first one resorts to a scalar function $f(x, y, z)$, and sets the function to 0, namely, $f(x, y, z) = 0$. In the plane example, $f(x, y, z) = x+2y+3z-4$, whereas in the sphere example, $f(x, y, z) = x^2+y^2+z^2-1$. It would be helpful to understand why $f(x, y, z) = 0$ is a surface in the following way. Take a point (x, y) in the xy plane, and solve the value of z from $f(x, y, z) = 0$. If z exists, think of z as the "elevation" of a mountain at the longitude x and altitude y. If you move (x, y) around, using z you will be tracing out the top of an undulating mountain. Note that sometimes multiple z may satisfy $f(x, y, z) = 0$, as is true for the sphere $x^2 + y^2 + z^2 = 1$.

Since a surface is intuitively two dimensional, we can also represent it using two real-valued parameters u, v. In this way, we should designate three functions: $x(u, v), y(u, v), z(u, v)$ which give the x, y, and z coordinates of a point on the surface. For example, the plane $x + 2y + 3z - 4 = 0$ can be alternatively represented with:

$$\begin{aligned} x(u,v) &= u \\ y(u,v) &= v \\ z(u,v) &= (4-u-2v)/3. \end{aligned}$$

Likewise, the sphere $x^2 + y^2 + z^2 = 1$ can be represented with:

$$\begin{aligned} x(u,v) &= \cos(u) \\ y(u,v) &= \sin(u)\cos(v) \\ z(u,v) &= \sin(u)\sin(v). \end{aligned}$$

2 Tangent Planes and Surface Normals

Consider a surface $f(x, y, z) = 0$. Fix a point $p(x_0, y_0, z_0)$ on the surface such that $\frac{\partial f}{\partial x}(x_0, y_0, z_0)$, $\frac{\partial f}{\partial y}(x_0, y_0, z_0)$, $\frac{\partial f}{\partial z}(x_0, y_0, z_0)$ exist, and are not all equal to 0. Take an *arbitrary* curve C on the surface passing p. We know that C can be described by functions $x(t), y(t)$, and $z(t)$, which take a real-valued parameter t, and give the x-, y-, and z-coordinates of a point on C. Let t_0 be the value of t corresponding to p (hence, $x_0 = x(t), y_0 = y(t)$, and $z_0 = z(t)$). We assume that $x'(t_0)$, $y'(t_0)$, $z'(t_0)$ exist, and are not all equal to 0.

As C is on the surface, we know that

$$f(x(t), y(t), z(t)) \quad = \quad 0.$$

Taking the derivative of both sides with respect to t gives:

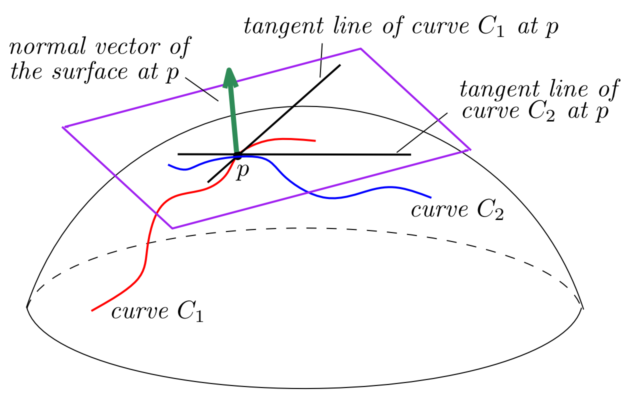
$$\begin{aligned}
\frac{d\Big(f(x(t), y(t), z(t))\Big)}{dt} &= 0 \quad \Rightarrow \\
\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} &= 0 \quad \text{(applied the chain rule)} \Rightarrow \\
\Big[\frac{\partial f}{\partial x}(x,y,z), \frac{\partial f}{\partial y}(x,y,z), \frac{\partial f}{\partial z}(x,y,z)\Big] \cdot \Big[x'(t), y'(t), z'(t)\Big] &= 0 \quad \Rightarrow \\
\nabla f(x,y,z) \cdot \Big[x'(t), y'(t), z'(t)\Big] &= 0.
\end{aligned}$$

Applying the above equation to point p results in

$$\nabla f(x_0, y_0, z_0) \cdot \Big[x'(t_0), y'(t_0), z'(t_0)\Big] \quad = \quad 0$$

The above equation tells us something interesting. Notice that $[x'(t_0), y'(t_0), z'(t_0)]$ is a tangent vector of C at p. From our assumptions, we have that neither $\nabla f(x_0, y_0, z_0)$ nor $[x'(t_0), y'(t_0), z'(t_0)]$ is $\mathbf{0}$. It thus follows that the direction of $\nabla f(x_0, y_0, z_0)$ is *perpendicular* to that of $[x'(t_0), y'(t_0), z'(t_0)]$.

Here is something even more interesting. Recall that we chose C as an *arbitrary* curve passing p whose tangent vector at p is not $\mathbf{0}$. There can be an infinite number of such curves (the figure below shows two examples). *All* their tangent lines must be perpendicular to the direction of $\nabla f(x_0, y_0, z_0)$! It thus follows that all those tangent lines must form a plane, and that the direction of $\nabla f(x_0, y_0, z_0)$ is perpendicular to this plane!



The plane aforementioned is therefore called the *tangent plane* of the surface at p. $\nabla f(x_0, y_0, z_0)$ is called a *normal vector* of the surface at p.