THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1020
Exercise 7
Produced by Jeff Chak-Fu WONG

Exercise 1 Establishing Properties of Logarithms

(a) Show that $\log_a 1 = 0$.

(b) Show that $\log_a a = 1$.

Solution:

(a) This fact was established when we graphed $y = \log_a x$. To show the result algebraically, let $y = \log_a 1$. Then

$$\begin{array}{rcll} y & = & \log_a 1 & \\ a^y & = & 1 & \text{Change to an exponential expression.} \\ a^y & = & a^0 & a^0 = 1 \text{ since } a > 0,\ a \neq 1 \\ y & = & 0 & \text{Solve for } y \\ \log_a 1 & = & 0 & y = \log_a 1. \end{array}$$

(b) Let $y = \log_a a$. Then

$$\begin{array}{rcll} y & = & \log_a a & \\ a^y & = & a & \text{Change to an exponential expression.} \\ a^y & = & a^1 & a = a^1 \\ y & = & 1 & \text{Solve for } y \\ \log_a a & = & 1 & y = \log_a a. \end{array}$$

To summarize:

$$\boxed{\log_a 1 = 0 \qquad\qquad \log_a a = 1.}$$

Exercise 2 Using Properties (1) and (2)

(a) $2^{\log_2 \pi} =$

(b) $\log_{0.2} 0.2^{-\sqrt{3}} =$

(c) $\ln e^{3kt} =$

Solution:

(a) $2^{\log_2 \pi} = \pi$.

(b) $\log_{0.2} 0.2^{-\sqrt{3}} = -\sqrt{3}$.

(c) $\ln e^{3kt} = 3kt$.

Exercise 3 Writing a Logarithmic Expression as a Sum of Logarithms

Write $\log_a(x\sqrt{x^2+1}),\ x > 0$, as a sum of logarithms. Express all powers as factors.

Solution:

$$\begin{aligned}
\log_a(x\sqrt{x^2+1}) &= \log_a x + \log_a(\sqrt{x^2+1}) \qquad && \log_a(MN) = \log_a M + \log_a N \\
&= \log_a x + \log_a(x^2+1)^{1/2} && \\
&= \log_a x + \frac{1}{2}\log_a(x^2+1) && \log_a M^r = r\log_a M.
\end{aligned}$$

Exercise 4 Writing a Logarithmic Expression as a Difference of Logarithms

Write

$$\ln \frac{x^2}{(x-1)^3} \qquad x > 1.$$

as a difference of logarithms. Express all powers as factors.

Solution: Using

$$\log_a \frac{M}{N} = \log_a M - \log_a N,$$

we have

$$\ln \frac{x^2}{(x-1)^3} = \ln x^2 - \ln(x-1)^3 = 2\ln x - 3\ln(x-1).$$

Exercise 5 Writing a Logarithmic Expression as a Sum and Difference of Logarithms

Write

$$\log_a \frac{\sqrt{x^2+1}}{x^3(x+1)^4} \qquad x > 0.$$

as a sum and difference of logarithms. Express all powers as factors.

Solution:

$$\begin{aligned}
\log_a \frac{\sqrt{x^2+1}}{x^3(x+1)^4} &= \log_a \sqrt{x^2+1} - \log_a[x^3(x+1)^4] && \text{Property (4)} \\
&= \log_a \sqrt{x^2+1} - [\log_a x^3 + \log_a (x+1)^4] && \text{Property (3)} \\
&= \log_a (x^2+1)^{1/2} - \log_a x^3 - \log_a (x+1)^4 \\
&= \frac{1}{2}\log_a (x^2+1) - 3\log_a x - 4\log_a (x+1) && \text{Property (5)}
\end{aligned}$$

Exercise 6 Writing Expressions as a Single Logarithm

Write each of the following as a single logarithm.

(a) $\log_a 5 + 4\log_a 3$;

(b) $\dfrac{2}{3}\ln 8 - \ln(3^4 - 5)$;

(c) $\log_a x + \log_a 7 + \log_a(x^2+1) - \log_a 3$.

Solution:

(a)

$$\begin{aligned} \log_a 5 + 4\log_a 3 &= \log_a 5 + \log_a 3^4 && r\log_a M = \log_a M^r \\ &= \log_a 5 + \log_a 81 \\ &= \log_a(5\cdot 81) && \log_a M + \log_a N = \log_a(M\cdot N) \\ &= \log_a 405. \end{aligned}$$

(b)

$$\begin{aligned} \frac{2}{3}\ln 8 - \ln(3^4-5) &= \ln 8^{2/3} - \ln(81-5) && r\log_a M = \log_a M^r \\ &= \ln 4 - \ln 76 && 8^{2/3} = (\sqrt[3]{8})^2 = 2^2 = 4 \\ &= \ln\left(\frac{4}{76}\right) && \log_a M - \log_a N = \log_a\left(\frac{N}{M}\right) \end{aligned}$$

(c)

$$\begin{aligned} \log_a x + \log_a 7 + \log_a(x^2+1) - \log_a 3 &= \log_a(7x) + \log_a(x^2+1) - \log_a 3 \\ &= \log_a[7x(x^2+1)] - \log_a 3 \\ &= \log_a\left[\frac{7x(x^2+1)}{3}\right] \end{aligned}$$

Exercise 7 Approximating a Logarithm Whose Base Is Neither 10 Nor e

Approximate $\log_2 7$. Round the answer to four decimal places.

Solution:

Let $y = \log_2 7$. Then $2^y = 7$, so

$$\begin{array}{rcll} 2^y & = & 7 & \\ \ln 2^y & = & \ln 7 & \text{Property (6)} \\ y \ln 2 & = & \ln 7 & \text{Property (5)} \\ y & = & \dfrac{\ln 7}{\ln 2} & \text{Exact value} \\ y & \approx & 2.8074 & \end{array}$$

Approximate value rounded to four decimal places

Example 7 shows how to approximate a logarithm whose base is 2 by changing to logarithms involving the base e. In general, we use the **Change-of-Base Formula.**

Exercise 8 Using the Change-of-Base Formula

Approximate: (a) $\log_5 89$ (b) $\log_{\sqrt{2}} \sqrt{5}$.

Round answers to four decimal places.

Solution:

(a)

$$\log_5 89 = \frac{\log 89}{\log 5} \approx \frac{1.949390007}{0.6989700043} \approx 2.7889$$

or

$$\log_5 89 = \frac{\ln 89}{\ln 5} \approx \frac{4.48863637}{1.609437912} \approx 2.7889$$

(b)

$$\log_{\sqrt{2}} \sqrt{5} = \frac{\log \sqrt{5}}{\log \sqrt{2}} = \frac{\frac{1}{2}\log 5}{\frac{1}{2}\log 2} = \frac{\log 5}{\log 2} \approx 2.3219$$

or

$$\log_{\sqrt{2}} \sqrt{5} = \frac{\ln \sqrt{5}}{\ln \sqrt{2}} = \frac{\frac{1}{2}\ln 5}{\frac{1}{2}\ln 2} = \frac{\ln 5}{\ln 2} \approx 2.3219$$

Exercise 9 Graphing a Logarithmic Function Whose Base Is Neither 10 Nor e

Use Property (9) and use MATLAB to graph $y = \log_2 x$ and $y = \log_{1/3} x$.