Exercises: Eigenvalues, Eigenvectors, and Similarity Transformation

Problem 1. Find all the eigenvalues and eigenvectors of $\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

Solution. Let λ be an eigenvalue of $\boldsymbol{A}$. To obtain all possible λ, we solve the characteristic equation of $\boldsymbol{A}$ (let $\boldsymbol{I}$ be the 3×3 identity matrix):

$$\begin{aligned} det(\boldsymbol{A} - \lambda \boldsymbol{I}) &= 0 \Rightarrow \\ \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} &= 0 \Rightarrow \\ (\lambda - 1)^2(\lambda + 1) &= 0 \end{aligned}$$

Hence, $\boldsymbol{A}$ has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$.

To find all the eigenvectors of $\lambda_1 = 1$, we need to solve $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ from:

$$\begin{aligned} (\boldsymbol{A} - \lambda_1 \boldsymbol{I})\boldsymbol{x} &= \boldsymbol{0} \Rightarrow \\ \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

The set of solutions to the above equation—*EigenSpace*(λ_1)—includes all $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ satisfying

$$\begin{aligned} x_1 &= u \\ x_2 &= v \\ x_3 &= u \end{aligned}$$

for any $u, v \in \mathbb{R}$. Any non-zero vector in *EigenSpace*(λ_1) is an eigenvector of $\boldsymbol{A}$ corresponding to λ_1.

Similarly, to find all the eigenvectors of $\lambda_2 = -1$, we need to solve $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ from:

$$\begin{aligned} (\boldsymbol{A} - \lambda_2 \boldsymbol{I})\boldsymbol{x} &= \boldsymbol{0} \Rightarrow \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

The set of solutions to the above equation—*EigenSpace*(λ_2)—includes all $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ satisfying

$$\begin{aligned} x_1 &= u \\ x_2 &= 0 \\ x_3 &= -u \end{aligned}$$

for any $u \in \mathbb{R}$. Any non-zero vector in *EigenSpace*(λ_2) is an eigenvector of $\boldsymbol{A}$ corresponding to λ_2.

Problem 2. Let $\boldsymbol{A}$ be an $n \times n$ square matrix. Prove: $\boldsymbol{A}$ and $\boldsymbol{A}^T$ have exactly the same eigenvalues.

Proof. Recall that an eigenvalue of a matrix is a root of the matrix's characteristic equation, which equates the matrix's characteristic polynomial to 0. It suffices to show that the characteristic polynomial of $\boldsymbol{A}$ is the same as that of $\boldsymbol{A}^T$. In other words, we want to show that $det(\boldsymbol{A} - \lambda \boldsymbol{I}) = det(\boldsymbol{A}^T - \lambda \boldsymbol{I})$. This is true because $\boldsymbol{A} - \lambda \boldsymbol{I} = (\boldsymbol{A}^T - \lambda \boldsymbol{I})^T$. □

Problem 3. Let $\boldsymbol{A}$ be an $n \times n$ square matrix. Prove: $\boldsymbol{A}^{-1}$ exists if and only if 0 is not an eigenvalue of $\boldsymbol{A}$.

Proof. *If-Direction.* The objective is to show that if 0 is not an eigenvalue of $\boldsymbol{A}$, then $\boldsymbol{A}^{-1}$ exists, namely, the rank of $\boldsymbol{A}$ is n. Suppose, on the contrary, that the rank of $\boldsymbol{A}$ is less than n. Consider the linear system $\boldsymbol{Ax} = \boldsymbol{0}$ where $\boldsymbol{x}$ is an $n \times 1$ matrix. The hypothesis that *rank* $\boldsymbol{A} < n$ indicates that the system has infinitely many solutions. In other words, there exists a non-zero $\boldsymbol{x}$ satisfying $\boldsymbol{Ax} = \boldsymbol{0x} = \boldsymbol{0}$. This, however, indicates that 0 is an eigenvalue of $\boldsymbol{A}$, which is a contradiction.

Only-If Direction. The objective is to show that if $\boldsymbol{A}^{-1}$ exists, then 0 is not an eigenvalue of $\boldsymbol{A}$. The existence of $\boldsymbol{A}^{-1}$ means that the rank of $\boldsymbol{A}$ is n, which in turn indicates that $\boldsymbol{Ax} = \boldsymbol{0}$ has a unique solution $\boldsymbol{x} = \boldsymbol{0}$. In other words, there is no non-zero $\boldsymbol{x'}$ satisfying $\boldsymbol{Ax'} = \boldsymbol{0x'}$, namely, 0 is not an eigenvalue of $\boldsymbol{A}$. □

Problem 4. Let $\boldsymbol{A}$ be an $n \times n$ square matrix such that $\boldsymbol{A}^{-1}$ exists. Prove: if λ is an eigenvalue of $\boldsymbol{A}$, then $1/\lambda$ is an eigenvalue of $\boldsymbol{A}^{-1}$.

Proof. Since λ is an eigenvalue of $\boldsymbol{A}$, there is a non-zero $n \times 1$ matrix $\boldsymbol{x}$ satisfying

$$\begin{aligned} \boldsymbol{Ax} &= \lambda \boldsymbol{x} \Rightarrow \\ \boldsymbol{A}^{-1}\boldsymbol{Ax} &= \lambda \boldsymbol{A}^{-1}\boldsymbol{x} \Rightarrow \\ \boldsymbol{x} &= \lambda \boldsymbol{A}^{-1}\boldsymbol{x} \Rightarrow \\ \boldsymbol{A}^{-1}\boldsymbol{x} &= (1/\lambda)\boldsymbol{x} \end{aligned}$$

which completes the proof. □

Problem 5. Diagonalize the following matrix:

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$$

Solution. Matrix $\boldsymbol{A}$ has two eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 2$. Since (i) $\boldsymbol{A}$ is a 2×2 matrix and (ii) it has 2 distinct eigenvalues, we can apply the diagonalization method we discussed in class.

Specifically, we obtain an arbitrary eigenvector $\boldsymbol{v_1}$ of λ_1, say $\boldsymbol{v_1} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and, and an arbitrary eigenvector $\boldsymbol{v_2}$ of λ_2, say $\boldsymbol{v_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Then, we form:

$$\boldsymbol{Q} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$$

by using $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ as the first and second columns, respectively. $\boldsymbol{Q}$ has the inverse:

$$\boldsymbol{Q}^{-1} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$$

We thus obtain the following diagonalization of $\boldsymbol{A}$:

$$\boldsymbol{A} = \boldsymbol{Q}\, \mathit{diag}[3, 2]\, \boldsymbol{Q}^{-1}.$$

Problem 6. Consider again the matrix $\boldsymbol{A}$ in Problem 5. Calculate $\boldsymbol{A}^t$ for any integer $t \geq 1$.

Solution. We already know that $\boldsymbol{A}$:

$$\boldsymbol{A} = \boldsymbol{Q}\, \mathit{diag}[3, 2]\, \boldsymbol{Q}^{-1}.$$

Hence:

$$\begin{aligned} \boldsymbol{A}^t &= \boldsymbol{Q}\, \mathit{diag}[3^t, 2^t]\, \boldsymbol{Q}^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 3^t & 0 \\ 0 & 2^t \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -3^t + 2^{t+1} & -3^t + 2^t \\ 2 \times 3^t - 2^{t+1} & 2 \times 3^t - 2^t \end{bmatrix} \end{aligned}$$

Problem 7. Diagonalize the matrix $\boldsymbol{A}$ in Problem 1.

Solution. Recall that all symmetric matrices are diagonalizable. $\boldsymbol{A}$ is a 3×3 matrix. The key is to find three linearly independent eigenvectors.

From the solution of Problem 1, we know that $\boldsymbol{A}$ has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$. $\mathit{EigenSpace}(\lambda_1)$ includes all $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ satisfying

$$\begin{aligned} x_1 &= u \\ x_2 &= v \\ x_3 &= u \end{aligned}$$

for any $u, v \in \mathbb{R}$. The vector space $\mathit{EigenSpace}(\lambda_1)$ has dimension 2 with a basis $\{\boldsymbol{v_1}, \boldsymbol{v_2}\}$ where $\boldsymbol{v_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ (given by $u = 1, v = 0$) and $\boldsymbol{v_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ (given by $u = 0, v = 1$).

Similarly, $EigenSpace(\lambda_2)$ includes all $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ satisfying

$$\begin{aligned} x_1 &= u \\ x_2 &= 0 \\ x_3 &= -u \end{aligned}$$

for any $u \in \mathbb{R}$. The vector space $EigenSpace(\lambda_2)$ has dimension 1 with a basis $\{\boldsymbol{v_3}\}$ where $\boldsymbol{v_3} = \begin{bmatrix} 1 \\ 0 \\ \text{-1} \end{bmatrix}$ (given by $u = 1$).

So far, we have obtained three linearly independent eigenvectors $\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}$ of $\boldsymbol{A}$. We can then apply the diagonalization method exemplified in Problem 5 to diagonalize $\boldsymbol{A}$. Specifically, we form:

$$\boldsymbol{Q} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

$\boldsymbol{Q}$ has the inverse:

$$\boldsymbol{Q}^{-1} = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & -1/2 \end{bmatrix}$$

We thus obtain the following diagonalization of $\boldsymbol{A}$:

$$\boldsymbol{A} = \boldsymbol{Q}\, diag[1, 1, -1]\, \boldsymbol{Q}^{-1}.$$