# Lecture Notes: Surface Integral by Coordinate

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## 1 Smooth, xy-Monotone, and Oriented Surfaces

Recall that one way to specify a surface in $\mathbb{R}^3$ is to give an equation $f(x, y, z) = 0$ over some legal ranges of $x, y, z$. We say that the surface is *smooth* if both of the following are satisfied:

- the gradient $\nabla f(p) = [\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}]$ changes continuously as point $p$ moves about on the surface;
- $\nabla f(p) \neq \mathbf{0}$.

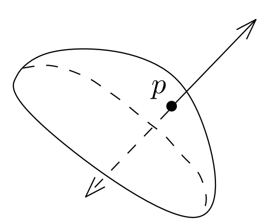
As discussed earlier, $\nabla f(p)$ gives a normal vector of the surface at point $p$. Hence, the first bullet essentially says that this normal vector changes continuously as $p$ moves on the surface. The second bullet implies that we can always obtain a *unit* normal vector at $p$ as $\frac{\nabla f(p)}{|\nabla f(p)|}$.

We say that a surface is *xy-monotone* if every line perpendicular to the xy-plane hits the surface at no more than one point. In other words, the surface can be represented as an equation $z = g(x, y)$. For example, the sphere $x^2 + y^2 + z^2 = 1$ is not xy-monotone, but the hemisphere

$$\begin{cases} x^2 + y^2 + z^2 = 1 \\ z \geq 0 \end{cases}$$

is, because we can represent the hemisphere as $z = \sqrt{1 - x^2 - y^2}$.

An xy-monotone surface $S$ usually has two "sides". For example, the cap-shaped surface below has two sides: facing outward and inward, respectively. To define surface integral by coordinate, we need to choose a side of the surface. Formally, we do so by choosing the directions of normal vectors. Specifically, for each point $p$ on the surface $S$, take a normal vector $\boldsymbol{u}$ at $p$. There are only two choices for $\boldsymbol{u}$, as shown in the example below. Denote by $\gamma(p)$ the angle between the direction of $\boldsymbol{u}$ and the positive direction of the z-axis. We require that either $\gamma(p) \in [0, \pi/2]$ for all $p$ on $S$, or $\gamma(p) \in [\pi/2, \pi]$ for all $p$ on $S$. In the former case, we say that we have chosen the *upper side* of $S$, where in the latter, we say that we have chosen the *lower side*. In both cases, $S$ is said to have been *oriented*.



## 2 Surface Integral by Coordinates x and y

Let $S$ be an oriented xy-monotone surface described by equation $z = g(x, y)$. Let $D$ be the projection of $S$ onto the xy-plane. We say that function $h(x, y, z)$ is *continuous* on $S$ if $h(x, y, g(x, y))$ is continuous in $D$. Then, we define *surface integral*

$$\iint_S h(x, y, z)\, dxdy \tag{1}$$

as a short form for

$$\begin{cases} \iint_D h(x, y, g(x, y))\, dxdy & \text{if } S \text{ is the upper side of } z = g(x, y) \\ -\iint_D h(x, y, g(x, y))\, dxdy & \text{otherwise} \end{cases}$$

**Example 1.** Let $S$ be the lower side of the plane $3x + 2y + z = 6$ with $0 \le x \le 1$ and $0 \le y \le 1$. Calculate $\iint_S x + y + z\, dxdy$.

*Solution.* Let $D$ be the area in the xy-plane corresponding to $0 \le x \le 1$ and $0 \le y \le 1$. $S$ can be described by the equation $z = 6 - 3x - 2y$.

$$\begin{aligned} \iint_S x + y + z\, dxdy &= -\iint_D x + y + 6 - 3x - 2y\, dxdy \\ &= -\iint_D 6 - 2x - y\, dxdy = -9/2. \end{aligned}$$

□

## 3 Evaluating Surface Integrals by Jacobian

Recall that a surface is inherently a 2D geometric object, even though it is embedded in $\mathbb{R}^3$. Besides using an equation $f(x, y, z) = 0$, we can also describe a surface by representing x-, y-, and z-coordinates as functions of two parameters $u$ and $v$, namely, $x(u, v)$, $y(u, v)$, $z(u, v)$. Accordingly, we can evaluate a surface integral by changing the integral variables from $x, y$ to $u, v$. However, since we are dealing with a double integral, the change of variables is more complicated than simply applying the chain rule; instead, we need to resorting to the Jacobian, as you should have learned in a prerequisite course. Next, we illustrate this using an example.

**Example 2.** Let $S$ be the upper side of the hemisphere $x^2 + y^2 + z^2 = 1$ with $z \ge 0$. Calculate $\iint_S z^2\, dxdy$.

*Solution.* Let $D$ be the projected region of $S$ onto the xy-plane, namely, $D$ is the disc $x^2 + y^2 \le 1$. Hence:

$$\iint_S z^2\, dxdy = \iint_D 1 - x^2 - y^2\, dxdy. \tag{2}$$

We can represent the x-, y-, and z-coordinates of each point $(x, y, z)$ on $S$ as functions of $u, v$:

$$\begin{aligned} x(u, v) &= \cos u \cdot \sin v \\ y(u, v) &= \sin u \cdot \sin v \\ z(u, v) &= \cos v \end{aligned}$$

with $0 \le u \le 2\pi$ and $0 \le v \le \pi/2$. The Jacobian $J$ equals:

$$\begin{aligned} J &= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \\ &= -\sin u \cdot \sin v \cdot \sin u \cdot \cos v - \cos u \cdot \cos v \cdot \cos u \cdot \sin v \\ &= -\sin v \cdot \cos v. \end{aligned}$$

Now we can change the variables $x, y$ in (2) to $u, v$ as:

$$\begin{aligned} \iint_R 1 - x^2 - y^2 \, dxdy &= \iint_R (1 - \cos^2 u \sin^2 v - \sin^2 u \sin^2 v) \cdot |J| \, dudv \\ &= \iint_R \cos^2 v \cdot |J| \, dudv \\ &= \iint_R \cos^2 v \cdot |\sin v \cdot \cos v| \, dudv \\ &= \int_0^{2\pi} \left( \int_0^{\pi/2} \cos^3 v \cdot \sin v \, dv \right) du \\ &= \frac{1}{4} \int_0^{2\pi} du = \pi/2. \end{aligned}$$

□

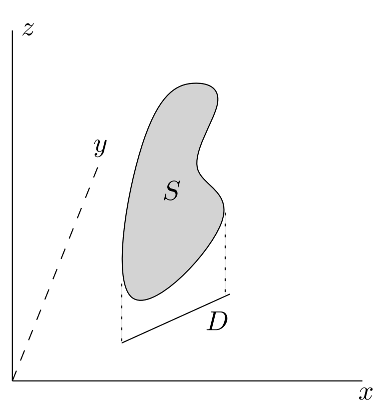
## 4 Surface Integrals on Regions Not xy-Monotone

So far our definition of surface integrals in (1) is limited to xy-monotone regions. Next, we extend the definition also to regions that are not xy-monotone. We achieve the purpose by (i) introducing a special case for vertical regions, and (ii) cutting a non-xy-monotone region into xy-monotone ones.

**A Special Case.** Let $S$ be a surface that is perpendicular to the xy-plane. Then, we define

$$\iint_S h(x, y, z) \, dxdy = 0.$$

The above definition is fairly intuitive. If $S$ is perpendicular to the xy-plane, its projection $D$ onto the xy-plane is a line segment whose area is 0; see below.



**Piecewise xy-Monotone Surfaces.** Let $S$ be a surface that can be cut into a sequence of surfaces $S_1, S_2, ..., S_m$, each of which is either an oriented surface, or perpendicular to the xy-plane. We refer to $S$ as a *piecewise xy-monotone surface*. Also, suppose that function $h(x, y, z)$ is continuous on each xy-monotone $S_i$ ($i \in [1, m]$). Then, we define

$$\iint_S h(x,y,z)\,dxdy \quad = \quad \sum_{i=1}^{m} \iint_{S_i} h(x,y,z)\,dxdy.$$

**Example 3.** Let $S$ be the outer side of the sphere $x^2 + y^2 + z^2 = 1$. Calculate $\iint_S z^2\,dxdy$.

*Solution.* Divide $S$ into two xy-monotone surfaces $S_1$ and $S_2$, where

- $S_1$ is the upper side of $x^2 + y^2 + z^2 = 1$ with $z \geq 0$;
- $S_2$ is the lower side of $x^2 + y^2 + z^2 = 1$ with $z \leq 0$.

Thus:

$$\iint_S h(x,y,z)\,dxdy \quad = \quad \iint_{S_1} h(x,y,z)\,dxdy + \iint_{S_2} h(x,y,z)\,dxdy.$$

We have seen in Example 2 that $\iint_{S_1} h(x,y,z)\,dxdy = \pi/2$. Next, we calculate $\iint_{S_2} h(x,y,z)\,dxdy$.

Let $D$ be the projected region of $S_2$ onto the xy-plane, namely, $D$ is the disc $x^2 + y^2 \leq 1$. Hence:

$$\iint_{S_2} z^2\,dxdy \quad = \quad -\iint_D 1 - x^2 - y^2\,dxdy. \tag{3}$$

We can represent the x-, y-, and z-coordinates of each point $(x, y, z)$ on $S$ as functions of $u, v$:

$$\begin{aligned} x(u,v) &= \cos u \cdot \sin v \\ y(u,v) &= \sin u \cdot \sin v \\ z(u,v) &= \cos v \end{aligned}$$

with $0 \leq u \leq 2\pi$ and $\pi/2 \leq v \leq \pi$. The Jacobian $J$ equals:

$$J \quad = \quad \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = -\sin v \cdot \cos v.$$

Now we can change the variables $x, y$ in (3) to $u, v$ as:

$$\begin{aligned} -\iint_D 1 - x^2 - y^2\,dxdy &= -\iint_R \cos^2 v \cdot |J|\,dudv \\ &= -\iint_R \cos^2 v \cdot |\sin v \cdot \cos v|\,dudv \\ &= \int_0^{2\pi} \left( \int_{\pi/2}^{\pi} \cos^3 v \cdot \sin v\,dv \right) du \\ &= -\frac{1}{4} \int_0^{2\pi} du = -\pi/2. \end{aligned}$$

Therefore, $\iint_S h(x,y,z)\,dxdy = \pi/2 - \pi/2 = 0$. □