Extensive Games

Sometimes games are played by n players, in the following manner:

- The game starts.
- Player i_1 chooses and takes an action.
- Player i_2 chooses and takes an action.
- Player i_3 chooses and takes an action.
- …

The game continues until there is an outcome. Such games are called **extensive games**.

DEFINITION. An **extensive game with perfect information** has the following components.

- A set N (the set of **players**).
- A set H of sequences (finite or infinite) that satisfies the following three properties.
  - The empty sequence $\emptyset$ is a member of H.
  - If $(a^k)_{k=1,\dots,K} \in H$ (where K may be infinite) and $L < K$ then $(a^k)_{k=1,\dots,L} \in H$.
  - If an infinite sequence $(a^k)_{k=1}^{\infty}$ satisfies $(a^k)_{k=1,\dots,L} \in H$ for every positive integer L then $(a^k)_{k=1}^{\infty} \in H$.

(H is the set of **histories**. A history $(a^k)_{k=1,\ldots,K} \in H$ is **terminal** if it is infinite, or if there is no a^{K+1} such that $(a^k)_{k=1,\ldots,K+1} \in H$. $Z \subseteq H$ is the set of terminal histories.)

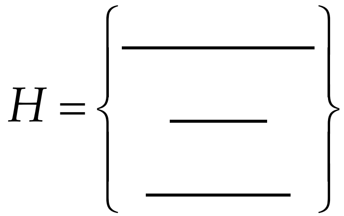
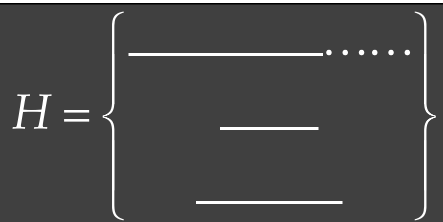
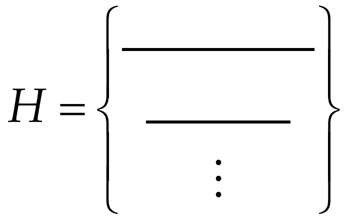
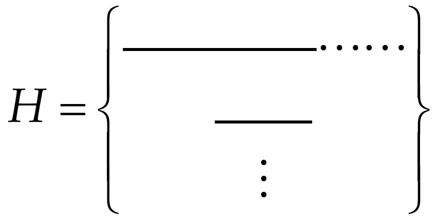
- A function P that assigns to each nonterminal sequence (each member of $H \setminus Z$) a member of N. (P is the **player function**, $P(h)$ being the player who takes an action *after* the history h.)
- For each player $i \in N$ a preference relation $\succsim_i$ on Z (the **preference relation** of player i on terminal histories).

An Extensive Game with Perfect Information:

$$\langle N, H, P, (\succsim_i) \rangle.$$

If H is finite, then the game is *finite*.

If the longest history is finite, then the game has a *finite horizon*.

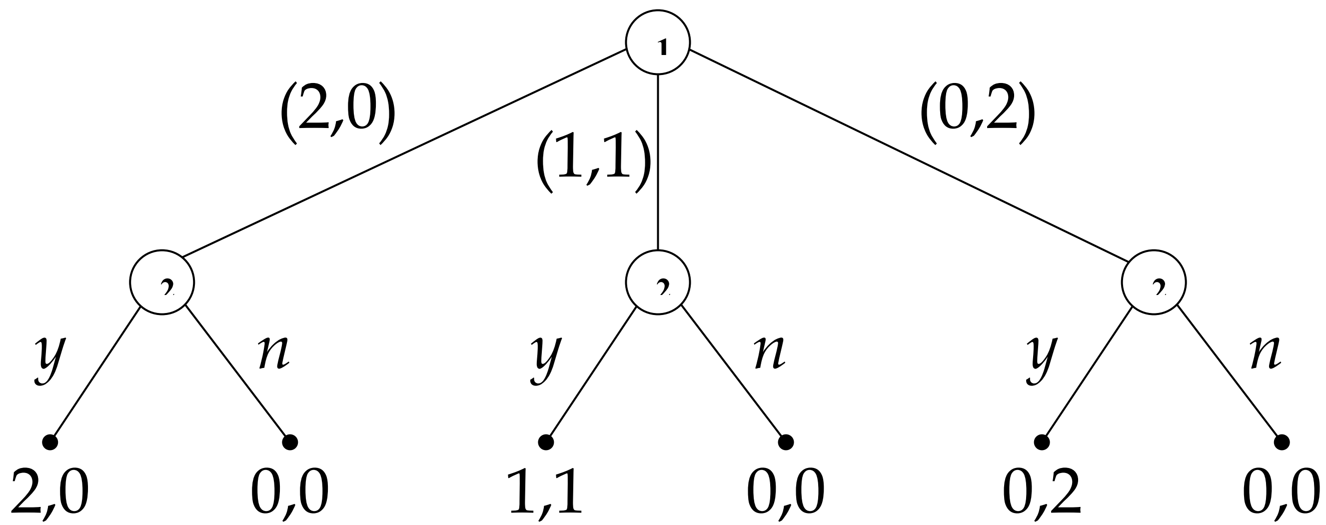
	… with Finite Horizon	*… with Infinite Horizon*
Finite Game		
Infinite Game		

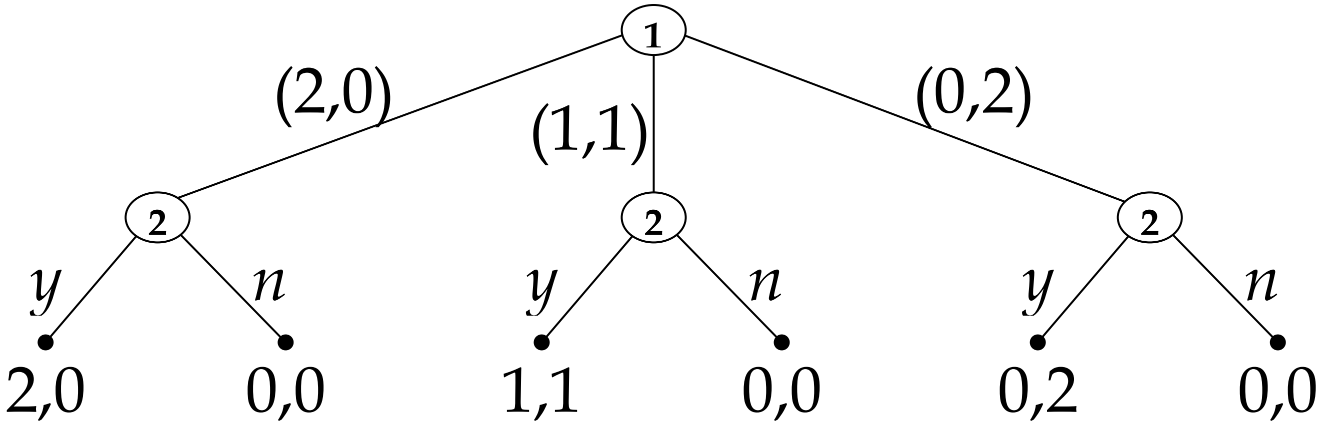
$\emptyset$ is the *initial history*.

$$\underbrace{\underbrace{\cdots\cdots\cdots\cdots\cdots\cdots\cdots}_{h \in H} \quad \overbrace{a}^{\substack{a \in A(h) \text{ by} \\ \text{player } P(h)}}}_{(h,a) \in H} \qquad\qquad A(h) = \{a : (h, a) \in H\}$$

EXAMPLE. Two people agree to use the following procedure to share two gold coins.

- Player 1 proposes an allocation, and Player 2 accepts or rejects.
- If Player 2 rejects, no one receives any gold coin.





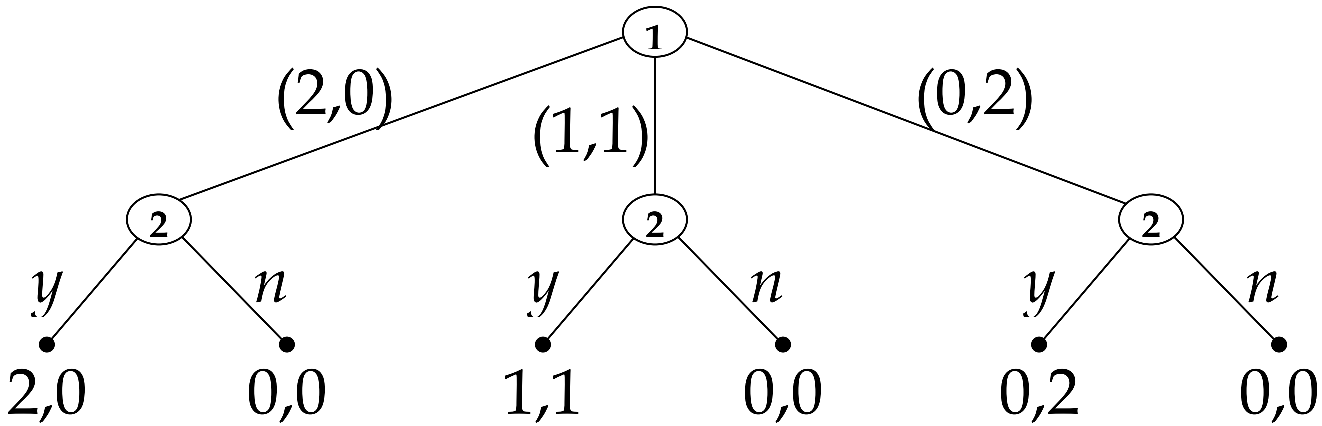
$N = \{1,2\}$

$H = \{\emptyset, (2,0), (1,1), (0,2), ((2,0), y), ((2,0), n),$
$((1,1), y), ((1,1), n), ((0,2), y), ((0,2), n)\}$

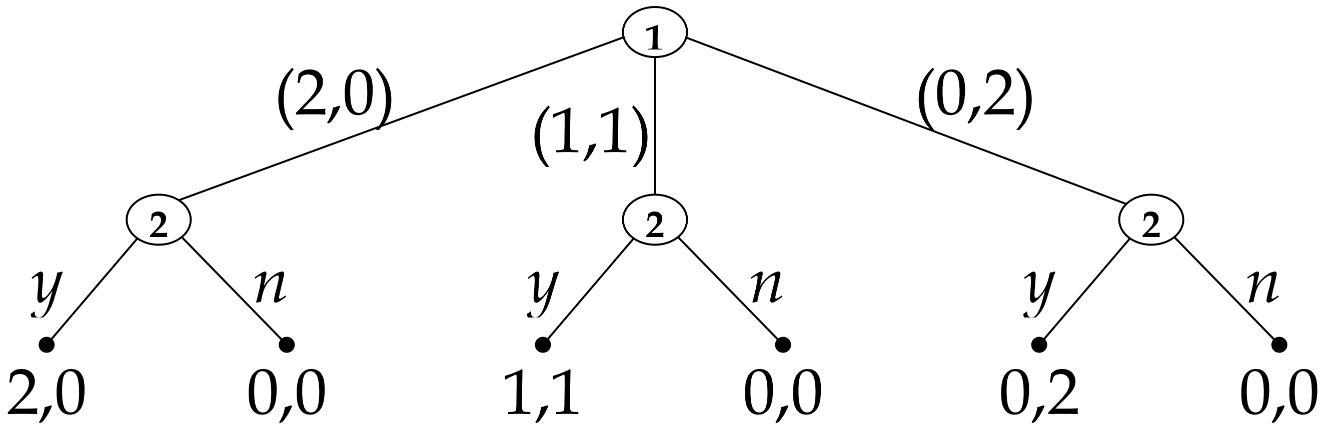
$P(\emptyset) = 1$; and $P(h) = 2$ for any nonterminal $h \neq \emptyset$.

$((2,0), y) \succ_1 ((1,1), y) \succ_1 ((0,2), y) \sim_1 ((2,0), n) \sim_1 ((1,1), n) \sim_1 ((0,2), n)$

$((0,2), y) \succ_2 ((1,1), y) \succ_2 ((2,0), y) \sim_2 ((2,0), n) \sim_2 ((1,1), n) \sim_2 ((0,2), n)$



A Possible Strategy for Player 1	A Possible Strategy for Player 2
$h = \emptyset$: (2,0)	$h = (2,0)\ : y$ $h = (1,1)\ : n$ $h = (0,2)\ : n$



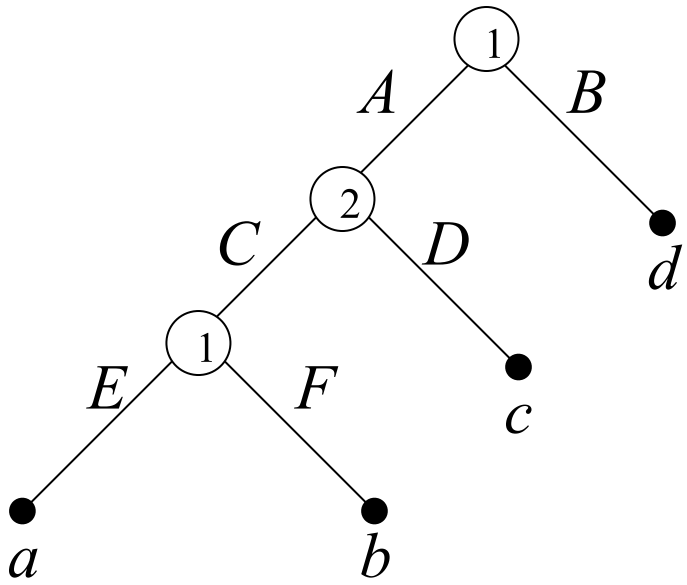
Is this a strategy for Player 1? N	***Is this a strategy for Player 2?*** N
$h = \emptyset$: (2,0) $h = (2,0)$: y	$h = (1,1)$: n $h = (0,2)$: n

A Possible Strategy for Player 1	***Is this a Strategy for Player 2?***
$h = \emptyset$: (2,0)	$h = (2,0)\ : y$ $h = (0,2)\ : n$

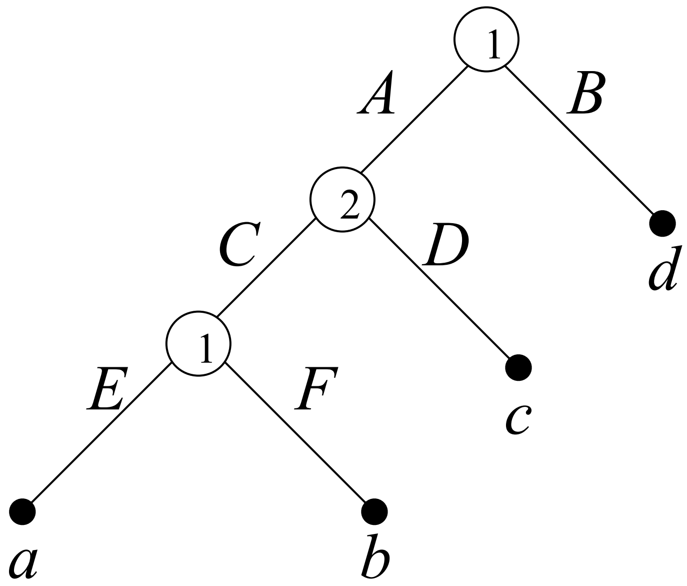
Strategies

DEFINITION. A **strategy of player** $i \in N$ in an extensive game with perfect information $\langle N, H, P, (\succsim_i) \rangle$ is a function that assigns an action in $A(h)$ to each nonterminal history $h \in H \backslash Z$ for which $P(h) = i$.

EXAMPLE. Consider the following extensive game with perfect information

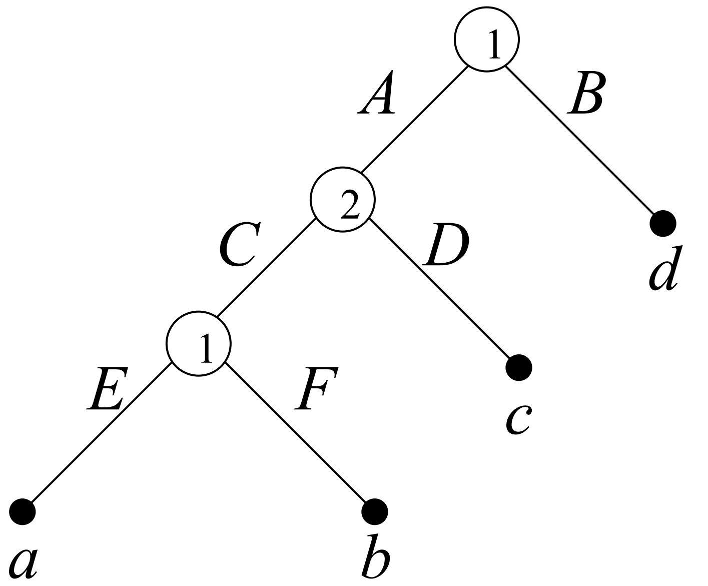


Possible strategies for Player 1: __, __, __, and __.
Possible strategies for Player 2: __ and __.



Consider the strategies of player 1.

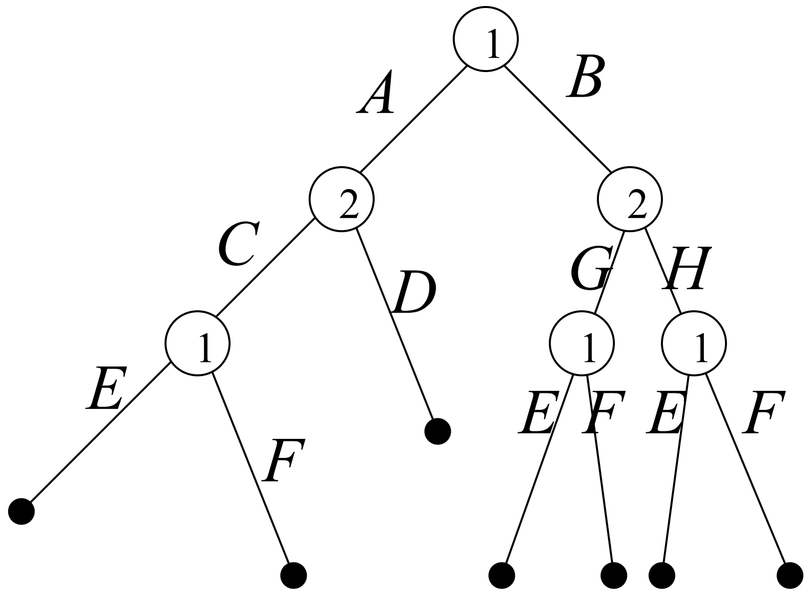
- The strategy AE is the following function: $\{\emptyset \mapsto A, (A, C) \mapsto E\}$.
- The strategy BF is the following function: $\{\emptyset \mapsto B, (A, C) \mapsto F\}$.



Suppose strategy profile is $s = (AE, D)$. The **outcome** $O(s)$ of this strategy profile s is the history $(A, D) \in H$.

What is the outcome of the strategy profile (AE, C)?

EXAMPLE. Consider the following extensive game with perfect information



Q: What are the possible strategies for Player 1?

Q: What are the possible strategies for Player 2?

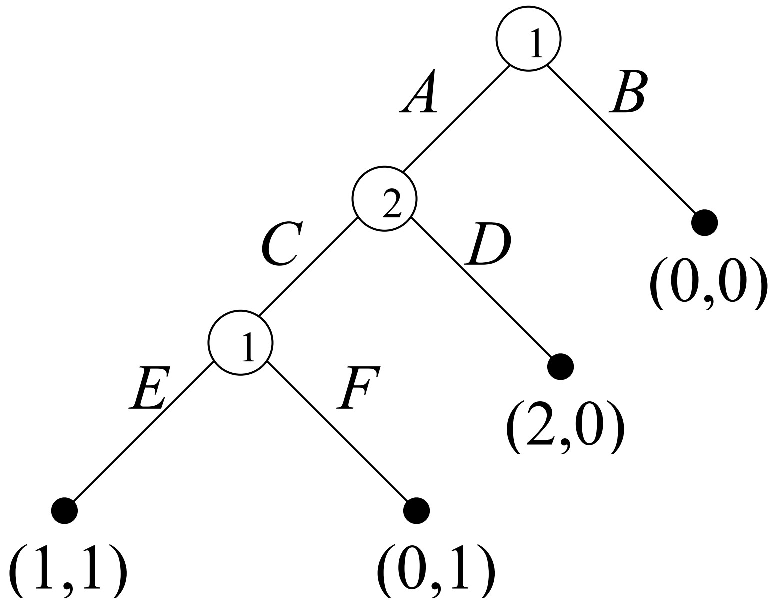
Nash Equilibrium

DEFINITION. A **Nash equilibrium of an extensive game with perfect information** $\Gamma = \langle N, H, P, (\succsim_i)\rangle$ is a strategy profile s^* such that for every player $i \in N$ we have

$$O(s^*_{-i}, s^*_i) \succsim_i O(s^*_{-i}, s_i)$$

for every strategy s_i of player i.

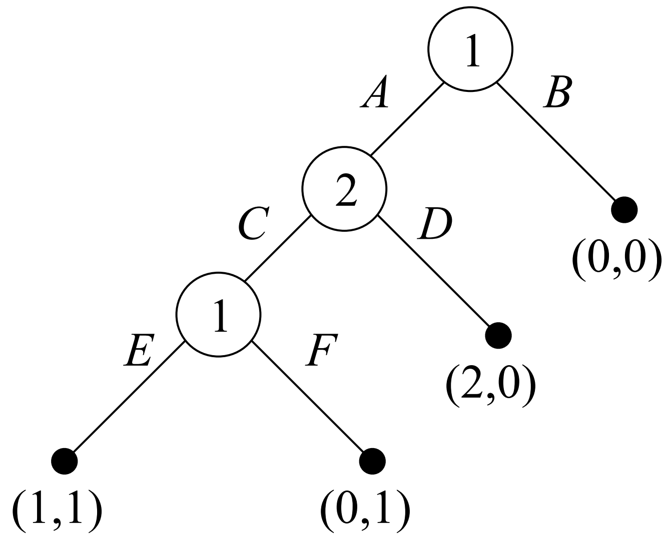
Class Discussion



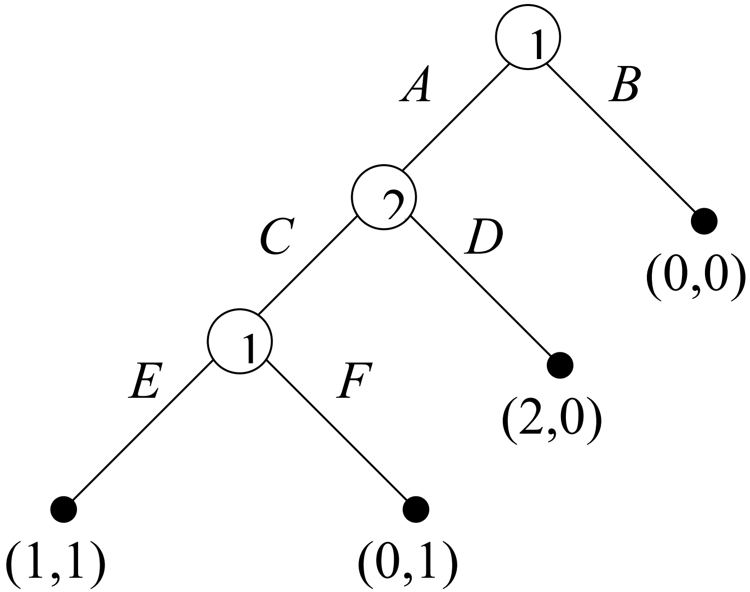
Q: Is (AE, D) a Nash equilibrium?

Q: Is (AE, C) a Nash equilibrium?

Preference Ordering on Strategy Profiles



- Player 1 has four strategies: $S_1 = \{AE, AF, BE, BF\}$.
- Player 2 has two strategies: $S_2 = \{C, D\}$.
- Player 1 evaluates each $s \in S_1 \times S_2$. For instances, $(AE, C) \succsim_1' (AF, C)$, $(AF, C) \succsim_1' (BF, D)$, and so on.
- Player 2 evaluates each $s \in S_1 \times S_2$. For instances, $(AE, C) \succsim_2' (AF, D)$, $(AF, C) \succsim_2' (BF, D)$, *etc*.



Player 2

Player 1		*C*	*D*
	AE	1,1	2,0
	AF	0,1	2,0
	BE	0,0	0,0
	BF	0,0	0,0

This reminds us of Strategic Games…

DEFINITION. The **strategic form of the extensive game with perfect information** $\Gamma = \langle N, H, P, (\succsim_i) \rangle$ is the strategic game $\langle N, (S_i), (\succsim_i') \rangle$ in which for each player $i \in N$

- S_i is the set of strategies of player i in Γ.
- $\succsim_i'$ is defined by $s \succsim_i' s'$ if and only if $O(s) \succsim_i O(s')$ for every $s \in \times_{i \in N} S_i$ and $s' \in \times_{i \in N} S_i$.

$\boxed{\text{A Nash equilibrium of an extensive game with perfect information is a Nash equilibrium of its strategic form.}}$

Reduced Strategies

Consider player 1's strategies

- $BE = \{\emptyset \mapsto B, (A, C) \mapsto E\}$ and
- $BF = \{\emptyset \mapsto B, (A, C) \mapsto F\}$.

The components $(A, C) \mapsto E$ and $(A, C) \mapsto F$ are in fact unnecessary because the history (A, C) is inconsistent with the action B taken.

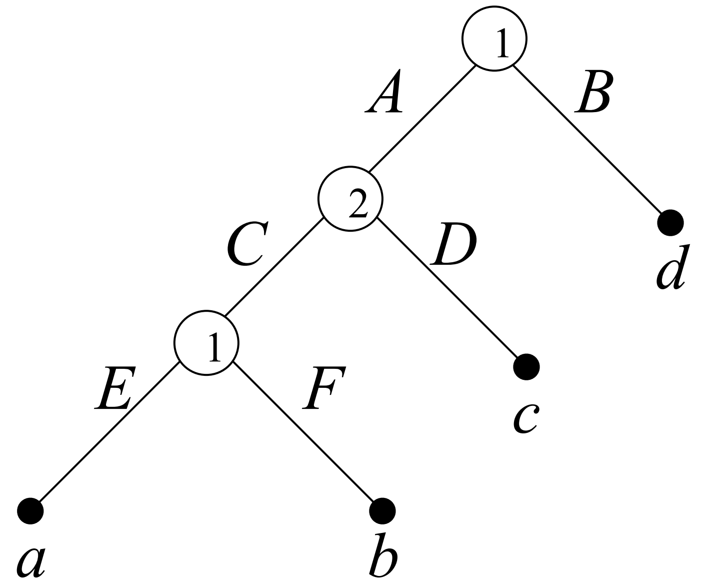
These strategies BE and BF can be reduced.

Reduced Strategies

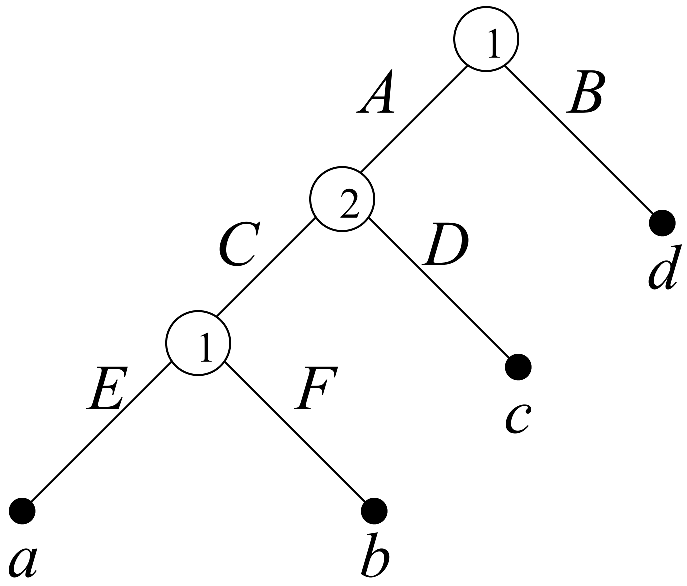
However, strategies

- $AE = \{\emptyset \mapsto A, (A, C) \mapsto E\}$
  and
- $AF = \{\emptyset \mapsto A, (A, C) \mapsto F\}$.

cannot be reduced.



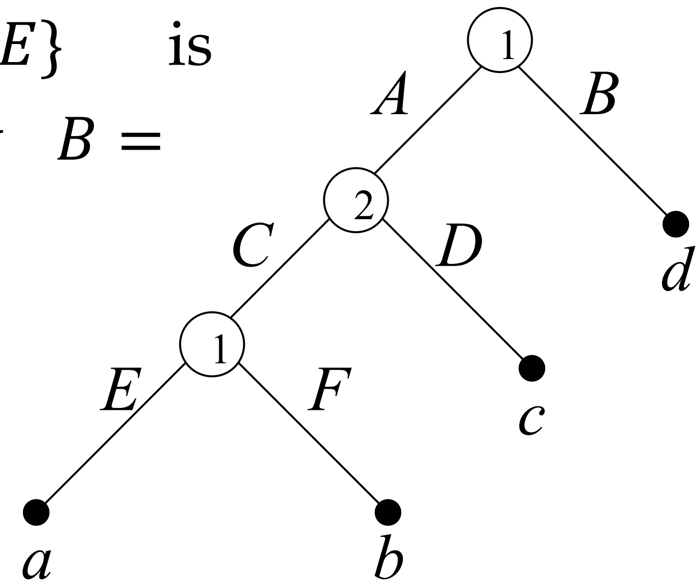
Reduced Strategies



Reduced strategies for Player 1: AE, AF, and B.

Reduced strategies for Player 2: C and D.

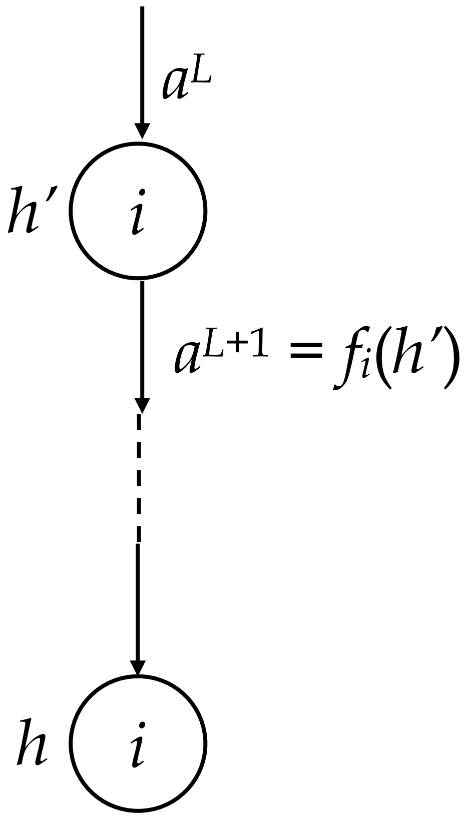
Strategy $BE = \{\emptyset \mapsto B, (A,C) \mapsto E\}$ is reduced to a **reduced strategy** $B = \{\emptyset \mapsto B\}$.



The domain of a strategy s_i of player i is the set $\{h \in H\colon P(h) = i\}$.

The domain of a reduced strategy f_i of player i is a subset of the set $\{h \in H\colon P(h) = i\}$.

The domain of a reduced strategy f_i of player i is a subset of the set $\{h \in H : P(h) = i\}$.



We include a history $h = (a^k)$ with $P(h) = i$ in the domain of a reduced strategy f_i if and only if all the actions of player i in h are those dictated by f_i:

If $h' = (a^k)_{k=1,\cdots L}$ is a subsequence of h with $P(h') = i$, then $f_i(h') = a^{L+1}$.

A reduced strategy of player i is a function f_i whose domain is a subset of $\{h \in H: P(h) = i\}$:

- f_i associates with each h in the domain of f_i an action in $A(h)$.
- A history $h = (a^k)$ is in the domain of f_i if and only if the following condition holds:

  If $h' = (a^k)_{k=1,\ldots,L}$ is a subsequence of h with $P(h') = i$, then $f_i(h') = a^{L+1}$.

Q: Is BE a reduced strategy of Player 1? Is BF?

A: Consider $h = (A, C)$ and $h' = \emptyset$.

We see that for other players, the strategies BE and BF are effectively equivalent, regardless of whatever strategies other players may take.

This reduced strategy
corresponds to
the equivalent class $\{BE, BF\}$.

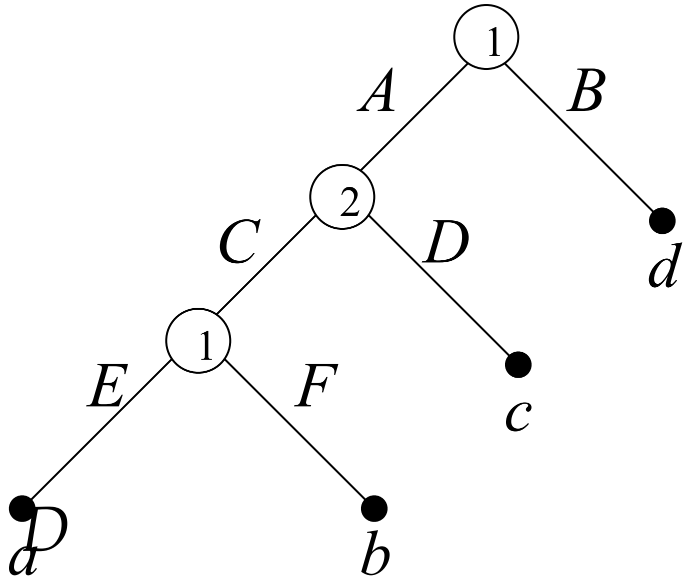
So we can actually call this reduced strategy $\{\emptyset \mapsto B\}$ by the name of BE, BF, or B.

Each reduced strategy of player i corresponds to a set of strategies of player i; for each vector of strategies of the other players each strategy in this set yields the same outcome (*i.e.*, these strategies are all *outcome-equivalent*).

Therefore,

The set of Nash equilibria of an extensive game with perfect information corresponds to the Nash equilibria of the strategic game in which the set of actions of each player is the set of its reduced strategies.

DEFINITION. Let $\Gamma = \langle N, H, P, (\succsim_i) \rangle$ be an extensive game with perfect information and let $\langle N, (S_i), (\succsim_i') \rangle$ be its strategic form. For any $i \in N$ define the strategies $s_i \in S_i$ and $s_i' \in S_i$ of player i be *equivalent* if for each $s_{-i} \in S_{-i}$ we have $(s_{-i}, s_i) \sim_j' (s_{-i}, s_i')$ for all $j \in N$. The **reduced strategic form of** Γ is the strategic game $\langle N, (S_i'), (\succsim_i'') \rangle$ in which for each $i \in N$ each set S_i' contains one member of each set of equivalent strategies in S_i, and $\succsim_i''$ is the preference ordering over $\times_{j \in N} S_j'$ induced by $\succsim_i'$.

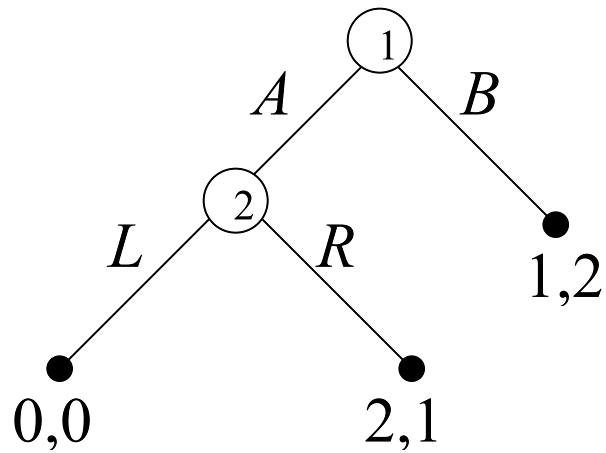


	C	D
AE	*a*	*c*
AF	*b*	*c*
BE	*d*	*d*
BF	*d*	*d*

	C	D
AE	*a*	*c*
AF	*b*	*c*
B	*d*	*d*

Strategic Form **Reduced Strategic Form**

Class Discussion

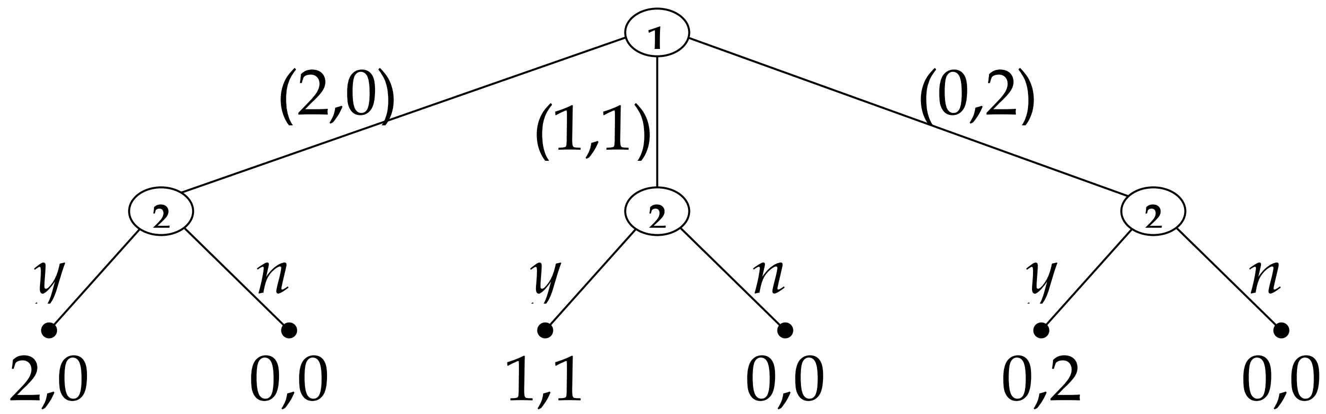


Q: What is its strategic form?

Q: What are the Nash equilibria?

Q: (B, L)?   What's that?

Class Discussion



Q: What is its strategic form?

Q: What are the Nash equilibria?

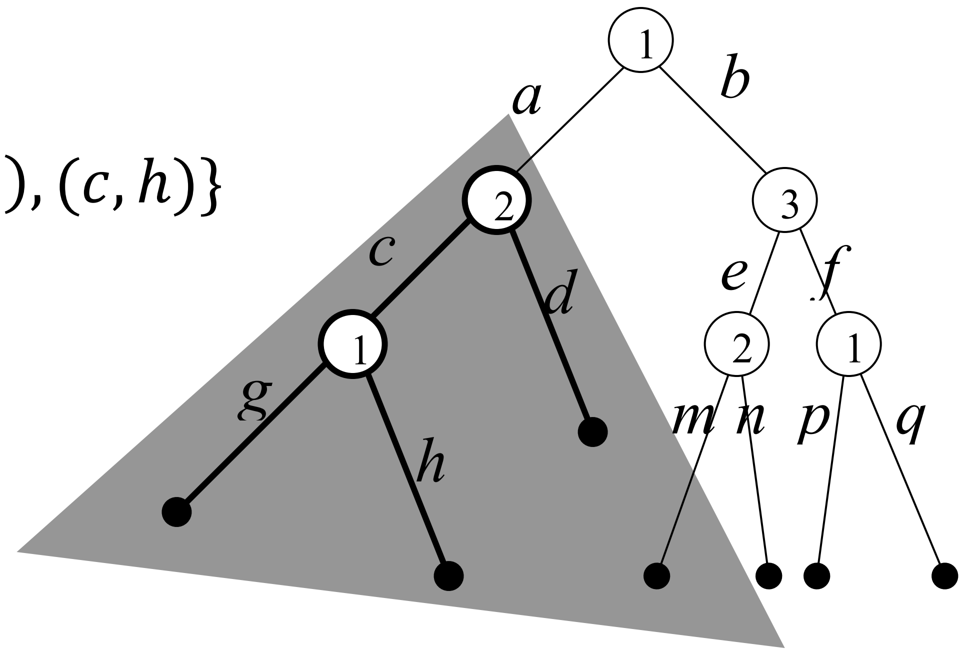
	yyy	*yyn*	*yny*	*ynn*	*nyy*	*nyn*	*nny*	*nnn*
(2,0)	**2,0**	**2,0**	**2,0**	**2,0**	0,0	0,0	**0,0**	**0,0**
(1,1)	1,1	1,1	0,0	0,0	**1,1**	**1,1**	0,0	0,0
(0,2)	0,2	0,0	0,2	0,0	0,2	0,0	**0,2**	0,0

A: ((2,0), *yyy*), ((2,0), *yyn*), ((2,0), *yny*), ((2,0), *ynn*), **((1,1), *nyy*)**, ((1,1), *nyn*), ((0,2), *nny*), ((2,0), *nny*), and ((2,0), *nnn*).

Subgames

Subgame $\Gamma(a)$

$H|_a = \{\emptyset, c, d, (c,g), (c,h)\}$
$P|_a(\emptyset) = 2$
$P|_a(c) = 1$

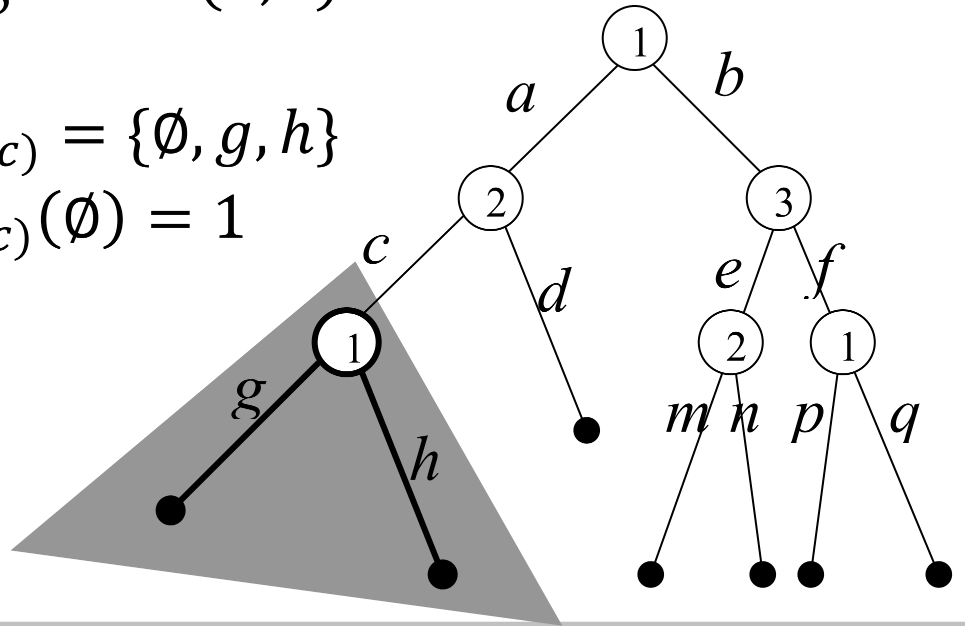


- $H|_a$ contains h if and only if H contains (a, h).
- $P|_a(h) = P(a, h)$.

Subgames

Subgame $\Gamma(a, c)$

$H|_{(a,c)} = \{\emptyset, g, h\}$
$P|_{(a,c)}(\emptyset) = 1$

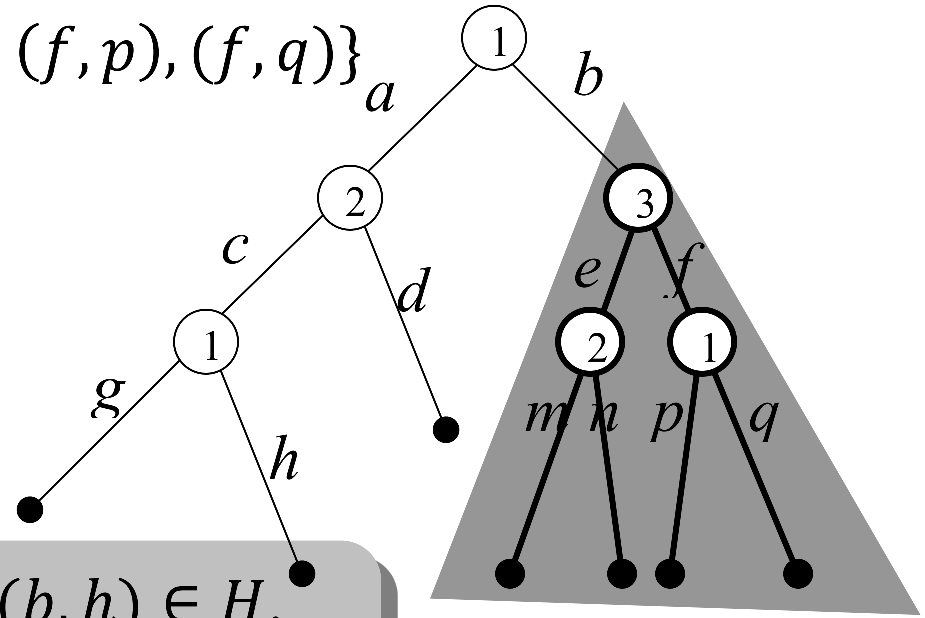


- $h \in H|_{(a,c)}$ if and only if $(a, c, h) \in H$.
- $P|_{(a,c)}(h) = P(a, c, h)$.

Subgames

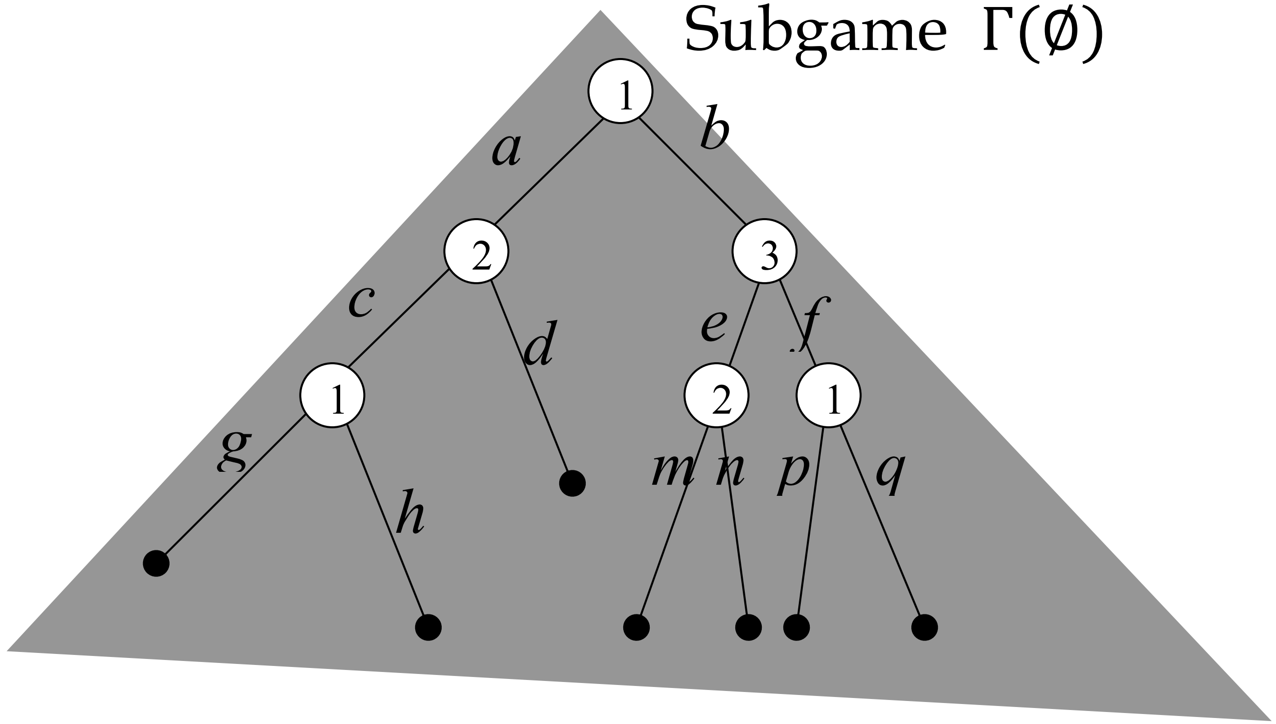
Subgame $\Gamma(b)$

$H|_b = \{\emptyset, e, f, (e,m), (e,n), (f,p), (f,q)\}$
$P|_b(\emptyset) = 3$
$P|_b(e) = 2$
$P|_b(f) = 1$



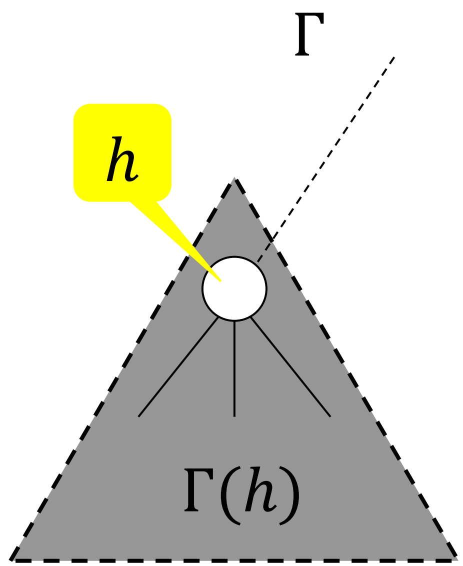
- $h \in H|_b$ if and only if $(b,h) \in H$.
- $P|_b(h) = P(b,h)$.

Subgames



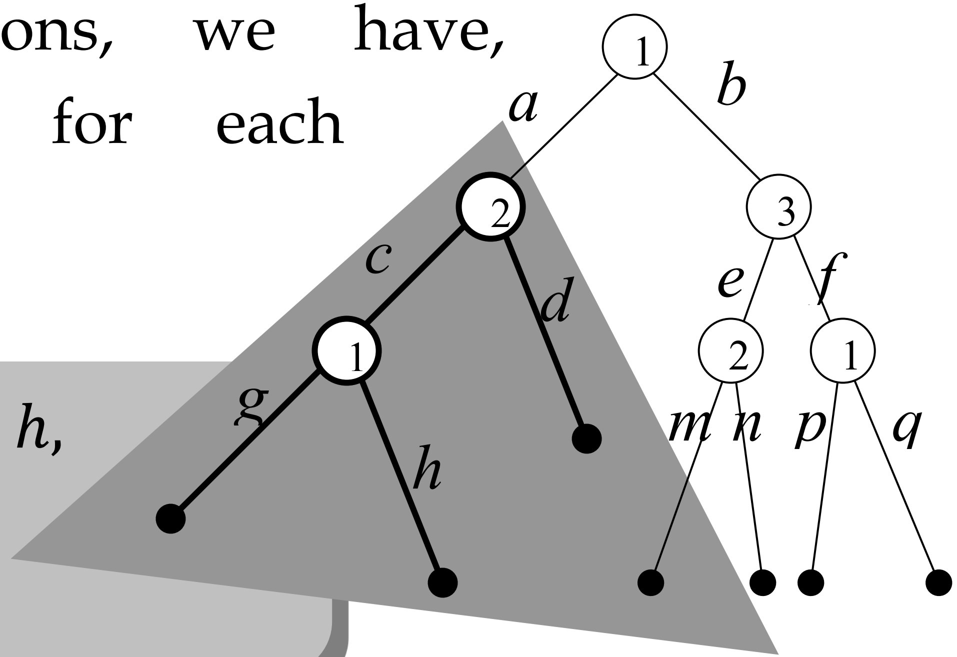
Subgames

DEFINITION. The **subgame of the extensive game with perfect information** $\Gamma = \langle N, H, P, (\succsim_i) \rangle$ **that follows the history** h is the extensive game $\Gamma(h) = \langle N, H|_h, P|_h, (\succsim_i|_h) \rangle$, where $H|_h$ is the set of sequences h' of actions for which $(h, h') \in H$, $P|_h$ is defined by $P|_h(h') = P(h, h')$ for each $h' \in H|_h$, and $\succsim_i|_h$ is defined by $h' \succsim_i|_h h''$ if and only if $(h, h') \succsim_i (h, h'')$.



For a strategy s_i of player i, denote by $s_i|_h$ the strategy that s_i induces in the subgame $\Gamma(h) = \langle N, H|_h, P|_h, (\succsim_i|_h) \rangle$.

Recall that a strategy is a function mapping from histories to actions, we have, $s_i|_h(h') = s_i(h, h')$ for each $h' \in H|_h$.



$$s_1 = \{\emptyset \mapsto a, (a, c) \mapsto h, (b, f) \mapsto p\}$$

$$s_1|_a = \{c \mapsto h\}$$

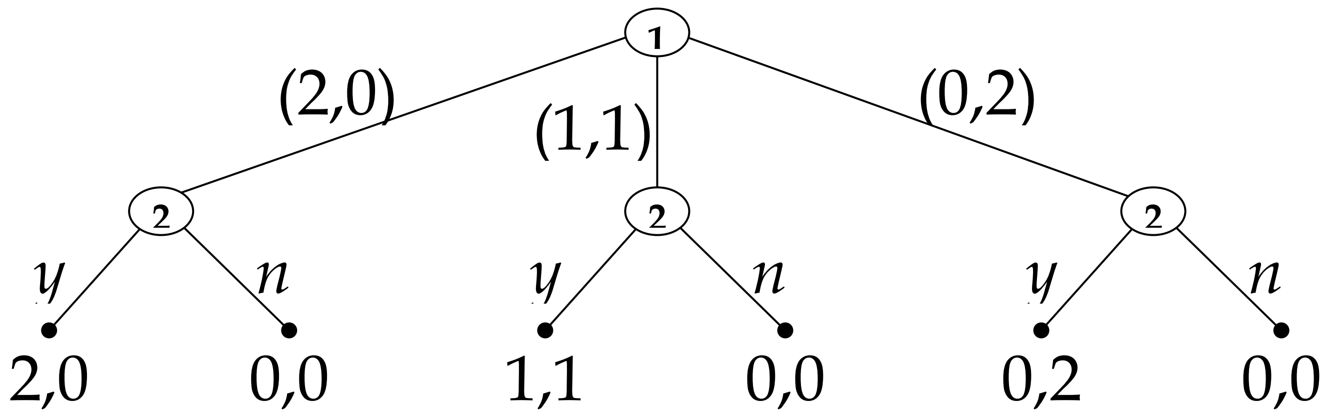
Subgame Perfect Equilibrium

A strategy profile s^* is a subgame perfect equilibrium of $\Gamma = \langle N, H, P, (\succsim_i) \rangle$,

if and only if

for <u>any</u> history h, $s^*|_h$ is a Nash equilibrium of the subgame $\Gamma(h) = \langle N, H|_h, P|_h, (\succsim_i|_h) \rangle$.

Subgame Perfect Equilibrium

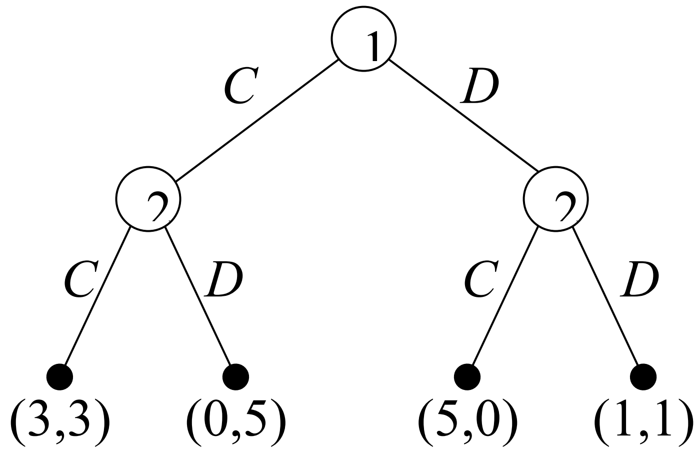


<u>Nash equilibria</u>

((2,0), *yyy*), ((2,0), *yyn*), ((2,0), *yny*), ((2,0), *ynn*),
((1,1), *nyy*), ((1,1), *nyn*), ((0,2), *nny*), ((2,0), *nny*)
((2,0), *nnn*)

Any of these are subgame perfect equilibria?

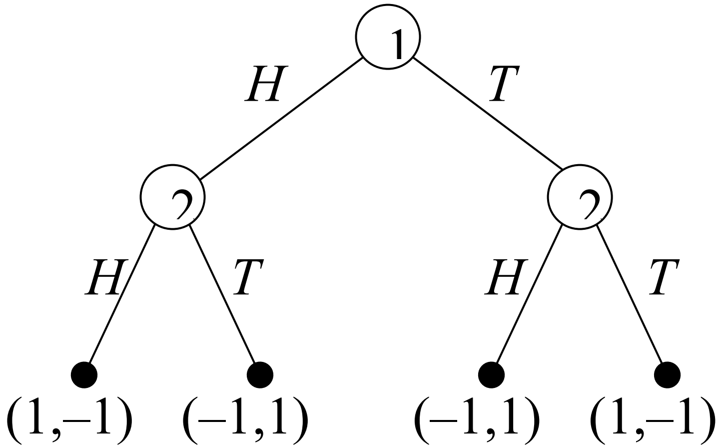
Subgame Perfect Equilibrium



Q: What are the Nash equilibria?

Q: What are the subgame perfect equilibria?

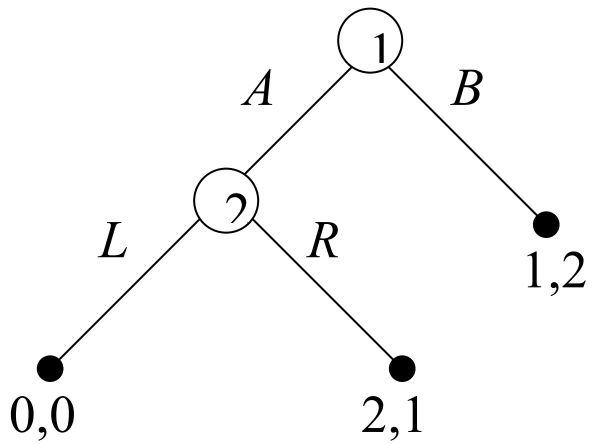
Subgame Perfect Equilibrium



Q: What are the Nash equilibria?

Q: What are the subgame perfect equilibria?

Class Discussion



Q: What are the Nash equilibria?

Q: What are subgame perfect equilibria?

A: Check for every player and every subgame.

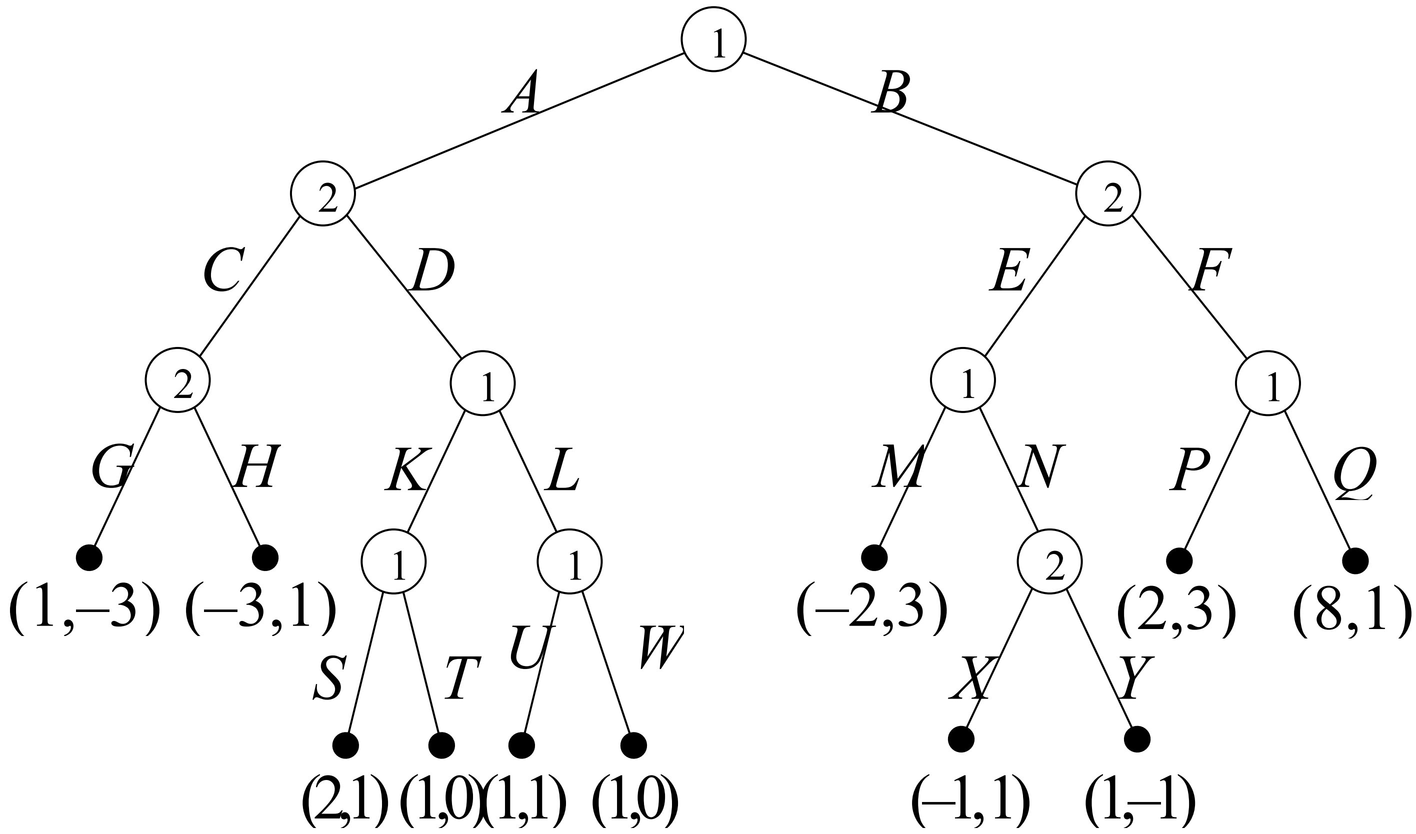
Subgame Perfect Equilibrium

DEFINITION. The **subgame perfect equilibrium of an extensive game with perfect information** $\Gamma = \langle N, H, P, (\succsim_i)\rangle$ is the strategy profile s^* such that for every player $i \in N$ and <u>every</u> nonterminal history $h \in H \backslash Z$ for which $P(h) = i$ we have

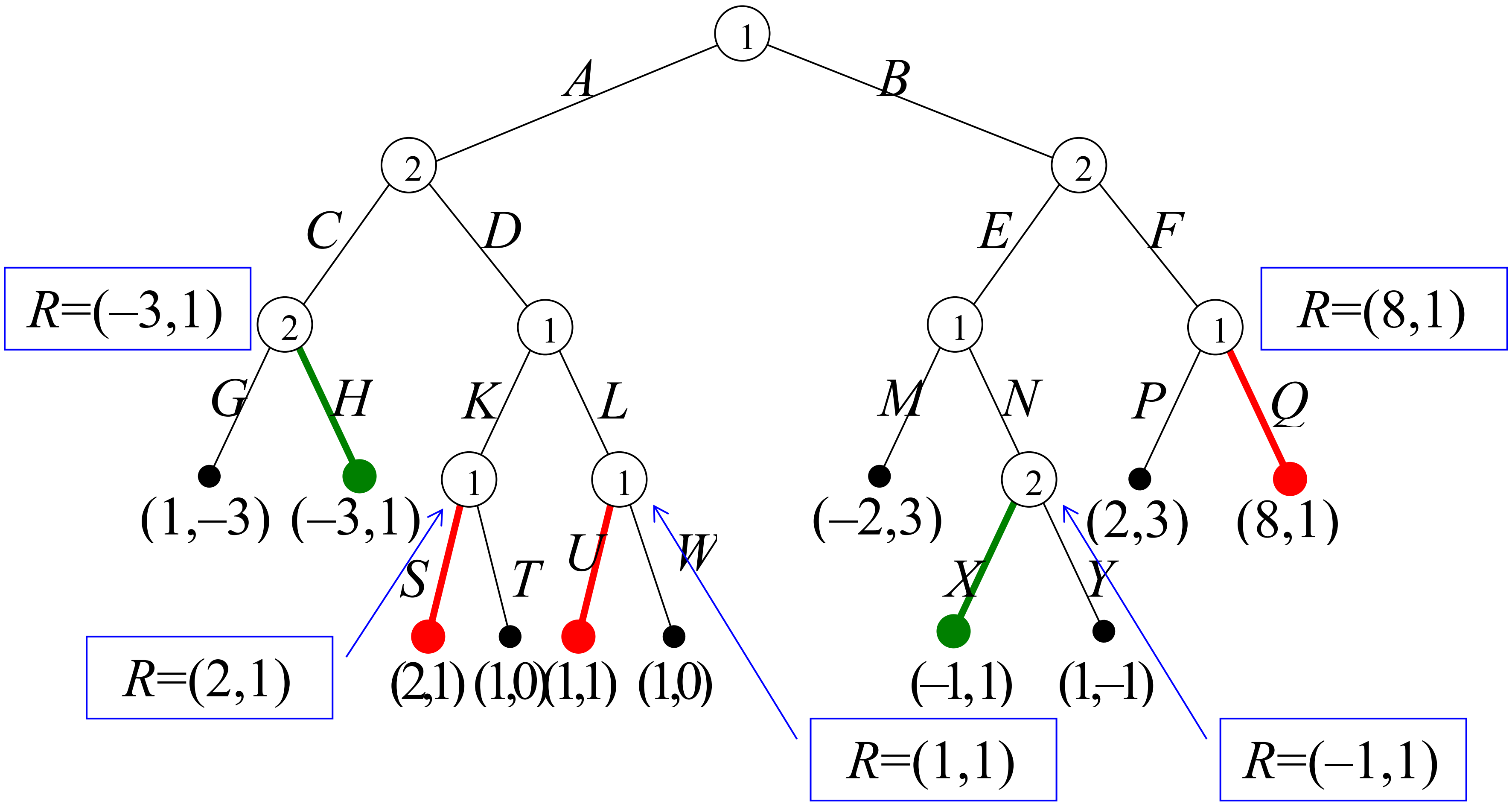
$$O_h(s^*_{-i}|_h, s^*_i|_h) \succsim_i|_h O_h(s^*_{-i}|_h, s_i)$$

for every strategy s_i of player i in the subgame $\Gamma(h)$. (*Note:* O_h *is the outcome function of* $\Gamma(h)$.)

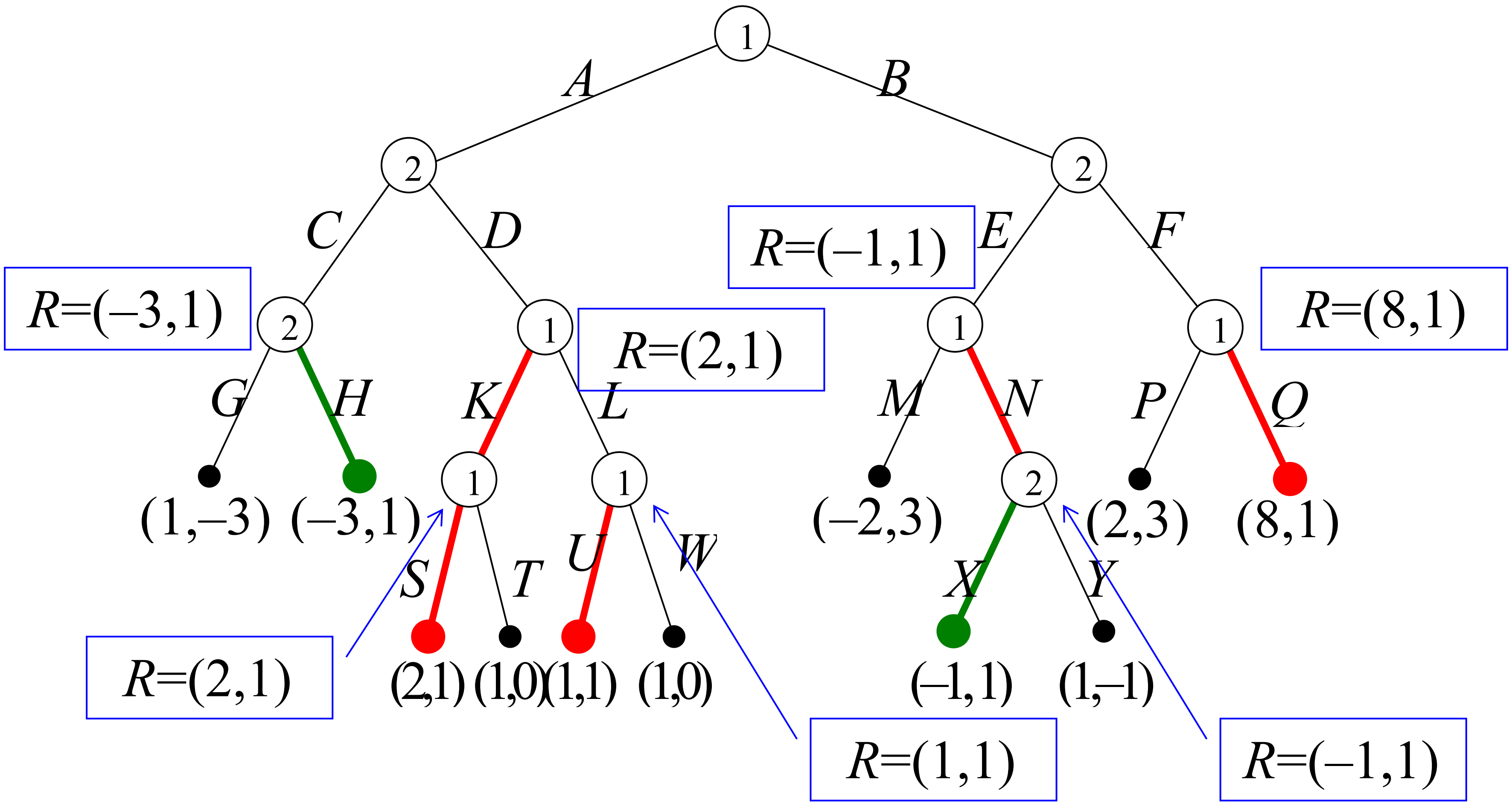
Subgame Perfect Equilibrium



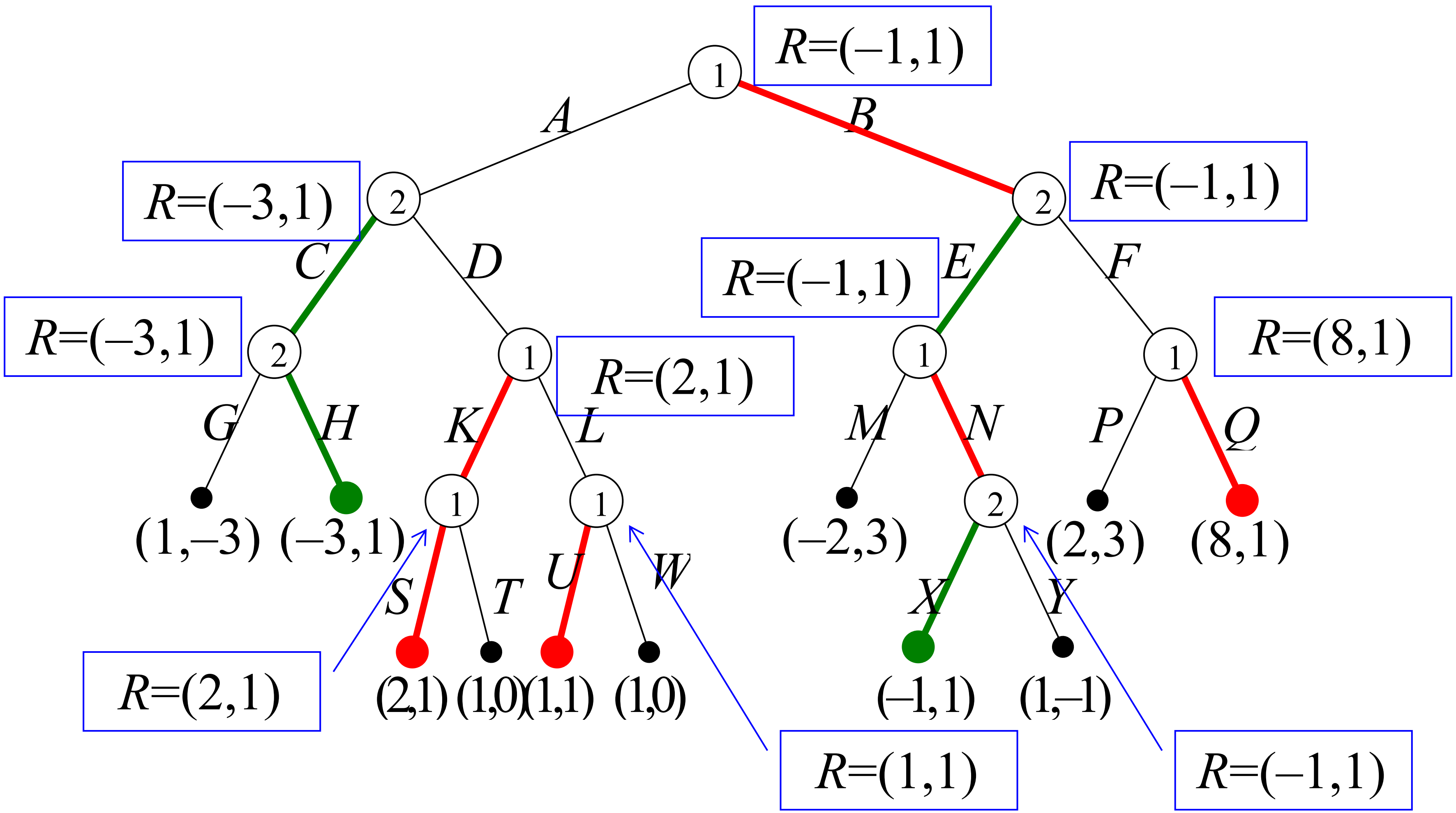
Subgame Perfect Equilibrium



Subgame Perfect Equilibrium



Subgame Perfect Equilibrium



Existence of Subgame Perfect Equilibrium

PROPOSITION. (Kuhn's theorem) Every finite extensive game with perfect information has a subgame perfect equilibrium.

Subgame Perfect Equilibrium

DEFINITION. The **subgame perfect equilibrium of an extensive game with perfect information** $\Gamma = \langle N, H, P, (\succsim_i)\rangle$ is the strategy profile s^* such that for every player $i \in N$ and <u>every</u> nonterminal history $h \in H \backslash Z$ for which $P(h) = i$ we have

$$O_h(s^*_{-i}|_h, s^*_i|_h) \succsim_i|_h O_h(s^*_{-i}|_h, s_i)$$

for every strategy s_i of player i in the subgame $\Gamma(h)$. (*Note:* O_h *is the outcome function of* $\Gamma(h)$.)

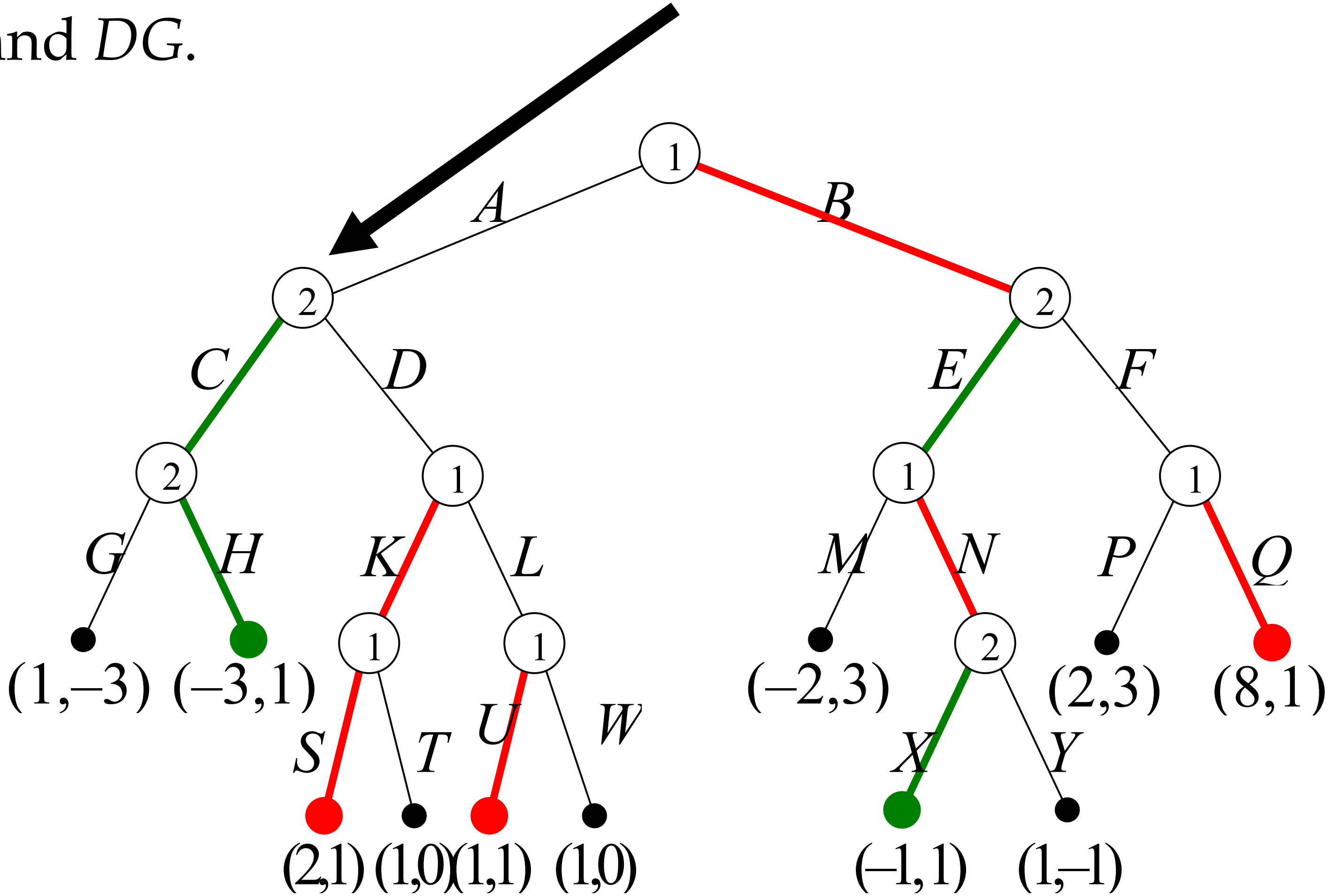
Subgame Perfect Equilibrium

In other words, the strategy profile s^* is a subgame perfect equilibrium of $\Gamma = \langle N, H, P, (\succsim_i) \rangle$,

if and only if

for <u>any</u> history h, $s^*|_h$ is a Nash equilibrium of the subgame $\Gamma(h) = \langle N, H|_h, P|_h, (\succsim_i|_h) \rangle$.

For player 2 in subgame $\Gamma(A)$, we check *CH*, *DH*, *CG*, and *DG*.



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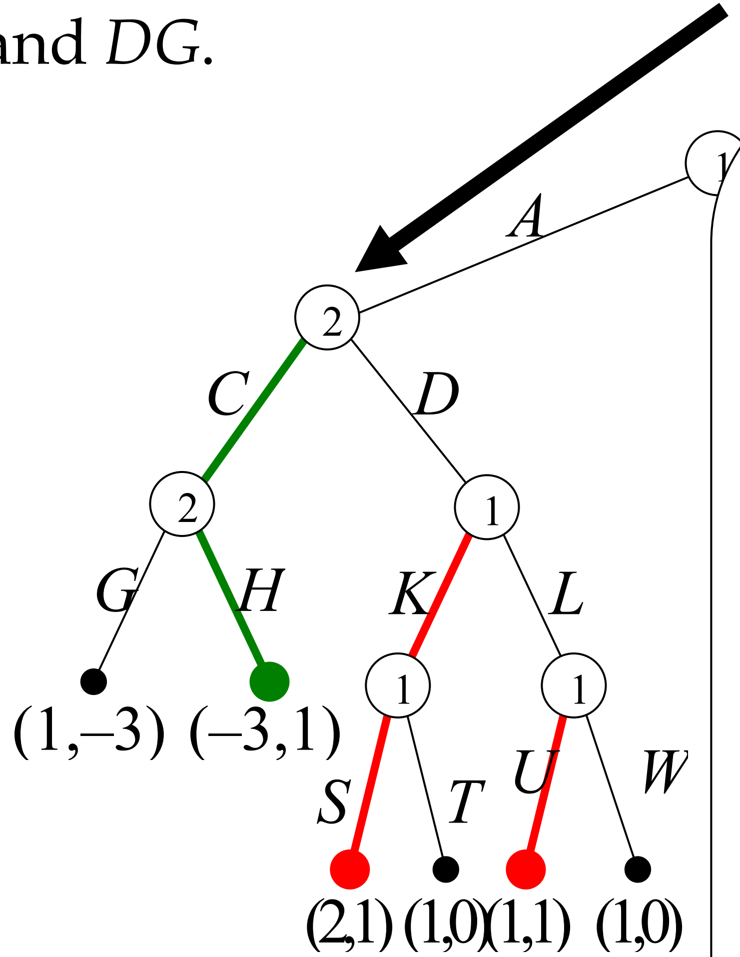


Q: Can we (be lazy and) <u>only</u> check, for each subgame, the player who makes the first move cannot obtain a better outcome by <u>*changing only his*</u> <u>**initial action**</u> in the subgame?

E.g., for player 2 in subgame $\Gamma(A)$, we check that taking D as the first action is not better than C.

(By definition) To verify a strategy profile is a subgame perfect equilibrium, we check for every subgame, the player who makes the first move cannot obtain a better outcome by *changing his* **strategy** in the subgame.

(By definition) To verify a strategy profile is a subgame perfect equilibrium, we check for every subgame, the player who makes the first move cannot obtain a better outcome by *changing his* **strategy** in the subgame.

However,

for a game Γ *with a finite horizon*, actually we only need to check for each subgame, the player who makes the first move cannot obtain a better outcome by *changing only his* **initial action** in the subgame.

LEMMA. (*The one deviation property*) Let $\Gamma = \langle N, H, P, (\succsim_i) \rangle$ be a finite horizon extensive game with perfect information.

The strategy profile s^* is a **subgame perfect equilibrium** of Γ if and only if for every player $i \in N$ and every history $h \in H$ for which $P(h) = i$ we have

$$O_h(s^*_{-i}|_h, s^*_i|_h) \succsim_i|_h O_h(s^*_{-i}|_h, s_i)$$

for every strategy s_i of player i in the subgame $\Gamma(h)$ ***that differs from*** $\boldsymbol{s}^*|_{\boldsymbol{h}}$ ***only in the action it prescribes after the initial history*** $\emptyset$ ***of*** $\Gamma(h)$.

In other words, when checking

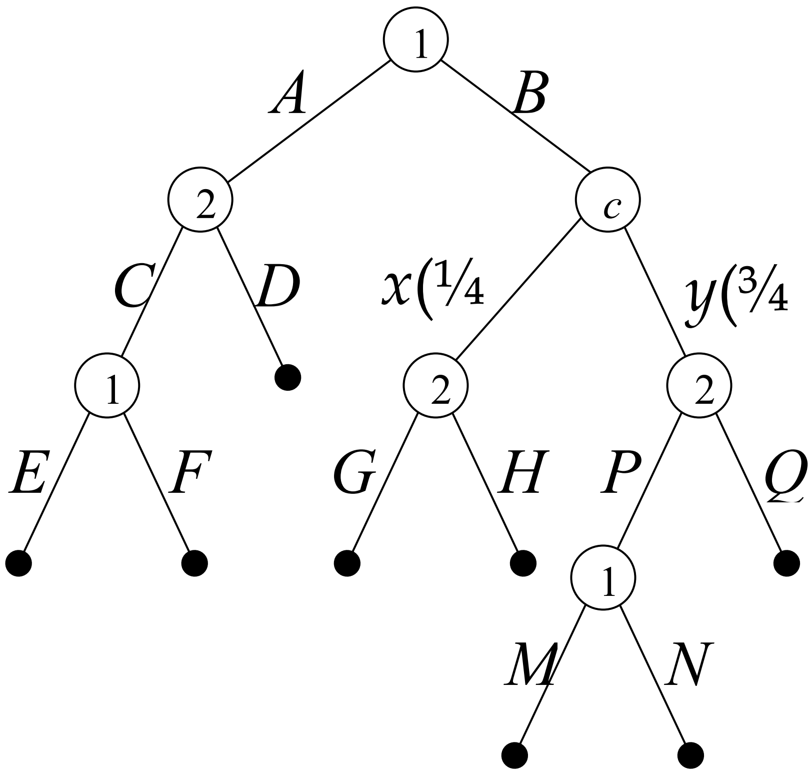
$$O_h(s^*_{-i}|_h, s^*_i|_h) \succsim_i|_h O_h(s^*_{-i}|_h, s_i),$$

we only need to verify that for player i**,**
$s^*_i|_h$ is better than every s_i with which player i should take a different initial action in the subgame $\Gamma(h)$,

instead of having to verify that for player i**,**
$s^*_i|_h$ is better than every s_i.

There are two extensions to extensive games with perfect information.

Extensive Games with Perfect Information and Chance Moves



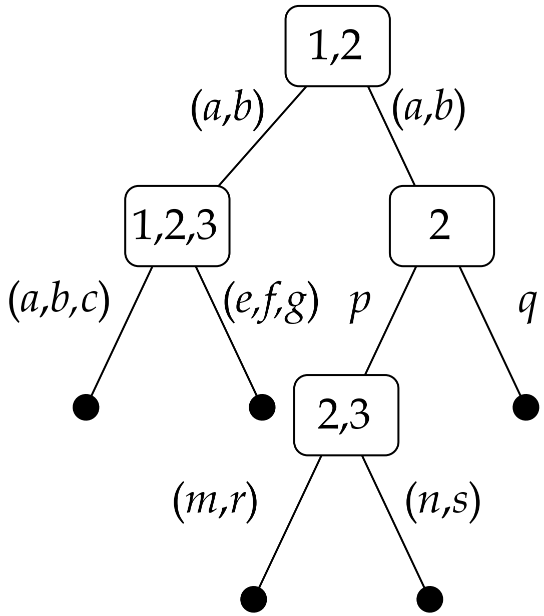
$f_c(x \mid B) = 1/4.$
$f_c(y \mid B) = 3/4.$

Extensive Games with Perfect Information and Chance Moves

$$\langle N, H, P, f_c, (\succsim_i) \rangle$$

Original	*With chance moves*
$P: (H \backslash Z) \to N$	$P: (H \backslash Z) \to N \cup \{c\}$
-	*If* $P(h) = c$, *then* $f_c(a\|h)$ *is the probability that a occurs after h.*
$\succsim_i$ *is a preference relation on* Z	$\succsim_i$ *is a preference relation on lotteries over* Z

Extensive Games with Perfect Information and Simultaneous Moves

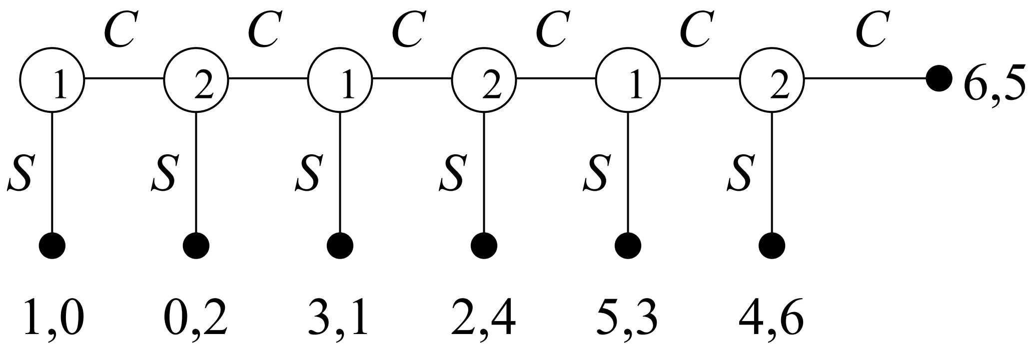


Extensive Games with Perfect Information and Simultaneous Moves

$$\langle N, H, P, (\succsim_i) \rangle$$

Original	*With simultaneous moves*
$P: (H \backslash Z) \rightarrow N$	$P: (H \backslash Z) \rightarrow \wp(N)$
A history is a sequence of actions.	*A history is a sequence of profiles of actions.*
For $h = (a_1, a_2, \dots a_{k-1})$, $a_k \in A_i(h)$, *where* $i = P(h)$.	*For* $h = (a^1, a^2, \dots a^{k-1})$, $a^k \in A(h) = \times_{i \in P(h)} A_i(h)$.

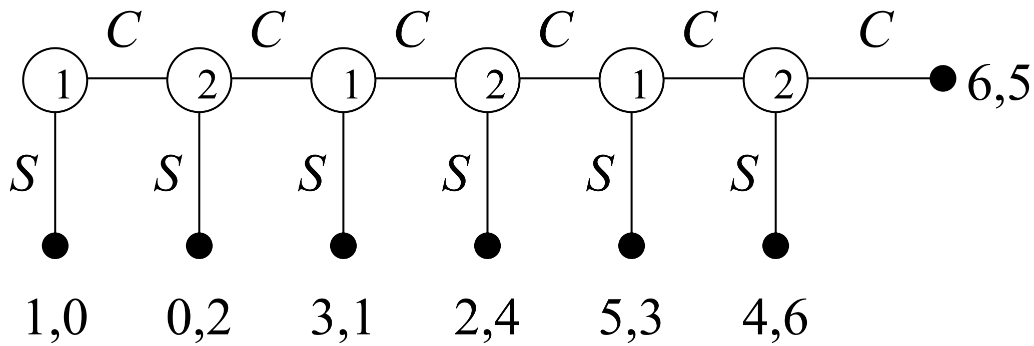
The Centipede Game



A centipede game with $T = 6$. A player prefers stopping **now** than the opponent stopping at **the next period**. The best outcome, however, is that the game stops after period T.

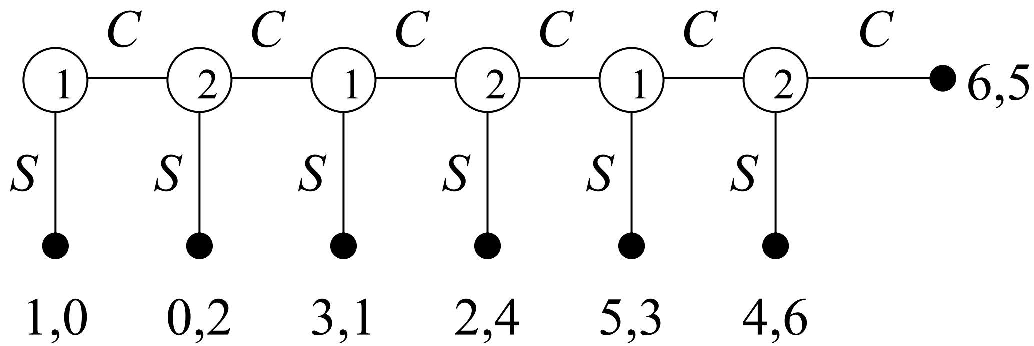
Each player alternatively decides whether to continue or stop. The whole game stops after T periods, where T is even.

Class Discussion



Q: What are the subgame perfect equilibria?

Class Discussion



Q: What are the subgame perfect equilibria?

A: Unique: $(S \cdots S, S \cdots S)$.

Q: Are there Nash equilibria that are not subgame perfect equilibria?