Exercises: Line Integral by Coordinate

Problem 1. Let C be the curve from point $p = (0, 0)$ to $q = (2, 4)$ on the parabola $y = x^2$. Calculate $\int_C (x^2 - y^2) dx$.

Solution: First, write C into its parametric form: $\boldsymbol{r}(t) = [x(t), y(t)]$ where $x(t) = t$, and $y(t) = t^2$. Points p and q are given by $t = 0$ and 2, respectively. Thus:

$$\begin{aligned}
\int_C (x^2 - y^2) dx &= \int_0^2 (t^2 - t^4) \frac{dx}{dt} dt \\
&= \int_0^2 (t^2 - t^4) dt \\
&= 8/3 - 32/5.
\end{aligned}$$

Problem 2. Let $\boldsymbol{r}(t) = [t, t^2, t^3]$ and $\boldsymbol{f}(\boldsymbol{r}) = [x - y, y - z, z - x]$. Let C be the curve from the point of $t = 0$ to the point of $t = 1$. Calculate $\int_C \boldsymbol{f}(\boldsymbol{r}) \cdot d\boldsymbol{r}$.

Solution:

$$\begin{aligned}
\int_C \boldsymbol{f}(\boldsymbol{r}) \cdot d\boldsymbol{r} &= \int_0^1 \boldsymbol{f}(\boldsymbol{r}) \cdot \boldsymbol{r}'(t)\, dt \\
&= \int_0^1 [t - t^2, t^2 - t^3, t^3 - t] \cdot [1, 2t, 3t^2]\, dt \\
&= \int_0^1 t - t^2 + 2t^3 - 2t^4 + 3t^5 - 3t^3\, dt \\
&= \int_0^1 t - t^2 - t^3 - 2t^4 + 3t^5\, dt \\
&= 1/60. \qquad (1)
\end{aligned}$$

Problem 3. Let $\boldsymbol{r}(t) = [x(t), y(t)]$ where $x(t) = \cos(t)$ and $y(t) = \sin(t)$. Let p be the point given by $t = \pi/4$. Calculate $\frac{dx}{ds}$ at p.

Solution:

$$\begin{aligned}
\frac{dx}{ds} &= \frac{dx/dt}{ds/dt} \\
&= \frac{dx/dt}{\sqrt{(dx/dt)^2 + (dy/dt)^2}} \\
&= \frac{x'(t)}{\sqrt{(x'(t))^2 + (y'(t))^2}} \\
&= \frac{-\sin(t)}{\sqrt{(-\sin(t))^2 + (\cos(t))^2}} \\
&= -\sin(t).
\end{aligned}$$

Hence, the value of $\frac{dx}{ds}$ at p is $-\sin(\pi/4) = -1/\sqrt{2}$.

Problem 4. Let $\boldsymbol{r}(t) = [x(t), y(t), z(t)]$. Let p be the point given by $t = t_0$. Prove that $[\frac{dx}{ds}(t_0), \frac{dy}{ds}(t_0), \frac{dz}{ds}(t_0)]$ is a unit tangent vector at p.

Proof:

$$\frac{dx}{ds} = \frac{dx/dt}{ds/dt} = \frac{dx/dt}{\sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2}}$$

Similarly:

$$\begin{aligned} \frac{dy}{ds} &= \frac{dy/dt}{ds/dt} = \frac{dy/dt}{\sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2}} \\ \frac{dz}{ds} &= \frac{dz/dt}{ds/dt} = \frac{dz/dt}{\sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2}}. \end{aligned}$$

Therefore:

$$\left[\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right] = \frac{[x'(t), y'(t), z'(t)]}{\sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2}}$$

which proves that $[\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}]$ is a tangent vector. Furthermore:

$$\left|\left[\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right]\right|^2 = \frac{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2}{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2} = 1$$

which means that $[\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}]$ is a unit vector. $\square$

Problem 5. This problem allows you to see the equivalence of line integral by length and line integral by coordinate. Let $\boldsymbol{r}(t) = [x(t), y(t)]$ where $x(t) = \cos(t)$ and $y(t) = \sin(t)$. Convert $\int_C x\,dx + \int_C y^2\,dy$ to line integral by length.

Solution:

$$\begin{aligned} \int_C x\,dx + \int_C y^2\,dy &= \int_C x\,\frac{dx}{ds}ds + \int_C y^2\,\frac{dy}{ds}ds \\ &= \int_C x\,\frac{dx}{ds} + y^2\,\frac{dy}{ds}\,ds \end{aligned} \tag{2}$$

In Problem 4, we have shown that $\frac{dx}{ds} = -\sin(t) = -y(t)$. Similarly:

$$\begin{aligned} \frac{dy}{ds} &= \frac{dy/dt}{ds/dt} \\ &= \frac{dy/dt}{\sqrt{(dx/dt)^2 + (dy/dt)^2}} \\ &= \frac{y'(t)}{\sqrt{(x'(t))^2 + (y'(t))^2}} \\ &= \frac{\cos(t)}{\sqrt{(-\sin(t))^2 + (\cos(t))^2}} \\ &= x(t). \end{aligned}$$

Hence:

$$(2) = \int_C -xy + y^2 x\,ds.$$