Probability 2017 Homework 4

Problem 1. (10 points) The random variables X, Y and Z are independent and uniformly distributed in $[0,1]$. Find the PDF of $X+Y+Z$.

Solution:
We first consider the sum of two random variables. Denote the random variable $A = X + Y$, then we have

$$f_A(a) = \int_{-\infty}^{\infty} f_X(x) f_Y(a-x) dx.$$

Since $f_X(x) = 1$ if $0 \leq x \leq 1$ and 0 otherwise, this becomes

$$f_A(a) = \int_0^1 f_Y(a-x) dx.$$

Now the integrand is 1 if $a - 1 \leq x \leq a$ and 0 otherwise. Therefore if $0 \leq a \leq 1$,

$$f_A(a) = \int_0^a dx = a,$$

while if $1 < a \leq 2$,

$$f_A(a) = \int_{a-1}^1 dx = 2 - a.$$

Hence,

$$f_A(a) = \begin{cases} a, & 0 \leq a \leq 1 \\ 2-a, & 1 < a \leq 2 \\ 0, & \text{otherwise.} \end{cases}$$

Next, denote $B = A + Z = X + Y + Z$. Then

$$f_B(b) = \int_{-\infty}^{\infty} f_Z(z) f_A(b-z) dz.$$

Noted that

(1) $f_A(b-z) = b - z$ if $0 \leq b - z \leq 1, i.e., b - 1 \leq z \leq b$, and $0 \leq z \leq 1$. Combining these two gives $\max(b-1, 0) \leq z \leq \min(b, 1)$.

(2) $f_A(b-z) = 2 - b + z$ if $1 < b - z \leq 2$, i.e., $b - 2 \leq z \leq b - 1$, and $0 \leq z \leq 1$. Combining these two gives $\max(b-2, 0) \leq z \leq \min(b-1, 1)$. We break the range of b into $0 \leq b \leq 1$, $1 < b \leq 2$ and $2 < b \leq 3$ and discuss the PDF of B within each range separately.

- $0 \leq b \leq 1$: Range in (1) reduces to $0 \leq z \leq b$; while that in (2) does not reduce to a feasible bound for z, therefore
$$f_B(b) = \int_0^b (b-z) dz = \frac{b^2}{2};$$
- $1 < b \leq 2$: Range in (1) reduces to $b - 1 \leq z \leq 1$; while that in (2) reduces to $0 \leq z \leq b - 1$, therefore
$$f_B(b) = \int_{b-1}^1 (b-z) dz + \int_0^{b-1} (2-b+z) dz = \frac{-2b^2 + 6b - 3}{2};$$
- $2 < b \leq 3$: Range in (1) does not reduce to a feasible bound for z; while that in (2) reduces to $b - 2 \leq z \leq 1$, therefore
$$f_B(b) = \int_{b-2}^1 (2-b+z) dz = \frac{(3-b)^2}{2};$$

In sum, the PDF of B is

$$f_B(b) = \begin{cases} \frac{b^2}{2}, & 0 \le b \le 1 \\ \frac{-2b^2+6b-3}{2}, & 1 < b \le 2 \\ \frac{(3-b)^2}{2}, & 2 < b \le 3 \\ 0, & \text{otherwise.} \end{cases}$$

Problem 2. (10 points) Suppose that X and Y are random variables with the same variance. Show that $X - Y$ and $X + Y$ are uncorrelated.

Solution:
The covariance of $X - Y$ and $X + Y$ is

$$\begin{aligned} \text{cov}(X-Y, X+Y) &= \text{cov}(X-Y, X) + \text{cov}(X-Y, Y) \\ &= \text{cov}(X, X) - \text{cov}(Y, X) + \text{cov}(X, Y) - \text{cov}(Y, Y) \\ &= \text{var}(X) - \text{var}(Y) \\ &= 0. \end{aligned}$$

Therefore, $X - Y$ and $X + Y$ are uncorrelated.

Problem 3. (10 points) Calculate $\mathbf{E}[X^3]$ and $\mathbf{E}[X^4]$ for a standard normal random variable X.

Solution:
Since the transform of a standard normal random variable X is

$$M_X(s) = e^{\frac{s^2}{2}},$$

then

$$\begin{aligned} \mathbf{E}[X^3] &= \frac{d^3}{ds^3} M_X(s)\Big|_{s=0} = 3se^{\frac{s^2}{2}} + s^3 e^{\frac{s^2}{2}}\Big|_{s=0} = 0, \\ \mathbf{E}[X^4] &= \frac{d^4}{ds^4} M_X(s)\Big|_{s=0} = 3e^{\frac{s^2}{2}} + 6s^2 e^{\frac{s^2}{2}} + s^4 e^{\frac{s^2}{2}}\Big|_{s=0} = 3. \end{aligned}$$

Problem 4. (10 points) At a certain time, the number of people that enter an elevator is a Poisson random variable with parameter λ. The weight of each person is independent of every other person's weight, and is uniformly distributed between 100 and 200 lbs. Let X_i be the fraction of 100 by which the ith person exceeds 100 lbs, e.g., if the 7th person weights 175 lbs., then X_7=0.75. Let Y be the sum of the X_i. What's the expectation of Y.

Solution:
Suppose N people enter the elevator, i.e., $Y = X_1, \ldots, X_N$. Then $\mathbf{E}[N] = \lambda$. Since X be the fraction of 100 by which the ith person exceeds 100 lbs, then X is uniformly distributed in $[0, 1]$ with $\mathbf{E}[X] = \frac{1}{2}$. We have

$$\mathbf{E}[Y] = \mathbf{E}[N]\mathbf{E}[X] = \frac{\lambda}{2}.$$

Therefore, the expectation of Y is $\frac{\lambda}{2}$.