1. Each of the 200 ENGG2430 students shows up to class independently with probability 0.9 and asks Poisson(0.05) questions in there. Let S be the number of students in class and Q the total number of questions asked. Find (a) $\mathrm{E}[S]$, (b) $\mathrm{E}[Q|S]$, (c) $\mathrm{E}[Q]$, (d) $\mathrm{Var}[\mathrm{E}[Q|S]]$, (e) $\mathrm{Var}[Q|S]$, (f) $\mathrm{E}[\mathrm{Var}[Q|S]]$, (g) $\mathrm{Var}[Q]$.

   Solution: Let Q_i be the number of question asked by the i-th student present in class; $Q = Q_1 + \cdots + Q_S$.

   (a) $\mathrm{E}[S] = 200 \cdot 0.9 = 180$.
   (b) $\mathrm{E}[Q|S] = \sum_{i=1}^{S} \mathrm{E}[Q_i] = S \cdot 0.05 = 0.05S$ by linearity of expectation.
   (c) $\mathrm{E}[Q] = \mathrm{E}[\mathrm{E}[Q|S]] = \mathrm{E}[0.05S] = 0.05 \cdot 180 = 9$ by (b).
   (d) $\mathrm{Var}[\mathrm{E}[Q|S]] = \mathrm{Var}[0.05S] = 0.05^2\,\mathrm{Var}[S] = 0.05^2 \cdot (200 \cdot 0.09 \cdot 0.01) = 0.00045$ by (b).
   (e) $\mathrm{Var}[Q|S] = \sum_{i=1}^{S} \mathrm{Var}[Q_i] = S \cdot 0.05 = 0.05S$ by independence of Q_i's.
   (f) $\mathrm{E}[\mathrm{Var}[Q|S]] = \mathrm{E}[0.05S] = 9$ by (e).
   (g) $\mathrm{Var}[Q] = \mathrm{Var}[\mathrm{E}[Q|S]] + \mathrm{E}[\mathrm{Var}[Q|S]] = 9.00045$ by (d) and (f).

2. You flip a coin with unknown probability of heads p. You want to learn the value of p.

   (a) Alice suggests the following estimator $\hat{P}_A$: Keep flipping the coin until you see the first head in the N-th flip. Set $\hat{P}_A = 1/N$.
   (b) Bob suggests another estimator $\hat{P}_B$: Flip the coin 10 times, count the number of heads Y and set $\hat{P}_B = Y/10$.

   What is the expectation of each estimator in terms of p? Which one is better?

   Solution: $\hat{P}_A$ has expectation

   $$\mathrm{E}[\hat{P}_A] = \sum_{n=1}^{\infty} \frac{1}{n} \cdot (1-p)^{n-1} p = \frac{p}{1-p} \sum_{n=1}^{\infty} \frac{(1-p)^n}{n} \approx \frac{p}{1-p} \cdot (-\log p),$$

   since the infinite series is the Taylor expansion of $-\log p$. $\hat{P}_B$ has expectation

   $$\mathrm{E}[\hat{P}_B] = \mathrm{E}\left[\frac{Y}{10}\right] = \frac{1}{10}\,\mathrm{E}[Y] = \frac{10p}{10} = p.$$

   The expectation of $\hat{P}_A \neq p$ in general while $\hat{P}_B$ does; $\hat{P}_A$ is said to be biased and $\hat{P}_B$ unbiased.

In the next two questions, estimate the quantity of your interest using the method of your choice: Markov's inequality, Chebyshev's inequality, or the Central Limit Theorem. Justify why the method is applicable and discuss the quality of the estimate.
Summary of the assumption and result of the three methods:

- Markov's inequality: $\mathrm{P}(X \geq a) \leq \mathrm{E}[X]/a$.
  - Applies for any **non-negative** random variable X, and any $a > 0$ ($a > \mathrm{E}[X]$ for a meaningful bound).
  - Requires only $\mathrm{E}[X]$, useful when it is small.
  - Is an upper bound to a one-sided (right tail) probability.

- Chebyshev's inequality: $\mathrm{P}(|X-\mu| \geq t\sigma) \leq 1/t^2$, where $\mu = \mathrm{E}[X], \sigma = \sqrt{\mathrm{Var}[X]}$.
  - Applies for any random variable X (with finite μ, σ), and any t ($t > 1$ for a meaningful bound).
  - Requires both expectation and variance of X.
  - Is an upper bound to a two-sided probability.
- Central Limit Theorem: $(X-\mu_X)/\sigma_X \approx \mathrm{Normal}(0,1)$, where $X = X_1 + \cdots + X_n$ are independent and have the same PDF/PMF, $\mu_X = \mathrm{E}[X] = n\,\mathrm{E}[X_i], \sigma_X = \sqrt{\mathrm{Var}[X]} = \sqrt{n\,\mathrm{Var}[X_i]}$.
  - Applies for X being sum of many (usually $n \geq 30$) independent random variables; no restriction on distribution of X_i's.
  - Requires both $\mathrm{E}[X_i]$ and $\mathrm{Var}[X_i]$.
  - Approximates the CDF of X. Using the axioms of probability, we can use it to approximate other events of interest (e.g. $\mathrm{P}(X < -5 \text{ or } X > 7)$).

Roughly speaking, in terms of generality, Markov's inequality > Chebyshev's inequality > CLT; and in terms of tightness, Markov's inequality < Chebyshev's inequality < CLT (if n is large enough).

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3. The following exam statistics are posted on the course website:

section	no. students	average grade	std. dev.
A	30	65	5
B	20	70	10

what can you say about the number of students whose exam grade was 30 or below?

Solution: Let X_A and X_B be the grade of a random student in section A and section B, respectively. The table tells us that $\mu_A = \mathrm{E}[X_A] = 65$, $\sigma_A = \sqrt{\mathrm{Var}[X_A]} = 5$, $\mu_B = \mathrm{E}[X_B] = 70$, $\sigma_B = \sqrt{\mathrm{Var}[X_B]} = 10$. By Chebyshev's inequality, for a random student in section A,

$$\mathrm{P}(X_A \leq 30) = \mathrm{P}(X_A \leq \mu_A - 7 \cdot \sigma_A) \leq \mathrm{P}(|X_A - \mu_A| \geq 7\sigma_A) \leq 1/49 \approx 0.0204.$$

Although we are only interested in the probability that X_A is 7 standard deviations smaller than its mean, Chebyshev's inequality only tells us the probability of the possibly larger event that X_A is either 7 standard deviations smaller or 7 standard deviations larger than its mean. This is already a tremendously small probability – about 2%.

Similarly, for a random student in section B,

$$\mathrm{P}(X_B \leq 30) = \mathrm{P}(X_B \leq \mu_B - 4 \cdot \sigma_B) \leq \mathrm{P}(|X_B - \mu_B| \geq 4\sigma_B) \leq 1/16 \approx 0.00625.$$

Since there are 30 students in section A, at most $1/49 \cdot 30$ students must have received 30 or below, so nobody got that kind of grade. In section B, at most $1/16 \cdot 20$ students got 30 or below, so at most one student in the whole class could have received 30 or below on the exam.

Alternative Solution: Alternatively, we can first calculate the statistics for a random student X in the whole course and then apply Chebyshev's inequality to X. Let X be a random student and Y their section (using 1 and 2 for sections A and B). Then $\mathrm{E}[X|Y]$ takes value 65 with probability 3/5 and 70 with probability 2/5.

By total expectation theorem,

$$\mu = \mathrm{E}[X] = \mathrm{E}[\mathrm{E}[X|Y]] = 65 \cdot 3/5 + 70 \cdot 2/5 = 67.$$

By total variance theorem $\mathrm{Var}[X] = \mathrm{Var}[\mathrm{E}[X|Y]] + \mathrm{E}[\mathrm{Var}[X|Y]]$, where

$$\mathrm{Var}[\mathrm{E}[X|Y]] = (65 - 67)^2 \cdot 3/5 + (70 - 67)^2 \cdot 2/5 = 6,$$

$$\mathrm{E}[\mathrm{Var}[X|Y]] = 5^2 \cdot 3/5 + 10^2 \cdot 2/5 = 55,$$

hence standard deviation of X is $\sigma = \sqrt{61} \approx 7.8103$. Chebyshev's inequality says

$$\mathrm{P}(X \leq 30) = \mathrm{P}(X \leq \mu - (37/\sqrt{61}) \cdot \sigma) \leq \mathrm{P}(|X - \mu| \geq (37/\sqrt{61}) \cdot \sigma) \leq 61/37^2 \approx 0.0445,$$

so the number of students who got 30 or below is at most $0.0445 \cdot 50 = 2.2225$, so at most 2.

4. You are collecting donations for a charity. Each donor gives you \$10 with probability half and \$20 with probability half. Assuming donors are independent, estimate the probability that you have collected at least \$1200 after taking in 100 donations.

   Solution: Let X be the total amount of money collected. We want to estimate $\mathrm{P}(X \geq 1200)$. X is the sum of 100 random variables with the same PMF so we can use the Central Limit Theorem. We have

$$\begin{aligned} \mu &= \mathrm{E}[X] = 100 \cdot (10 \cdot 1/2 + 20 \cdot 1/2) = 1500 \\ \sigma &= \sqrt{\mathrm{Var}\, X} = \sqrt{100 \cdot ((10 - 15)^2 \cdot 1/2 + (20 - 15)^2 \cdot 1/2)} = \sqrt{100 \cdot 25} = 50. \end{aligned}$$

   Therefore,

$$\mathrm{P}(X \geq 1200) \approx \mathrm{P}(X \geq \mu - 6\sigma) \approx \mathrm{P}(N \geq -6) \approx 1 - 9.86 \cdot 10^{-10},$$

   where N is a Normal$(0, 1)$ random variable.

5. You randomly divide 48 boys and 48 girls into teams of equal size. Show that if you divide them into 12 teams of 8 then there are no same-sex teams with probability at least 90%.

   Solution: For $12 \geq i \geq 1$, let X_i be the indicator variable that the i-th team consists of all boys or all girls, then $X = \sum_{i=1}^{12} X_i$. The X_i's are not independent, so the Central Limit Theorem doesn't apply.

   The probability that any given team is all-boys is

$$p = \frac{48}{96} \cdot \frac{47}{95} \cdots \frac{41}{89}$$

   using the formula for conditional probabilities (the first member is a boy, the second member is a boy given the first one is etc.). As boys and girls are symmetric, the probability the team is same-sex is $2p$. By linearity of expectation,

$$\mathrm{E}[X] = \mathrm{E}[X_1] + \cdots + \mathrm{E}[X_{12}] = 12 \cdot (2p) \approx 0.068.$$

   At this point we can proceed in two ways. We can use Markov's inequality to conclude that $\mathrm{P}(X \geq 1) \leq \mathrm{E}[X]/1 \approx 0.068$, so the probability of having no same-sex teams is $\mathrm{P}(X = 0) \approx 1 - 0.068 = 0.922$. This meets the requirement and we are done.

   Alternatively, we can calculate $\mathrm{Var}[X]$ and apply Chebyshev's inequality, which could result in a better bound. This is feasible but a bit difficult since $X_1, \ldots, X_{12}$ are not independent so we need their covariances.