THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1020
Exercise 4
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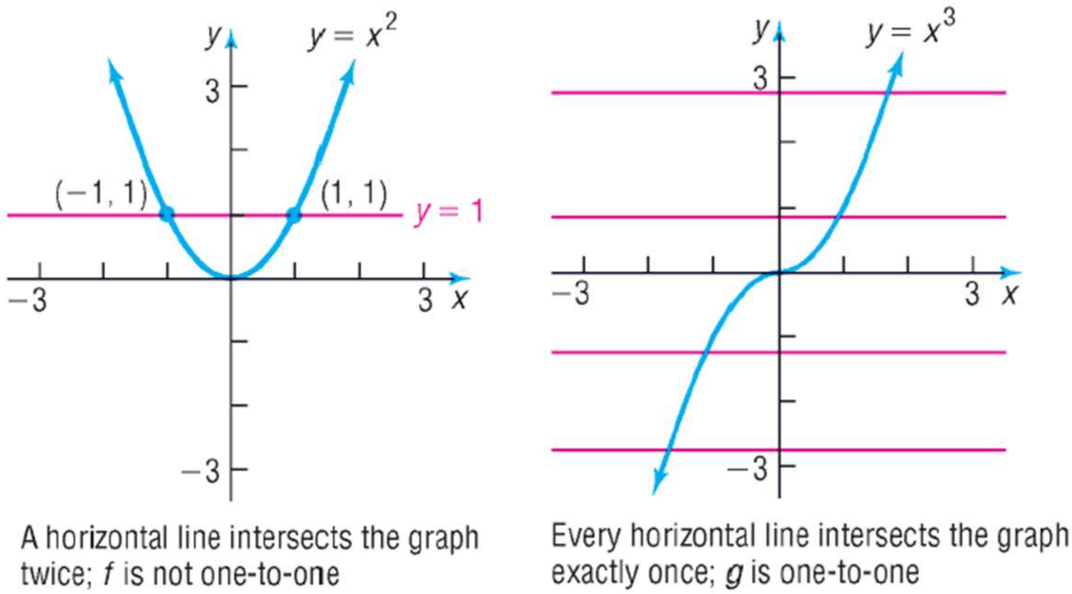
1. **Exercise 1** For each function, use its graph to determine whether the function is one–to–one.

   (a) $f(x) = x^2$ (b) $g(x) = x^3$.



(a) $f(x) = x^2$ (b) $g(x) = x^3$

Figure 1:

Solution:

(a) Figure 1(a) illustrates to horizontal–line test for $f(x) = x^2$. $f(x) = x^2$ is an even function. The horizontal line $y = 1$ intersects the graph of f twice, at $(1, 1)$ and at $(-1, 1)$. So f is not one–to–one.

(b) Figure 1(b) illustrates the horizontal–line test for $g(x) = x^3$. $g(x) = x^3$ is an odd function. Since every horizontal line intersects the graph of g exactly once, it follows that g is one–to–one.

2. **Exercise 2 Verify Inverse Function**

(a) Verify the inverse of $g(x) = x^5$ is $g^{-1}(x) = \sqrt[5]{x}$.

(b) Verify the inverse of $f(x) = 3x + 5$ is $f^{-1} = \dfrac{1}{3}(x - 5)$.

Solution:

(a) We verify that the inverse of $g(x) = x^5$ is $g^{-1}(x) = \sqrt[5]{x}$ by showing that

$$\begin{array}{ll} g^{-1}(g(x)) = g^{-1}(x^5) = \sqrt[5]{x^5} = \left(x^5\right)^{1/5} = x & \text{for all } x \text{ in the domain of } g; \\ g(g^{-1}(x)) = g(\sqrt[5]{x}) = (\sqrt[5]{x})^5 = \left(x^{1/5}\right)^5 = x & \text{for all in the domain of } g^{-1}. \end{array}$$

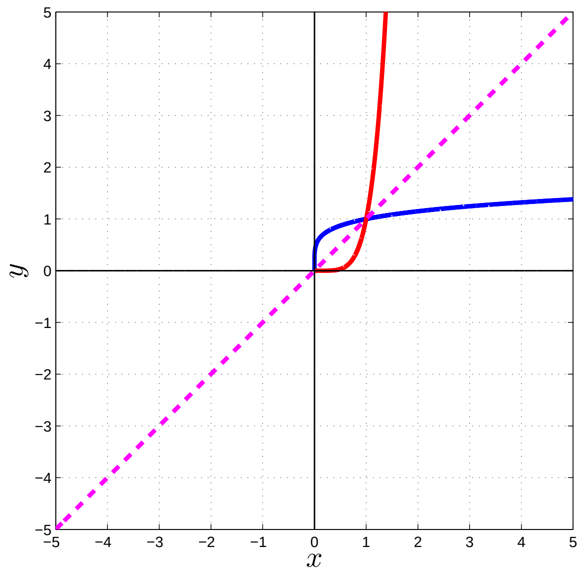


Figure 2: $g(x) = x^5$ (in red) and $g^{-1}(x) = \sqrt[5]{x}$ (in blue).

(b) We verify that the inverse of $f(x) = 3x + 5$ is $f^{-1} = \frac{1}{3}(x - 5)$ by showing that

$$f^{-1}(f(x)) = f^{-1}(3x+5) = \frac{1}{3}[(3x+5) - 5] = \frac{1}{3}(3x) = x \quad \text{for all } x \text{ in the domain of } f;$$
$$f(f^{-1}(x)) = f\left(\frac{1}{3}(x-5)\right) = 3\left[\frac{1}{3}(x-5)\right] + 5 = (x-5) + 5 = x \quad \text{for all } x \text{ in the domain of } f^{-1}.$$

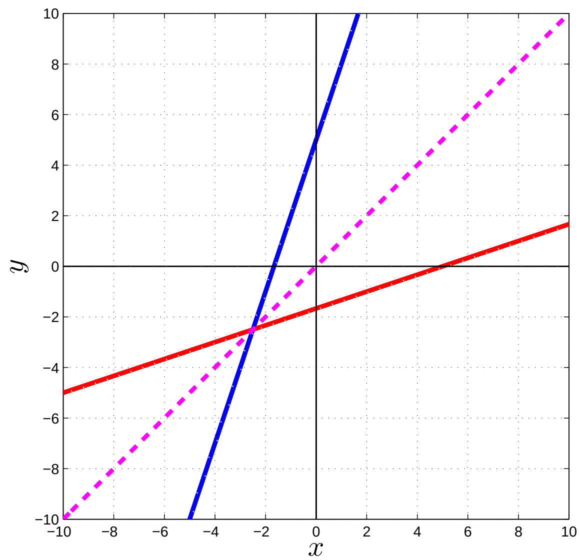


Figure 3: $f(x) = 3x + 5$ (in blue) and $f^{-1} = \frac{1}{3}(x - 5)$ (in red).

3. **Exercise 3 Verify Inverse Function**

Verify that the inverse of $f(x) = \dfrac{1}{x-1}$ is $f^{-1}(x) = \dfrac{1}{x} + 1$. For what values of x is $f^{-1}(f(x)) = x$? For what values of f is $f(f^{-1}(x)) = x$?

Solution:

The domain of f is $\{x|x \neq 1\} = \mathbb{R}\backslash\{1\}$ and the domain of f^{-1} is $\{x|x \neq 0\} = \mathbb{R}\backslash\{0\}$. Now

$$f^{-1}(f(x)) = f^{-1}\left(\frac{1}{x-1}\right) = \frac{1}{\dfrac{1}{x-1}} + 1 = x - 1 + 1 = x \qquad \text{provided} \quad x \neq 1;$$

$$f(f^{-1}(x)) = f\left(\frac{1}{x} + 1\right) = \frac{1}{\dfrac{1}{x} + 1 - 1} = \frac{1}{\dfrac{1}{x}} = x \qquad \text{provided} \quad x \neq 0.$$

4. **Exercise 4 Finding the Inverse Function**

The function

$$f(x) = \frac{2x+1}{x-1} \qquad x \neq 1$$

is one−to−one. Find its inverse and check the result.

Solution:

STEP 1: Replace $f(x)$ with y and interchange the variable x and y in

$$y = \frac{2x+1}{x-1}$$

to obtain

$$x = \frac{2y+1}{y-1}.$$

STEP 2: Solve for y

$$\begin{array}{rcll} x &=& \dfrac{2y+1}{y-1} & \\ x(y-1) &=& 2y+1 & \text{Multiply both sides by } y=-1 \\ xy-x &=& 2y+1 & \text{Apply the Distributive Property} \\ xy-2y &=& x+1 & \text{Subtract } 2y \text{ from both sides add } x \text{ to both sides} \\ (x-2)y &=& x+1 & \text{Factor} \\ y &=& \dfrac{x+1}{x-2} & \text{Divide by } x-2 \end{array}$$

The inverse is

$$f^{-1}(x) = \frac{x+1}{x-2}, \; x \neq 2 \qquad \text{Replace } y \text{ by} f^{-1}(x)$$

STEP 3:

<u>Check:</u>

$$f^{-1}(f(x)) = f^{-1}\left(\frac{2x+1}{x-1}\right) = \frac{\dfrac{2x+1}{x-1}+1}{\dfrac{2x+1}{x-1}-2} = \frac{2x+1+x-1}{2x+1-2(x-1)} = \frac{3x}{3} = x \qquad x \neq 1;$$

$$f(f^{-1}(x)) = f\left(\frac{x+1}{x-2}\right) = \frac{2\left(\dfrac{x+1}{x-2}\right)+1}{\dfrac{x+1}{x-2}-1} = \frac{2(x+1)+x-2}{x+1-(x-2)} = \frac{3x}{3} = x \qquad x \neq 2.$$

5. **Exercise 5 Find the Range of a Function**

Find the domain and the range of

$$f(x) = \frac{2x+1}{x-1}.$$

Solution: The domain of f is $\{x|x \neq 1\}$. To find the range of f, we use the fact the domain of f^{-1} equals the range of f. Based on Example 4,we have

$$f^{-1}(x) = \frac{x+1}{x-2}.$$

The domain of f^{-1} is $\{x|x \neq 2\}$, so the range of f is $\{y|y \neq 2\}$. Also, because the domain of f is $\{x|x \neq 1\}$, the range of f^{-1} is $\{y|y \neq 2\}$.

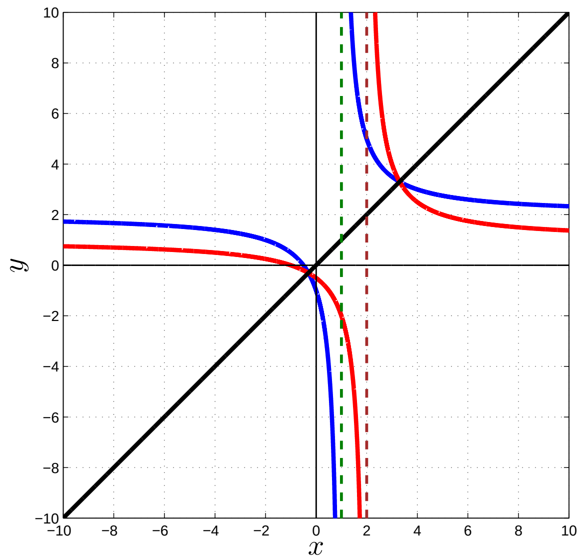


Figure 4: $f(x) = \frac{2x+1}{x-1}$ (in blue) and $f^{-1} = \frac{2x+1}{x-1}$ (in red).

6. **Exercise 6 Finding the Inverse of a Domain-restricted Function**

   Find the inverse of $y = f(x) = x^2$ if $x \geq 0$.

   Solution The function $y = x^2$ is not one–to–one. [Refer to Example 1(a)] However, if we restrict the domain of this function to $x \geq 0$, as indicated. We have a new function that is increasing and therefore is one–to–one (why?). As a result, the function defined by $y = f(x) = x^2, x \geq 0$, has an inverse function, f^{-1}.