Exercises: Path Independence of Line Integral

Problem 1. Calculate $\int_C d\boldsymbol{r} = \int_C dx + \int_C dy$ where C is a smooth curve from point $p = (1, 2)$ to $q = (3, 4)$.

Solution: Introduce $g(x, y) = x + y$. Clearly, $\frac{\partial g}{\partial x} = 1$ and $\frac{\partial g}{\partial y} = 1$. Hence, $\int_C dx + \int_C dy = g(3, 4) - g(1, 2) = 4$.

Problem 2. Calculate $\int_C 2xy\, dx + \int_C x^2\, dy$ where C is a smooth curve from point $p = (1, 2)$ to $q = (3, 4)$.

Solution: Introduce $g(x, y) = x^2 y$. Clearly, $\frac{\partial g}{\partial x} = 2xy$ and $\frac{\partial g}{\partial y} = x^2$. Hence, $\int_C 2xy\, dx + \int_C x^2\, dy = g(3, 4) - g(1, 2) = 34$.

Problem 3. Calculate $\int_C yz\, dx + \int_C xz\, dy + \int_C xy\, dz$ where C is a smooth curve from point $p = (1, 2, 3)$ to $q = (3, 4, 5)$.

Solution: Introduce $g(x, y, z) = xyz$. Clearly, $\frac{\partial g}{\partial x} = yz$, $\frac{\partial g}{\partial y} = xz$, and $\frac{\partial g}{\partial z} = xy$. Hence, $\int_C yz\, dx + \int_C xz\, dy + \int_C xy\, dz = g(3, 4, 5) - g(1, 2, 3) = 54$.

Problem 4. Calculate $\int_C yz\, dx + \int_C xz\, dy + \int_C xy\, dz$ where C is the curve given by $\boldsymbol{r}(t) = [\cos(t), \sin(t), 1]$ with $t \in [0, 2\pi]$.

Solution: We already know that $\int_C yz\, dx + \int_C xz\, dy + \int_C xy\, dz$ is path independent. Also observe that C is a closed curve (because $\boldsymbol{r}(0) = \boldsymbol{r}(2\pi)$). In this case, it must hold that $\int_C yz\, dx + \int_C xz\, dy + \int_C xy\, dz = 0$. The following is a detailed proof.

Let p, q, u be the points given by $\boldsymbol{r}(0)$, $\boldsymbol{r}(\pi)$, and $\boldsymbol{r}(2\pi)$, respectively. Note that $p = u$ Let C_1 be the curve from p to q, and C_2 be the curve from q to u. Therefore:

$$
\begin{aligned}
& \int_C yz\, dx + \int_C xz\, dy + \int_C xy\, dz \\
= & \left(\int_{C_1} yz\, dx + \int_{C_1} xz\, dy + \int_{C_1} xy\, dz \right) + \left(\int_{C_2} yz\, dx + \int_{C_2} xz\, dy + \int_{C_2} xy\, dz \right). \qquad (1)
\end{aligned}
$$

Let $C_{2'}$ be the curve from u to q. By path independence, we have:

$$
\int_{C_1} yz\, dx + \int_{C_1} xz\, dy + \int_{C_1} xy\, dz \quad = \quad \int_{C_2'} yz\, dx + \int_{C_2'} xz\, dy + \int_{C_2'} xy\, dz. \qquad (2)
$$

On the other hand:

$$
\int_{C_2} yz\, dx + \int_{C_2} xz\, dy + \int_{C_2} xy\, dz \quad = \quad -\left(\int_{C_2'} yz\, dx + \int_{C_2'} xz\, dy + \int_{C_2'} xy\, dz \right). \qquad (3)
$$

Combining (2) and (3) shows that (1) equals 0.

Problem 5. Suppose that $\int_C f_1(x, y, z) dx + \int_C f_2(x, y, z)\, dy + \int_C f_3(x, y, z)\, dz$ equals 0 for *any* closed curve C. Prove that the integral is path independent.

Proof: Fix two arbitrary points p, q. Let C_1 and C_2 be two difference curves from p to q. Let $C_{1'}$ be the same curve from q to p by reversing the direction of C_1. Then, $C_{1'}$ followed by C_2 forms a closed curve C. It holds that

$$\begin{aligned} 0 &= \int_C f_1(x,y,z)dx + \int_C f_2(x,y,z)\,dy + \int_C f_3(x,y,z)\,dz \\ &= \left(\int_{C_1'} yz\,dx + \int_{C_1'} xz\,dy + \int_{C_1'} xy\,dz\right) + \left(\int_{C_2} yz\,dx + \int_{C_2} xz\,dy + \int_{C_2} xy\,dz\right) \end{aligned} \tag{4}$$

On the other hand:

$$\int_{C_1} yz\,dx + \int_{C_1} xz\,dy + \int_{C_1} xy\,dz = -\left(\int_{C_1'} yz\,dx + \int_{C_1'} xz\,dy + \int_{C_1'} xy\,dz\right). \tag{5}$$

From (4) and (5), we know that

$$\int_{C_1} yz\,dx + \int_{C_1} xz\,dy + \int_{C_1} xy\,dz = \int_{C_2} yz\,dx + \int_{C_2} xz\,dy + \int_{C_2} xy\,dz$$

$\square$