7.4 Linear Independence. Rank of a Matrix. Vector Space

Linear Independence/Dependence of Vectors

- Given any set of m vectors $\mathbf{a}_{(1)}, \dots, \mathbf{a}_{(m)}$ (of same size), a **linear combination** of these vectors is of the form

$$c_1\mathbf{a}_{(1)} + c_2\,\mathbf{a}_{(2)} + \dots + c_m\mathbf{a}_{(m)}$$

where $c_1, c_2, \dots, c_m$ are any scalars.

- Now consider the solution to the equation

$$\textbf{(1)} \qquad c_1\mathbf{a}_{(1)} + c_2\,\mathbf{a}_{(2)} + \dots + c_m\mathbf{a}_{(m)} = \mathbf{0}$$

Zero vector of same size as $\mathbf{a}_{(j)}$

  - Trivial solution: $c_j=0$ for $j=1,\dots,m$, NOT interesting
  - **Interested in non-trivial solution:** some of the scalars $c_1, c_2, \dots, c_m$, nonzero

- **Question**: can we find non-trivial solution for scalars $c_1, c_2, \dots, c_m$, to satisfy (1)?

7.4 Linear Independence. Rank of a Matrix. Vector Space

Question: Can we find non-trivial solution for scalars $c_1, c_2, \ldots, c_m$, to satisfy (1)?

No

- Vectors $\mathbf{a}_{(1)}, \ldots, \mathbf{a}_{(m)}$ are **linearly independent** (or they form a *linear independent set*).
- The trivial solution, i.e., all $c_j{}'=0$, is the only solution to (1)

Yes

- Vectors $\mathbf{a}_{(1)}, \ldots, \mathbf{a}_{(m)}$ are **linearly dependent** (or they form a *linearly dependent set*).
- A set of vectors that includes the **zero vector** must be **linearly dependent**. (Why?)

- For **linearly dependent** case, i.e., non-trivial solution exists, we have

$$c_1\mathbf{a}_{(1)} + c_2\,\mathbf{a}_{(2)} + \dots + c_m\mathbf{a}_{(m)} = \mathbf{0} \qquad \text{(for some } c_j \neq 0\text{)}$$

  - Then at least one of the vectors $\mathbf{a}_{(1)}, \dots, \mathbf{a}_{(m)}$ can be expressed as a linear combination of other vectors, e.g., if $c_1 \neq 0$, then we can express $\mathbf{a}_{(1)}$ as

$$\mathbf{a}_{(1)} = -\frac{c_2}{c_1}\,\mathbf{a}_{(2)} - \dots - \frac{c_m}{c_1}\,\mathbf{a}_{(m)}$$

(Hence the term linearly *dependent*)

  - Vector $\mathbf{a}_{(1)}$ can thus be removed from the set without losing "content" or "information".
- Hence, a **linearly dependent** set of vectors can be reduced in number (get rid of at least one, or perhaps more) until we get a **linearly independent** set.
- A **linearly independent** set is the smallest "truly essential" set – it cannot be reduced further without losing "content" or "information".

- **Example 1.5**: Given a set of vectors $\mathbf{a}_{(1)}, \ldots, \mathbf{a}_{(4)}$

$$\mathbf{a}_{(1)} = [1 \quad 1 \quad 0 \quad 1];$$

$$\mathbf{a}_{(2)} = [0 \quad 1 \quad 1 \quad 1];$$

$$\mathbf{a}_{(3)} = [1 \quad 0 \quad 0 \quad 2];$$

$$\mathbf{a}_{(4)} = [1 \quad 3 \quad 2 \quad 3]$$

  - Is the set of vectors $\{\mathbf{a}_{(1)}, \mathbf{a}_{(2)}, \mathbf{a}_{(3)}, \mathbf{a}_{(4)}\}$ **linearly dependent or not?**
    - Try finding scalars c_1, c_2, c_3, c_4 such that

$$c_1\,\mathbf{a}_{(1)} + c_2\,\mathbf{a}_{(2)} + c_3\,\mathbf{a}_{(3)} + c_4\,\mathbf{a}_{(4)} = \mathbf{0}$$

    - This leads to a system of linear equations:

$$\begin{array}{rcrcrcrcl} c_1 & + & & + & c_3 & + & c_4 & = & 0 \\ c_1 & + & c_2 & + & & + & 3c_4 & = & 0 \\ & & c_2 & + & & + & 2c_4 & = & 0 \\ c_1 & + & c_2 & + & 2c_3 & + & 3c_4 & = & 0 \end{array}$$

  - Consistent, multiple solution case:

$$
\begin{aligned}
c_4 &= t \quad \text{is free parameter} \\
c_3 &= 0 \\
c_2 &= -2c_4 = -2t \\
c_1 &= -c_4 = -t
\end{aligned}
$$

  - One set of non-trivial solution for every value of *t*.

- The set of vectors $\{\mathbf{a}_{(1)}, \mathbf{a}_{(2)}, \mathbf{a}_{(3)}, \mathbf{a}_{(4)}\}$ is **linearly dependent.**
- Furthermore, picking t=-1, we have c_1=1, c_2=2, c_3=0 , c_4=-1, i.e.,

(1′)
$$\mathbf{a}_{(1)} + 2\,\mathbf{a}_{(2)} - \mathbf{a}_{(4)} = \mathbf{0}$$

- From (1′), we can express of one of the vectors of $\mathbf{a}_{(1)}$, $\mathbf{a}_{(2)}$, or $\mathbf{a}_{(4)}$ in terms of the other two.
- We can take out one of them without losing any "content" or "information."

- For example, if we choose to take out $\mathbf{a}_{(4)}$,
  - The reduced set $\{\mathbf{a}_{(1)}, \mathbf{a}_{(2)}, \mathbf{a}_{(3)}\}$ contain the same “content” or “information” as the original set $\{\mathbf{a}_{(1)}, \mathbf{a}_{(2)}, \mathbf{a}_{(3)}, \mathbf{a}_{(4)}\}$
  - If needed, we can always reproduce $\mathbf{a}_{(4)}$ as

$$\mathbf{a}_{(4)} = \mathbf{a}_{(1)} + 2\,\mathbf{a}_{(2)}.$$

- Question: is the reduced set $\{\mathbf{a}_{(1)}, \mathbf{a}_{(2)}, \mathbf{a}_{(3)}\}$ **linearly dependent or not**? Repeat the process:
  - Try finding scalars c_1, c_2, c_3 such that

$$c_1\mathbf{a}_{(1)} + c_2\,\mathbf{a}_{(2)} + c_3\mathbf{a}_{(3)} = \mathbf{0}$$

  - This leads to a system of linear equations:

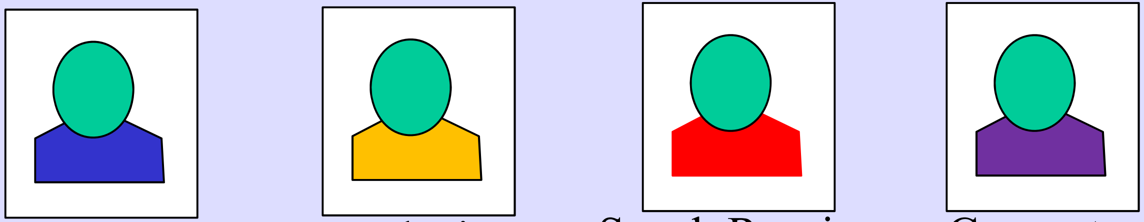
$$\left.\begin{aligned} c_1 + \quad\; + c_3 &= 0 \\ c_1 + c_2 + \quad\; &= 0 \\ c_2 + \quad\; &= 0 \\ c_1 + c_2 + 2c_3 &= 0 \end{aligned}\right\} \Rightarrow$$

Only trivial solution possible: $c_1=0$ $c_2=0$, $c_3=0$.

  - The reduced set $\{\mathbf{a}_{(1)}, \mathbf{a}_{(2)}, \mathbf{a}_{(3)}\}$ **linearly independent.**

- Or we can also choose to take out $\mathbf{a}_{(1)}$ and keep $\mathbf{a}_{(4).}$ So the reduced set $\{\mathbf{a}_{(2)}, \mathbf{a}_{(3)}, \mathbf{a}_{(4)}\}$ is **linearly independent**.
- Or to take out $\mathbf{a}_{(2).}$ So the reduced set $\{\mathbf{a}_{(1)}, \mathbf{a}_{(3)}, \mathbf{a}_{(4)}\}$ is **linearly independent**.
- Note that c_3=0 so $\mathbf{a}_{(3)}$ cannot be expressed in terms of other vectors. So $\mathbf{a}_{(3)}$ must be kept in the linearly independent set of 3 vectors.
- **Conclusion**: one of the $\mathbf{a}_{(1)}, \mathbf{a}_{(2)}, \mathbf{a}_{(4)}$ can be taken out; keeping the remaining two with $\mathbf{a}_{(3)}$ yield a **linearly independent** set.
- Further reduction from the linearly independent set leads to losing "content" or "information" of the original set of 4 vectors.
- Here, 3 is the **smallest** number of vectors (properly selected) to retains all "content" or "information" of the original set of 4 vectors.

- Idea similar to picking agents of MI team:



Top Secret:
Faces not to
be shown

	Agent #1	Agent #2	Agent #3	Agent #4
Expertise:	Speed car Explosive	Explosive Computer	Speak Russian Fast with knife	Computer Speed car

- **Linearly independent** set is the team with smallest number of members that keeps all the “skill” set (to achieve the mission)
- In this case, we do not need all 4 agents. We can just pick any two out of Agent #1, #2, and #4.
- However, Agent #3 must be kept.

- Another perspective to look at the **Example 1.5**: Given the 4 vectors $\{\mathbf{a}_{(1)}, \mathbf{a}_{(2)}, \mathbf{a}_{(3)}, \mathbf{a}_{(4)}\}$, we ask:
  - Can we extract 1 **linearly independent** vector from the set? **Yes!** Actually, we have 4 such choices: $\{\mathbf{a}_{(1)}\}$, or $\{\mathbf{a}_{(2)}\}$, or $\{\mathbf{a}_{(3)}\}$, or $\{\mathbf{a}_{(4)}\}$.
  - Can we extract 2 **linearly independent** vectors from the set? **Yes!** Actually, we have 6 such choices: $\{\mathbf{a}_{(1)}, \mathbf{a}_{(2)}\}$, or $\{\mathbf{a}_{(1)}, \mathbf{a}_{(3)}\}$, or $\{\mathbf{a}_{(1)}, \mathbf{a}_{(4)}\}$, or $\{\mathbf{a}_{(2)}, \mathbf{a}_{(3)}\}$, or $\{\mathbf{a}_{(2)}, \mathbf{a}_{(4)}\}$, or $\{\mathbf{a}_{(3)}, \mathbf{a}_{(4)}\}$.
  - Can we extract 3 **linearly independent** vectors from the set? **Yes!** In this case, we can 3 possible choices: $\{\mathbf{a}_{(1)}, \mathbf{a}_{(2)}, \mathbf{a}_{(3)}\}$, or $\{\mathbf{a}_{(1)}, \mathbf{a}_{(3)}, \mathbf{a}_{(4)}\}$, or $\{\mathbf{a}_{(2)}, \mathbf{a}_{(3)}, \mathbf{a}_{(4)}\}$.
  - Going further, can we extract 4 **linearly independent** vectors from the set? Answer is **NO**.
  - Hence 3 is the maximum number of **linearly independent** vectors possibly extracted from the original 4 vectors.

- **Example 2**: Given a set of vectors $\mathbf{a}_{(1)}$, $\mathbf{a}_{(2)}$, $\mathbf{a}_{(3)}$

$$\mathbf{a}_{(1)} = [3 \quad 0 \quad 2 \quad 2];$$

$$\mathbf{a}_{(2)} = [-6 \quad 42 \quad 24 \quad 54];$$

$$\mathbf{a}_{(3)} = [21 \quad -21 \quad 0 \quad -15]$$

  - The set of vectors is **linearly dependent** because we can find multiple non-trivial solutions
  - Picking free parameter c_3=-1 yields c_1=6, c_2=-0.5, and

$$6\mathbf{a}_{(1)} - \tfrac{1}{2}\,\mathbf{a}_{(2)} - \mathbf{a}_{(3)} = \mathbf{0}$$

  - We can express of one of the vectors of $\mathbf{a}_{(1)}$, $\mathbf{a}_{(2)}$, or $\mathbf{a}_{(3)}$ in terms of the other two.
  - Maximum number of **linearly independent** vector is 2.

Definition: Rank of a Matrix

The **rank** of a matrix **A** is the *maximum* number of linearly independent row vectors of **A**. It is denoted by rank **A**.

- Example: Let the vectors $\mathbf{a}_{(1)}, \ldots, \mathbf{a}_{(4)}$ of Example 1.5 form the rows of matrix **A**, i.e.,

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_{(1)} \\ \mathbf{a}_{(2)} \\ \mathbf{a}_{(3)} \\ \mathbf{a}_{(4)} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 2 \\ 1 & 3 & 2 & 3 \end{bmatrix}$$

  - From Example 1.5, we can extract 1, 2, or 3 **linearly independent** row vectors from matrix **A**, but 3 is the *maximum* number of linearly independent row vectors possible!
  - Rank **A** = 3

- Recall: a matrix $\mathbf{A}_1$ is **row-equivalent** to another matrix $\mathbf{A}_2$ if $\mathbf{A}_1$ can be obtained from $\mathbf{A}_2$ by (finitely many!) elementary row operations.
- Elementary row operations do **not** change the rank of a matrix.

Theorem 1:

Row-equivalent matrices have the same rank.

- **Idea** to (easier) determine the rank of a matrix:
  - Convert matrix to row echelon form
  - Determine the matrix rank as the number of nonzero rows in the row echelon form

- **Example 1.5 revisited**: From Matrix **A** with row vectors $\mathbf{a}_{(1)}, \ldots, \mathbf{a}_{(4)}$

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_{(1)} \\ \mathbf{a}_{(2)} \\ \mathbf{a}_{(3)} \\ \mathbf{a}_{(4)} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 2 \\ 1 & 3 & 2 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 2 & 2 & 2 \end{bmatrix} \begin{matrix} \\ \\ \text{Row 3 – Row 1} \\ \text{Row 4 – Row 1} \end{matrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{R} \begin{matrix} \\ \\ \text{Row 3 + Row 2} \\ \text{Row 4 – 2 Row 2} \end{matrix}$$

  - **R** is in row echelon form, rank **R=3** (Why?)
  - rank **A** = rank **R** =3.
  - Consistent with previous Example 1.5 – Maximum number of linearly independent rows in **A** (and hence **R)** is 3.

- Why rank **R**=3?

$$\mathbf{R} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

  - Discard the zero row as including it will make any set of vectors linearly dependent
  - Are the remaining three nonzero row vectors **linearly independent**? Try finding constant c_1, c_2 and c_3 such that

$$\begin{aligned} & c_1[1 \quad 1 \quad 0 \quad 1] \\ + & c_2[0 \quad 1 \quad 1 \quad 1] \\ + & c_3[0 \quad 0 \quad 1 \quad 2] \\ = & \ [0 \quad 0 \quad 0 \quad 0] \end{aligned} \Longrightarrow$$

Easy to see

c_1 =0, c_2 =0 **and**

c_3 =0

  - Hence rank **R** = 3.

EXAMPLE 3 Determination of Rank

Given the matrix (recall previous Example 2):

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix} \qquad \text{(given)}$$

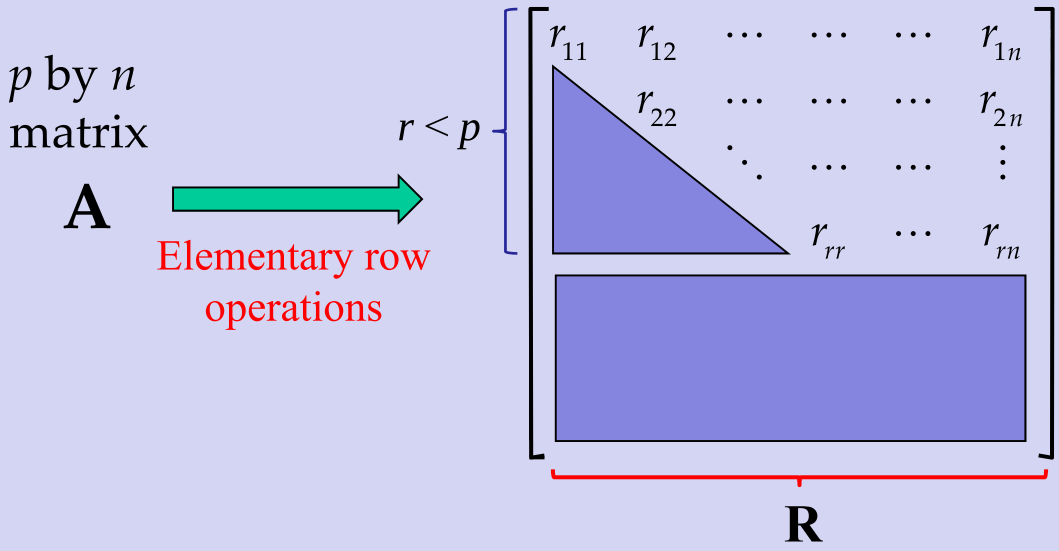
$$\Rightarrow \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & -21 & -14 & -29 \end{bmatrix} \qquad \begin{matrix} \text{Row 2} + 2\ \text{Row 1} \\ \text{Row 3} - 7\ \text{Row 1} \end{matrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{R} \qquad \text{Row 3} + \frac{1}{2}\ \text{Row 2.}$$

- **R** is in row echelon form, by number of nonzero rows rank **A** = rank **R** =2.

Implication of Row Echelon Form

- Given a matrix **A** of size *p* by *n*. Convert into Row Echelon Form (REF):



- Number of nonzero rows r: $r \le p$, depending on whether there is zero row(s) or not
- Hence, rank $\mathbf{A} = r \le p$, the rank of a matrix must be equal or smaller than the number of its rows

Theorem 2: Using matrix rank to determine Linearly Independency or Dependency of vectors

- Given a set of row vectors, to determine if they are linearly dependent (LD) or linearly independent (LI), we can
  - put them as rows of a matrix and
  - determine the rank of the resulting matrix
- Specifically, consider p row vectors $\{\mathbf{a}_{(1)}, \mathbf{a}_{(2)}, \ldots.. , \mathbf{a}_{(p)}\}$ each having n components.
  - If the matrix formed has rank = p → The set of p vectors are **linearly independent**.
  - If the matrix formed has rank < p → The set of p vectors are **linearly dependent**

Theorem 3 Matrices **A** and $\mathbf{A}^{\mathsf{T}}$ have same rank.

- The maximum number of linearly independent rows and columns in a matrix are the same.
- Hence, rank **A** also equals the maximum number of linearly independent column vectors of **A**.

- Why? **Example 2 again:**
  - Matrix $\mathbf{A} = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$ has rank 2
  - This means that the maximum number of linearly independent row vectors of **A** is 2, in fact,

$$6 \text{ Row1} - 0.5 \text{ Row2} - \text{Row3} = 0$$

Example *(continued)* **:**

- Choose Row1 and Row2 as linearly independent set.
- All rows of **A** can be expressed in terms of Row1 and Row2:

$$\begin{bmatrix}3 & 0 & 2 & 2\end{bmatrix} = 1 \times \begin{bmatrix}3 & 0 & 2 & 2\end{bmatrix} + 0 \times \begin{bmatrix}-6 & 42 & 24 & 54\end{bmatrix}$$

$$\begin{bmatrix}-6 & 42 & 24 & 54\end{bmatrix} = 0 \times \begin{bmatrix}3 & 0 & 2 & 2\end{bmatrix} + 1 \times \begin{bmatrix}-6 & 42 & 24 & 54\end{bmatrix}$$

$$\begin{bmatrix}21 & -21 & 0 & -15\end{bmatrix} = 6 \times \begin{bmatrix}3 & 0 & 2 & 2\end{bmatrix} - \tfrac{1}{2} \times \begin{bmatrix}-6 & 42 & 24 & 54\end{bmatrix}$$

- Now consider the column of matrix **A**:

$$\begin{bmatrix}3\\-6\\21\end{bmatrix} = 3 \times \begin{bmatrix}1\\0\\6\end{bmatrix} - 6 \times \begin{bmatrix}0\\1\\-\tfrac{1}{2}\end{bmatrix} \qquad \begin{bmatrix}2\\24\\0\end{bmatrix} = 2 \times \begin{bmatrix}1\\0\\6\end{bmatrix} + 24 \times \begin{bmatrix}0\\1\\-\tfrac{1}{2}\end{bmatrix}$$

$$\begin{bmatrix}0\\42\\-21\end{bmatrix} = 0 \times \begin{bmatrix}1\\0\\6\end{bmatrix} + 42 \times \begin{bmatrix}0\\1\\-\tfrac{1}{2}\end{bmatrix} \qquad \begin{bmatrix}2\\54\\-15\end{bmatrix} = 2 \times \begin{bmatrix}1\\0\\6\end{bmatrix} + 54 \times \begin{bmatrix}0\\1\\-\tfrac{1}{2}\end{bmatrix}$$

- Maximum number of linearly independent columns is also 2.

7.4 Linear Independence. Rank of a Matrix. Vector Space

- If a given matrix **A** of size *m* by *n:*

$$\operatorname{rank} \mathbf{A} \le m$$

$$\operatorname{rank} \mathbf{A}^{\mathrm{T}} \le n$$

(previous result: matrix rank smaller or equal row number)

$$\operatorname{rank} \mathbf{A} = \operatorname{rank} \mathbf{A}^{\mathrm{T}} \le \min(m, n)$$

- Example: *m*=3, *n*=5

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 & 2 & 7 \\ 1.2 & 3 & 1 & 8 & 6 \\ 2 & 3 & 4 & 3 & 5 \end{bmatrix} \rightarrow \operatorname{rank} \mathbf{A} \le 3$$

- Example: *m*=4, *n*=2

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 1 & 4 \\ 2.2 & 4 \\ 6 & 3 \end{bmatrix} \rightarrow \operatorname{rank} \mathbf{A} \le 2$$

Theorem 4: Linear Dependence of Vectors

Consider p vectors each having n components. If $n < p$, then these vectors must be **linearly dependent**.

- Reasoning:
  - Form matrix **A** with the p vectors
  - **A** is either p by n or n by p, depending on whether the given vectors are row or columns
  - Matrix **A** has rank(**A**)<$min(n,p)=n$ (given that $n < p$)
  - **A** has at most n linearly independent vectors
  - The given $p>n$ vectors must be linearly dependent
- **Example:** Given p =5 vectors with n=3 components.

$$\mathbf{a}_{(1)} = [1 \quad 1 \quad 2]; \ \mathbf{a}_{(2)} = [3 \quad 2 \quad 1]; \ \mathbf{a}_{(3)} = [3 \quad 4 \quad 9];$$

$$\mathbf{a}_{(4)} = [100 \quad 23 \quad 4]; \ \mathbf{a}_{(5)} = [2 \quad 8 \quad 2];$$

$n < p$ → Given set of vectors linearly dependent!

Definition: Vector Space

For a nonempty set V of vectors (of same size), if it satisfies the following conditions:

- Given any two vectors **a** and **b** in V, then all their linear combinations $\alpha\mathbf{a} + \beta\mathbf{b}$ (α, β any real numbers) are also elements of V,
- Addition between vectors **a**, **b**, **c** in V satisfying (see the laws (3) in Sec. 7.1):

  (a) $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$

  (b) $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$

  (c) $\mathbf{a} + \mathbf{0} = \mathbf{a}$ (Existence of **0** vector)

  (d) $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$ (Existence of (**-a**) vector)
- Scalar multiplication of vectors **a**, **b** in V satisfying (see the laws (4) in Sec. 7.1):

  (a) $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$ c, k any finite real number

  (b) $(c+k)\mathbf{a} = c\mathbf{a} + k\mathbf{a}$

  (c) $c(k\mathbf{a}) = (ck)\mathbf{a}$

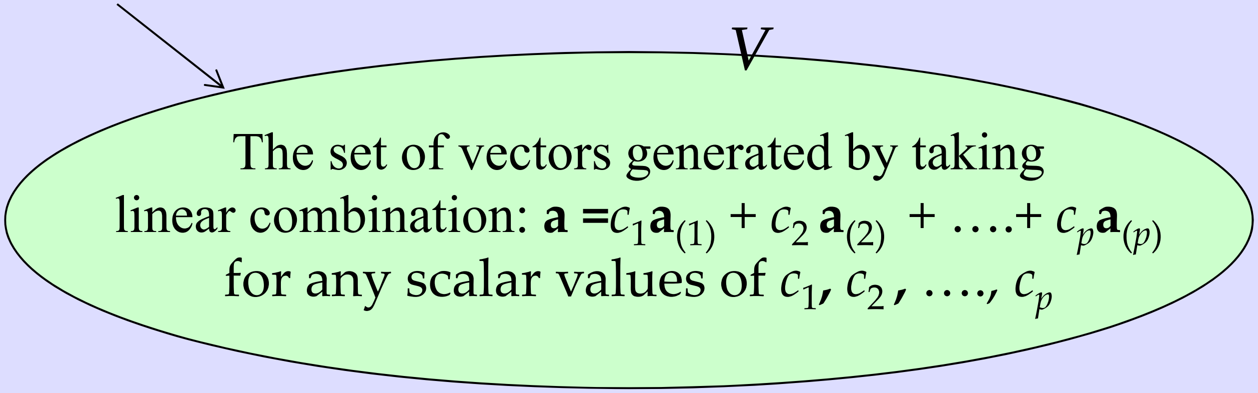
  (d) $1\mathbf{a} = \mathbf{a}$

then V is a vector space.

Span of Vector Set

Given a Vector Set

$(\mathbf{a}_{(1)}, \ldots , \mathbf{a}_{(p)})$



V

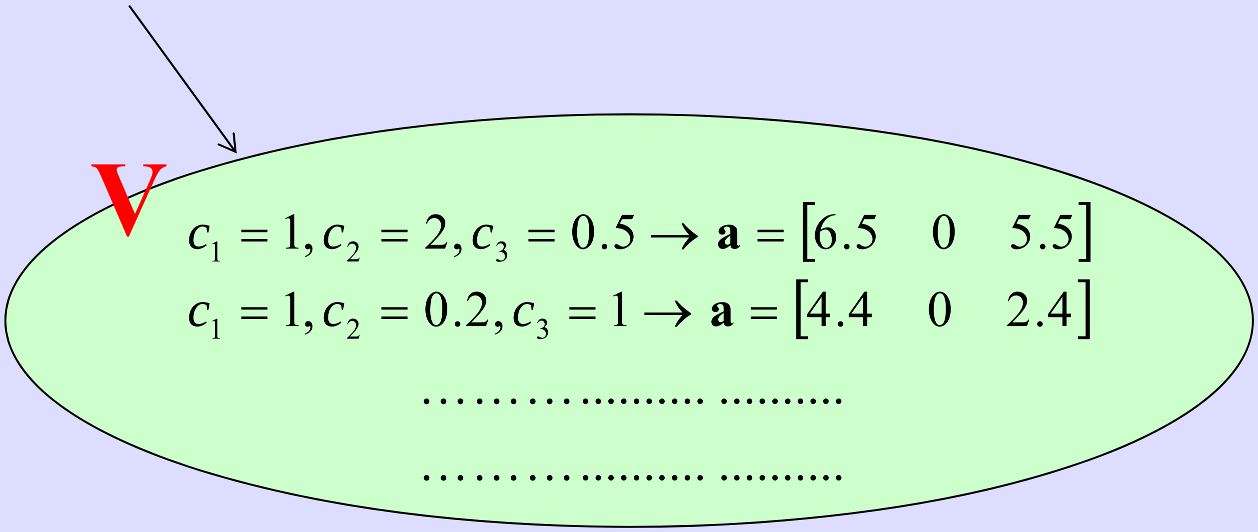
The set of vectors generated by taking linear combination: $\mathbf{a} = c_1\mathbf{a}_{(1)} + c_2\,\mathbf{a}_{(2)} + \ldots + c_p\mathbf{a}_{(p)}$ for any scalar values of $c_1, c_2, \ldots, c_p$

The Span of the Vector Set $(\mathbf{a}_{(1)}, \ldots, \mathbf{a}_{(p)})$

- We say that the given set of vectors *spans* V.
- V, the set of all linear combinations of given vectors $\mathbf{a}_{(1)}, \ldots, \mathbf{a}_{(p)}$, is called the **span** of these vectors.
- A span is a **Vector Space**.

Example X:

Given **Vector Set** ($\mathbf{a}_{(1)}$=[1 0 1], $\mathbf{a}_{(2)}$=[2 0 2], $\mathbf{a}_{(3)}$=[3 0 1])



V

$$c_1 = 1, c_2 = 2, c_3 = 0.5 \rightarrow \mathbf{a} = \begin{bmatrix} 6.5 & 0 & 5.5 \end{bmatrix}$$

$$c_1 = 1, c_2 = 0.2, c_3 = 1 \rightarrow \mathbf{a} = \begin{bmatrix} 4.4 & 0 & 2.4 \end{bmatrix}$$

……………….. ……….

……………….. ……….

The Span of the Vector Set ($\mathbf{a}_{(1)}$, …, $\mathbf{a}_{(3)}$)

- V contains vectors with special structure [α 0 β], where scalars α and β can be any value
- V satisfies all conditions in the definition

  → V is a vector space

- **Note:** Not all set of vectors are vector space. A set must satisfy the conditions in the definition to qualify as Vector Space.

- **Example:** The space of row vector of 3 components:

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \right\}$$

Check: V satisfies all the conditions → Vector space!
Note: this means R^3 is a Vector Space

- **Example:** The space of row vector of 3 components:

$$V_C = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}, \text{with } a_1 + a_2 + a_3 = 0 \right\}$$

Check: V_C satisfies all the conditions → Vector space!
In this case, V_C is a subset of R^3**.** We called V_C a subspace of R^3. (More later)

- **Example:** The space of row vector of 3 components:

$$V = \{[a_1 \quad a_2 \quad a_3], \text{with } a_1, a_2, a_3 \text{ all integers }\}$$

  - Check: V not satisfy the first condition:
    *Given any two vectors **a** and **b** in V, then their linear combinations α**a** + β**b** (α, β any real numbers) are also elements of V*
  - V is NOT a vector space.

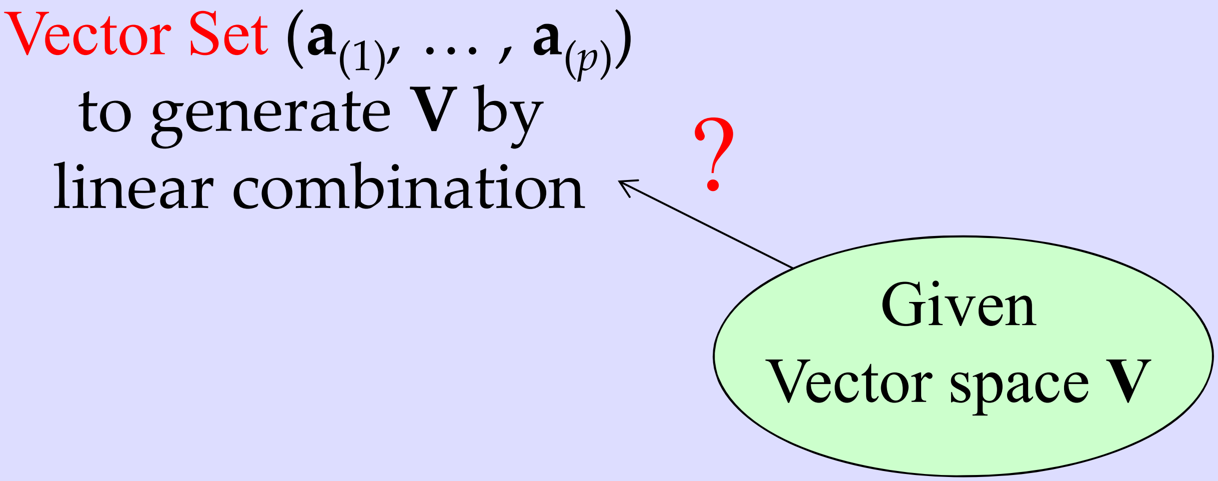
- Example: $V = \{[a_1 \quad a_2 \quad a_3], \text{with } a_1 + a_2 = 9\}$

- Example: $V = \{[a_1 \quad a_2 \quad a_3], \text{with } a_1^2 = a_2^2\}$

- Example:

$$V = \{[a_1 \quad a_2 \quad a_3], \text{with } a_1 + 3a_2 + 3a_3 = 0\}$$

Reverse question: given a vector space **V**, find the vector set that spans **V?**



Vector Set ($\mathbf{a}_{(1)}, \ldots, \mathbf{a}_{(p)}$)
to generate **V** by
linear combination

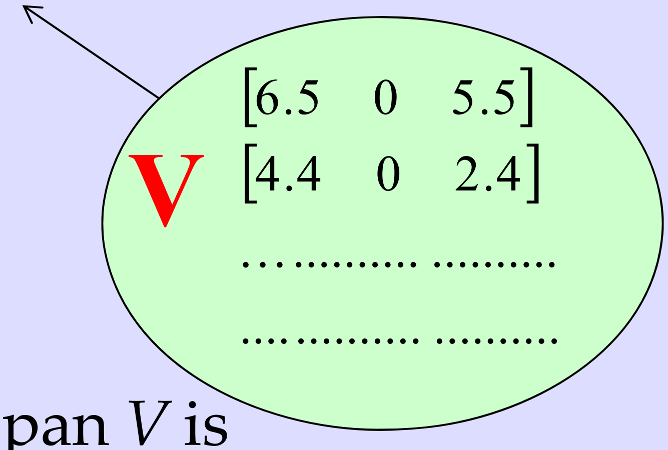
?

Given
Vector space **V**

Answers: Vector set to generate **V** is not unique
(both in values and number)

Example X Revisited:

Given V containing vectors with special structure $[\ \alpha \quad 0 \quad \beta\ ]$, where scalars α and β can be any value


Vector Set to span V?

-- One possible vector set to span V is
S1= ($\mathbf{a}_{(1)=}[1\ 0\ 1]$, $\mathbf{a}_{(2)=}[2\ 0\ 2]$, $\mathbf{a}_{(3)=}[3\ \ 0\ \ 1]$)

-- Another set is S2= ($\mathbf{a}_{(1)=}[1\ 0\ 1]$, $\mathbf{a}_{(2)=}[2\ 0\ 1]$)

-- Or we have S3= ($\mathbf{a}_{(1)=}[1\ 0\ 0]$, $\mathbf{a}_{(2)=}[0\ 0\ 1]$)

-- There are actually infinite # of choices for this vector set!

Observations from Example X:

-- The set S1= ($\mathbf{a}_{(1)=}$[1 0 1], $\mathbf{a}_{(2)=}$[2 0 2], $\mathbf{a}_{(3)=}$[3 0 1]) is not a linear independent set. There is some redundant "information" contained among the three vectors.

-- On the other hand, both S2= ($\mathbf{a}_{(1)=}$[1 0 1], $\mathbf{a}_{(2)=}$[2 0 1]) and S3= ($\mathbf{a}_{(1)=}$[1 0 0], $\mathbf{a}_{(2)=}$[0 0 1]) are linearly independent set. S2 and S3 are called **basis** of **V**.

-- Both S2 and S3 need only 2 vectors to span **V** compared to 3 in S1.

-- We call the dimension of **V**: dim **V**=2, meaning:

  * Vector space **V** has two "degrees" of freedom
  * Maximum number of vectors in **V** that can be linearly independent to each other is 2. Pick any 3 vectors in **V** must be linearly dependent.
  * Pick any two linearly independent vectors from **V** produces a **basis** that spans **V**.

The Basis of Vector Space

- The maximum number of **linearly independent** vectors in V is called the **dimension** of V and is denoted by dim V. (Consider dim V finite; dim V can also be infinite)
- A **linearly independent** set in V consisting of the maximum possible number of vectors in V forms a **basis** for V.
- In other words, any largest possible set of linearly independent vectors in V is a basis for V – adding one or more vector to it makes the set linearly dependent.
- The number of vectors of a **basis** for V equals dim V.

Example X Revisited:

Given **V** containing vectors with special structure
[α 0 β], where scalars α and β can be any value

- Maximum # of LI vectors in **V**?
  - Pick 1 vector: $\mathbf{a}_{(1)=}$[5 0 1] → 1 vector is LI
  - Add 1 more to $\mathbf{a}_{(1)}$:
    - Try **x**=[10 0 2]? → $\mathbf{a}_{(1)}$ and **x** are LD
    - Try again, $\mathbf{a}_{(2)=}$[4 0 1] → $\mathbf{a}_{(1)}$ and $\mathbf{a}_{(2)}$ are LI
  - **V** can have 2 LI vectors!
  - Add 1 more to $\mathbf{a}_{(1)}$ and $\mathbf{a}_{(2)}$?
    - Try **x'**=[0 0 2]? → $\mathbf{a}_{(1),}$ $\mathbf{a}_{(2)}$ and **x'** are LD
    - Cannot find 3rd vector and still maintain LI set
- Maximum # of LI vector from **V** is 2 →dim V =2
- $\mathbf{a}_{(1)}$ and $\mathbf{a}_{(2)}$ form a basis for **V**
- Actually, any 2 LI vectors from **V** is a basis for **V**

Vector Subspace

- Definition: **subspace** of a vector space *V*
  An nonempty subset of *V* that forms a vector space with respect to the two algebraic operations (addition and scalar multiplication) defined for the vectors of *V*.

- **Example.** The space of row vector of 3 components:

$$\mathbf{V_0} = \{[a_1 \quad a_2 \quad a_3], \text{with } a_1 + a_2 + a_3 = 0\}$$

is a vector space. The space of row vector:

$$\mathbf{V_1} = \{[a_1 \quad a_2 \quad a_3], \text{with } a_1 + a_2 = 0, a_3 = 0\}$$

is a vector space (conditions checked). V_1 is a <u>subspace</u> of V_0 . Both are subspace of R^3.

Example X Revisited:

Given **V** containing vectors with special structure $[\ \alpha \quad 0 \quad \beta\ ]$, where scalars α and β can be any value

-- Subspaces of **V**:

$V1 = [\ \alpha \quad 0 \quad 0\ ]$,

$V2 = [\ 0 \quad 0 \quad \beta\ ]$,

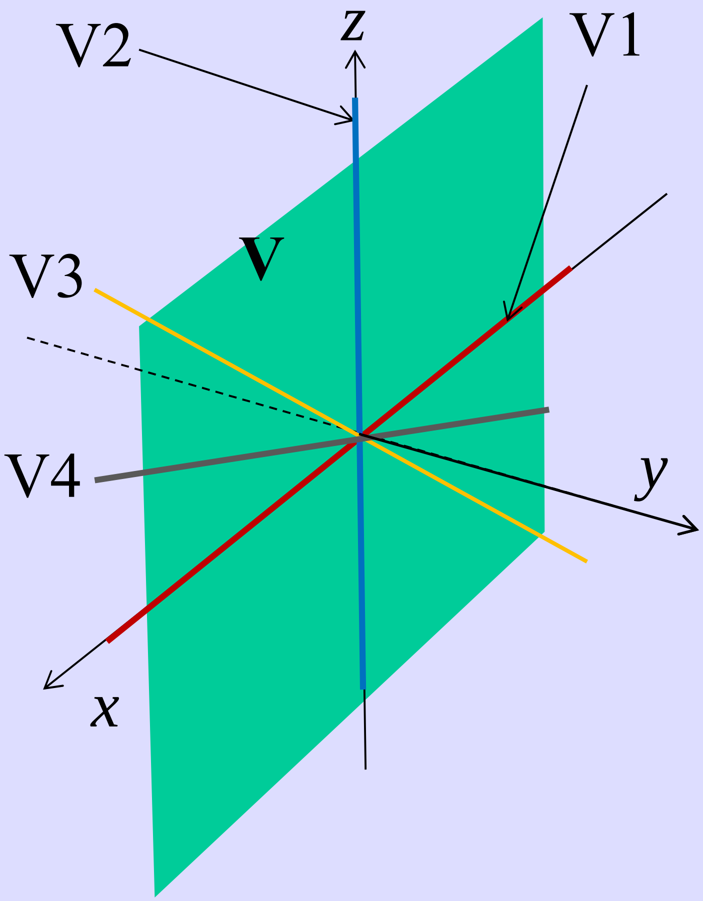
$V3 = \alpha [\ 1 \quad 0 \quad 1\ ]$,

where $a_1 = a_3$,

$V4 = \alpha [\ 2 \quad 0 \quad 1\ ]$,

where $a_1 = 2a_3$,

…………………..



Theorem 5

Vector Space R^n

The space consisting of all vectors with n components (n real numbers) has dimension n. is a vector space and denoted by space $\boldsymbol{R}^n$.

$$R^n = \{\mathbf{a} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}, a_j,\ j = 1, \cdots, n \text{ can be any real numbers}\}$$

or

$$R^n = \{\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, a_j,\ j = 1, \cdots, n \text{ can be any real numbers}\}$$

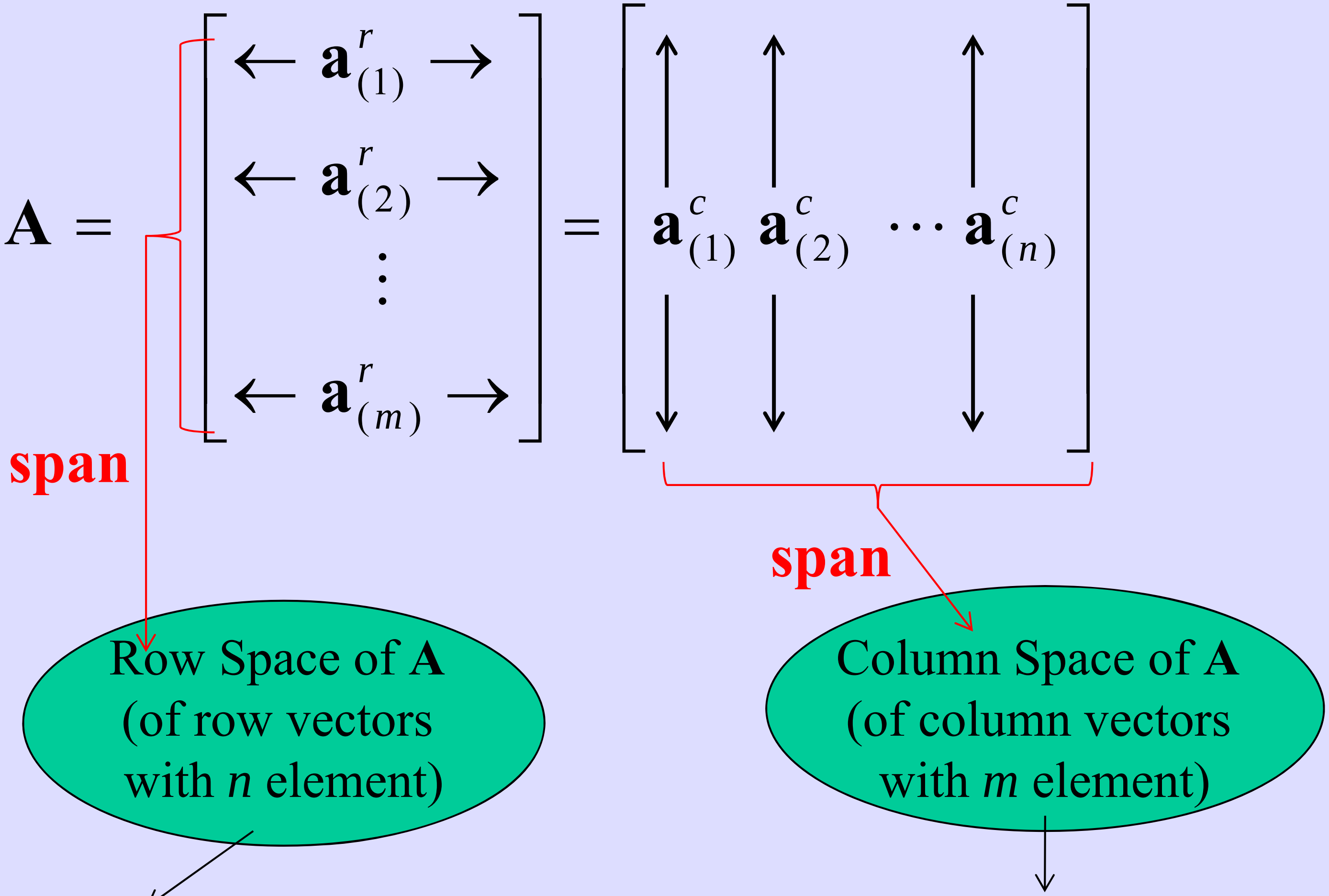
Theorem 6

Row Space and Column Space

Given a matrix *A*,

- The row space of **A** is the vector space spanned by the rows of *A*.
- The column space of **A** is the vector space spanned by the columns of *A*.
- Because rank A= rank A^T, the maximum numbers of linearly independent row and column vectors are the same.
- The row and the column spaces of *A* thus have the same dimension and both equal to rank *A*.

Given a *m* x *n* matrix **A,**



$$\mathbf{A} = \begin{bmatrix} \leftarrow \mathbf{a}^{r}_{(1)} \rightarrow \\ \leftarrow \mathbf{a}^{r}_{(2)} \rightarrow \\ \vdots \\ \leftarrow \mathbf{a}^{r}_{(m)} \rightarrow \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \mathbf{a}^{c}_{(1)} & \mathbf{a}^{c}_{(2)} & \cdots & \mathbf{a}^{c}_{(n)} \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

span

span

Row Space of **A**
(of row vectors
with *n* element)

Column Space of **A**
(of column vectors
with *m* element)

Dim (Row Space) = rank **A** = rank $\mathbf{A}^{T}$ = Dim (Column Space)

Null Space

Given a matrix **A,** the space of column vectors **x** such that $\mathbf{Ax} = \mathbf{0}$ is called the **null space** of **A** and has the following properties,

- It is a vector space.
- It is the solution space of the homogeneous equation

  $\mathbf{Ax} = \mathbf{0}$.
- Its dimension is called the **nullity** of **A**.
- It satisfies the basic relation (later)

(6) $\quad \text{rank } \mathbf{A} + \text{nullity } \mathbf{A} = \text{Number of columns of } \mathbf{A}.$

7.5 Solutions of Linear Systems: Existence, Uniqueness

Recall: A linear system of *m* equations in *n* unknowns $x_1, \ldots, x_n$

(1)

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

can be expressed as

Ax=b,

where the coefficient matrix **A** is defined as

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ a_{21} & \cdots & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{bmatrix}$$

7.5 Solutions of Linear Systems: Existence, Uniqueness

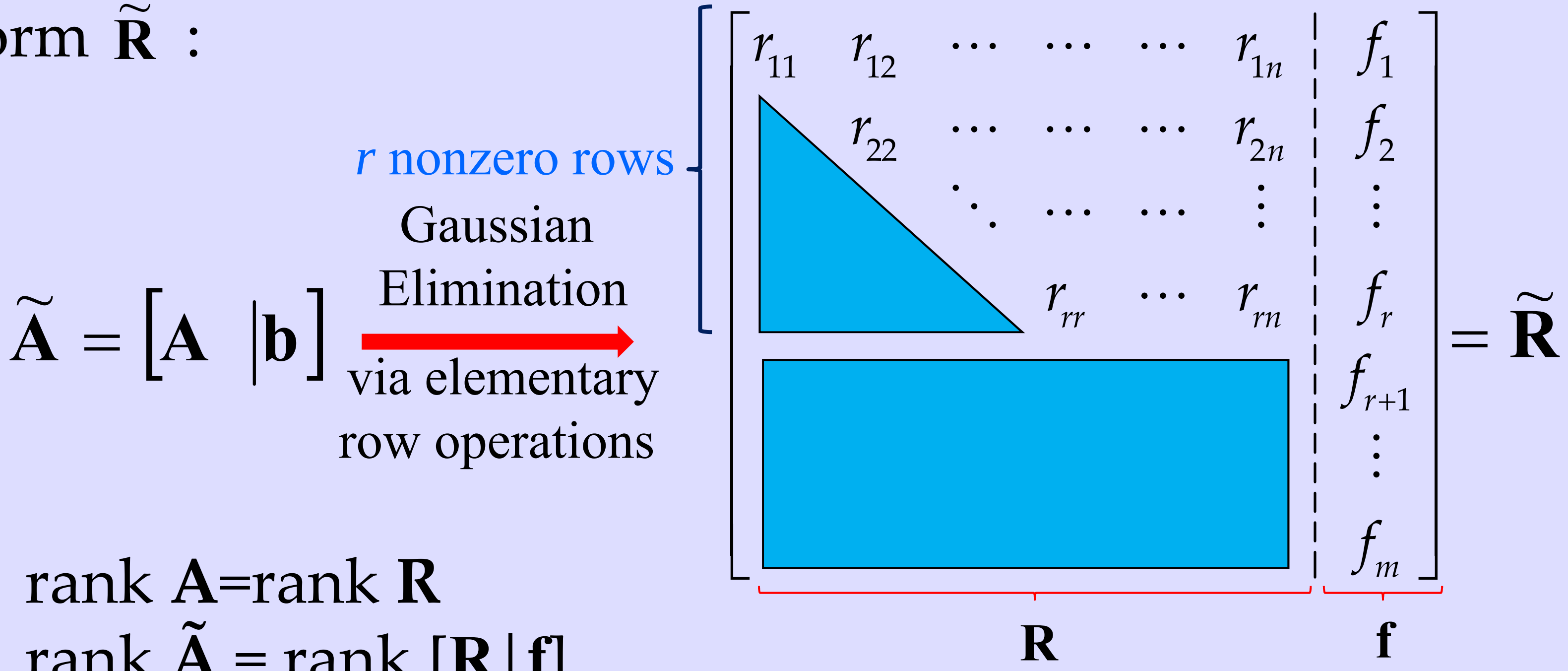
Existence of solution depend on whether (1) is **consistent** or **inconsistent**.

- Previously, we determine **consistency** or **inconsistency** via Gaussian elimination conducted through elementary row operations on the augmented matrix **Ã** defined as

$$\tilde{\mathbf{A}} = \left[\begin{array}{cccc:c} a_{11} & \cdots & \cdots & a_{1n} & b_1 \\ a_{21} & \cdots & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} & b_m \end{array}\right]$$

- Elementary row operations will not change the rank of a matrix
- Here, we formulate the **consistency/inconsistency** problem using the concept of **matrix rank!**

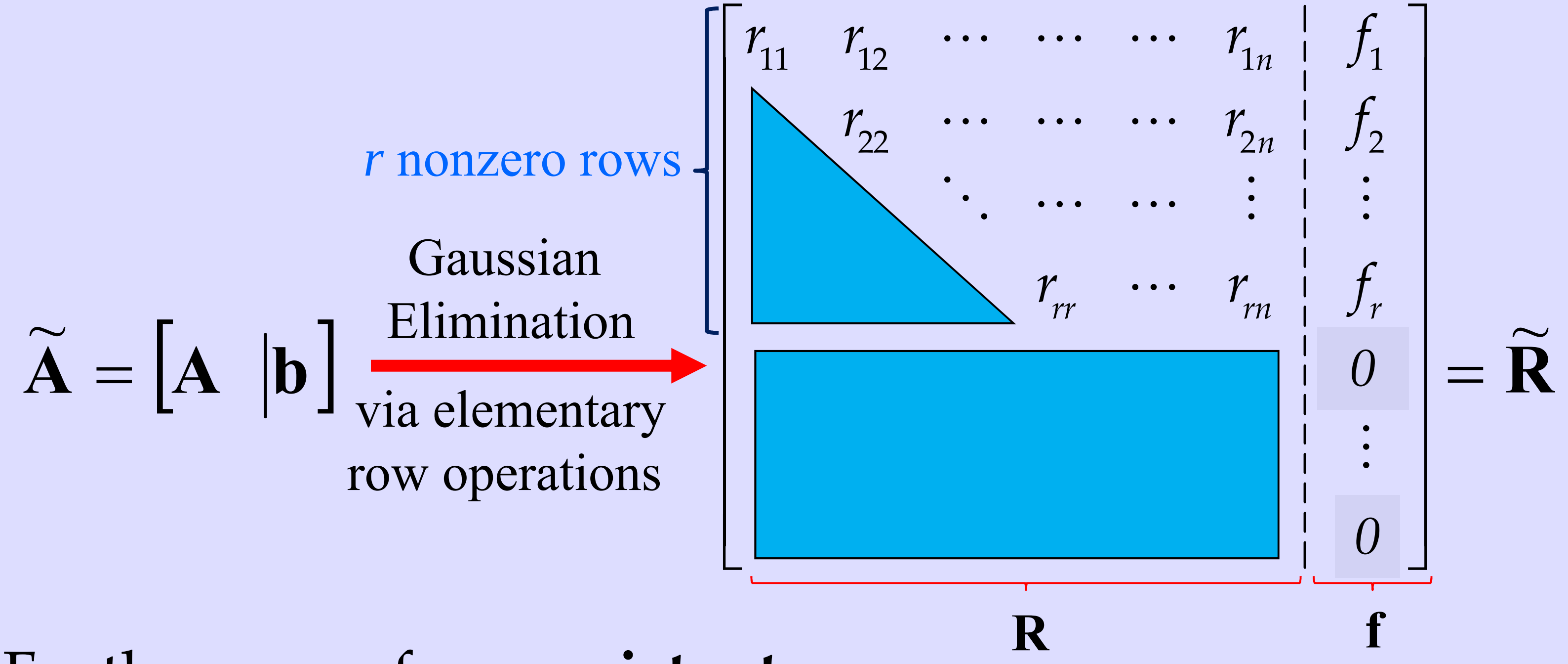
Upon converting augmented matrix $\tilde{\mathbf{A}}$ into row echelon form $\tilde{\mathbf{R}}$:



$$\tilde{\mathbf{A}} = [\mathbf{A} \;|\mathbf{b}] \xrightarrow[\text{via elementary row operations}]{\text{Gaussian Elimination}} \left[\begin{array}{cccccc|c} r_{11} & r_{12} & \cdots & \cdots & \cdots & r_{1n} & f_1 \\ & r_{22} & \cdots & \cdots & \cdots & r_{2n} & f_2 \\ & & \ddots & \cdots & \cdots & \vdots & \vdots \\ & & & r_{rr} & \cdots & r_{rn} & f_r \\ & & & & & & f_{r+1} \\ & & & & & & \vdots \\ & & & & & & f_m \end{array}\right] = \tilde{\mathbf{R}}$$

- rank **A**=rank **R**
- rank $\tilde{\mathbf{A}}$ = rank [**R** | **f**]
- Previously, (1) is **inconsistent**, i.e., no solution exist, if any of the $f_{r+1}, f_{r+2}, \ldots, f_m$ is nonzero.
- Any of the $f_{r+1}, f_{r+2}, \ldots, f_m$ being nonzero means that matrix $\tilde{\mathbf{R}}$ into have more nonzero rows that **R.**
- Hence, (1) is **inconsistent** iff $\text{rank } \mathbf{A} \neq \text{rank } \tilde{\mathbf{A}}$.

- Previously, (1) is **consistent**, i.e., solution exist, if f_{r+1}, $f_{r+2}, \ldots , f_m$ are all zero, i.e., **R** and $\tilde{\mathbf{R}}$ have same number of nonzero rows.
- Equations (1) is **consistent** iff rank **A** = rank $\tilde{\mathbf{A}}$ = r.



- Furthermore, for **consistent** case,
  (i) Unique solution iff # of nonzero equation $r = n$.
  (ii) Multiple solutions iff # of nonzero equation $r < n$.

- Expressing the above results in terms of matrix rank:

  (a) System (1) is **inconsistent** (i.e., no solution exist) iff $\text{rank } \mathbf{A} \neq \text{rank } \tilde{\mathbf{A}}$.

  (b) System (1) is **consistent** (i.e., solution exist) iff $\text{rank } \mathbf{A} = \text{rank } \tilde{\mathbf{A}}$. Furthermore,

    (i) Unique Solution: System (1) has precisely one solution iff rank $\mathbf{A} = n$.

    (ii) Multiple solutions: System (1) has infinitely many solutions iff rank $\mathbf{A} < n$.

- With rank $\mathbf{A}=r$, the result here is the same as we have previously.

Further Characterization of solutions to Nonhomogeneous Linear System using Matrix Rank

If a nonhomogeneous linear system (1) **Ax=b** is **consistent**, its solutions can be divided into two components:

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{x}_h$$

where

- $\mathbf{x}_0$ *is any* (fixed) solution of nonhomogeneous system

  Ax=b, and

- $\mathbf{x}_h$ correspond to the solution(s) of the corresponding homogeneous system **Ax=0**.

The above applies to both the unique solution case and multiple solution case.

Recall EXAMPLE 3 Consistent (Multiple Solution) Case of Section 7.3

$n = 4$

$m = 3$

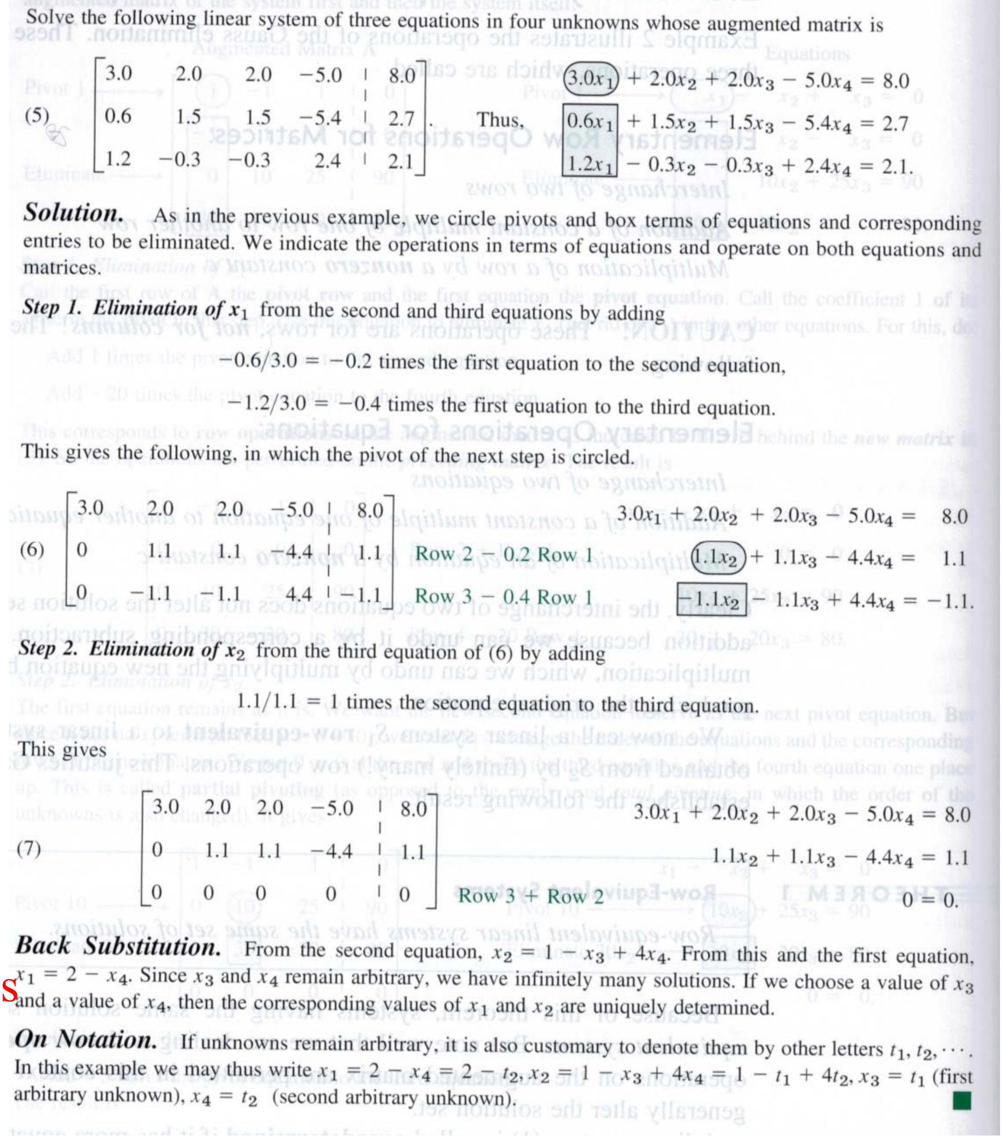
Free variables of x_3 and x_4

Solution is:

$$\mathbf{x} = \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{x}_0} + \underbrace{\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \alpha + \begin{bmatrix} -1 \\ 4 \\ 0 \\ 1 \end{bmatrix} \beta}_{\mathbf{x}_h}$$

Solution to homogeneous equation $\mathbf{Ax}_h\mathbf{=0}$

Any solution to given inhomogeneous equation $\mathbf{Ax}_0\mathbf{=b}$



Solve the following linear system of three equations in four unknowns whose augmented matrix is

$$(5)\quad \left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{array}\right]. \qquad \text{Thus,} \qquad \begin{aligned} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 &= 8.0 \\ \boxed{0.6x_1} + 1.5x_2 + 1.5x_3 - 5.4x_4 &= 2.7 \\ \boxed{1.2x_1} - 0.3x_2 - 0.3x_3 + 2.4x_4 &= 2.1. \end{aligned}$$

Solution. As in the previous example, we circle pivots and box terms of equations and corresponding entries to be eliminated. We indicate the operations in terms of equations and operate on both equations and matrices.

Step 1. Elimination of x_1 from the second and third equations by adding

$-0.6/3.0 = -0.2$ times the first equation to the second equation,

$-1.2/3.0 = -0.4$ times the first equation to the third equation.

This gives the following, in which the pivot of the next step is circled.

$$(6)\quad \left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & -1.1 & -1.1 & 4.4 & -1.1 \end{array}\right] \begin{array}{l} \\ \text{Row 2} - 0.2\text{ Row 1} \\ \text{Row 3} - 0.4\text{ Row 1} \end{array} \qquad \begin{aligned} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 &= 8.0 \\ 1.1x_2 + 1.1x_3 - 4.4x_4 &= 1.1 \\ \boxed{-1.1x_2} - 1.1x_3 + 4.4x_4 &= -1.1. \end{aligned}$$

Step 2. Elimination of x_2 from the third equation of (6) by adding

$1.1/1.1 = 1$ times the second equation to the third equation.

This gives

$$(7)\quad \left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ \text{Row 3} + \text{Row 2} \end{array} \qquad \begin{aligned} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 &= 8.0 \\ 1.1x_2 + 1.1x_3 - 4.4x_4 &= 1.1 \\ 0 &= 0. \end{aligned}$$

Back Substitution. From the second equation, $x_2 = 1 - x_3 + 4x_4$. From this and the first equation, $x_1 = 2 - x_4$. Since x_3 and x_4 remain arbitrary, we have infinitely many solutions. If we choose a value of x_3 and a value of x_4, then the corresponding values of x_1 and x_2 are uniquely determined.

On Notation. If unknowns remain arbitrary, it is also customary to denote them by other letters $t_1, t_2, \cdots$. In this example we may thus write $x_1 = 2 - x_4 = 2 - t_2$, $x_2 = 1 - x_3 + 4x_4 = 1 - t_1 + 4t_2$, $x_3 = t_1$ (first arbitrary unknown), $x_4 = t_2$ (second arbitrary unknown). ■

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Homogeneous Linear System

A homogeneous linear system

(4)
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ \cdots\cdots\cdots&\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

or in matrix form,

$$\mathbf{Ax=0},$$

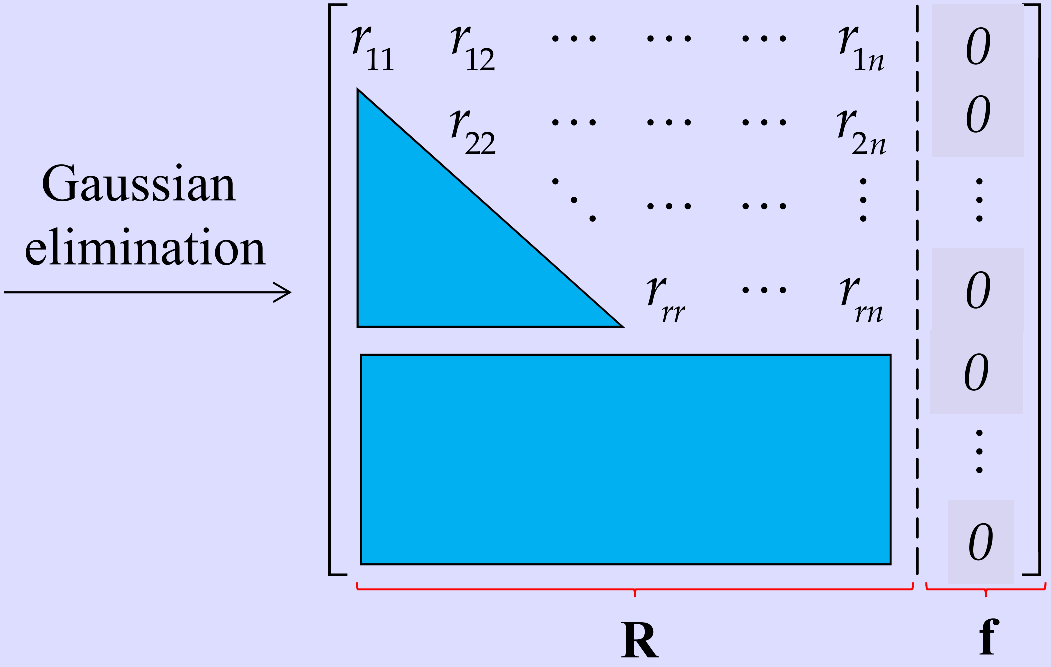
always has the **trivial solution** $x_1 = 0, \ldots, x_n = 0$, i.e., $\mathbf{x=0}$.

Question is: when will **(4)** have **nontrivial solutions**, i.e., all or some of the $x_1, \ldots, x_n$, are nonzero?

Using Gaussian elimination, i.e., elementary row operations to solve homogeneous equation $\mathbf{Ax=0}$:

Augmented matrix

$$\tilde{\mathbf{A}} = [\mathbf{A} \;|\mathbf{0}]$$



Homogeneous linear equations always **consistent**, as f_{r+1}, f_{r+2}, … , f_m must all be zero, hence:

- if rank $\mathbf{A}=n$, $\mathbf{Ax=0}$ has unique solution, i.e., $\mathbf{x=0}$!
- if rank $\mathbf{A} < n$, $\mathbf{Ax=0}$ have multiple solutions

In more details:

- If rank **A**=n, **Ax=0** has unique solution
  - **trivial solution**, **x=0** (the only solution!)
- If rank **A**=$r < n$, **Ax=0** have multiple solutions
  - **nontrivial solution**, obtained by solving r of the variables $x_1, \ldots, x_n$, in terms of the remaining (n-r) ***free*** variables (taking on arbitrary values).
  - The solutions of **Ax=0** form a vector space, called the Null Space of **A.**
  - The Null Space as formed is of dimension ($n - r$). (Why? There are (n-r) ***free*** variables).
  - The Null Space has at most (n-r) linearly independent vectors.
  - Any (n-r) linearly independent vectors in the Null Space is a basis to span itself.

7.5 Solutions of Linear Systems: Existence, Uniqueness

- The dimension of the **null space** of **A,** called **nullity** of **A** as defined previously, is $(n - r)$. Hence,

$$\text{nullity } \mathbf{A} = n - \text{rank } \mathbf{A}$$

or

(5) $$\text{rank } \mathbf{A} + \text{nullity } \mathbf{A} = n$$

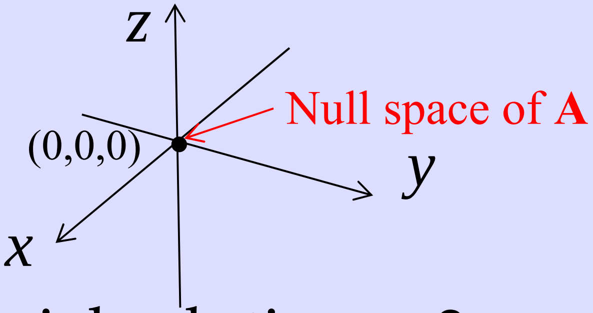
where n is the number of unknowns (number of columns of **A**).

- Note: solutions of nonhomogeneous systems do not form a vector space.)

Null Space Examples

Example N1:

$m = 3, n = 3$ $\quad \mathbf{A} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$



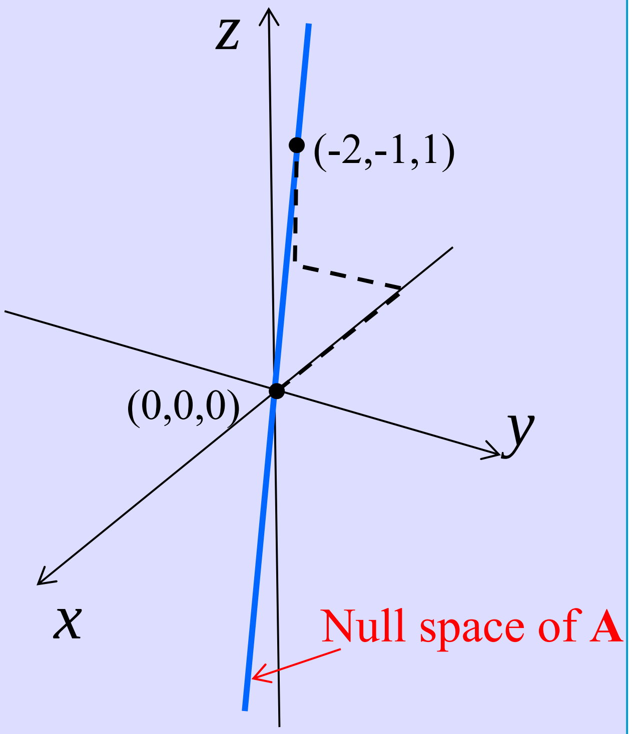
rank A=3=n, solution of **Ax=0** is the trivial solution **x=0**.

Example N2:

$m = 2$ $n = 3$ $\quad \mathbf{A} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 3 \end{bmatrix}$

rank **A**=2 < n, nontrivial solutions of **Ax=0** is the Null Space of dimension (n-r)=1 given by:

$$\mathbf{x}_h = \beta \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$

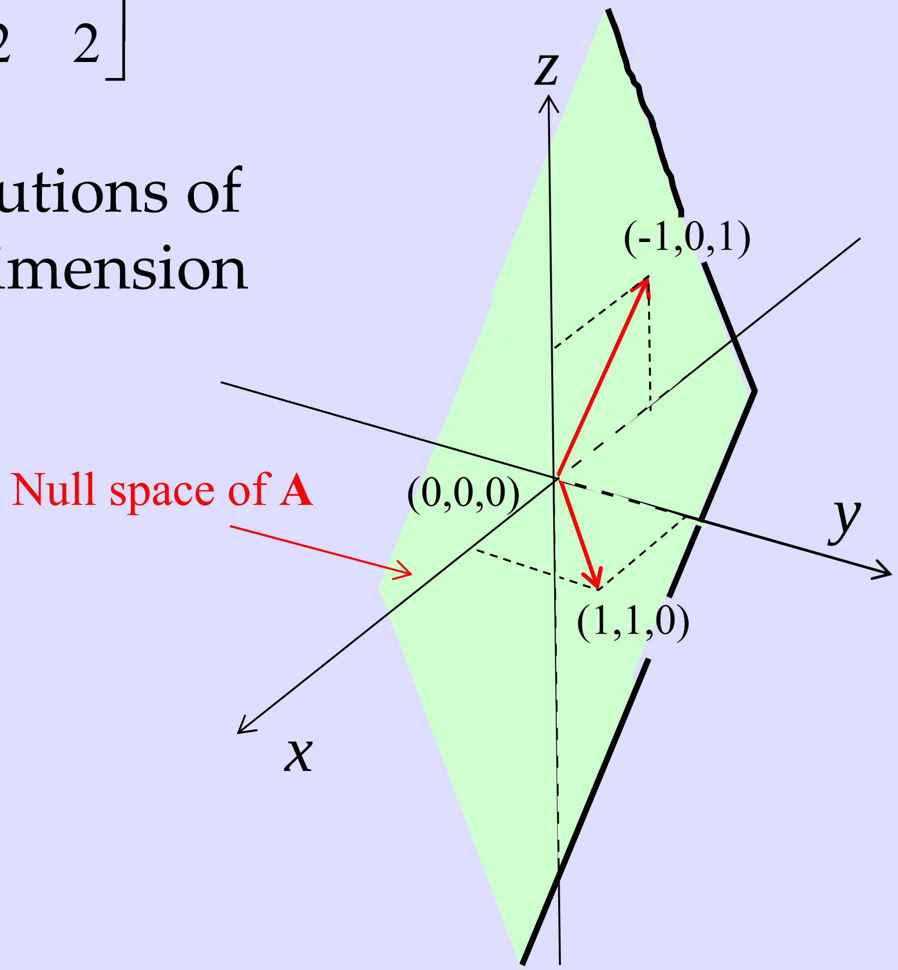


Example N3:

$$m = 2 \; n = 3 \qquad \mathbf{A} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \end{bmatrix}$$

rank **A**=1 < n, nontrivial solutions of **Ax=0** is the Null Space of dimension (n-r)=2 given by:

$$\mathbf{x}_h = \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$



Theorem 3

Homogeneous Linear System with Fewer Equations Than Unknowns

A homogeneous linear system **Ax=0** with fewer equations than unknowns always has nontrivial solutions.

Reasoning:
Let **A** be m by n with $m < n$ (fewer equations than unknowns). Then, we have rank **A** $\leq m < n$, which yields the **consistent** (multiple solution) case.

Combining the previous results, we have the final characterization of the solution of the nonhomogeneous linear equations (1) in the **consistent** case: $\mathbf{x} = \mathbf{x}_0 + \mathbf{x}_h$

- If rank $\mathbf{A}=n$, $\mathbf{Ax=b}$ has unique solution; $\mathbf{Ax=0}$ has **trivial solution**, $\mathbf{x}_h=\mathbf{0}$.
- If rank $\mathbf{A}=r<n$, $\mathbf{Ax=b}$ have multiple solutions, with
  - $\mathbf{Ax=0}$ has **nontrivial solution** $\mathbf{x}_h$ obtained by solving r of the variables $x_1, \ldots, x_n$, in terms of the remaining $(n\text{-}r)$ ***free*** variables in the row echelon form $\mathbf{Rx=f}$.
  - $\mathbf{x}_0$ is any solution of $\mathbf{Ax=b}$ or $\mathbf{Rx=f}$. Note that one can conveniently obtain $\mathbf{x}_0$ by assigning zero values to all free parameter.

Redo EXAMPLE 3 Consistent (Multiple Solution) Case of Section 7.3

$n = 4\,,\, m = 3$

Solving for $\mathbf{x}_0$:

set free variables of x_3=0 and x_4=0

$$\left[\begin{array}{cccc:c} 3 & 2 & 2 & -5 & 8 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow x_2 = 1 \Rightarrow x_1 = 2 \Rightarrow \mathbf{x}_0 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Solving for $\mathbf{x}_h$:

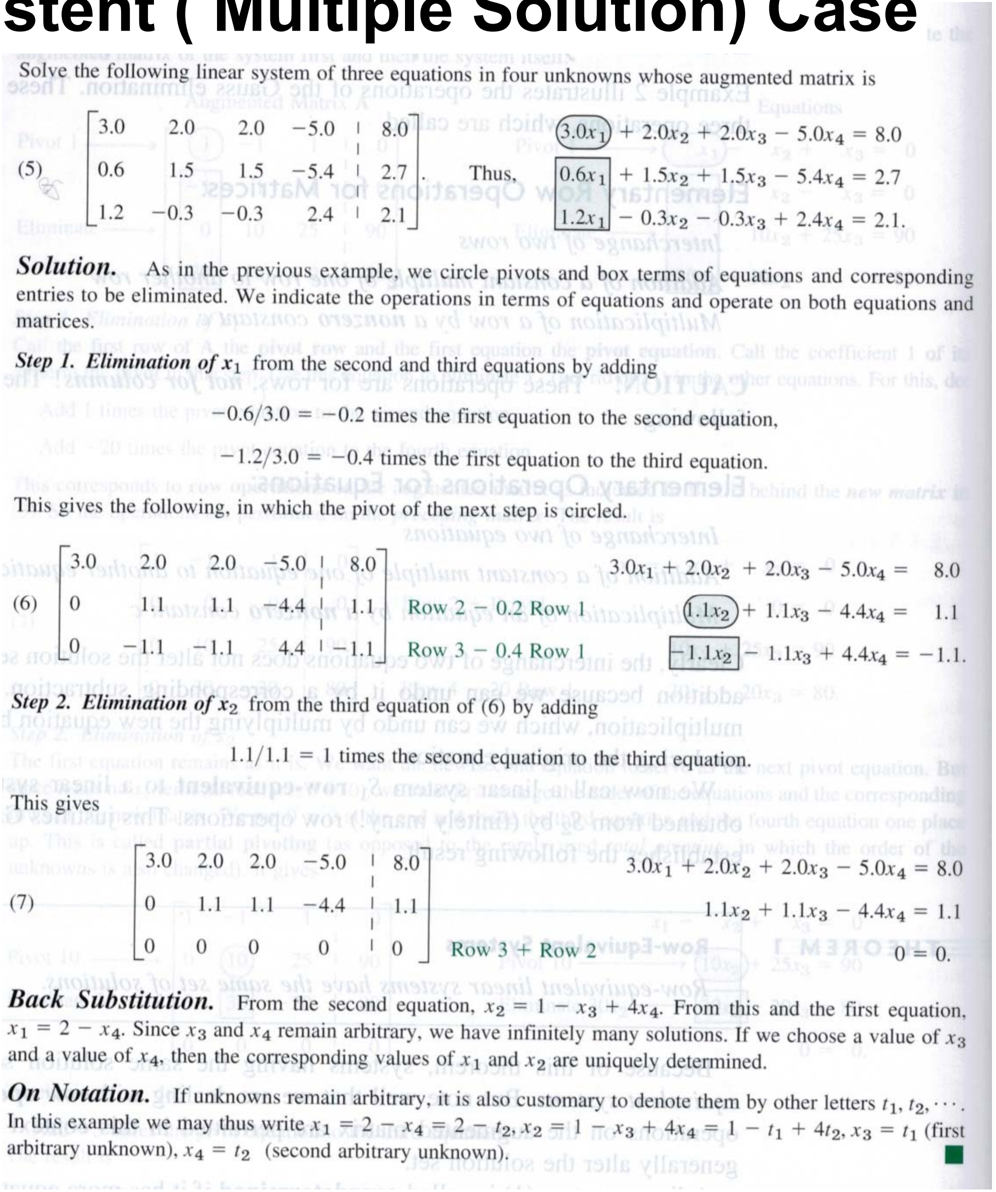
Let **f**=**0** and express x_1 and x_2 in terms of free variables $x_3=\alpha$ and $x_4=\beta$

$$\left[\begin{array}{cccc:c} 3 & 2 & 2 & -5 & 0 \\ 0 & 1.1 & 1.1 & -4.4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \mathbf{x}_h = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \alpha + \begin{bmatrix} -1 \\ 4 \\ 0 \\ 1 \end{bmatrix} \beta$$

In this case, Rank **A**=2 <n=4

→ nontrivial solution for $\mathbf{x}_h$



Solve the following linear system of three equations in four unknowns whose augmented matrix is

$$\text{(5)} \quad \left[\begin{array}{cccc:c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{array}\right]. \qquad \text{Thus,} \qquad \begin{aligned} \enclose{circle}{3.0x_1} + 2.0x_2 + 2.0x_3 - 5.0x_4 &= 8.0 \\ \boxed{0.6x_1} + 1.5x_2 + 1.5x_3 - 5.4x_4 &= 2.7 \\ \boxed{1.2x_1} - 0.3x_2 - 0.3x_3 + 2.4x_4 &= 2.1. \end{aligned}$$

Solution. As in the previous example, we circle pivots and box terms of equations and corresponding entries to be eliminated. We indicate the operations in terms of equations and operate on both equations and matrices.

Step 1. Elimination of x_1 from the second and third equations by adding

$-0.6/3.0 = -0.2$ times the first equation to the second equation,

$-1.2/3.0 = -0.4$ times the first equation to the third equation.

This gives the following, in which the pivot of the next step is circled.

$$\text{(6)} \quad \left[\begin{array}{cccc:c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & -1.1 & -1.1 & 4.4 & -1.1 \end{array}\right] \begin{array}{l} \\ \text{Row 2} - 0.2 \text{ Row 1} \\ \text{Row 3} - 0.4 \text{ Row 1} \end{array} \qquad \begin{aligned} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 &= 8.0 \\ \enclose{circle}{1.1x_2} + 1.1x_3 - 4.4x_4 &= 1.1 \\ \boxed{-1.1x_2} - 1.1x_3 + 4.4x_4 &= -1.1. \end{aligned}$$

Step 2. Elimination of x_2 from the third equation of (6) by adding

$1.1/1.1 = 1$ times the second equation to the third equation.

This gives

$$\text{(7)} \quad \left[\begin{array}{cccc:c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ \text{Row 3} + \text{Row 2} \end{array} \qquad \begin{aligned} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 &= 8.0 \\ 1.1x_2 + 1.1x_3 - 4.4x_4 &= 1.1 \\ 0 &= 0. \end{aligned}$$

Back Substitution. From the second equation, $x_2 = 1 - x_3 + 4x_4$. From this and the first equation, $x_1 = 2 - x_4$. Since x_3 and x_4 remain arbitrary, we have infinitely many solutions. If we choose a value of x_3 and a value of x_4, then the corresponding values of x_1 and x_2 are uniquely determined.

On Notation. If unknowns remain arbitrary, it is also customary to denote them by other letters $t_1, t_2, \cdots$. In this example we may thus write $x_1 = 2 - x_4 = 2 - t_2$, $x_2 = 1 - x_3 + 4x_4 = 1 - t_1 + 4t_2$, $x_3 = t_1$ (first arbitrary unknown), $x_4 = t_2$ (second arbitrary unknown). ■

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MORE EXAMPLES

Example M1: Consistent case: unique solution

Solve the linear system

$$\begin{aligned} x_1 - x_2 + x_3 &= 2 \\ 2x_1 - x_2 + 3x_3 &= 5 \\ x_1 + x_2 + 4x_3 &= 6 \end{aligned}$$

$m = 3\,,\, n = 3$

Augmented matrix **Ã**

$$\tilde{\mathbf{A}} = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 2 & -1 & 3 & 5 \\ 1 & 1 & 4 & 6 \end{array}\right]$$

$\underbrace{\qquad\qquad}_{\mathbf{A}}$ $\underbrace{\quad}_{\mathbf{b}}$

Elementary row operations $\Longrightarrow$

Row echelon form

$$r = 3 \left\{ \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \right.$$

$\underbrace{\qquad\qquad}_{\mathbf{R}}$ $\underbrace{\quad}_{\mathbf{f}}$

$\Downarrow$

rank **A** = rank **Ã**

→ **Consistent**

rank **A** = r =3= n

→ case (b) (i):

Unique solution

Refer to Example N1 for $\mathbf{x}_\mathrm{h}$ $\Longleftarrow$

Solution:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}}_{\mathbf{x}_\mathrm{o}} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{x}_\mathrm{h}} \quad \mathbf{x}_\mathrm{h} = \mathbf{0}$$

Example M2: Consistent case: multiple solutions

Solve the linear system

$$\begin{aligned} x_1 - x_2 + x_3 &= 2 \\ 2x_1 - x_2 + 3x_3 &= 5 \\ x_1 + x_2 + 3x_3 &= 4 \end{aligned}$$

$m = 3\ ,\ n = 3$

Augmented matrix Ã

$$\tilde{\mathbf{A}} = \left[\begin{array}{ccc:c} 1 & -1 & 1 & 2 \\ 2 & -1 & 3 & 5 \\ 1 & 1 & 3 & 4 \end{array}\right]$$

$\underbrace{\qquad\qquad}_{\mathbf{A}}\ \underbrace{\quad}_{\mathbf{b}}$

$\Longrightarrow$ Elementary row operations

Row echelon form

$$r = 2 \left\{ \left[\begin{array}{ccc:c} 1 & -1 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \right.$$

$\underbrace{\qquad\qquad}_{\mathbf{R}}\ \underbrace{\quad}_{\mathbf{f}}$

$\Downarrow$

rank **A** = rank **Ã**

→ **Consistent**

rank **A** = r =2 < n=3

→ case (b) (ii): multiple solutions

Free variables:

$x_3 = \beta$

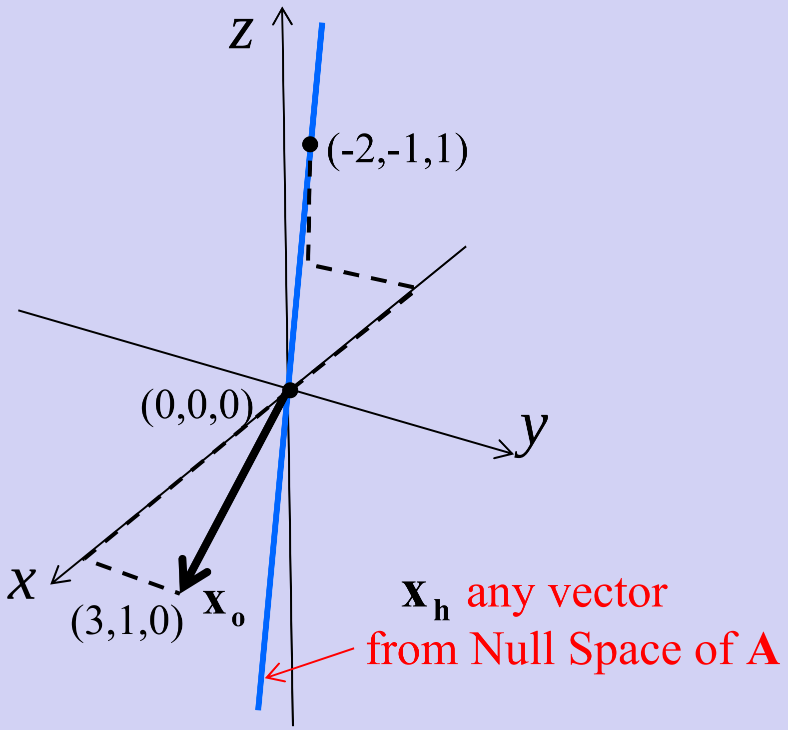
$\Longleftarrow$

Refer to Example N2 for x_h

Solution:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{x}_\mathbf{o}} + \beta \underbrace{\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}}_{\mathbf{x}_\mathbf{h}}$$

Solution space of Example M2: $\mathbf{x} = \mathbf{x}_0 + \mathbf{x}_h$



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Example M3: Consistent case: multiple solutions

Solve the linear system

$$\begin{aligned} x_1 - x_2 + x_3 &= 2 \\ 2x_1 - 2x_2 + 2x_3 &= 4 \\ -x_1 + x_2 - x_3 &= -2 \end{aligned}$$

$m = 3, n = 3$

Augmented matrix $\tilde{\mathbf{A}}$

$$\tilde{\mathbf{A}} = \left[\begin{array}{c:c} \underbrace{\begin{matrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ -1 & 1 & -1 \end{matrix}}_{\mathbf{A}} & \underbrace{\begin{matrix} 2 \\ 4 \\ -2 \end{matrix}}_{\mathbf{b}} \end{array}\right] \xrightarrow{\text{Elementary row operations}} \left[\begin{array}{c:c} \underbrace{\begin{matrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}}_{\mathbf{R}} & \underbrace{\begin{matrix} 2 \\ 0 \\ 0 \end{matrix}}_{\mathbf{f}} \end{array}\right]$$

Row echelon form

$r = 1$

rank $\mathbf{A}$ = rank $\tilde{\mathbf{A}}$

→ **Consistent**

rank $\mathbf{A} = r = 1 < n = 3$

→ case (b) (i): multiple solution

Free variables:

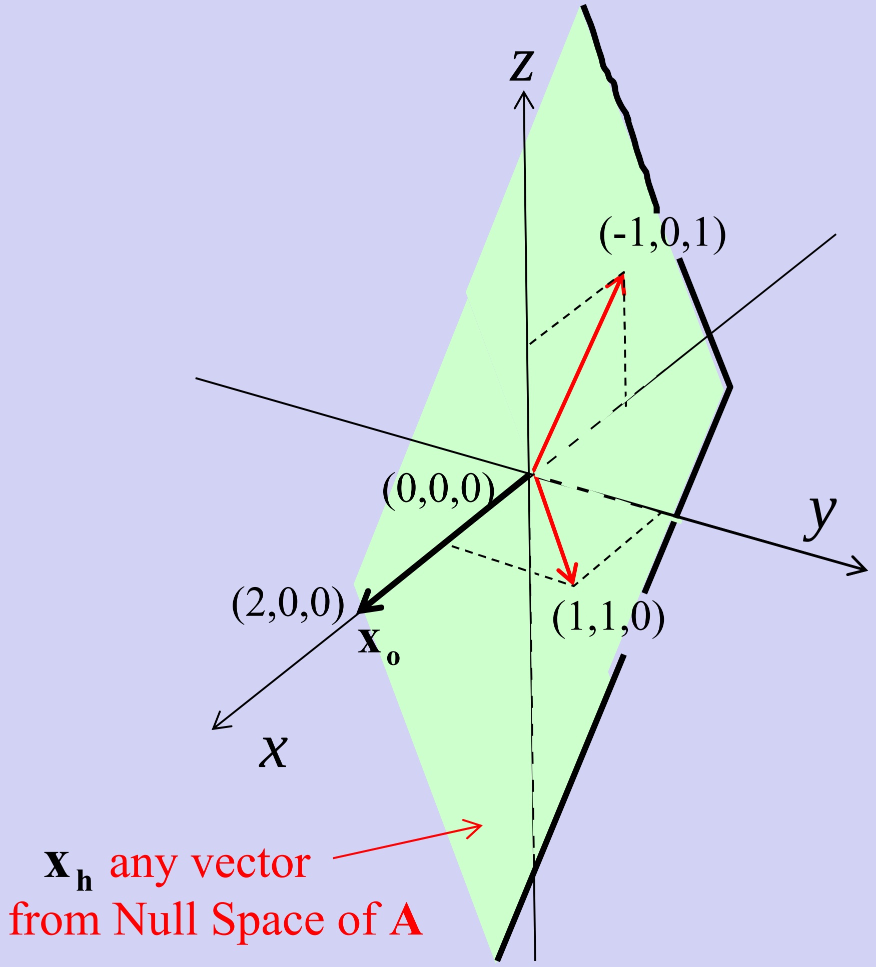
$x_2 = \alpha, x_3 = \beta$

Refer to Example N3 for x_h

Solution:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{x}_o} + \underbrace{\alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{x}_h}$$

Solution space of Example M3: $\mathbf{x} = \mathbf{x}_0 + \mathbf{x}_h$



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7.6 For Reference: Second- and Third-Order Determinants

7.6 For Reference: Second- and Third-Order Determinants

- A **determinant of second order** is denoted and defined for 2 by 2 matrix by

$$(1) \qquad D = \det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

- Determinant denoted with ***bars*** (whereas a matrix uses ***brackets***).

7.6 For Reference:
Second- and Third-Order Determinants

- Given a linear systems of **2** equations in **2** unknowns

$$(2) \qquad \begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

Let D be the determinant as given in (1).

- If $D \neq 0$, then system (2) is **consistent** with unique solution (later!). This unique solution is given by the **Cramer's rule:**

$$(3) \qquad x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{D} = \frac{b_1a_{22} - a_{12}b_2}{D},$$

$$x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{D} = \frac{a_{11}b_2 - b_1a_{21}}{D}.$$

Third-Order Determinants

A **determinant of third order** can be defined for 3 by 3 matrix by

$$(4)\quad D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\underbrace{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}} - a_{21}\underbrace{\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}} + a_{31}\underbrace{\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}}.$$

Minor corresponding to entry a_{11}, a_{21}, and a_{31}

Note that

- The signs on the right are + − +.
- Each of the three terms on the right is an entry in the first column of D times its corresponding **minor**, which is the second-order determinant obtained from D by deleting the row and column of that entry;
- If we write out the minors in (4), we obtain

$$(4^*)\quad D = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{21}a_{13}a_{32} - a_{21}a_{12}a_{33} + a_{31}a_{12}a_{23} - a_{31}a_{13}a_{22}.$$

7.6 For Reference: Second- and Third-Order Determinants

- Given a linear systems of **3** equations in **3** unknowns

$$(5)\qquad \begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

  Let D be the determinant as given in (4).

- If $D \neq 0$, system (5) is **consistent** with unique solution (later!). This unique solution is given by the **Cramer's rule:**

$$x_1 = \frac{D_1}{D}, \qquad x_2 = \frac{D_2}{D}, \qquad x_3 = \frac{D_3}{D}$$

, where

$$(6)\qquad D_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}, \qquad D_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}, \qquad D_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}.$$

- Note: D_1, D_2, D_3 are obtained by replacing Columns 1, 2, 3, respectively, by the column of the right sides of (5) in (4).

7.7 Determinants. Cramer's Rule

Generalization to *n* by *n* Matrix

A **determinant of order** n is a scalar associated with an $n \times n$ (***square***) matrix $\mathbf{A} = [a_{jk}]$, and is denoted by

(1)
$$D = \det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix}.$$

For $n = 1$, this determinant is defined by

(2)
$$D = a_{11}.$$

7.7 Determinants. Cramer's Rule

- **Def:** M_{jk}, called the Minor of entry a_{jk}, is a determinant of order $n - 1$, namely, the determinant of the submatrix of **A** obtained from **A** by omitting the row and column of the entry a_{jk}, that is, the jth row and the kth column.

$$\mathbf{A} = \begin{array}{c|ccccccc|} & 1^{\text{st}} & \cdots & (k\text{-}1)^{\text{th}} & k^{\text{th}} & (k+1)^{\text{th}} & \cdots & n^{\text{th}}\text{ column} \\ 1^{\text{st}} & a_{1,1} & \cdots & a_{1,k-1} & a_{1,k} & a_{1,k+1} & \cdots & a_{1,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (j\text{-}1)^{\text{th}} & a_{j-1,1} & \cdots & a_{j-1,k-1} & a_{j-1,k} & a_{j-1,k+1} & \cdots & a_{j-1,n} \\ j^{\text{th}} & a_{j,1} & \cdots & a_{j,k-1} & a_{j,k} & a_{j,k+1} & \cdots & a_{j,n} \\ (j+1)^{\text{th}} & a_{j+1,1} & \cdots & a_{j+1,k-1} & a_{j+1,k} & a_{j+1,k+1} & \cdots & a_{j+1,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ n^{\text{th}}\text{ row} & a_{n,1} & \cdots & a_{n,k-1} & a_{n,k} & a_{n,k+1} & \cdots & a_{n,n} \end{array}$$

$$\begin{array}{c|cccccc|l}
 & 1^{\text{st}} & \cdots\cdots & (k\text{-}1)^{\text{th}} & (k+1)^{\text{th}} & \cdots\cdots & n^{\text{th}}\text{ column} & \\
1^{\text{st}} & a_{1,1} & \cdots & a_{1,\,k-1} & a_{1,\,k+1} & \cdots & a_{1,n} & \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \\
(j\text{-}1)^{\text{th}} & a_{j-1,1} & \cdots & a_{j-1,\,k-1} & a_{j-1,\,k+1} & \cdots & a_{j-1,\,n} & \\
(j+1)^{\text{th}} & a_{j+1,1} & \cdots & a_{j+1,\,k-1} & a_{j+1,\,k+1} & \cdots & a_{j+1,\,n} & = M_{jk} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \\
n^{\text{th}}\text{ row} & a_{n,1} & \cdots & a_{n,\,k-1} & a_{n,\,k+1} & \cdots & a_{n,n} &
\end{array}$$

- Take submatrix of **A** by omitting its *j*th row and the *k*th column
- M_{jk} is the determinant of the resulting $(n-1) \times (n-1)$ submatrix

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- For $n \geq 2$, the determinant of the $n \times n$ matrix A can be put in terms of the Minors M_{jk} about the entries of the 1st row as follows:

$$D = a_{1,1}M_{1,1} - a_{1,2}M_{1,2} + a_{1,3}M_{1,3} + \cdots + (-1)^{n+1}a_{1,n}M_{1,n}$$

or about the entries of the 1st column,

$$D = a_{1,1}M_{1,1} - a_{2,1}M_{2,1} + a_{3,1}M_{3,1} + \cdots + (-1)^{n+1}a_{n,1}M_{n,1}$$

or about the entries of the jth row, j=1,2,…, n

$$D = (-1)^{j+1}a_{j1}M_{j1} + (-1)^{j+2}a_{j2}M_{j2} + (-1)^{j+3}a_{j3}M_{j3} + \cdots + (-1)^{j+n}a_{jn}M_{jn}$$

$$= \sum_{k=1}^{n}(-1)^{j+k}a_{jk}M_{jk}$$

or about the entries of the kth column, k=1,2,…, n

$$D = (-1)^{k+1}a_{1k}M_{1k} + (-1)^{k+2}a_{2k}M_{2k} + (-1)^{k+3}a_{3k}M_{3k} + \cdots + (-1)^{k+n}a_{nk}M_{nk}$$

$$= \sum_{j=1}^{n}(-1)^{j+k}a_{jk}M_{jk}$$

- Can also define Cofactor C_{jk} of the entry a_{jk} as

$$C_{jk} = (-1)^{j+k} M_{jk}$$

i.e., C_{jk} absorbs the factor $(-1)^{j+k}$ in its definition so that the determinant can be expressed as:

(3a) $D = a_{j1}C_{j1} + a_{j2}C_{j2} + \dots + a_{jn}C_{jn}$
(about jth row, $j = 1, 2, \dots$, or n)

or

(3b) $D = a_{1k}C_{1k} + a_{2k}C_{2k} + \dots + a_{nk}C_{nk}$
(about kth column $k = 1, 2, \dots$, or n).

- As such, D is defined in terms of n determinants of order $n - 1$, each of which is, in turn, defined in terms of $n - 1$ determinants of order $n - 2$ and so on—until we finally arrive at first-order determinants.
- We may **expand** D by any row or column, that is, one can choose to use in (3) the entries in any row or column.
- Result is same for D no matter which columns or rows we choose in expanding the determinant.
- Definition is consistent with that of second and third order determinants before in (1) and (4) of Section 7.6.

EXAMPLE 1

Minors and Cofactors of a Third-Order Determinant

In (4) of the previous section the minors and cofactors of the entries in the first column can be seen directly. For the entries in the second row the minors are

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}, \qquad M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, \qquad M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

and the cofactors are $C_{21} = -M_{21}$, $C_{22} = +M_{22}$, and $C_{23} = -M_{23}$.

Note also that the signs in C_{jk} form a **checkerboard pattern**

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

EXAMPLE 2

Expansions of a Third-Order Determinant

$$D = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix} = 1\begin{vmatrix} 6 & 4 \\ 0 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} + 0\begin{vmatrix} 2 & 6 \\ -1 & 0 \end{vmatrix}$$

$$= 1(12 - 0) - 3(4 + 4) + 0(0 + 6) = -12.$$

This is the expansion by the first row. The expansion by the third column is

$$D = 0\begin{vmatrix} 2 & 6 \\ -1 & 0 \end{vmatrix} - 4\begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix} + 2\begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 0 - 12 + 0 = -12.$$

Verify that the other four expansions also give the value −12.

- Obtaining determinant using minors and cofactors are for the definition purposes.
- The formula are extremely computationally intensive to perform for large matrix.

e=2.7183

  - Number of computations nxn matrix: $N(n) \sim e\,(n!)$
  - Computing time for matrices of different sizes:

Matrix size nxn	Number of computations N(n)	Computing Time of the Determinant
5x5	~324	~0.0003 sec
10x10	$\sim 9\text{x}10^{6}$	~10 sec
15x15	$\sim 3.6\text{x}10^{12}$	~40 days
20x20	$\sim 6.6\text{x}10^{18}$	~200,000 years
25x25	$\sim 4.2\text{x}10^{25}$	$\sim 10^{12}$ years

* From Greenberg, Advanced Engineering Mathematics

- Scientific/engineering calculation easily involve matrices of size 250x250 → need more efficient method in computing determinant of matrix!

Application of Row elementary operations to Determinants

- Seek to find determinant by applying elementary row operations to convert (1) to an row echelon form.
- Given that determinant involves only square matrices, row echelon form in this case means "upper triangular" matrix.
- "Upper triangular" matrix determinant value very easily computed as product of its diagonal entries.
- Approach **similar** to Sec. 7.3. **but not the same** because determinant values **can be changed** by elementary row operations.

THEOREM 1 Behavior of an *n*th-Order Determinant under Elementary Row Operations

(a) Interchange of two rows changes the value of the determinant by −1.

$$\det\begin{bmatrix} \text{---}\ \mathbf{a}_{(1)}\ \text{---} \\ \vdots \\ \text{---}\ \mathbf{a}_{(j)}\ \text{---} \\ \vdots \\ \text{---}\ \mathbf{a}_{(k)}\ \text{---} \\ \vdots \\ \text{---}\ \mathbf{a}_{(n)}\ \text{---} \end{bmatrix} = \begin{vmatrix} \text{---}\ \mathbf{a}_{(1)}\ \text{---} \\ \vdots \\ \text{---}\ \mathbf{a}_{(j)}\ \text{---} \\ \vdots \\ \text{---}\ \mathbf{a}_{(k)}\ \text{---} \\ \vdots \\ \text{---}\ \mathbf{a}_{(n)}\ \text{---} \end{vmatrix} = (-1)\begin{vmatrix} \text{---}\ \mathbf{a}_{(1)}\ \text{---} \\ \vdots \\ \text{---}\ \mathbf{a}_{(k)}\ \text{---} \\ \vdots \\ \text{---}\ \mathbf{a}_{(j)}\ \text{---} \\ \vdots \\ \text{---}\ \mathbf{a}_{(n)}\ \text{---} \end{vmatrix} \begin{array}{l} \\ \\ \leftarrow \\ \text{swap} \\ \leftarrow \\ \\ \\ \end{array}$$

- Proof (Optional):

i. For example: consider **M′** obtained by interchanging, say, the 1st and 2nd row of **M**:

$$\mathbf{M} = \begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & \cdots & a_{n,n} \end{vmatrix}$$

Determinant D (expanding about 1st row) of **M**:

$$D = a_{1,1}M_{1,1} - a_{1,2}M_{1,2} + \cdots$$
$$+ (-1)^{n+1} a_{1,n} M_{1,n}$$

$$\mathbf{M}' = \begin{vmatrix} a_{2,1} & a_{2,2} & a_{2,3} & \cdots & \cdots & a_{2,n} \\ a_{1,1} & a_{1,2} & a_{1,3} & \cdots & \cdots & a_{1,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & \cdots & a_{n,n} \end{vmatrix}$$

Determinant D' (expanding about 2nd row) of **M′**:

$$D' = -a_{1,1}M'_{2,1} + a_{1,2}M'_{2,2} - \cdots$$
$$+ (-1)^{n+2} a_{1,n} M'_{2,n}$$

ii. Noting that $M_{1,1}=M'_{2,1}$, $M_{1,2}=M'_{2,2}$, or $M_{1,j}=M'_{2,j}$ in general, hence we have $D'=(-1)D$.

Optional

iii. By same reasoning, interchanging any adjacent row, say, r^{th} and $(r+1)^{th}$ row of a matrix **M** changes its determinant value by (-1).

iii. Consider swapping the j^{th} and k^{th} row of **M**: with $j < k$, then
- Moving the j^{th} row to the j^{th} row location involves $(k\text{-}j)$ interchangings of adjacent row
- Then moving the k^{th} row to the j^{th} row position involves $(k\text{-}j)\text{-}1$ interchangings of adjacent row

iv. Swapping the j^{th} and k^{th} row hence involves a total of involves $2(k\text{-}j)\text{-}1$ interchangings of adjacent row

v. Swapping any j^{th} and k^{th} row of a matrix **M** then changes its determinant value by $(-1)^{2(k-j)-1} =(-1)$.

(b) Addition of a multiple of a row to another row does not alter the value of the determinant.

$$\begin{vmatrix} \text{---}\ \mathbf{a}_{(1)}\ \text{---} \\ \vdots \\ \text{---}\ \mathbf{a}_{(j)}\ \text{---} \\ \vdots \\ \text{---}\ \mathbf{a}_{(k)}\ \text{---} \\ \vdots \\ \text{---}\ \mathbf{a}_{(n)}\ \text{---} \end{vmatrix} = \begin{vmatrix} \text{---}\ \mathbf{a}_{(1)}\ \text{---} \\ \vdots \\ -\ \mathbf{a}_{(j)} + c\mathbf{a}_{(k)} - \\ \vdots \\ \text{---}\ \mathbf{a}_{(k)}\ \text{---} \\ \vdots \\ \text{---}\ \mathbf{a}_{(n)}\ \text{---} \end{vmatrix}$$

Proof (Optional): Expanding RHS about the j^{th} row:

$$(-1)^{j+1}(a_{j,1} + ca_{k,1})M_{j,1} + (-1)^{j+2}(a_{j,2} + ca_{k,2})M_{j,2} + \cdots + (-1)^{j+n}(a_{j,n} + ca_{k,n})M_{j,n}$$

$$= (-1)^{j+1}a_{j,1}M_{j,1} + (-1)^{j+2}a_{j,2}M_{j,2} + \cdots + (-1)^{j+n}a_{j,n}M_{j,n}$$

$$+ \cancel{(-1)^{j+1}ca_{k,1}M_{j,1} + (-1)^{j+2}ca_{k,2}M_{j,2} + \cdots + (-1)^{j+n}ca_{k,n}M_{j,n}} \longrightarrow 0 \text{ by (c)}$$

=LHS

(c) Multiplication of a row by a nonzero constant c multiplies the value of the determinant by c.

$$\underbrace{\begin{vmatrix} \text{---} & \mathbf{a}_{(1)} & \text{---} \\ & \vdots & \\ \text{---} & c\mathbf{a}_{(j)} & \text{---} \\ & \vdots & \\ \text{---} & \mathbf{a}_{(k)} & \text{---} \\ & \vdots & \\ \text{---} & \mathbf{a}_{(n)} & \text{---} \end{vmatrix}}_{\substack{(-1)^{j+1} c a_{j,1} M_{j,1} + (-1)^{j+2} c a_{j,2} M_{j,2} + \\ \cdots + (-1)^{j+n} c a_{j,n} M_{j,n}}} = c \underbrace{\begin{vmatrix} \text{---} & \mathbf{a}_{(1)} & \text{---} \\ & \vdots & \\ \text{---} & \mathbf{a}_{(k)} & \text{---} \\ & \vdots & \\ \text{---} & \mathbf{a}_{(j)} & \text{---} \\ & \vdots & \\ \text{---} & \mathbf{a}_{(n)} & \text{---} \end{vmatrix}}_{c\left(\substack{(-1)^{j+1} a_{j,1} M_{j,1} + (-1)^{j+2} a_{j,2} M_{j,2} + \\ \cdots + (-1)^{j+n} a_{j,n} M_{j,n}}\right)}$$

Proof (Optional): Expanding both side about the j^{th} row

EXAMPLE 4

Evaluation of Determinants by converting to Upper Triangular Form via elementary row operations

$$D = \begin{vmatrix} 2 & 0 & -4 & 6 \\ 4 & 5 & 1 & 0 \\ 0 & 2 & 6 & -1 \\ -3 & 8 & 9 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 0 & -4 & 6 \\ 0 & 5 & 9 & -12 \\ 0 & 2 & 6 & -1 \\ 0 & 8 & 3 & 10 \end{vmatrix}$$

Row 2 − 2 Row 1
(No change in D value)

Row 4 + 1.5 Row 1
(No change in D value)

EXAMPLE 4 (continued)

$$= \begin{vmatrix} 2 & 0 & -4 & 6 \\ 0 & 5 & 9 & -12 \\ 0 & 0 & 2.4 & 3.8 \\ 0 & 0 & -11.4 & 29.2 \end{vmatrix}$$

Row 3 – 0.4 Row 2
(No change in *D* value)

Row 4 – 1.6 Row 2
(No change in *D* value)

$$= \begin{vmatrix} 2 & 0 & -4 & 6 \\ 0 & 5 & 9 & -12 \\ 0 & 0 & 2.4 & 3.8 \\ 0 & 0 & -0 & 47.25 \end{vmatrix}$$

Row 4 + 4.75 Row 3
(No change in *D* value)

$$= 2 \cdot 5 \cdot 2.4 \cdot 47.25 = 1134.$$

Note: Blue explanations always referring to the ***preceding determinant***

EXAMPLE

Evaluation of Determinants by converting to Upper Triangular Form via elementary row operations

$$D = \begin{vmatrix} 2 & 4 & -2 & 6 \\ 1 & 2 & 5 & 4 \\ 1 & 1 & 2 & 4 \\ 0 & 2 & -6 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 4 & -2 & 6 \\ 0 & 0 & 6 & 1 \\ 0 & -1 & 3 & 1 \\ 0 & 2 & -6 & 3 \end{vmatrix}$$

Row 2 - 0.5 Row 1
(No change in *D* value)
Row 3 - 0.5 Row 1
(No change in *D* value)

EXAMPLE

$$= (-1)\begin{vmatrix} 2 & 4 & -2 & 6 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & 6 & 1 \\ 0 & 2 & -6 & 3 \end{vmatrix}$$

Row 2 swap with Row 3

(Change in *D* value by -1)

$$= -\begin{vmatrix} 2 & 4 & -2 & 6 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & 5 \end{vmatrix}$$

Row 4 +2 Row 2

(No change in *D* value)

$$= 60$$

THEOREM 2

Further Properties of *n*th-Order Determinants

(a)–(c) in Theorem 1 hold also for columns.

(d) Transposition leaves the value of a determinant unaltered, i.e., $\det \mathbf{A} = \det(\mathbf{A}^T)$

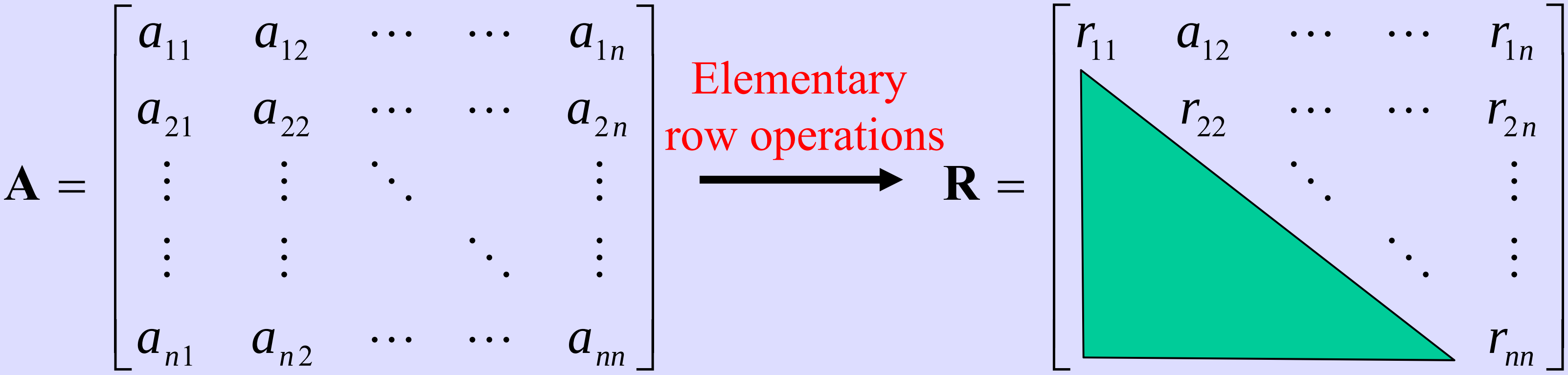
(e) A zero row or column renders the value of a determinant zero.

(f) Proportional rows or columns render the value of a determinant zero. In particular, a determinant with two identical rows or columns has the value zero.

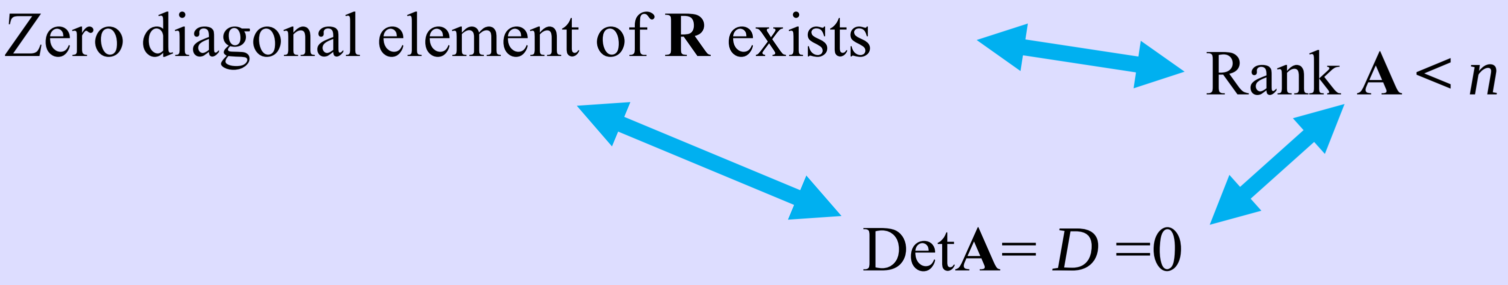
Determinants and Rank

An $n \times n$ square matrix $\mathbf{A}$ has rank n iff det $\mathbf{A} \neq 0$.

The above is readily concluded from the elementary row operations on a matrix.



$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{bmatrix} \xrightarrow{\text{Elementary row operations}} \mathbf{R} = \begin{bmatrix} r_{11} & a_{12} & \cdots & \cdots & r_{1n} \\ & r_{22} & \cdots & \cdots & r_{2n} \\ & & \ddots & & \vdots \\ & & & \ddots & \vdots \\ & & & & r_{nn} \end{bmatrix}$$



Zero diagonal element of $\mathbf{R}$ exists

Rank $\mathbf{A} < n$

Det$\mathbf{A}$= D =0

Theorem 3: Rank in Terms of Determinants

- A **submatrix** is a matrix formed by selecting certain rows and columns from a bigger matrix.
- Consider an $m \times n$ matrix $\mathbf{A} = [a_{jk}]$:

  1) **A** has rank r iff **A** has an $r \times r$ submatrix with a nonzero determinant. Here, $r \geq 1$.
  2) In the above case, any square submatrix of **A** that has more than r rows, if exists, has a zero determinant value.

- Example:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 2 \\ 1 & 3 & 2 & 3 \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} 3 & 2 & 2 & -5 & 8 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{bmatrix}$$

Rank(**A**)=3 ↔ Max. 3x3 submatrix with $D \neq 0$

Rank(**A**)=2 ↔ Max. 2x2 submatrix with $D \neq 0$

THEOREM 4 Cramer's Rule

Cramer's Theorem
(Solution of Linear Systems by Determinants)

- Given a linear system of n equations in the same number of unknowns $x_1, \dots, x_n$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned} \tag{6}$$

If D = det **A** is nonzero, then rank(**A**)=n

→ system **consistent** with **unique solution**.

- This unique solution can be obtained using Gaussian elimination (as before), or using the Cramer's rule!

THEOREM 4 (continued)

- The Cramer's rule solution is given by the formulas

$$\textbf{(7)} \quad x_1 = \frac{D_1}{D}, \qquad x_2 = \frac{D_2}{D}, \quad \cdots, \quad x_n = \frac{D_n}{D} \qquad \textbf{(Cramer's rule)}$$

where D_k is the determinant obtained from D by replacing the k^{th} column in D with a column having entries $b_1, \ldots, b_n$.

- In case that system (6) *is **homogeneous***, i.e., **b=0**, then:

-- *If* $D \neq 0$, Cramer's rule yield the trivial solution $x_1 = 0$, $x_2 = 0, \ldots, x_n = 0$ for (6).

-- *If* $D = 0$, cannot use Cramer's rule, and the ***homogeneous*** system has nontrivial (multiple) solutions.

- **Proof** (Optional): Taking linear combination of the equations,

Cofactor

$$\begin{aligned} & C_{11} \times (\ a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1\ ) \\ + & C_{21} \times (\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2\ ) \\ & \vdots \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ + & C_{n1} \times (a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n) \end{aligned}$$

- With the result:

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{vmatrix} \qquad 0 = \begin{vmatrix} a_{12} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{22} & a_{22} & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ a_{n2} & a_{n2} & \cdots & \cdots & a_{nn} \end{vmatrix}$$

$$x_1\boxed{(a_{11}C_{11} + a_{21}C_{21} + \cdots \ + a_{n1}C_{n1})} + x_2\boxed{(a_{12}C_{11} + a_{22}C_{21} + \cdots \ + a_{n2}C_{n1})}$$

$$+ \cdots + x_n\boxed{(a_{1n}C_{11} + a_{2n}C_{21} + \cdots \ + a_{nn}C_{n1})} = \boxed{(b_1C_{11} + b_2C_{21} + \cdots \ + b_nC_{n1})}$$

$$0 = \begin{vmatrix} a_{1n} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{2n} & a_{22} & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ a_{nn} & a_{n2} & \cdots & \cdots & a_{nn} \end{vmatrix} \qquad D_1 = \begin{vmatrix} b_1 & a_{12} & \cdots & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ b_n & a_{n2} & \cdots & \cdots & a_{nn} \end{vmatrix}$$

- Hence, $x_1 = \dfrac{D_1}{D}$ → Similar procedures for x_2, …, x_n