Exercises: Green's Theorem

For Problems 1-3, use the Green's Theorem to evaluate the following line integrals as double integrals. The curve C in each case is always in the positive direction.

Problem 1. $\int_C \boldsymbol{f}(\boldsymbol{r}) \cdot d\boldsymbol{r}$, where $\boldsymbol{f} = [y, -x]$ and C is the circle $x^2 + y^2 = 1$.

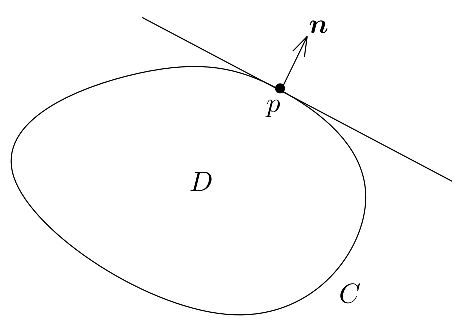
Problem 2. $\int_C \boldsymbol{f}(\boldsymbol{r}) \cdot d\boldsymbol{r}$, where $\boldsymbol{f} = [6y^2, 2x - 2y^4]$, and C is the boundary of the square with $(0, 0)$ and $(1, 1)$ as the opposite corners.

Problem 3. $\int_C \boldsymbol{f}(\boldsymbol{r}) \cdot d\boldsymbol{r}$, where $\boldsymbol{f} = [x^2e^y, y^2e^x]$, and C is the boundary of the square with $(0, 0)$ and $(1, 1)$ as the opposite corners.

Problem 4. Consider the set S of line integrals of the form $\int_C (f_1\, dx + f_2\, dy)$. Prove that if (i) $\frac{\partial f_1}{\partial y}$ and $\frac{\partial f_2}{\partial x}$ are continuous in $\mathbb{R}^2$ and (ii) $\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x}$, then S is path independent.

Problem 5* (Hard). Let C be a closed piecewise smooth curve such that the region D enclosed by C is monotone. Consider an arbitrary point p on C. We call $\boldsymbol{n}(x, y)$ a *unit outer normal vector* at $p = (x, y)$ if it satisfies all the following conditions:

- $|\boldsymbol{n}| = 1$;
- the direction of $\boldsymbol{n}$ is perpendicular to the tangent line of C at p;
- the direction of $\boldsymbol{n}$ points towards the outer area of D at p.



Define $\boldsymbol{f}(x, y) = [f_1(x, y), f_2(x, y)]$ such that $\frac{\partial f_1}{\partial x}$ and $\frac{\partial f_2}{\partial y}$ are continuous in D. Prove:

$$\iint_D \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}\, dxdy \quad = \quad \int_C \boldsymbol{f} \cdot \boldsymbol{n}\, ds.$$