# Solution to Homework 1

**Problem 1** Prove the set identity

$$(A \cap B)^c = (A^c \cap B) \cup (A^c \cap B^c) \cup (A \cap B^c)$$

**Solution**

$$\begin{aligned}
&(A^c \cap B) \cup (A^c \cap B^c) \cup (A \cap B^c) \\
&= ((A^c \cap A^c) \cup (A^c \cap B^c) \cup (B \cap A^c) \cup (B \cap B^c)) \cup (A \cap B^c) \\
&= (A^c \cap B) \cup (A \cap B^c) \\
&= (A \cap B)^c
\end{aligned} \tag{1}$$

**Problem 2** A six-sided die is loaded in a way that each even face is twice as likely as each odd face. All even faces are equally likely, as are all odd faces. Construct a probabilistic model for a single roll of this die and find the probability that the outcome is less than 4

**Solution** Let $A$ be the outcome. We have then $\mathbb{P}(A = 1) = \mathbb{P}(A = 3) = \mathbb{P}(A = 5) = \frac{1}{9}$ and $\mathbb{P}(A = 2) = \mathbb{P}(A = 4) = \mathbb{P}(A = 6) = \frac{2}{9}$. Hence $\mathbb{P}(A < 4)\mathbb{P}(A = 1) + \mathbb{P}(A = 2) + \mathbb{P}(A = 3) = \frac{4}{9}$.

**Problem 3** A batch of one hundred items is inspected by testing four randomly selected items. If one of the four is defective, the batch is rejected. What is the probability that the batch is accepted if it contains five defectives

**Solution** The probability the first inspected item being non-defective is $\frac{95}{100}$. Condition on that, the second item is non-defective is $\frac{94}{99}$. Third $\frac{93}{98}$ and fourth $\frac{92}{97}$. Hence the probability of acceptance is

$$\frac{95 \cdot 94 \cdot 93 \cdot 92}{100 \cdot 99 \cdot 98 \cdot 97} \approx 0.812 \tag{2}$$

**Problem 4** Alice and Bob have $2n+1$ coins, each coin with probability of heads equal to $\frac{1}{2}$. Bob tosses $n + 1$ coins, while Alice tosses the remaining $n$ coins. Assuming independent coin tosses, show that the probability that after all coins have been tossed, Bob will have gotten more heads than Alice is $\frac{1}{2}$.

**Solution** Let $A$ be the number of heads in Bob's *last* $n$ tosses minus the number of heads in Alice's $n$ tosses. Then it is easily seen that the PMF of $A$ is symmetric about 0. Therefore, $\mathbb{P}(A > 0) = \mathbb{P}(A < 0)$. Considering that $\mathbb{P}(A > 0) = \mathbb{P}(A < 0) + \mathbb{P}(A = 0) = 1$, we have that

$$\mathbb{P}(A > 0) + \frac{1}{2}\mathbb{P}(A = 0) = 1/2.$$

Define $B = 1$ if Bob's first tossed coin is a head, and 0 if it is a tail. Then the probability that

$$\begin{aligned}
&\mathbb{P}[\text{Bob finally gets more heads than Alice}] \\
&= \mathbb{P}(B = 0)\mathbb{P}(A > 0) + \mathbb{P}(B = 1)\mathbb{P}(A \geq 0) \\
&= \frac{1}{2}\mathbb{P}(A > 0) + \frac{1}{2}\mathbb{P}(A \geq 0) \\
&= \frac{1}{2}\mathbb{P}(A > 0) + \frac{1}{2}\Big(\mathbb{P}(A > 0) + \mathbb{P}(A = 0)\Big) \\
&= \mathbb{P}(A > 0) + \frac{1}{2}\mathbb{P}(A = 0) \\
&= \frac{1}{2}
\end{aligned}$$

**Problem 5** A power utility can supply electricity to a city from $n$ different power plants. Power plant $i$ fails with probability $p_i$, independent of the others. Suppose that two power plants are sufficient and necessary to keep the city from a black-out. Find the probability that the city will experience a black-out.

**Solution** The probability that no plant stands well is

$$E_1 = p_1 p_2 \cdots p_n \tag{3}$$

The probability that one and only one plant stands well is

$$E_2 = (1 - p_1) p_2 p_3 \cdots p_n + p_1 (1 - p_2) p_3 \cdots p_n + \cdots + p_1 p_2 \cdots p_{n-1} (1 - p_n) \tag{4}$$

Hence the probability that at least 2 plants stand well is

$$1 - E_1 - E_2 \tag{5}$$