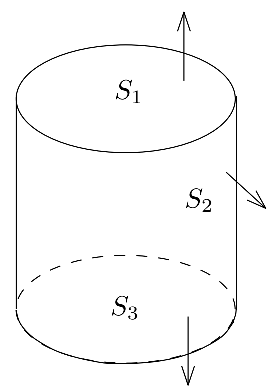
# Exercises: Divergence Theorem and Stokes' Theorem

**Problem 1.** This exercise allows you to see the main idea behind the proof of the Divergence Theorem. Suppose that $T$ is a closed region in $\mathbb{R}^3$ whose boundary surface $S$ can be divided into xy-monotone surfaces: $S_1$ and $S_2$, whose projections onto the xy-plane are the same region $D$. (For example, the ball $x^2 + y^2 + z^2 \leq 1$ is such a region because we can divide its boundary into two xy-monotone surfaces (i) $S_1$: $x^2 + y^2 + z^2 = 1$ with $z \geq 0$, and (ii) $S_2$: $x^2 + y^2 + z^2 = 1$ with $z \leq 0$.) Let $f(x, y, z)$ be a function that is continuous on $S$. Orient $S$ by taking its outer side. Prove that

$$\iiint_T \frac{\partial f}{\partial z}\, dxdydz \quad = \quad \iint_S f\, dxdy.$$

**Problem 2.** Consider the cylinder $x^2 + y^2 \leq 1$ and $0 \leq z \leq 1$. Let $S$ be the boundary of the cylinder; see below. Use the Divergence Theorem to calculate $\iint_S xy\, dydz + y^2\, dxdz + z\, dxdy$.




**Problem 3.** This exercise allows you to derive another popular form of the Divergence theorem. Let $T$ be a closed region in $\mathbb{R}^3$ that is bounded by a surface $S$, which is the union of a finite number of smooth surfaces $S_1, ..., S_k$. Define $\boldsymbol{f}(x, y, z)$ to be a vector function that is continuous on each $S_i$ ($1 \leq i \leq k$). For each point $p = (x, y, z)$, define $\boldsymbol{n}(x, y, z)$ to be the unit vector of $S$ at $p$ pointing towards the outside of $S$. Prove:

$$\iiint_T \text{div } \boldsymbol{f}\, dxdydz \quad = \quad \iint_S \boldsymbol{f} \cdot \boldsymbol{n}\, dA.$$

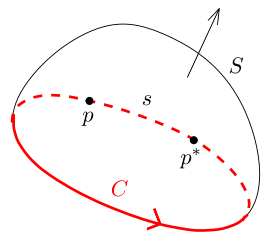
**Problem 4.** Let $\boldsymbol{f} = [e^x, e^y, e^z]$. Let $S$ be the boundary of the cube with $|x| \leq 1$, $|y| \leq 1$, and $|z| \leq 1$. Let $\boldsymbol{n}$ be defined as in the previous problem. Calculate $\iint_S \boldsymbol{f} \cdot \boldsymbol{n}\, dA$.

**Problem 5.** Let $C$ be the curve that is the intersection of

$$\begin{aligned} x^2 + y^2 &= 2z \\ z &= 2 \end{aligned}$$

Designate the direction of $C$ as passing points $(2, 0, 2)$, $(0, 2, 2)$, and $(-2, 0, 2)$ in this sequence. Use the Stokes' theorem to calculate $\int_C y\, dx - xz\, dy + yz^2\, dz$.

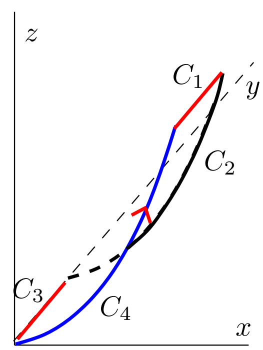
**Problem 6.** This exercise allows you to see an alternative form of the Stokes' theorem. Let $S$ be a piecewise surface and $C$ its boundary curve, both oriented in the way described in the Stokes theorem (see lecture notes). Also, let $f_1, f_2, f_3$ be functions that have continuous partial derivatives on each smooth surface that constitutes $S$.




Define $\boldsymbol{f}(x, y, z) = [f_1, f_2, f_3]$, and $\boldsymbol{n}(x, y, z)$ be the unit normal vector of $S$ at point $(x, y, z)$, emanating from the side of $S$ chosen. Fix any point $p^*$ on $C$. Given any point $p$ on $C$, denote by $s$ the length of the curve from $p^*$ to $p$, following the direction of $C$. Let $\boldsymbol{r}(s) = [x(s), y(s), z(s)]$ be a parametric form of $C$. Prove:

$$\iint_S \mathrm{curl} \boldsymbol{f} \cdot \boldsymbol{n} \, dA \quad = \quad \int_C \boldsymbol{f} \cdot \boldsymbol{r}'(s) \, ds.$$

**Problem 7.** Let $S$ be the surface $z = x^2$ with $0 \le x \le 2$ and $0 \le y \le 1$. Orient $S$ by taking its upper side. Define $\boldsymbol{f} = [e^y, e^z, e^x]$. Calculate $\iint_S \mathrm{curl} \boldsymbol{f} \cdot \boldsymbol{n} \, dA$, where $\boldsymbol{n}$ is as defined in the previous problem. Calculate $\iint_S \mathrm{curl} \boldsymbol{f} \cdot \boldsymbol{n} \, dA$.




**Problem 8.** Fix a vector function $\boldsymbol{f}(x, y, z) = [f_1, f_2, f_3]$. Prove that if $\mathrm{curl} \boldsymbol{f} = \boldsymbol{0}$, then the class of line integrals $\int_C f_1 \, dx + f_2 \, dy + f_3 \, dz$ is path independent.