Lecture Note 15

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MATH1020
General Mathematics

The Parabola

Theorem 1 A **parabola** is the collection of all points P in the plane that the same distance from a fixed point F as they are from a fixed line D. The point F is called the **focus** of the parabola, and the line D is its **directrix.** As a result, a parabola is the set of points P for which

$$d(F, P) = d(P, D). \tag{1}$$

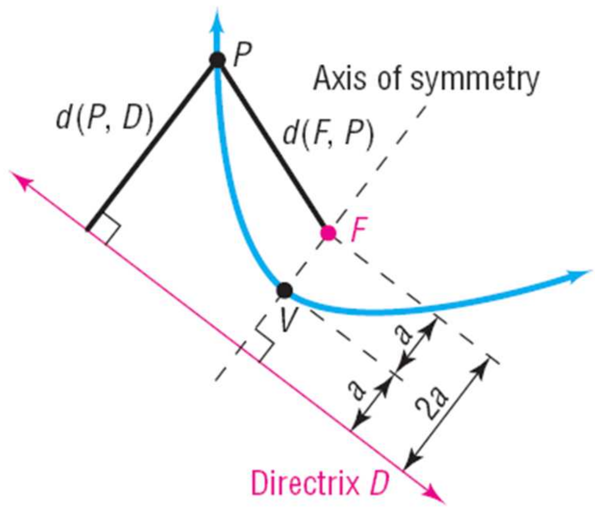
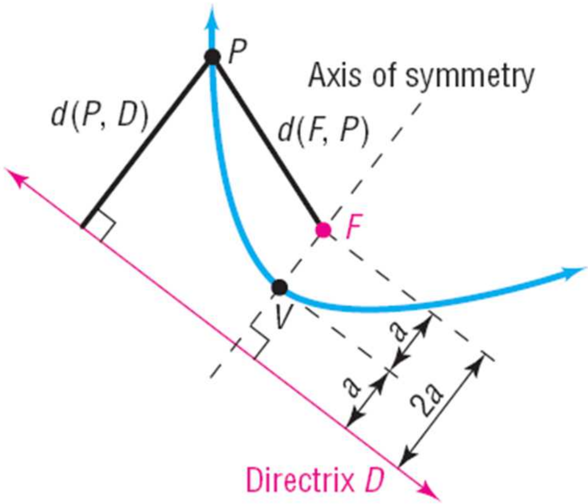
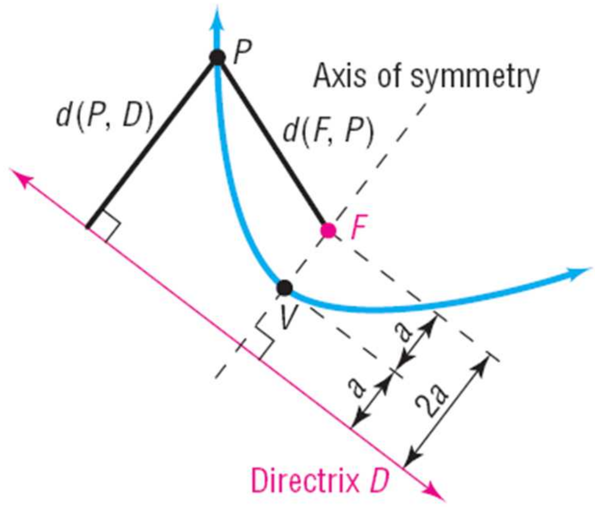


Figure 1:



Remark 1 Figure 1 shows a parabola (in blue).

- The line through the focus F and perpendicular to the directrix D is called the **axis of symmetry** of the parabola.
- The point of intersection of the parabola with its axis of symmetry is called the **vertex** V.



Because the vertex V lies on the parabola, it must satisfy equation (1):

$$d(F, V) = d(V, D).$$

The vertex is midway between the focus and the directrix. We shall let a equal the distance $d(F, V)$ from F to V.

To derive an equation for a parabola, we use a Cartesian (or rectangular) system of coordinates position so that the vertex V, focus F, and directrix D of the parabola are conveniently located.

Example 1 Analyze Parabolas with Vertex at the Origin

If we select to locate the vertex V at the origin $(0, 0)$, we can conveniently position the focus F on either the $x-$axis or the $y - a$xis.

First, consider the case where the focus F is on the positive $x-$axis, as shown in Figure 2.

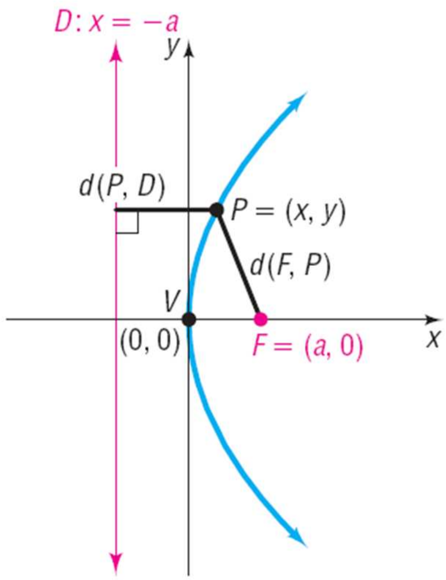
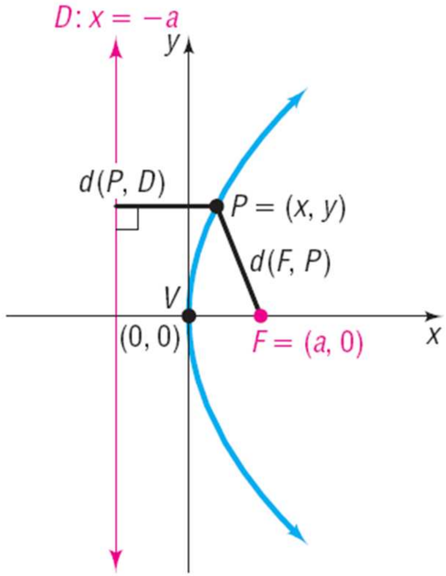


Figure 2:

Because the distance from F to V is a, the coordinates of F will be $(a, 0)$ with $a > 0$.

Similarly, because the distance from V to the directrix D is also a and, because D must be perpendicular to $\perp$ the $x-$axis (since the $x-$axis is the axis of symmetry), the equation of the directrix D must be $x = -a$.

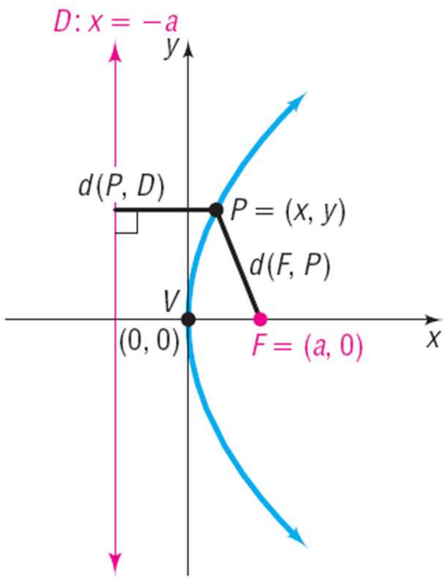


Now, if $p = (x, y)$ is any point on the parabola, P must equation (1):

$$d(F, P) = d(P, D).$$

So we have

$$\begin{aligned}
\sqrt{(x-a)^2+(y-0)^2} &= |x+a| && \text{Use the Distance Formula.} \\
(x-a)^2+(y-0)^2 &= (x+a)^2 && \text{Square both sides.} \\
x^2-2ax+a^2+y^2 &= x^2+2ax+a^2 && \text{Remove parentheses.} \\
y^2 &= 4ax && \text{Simplify.}
\end{aligned}$$



Theorem 2 **Equation of a Parabola: Vertex at $(0,0)$, Focus at $(a,0)$, $a>0$**

The equation of a parabola with vertex at $(0,0)$, focus at $(a,0)$, and directrix $x=-a, a>0$, is

$$y^2 = 4ax. \tag{2}$$

Example 2 Finding the Equation of a Parabola and Graphing It

Find an equation of the parabola with vertex at $(0,0)$ and focus at $(3,0)$. Graph the equation.

Solution:

The distance from the vertex $(0,0)$ to the focus $(3,0)$ is $a = 0$. Based on equation (2), the equation of this parabola is

$$\begin{aligned} y^2 &= 4ax \\ y^2 &= 12x \qquad (a = 3) \end{aligned}$$

To graph this parabola, it is helpful to plot the points on the graph directly above or below the focus. To locate these two points, we let $x = 3$. Then

$$\begin{aligned} y^2 &= 12x = 12(3) = 36 \\ y &= \pm 6 \qquad \text{(Solve for } y\text{)} \end{aligned}$$

The points on the parabola directly <u>above</u> or <u>below</u> the focus are $(3,6)$ and $(3,-6)$. These points help in graphing the parabola because they determine the "opening", See Figure 3.

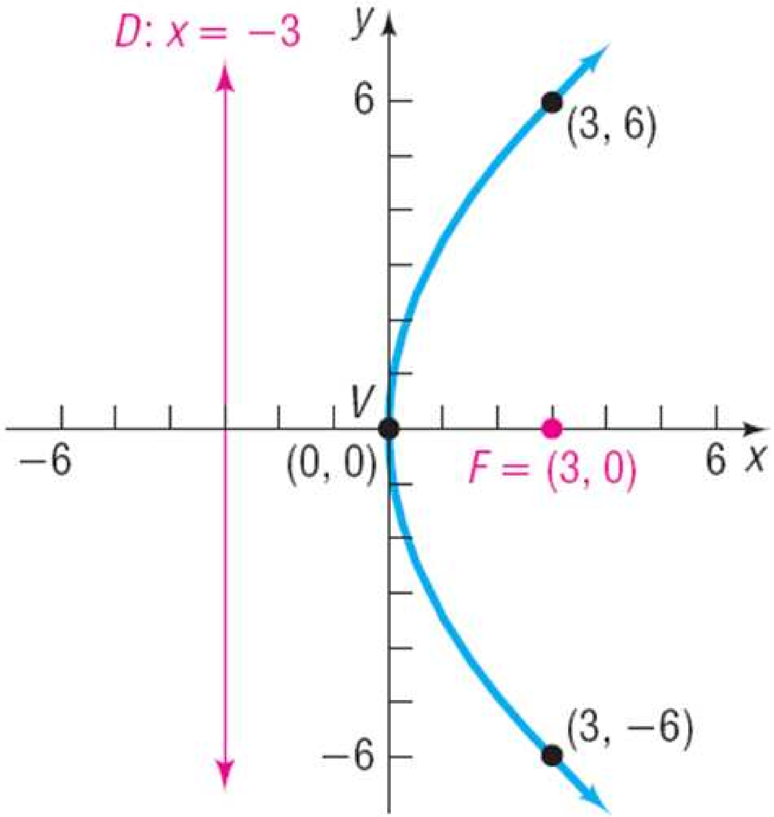


Figure 3:

Remark 2 In general, the points on a parabola $y^2 = 4ax$ that lie above and below the focus $(a, 0)$ are each at a distance $2a$ from the focus. This follows from the fact that if $x = a$ then $y^2 = 4ax = 4a^2$, so $y = \pm 2a$. The line segment joining these two points is called the **latus rectum;** its length is $4a$.

Example 3 Graphing a Parabola Using a Graphing Utility

Graph the parabola $y^2 = 12x$.

Solution:

To graph the parabola $y^2 = 12x$, we need to graph the two function $Y_1 = \sqrt{12x}$ and $Y_2 = -\sqrt{12x}$ on a square screen.

Figure 4 shows the graph of $y^2 = 12x$.

Notice that the graph fails the vertical line test, so $y^2 = 12x$ is not a function.

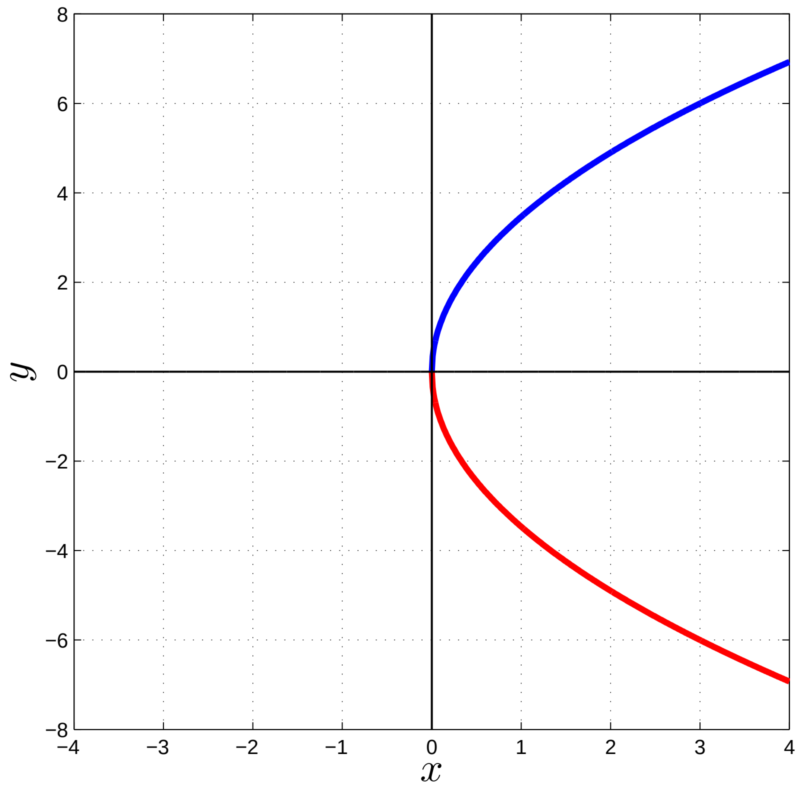


Figure 4:

Remark 3 By reversing the steps we used to obtain equation (2), $y^2 = 4ax$, is a parabola; its vertex is at $(0, 0)$, its focus is at $(a, 0)$, its direction "**Analyze the equation**" will mean to find the vertex, focus, and directrix of the parabola and graph it.

Example 4 Analyzing the Equation of a Parabola

Analyze the equation: $y^2 = 8x$.

Solution:

The equation $y^2 = 8x$ is of the form $y^2 = 4ax$, where $4a = 8$, so $a = 2$.

Consequently, the graph of the equation is a parabola with vertex at $(0, 0)$ and force on the positive $x-$axis at $(2, 0)$.

The directrix is the vertical line $x = -2$.

The two points defining the latus rectum are obtained by letting $x = 2$. Then $y^2 = 16$, so $y = \pm 4$.

See Figure 5 for the graph drawn by hand.

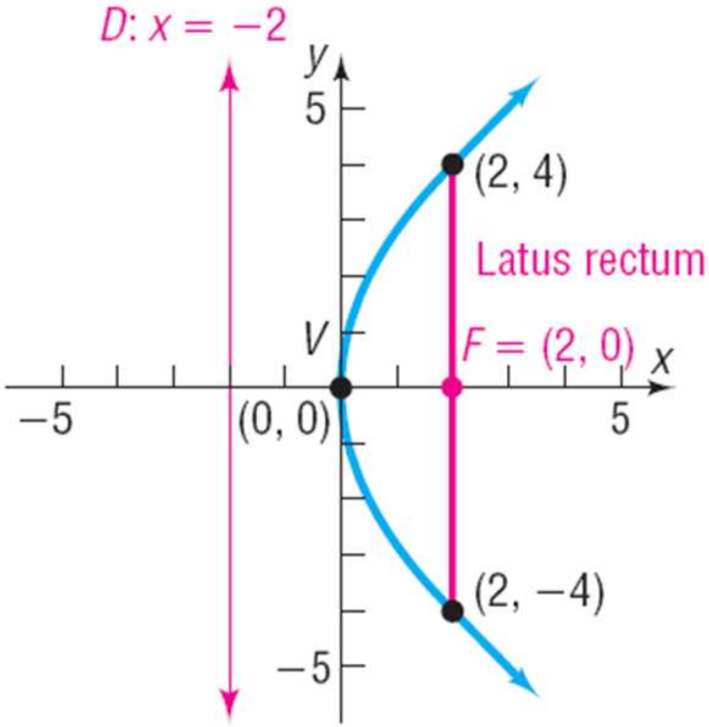


Figure 5:

Figure 6 shows the graph obtained using MATLAB.

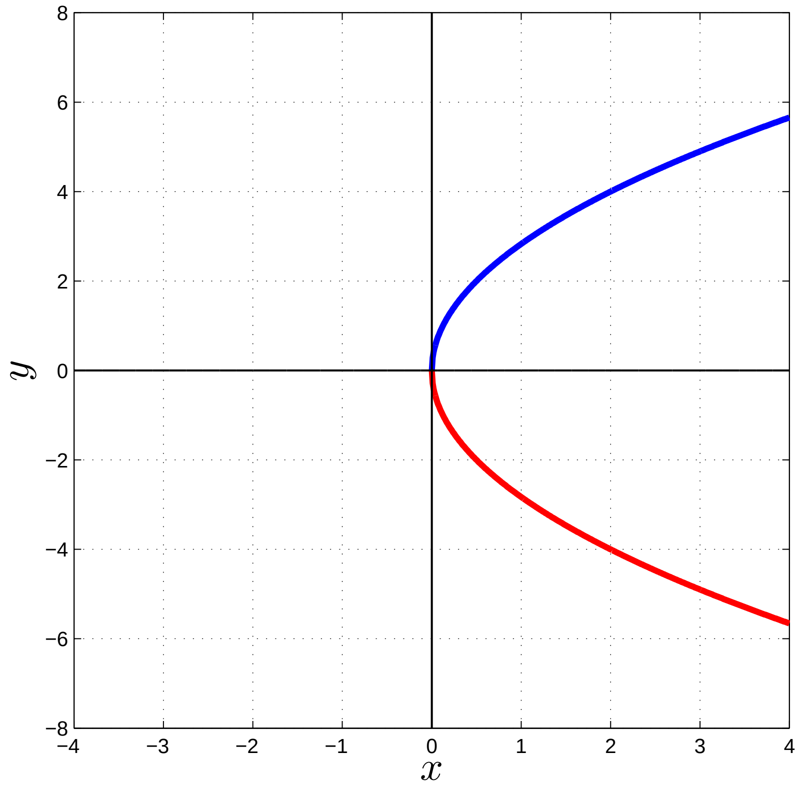


Figure 6:

Remark 4 Recall that we reached at equation (2) after locating the focus on the positive $x-$axis. If the focus is located on the positive $y-$axis, or negative $y-$axis, a different form of the equation for the parabola results. The four forms of the equation of a parabola with vertex at $(0,0)$ and focus on a coordinate axis a distance a form $(0,0)$ are given in Table 1, and their graphs are given in Figure 7 and Figure 8. Notice that each graph is symmetric with respect to its axis of symmetry.

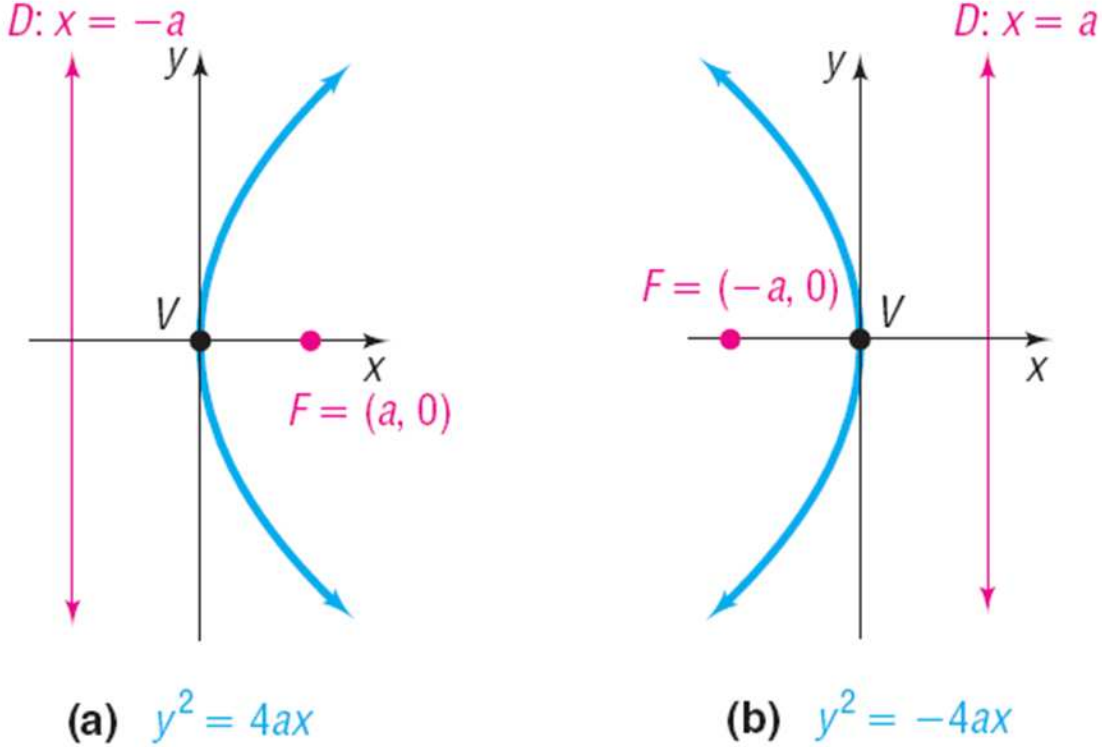


Figure 7:

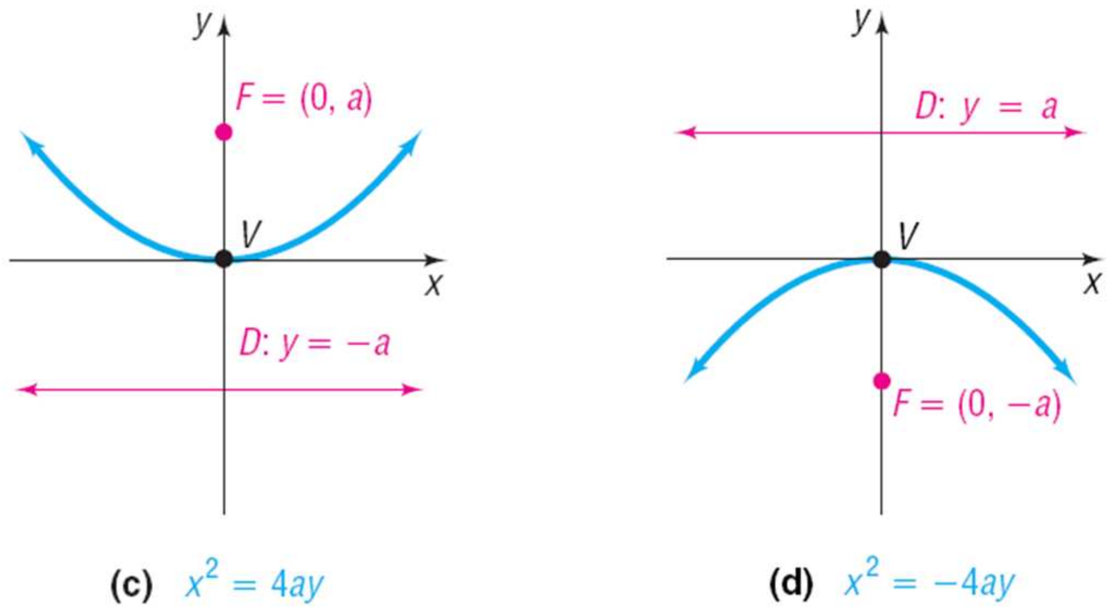


Figure 8:

Table 1. EQUATIONS OF A PARABOLA VERTEX AT $(0, 0)$; FOCUS ON AN AXIS; $A > 0$

Vertex	**Focus**	**Directrix**	**Equation**	**Description**
$(0, 0)$	$(a, 0)$	$x = -a$	$y^2 = 4ax$	Parabola, axis of symmetry is the $x-$axis, opens right
$(0, 0)$	$(-a, 0)$	$x = a$	$y^2 = -4ax$	Parabola, axis of symmetry is the $x-$axis, opens left
$(0, 0)$	$(0, a)$	$y = -a$	$x^2 = 4ay$	Parabola, axis of symmetry is the $y-$axis, opens up
$(0, 0)$	$(0, -a)$	$y = a$	$x^2 = -4ay$	Parabola, axis of symmetry is the $y-$axis, opens down

Example 5 Analyzing the Equation of a Parabola

Analyze the equation: $x^2 = -12y$.

Solution:

The equation $x^2 = -12y$ is of the form $x^2 = -4ay$, with $a = 3$.

Consequently, the graph of the equation is a parabola with vertex at $(0, 0)$, focus at $(0, -3)$ and directrix the line $y = 3$.

The parabola opens down, and its axis of symmetry is the $y-$axis.

To obtain the points defining the latus rectum, let $y = -3$. Then $x^2 = 36$, so $x = \pm 6$.

See Figure 9 for the graph drawn by hand. Figure 10 shows the graph using MATLAB.

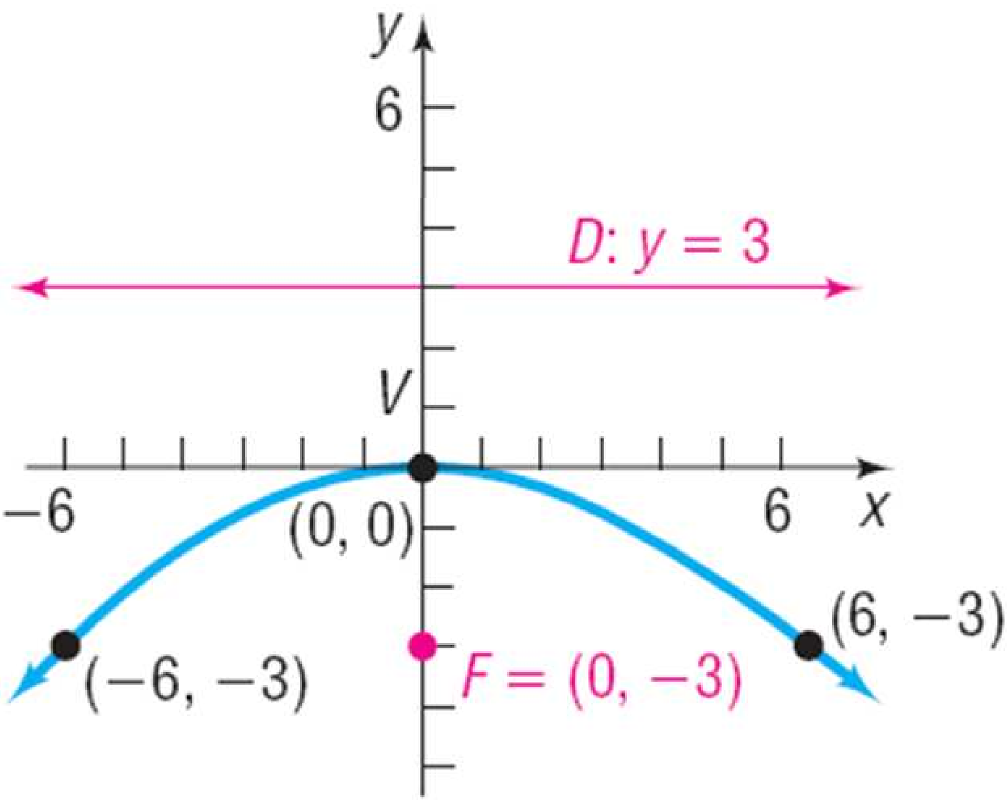


Figure 9:

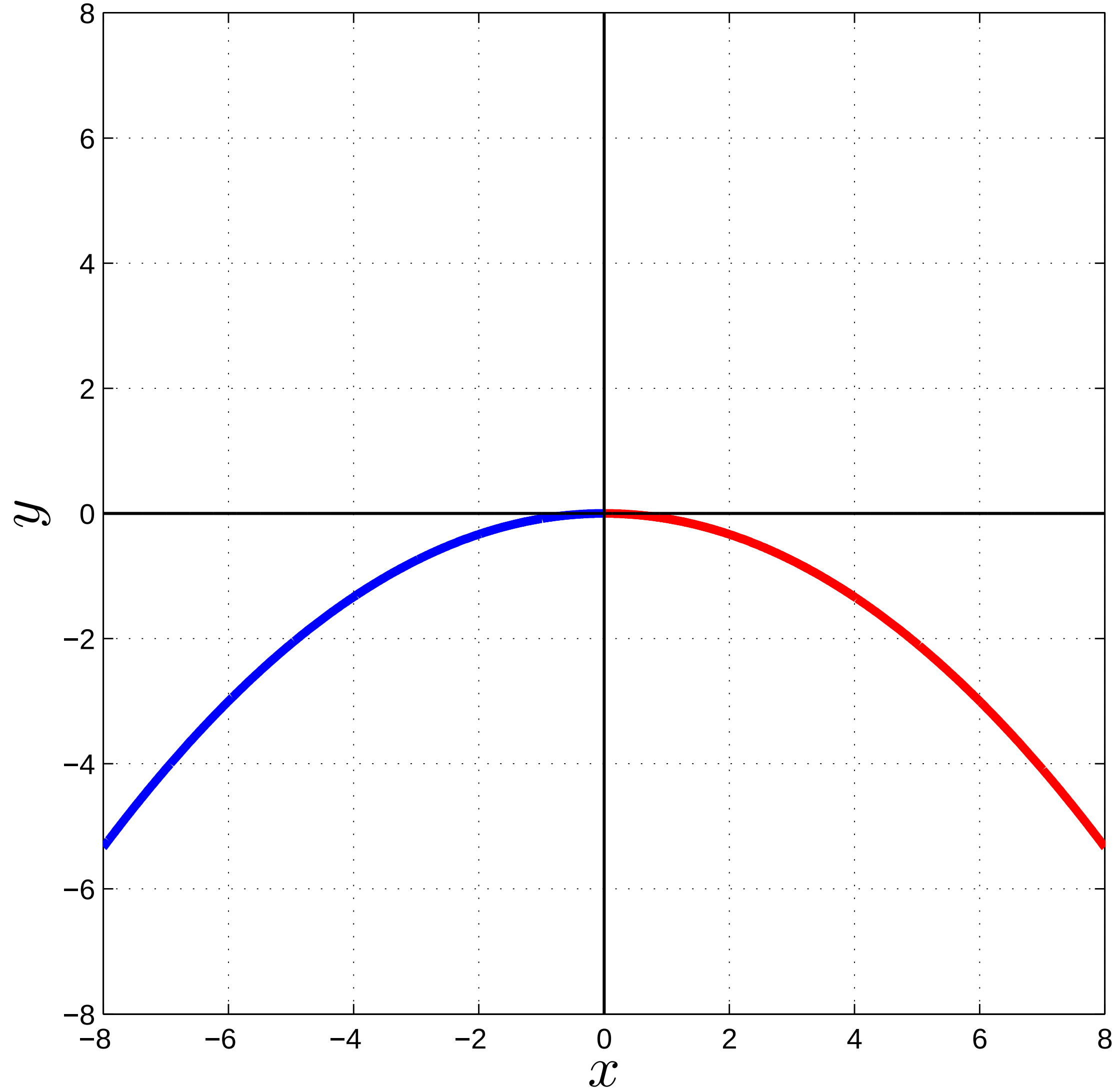


Figure 10:

Example 6 Finding the Equation of a Parabola

Find the equation of the parabola with focus at $(0, 4)$ and directrix the line $y = -4$. Graph the equation.

Solution:

A parabola whose focus is at $(0, 4)$ and whose directrix is the horizontal line $y = -4$ will have its vertex at $(0, 0)$.

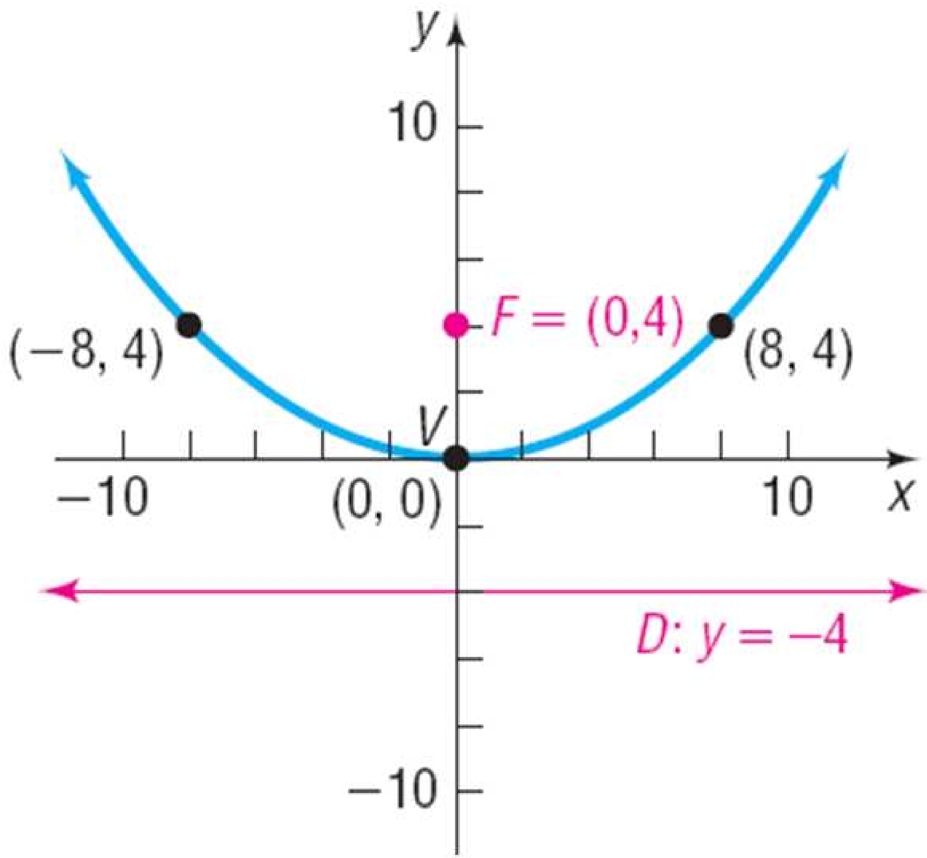
(Do you see why? The vertex is midway between the focus and the directrix.)

Since the focus is on the positive $y-$axis at $(0, 4)$, the equation of this parabola is of the form $x^2 = 4ay$, with $a = 4$; that is,

$$x^2 = 4ay \underset{\substack{\uparrow \\ a=4}}{=} 4(4y) = 16y.$$

Letting $y = 4$, we find $x^2 = 64$, so $x = \pm 8$.

The points $(8, 4)$ and $(-8, 4)$ determine the latus rectum. Figure 11 shows the graph of $x^2 = 16y$.



$$x^2 = 16y$$

Figure 11:

Example 7 Finding the Equation of a Parabola

Find the equation of a parabola with vertex at $(0, 0)$ if its axis of symmetry is the $x-$axis and its graph contains the point $(-\frac{1}{2}, 2)$. Find its focus and directrix, and graph the equation.

Solution:

The vertex is at the origin, the axis of symmetry is the $x-$axis, and the graph contains a point in the second quadrant, so the parabola opens to the left.

We see from Table 1 that the form of the equation is

$$y^2 = -4ax$$

Because the point $(-\frac{1}{2}, 2)$ is on the parabola, the coordinates $x = -\frac{1}{2}, y = 2$ must satisfy the equation $y^2 = -4ax$.

Substituting $x = -\frac{1}{2}$ and $y = 2$ into the equation, we find that

$$\begin{aligned} 2^2 &= -4a(-\tfrac{1}{2}) && (y^2 = -4ax, x = -\tfrac{1}{2}, y = 2) \\ a &= 2. \end{aligned}$$

The equation of the parabola is

$$y^2 = -4(2x) = -8x.$$

The focus is at $(-2, 0)$ and the directrix is the line $x = 2$, we find $y^2 = 16$, so $y = \pm 4$. The points $(-2, 4)$ and $(-2, -4)$ define the latus rectum. See Figure 12.

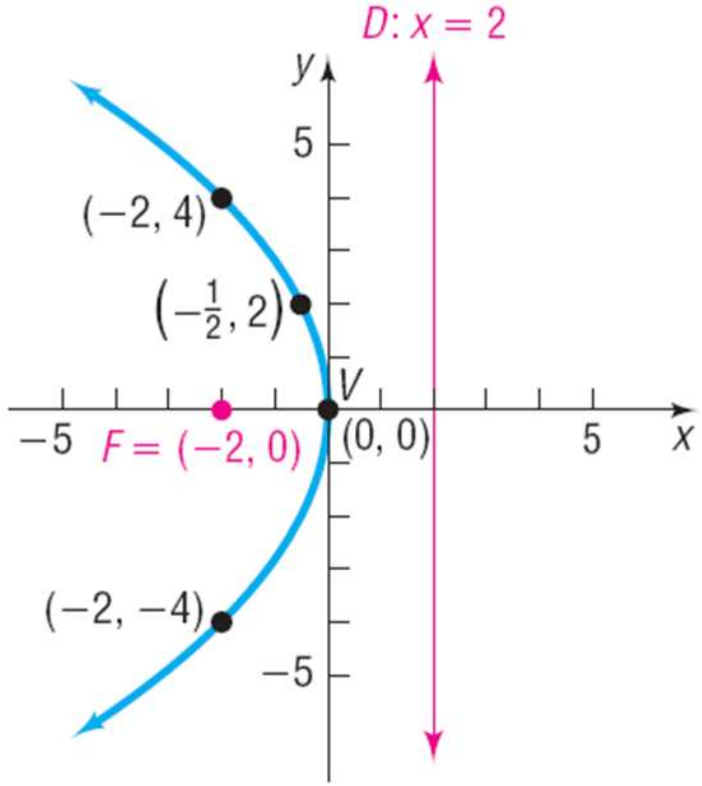


Figure 12:

Analyze Parabolas with Vertex at (h, k)

If a parabola with vertex at the origin and axis of symmetry along a coordinate axis is shifted horizontally h units and then vertically k units, the result is a parabola with vertex at (h, k) and axis of symmetry parallel to a coordinate axis.

The equations of such parabolas have the same forms as those in Table 1, but with x replaced by $x - h$ (the horizontal shift) and y replaced by $y - k$ (the vertical shift).

Table 2 gives the forms of the equations of such parabolas. Figures 13 and Figures 14 illustrate the graphs for $h > 0, k > 0$.

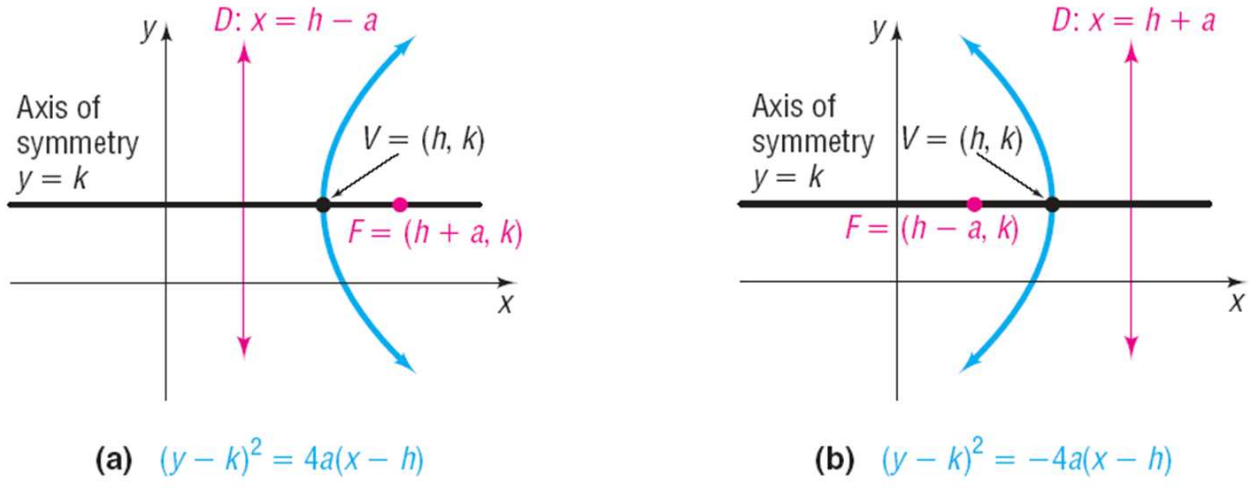


Figure 13:

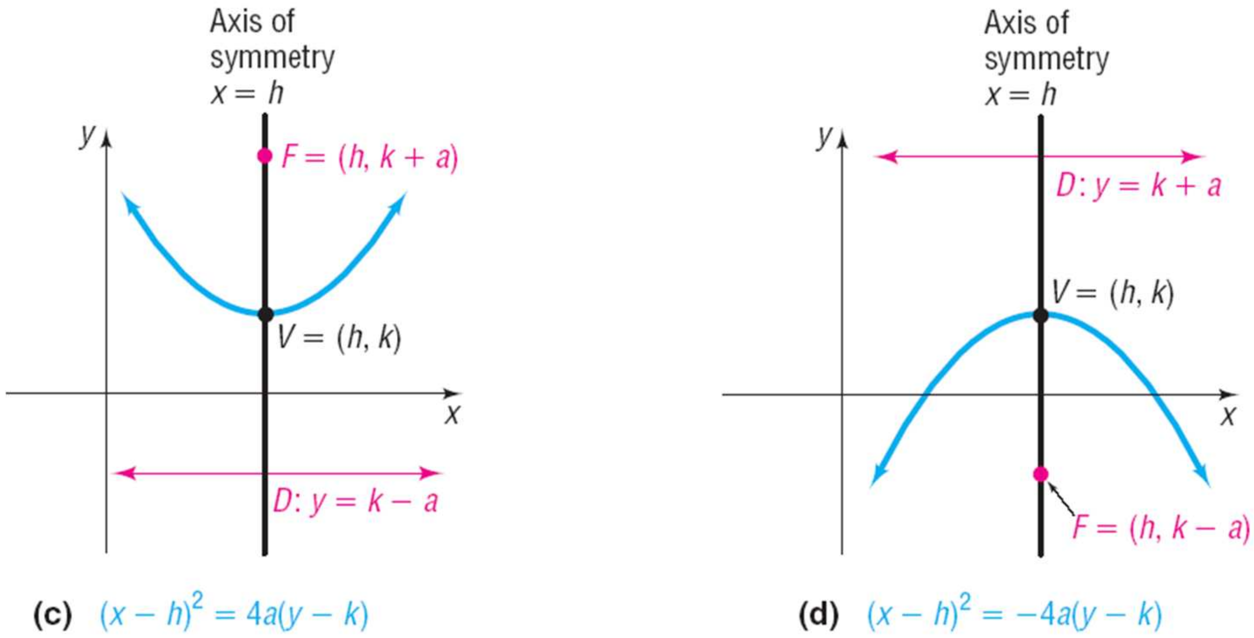

Axis of
symmetry
x = h
y
F = (h, k + a)
V = (h, k)
x
D: y = k − a
(c) (x − h)^2 = 4a(y − k)
Axis of
symmetry
x = h
y
D: y = k + a
V = (h, k)
x
F = (h, k − a)
(d) (x − h)^2 = −4a(y − k)

Figure 14:

Table 2. PARABOLAS WITH VERTEX AT (h, k);AXIS OF SYMMETRY PARALLEL TO A COORDINATE AXIS $a > 0$

Vertex	Focus	Directrix	Equation	Description
(h, k)	$(h + a, k)$	$x = h - a$	$(y - k)^2 = 4a(x - h)$	Parabola, axis of symmetry parallel to $x-$axis, opens right
(h, k)	$(h - a, k)$	$x = h + a$	$(y - k)^2 = -4a(x - h)$	Parabola, axis of symmetry parallel to $x-$axis, opens left
(h, k)	$(h, k + a)$	$y = k - a$	$(y - k)^2 = 4a(y - k)$	Parabola, axis of symmetry parallel to $y-$axis, opens up
(h, k)	$(h, k - a)$	$y = k = a$	$(y - k)^2 = -4a(y - k)$	Parabola, axis of symmetry parallel to $y-$axis, opens down

Example 8 **Finding the Equation of a Parabola, Vertex Not at Origin**

Find an equation of the parabola with vertex at $(-2, 3)$ and focus at $(0, 3)$. Graph the equation.

Solution:

The vertex $(-2, 3)$ and focus $(0, 3)$ both lie on the horizontal line $y = 3$ (the axis of symmetry).

The distance a from the vertex $(2, -3)$ to the focus $(0, 3)$ is $a = 2$.

Also, because the focus lies to the right of the vertex, we know that the parabola opens to the right.

Consequently, the form of the equation is

$$(y - k)^2 = 4a(x - h)$$

where $(h, k) = (-2, 3)$ and $a = 2$.

Therefore, the equation is

$$(y - 3)^2 = 4 \cdot 2[x - (-2)]$$
$$(y - 3)^2 = 8(x + 2)$$

If $x = 0$, then $(y-3)^2 = 16$. Then $y - 3 = \pm 4$, so $y = -1$ or $y = 7$. The points $(0,-1)$ and $(0,7)$ define the latus rectum; the line $x = -4$ is the directrix. See Figure 15.

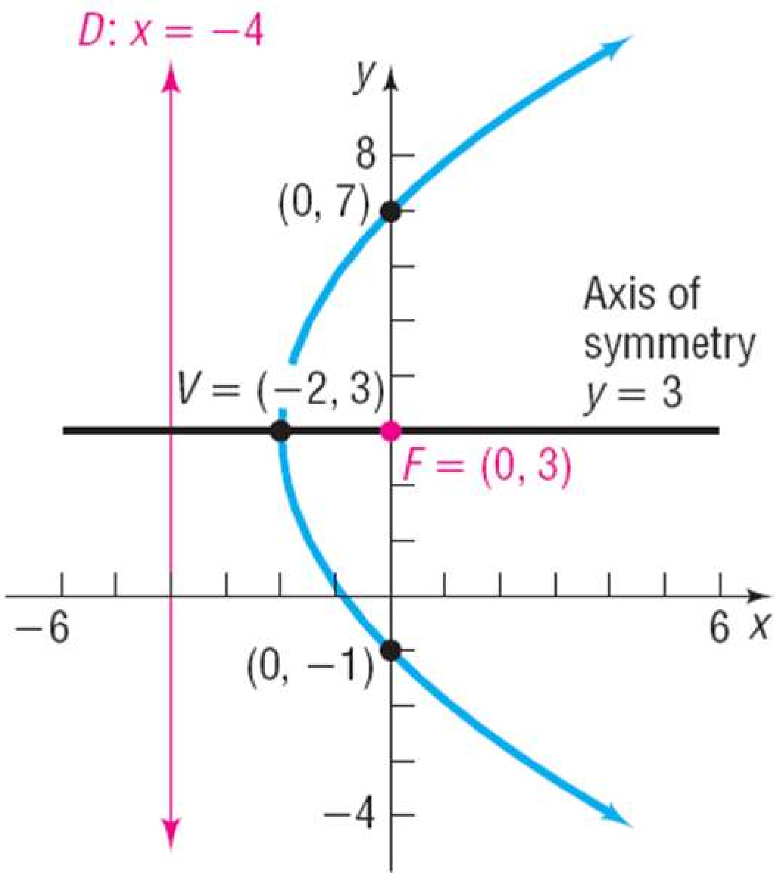


Figure 15:

Example 9 Using a Graphing Utility to Graph a Parabola, Vertex Not at Origin

Using a graphing utility, graph the equation $(y-3)^2 = 8(x-3)$.

Solution: First, we must solve the equation for y.

$$\begin{aligned}
(y-3)^2 &= 8(x+2) & \\
y-3 &= \pm\sqrt{8(x+2)} & \text{Use the Square Root Method.} \\
y &= 3 \pm \sqrt{8(x+2)} & \text{Add 3 to both sides.}
\end{aligned}$$

Figure 16 shows the graphs of the equations $Y_1 = 3 + \sqrt{8(x+2)}$ and $Y_2 = 3 - \sqrt{8(x+2)}$.

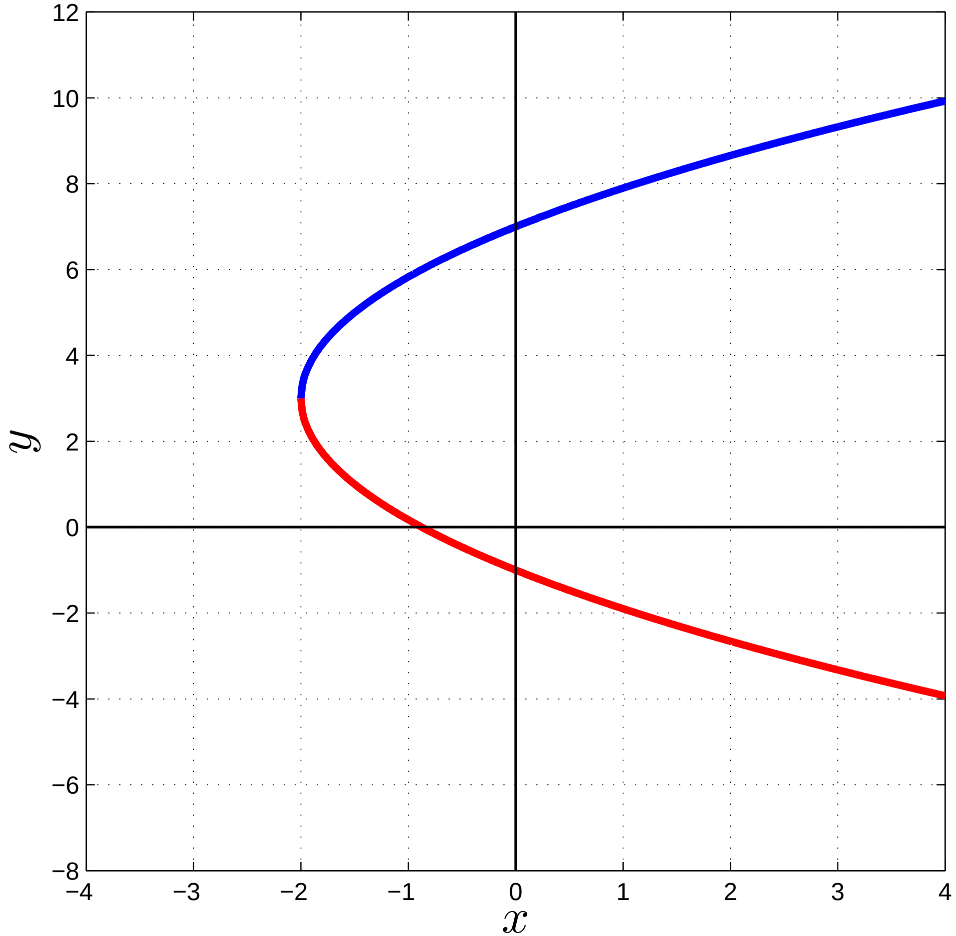


Figure 16: