Lecture 5.5 – Circular Motion

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Forces in Two Dimensions

- Newton's laws of motion describe all motion, not just motion in a straight line.

- Newton's Second Law can be applied to 2D motion as well. Each direction will have its own acceleration.

$$a_x = \frac{(F_{\text{net}})_x}{m} \text{ and } a_y = \frac{(F_{\text{net}})_y}{m}$$

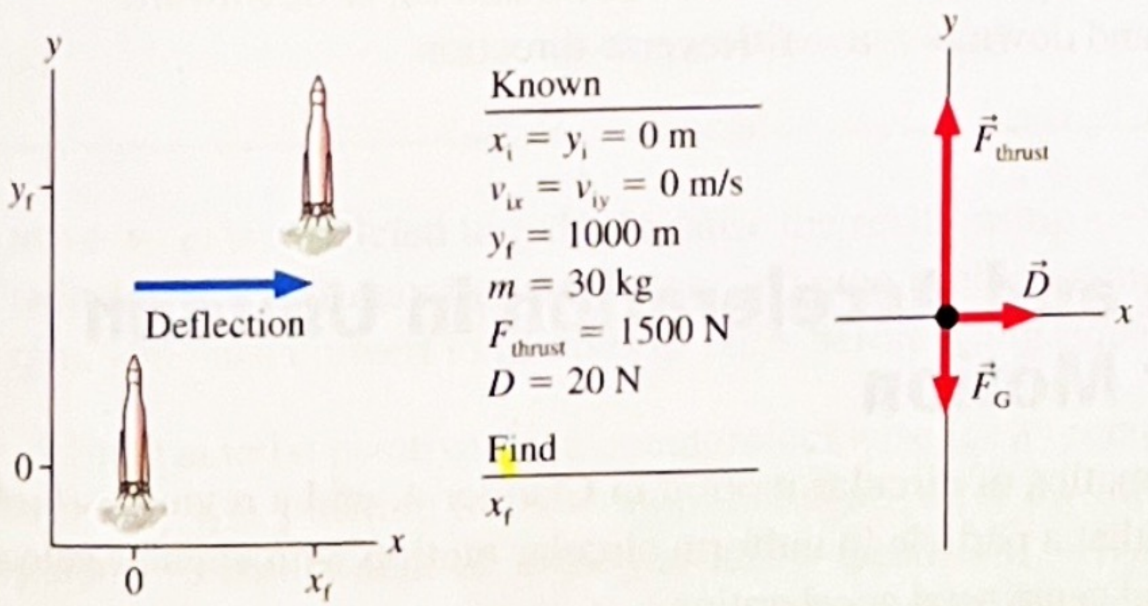
- If the x- and y-components of acceleration are *independent* of one another, then our method of solving 2D kinematics problems is still valid.

Example #1

EXAMPLE 8.1 Rocketing in the wind

A small rocket for gathering weather data has a mass of 30 kg and generates 1500 N of thrust. On a windy day, the wind exerts a 20 N horizontal force on the rocket. If the rocket is launched straight up, what is the shape of its trajectory, and by how much has it been deflected sideways when it reaches a height of 1.0 km? Because the rocket goes much higher than this, assume there's no significant mass loss during the first 1.0 km of flight.

FIGURE 8.1 Pictorial representation of the rocket launch.



$$\vec{F}_{net} = \sum_i \vec{F}_i = \vec{F}_{thrust} + \vec{D} + \vec{F}_G = m\vec{a}$$

$$(\sum F_i)_x = D = ma_x$$

$$a_x = \frac{D}{m}$$

$$(\sum F_i)_y = F_{thrust} - F_G = ma_y$$

$$a_y = \frac{F_{thrust} - F_G}{m} = \frac{F_{thrust} - mg}{m}$$

$$x(t) = x_i + v_{i,x}\Delta t + \frac{1}{2}a_x(\Delta t)^2 = \frac{1}{2}a_x(\Delta t)^2 = \frac{1}{2}\left(\frac{D}{m}\right)(\Delta t)^2$$

$$x(t) = \frac{D}{2m}(\Delta t)^2$$

$$y(t) = y_i + v_{i,y}\Delta t + \frac{1}{2}a_y(\Delta t)^2 = \frac{1}{2}a_y(\Delta t)^2 = \frac{1}{2}\left(\frac{F_{thrust} - mg}{m}\right)(\Delta t)^2$$

$$y(t) = \left(\frac{F_{thrust} - mg}{2m}\right)(\Delta t)^2$$

$$\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} = \left[\frac{D}{2m}(\Delta t)^2\right]\hat{\imath} + \left[\left(\frac{F_{thrust} - mg}{2m}\right)(\Delta t)^2\right]\hat{\jmath}$$

Example #1

- Now, what about its trajectory? (A plot of its actual position.)
- Both its x- and y-components demonstrate acceleration since t is squared. This will create a parabolic plot of $x(t)$ and $y(t)$.
- However, we want $y(x)$. So, let's combine $x(t)$ and $y(t)$ such that $y(x) = y(x(t))$. (Let's put $x(t)$ "inside" $y(t)$)

Solving $x(t) = \frac{D}{2m}(\Delta t)^2$ for $(\Delta t)^2$ give us $\frac{2mx}{D} = (\Delta t)^2$.

$$y(x) = \left(\frac{F_{\text{thrust}} - mg}{2m}\right)(\Delta t)^2 = \left(\frac{F_{\text{thrust}} - mg}{2m}\right)\left(\frac{2mx}{D}\right) = \left(\frac{F_{\text{thrust}} - mg}{D}\right)x$$

This is just a straight line! $y = mx$

Although the individual components *(x,y)* are quadratic, the trajectory of the rocket appears as a diagonal line to the right.

What about the deflection after 1km?

"Flip" the equation:

$$y(x) = \left(\frac{F_{\text{thrust}} - mg}{D}\right)x$$

Becomes

$$x = \left(\frac{D}{F_{\text{thrust}} - mg}\right)y$$

$$x = \left(\frac{20\text{N}}{1500\text{N} - 30\text{kg} \cdot 9.8\frac{\text{m}}{\text{s}^2}}\right)(1000\text{m}) = 17\text{m}$$

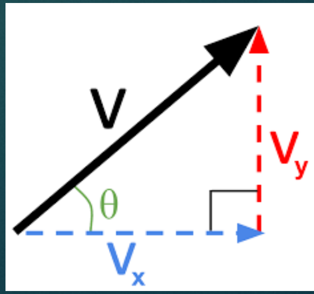
After 1km of vertical distance, the wind has moved the rocket 17m to the right.

A word of caution!

- Careful! Not all accelerations have independent components!
- Consider the simplified drag model where $\vec{D} = \frac{1}{4}Av^2$.
- If we apply this to projectile motion, we could try to break up the force:

$$D_x = -\frac{1}{4}Av^2\cos\theta \text{ and } D_y = -\frac{1}{4}Av^2\sin\theta$$

HOWEVER, the original v is still a part of both equations. Remember that v is made up of the x-component AND y-component of velocity!



A New Coordinate System

- A Cartesian grid is not the best coordinate system to use for circular motion.
- For a particle undergoing circular motion, lets define a new coordinate system that "rides along" with the particle.
- Three axes:
  - The *r-axis*, which points from the particle toward the center of the circle.
  - The *t-axis*, which is tangent to the circle.
  - The *z-axis*, which is perpendicular to the plane of motion. (elevation)
- For any vector $\vec{A}$ in this coordinate system:

$$A_r = A\cos\phi$$
$$A_t = A\sin\phi$$

FIGURE 8.3 The *rtz*-coordinate system.

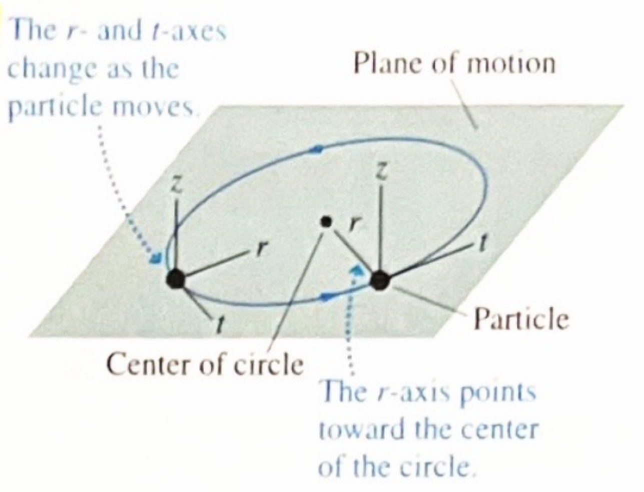
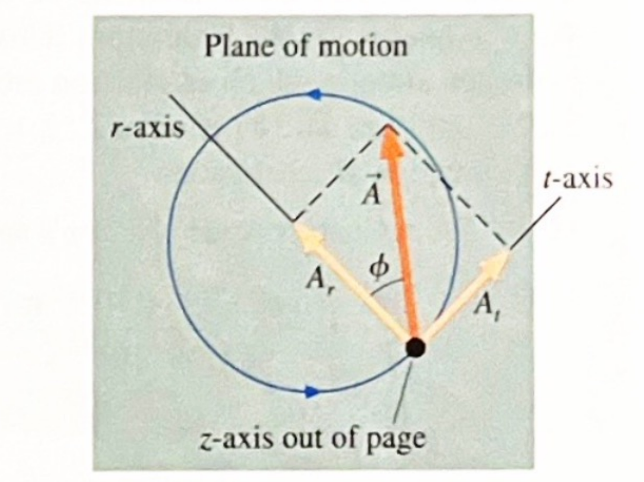


FIGURE 8.4 Vector $\vec{A}$ can be decomposed into radial and tangential components.



A New Coordinate System

For a particle in uniform circular motion:

- The velocity vector is tangent to the circle.

$$v_r = 0\,\frac{\mathrm{m}}{\mathrm{s}}$$

$$v_t = \frac{ds}{dt} = r\frac{d\theta}{dt} = r\omega$$

$$v_z = 0\,\frac{\mathrm{m}}{\mathrm{s}}$$

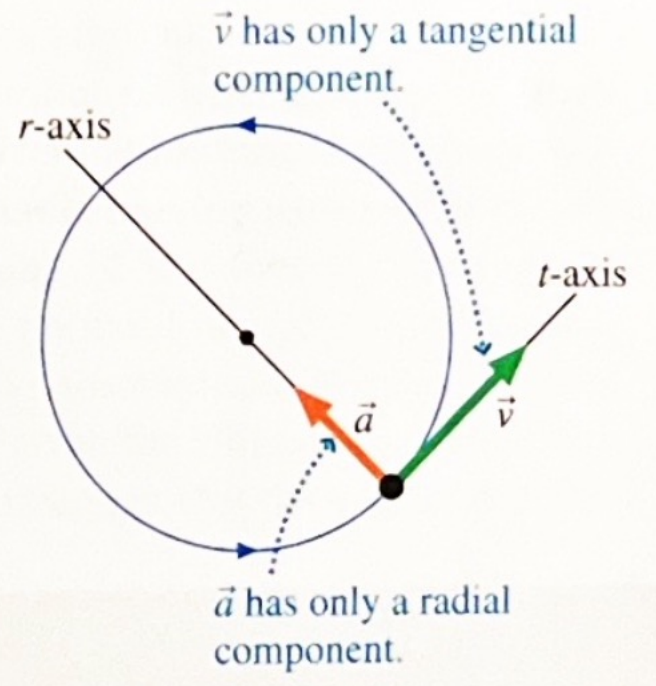
- The acceleration of uniform circular motion, points toward the center of the circle.

$$a_r = \frac{v_t^2}{r} = r\omega^2$$

$$a_t = 0\,\frac{\mathrm{m}}{\mathrm{s}^2}$$

$$a_z = 0\,\frac{\mathrm{m}}{\mathrm{s}^2}$$

FIGURE 8.5 The velocity and acceleration vectors in the *rtz*-coordinate system.



What about forces in uniform circular motion?

Newton's Second Law still applies:

$$\vec{F}_{\text{net}} = m\vec{a} = m\vec{a}_c = m\left(\frac{v^2}{r}\right) = \frac{mv^2}{r}$$

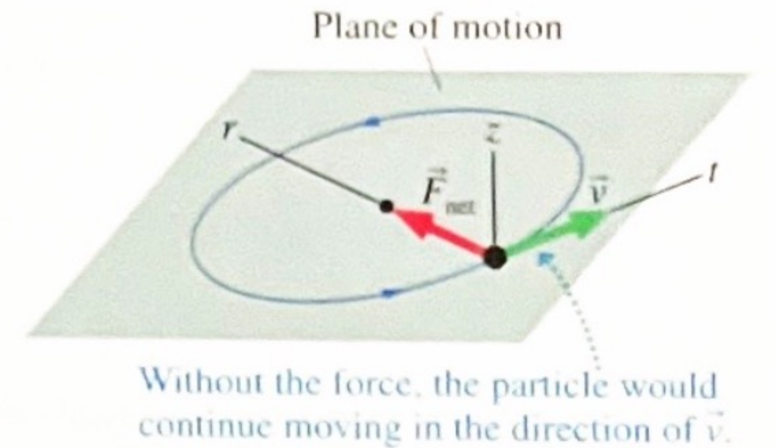
A particle of mass *m* moving at constant speed v around a circle of radius *r* must have a net force of magnitude $\frac{mv^2}{r}$ pointing toward the center of the circle to maintain its circular trajectory.

In *rtz-coordinates*:

$$(\sum F_i)_r = ma_r = \frac{mv^2}{r} = m\omega^2 r$$
$$(\sum F_i)_t = ma_t = 0$$
$$(\sum F_i)_z = ma_z = 0$$

For uniform circular motion, the sum of the forces along the *t*- and *z-axes* must be zero.

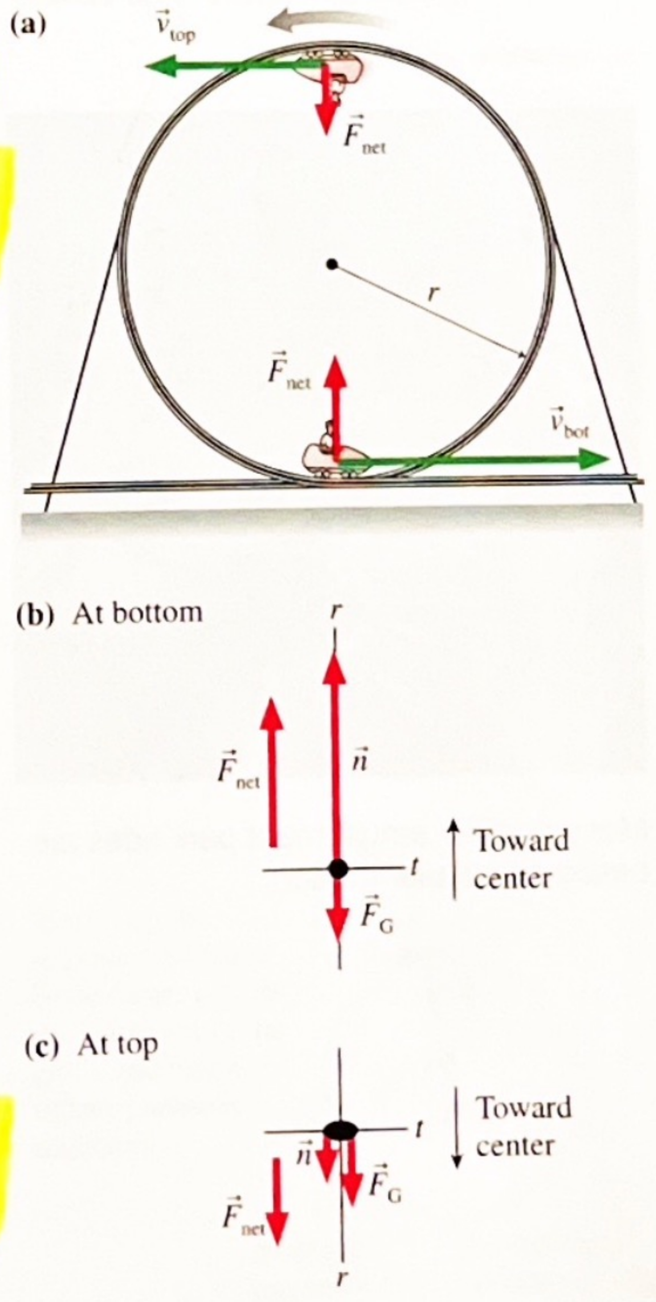
FIGURE 8.6 The net force points in the radial direction, toward the center of the circle.



What about a vertical circle?

- Motion in a vertical circle is NOT uniform circular motion. Consider the roller coaster car here.
- In this case, the car slows down as it goes up one side and speeds up as it comes back down on the other.
- Only at the top and bottom of the loop is the car's tangential acceleration zero. Thus, these are the only two places where the acceleration is purely centripetal.
- Newton's Second Law applies at these two points.
- The only forces acting here are gravity and the normal force.
  - At the bottom of the loop, the normal force is responsible for holding the car in the circular trajectory. (This "extra" normal force comes from the curvature of the track.)
  - At the top of the circle, the normal force AND gravity hold the car in circular motion.

FIGURE 8.17 A roller coaster car going around a loop-the-loop.



Summing the forces in a vertical circle

Bottom of the loop:

$$\sum_i F_r = ma_r = \frac{m(v_{\text{bot}})^2}{r} = n_r + F_G = n - mg$$

$$n_{\text{bot}} = mg + \frac{m(v_{\text{bot}})^2}{r}$$

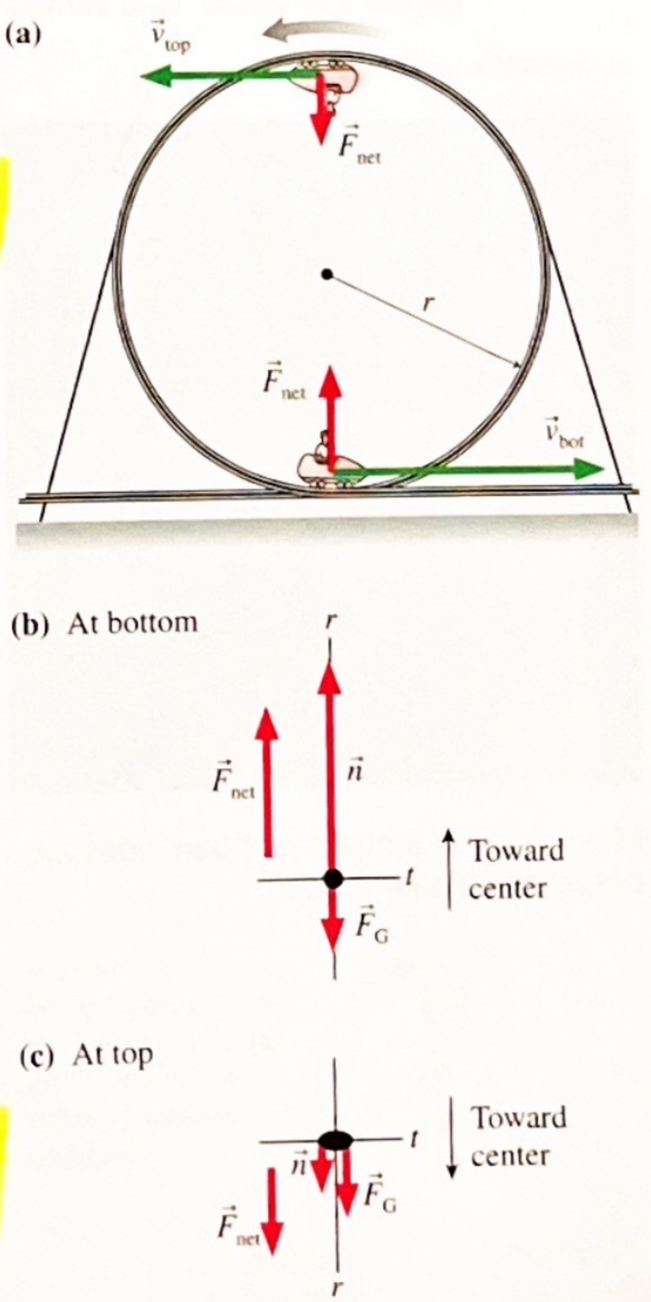
Top of the loop:

$$\sum_i F_r = ma_r = \frac{m(v_{\text{bot}})^2}{r} = n_r + F_G = n + mg$$

$$n_{\text{top}} = \frac{m\left(v_{\text{top}}\right)^2}{r} - mg$$

Since $n_{\text{bot}} > n_{\text{top}}$, a rider weighs more at the bottom of the loop than the top!

FIGURE 8.17 A roller coaster car going around a loop-the-loop.



The Critical Speed

We can see from this example that the force of gravity and the normal force help to hold the car on a circular path.

What happens when we lose our assistance from the track? This happens when $n = 0\text{N}$. When there is no normal force, there is no contact with the track!

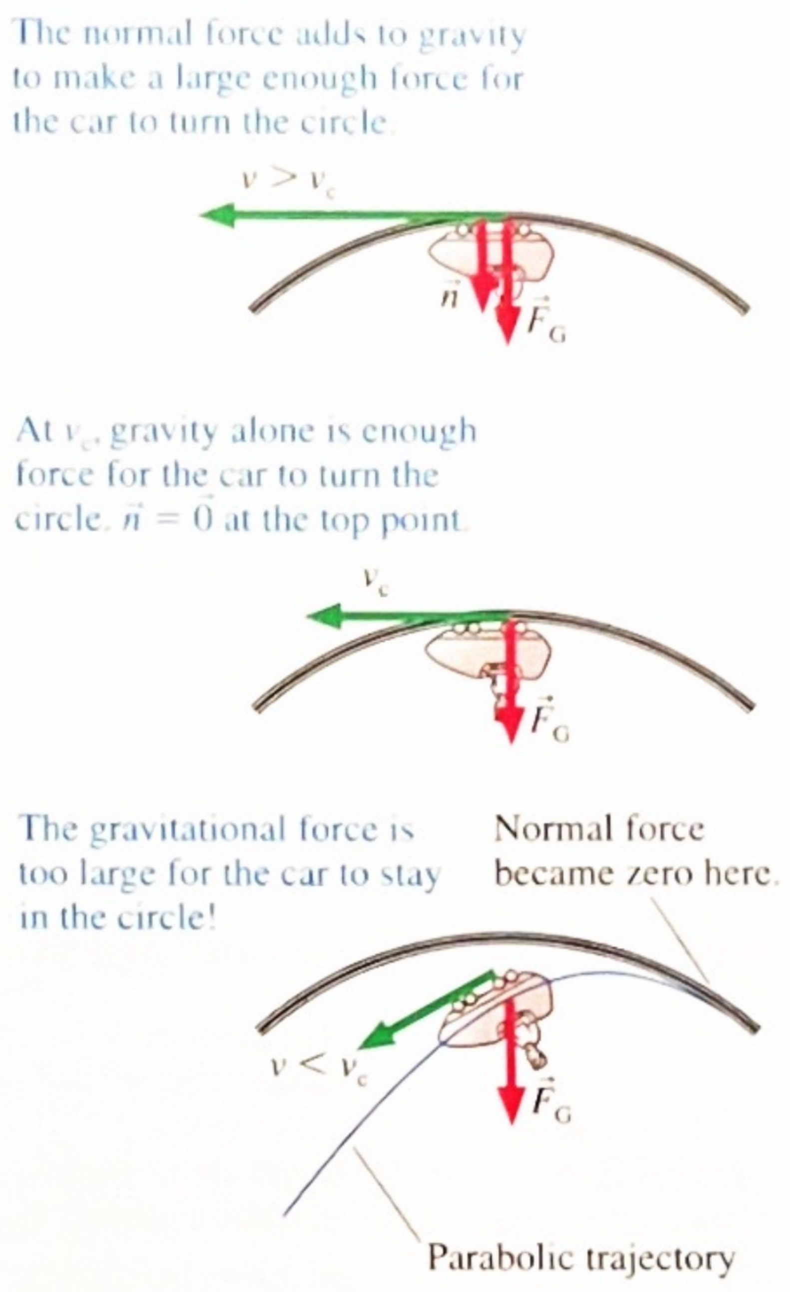
This means that $\frac{m(v_{\text{top}})^2}{r} = mg$. ONLY gravity is supplying the centripetal force. Any speed slower than this will cause the car to leave the track!

We can solve for v_{top} in this scenario.

$$v_{\text{crit}} = \sqrt{rg}$$

This is the *critical speed*. A speed slower than v_{crit} will not sustain the car through the loop.

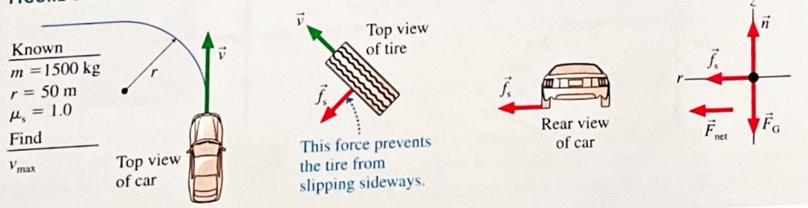
FIGURE 8.18 A roller coaster car at the top of the loop.



Setting up Problems

EXAMPLE 8.4 Turning the corner I
What is the maximum speed with which a 1500 kg car can make a left turn around a curve of radius 50 m on a level (unbanked) road without sliding?

FIGURE 8.8 Pictorial representation of a car turning a corner.

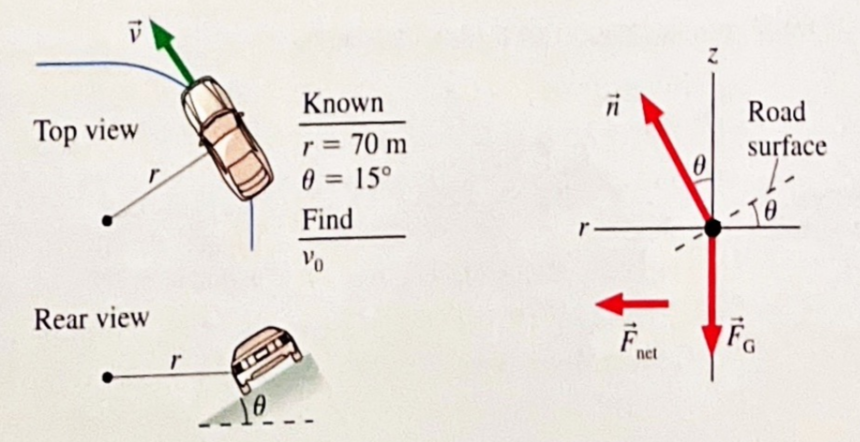


$$\sum F_r = f_s = \frac{mv^2}{r}$$

$$\sum F_z = n - mg = 0$$

EXAMPLE 8.5 Turning the corner II
A highway curve of radius 70 m is banked at a 15° angle. At what speed v_0 can a car take this curve without assistance from friction?

FIGURE 8.9 Pictorial representation of a car on a banked curve.

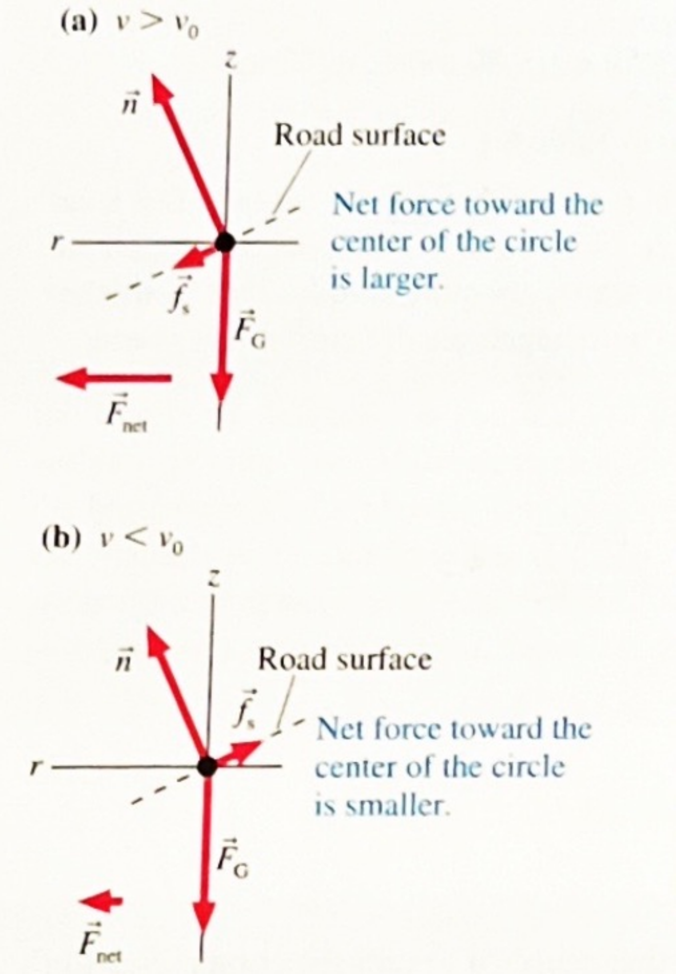


$$\sum F_r = n\sin\theta = \frac{mv_0^2}{r}$$

$$\sum F_z = n\cos\theta - mg = 0$$

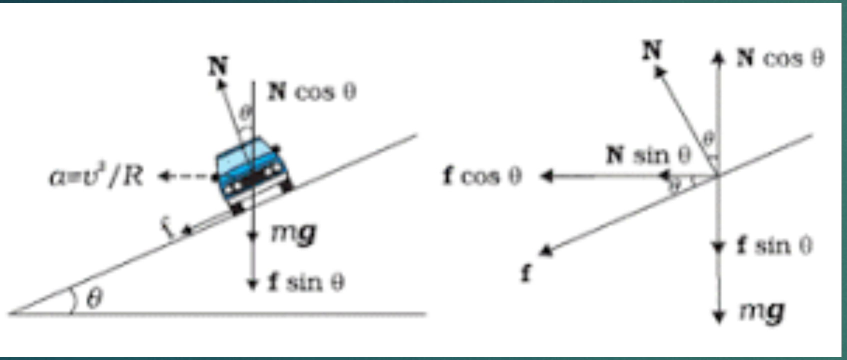
The exact velocity at which friction is not needed is v_0. Any speed faster or slower will required friction to make the turn!

FIGURE 8.10 Free-body diagrams showing the static friction force when $v > v_0$ and when $v < v_0$.



Example #2

A curved ramp of an interchange is built such that it is sloped 10 degrees into the turn with a radius of 60m. Given that the coefficient of static friction of a typical car tire is 0.7 (normal conditions), what is the maximum speed a 2000kg car can travel around this turn?



$$\sum F_r = n\sin\theta + \mu_s n\cos\theta = \frac{mv^2}{r}$$

$$\sum F_z = n\cos\theta - \mu_s n\sin\theta - mg = 0$$

v is needed from $\sum F_r$ but we need n.

From $\sum F_z$:

$$n = \frac{mg}{\cos\theta - \mu_s\sin\theta}$$

From $\sum F_r$:

$$n(\sin\theta + \mu_s\cos\theta) = \frac{mv^2}{r}$$

Combining the two equations:

$$\left(\frac{\sin\theta + \mu_s\cos\theta}{\cos\theta - \mu_s\sin\theta}\right) g = \frac{v^2}{r}$$

$$v = \sqrt{rg\left(\frac{\sin\theta + \mu_s\cos\theta}{\cos\theta - \mu_s\sin\theta}\right)}$$

$$v \approx 24\,\frac{\mathrm{m}}{\mathrm{s}}$$