LECTURE 4.4 – CIRCULAR MOTION

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UNIFORM CIRCULAR MOTION

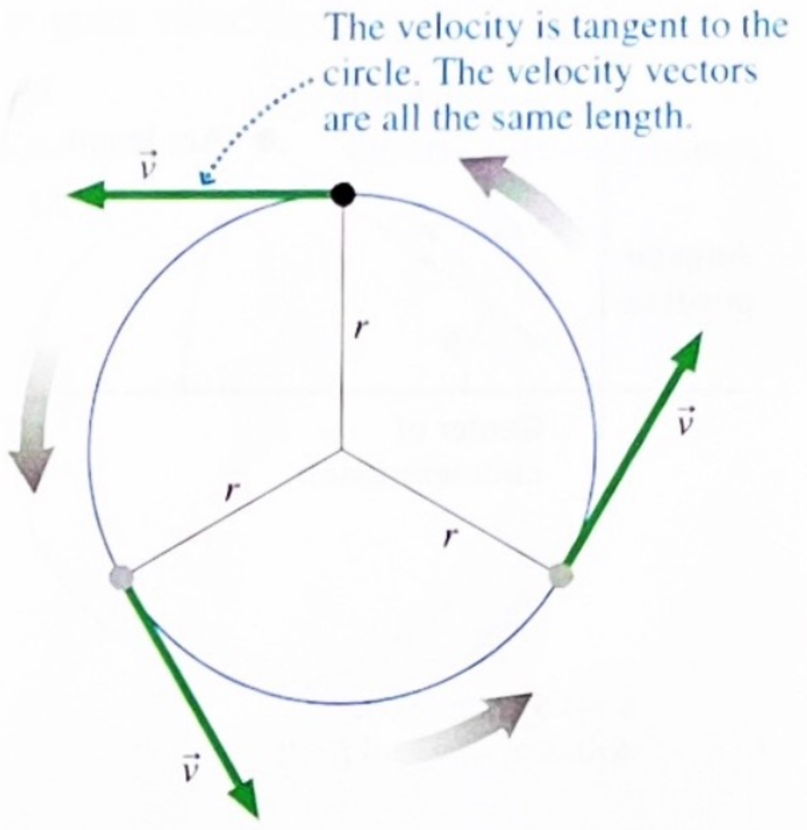


FIGURE 4.32 A particle in uniform circular motion.

Consider a particle that moves at a *constant speed* around a circle of radius *r*.

The time it takes for the particle to go around the circle once (one revolution) is called the *period, T*.

Therefore, $v = \dfrac{\text{distance}}{\text{time}} = \dfrac{\text{circumference}}{\text{period}} = \dfrac{2\pi r}{T}$

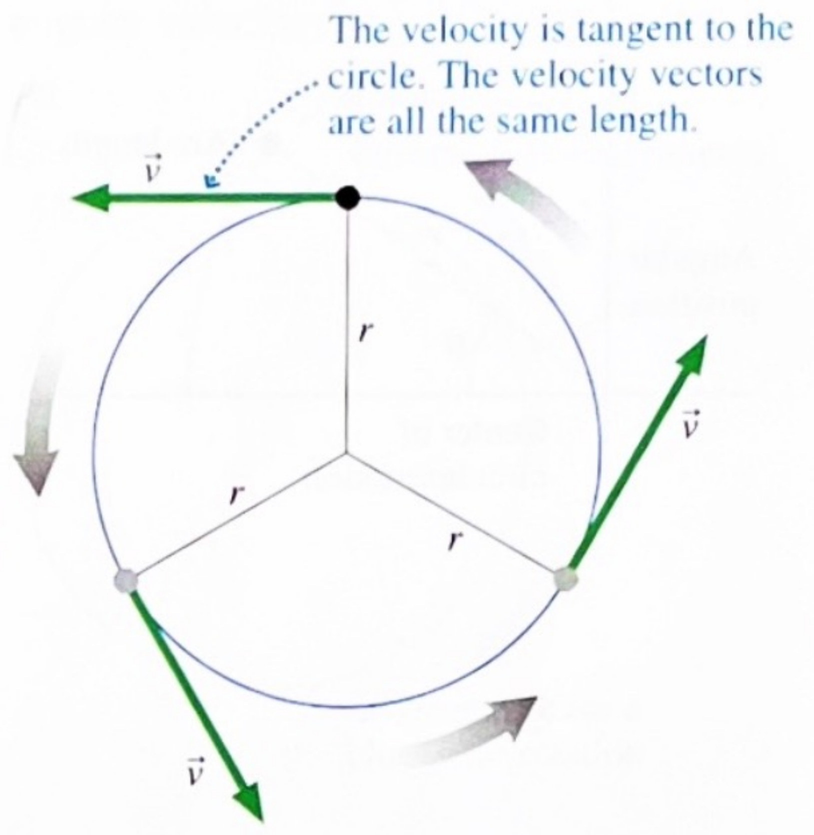
Putting an origin at the center of the circle would make it possible to track the particle using *(x,y)* coordinates but there is something more efficient.

POLAR COORDINATES

- Instead of recording the particle's position on a Cartesian grid, we can use *polar coordinates* to express its location.
  - Using the radius of the particle's motion, *r*
  - Using its angle, θ, from the (+) x-axis. This is the *angular position* of the particle.

- For angular positions, counterclockwise is (+), clockwise is (-)

FIGURE 4.32 A particle in uniform circular motion.



ANGULAR POSITION

The distance traveled by a particle undergoing circular motion is often represented as an *arc length*.

The unit of *radian* is a measure of such a distance:

$$\theta_{rad} \equiv \frac{s}{r}$$

The radian is the distance a particle has traveled *around* the circle relative to the *radius* of the circle.

1 rad on a circle is where the arc length equals the radius.

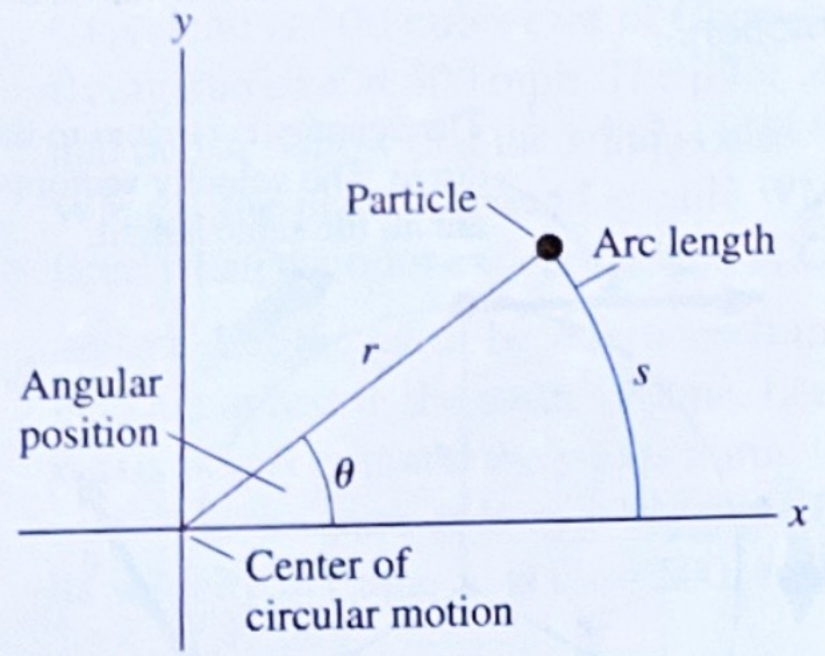
This equation can be rearranged to calculate the distance (arc length) that a particle has traveled.

$$s = r\theta$$

Thus, we can define *angular displacement* as:

$$\Delta\theta = \theta_f - \theta_i$$

FIGURE 4.33 A particle's position is described by distance r and angle θ.



SOME NOTES ABOUT UNITS

How many radians are in a full circle?

$$\theta_{\text{circle}} = \frac{s}{r} = \frac{\text{distance traveled}}{\text{r}} = \frac{2\pi r}{r} = 2\pi \text{ rad}$$

We also know that 1 full revolution is 360° so:

$$1 \text{ rev} = 360° = 2\pi \text{ rad}$$

$$1 \text{ rad} = 1 \text{ rad} \times \frac{360°}{2\pi \text{ rad}} = 57.3°$$

The SI unit for angles is the radian.

ANGULAR VELOCITY

We know *where* and *when* a particle is positioned on a circle. From this we can define and *angular velocity*.

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

Note that ω is the Greek letter "omega", not "w". This is the rate of change of *angular position*.

If a particle is moving in uniform circular motion, ω is not changing. Therefore,

$$\theta_f = \theta_i + \omega\Delta t$$

Notice that this is the same equation for uniform linear motion!

$$x_f = x_i + a\Delta t$$

A useful method for calculating ω:

$$|\omega| = \frac{\text{distance}}{\text{time}} = \frac{2\pi}{T}$$

ANGULAR VS. TANGENTIAL VELOCITY

Consider a solid wheel with a green sticker on its edge (a distance *r* from the radius) and a orange sticker between the green sticker and the center of the circle ($\frac{1}{2}r$). Suppose the wheel turns such that the stickers are moved through $90°$ in 1 second.

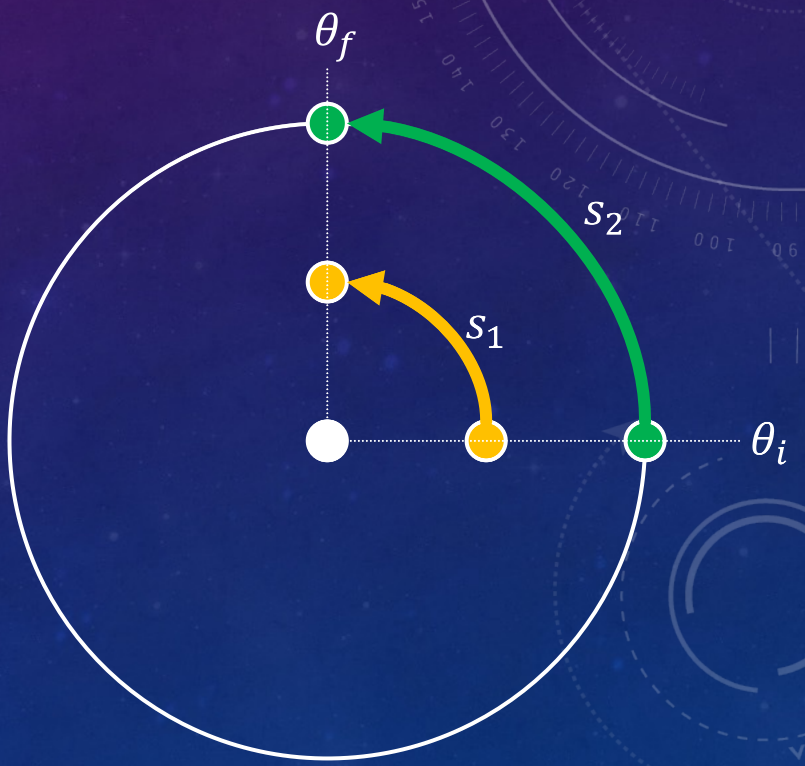
The *angular velocity* is the same since $\Delta\theta_1 = \Delta\theta_2$:

$$\frac{\Delta\theta_1}{\Delta t} = \frac{\Delta\theta_2}{\Delta t} = \frac{\Delta\theta}{\Delta t}$$

However, the green sticker traveled over a much larger arc length than the orange sticker. Thus, the green sticker had a larger *tangential velocity*. (It covered a greater distance in the same amount of time.)

How do we calculate the velocity *tangential* to the curve?

$$v_t = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt} = \frac{d}{dt}(r\theta) = r\frac{d\theta}{dt} = r\omega$$



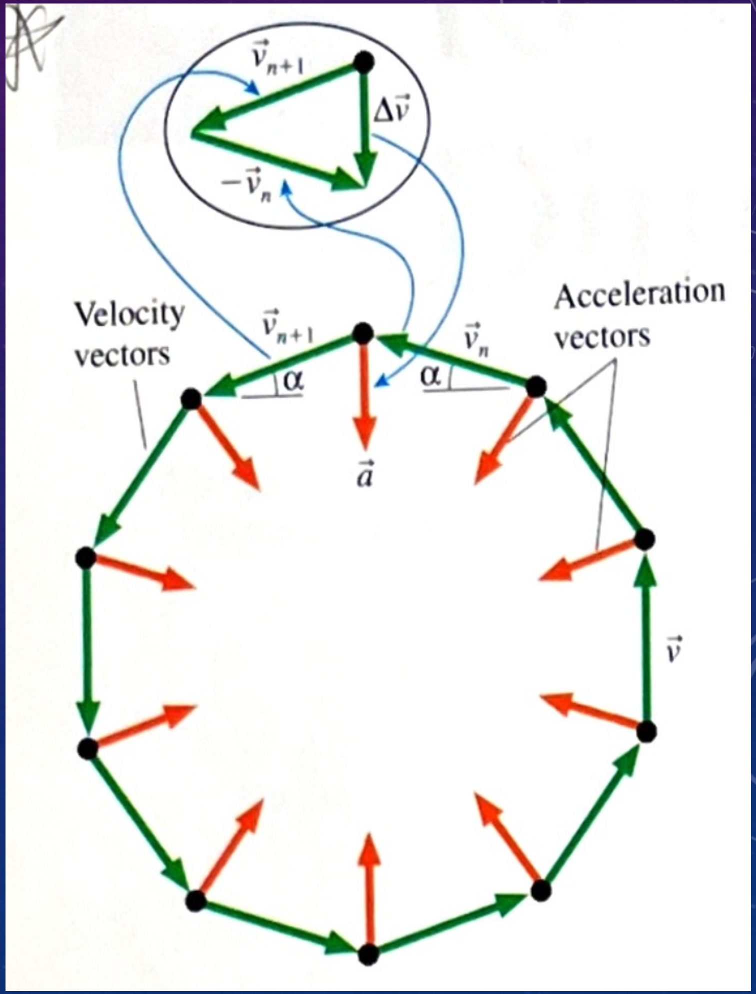
ACCELERATION

Let's return to the example of the Ferris wheel. Remember that a rider is experiencing *uniform circular motion* (constant speed).

However, the motion diagram shows that there is still an acceleration toward the center of the circle since the velocity vector is always changing.

This is the *centripetal acceleration* (center seeking).

How do we know the *magnitude* of this centripetal acceleration?



DERIVING THE MAGNITUDE OF CENTRIPETAL ACCELERATION

Consider what we know already:

$$s = r\theta \text{ then } ds = rd\theta$$

This means that an *infinitesimally small* change in *s* is equal to the radius times an infinitesimally small change in θ. (r is always constant)

We can apply this same logic to $\vec{v}$ since it is also moving in a complete circle over time.

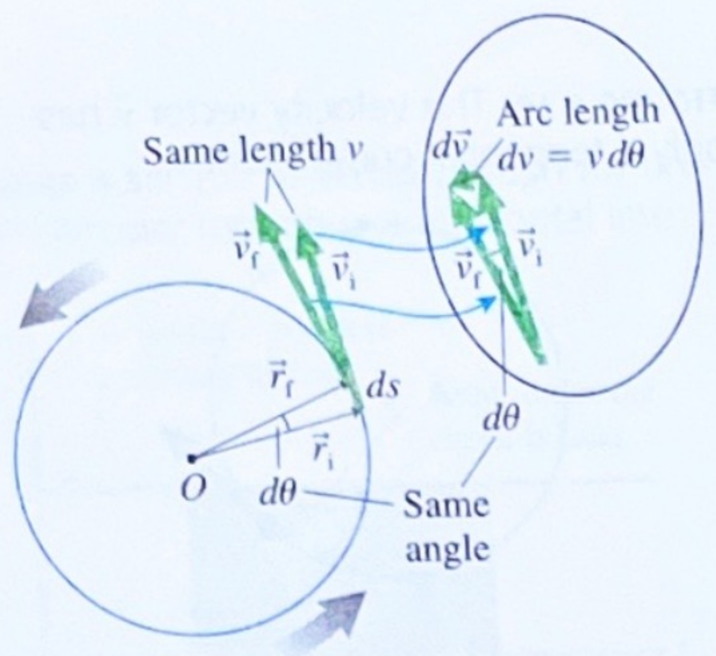
$$dv = vd\theta \text{ so } \frac{dv}{d\theta} = v$$

Really small tangential changes can be treated like really small arc lengths.

We also know that:

$$v_t = r\omega = r\frac{d\theta}{dt} \text{ therefore } \frac{d\theta}{dt} = \frac{v}{r}$$

FIGURE 4.41 Finding the acceleration of circular motion.



PUTTING IT ALL TOGETHER

We know that $\vec{a} = \frac{d\vec{v}}{dt}$. So how can we construct a formula for the centripetal acceleration, $\vec{a}_C$?

Notice that we can use relationships that we can use the chain rule to build a definition from relationships that we already know:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d\theta}{dt}\frac{dv}{d\theta} = \frac{v}{r} \cdot v = \frac{v^2}{r}$$

Remember that $v = r\omega$. So,

$$\vec{a} = \frac{v^2}{r} = \frac{(r\omega)^2}{r} = \frac{r^2\omega^2}{r} = r\omega^2$$

Centripetal acceleration is NOT constant acceleration. Therefore, constant-acceleration kinematics will not work here.

WHEN DO KINEMATIC EQUATIONS APPLY?

From a previous lecture, you will recall:

$\vec{a}_{\parallel}$ indicates the *parallel* component that changes the speed of the object.

(Tangential Acceleration)

$\vec{a}_{\perp}$ indicates the *perpendicular* component that changes the direction of the object.

(Centripetal Acceleration)

In the case of *nonuniform circular motion*, where $\vec{a}_{\parallel}$ is changing, $\vec{a}$ no longer points toward the center of the circle.

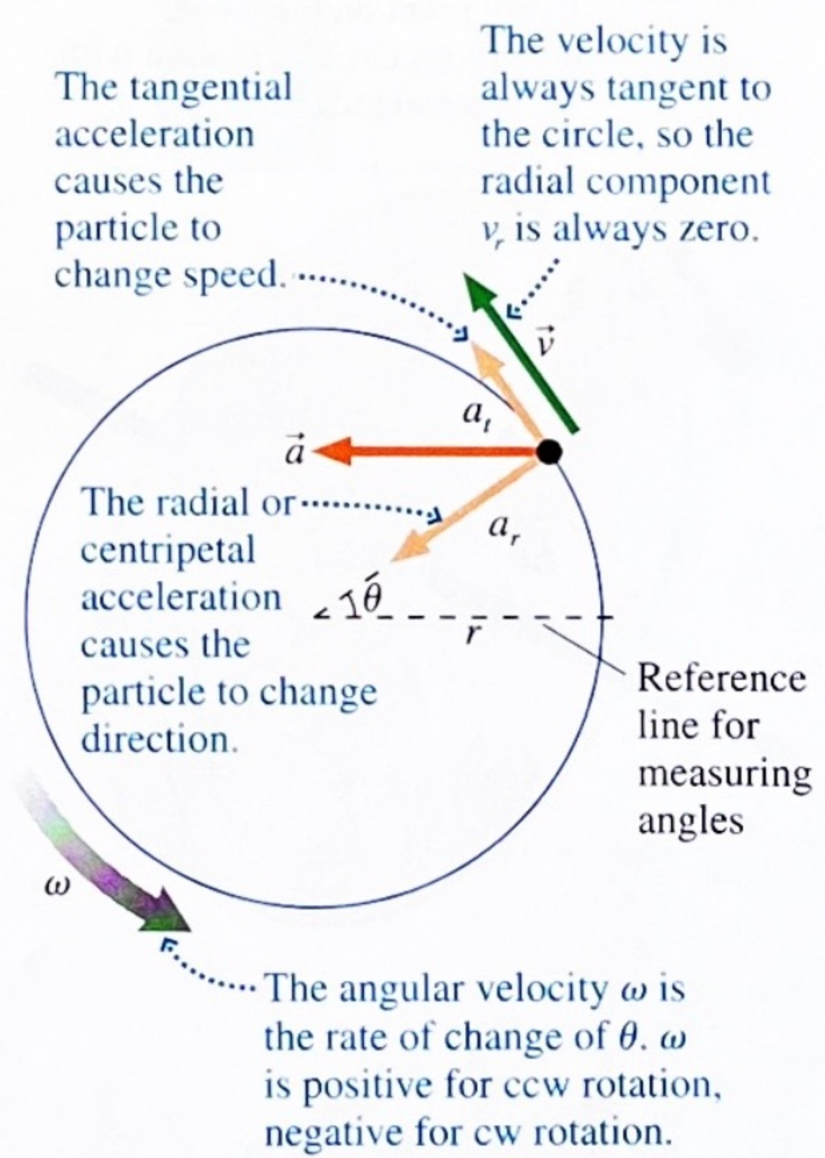
- It points ahead of the center if the particle is speeding up
- It points behind of the center if the particle is slowing down

$$|\vec{a}| = \sqrt{\vec{a}_{\parallel}^2 + \vec{a}_{\perp}^2} = \sqrt{a_t^2 + a_c^2}$$

$$a_t = \frac{d}{dt}(v_t) = \frac{d}{dt}(r\omega) = r\frac{d\omega}{dt} = r\frac{d^2\theta}{dt^2} = r\alpha$$

α is the angular acceleration (units: rad/s^2)

FIGURE 4.42 Nonuniform circular motion.



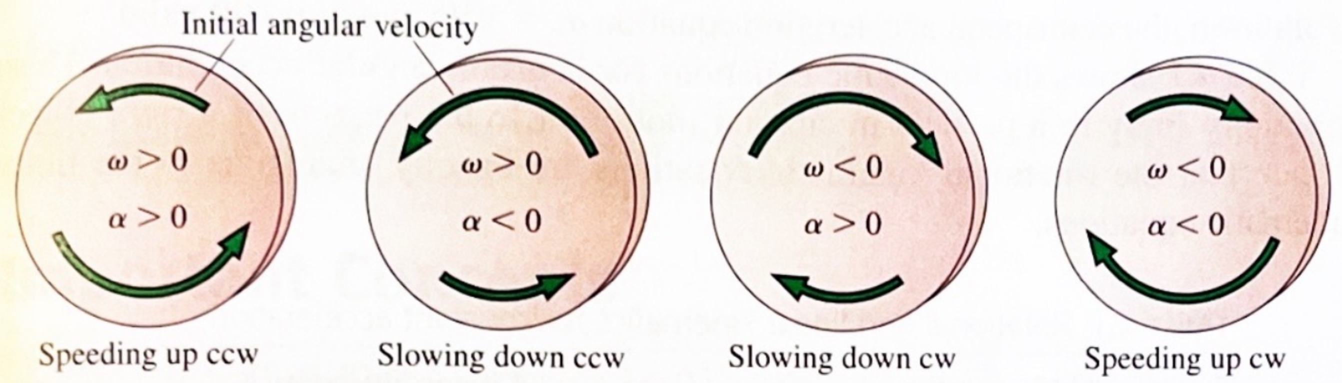
ANGULAR ACCELERATION

α is (+) if ω is increasing ccw or decreasing cw.

α is (-) if ω is increasing cw or decreasing ccw.

Two points on a rotating object will have the *same* angular acceleration but *different* tangential acceleration.

FIGURE 4.46 The signs of angular velocity and acceleration.



JUST LIKE CARTESIAN KINEMATICS!

Notice that only *angular* velocity and acceleration are used here! Tangential velocities and accelerations are different!

Centripetal acceleration is not included here.

TABLE 4.1 Rotational and linear kinematics for constant acceleration

Rotational kinematics	Linear kinematics
$\omega_f = \omega_i + \alpha \Delta t$	$v_{fs} = v_{is} + a_s \Delta t$
$\theta_f = \theta_i + \omega_i \Delta t + \frac{1}{2}\alpha(\Delta t)^2$	$s_f = s_i + v_{is} \Delta t + \frac{1}{2}a_s(\Delta t)^2$
$\omega_f^2 = \omega_i^2 + 2\alpha \Delta\theta$	$v_{fs}^2 = v_{is}^2 + 2a_s \Delta s$

AN EXAMPLE

A well-lubricated bicycle wheel spins a long time before stopping. Suppose a wheel initially rotating at 100rpm takes 60s to stop. If the angular acceleration is constant, how many revolutions does the wheel make while stopping? What is the initial centripetal acceleration felt by a sticker on the wheel (the radius of the wheel is 0.3m)?