LECTURE 4.1 – 2D KINEMATICS PARAMETRIC EQUATIONS

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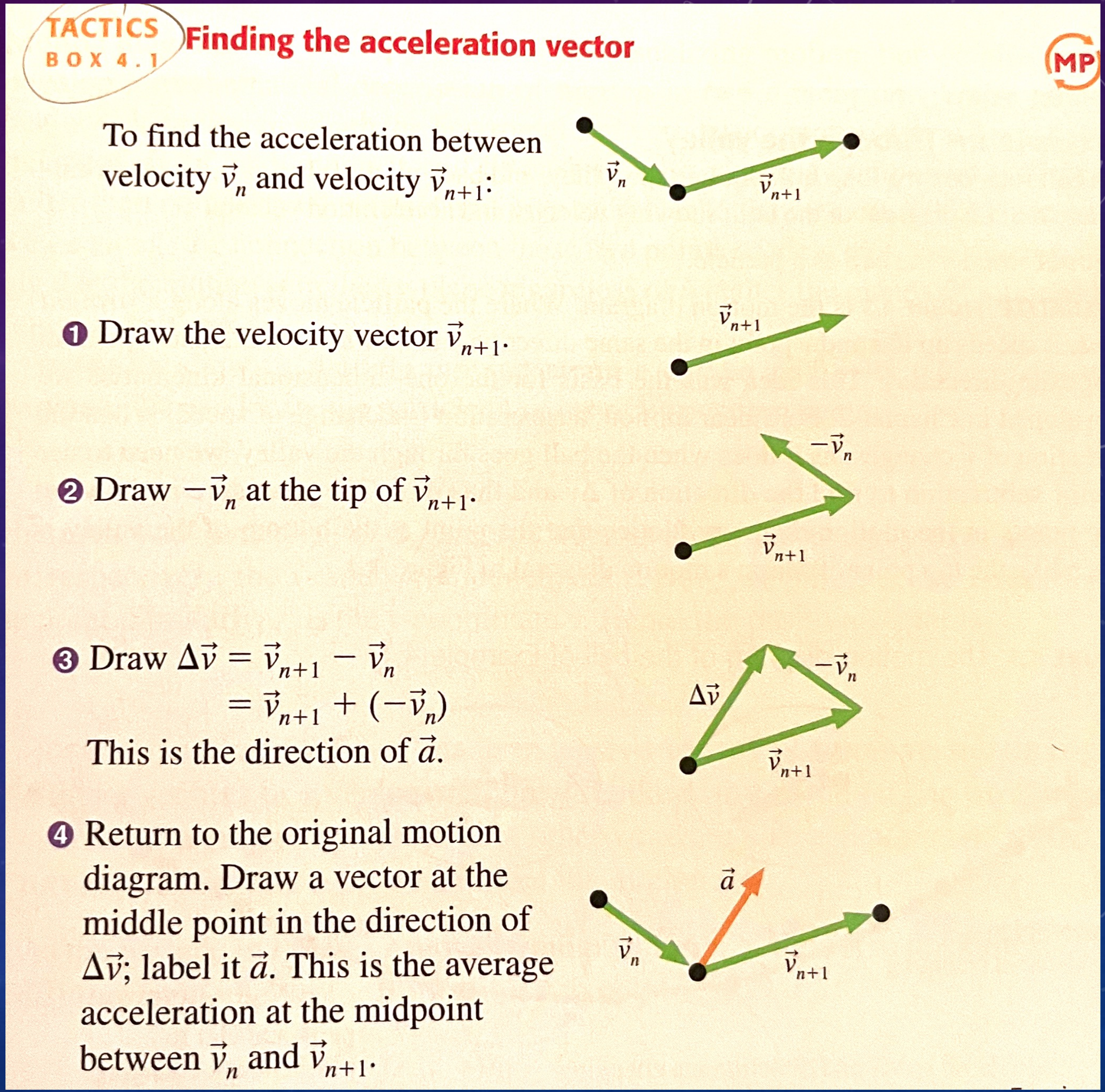
REVIEW OF VELOCITY

As an object moves, its velocity vector can change in two possible ways:

1. The *magnitude* of $\vec{v}$ can change, indicating a change in *speed*
2. The *direction* of $\vec{v}$ can change, indicating a change in direction

These changes manifest as *acceleration*.



TACTICS BOX 4.1 **Finding the acceleration vector**

MP

To find the acceleration between velocity $\vec{v}_n$ and velocity $\vec{v}_{n+1}$:

❶ Draw the velocity vector $\vec{v}_{n+1}$.

❷ Draw $-\vec{v}_n$ at the tip of $\vec{v}_{n+1}$.

❸ Draw $\Delta\vec{v} = \vec{v}_{n+1} - \vec{v}_n$
$= \vec{v}_{n+1} + (-\vec{v}_n)$
This is the direction of $\vec{a}$.

❹ Return to the original motion diagram. Draw a vector at the middle point in the direction of $\Delta\vec{v}$; label it $\vec{a}$. This is the average acceleration at the midpoint between $\vec{v}_n$ and $\vec{v}_{n+1}$.

DECOMPOSITION OF THE ACCELERATION VECTOR

In a Cartesian grid, an acceleration vector can be decomposed into two components:

- (1) parallel and
- (2) perpendicular

to the direction of motion.

$\vec{a}_{\parallel}$ indicates the *parallel* component that changes the speed of the object.

$\vec{a}_{\perp}$ indicates the *perpendicular* component that changes the direction of the object.

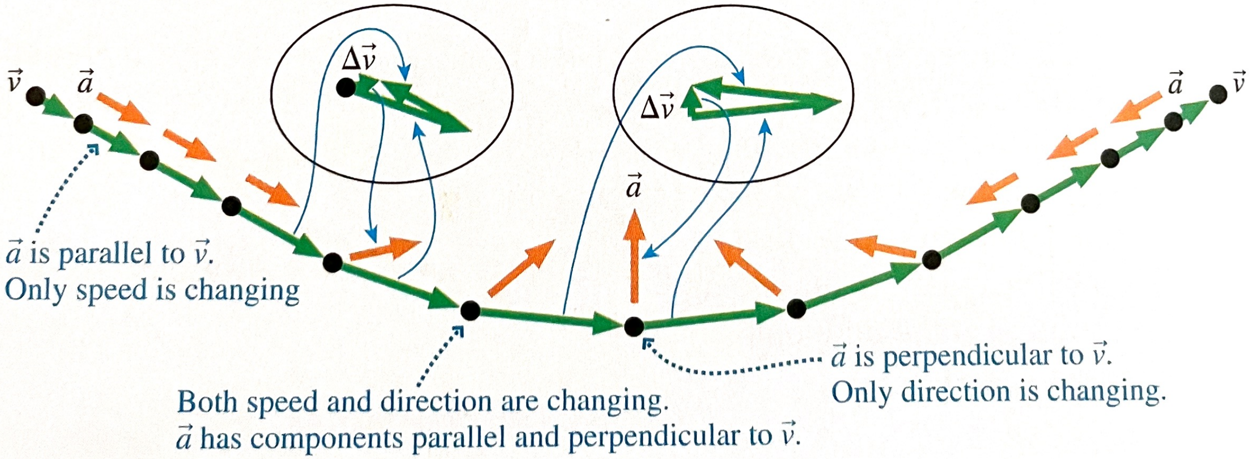
EXAMPLE 4.1 Through the valley

A ball rolls down a long hill, through the valley, and back up the other side. Draw a complete motion diagram of the ball, showing velocity and acceleration vectors.

MODEL Model the ball as a particle.

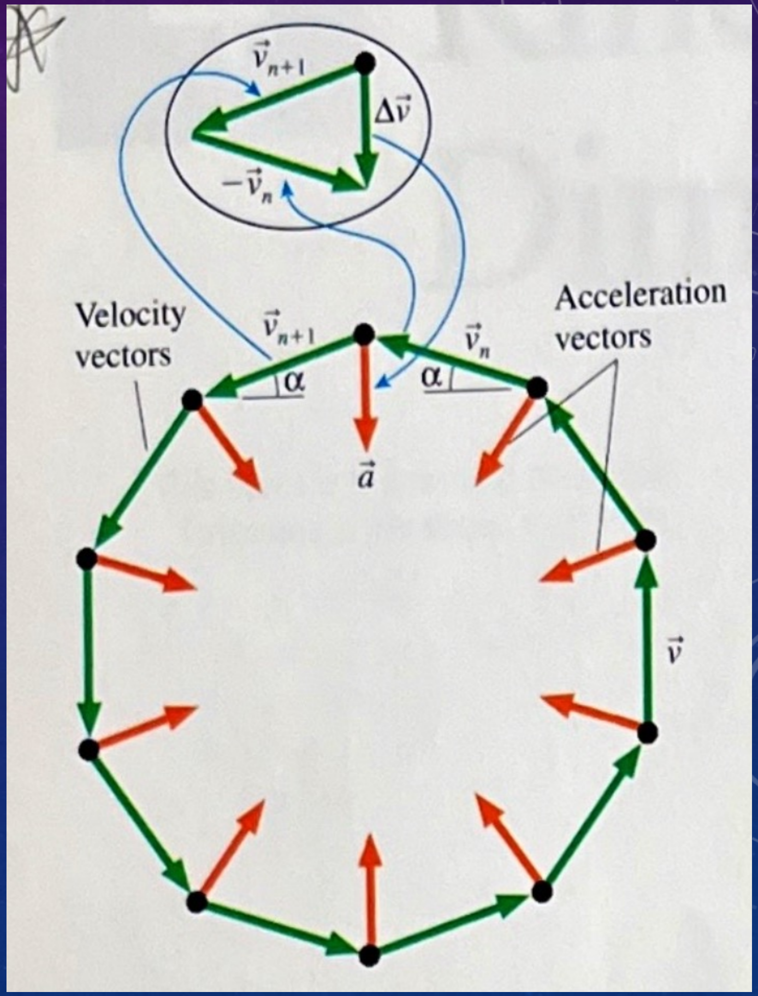
VISUALIZE **FIGURE 4.3** is the motion diagram. Where the particle moves along a *straight line*, it speeds up if $\vec{a}$ and $\vec{v}$ point in the same direction and slows down if $\vec{a}$ and $\vec{v}$ point in opposite directions. This idea was the basis for the one-dimensional kinematics we developed in Chapter 2. For linear motion, acceleration is a change of speed. When the direction of $\vec{v}$ changes, as it does when the ball goes through the valley, we need to use vector subtraction to find the direction of $\Delta\vec{v}$ and thus of $\vec{a}$. The procedure is shown at two points in the motion diagram. Notice that the point at the bottom of the valley is much like the top point of Maria's motion diagram in Figure 4.2.

FIGURE 4.3 The motion diagram of the ball of Example 4.1.



CENTRIPETAL ACCELERATION

- CENTRIPETAL AND CENTRIFUGAL ~~ACCELERATIONS~~ FORCES ARE DIFFERENT
  - Centrifugal forces “aren’t real”!
- An acceleration that always points toward the center of a circle is called a *centripetal acceleration*.
- Examples:
  - Your acceleration on a Ferris wheel (constant speed, changing direction)
  - Your acceleration in a car around a smooth curve (constant speed, changing direction)



2D-KINEMATICS

The position of a particle from the origin at a particular time is given by the position vector:

$$\vec{r} = x\hat{\imath} + y\hat{\jmath}$$

Notice that this representation is *static* in time. (One moment is represented.)

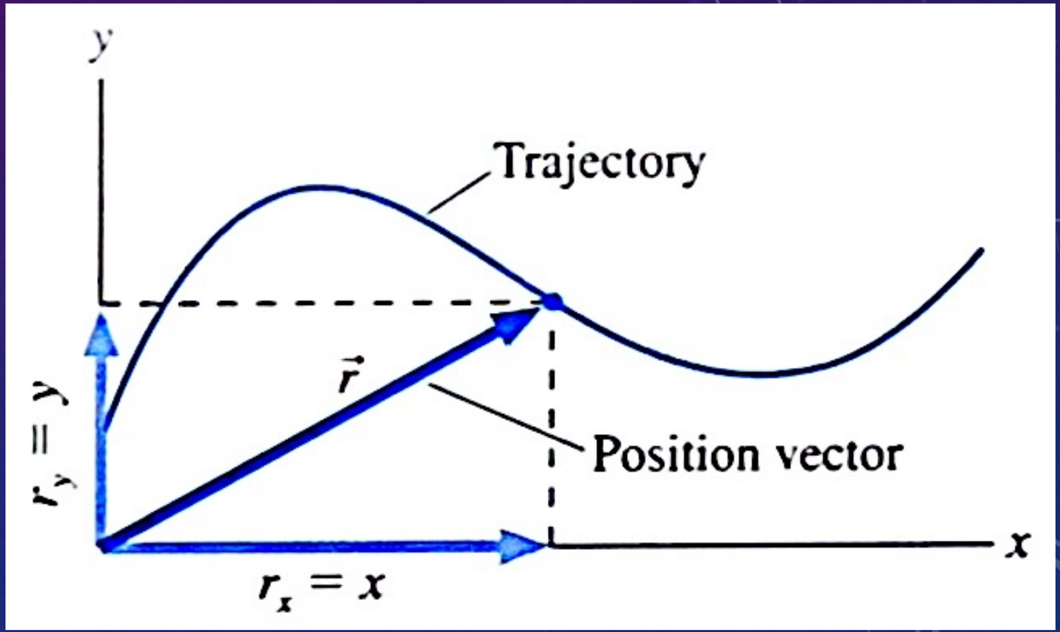
What if we wanted to represent this position of the object over time?

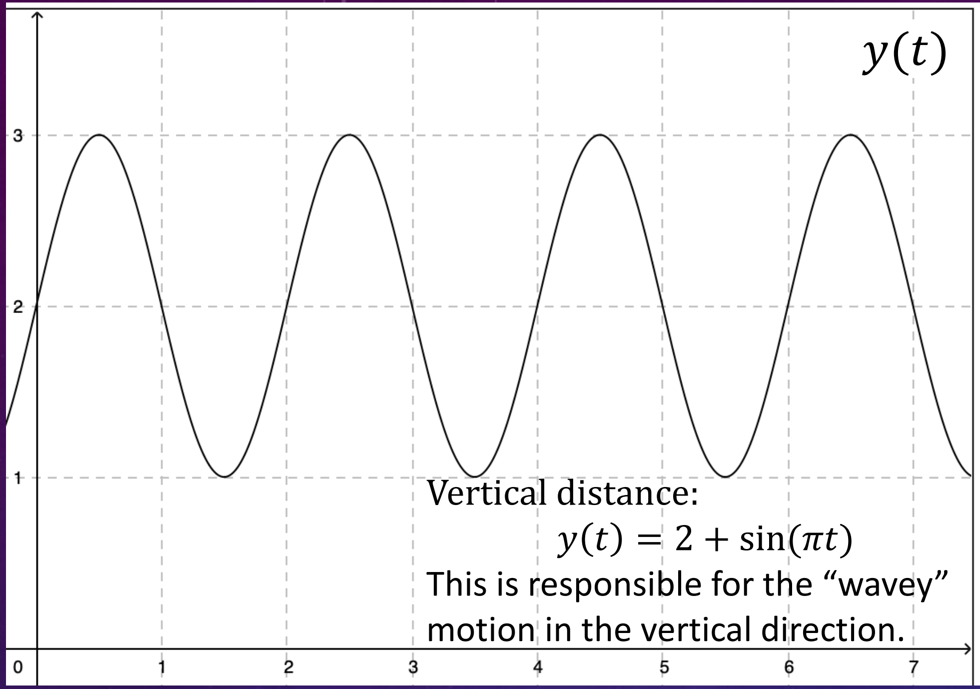
$\vec{r}$, x, and y all *change with time*. So, $\vec{r}$ can be represented as a *parametric equation*.

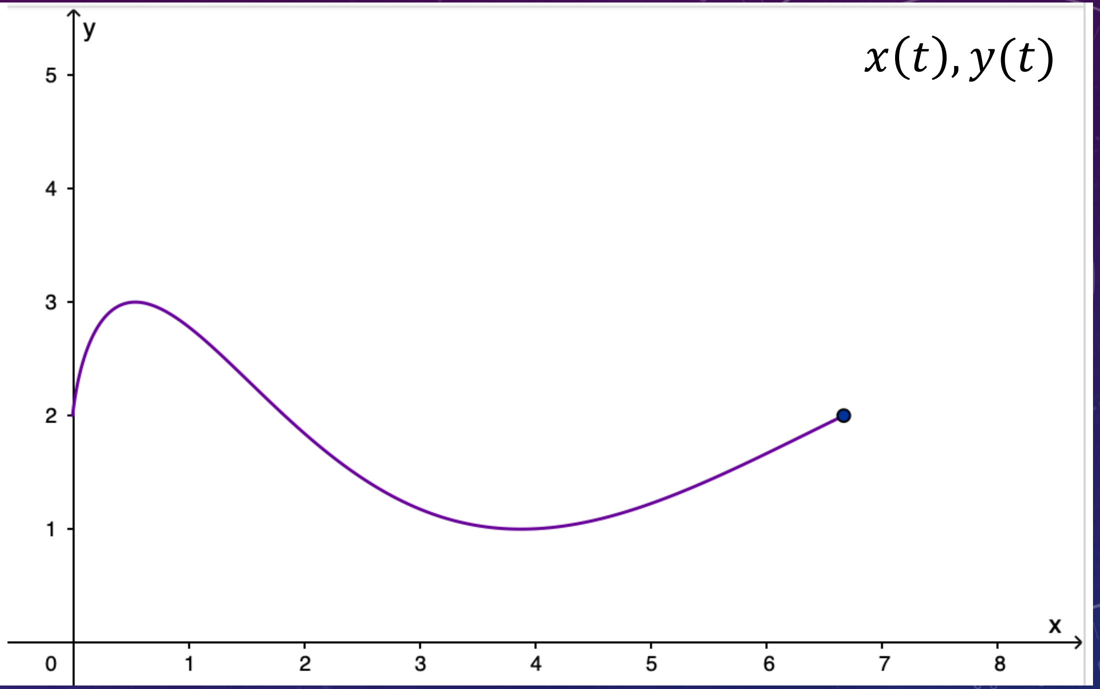
$$\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$$

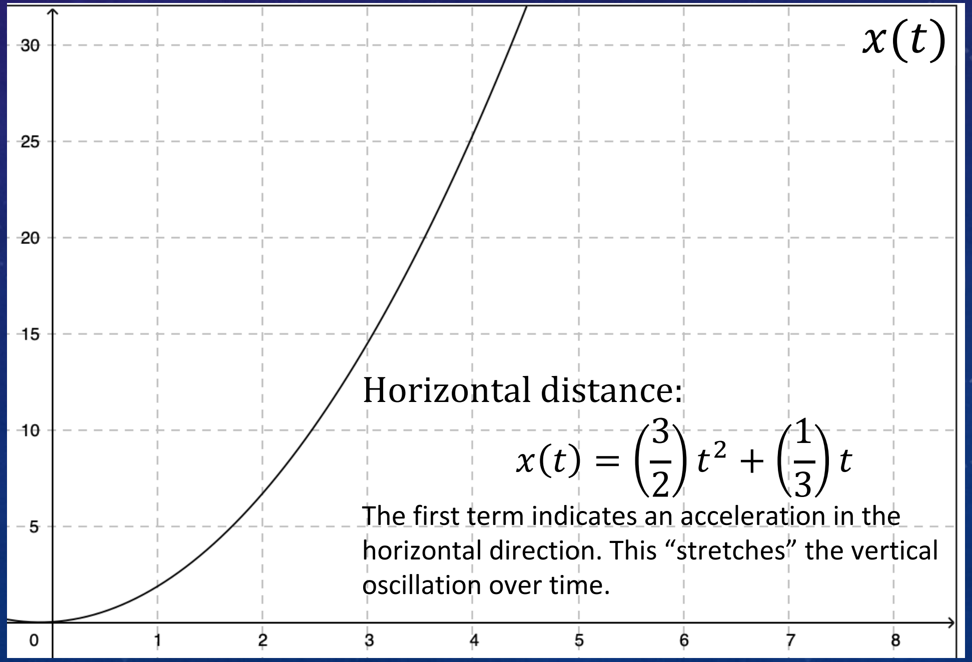
$x(t)$ only describes the object's movement in the *x-direction*.

$y(t)$ only describes the object's movement in the *y-direction*.









EXAMPLE OF A PARAMETRIC EQUATION

$$\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} = \left[\left(\frac{3}{2}\right)t^2 + \left(\frac{1}{3}\right)t\right]\hat{\imath} + [2 + \sin(\pi t)]\hat{\jmath}$$

$0 \leq t \leq 2$

DISPLACEMENT IN 2D

Just like with 1D kinematics, we can calculate a displacement vector:

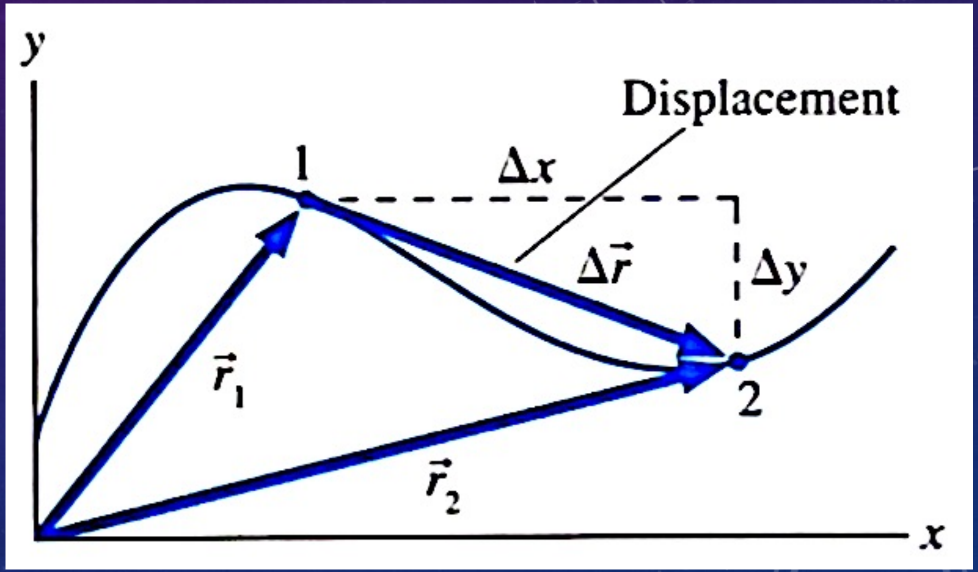
$$\Delta\vec{r} = \vec{r}_2(t_2) - \vec{r}_1(t_1) = x_2(t_2)\hat{\imath} + y_2(t_2)\hat{\jmath} - x_1(t_1)\hat{\imath} + y_1(t_1)\hat{\jmath}$$

These terms can be rearranged:

$$\Delta\vec{r} = [x_2(t_2) - x_1(t_1)]\hat{\imath} + [y_2(t_2) - y_1(t_1)]\vec{\jmath} = (\Delta x)\hat{\imath} + (\Delta y)\hat{\jmath}$$

Thus,

$$\Delta\vec{r} = \Delta x\hat{\imath} + \Delta y\hat{\jmath}$$



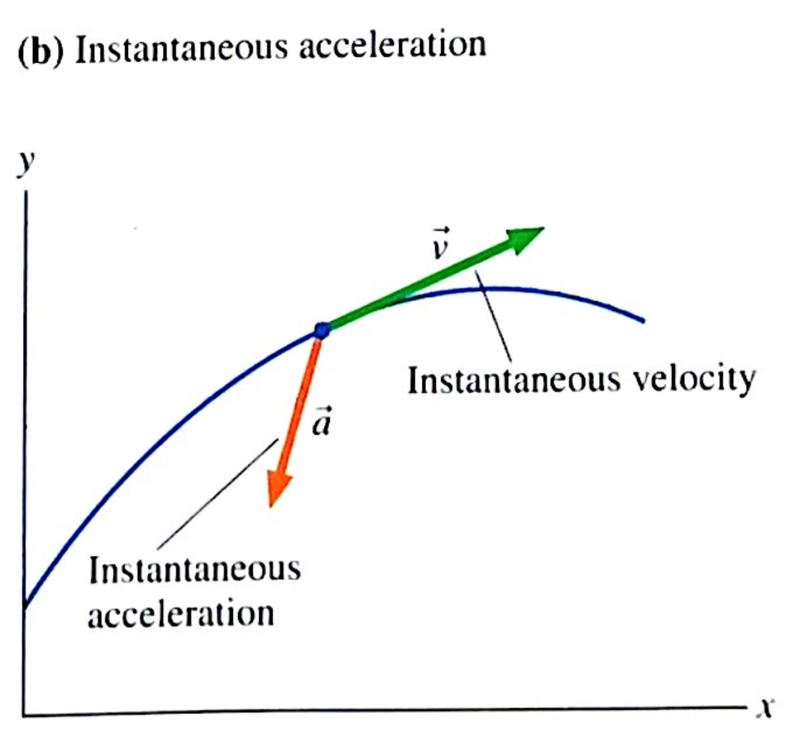
WHAT ABOUT VELOCITY AND ACCELERATION?

$$\vec{v}(t) = \lim_{\Delta t \to 0}\left(\frac{\Delta \vec{r}}{\Delta t}\right) = \frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0}\frac{\Delta x}{\Delta t}\hat{\imath} + \lim_{\Delta t \to 0}\frac{\Delta y}{\Delta t}\hat{\jmath} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} = v_x(t)\hat{\imath} + v_y(t)\hat{\jmath}$$

The instantaneous velocity vector is *tangent* to the trajectory!

$$\vec{a}(t) = \lim_{\Delta t \to 0}\left(\frac{\Delta \vec{v}}{\Delta t}\right) = \frac{d\vec{v}}{dt} = \lim_{\Delta t \to 0}\frac{\Delta v_x}{\Delta t}\hat{\imath} + \lim_{\Delta t \to 0}\frac{\Delta v_y}{\Delta t}\hat{\jmath} = \frac{dv_x}{dt}\hat{\imath} + \frac{dv_y}{dt}\hat{\jmath} = a_x(t)\hat{\imath} + a_y(t)\hat{\jmath}$$

The instantaneous acceleration vector MAY NOT be tangent to the trajectory!



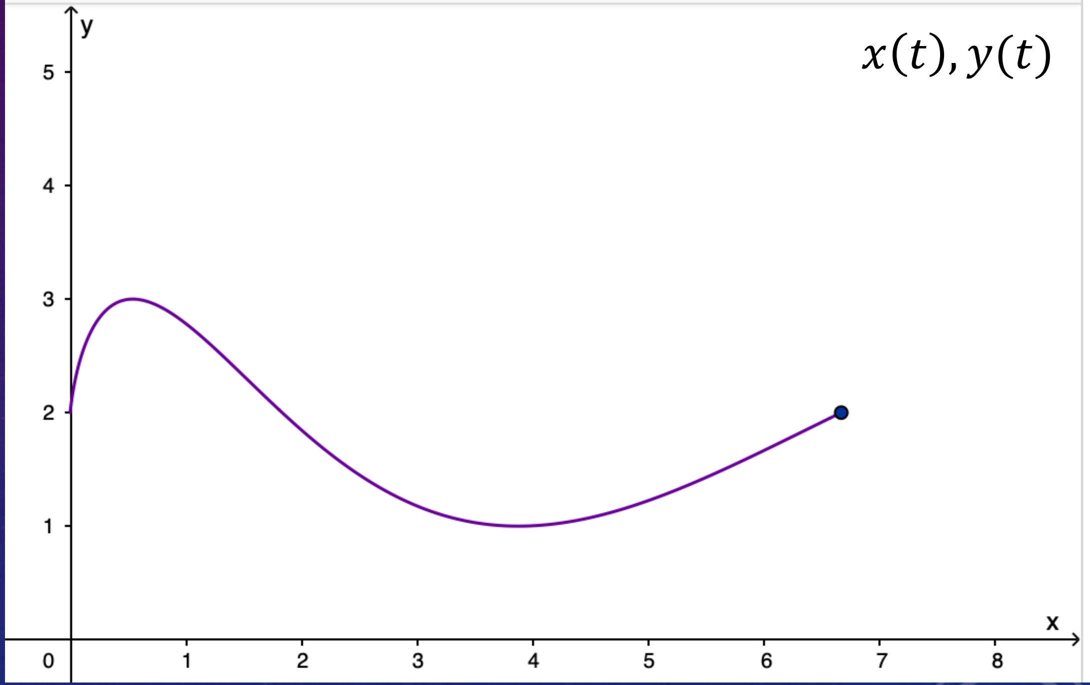
EXAMPLE

Consider the previous particle where the trajectory is given by:

$$\vec{r}(t) = \left[\left(\frac{3}{2}\right)t^2 + \left(\frac{1}{3}\right)t\right]\hat{\imath} + [2 + \sin(\pi t)]\,\hat{\jmath}$$

On the interval $t = [0s, 2s]$.

- Where are its turning points?
- Where does this particle experience the most acceleration?



TURNING POINTS

WHERE $v = 0\,\frac{\text{M}}{\text{S}}$ FOR AN INSTANT WHILE THE OBJECTS DIRECTION CHANGES

X-direction turning points

$$x(t) = \left(\frac{3}{2}\right)t^2 + \left(\frac{1}{3}\right)t$$

$$v_x(t) = 3t + \frac{1}{3}$$

Where does $v_x(t) = 0\,\frac{\text{m}}{\text{s}}$?

$$v_x\left(t = -\frac{1}{9}\text{s}\right) = 0\,\frac{\text{m}}{\text{s}}$$

This time is outside our range of interest (0s to 2s) so there are no turning points in the x-direction.

Y-direction turning points

$$y(t) = 2 + \sin(\pi t)$$

$$v_y(t) = \pi\cos(\pi t)$$

Where does $v_y(t) = 0\,\frac{\text{m}}{\text{s}}$?

$$v_y\left(t = \frac{1}{2}\text{s}, \frac{3}{2}\text{s}, \frac{5}{2}\text{s}, \ldots\right) = 0\,\frac{\text{m}}{\text{s}}$$

Some of these are in our range of interest (0s to 2s). Which ones are turning points?

CONSIDERING $v_y(t)$

$v_y(t) = \pi\cos(\pi t)$

Since a periodic function is involved, there will be an oscillation.

Since $v_y\left(t = \frac{1}{2}\mathrm{s}, \frac{3}{2}\mathrm{s}\right) = 0\,\frac{\mathrm{m}}{\mathrm{s}}$ and there is a transition from (+) to (-) and (-) to (+), these are turning points.

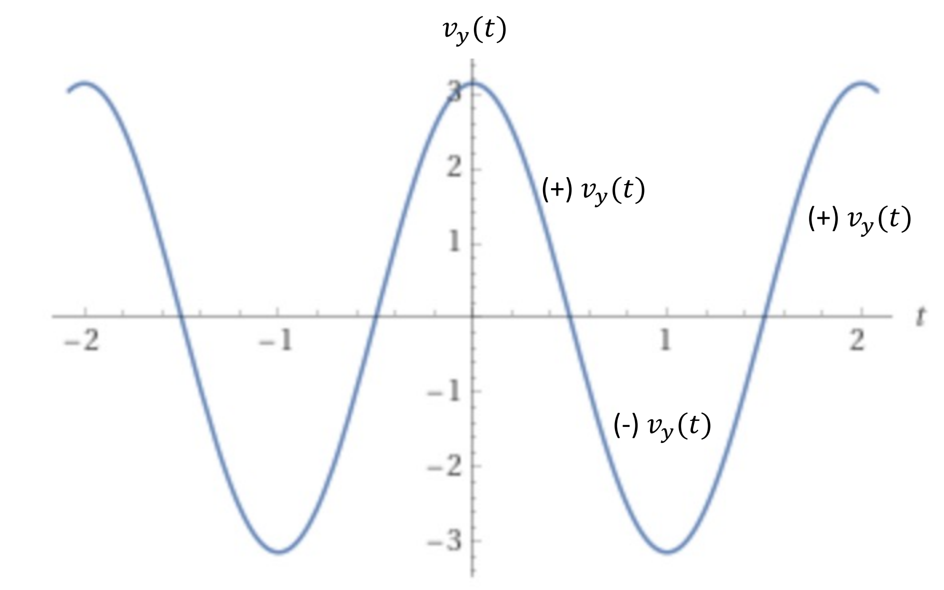
But where is that on the trajectory?

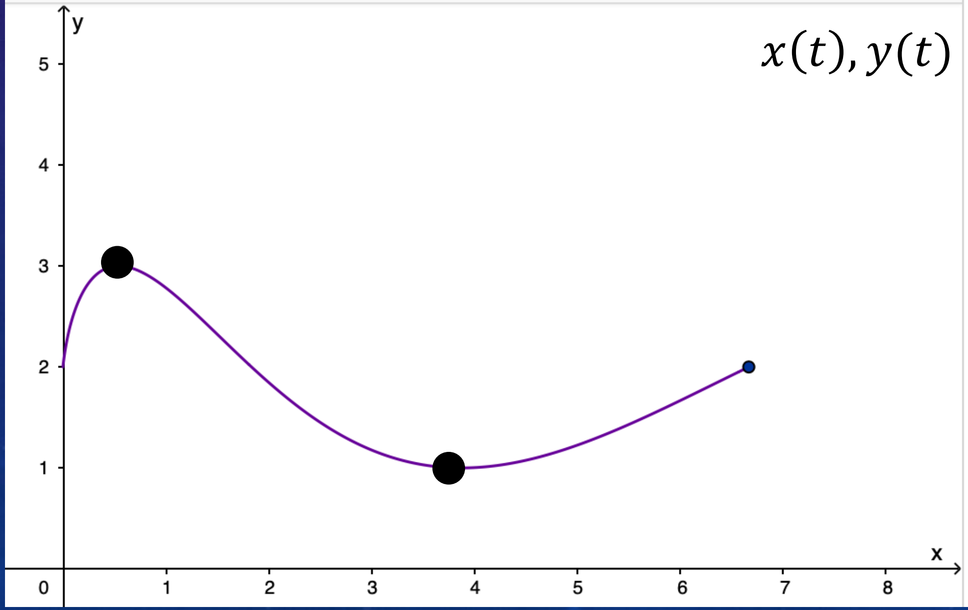
$$\vec{r}(t) = \left[\left(\frac{3}{2}\right)t^2 + \left(\frac{1}{3}\right)t\right]\hat{\imath} + [2 + \sin(\pi t)]\,\hat{\jmath}$$

$$\vec{r}_1\left(t_1 = \frac{1}{2}\right) = \left[\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\right]\hat{\imath} + \left[2 + \sin\left(\pi\frac{1}{2}\right)\right]\hat{\jmath} = (0.542\mathrm{m})\hat{\imath} + (3\mathrm{m})\hat{\jmath}$$

$$\vec{r}_2\left(t_2 = \frac{3}{2}\right) = \left[\left(\frac{3}{2}\right)\left(\frac{3}{2}\right)^2 + \left(\frac{1}{3}\right)\left(\frac{3}{2}\right)\right]\hat{\imath} + \left[2 + \sin\left(\pi\frac{3}{2}\right)\right]\hat{\jmath} = (3.875\mathrm{m})\hat{\imath} + (1\mathrm{m})\hat{\jmath}$$

Remember that the velocity vector is tangent to the trajectory!





WHERE DOES THE PARTICLE EXPERIENCE THE MOST ACCELERATION?

We know how to write the acceleration of a parametric equation:

$$\vec{a}(t) = \frac{dv_x}{dt}\hat{\imath} + \frac{dv_y}{dt}\hat{\jmath} = \frac{d}{dt}\left[3t + \frac{1}{3}\right]\hat{\imath} + \frac{d}{dt}[\pi\cos(\pi t)]\hat{\jmath} = 3\,\hat{\imath} - \pi^2\sin(\pi t)\,\hat{\jmath}$$

Where is the "most" acceleration? (Where is the *magnitude of* $\vec{a}(t)$ in the range of $0 \leq t \leq 2$?)

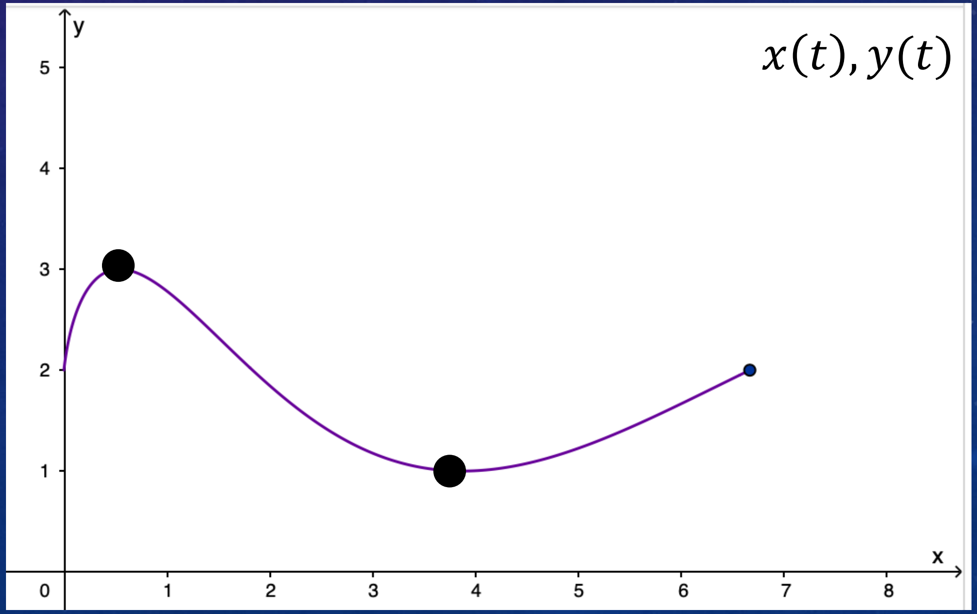
Calculate the magnitude of the acceleration vector:

$$|\vec{a}(t)| = \sqrt{3^2 + (\pi^2\sin(\pi t))^2}$$

Where is this at its maximum value?

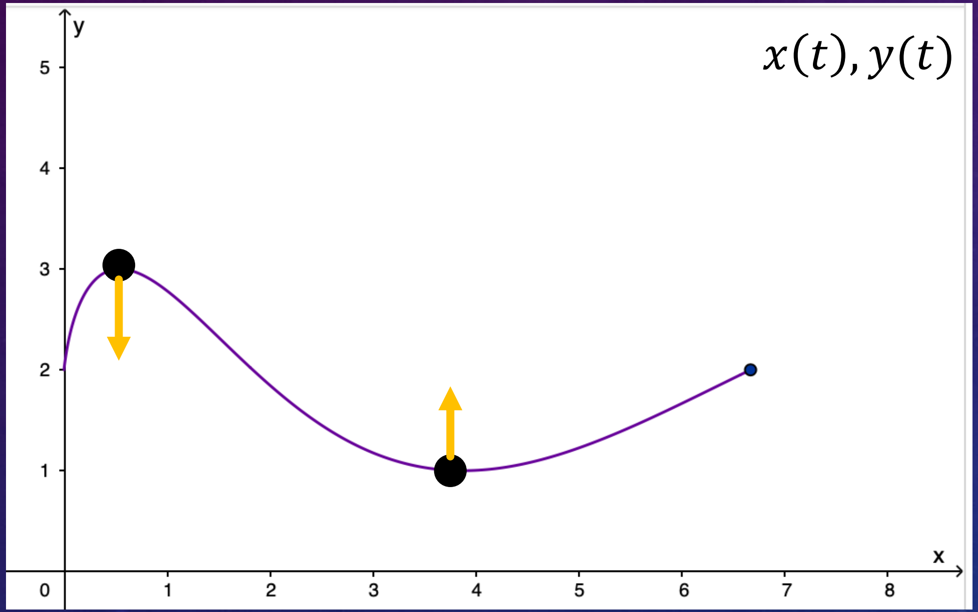
(Hint: What values make $\sin(\pi t) = \pm 1$?)

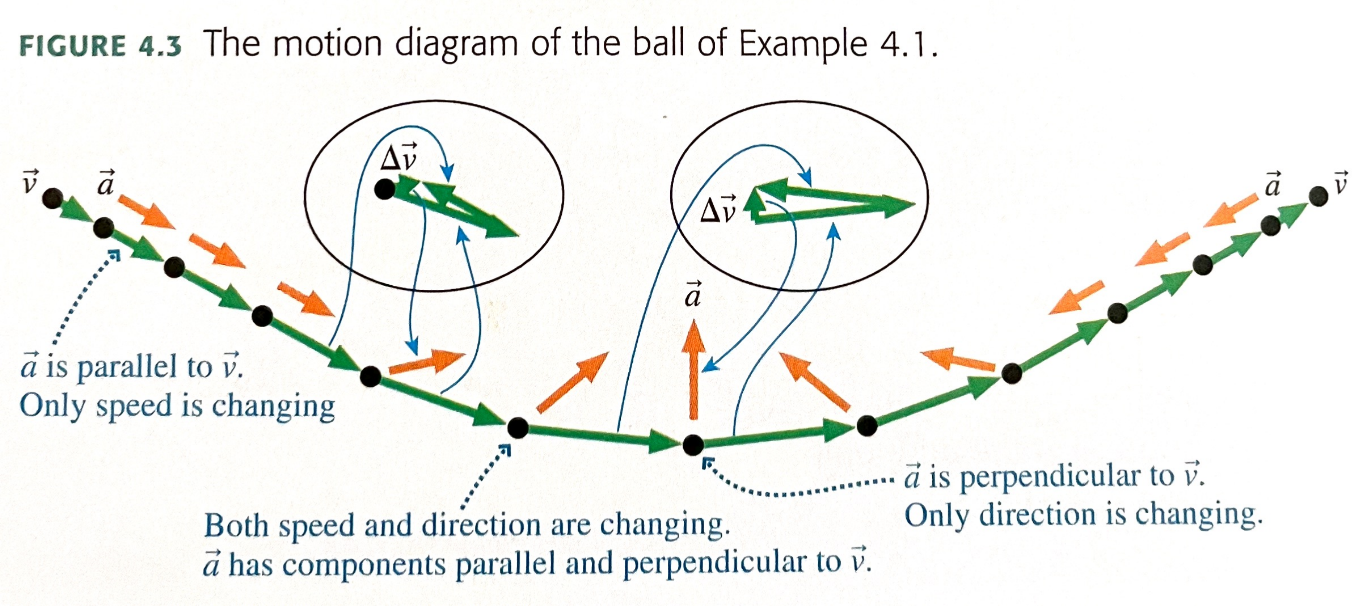
$t = \frac{1}{2}\,\mathrm{s}, \frac{3}{2}\,\mathrm{s}$



CENTRIPETAL ACCELERATION

- In both examples, the acceleration vector has the largest magnitude in the curves.





SUMMARY

As an object moves, its velocity vector can change in two possible ways:

1. The *magnitude* of $\vec{v}$ can change, indicating a change in *speed*
2. The *direction* of $\vec{v}$ can change, indicating a change in direction

These changes manifest as *acceleration*.

$\vec{a}_{\parallel}$ indicates the *parallel* component that changes the speed of the object.

$\vec{a}_{\perp}$ indicates the *perpendicular* component that changes the direction of the object.

An acceleration that always points toward the center of a circle is called a *centripetal acceleration*.

Modeling motion with parametric equations:

$$\Delta\vec{r} = (\Delta x)\hat{\imath} + (\Delta y)\hat{\jmath}$$

$$\vec{v}(t) = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} = v_x(t)\hat{\imath} + v_y(t)\hat{\jmath}$$

$$\vec{a}(t) = \frac{dv_x}{dt}\hat{\imath} + \frac{dv_y}{dt}\hat{\jmath} = a_x(t)\hat{\imath} + a_y(t)\hat{\jmath}$$