CONTINUOUS RANDOM VARIABLES

MAT 152 – Statistical Methods I

Lecture 3

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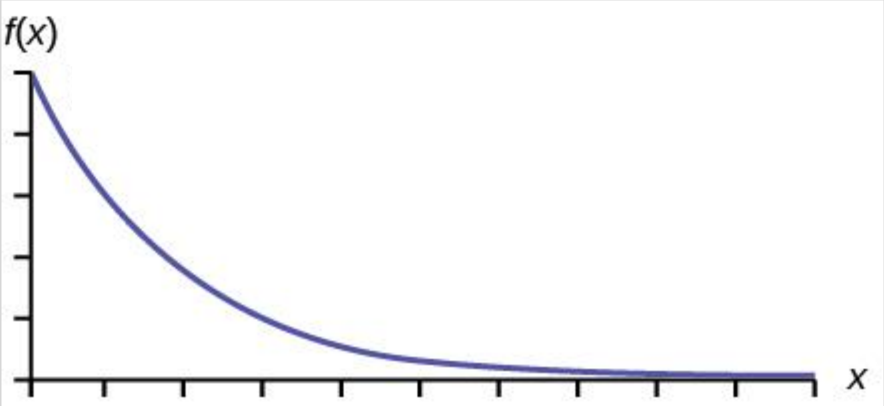
Fall 2020

The Exponential Distribution

The exponential distribution is often used to model the amount of time until some specific event occurs.



Applications include modeling the longevity of a physical device.

It is used to model events that are equally likely.

There are many small values and only a few large values.

The PDF is given by:

$$f(x) = \lambda e^{-\lambda x}$$

$\lambda = \frac{1}{\mu}$ (decay parameter)

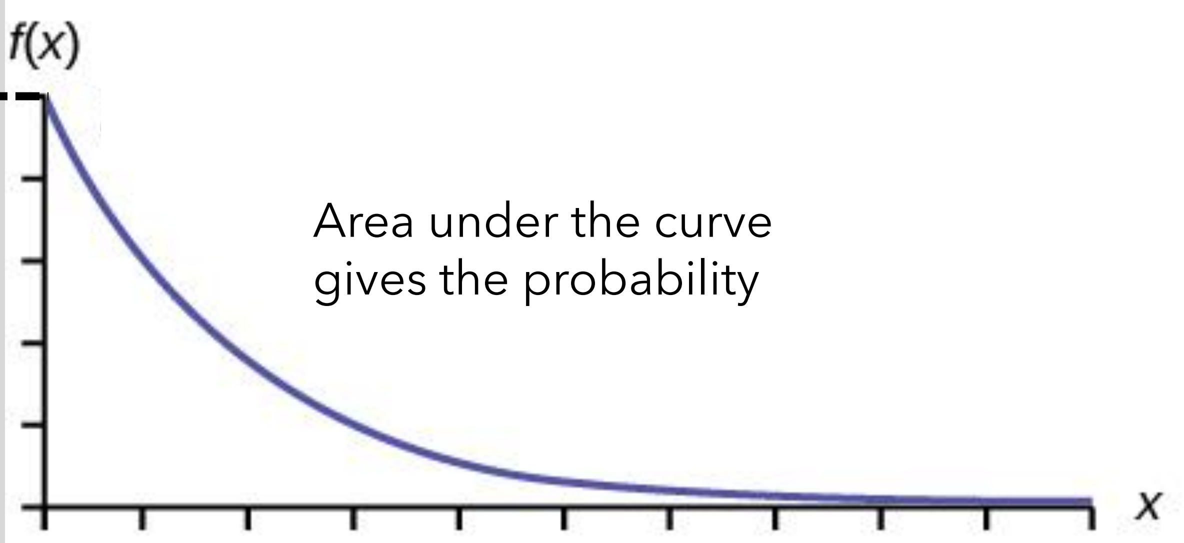
$$\boxed{\mu = \sigma}$$

If a continuous random variable follows an exponential distribution, it can be written as $X \sim \mathrm{Exp}(\lambda)$.

The Exponential Distribution

λ is the maximum value

$$f(x) = \lambda e^{-\lambda x}$$

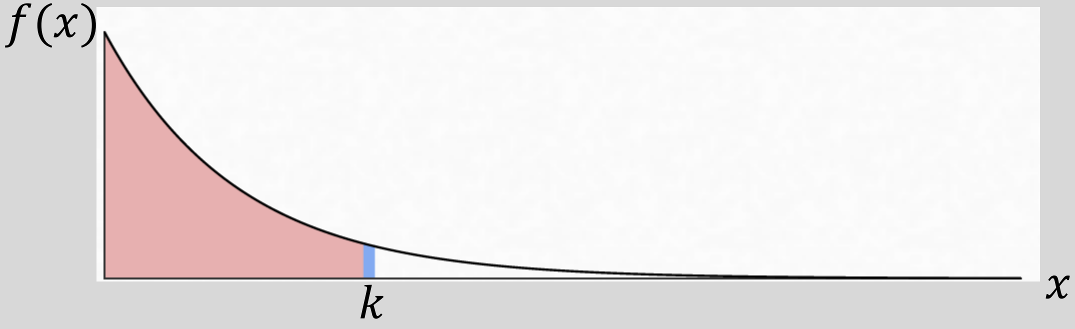


Area under the curve gives the probability

The Exponential Distribution

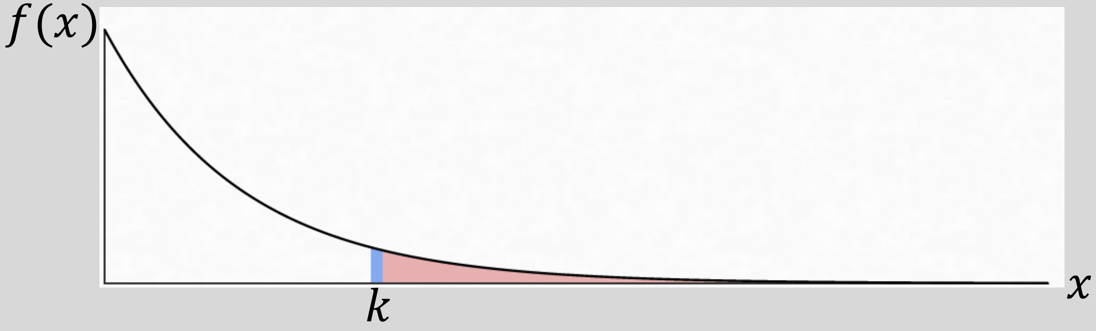
Area to the left (cdf)

$$P(x < k) = 1 - e^{-\lambda k}$$

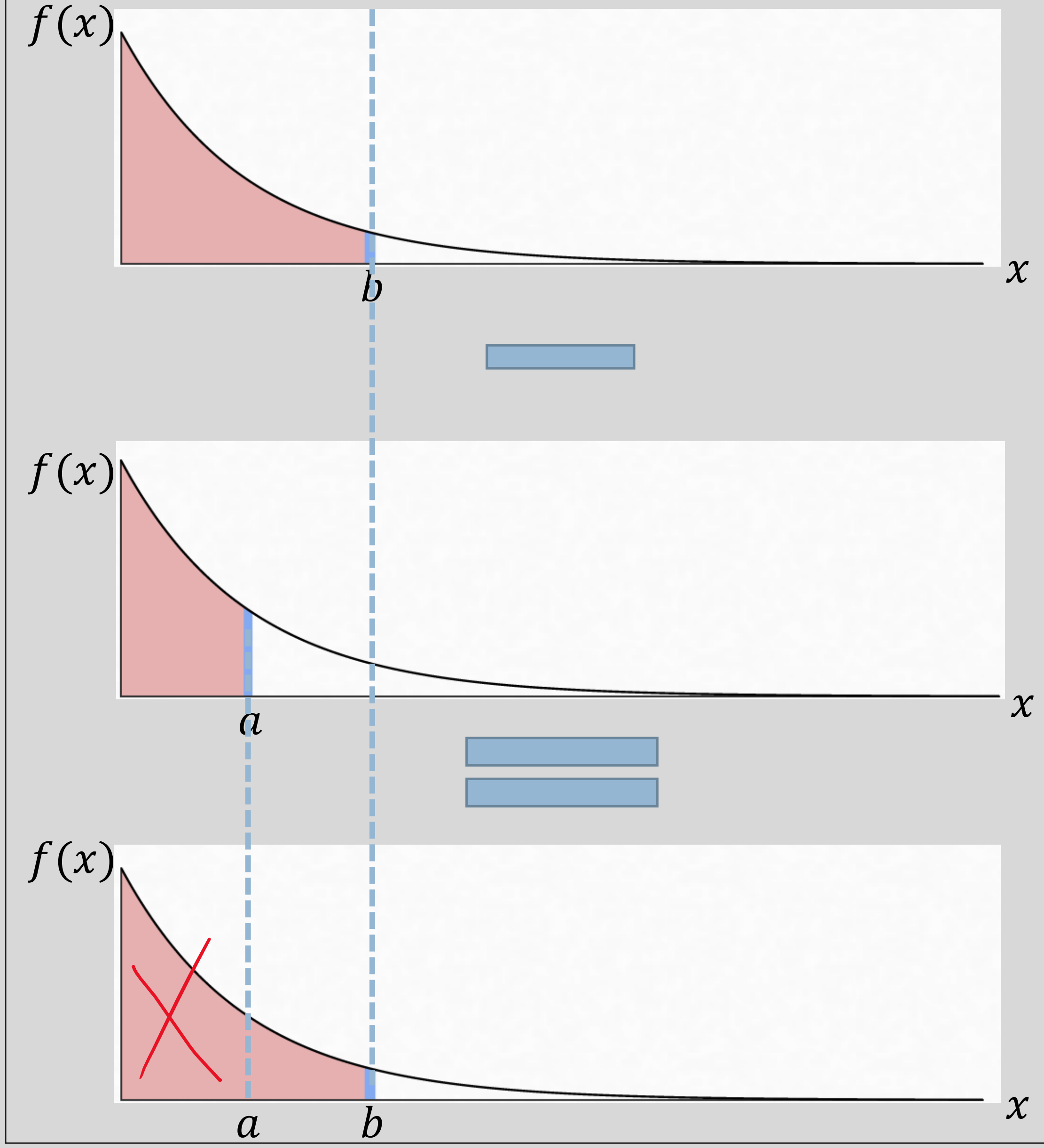


Area to the right

$$P(x < k) = e^{-\lambda k}$$



The Exponential Distribution



$$P(a < x < b)$$

To find the probability along an interval, use the cdf to calculate the probability of $P(x < a)$ and $P(x < b)$. Subtract off the lower interval.

$$P(a < x < b) = P(x < b) - P(x < a) = \left(1 - e^{-\lambda b}\right) - \left(1 - e^{-\lambda a}\right) =$$

$$P(a < x < b) = e^{-\lambda a} - e^{-\lambda b}$$

Example

The time spent waiting between events is often modeled using the exponential distribution. Suppose that an average of 30 customers per hour arrive at a store and the time between arrivals is exponentially distributed.

1) On average, how many minutes elapse between two successive arrivals?

   30 customers per hour = 30 customers per 60 minutes = 0.5 customers per minute (Or, 1 customer every 2 minutes)

2) When the store first opens, how long (on average) does it take for three customers to arrive?

   Since 1 customer arrives every 2 minutes, it will take ~6 minutes for three customers to arrive. Each arrival "resets" the clock.

3) After a customer arrives, find the probability that it takes less than 1 minute for the next customer to arrive.

   $\mu = 2 \text{ minutes}; \lambda = \frac{1}{\mu} = 0.5$

   Let $X \sim \text{Exp}(0.5)$ and $X = \text{the time between arrivals (in minutes)}.$
   $P(x < 1) = 1 - e^{-0.5 \cdot 1} \approx 0.3935$

Example (cont.)

The time spent waiting between events is often modeled using the exponential distribution. Suppose that an average of 30 customers per hour arrive at a store and the time between arrivals is exponentially distributed.

4) After a customer arrives, find the probability that it takes more than 5 minutes for the next customer to arrive.

   $P(x > 5) = e^{-0.5 \cdot 5} \approx 0.0821$

5) 70% of the customers arrive within k minutes of the previous customer. (Find the 70th percentile.)

   $0.7 = P(x < k)$
   $0.7 = 1 - e^{-0.5k}$
   $k = 2.41$ minutes

“Memorylessness” of the Exponential Distribution

$$P(X > r + t \mid X > r) = P(X > t)\ \forall\, r, t \geq 0$$

The memoryless property says that knowledge of what has occurred in the past has no effect on future probabilities.

Using the previous example:

If 5 minutes have elapsed since the last customer arrived, then the probability of that more than one minute will elapse before the next customer arrives is…
$P(X > 6 \mid X > 5)$
$P(X > 5 + 1 \mid X > 5) = P(X > 1) = e^{-0.5 \cdot 1} \approx 0.6065$

This conditional probability is the same as a “simple” probability. What already happened doesn’t affect the future outcomes.

Relationship Between the Poisson and the Exponential Distribution

If the time between events follows an exponential distribution with a mean of μ and the times are independent, then the number of events per unit time follows a Poisson distribution with mean λ.

Exponential Distribution: time between events

Poisson Distribution: number of events per time

Example

At a police station in a large city, calls come in at an average rate of 4 calls per minute. Assume that the time that elapses from one call to the next follows an exponential distribution. Thus, the total number of calls received during a time period follows a Poisson distribution.

Find the probability that after a call is received, the next call that comes in occurs in less than 10 seconds.

This is asking about the time between events. (exponential distribution)

$T = \text{time between calls}$

4 calls per minute means ~0.25 minutes between calls

$\mu = 0.25; \lambda = \frac{1}{\mu} = \frac{1}{0.25} = 4$

Thus $T \sim \text{Exp}(4)$.

$P\left(T < \frac{1}{6}\right) = 1 - e^{-4\cdot\left(\frac{1}{6}\right)} \approx 0.4866$

Example

At a police station in a large city, calls come in at an average rate of 4 calls per minute. Assume that the time that elapses from one call to the next follows an exponential distribution. Thus, the total number of calls received during a time period follows a Poisson distribution.

Find the probability that 5 calls occur within a minute.

This is asking about the number of events per time. (Poisson distribution)

X = The number of calls per minute

Thus $X \sim \mathrm{P_d}(4)$.

$P(x = 5) = 0.1563$

A Quick Review

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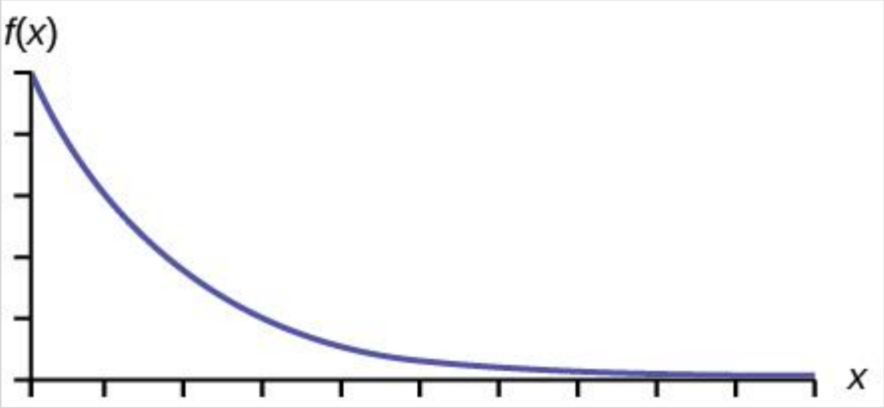

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