Lecture 7.2 – Hooke's Law

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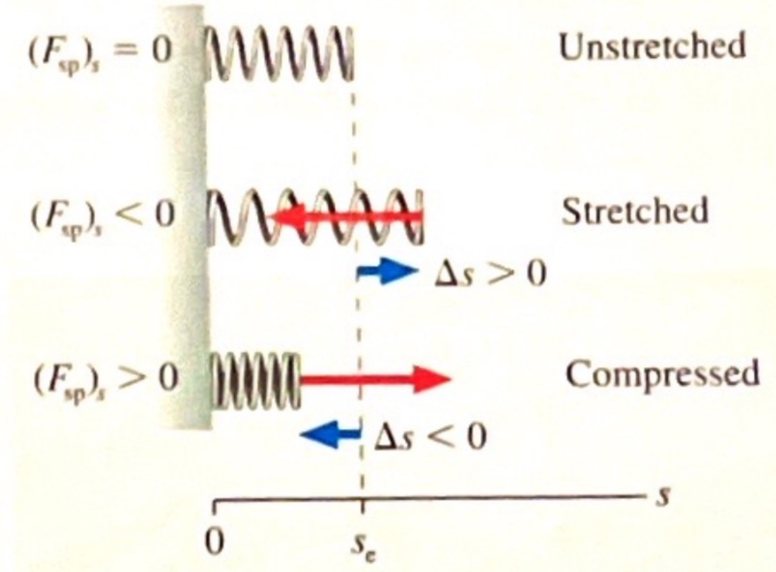
Restoring Forces

- A force that restores a system to an equilibrium position is called a *restoring force*.
- Systems that demonstrate restoring forces are called *elastic*.
- Let's examine an example with a spring. Suppose there is a spring that naturally sets at length s_e. This is the length when the spring is neither stretched or compressed.
- If you *stretch* the spring to length $s_{(+)}$, it will pull back toward equilibrium s_e, if you *compress* the string to $s_{(-)}$, it will push back to s_e.

FIGURE 10.16 The direction of $\vec{F}_{sp}$ is always opposite the displacement $\Delta\vec{s}$.



Example

Suppose we hang a "massless" spring vertically such that its equilibrium position is L_0.

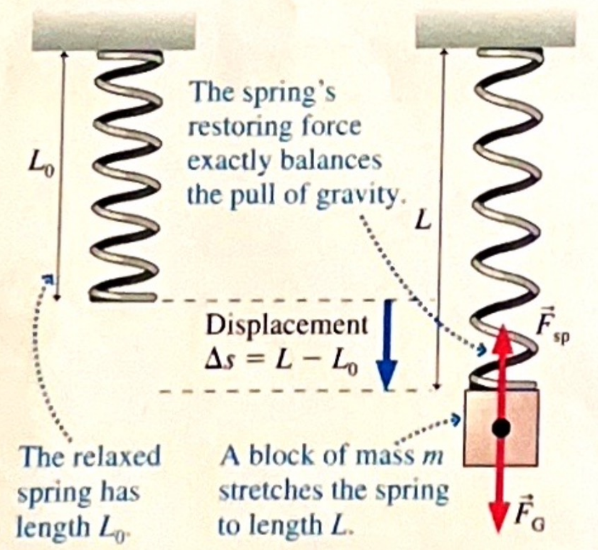
If we attach a mass to it, the string will stretch downward. The system will reach a new equilibrium position at L.

This means that the weight of the mass is causing the spring to resist with an equal and opposite force: $\sum F_{\text{net}} = F_{sp} - F_g = 0$. So, $F_{sp} = F_g$.

The mass is responsible for a certain amount of displacement: $\Delta s = L - L_0$

Increasing the mass will increase the displacement: $F_{sp} = F_g = mg$.

FIGURE 10.14 A hanging mass stretches a spring of equilibrium length L_0 to length L.



Discovering Hooke's Law

If we increase the mass in increments, we find that the displacement also increases.

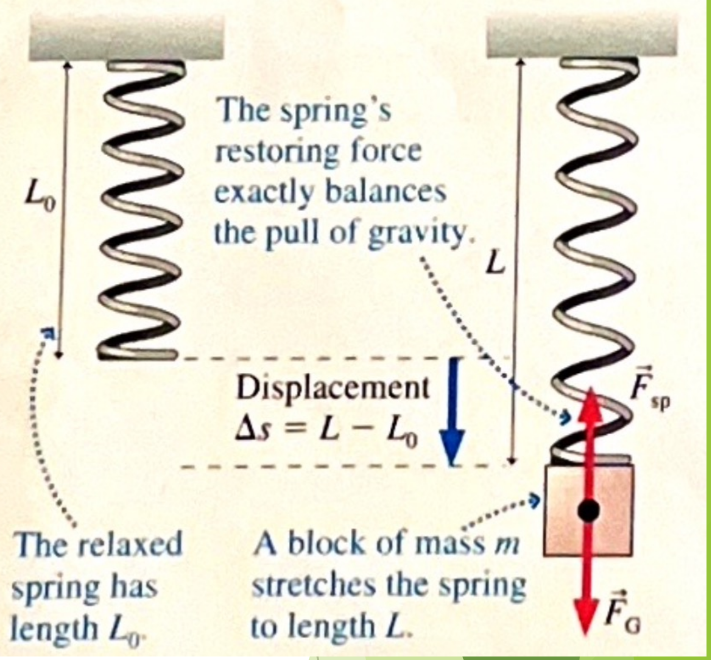
If we plot the force exerted by the spring vs. the displacement, we will find a linear relationship.

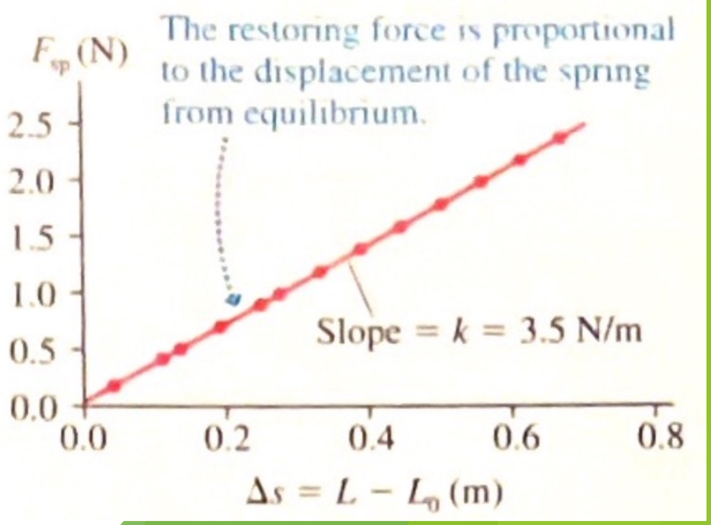
$$y = mx + b$$
$$F_g = F_{sp} = k\Delta s + 0$$

The restoring force provided by the spring is proportional to the amount the spring is displaced.

Here, k is the *spring constant*. [units: $\mathrm{N/m}$]

FIGURE 10.14 A hanging mass stretches a spring of equilibrium length L_0 to length L.





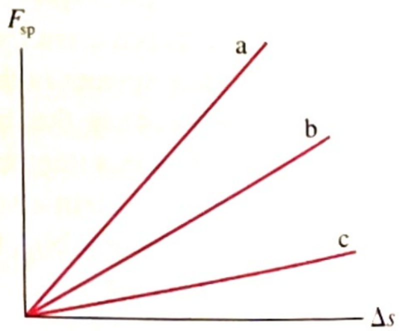
Discovering Hooke’s Law

Note: the force does not depend on the spring’s equilibrium length (L_0) but its *displacement* from equilibrium (Δs).

If k is large, the spring is “stiff” and it is hard to stretch.

If k is small, the spring is “soft” and it is easy to stretch.

STOP TO THINK 10.4 The graph shows force versus displacement for three springs. Rank in order, from largest to smallest, the spring constants k_a, k_b, and k_c.



Remember, the direction of $\vec{F}_{sp}$ always opposes the direction of displacement!

A stretched spring (+ direction) wants to return to equilibrium (- direction).

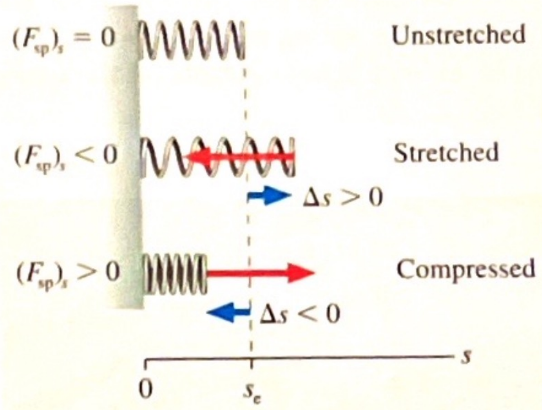
A compressed spring (- direction) wants to return to equilibrium (+ direction).

We can represent this by adding a negative sign:

$$\vec{F}_{sp} = -k\Delta s$$

This is *Hooke’s Law*.

FIGURE 10.16 The direction of $\vec{F}_{sp}$ is always opposite the displacement $\Delta\vec{s}$.



Elastic Potential Energy

- A spring force is a *variable* force. The force is zero if there is no displacement and it steadily increases as the stretching/compressing increases.

- Using Newton's Second Law:

$$F_{net} = ma_s = m\frac{dv_s}{dt} = -k\Delta s = -k(s - s_e)$$

$$m\frac{dv_s}{dt} = -k(s - s_e)$$

Using the chain rule again: $\frac{dv_s}{dt} = \frac{dv_s}{ds}\frac{ds}{dt} = v_s\frac{dv_s}{ds}$

$$mv_s\frac{dv_s}{ds} = -k(s - s_e)$$

$$mv_s dv_s = -k(s - s_e)ds$$

Integration

$$\int_{v_i}^{v_f} mv_s\, dv_s = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = -k\int_{s_i}^{s_f}(s - s_e)\, ds$$

Using a u-substitution: $u = s - s_e$ and $du = ds$

New limits: $u_f = s_f - s_e = \Delta s_f$ and $u_i = s_i - s_e = \Delta s_i$

$$-k\int_{s_i}^{s_f}(s - s_e)\, ds = -k\int_{\Delta s_i}^{\Delta s_f} u\, du = -\frac{1}{2}ku^2\Big|_{\Delta s_i}^{\Delta s_f}$$

$$-\frac{1}{2}ku^2\Big|_{\Delta s_i}^{\Delta s_f} = -\frac{1}{2}k\left(\Delta s_f\right)^2 - \left(-\frac{1}{2}k(\Delta s_i)^2\right)$$

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = -\frac{1}{2}k\left(\Delta s_f\right)^2 + \frac{1}{2}k(\Delta s_i)^2$$

Set up in "before" and "after" format:

$$\frac{1}{2}mv_i^2 + \frac{1}{2}k(\Delta s_i)^2 = \frac{1}{2}mv_f^2 + \frac{1}{2}k\left(\Delta s_f\right)^2$$

$U_s = \frac{1}{2}k(\Delta s)^2$ is defined as *elastic potential energy*.

$$K_i + U_{s,i} = K_f + U_{s,f}$$

Potential Energy Summary

An object moving on a spring obeys the equation:

$$K_i + U_{s,i} = K_f + U_{s,f}$$

Mechanical energy is conserved for an object moving *without friction* on an ideal spring.

An object on a spring can move vertically, thus:

$$\mathrm{E}_{\mathrm{mech}} = K + U_g + U_s$$

Example #1

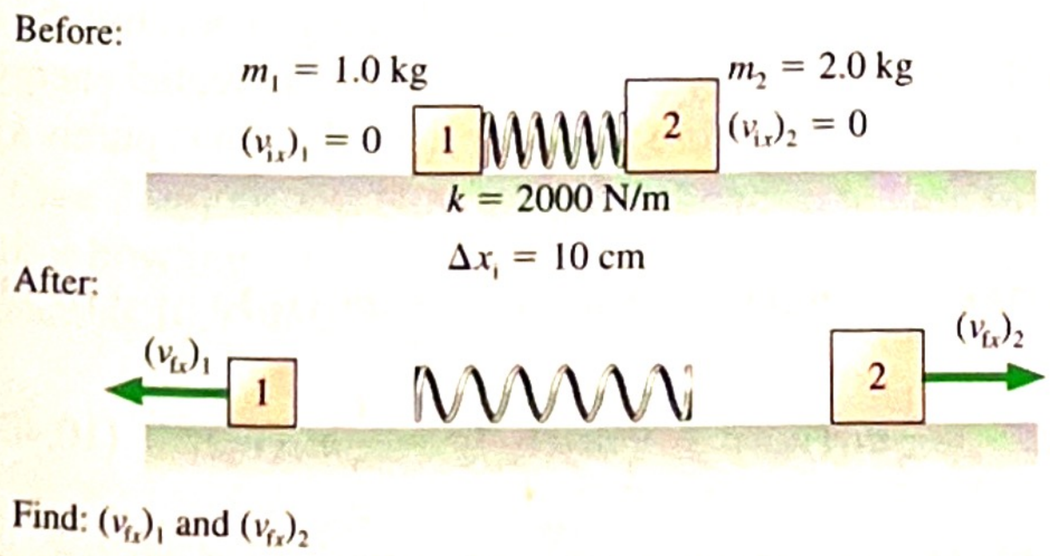
EXAMPLE 10.8 Pushing apart

A spring with spring constant 2000 N/m is sandwiched between a 1.0 kg block and a 2.0 kg block on a frictionless table. The blocks are pushed together to compress the spring by 10 cm, then released. What are the velocities of the blocks as they fly apart?

MODEL Assume an ideal spring that obeys Hooke's law. There's no friction; hence the mechanical energy $K + U_s$ is conserved. In addition, because the blocks and spring form an isolated system, their total momentum is conserved.

VISUALIZE **FIGURE 10.23** is a pictorial representation.

FIGURE 10.23 Pictorial representation of the blocks and spring.



- The system is defined as: m_1, m_2, and the spring.
- There is no friction.
- We are finding two velocities: $v_{f,1}$ and $v_{f,2}$.
- Since there are two unknowns, two equations are needed!
- In this system, momentum AND energy are conserved!

$$\vec{P}_i = \vec{P}_f$$
$$p_{1,i} + p_{2,i} = p_{1,f} + p_{2,f}$$

$$K_i + U_{s,i} = K_f + U_{s,f}$$
$$\frac{1}{2}m_1 v_{1,i}^2 + \frac{1}{2}m_2 v_{2,i}^2 + \frac{1}{2}k(\Delta x_i)^2 = \frac{1}{2}m_1 v_{1,f}^2 + \frac{1}{2}m_2 v_{2,f}^2 + \frac{1}{2}k(\Delta x_f)^2$$

Example #1

Momentum

$$\vec{P}_i = \vec{P}_f$$
$$p_{1,i} + p_{2,i} = p_{1,f} + p_{2,f}$$
$$m_1 v_{1,i} + m_2 v_{2,i} = m_1 v_{1,f} + m_2 v_{2,f}$$

Initial momentum is zero for both blocks.

$$0\ \mathrm{kg}\frac{\mathrm{m}}{\mathrm{s}} = m_1 v_{1,f} + m_2 v_{2,f}$$

Solve for one of the velocities:

$$v_{1,f} = -\frac{m_2}{m_1}\left(v_{2,f}\right)$$

Energy

$$\mathrm{E}_{\mathrm{mech},i} = \mathrm{E}_{\mathrm{mech},f}$$
$$K_i + U_{s,i} = K_f + U_{s,f}$$
$$\frac{1}{2}m_1 v_{1,i}^2 + \frac{1}{2}m_2 v_{2,i}^2 + \frac{1}{2}k(\Delta x_i)^2 = \frac{1}{2}m_1 v_{1,f}^2 + \frac{1}{2}m_2 v_{2,f}^2 + \frac{1}{2}k\left(\Delta x_f\right)^2$$

Initial kinetic energy is zero for both blocks.

$$\frac{1}{2}k(\Delta x_i)^2 = \frac{1}{2}m_1 v_{1,f}^2 + \frac{1}{2}m_2 v_{2,f}^2$$
$$k(\Delta x_i)^2 = m_1 v_{1,f}^2 + m_2 v_{2,f}^2$$

Use the definition of $v_{1,f}$ from momentum conservation to solve for $v_{2,f}$:

$$k(\Delta x_i)^2 = m_1\left(\frac{m_2}{m_1}\left(v_{2,f}\right)\right)^2 + m_2 v_{2,f}^2 = \left(\frac{m_2^2}{m_1} + m_2\right)v_{2,f}^2$$
$$v_{2,f} = \sqrt{\frac{k(\Delta x_i)^2}{m_2\left(1 + \frac{m_2}{m_1}\right)}}$$

Example #1

Energy

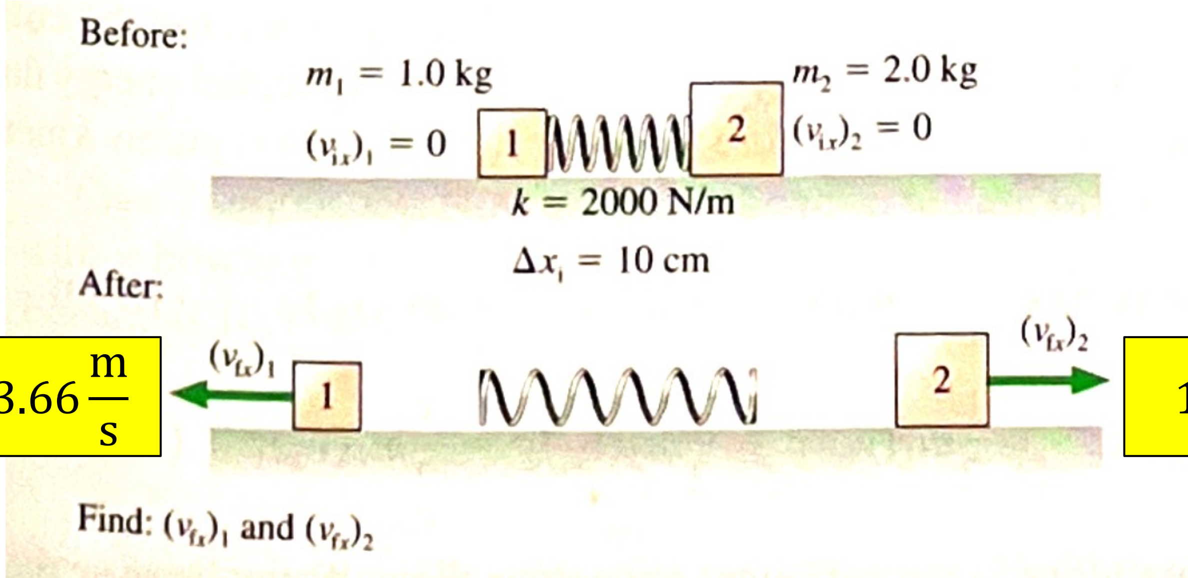
$$v_{2,f} = \sqrt{\frac{k(\Delta x_i)^2}{m_2\left(1+\frac{m_2}{m_1}\right)}}$$

$$v_{2,f} = \sqrt{\frac{2000\,\frac{\text{N}}{\text{m}}\cdot(0.1\text{m})^2}{2.0\text{kg}\left(1+\frac{2.0\text{kg}}{1.0\text{kg}}\right)}} \approx 1.83\,\frac{\text{m}}{\text{s}}$$

Momentum

$$v_{1,f} = -\frac{m_2}{m_1}\left(v_{2,f}\right)$$

$$v_{1,f} = -\frac{2.0\text{kg}}{1.0\text{kg}}\left(1.83\,\frac{\text{m}}{\text{s}}\right) = -3.66\,\frac{\text{m}}{\text{s}}$$



$\boxed{-3.66\,\frac{\text{m}}{\text{s}}}$

$\boxed{1.83\,\frac{\text{m}}{\text{s}}}$

Summary

- Restoring forces seek to bring a system back to equilibrium.
- Hooke's Law (springs):

$$\vec{F}_{sp} = -k\Delta s$$

- Spring constant: k
  - Large k is a "stiff" spring
  - Small k is a "soft" spring
- Elastic Potential Energy:

$$U_s = \frac{1}{2}k(\Delta s)^2$$

- Mechanical Energy:

$$\mathrm{E}_{\mathrm{mech}} = K + U_g + U_s$$