Topics in Probability

MAT 152 – STATISTICAL METHODS I
LECTURE 1
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Definitions

Probability is a measure that is associated with how certain we are of outcomes of an experiment.

An experiment is a planned operation carried out under controlled conditions.

- Ex: rolling a dice, flipping a coin, etc.
- A chance-based experiment does not have a predetermined outcome.
- Outcomes can be equally likely: fair coin
- Or, there can be a bias: trick coin

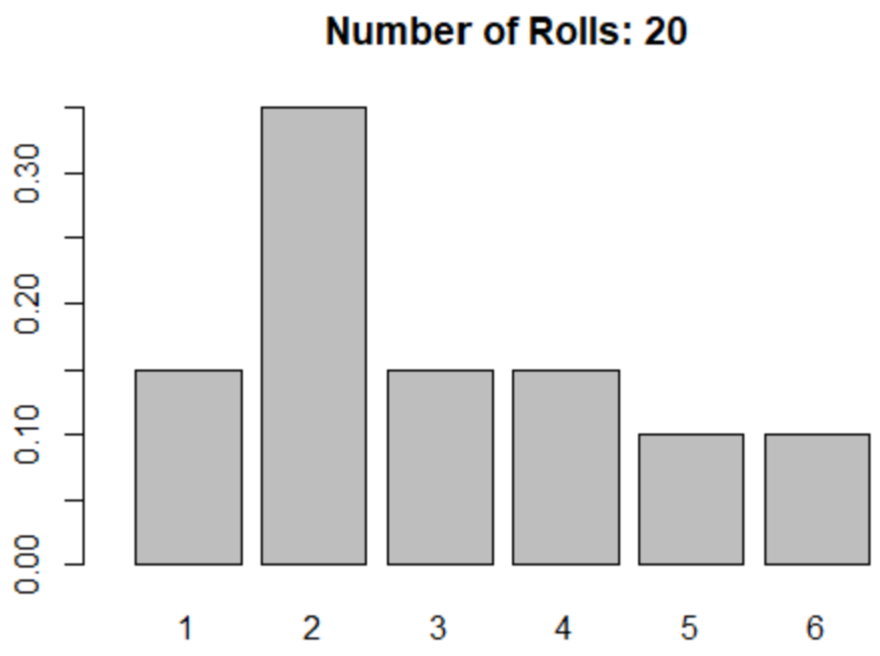
The probability of an outcome is the long-term relative frequency of that outcome. (Probabilities are between 0 and 1).

Rolling a Dice…

A fair dice was rolled 20 times. The relative frequency distribution is included in the graph.

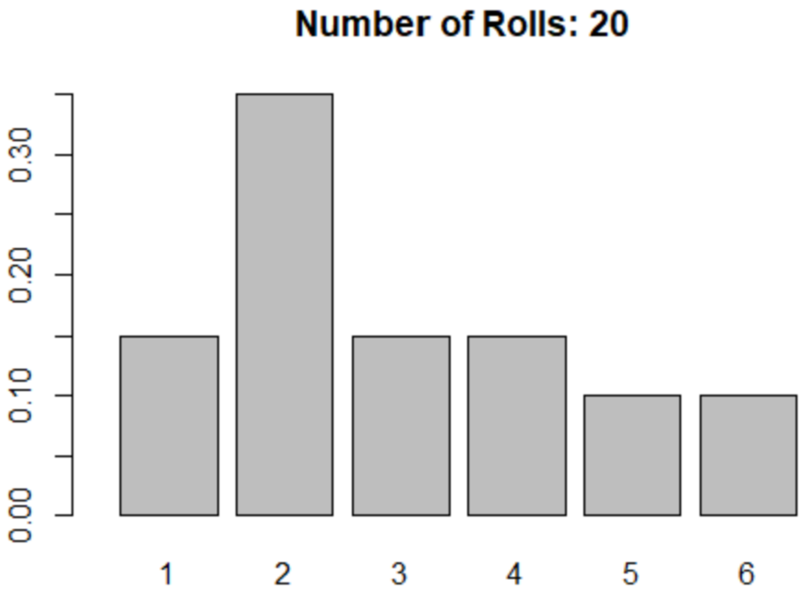
Clearly, the dice appears to be biased, producing a “2” more frequently than any other value.

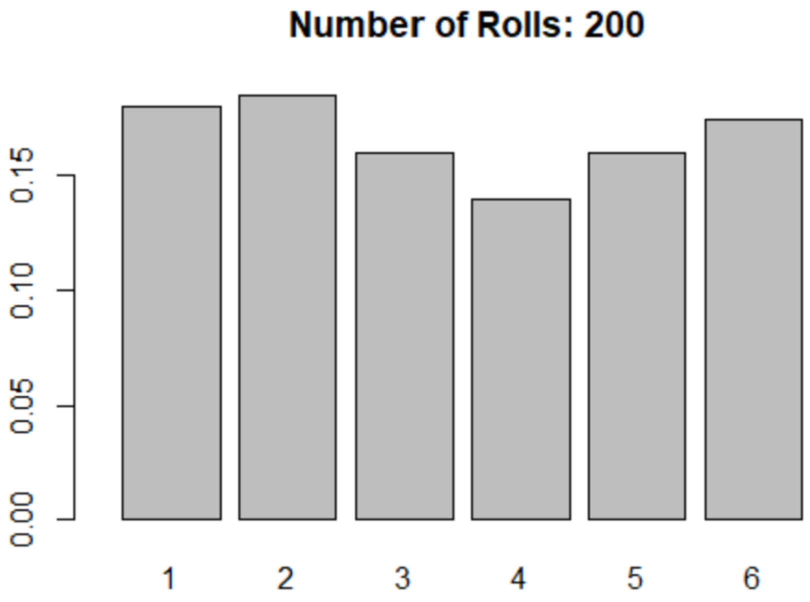
Number Rolled	Frequency	Relative Frequency
1	3	3/20 = 0.15 (15%)
2	7	7/20 = 0.35 (35%)
3	3	3/20 = 0.15 (15%)
4	3	3/20 = 0.15 (15%)
5	2	2/20 = 0.1 (10%)
6	2	2/20 = 0.1 (10%)
Total =	20	1

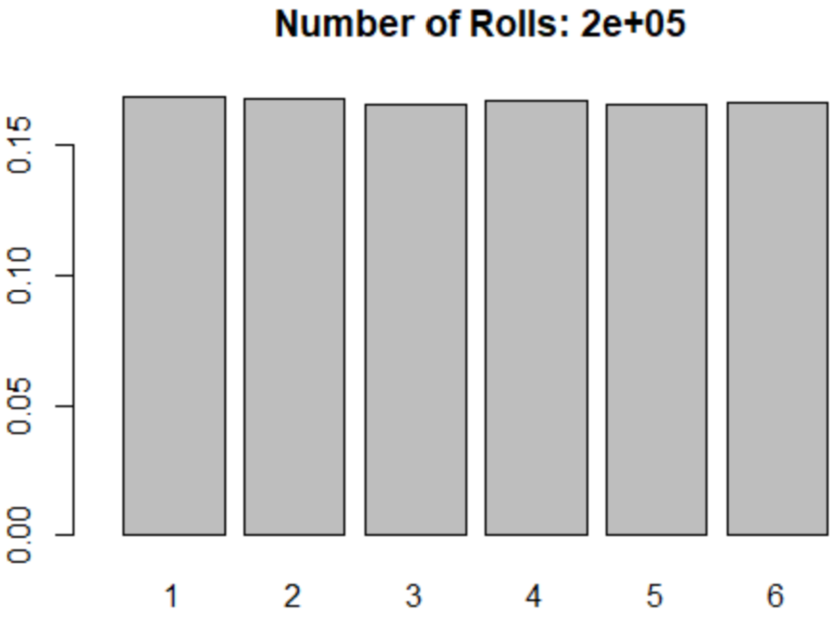


Rolling a Dice…

What if the “fair” dice was rolled more than 20 times? (20, 200, 200000 times)





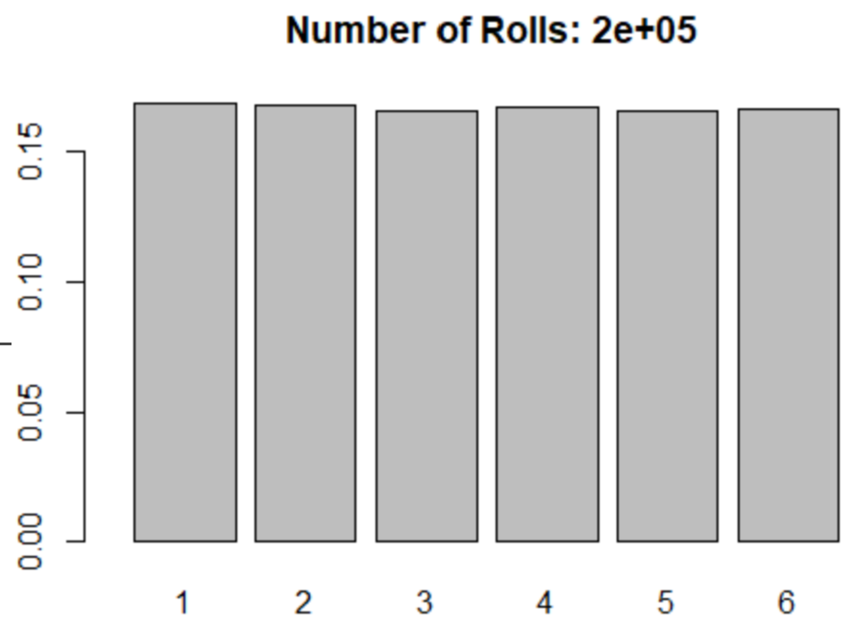


Rolling a Dice

For a fair dice, the probability of rolling any number (1-6) is 0.166667. As the rolls pass 200,000 this value becomes more obvious in the data.

The probability is the long-term relative frequency of an outcome.

As an experiment is repeated, the relative frequency gets closer to the theoretical probability. This is known as the Law of Large Numbers.



Number Rolled	Frequency	Relative Frequency
1	33,645	0.168225
2	33,470	0.167350
3	33,100	0.165500
4	33,426	0.167130
5	33,158	0.165790
6	33,201	0.166005
Total =	200,000	1

A Quick Aside: Set Theory

In mathematics, a set is a collection of objects. For example, consider the set of the numbers 1-5.

$$S = \{1, 2, 3, 4, 5\}$$

Other sets can be defined such as:

$$A = \{1, 2\} \text{ and } B = \{5, 6, 7, 8\}$$

How can we talk about the sets *S*, *A*, and *B* together?

Here, intersections and unions can be calculated.

A Quick Aside: Set Theory

The <u>intersection</u> of two sets considers ONLY the values that are present in both sets.

The intersection is notated as “A and B” or $A \cap B$.

From the previous slide, $S = \{1, 2, 3, 4, 5\}$, $A = \{1, 2\}$, and $B = \{5, 6, 7, 8\}$.

$S \cap A = \{1, 2\}$ (1 and 2 are in both sets)

$S \cap B = \{5\}$ (5 is the only value in both sets)

$A \cap B = \emptyset$ ($\emptyset$ is the empty set. *A* and *B* do not share any values.)

The <u>union</u> of two sets considers the COMBINATION of the sets.

The union is notated as “A or B” or $A \cup B$.

$S \cup A = \{1, 2, 3, 4, 5\}$ (A is already contained in S. The values 1 and 2 are not repeated.)

$S \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}$ (S and B are combined. 5 is not repeated.)

$A \cup B = \{1, 2, 5, 6, 7, 8\}$

Sample Spaces and Events

Consider the fair dice from earlier. What is the set of all possible outcomes?

$$S = \{1, 2, 3, 4, 5, 6\}$$

This is the sample space.

Consider the two events: A and B

A = Rolling a 1 or 2; $A = \{1, 2\}$.

B = Rolling an odd number; $B = \{1, 3, 5\}$.

What is the probability of event A? P(A)=?

The events in event A divided by the total number of events.

$$P(A) = \frac{\{1,2\}}{\{1,2,3,4,5,6\}} = \frac{2}{6}$$

There is a 2/6 or 1/3 chance of event A occurring.

Sample Spaces and Events

Sample Space: $S = \{1, 2, 3, 4, 5, 6\}$
Rolling a 1 or 2: $A = \{1, 2\}$
Rolling an odd number: $B = \{1, 3, 5\}$

What is the probability of rolling a {1, 2} OR an odd number {1, 3, 5}? $A \cup B$

$$P(A \cup B) = \frac{\{1, 2, 3, 5\}}{\{1, 2, 3, 4, 5, 6\}} = \frac{4}{6} = \frac{2}{3}$$

What is the probability of rolling a value that satisfies events A and B? $A \cap B$
The value 1 is the only way to satisfy both requirements.

$$P(A \cap B) = \frac{\{1\}}{\{1, 2, 3, 4, 5, 6\}} = \frac{1}{6}$$

Conditional Probability

Suppose we roll the fair dice and it lands on an odd number. That means the event "B" has just occurred. What is the probability of A (landing on a 1 or 2) given that B just happened?

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{\{1\}}{\{1,2,3,4,5,6\}}}{\frac{\{1,3,5\}}{\{1,2,3,4,5,6\}}} = \frac{\frac{1}{6}}{\frac{3}{6}} = \frac{1}{3}$$

$P(A|B)$ is read as "what is the probability of A given B?"

Given that B just happened (a 1, 3, or 5 was rolled), only one of the three values meets the requirements for A (the value 1). Since B already happened, the sample space is reduced to {1, 3, 5}.

This is the <u>conditional probability</u>. A conditional reduces the sample space.

The Complement

Consider the sample space from earlier:

$S = \{1, 2, 3, 4, 5, 6\}$.

If $A = \{1, 2\}$. Then the complement of A is everything that is NOT in A.

$A' = \{3, 4, 5, 6\}$; A' is called "A prime".

Thus, the complement of B is all even numbers.

$B = \{1, 3, 5\}$; $B' = \{2, 4, 6\}$

Note the following:

$P(E) + P(E') = 1$ for any event.

$P(E \cap E') = 0$ for any event.

$P(E \cup E') = 1$ for any event.

The Addition Rule

Suppose $S = \{1, 2, 3, 4, 5\}$, $A = \{1, 2, 3\}$, and $B = \{3, 4, 5\}$.

If $P(A \cup B)$ is to be calculated, special care must be taken.

Remember that $A \cup B = \{1, 2, 3, 4, 5\}$ but "3" was only listed once even though it appears in both events. Therefore, this duplicate must be accounted for when finding $P(A \cup B)$.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Here, the probability of A OR B happening is P(A) plus P(B) minus any overlapping values. In our case, it is the value 3.

This is the <u>Addition Rule</u>.

The Multiplication Rule

Recall that the conditional probability of A given B is expressed as:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Using algebra, this statement can be rewritten as:

$$P(B) \cdot P(A|B) = P(A \cap B)$$

$P(A \cap B)$ is the probability of an outcome satisfying both events at the same time. Here, the probability of B occurring must first be addressed. Thus, the first part of this equation is $P(B)$. This is multiplied by the probability of A occurring on the reduced sample space. This reduced sample space represents the intersection of the two events.

This is the <u>Multiplication Rule</u>.

Independent Events

Two events, A and B, are said to be independent if the occurrence of one does not affect the occurrence of the other.

Suppose two coins are tossed. The sample space is given by: $S = \{HH, HT, TT, TH\}$

A = A heads on the first flip; $A = \{HH, HT\}$

B = A heads on the second flip; $B = \{HH, TH\}$

$$P(A) = \frac{\{HH,HT\}}{\{HH,HT,TT,TH\}} = \frac{2}{4} = \frac{1}{2} \text{ and } P(B) = \frac{\{HH,TH\}}{\{HH,HT,TT,TH\}} = \frac{2}{4} = \frac{1}{2}$$

What about $P(B|A)$? That is, what is the probability of B given that A has already happened?

If a heads occurred on the first flip, does this influence the probability of a second heads?

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{4}}{\frac{2}{4}} = \frac{1}{2}$$

Since $P(B|A) = P(B)$, the outcome of event A has no effect on B. Thus, the events are independent.

Mutually Exclusive Events

Events A and B are said to be mutually exclusive if no outcome can satisfy both requirements.

Consider rolling a fair dice. $S = \{1, 2, 3, 4, 5, 6\}$

Event A = Rolling an odd number; $A = \{1, 3, 5\}$

Event B = Rolling an even number; $B = \{2, 4, 6\}$

Since $A \cap B = \emptyset$, $P(A \cap B) = 0$. These events are mutually exclusive.

Summary

INDEPENDENT EVENTS

The occurrence of one event does not affect the outcome of another.

For independent events, the following are true:

$P(A|B) = P(A)$
$P(B|A) = P(B)$
$P(A \cap B) = P(A)P(B)$

MUTUALLY EXCLUSIVE EVENTS

If events A and B cannot occur at the same time, then $P(A \cap B) = 0$.

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ is reduced to the following:

$P(A \cup B) = P(A) + P(B)$