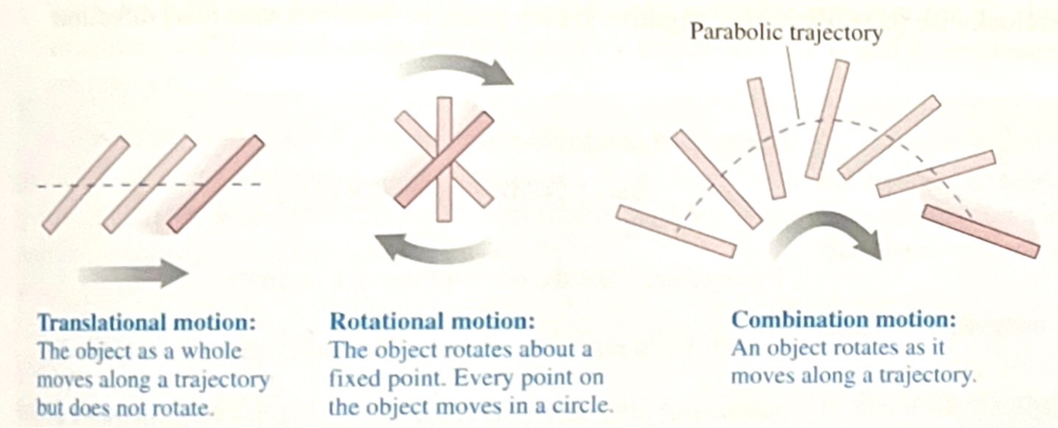
# Lecture 9.1 – Rotational Dynamics

Dustin Roten, Ph.D.
Wilkes Community College
Fall 2023

# Rotational Motion

- The particle model does not incorporate characteristics of the object itself.
- For extended objects, their size and shape do matter.
- A *rigid body* is an extended object whose size and shape do not change during movement.
- There are three basic types of rigid body motion:
  - Translational
  - Rotational
  - Combination

FIGURE 12.2 Three basic types of motion of a rigid body.



# Rotation About the Center of Mass

- If an object is spun without any translational motion, it will naturally rotate around its *center of mass*.

- The center of mass (C.O.M.) remains motionless while every other point on the object undergoes circular motion around it.

- How is the location of the C.O.M. calculated?
  1. Treat the object like a collection of small particles where each one has a mass, $m_i$, at positions $x_i$.
  2. Calculate a weighted average of these particles.

$$x_{\mathrm{cm}} = \frac{1}{M}\sum_{i=1}^{N} m_i x_i = \frac{m_1 x_1 + \cdots + m_N x_N}{m_1 + \cdots + m_N}$$

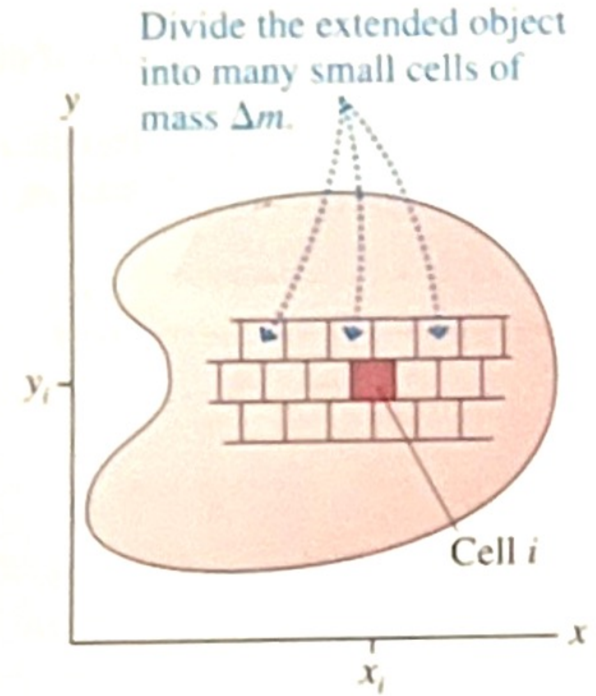
$$y_{\mathrm{cm}} = \frac{1}{M}\sum_{i=1}^{N} m_i y_i = \frac{m_1 y_1 + \cdots + m_N y_N}{m_1 + \cdots + m_N}$$

- $M = \sum_i m = m_1 + \cdots + m_N$ is the total mass of the object.

$$(x_{\text{cm}}, y_{\text{cm}}) = \left(\frac{1}{M}\sum_{i=1}^{N} m_i x_i \,, \frac{1}{M}\sum_{i=1}^{N} m_i y_i\right)$$

# Rotation About the Center of Mass

FIGURE 12.7 Calculating the center of mass of an extended object.




- Particles of larger mass count more than particles of smaller mass.
- C.O.M. is the *mass-weighted center* of the object.

- For a continuous object, it can be made up of infinitesimally small masses, $\Delta m_i$. Thus, $x_{\text{cm}}$ and $y_{\text{cm}}$ become integrals:

$$x_{\text{cm}} = \lim_{\Delta m_i \to 0} \frac{1}{M} \sum_i x_i \Delta m_i = \frac{1}{M}\int x\, dm$$

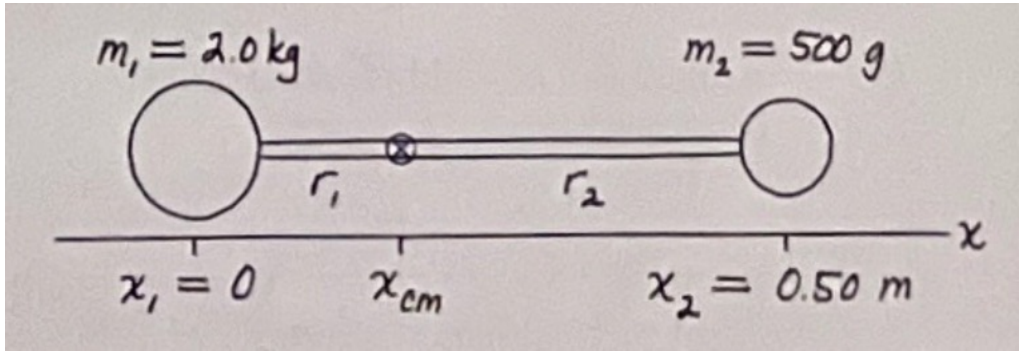
$$y_{\text{cm}} = \lim_{\Delta m_i \to 0} \frac{1}{M} \sum_i y_i \Delta m_i = \frac{1}{M}\int y\, dm$$

- This is the formal definition of C.O.M. but the $dm$ must be changed. Integrals are carried out over coordinates, NOT masses.

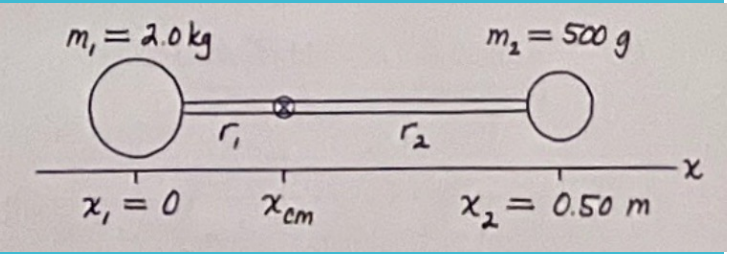
# Example #1

**A $500\text{g}$ ball and a $2.0\text{kg}$ ball are connected by a massless 50-cm-long rod.**

- Where is the center of mass?
- What is the speed of each ball if they rotate about the center of mass at $40\text{rpm}$?



# Example #1




(a) Where is the center of mass?

$$x_{\text{cm}} = \frac{1}{M}\sum_{i=1}^{2} m_i x_i = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{(2.0\text{kg})(0\text{m}) + (0.5\text{kg})(0.5\text{m})}{2.5\text{kg}} = 0.1\text{m}$$

The center of mass is closer to the heavier ball.

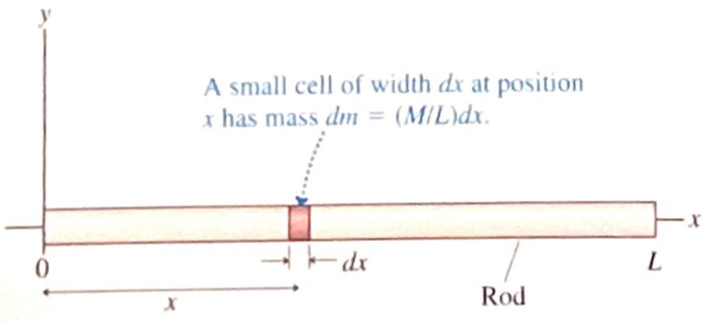
(b) First, convert 40rpm to rad/s.

$$\omega = \frac{40\text{ rev}}{1\text{ min}} \cdot \frac{1\text{ min}}{60\text{ sec}} \cdot \frac{2\pi\text{ rad}}{1\text{ rev}} \cdot 4.19\frac{\text{rad}}{\text{s}}$$

$$\text{v}_{\text{t},1} = \text{r}_1\omega = (0.1\text{m})\left(4.19\frac{\text{rad}}{\text{s}}\right) \approx 0.42\frac{\text{m}}{\text{s}}$$

$$\text{v}_{\text{t},2} = \text{r}_2\omega = (0.4\text{m})\left(4.19\frac{\text{rad}}{\text{s}}\right) \approx 1.68\frac{\text{m}}{\text{s}}$$

# Example #2




$$\lambda(x) = \lambda_0\left(1 - \frac{x}{2L}\right)$$

$$\lambda(0) = \lambda_0$$

$$\lambda(L) = \frac{\lambda_0}{2}$$

**Consider a thin non-uniform rod of length *L* that loses mass such that its linear mass density ($\lambda$) is $\lambda_0$ on one end and $\frac{\lambda_0}{2}$ on the other. Determine the location of its center of mass.**

The linear mass density [$\lambda(x)$] tells us how the density/mass of the rod changes across its length: $\lambda(x) = \frac{\Delta m}{\Delta x} = \frac{dm}{dx}$ [units: $\mathrm{kg/m}$]

First, we need the total mass of the rod. Since the density is always changing across the rod, we can sum small "chunks" to get the total mass, *M*.

$$M = \lim_{\Delta m_i \to 0} \sum_{i=1}^{N} \Delta m_i = \lim_{\Delta x_i \to 0} \sum_{i=1}^{N} \lambda(x_i)\Delta x_i = \int_0^L \lambda(x)dx$$

# Example #2

Calculate the total mass of the object:

$$M = \int_0^L \lambda(x)dx = \lambda_0 \int_0^L \left(1 - \frac{x}{2L}\right) dx = -2L\lambda_0 \int_1^{\frac{1}{2}} u\, du = -L\lambda_0 u^2 \Big|_1^{\frac{1}{2}} = \frac{3}{4} L\lambda_0$$

$$u = 1 - \frac{x}{2L}$$

$$-2L du = dx$$

Now, we calculate the center of mass:

$$x_{\mathrm{cm}} = \frac{1}{M} \int x\, dm = \frac{\lambda_0}{M} \int_0^L \left(x - \frac{x^2}{2L}\right) dx = \frac{\lambda_0}{M}\left[\int_0^L x\, dx - \frac{1}{2L}\int_0^L x^2\, dx\right]$$

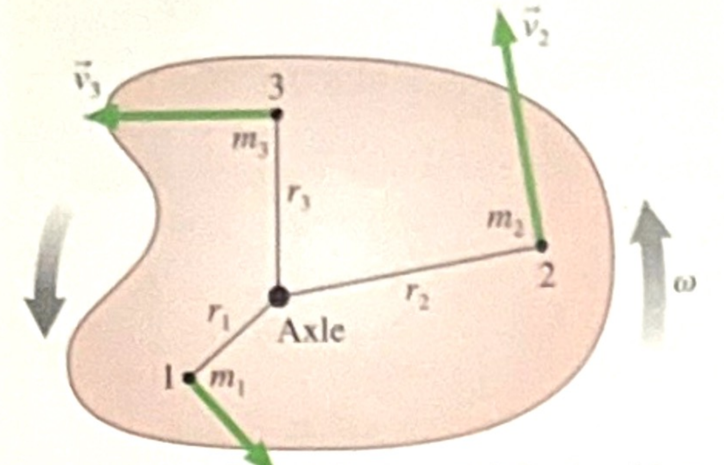
$$\frac{\lambda_0}{M}\left[\int_0^L x\, dx - \frac{1}{2L}\int_0^L x^2\, dx\right] = \frac{\lambda_0}{2M} x^2 \Big|_0^L - \frac{\lambda_0}{6ML} x^3 \Big|_0^L = \frac{\lambda_0 L^2}{3M} = \frac{\lambda_0 L^2}{3\left(\frac{3}{4} L\lambda_0\right)} = \frac{4}{9} L$$

(ALMOST in the center but weighted toward the denser end.)

For any symmetrical object of uniform density, the C.O.M. is at the physical center of the object.

The kinetic energy due to rotation is called *rotational kinetic energy*.

# Rotational Energy




Each particle in the object has kinetic energy as the object rotates.

Consider a nonuniform object rotating about an axle:

- Although it is a continuous object, lets first consider three "pieces" of it.
- The object's rotational kinetic energy is the sum of the kinetic energies of ALL its pieces. $K_{\text{rot}} = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}m_3v_3^2 + \cdots$

We see from the figure that each "piece" has a tangential velocity: $v_i = r_i\omega$. Note that $\omega$ is the same for all "pieces".

$$K_{\text{rot}} = \frac{1}{2}m_1r_1^2\omega^2 + \frac{1}{2}m_2r_2^2\omega^2 + \frac{1}{2}m_3r_3^2\omega^2 + \cdots$$

This can be simplified by pulling out the $\frac{1}{2}$ and $\omega$.

$$K_{\text{rot}} = \frac{1}{2}\left(\sum_{i=1}^{N} m_ir_i^2\right)\omega$$

$\sum m_ir_i^2$ is the object's *moment of inertia*. It has units of $kg\ m^2$. An object's moment of inertia depends on the location of the axis of rotation (*r* dependence).

# Rotational Energy

The moment of inertia $I = \sum m_i r_i^2$ is the rotational equivalent of mass. Objects with large values of $I$ are more "resistant" to rotation. This is like objects with large mass that are hard to accelerate.

An object's moment of inertia depends not only on the object's mass but also on how the mass is *distributed about the axis of rotation*.

Using $I = \sum m_i r_i^2$, the rotational kinetic energy can be written as $K_{\text{rot}} = \frac{1}{2}\left(\sum m_i r_i^2\right)\omega^2 = \frac{1}{2}I\omega^2$.

Moments of inertia for many solid objects can be easily looked up. $I$ must be manually calculated for uncommon objects.

# Calculating $I$

While we can guess that the C.O.M. of a symmetrical object is at its physical center, "$I$" cannot be guessed.

If we take a solid object and break it up into many small pieces, the equation becomes:

$$I = \lim_{\Delta m \to 0} \sum_i r_i^2 \, \Delta m = \int r^2 \, dm$$

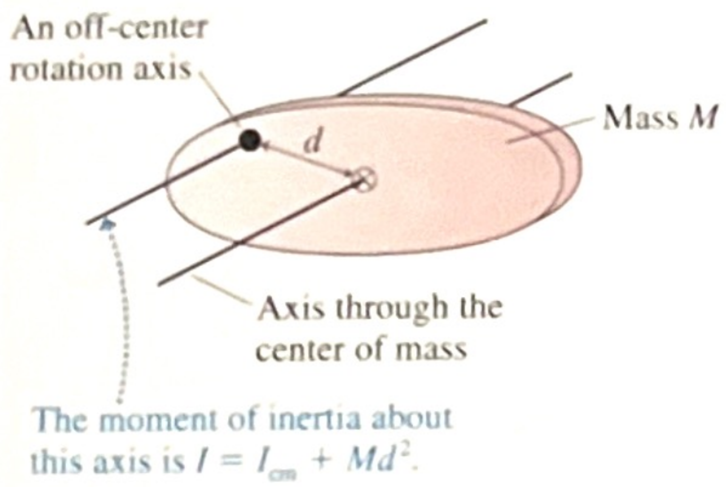
Here, $r$ is the distance from the axis of rotation and $dm$ must be rewritten in terms of coordinate differentials ($dx$ and/or $dy$).

If a complex object is divided into multiple pieces with moments of inertia $I_1, I_2, I_3, \ldots$ then they can be summed.

$$I_{\text{object}} = \sum m_i r_i^2$$

# Parallel Axis Theorem

FIGURE 12.16 Rotation about an off-center axis.




Suppose you need to know the moment of inertia for a rotation about an off-center axis. This can be found with the *parallel axis theorem*.

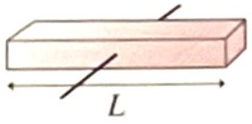
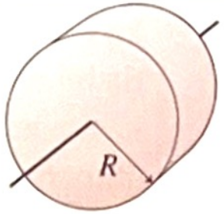
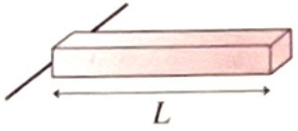
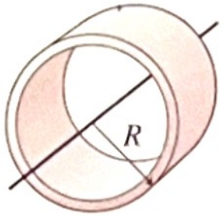
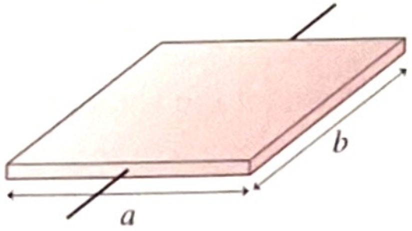
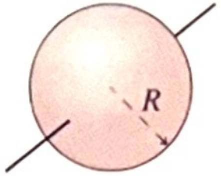
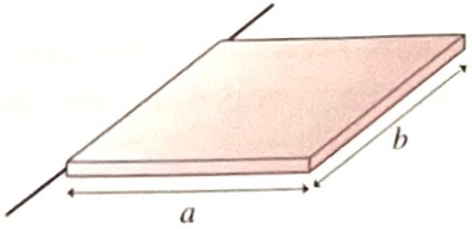
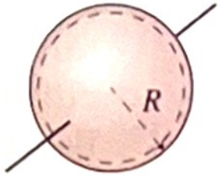
If the axis of rotation is parallel but offset from the axis of rotation through the C.O.M., then the new moment of inertia is simply:

$$I = I_{\mathrm{cm}} + Md^2$$

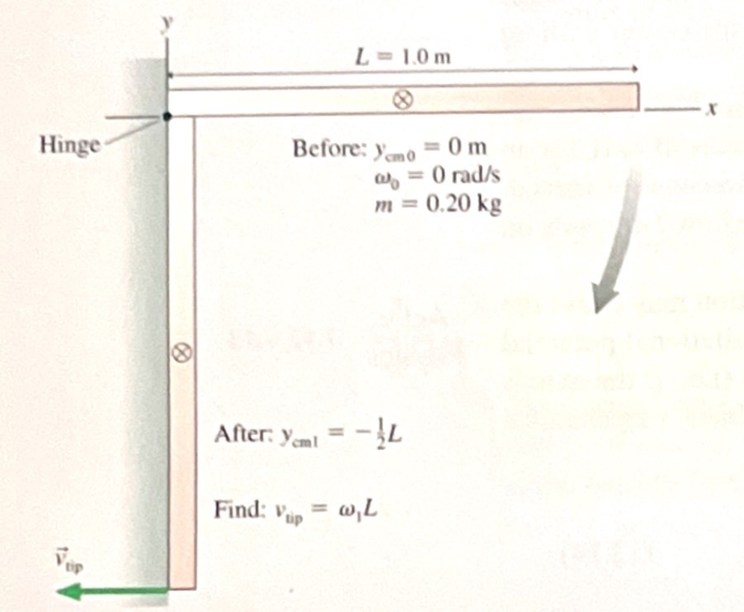
where $I_{\mathrm{cm}}$ is the known moment of inertia *if* the rotation is through the C.O.M., $M$ is the mass of the object and $d$ is the distance to the *actual* axis of rotation.

# Common Values of $I$

**TABLE 12.2** Moments of inertia of objects with uniform density

| Object and axis | Picture | $I$ | Object and axis | Picture | $I$ |
|---|---|---|---|---|---|
| Thin rod, about center |  | $\frac{1}{12}ML^2$ | Cylinder or disk, about center |  | $\frac{1}{2}MR^2$ |
| Thin rod, about end |  | $\frac{1}{3}ML^2$ | Cylindrical hoop, about center |  | $MR^2$ |
| Plane or slab, about center |  | $\frac{1}{12}Ma^2$ | Solid sphere, about diameter |  | $\frac{2}{5}MR^2$ |
| Plane or slab, about edge |  | $\frac{1}{3}Ma^2$ | Spherical shell, about diameter |  | $\frac{2}{3}MR^2$ |

# $K_{\text{rot}}$ and $E_{\text{mech}}$

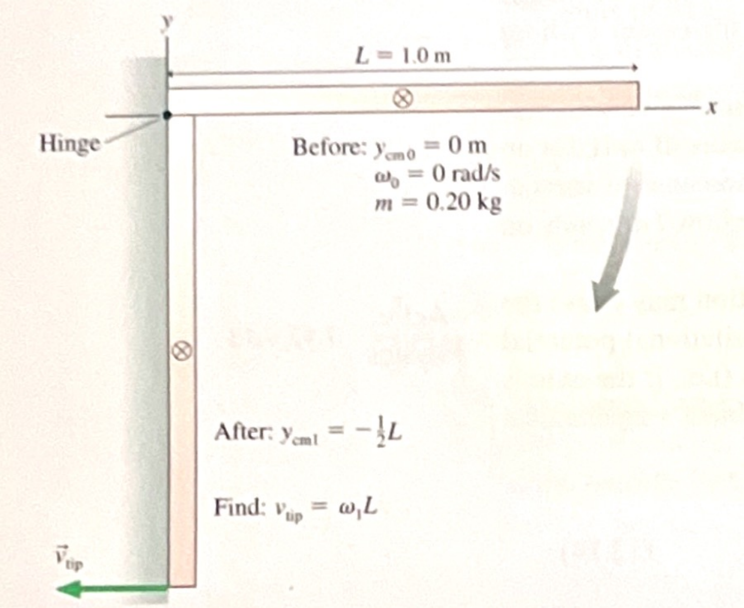



If there are no dissipative or external forces, then an object's mechanical energy is conserved.

$$E_{\text{mech}} = K_{\text{rot}} + U_g = \frac{1}{2}I\omega^2 + Mgy_{\text{cm}}$$

If the axis of rotation is not through the C.O.M. then the rotation may cause the C.O.M. to move up or down. In this case, the object's gravitational potential energy will change.

# Example #3




**A $1.0\text{m}$ long, $200\text{g}$, rod is hinged at one end and connected to a wall. It is held out horizontally then released. What is the speed of the rod when it hits the wall?**

This rotation is not about the center of mass so the gravitational potential energy changes: $\Delta K_{\text{rot}} + \Delta \text{U}_{\text{g}} = \Delta E = 0$.

The parallel-axis theorem must be used since the rotation is not through the C.O.M.

From $K_{\text{rot},i} + U_{\text{g},i} = K_{\text{rot},f} + U_{\text{g},f}$ we know that $K_{\text{rot},f} + U_{\text{g},f} = 0$ since $K_{\text{rot},i}, U_{\text{g},i} = 0$

So, $K_{\text{rot},f} + U_{\text{g},f} = \frac{1}{2} I\omega_f^2 + Mgy_{cm,f} = 0$. We must solve for $\omega_f$.

# Example #3

From $K_{rot,f} + U_{g,f} = \frac{1}{2} I\omega_f^2 + Mgy_{cm,f} = 0$,

$$\omega_f = \sqrt{-\frac{2mg}{I}\left(y_{cm,f}\right)}$$

Now, we need $I$. Since we know that $\lambda = \frac{dm}{dx} = \frac{M}{L}$, we can calculate it. First, let's calculate $I_{cm}$:

$$I_{cm} = \int r^2 \, dm = \int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 \left(\frac{M}{L} dx\right) = \frac{M}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 \, dx = \frac{M}{3L} x^3 \Big|_{-\frac{L}{2}}^{\frac{L}{2}} = \frac{1}{12} ML^2$$

Now, we must apply the parallel axis theorem:

$$I = I_{cm} + Md^2$$

The distance from the C.O.M. is $d = \frac{L}{2}$, so:

$$I = I_{cm} + Md^2 = \frac{1}{12} ML^2 + \frac{1}{4} ML^2 = \frac{1}{3} ML^2$$

# Example #3

Finally, we put all the pieces together:

$$\omega = \sqrt{-\frac{2Mg}{I} y_{cm,f}} = \sqrt{-\frac{2Mg}{\frac{1}{3}ML^2}\left(-\frac{L}{2}\right)} = \sqrt{\frac{3g}{L}}$$

$$\vec{v}_t = r\omega = L\sqrt{\frac{3g}{L}} = \sqrt{3gL} = 5.4\,\frac{\mathrm{m}}{\mathrm{s}}$$

# Summary

An object naturally wants to rotate about its center of mass.

The center is the mass-weighted center of the object:

$$(x_{cm}, y_{cm}) = \left(\frac{1}{M}\sum m_i x_i, \frac{1}{M}\sum m_i y_i\right) = \left(\frac{1}{M}\int x\, dm, \frac{1}{M}\int y\, dm\right)$$

Rotational energy is given by $K_{rot} = \frac{1}{2}I\omega^2$ where $I$ is the moment of inertia:

$$I = \sum_i m_i r_i^2 = \int r^2\, dm$$

The parallel axis theorem shifts the moment of inertia calculation to account for off-axis rotations:

$$I = \sum_i m_i r_i^2 = \int (r_{cm} + d)^2\, dm = I_{cm} + Md^2$$

For rotational motion, $E_{mech} = K_{rot} + U_g = \frac{1}{2}I\omega^2 + Mgy_{cm}$