9.2 – Torque

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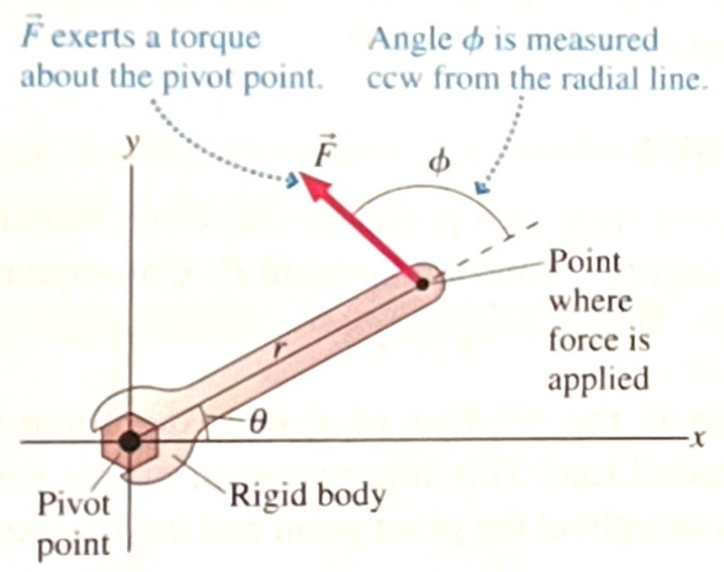
Torque



The ability of a force to cause a rotation depends on three factors:

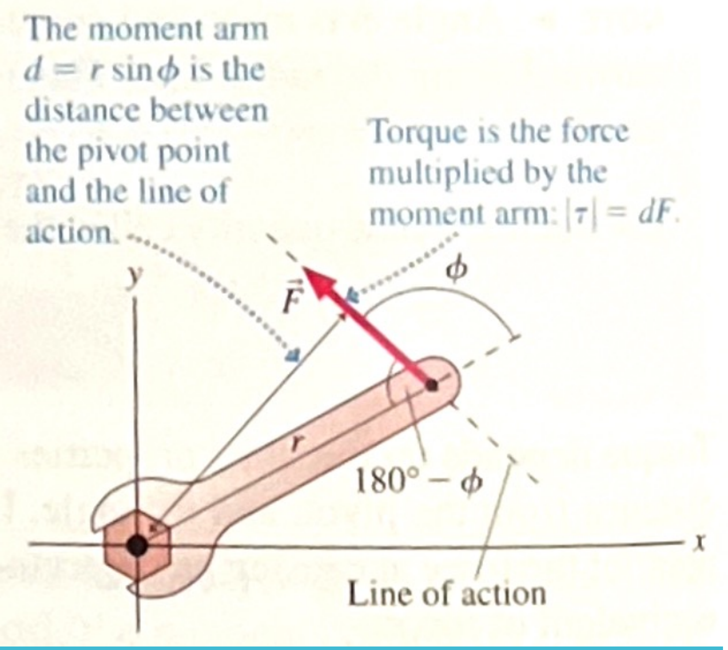
1. the magnitude of the force ($|\vec{F}|$).
2. the distance, r, from the point of application to the pivot ($|\vec{r}|$).
3. The angle at which the force is applied (ϕ).

A force causing an object to rotate about a pivot is called a *torque*. Torque depends on the three factors above.

Torque is the rotational equivalent of force and has a sign. A torque causing a counter-clockwise rotation is positive. A torque causing a clockwise rotation is negative.

Torque is calculated *about a pivot point*.

Calculating Torque



Suppose a force is applied to a bar attached to a pivot point on one side.

The force applied to the opposite end of the bar can be applied at any angle, ϕ.

There are two components in this situation that must be calculated: (1) the *line of action* and (2) the *moment arm*.

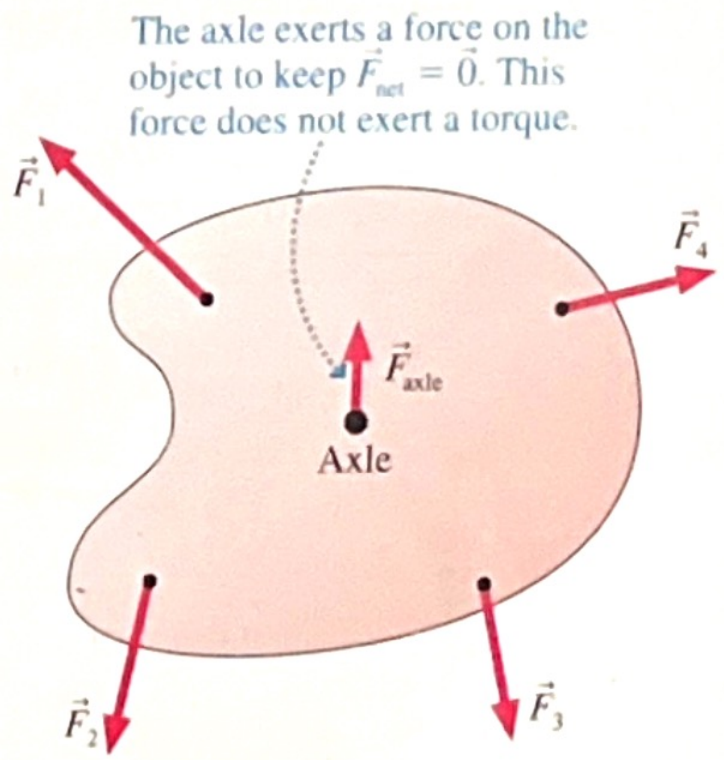
1. The *line of action* runs through the applied force.
2. The *momentum arm* is the shortest distance between the pivot point and the line of action. This distance can be calculated as $d = r\sin(\alpha) = r\sin(\phi)$

The magnitude of the torque (τ) can now be calculated: $|\tau| = dF$.

Thus, $|\tau| = |\vec{F}||\vec{r}|\sin(\phi)$.

This can also be thought of as the length of the lever times the tangential component of the applied force: $|\tau| = |\vec{r}||\vec{F}|\sin(\phi)$

Net Torque

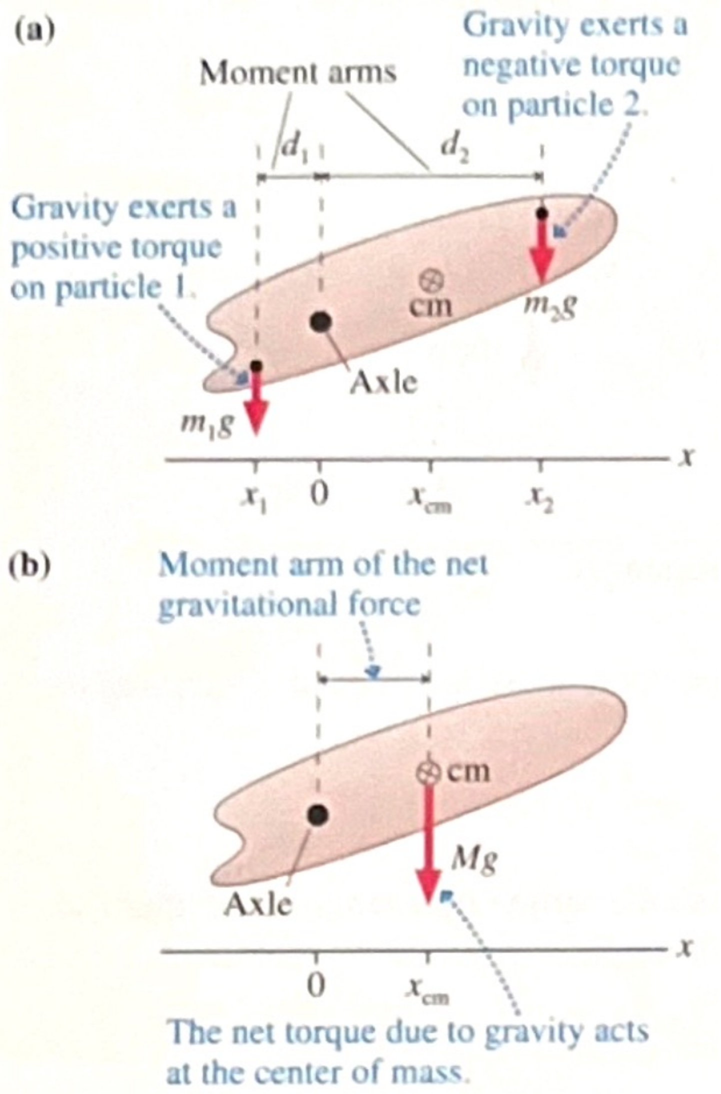


Consider an extended object free to rotate about an axle. There are multiple forces applying torques to the object. These forces may be pushing in different directions but τ_{net} is what matters.

$$\tau_{\text{net}} = \tau_1 + \tau_2 + \cdots = \sum_i \tau_i$$

Although these forces may have unbalanced components in a non-radial direction, the axle acts as a normal force to "cancel out" any translational motion of the object.

Gravitational Torque



Consider a rod with an off-axis axle such that its center of mass is higher than the axle's location.

Gravity acts on *all* particles in the object, exerting a downward force of *magnitude* $F_i = m_i g$ on particle i. The *magnitude* of gravitational torque on each particle is $|\tau_i| = d_i m_i g$ where d_i is the *moment arm*.

A particle to the right of the axle will experience a negative (cw) torque while a particle to the left of the axle will experience a positive (ccw) torque.

The torque is opposite in sign to x_i, so particle i will experience:

$$\tau_i = -m_i x_i g = -(m_i x_i)g$$

Gravitational Torque

The net torque due to gravity is found by summing the individual torques:

$$\tau_{\text{grav}} = \sum_i \tau_i = \sum_i (-m_i x_i g) = -\left(\sum_i m_i x_i\right) g$$

By considering the C.O.M. calculation, $x_{\text{cm}} = \frac{1}{M}\sum_i m_i x_i$, we see that the gravitational torque is given by:

$$\tau_{\text{grav}} = -Mgx_{\text{cm}}$$

(x_{cm} is the position of the C.O.M. relative to the axis of rotation.)

Mg is the net gravitational force on the entire object and x_{cm} is the moment arm. Thus, *the gravitational torque is found by treating the object as if all its mass were concentrated at the center of mass*.

Note that, if $x_{\text{cm}} = 0\text{m}$, there will be no gravitational torque exerted on an object. Thus, if the C.O.M. is directly above the pivot point, an object will balance.

Rotational Dynamics

Torque is the rotational equivalent of force.

How can Newton's Second Law incorporate rotational kinematics?

All points in a rotating object have the same angular acceleration. The angular acceleration is related to the tangential acceleration by $a_t = r\alpha$.

By rewriting Newton's Second Law:

$$F_t = ma_t = mr\alpha$$

Multiplying both sides by r shows that $rF_t = mr^2\alpha$. Noting that rF_t is the torque τ on the particle, $\tau = mr^2\alpha$.

Here, we see that *a torque causes an angular acceleration*.

Rotational Dynamics

If the torques acting on all the "pieces" of an object are summed:

$$\tau_{\text{net}} = \sum_i \tau_i = \sum_i (m_i r_i^2 \alpha) = \left(\sum_i m_i r_i^2\right) \alpha$$

Noting that $I = \sum_i m_i r_i^2$, $\tau_{\text{net}} = I\alpha$.

If $\tau_{\text{net}} = 0\text{Nm}$, an object either does not rotate or rotates at constant angular velocity.

TABLE 12.3 Rotational and linear dynamics

Rotational dynamics		Linear dynamics	
torque	τ_{net}	force	$\vec{F}_{\text{net}}$
moment of inertia	I	mass	m
angular acceleration	α	acceleration	$\vec{a}$
second law	$\alpha = \tau_{\text{net}}/I$	second law	$\vec{a} = \vec{F}_{\text{net}}/m$

Rotation About a Fixed Axis

When solving rotational dynamics problems, one must:

1. Model and draw the object as a simple shape to clarify the situation.
2. Identify the axis about which the object rotates.
3. Identify forces and determine the distances from the axis.
4. Identify all torques and their signs.
5. Apply Newton's Second Law: $\tau_{\text{net}} = I\alpha$.

In some cases, linear motion may be translated into rotational motion. Consider a rope across a pulley where the end of the rope is attached to an object in linear motion.

- If the rope turns on the pulley without slipping, then the rope's speed will match the speed of the rim of the pulley ($v_{\text{rope}} = v_t$).
- If the object is accelerating, then $a_{\text{rope}} = \alpha_t$.

Static Equilibrium

For a rigid body to be in static equilibrium, $\vec{F}_{\text{net}}, \tau_{\text{net}} = 0$.

An entire branch of engineering called *statics* analyzes building, dams, and bridges in *total static equilibrium*.

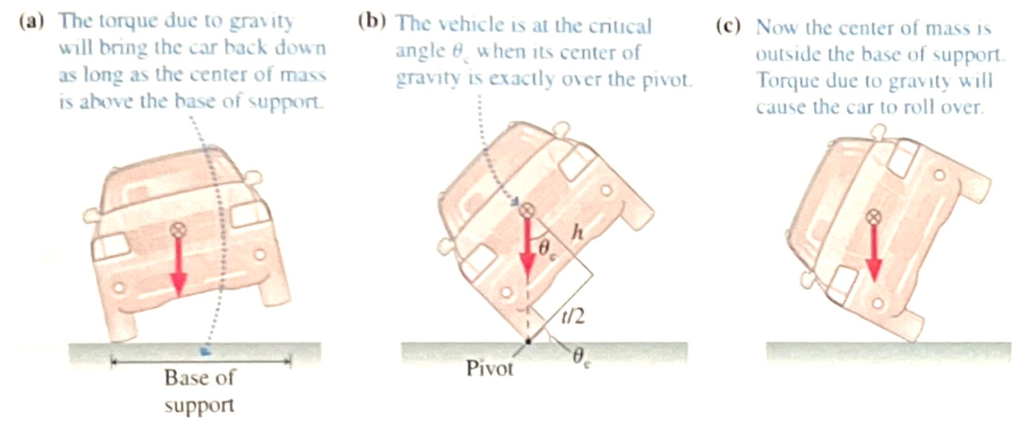
For a rigid body in total equilibrium, there is no net torque about *any* point.

When analyzing static equilibrium problems, (1) model and draw the object as a simple shape, (2) include all forces, (3) pick *any* point as a pivot point ($\tau_{\text{net}} = 0\text{Nm}$), (4) determine the moment arms of all forces about the pivot point and the signs of their associated torques.

Clever selection of a pivot point where several forces act will reduce their torques to zero, simplifying the problem.

Balance and Stability

FIGURE 12.40 Stability depends on the position of the center of mass.



An extended object has a *base of support* on which it rests when in static equilibrium. If the object is tilted, one edge becomes a *pivot point*.

If the object's center of mass remains over the base of support, gravitational torque will rotate the object back to equilibrium.

If the C.O.M. gets outside of the base of support, gravitational torque causes a rotation in the opposite direction.

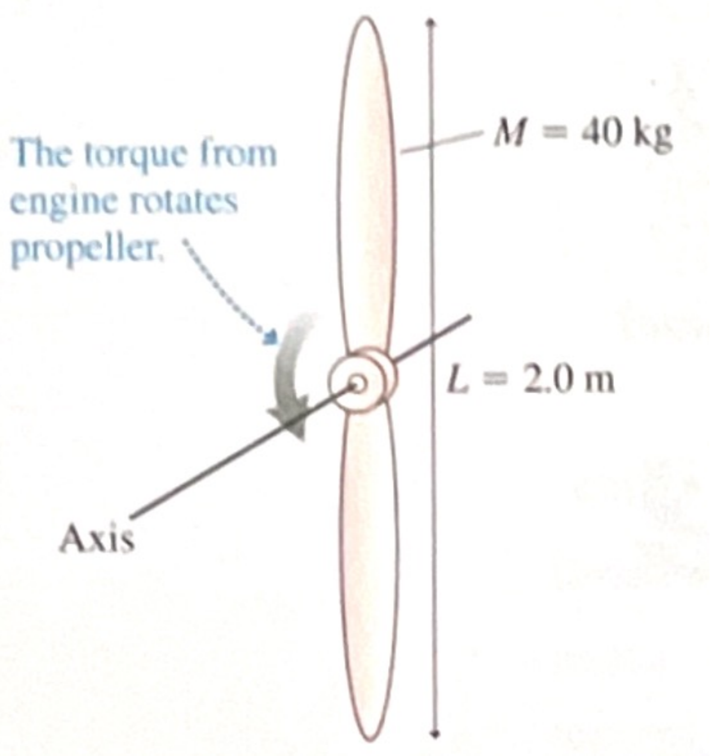
A critical angle θ_c is reached when the C.O.M. is directly over the pivot point. At θ_c, there is no net torque. θ_c can be calculated such that:

$$\theta_c = \tan^{-1}\left(\frac{t}{2h}\right)$$

Here, t is the base width and h is the height of the C.O.M.

Example #1 Newton's Second Law

FIGURE 12.31 A rotating airplane propeller.



The engine in a small airplane is specified to have a torque of 60Nm. This engine drives a 2.0m long, 40kg propeller. On startup, how long does it take the propeller to reach 200rpm?

Modeled as a simple bar of length $L = 2.0\text{m}$ and $m = 40\text{kg}$. The axis of rotation goes through the C.O.M.

First, the moment of inertia (I) must be calculated. Using the previous table:

$$I = \frac{1}{12}ML^2 = \frac{1}{12}(40\text{kg})(2.0\text{m})^2 \approx 13.33\ \text{kg}\cdot\text{m}^2$$

Next, the acceleration can be calculated: $\alpha = \frac{\tau}{I} = \frac{60Nm}{13.33kg{\cdot}m^2} \approx 4.50\frac{rad}{s^2}$

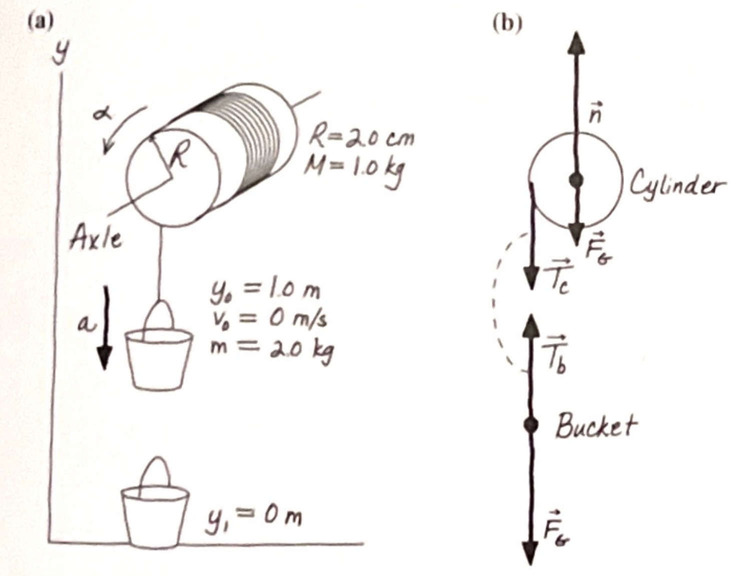
Finally, what is the time needed to reach $\omega_f = 200rpm \approx 20.9\frac{rad}{s}$?

$$\Delta t = \frac{\Delta\omega}{\alpha} = \frac{\omega_f - \omega_i}{\alpha} = \frac{20.9\ \text{rad/s} - 0\ \text{rad/s}}{4.5\ \text{rad/s}^2} \approx 4.6\text{s}$$

Assess: seems reasonable while the engine is spinning up. The effects of friction and drag cancels the torque when full speed is reached.

Example #2 Linear/Rotational Dynamics

FIGURE 12.34 The falling bucket turns the cylinder.



A 2.0kg bucket is attached to a massless string that is wrapped around a 1.0kg, 4.0cm diameter cylinder. The cylinder rotates on an axle through the center. If the bucket is released from 1.0m above the floor, how long does it take to reach the floor?

Assuming the string is massless and doesn't slip: $\vec{T}_b = \vec{T}_c = \vec{T}$.

First, let's apply NSL to the bucket: $\sum F_{\text{net}} = ma_y = T - mg$.

The bucket applies a tension force to the edge of the cylinder, making the moment arm $d = R$. This torque causes a ccw rotation, so it is (+).

For the cylinder, $\tau_{\text{net}} = rF\sin\phi = RT\sin(90°) = TR$.

$$\alpha = \frac{\tau_{\text{net}}}{I} = \frac{TR}{\frac{1}{2}MR^2} = \frac{2T}{MR}$$

Since the string doesn't slip, $-a_y = a_t = \alpha R$, so $a_y = -\alpha R$.

Example #2 Linear/Rotational Dynamics

We can now combine the *linear* motion of the bucket with the *rotational* motion of the cylinder:

$$a_y = -\alpha R = -\left(\frac{2T}{MR}\right)R = -\frac{2T}{M}$$

Solving for T, $T = -\frac{1}{2}Ma_y$. Using this value in the bucket's equation:

$$ma_y = -\frac{1}{2}Ma_y - mg$$

Now, $a_y = -\frac{g}{1+\frac{M}{2m}} \approx -7.84\,\frac{m}{s^2}$

From basic kinematics ($\Delta y = \frac{1}{2}a_y(\Delta t)^2$),

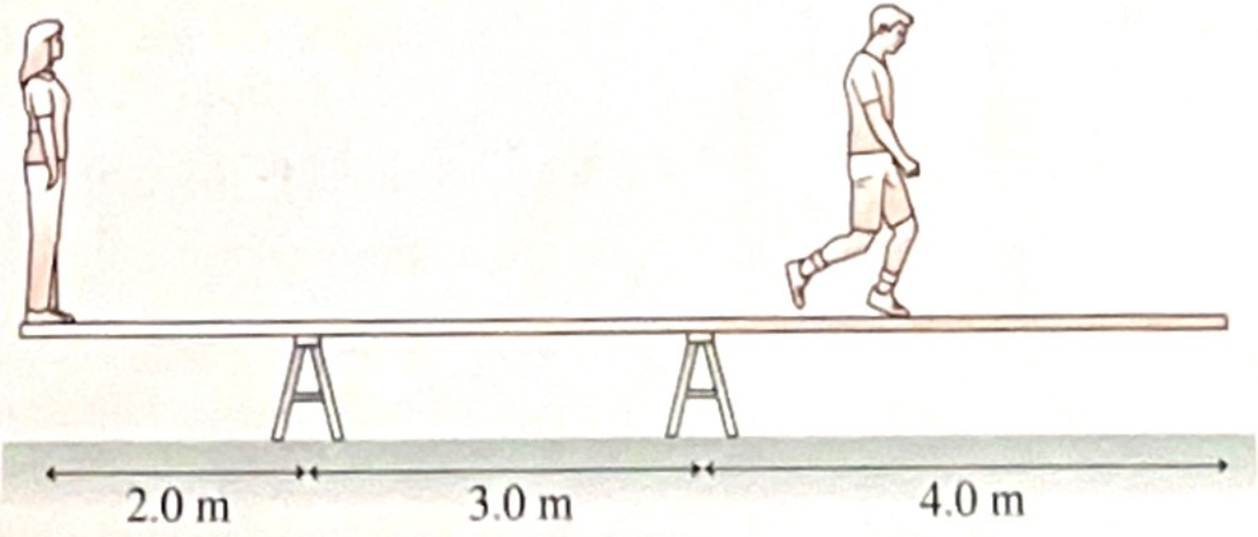
$$\Delta t = \sqrt{\frac{2\Delta y}{a_y}} = \sqrt{\frac{2(-1.0m)}{-7.84\,\frac{m}{s^2}}} = 0.50s$$

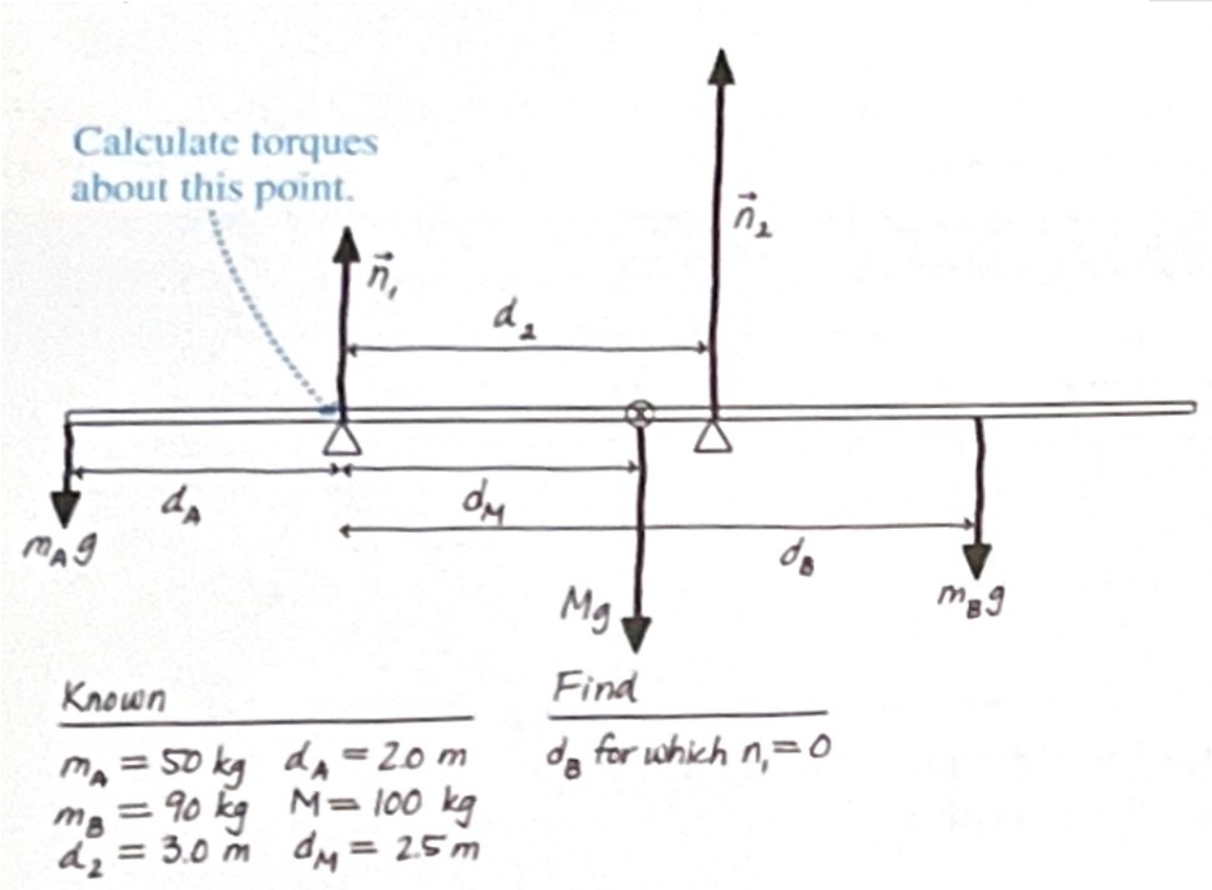
With the cylinder present, the downward force of gravity must accelerate the bucket AND rotate the cylinder.

Example #3 Equilibrium

Adrienne (50kg) and Bo (90kg) are playing on a 100kg rigid plank resting on two supports. If Adrienne stands on the left end, can Bo walk all the way to the right end without the plank tipping over?

FIGURE 12.37 Adrienne and Bo on the plank.





Example #3 Equilibrium

Since we are considering the system to be in equilibrium, we can pick any pivot point we want. Thus, the left support was chosen. This choice removes any torque from $\vec{n}_1$.

Now, we sum the torques, being mindful of the sign of each term:

$$\tau_{\text{net}} = \tau_A + \tau_1 - \tau_{cm} + \tau_2 - \tau_B$$

$$\tau_{\text{net}} = d_A m_A g - d_M M g + d_2 n_2 - d_B m_B g$$

$$\tau_{\text{net}} = d_A m_A g - d_M M g + d_2 (m_A + m_B + M) g - d_B m_B g$$

$$\tau_{\text{net}} = g(d_A m_A - d_M M + d_2 (m_A + m_B + M) - d_B m_B)$$

$$\tau_{\text{net}} = \left(9.8 \frac{\text{m}}{\text{s}^2}\right)(2.0\text{m} \cdot 50\text{kg} - 2.5\text{m} \cdot 100\text{kg} + 3.0\text{m} \cdot 240\text{kg} - 7.0\text{m} \cdot 90\text{kg}) = -588\text{Nm}$$

If Bo can stand on the end, this equation would equal zero.