MAT 2240 - Linear Algebra
Project #2

Directions: Complete the following exercises *after* the accompanying lecture has been given by instructor. A combination of computer work and hand-written work may be required. All solutions should be printed and/or written neatly and submitted.

1. Use Theorem 2 from Chapter 2 and the following *Powers of a Matrix* section to construct a single linear transformation matrix that applies a $90°$ rotation, a reflection across the $y = x$ line, and a reflection across the *x-axis* (in this particular order). Use a trivial shape to sketch each step of the transformation and include your constructed matrix. What are the properties of this unique matrix?

2. Construct a figure using the **fill();** function. You **must** use at least 20 points and 2 different layers (colors). All relevant graphs must include the original and transformed images.

   (a) Apply a reflection through the $y = -x$ line to your original image.

   (b) Scale your original image by a factor of 0.8

   (c) Use a randomly generated parameter between 0 and 1 to apply a horizontal shear to your result from part (b).

   (d) Stretch your original figure by a factor of 2 in the vertical direction and rotate it by $90°$.

   (e) Plot the span of the height and width of your original figure on their respective axes.

3. Cryptography is a field of applied mathematics that uses concepts from combinatorics, number theory, graph theory, linear algebra, etc. to encode messages. A "Hill Cipher" (developed by Lester S. Hill) is a device that places a message inside of a matrix and then multiplies it by a "secret" encoding matrix. The result of this matrix multiplication is the encoded message. Once received, multiplying by the inverse of the encoding matrix reveals the original message. For a better understanding of this process, consider the following example...

Example:
Suppose Dr. Marland wants to send the message "The limit does not exist!" to Dr. Salinas. He would first place the message into a matrix. Although a matrix of any size will work, he chooses a 3-row matrix. The message is placed into the matrix by using 1-26 for the letters A-Z, 27 for spaces, and 28+ for any special characters (In this case, 28 will be used for !).

$$M = \begin{bmatrix} T & & M & & E & N & & I & \\ H & L & I & D & S & O & E & S & \\ E & I & T & O & & T & X & T & ! \end{bmatrix} = \begin{bmatrix} 20 & 27 & 13 & 27 & 5 & 14 & 27 & 9 & 27 \\ 8 & 12 & 9 & 4 & 19 & 15 & 5 & 19 & 27 \\ 5 & 9 & 20 & 15 & 27 & 20 & 24 & 20 & 28 \end{bmatrix}$$

Dr. Marland then encodes the message using his "secret" matrix, A, such that AM provides the coded message:

$$AM = \begin{bmatrix} 3 & 2 & 7 \\ 9 & 0 & 1 \\ 4 & 9 & 2 \end{bmatrix} \begin{bmatrix} 20 & 27 & 13 & 27 & 5 & 14 & 27 & 9 & 27 \\ 8 & 12 & 9 & 4 & 19 & 15 & 5 & 19 & 27 \\ 5 & 9 & 20 & 15 & 27 & 20 & 24 & 20 & 28 \end{bmatrix} =$$

$$\begin{bmatrix} 111 & 168 & 197 & 194 & 242 & 212 & 259 & 205 & 331 \\ 185 & 252 & 137 & 258 & 72 & 146 & 267 & 101 & 271 \\ 162 & 234 & 173 & 174 & 245 & 231 & 201 & 247 & 407 \end{bmatrix}$$

This would be the matrix that Dr. Salinas receives. In order to decode this message, the inverse of A must be applied:

$$A^{-1}(AM) = (A^{-1}A)M = IM = M.$$

The application of A^{-1} will provide the original message, M.

(a) Select an encoding matrix of any (reasonable) size. Use it to encode the message "PAYDAY IS ON FRIDAY". Record your encoded matrix.

(b) Find the inverse of your encoding matrix, A, using MATLAB/Octave. (Use the inverse function **inv(A)**) and verify that you can decode your message. Record A^{-1}.

(c) Suppose that Drs. Marland and Salinas realize that you have been intercepting their secret messages. So, they decide to add an extra layer of security by using a new encoding algorithm: $M_{encrypted} = A(M_{original} + \beta)$. If the encoding matrix, A, from the example above is used and you know that the message from part (b) is transmitted, what is β if you intercept the coded message below?

$$M_{encrypted} = \begin{bmatrix} 413 & 367 & 355 & 501 & 468 & 473 \\ 415 & 313 & 553 & 759 & 908 & 837 \\ 368 & 354 & 388 & 600 & 664 & 477 \end{bmatrix}$$