The wavefrom of GW beats emited by superradiant scalar condensate

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I. SCALAR SUPERRADIANCE AND GW BEAT

Superradiance allows ultra-light bosons to extract energy and angular momentum from rotating black holes (BHs), consequently leading to the condensation of these bosons and the formation of a BH-condensate system. In this work, we investigate a scalar field in a bound state around a Kerr black hole. The energy eigenvalues of this system are generally complex and characterized by three numbers $\{n, l, m\}$. We also define another index, $\bar{n} = n + l + 1$, to further describe the system.

The eigenvalues can be expressed in terms of their real (ω) and imaginary (Γ) parts. For bounded scalar fields, the analytical approximation of the real part in the $M\mu \ll 1$ limit is provided in Ref. [1],

$$\omega_{nlm} = \mu\Big(1 - \frac{\alpha^2}{2\bar{n}^2} - \frac{\alpha^4}{8\bar{n}^2} + \frac{f_{\bar{n}l}}{\bar{n}^3}\alpha^4 + \frac{h_l a_* m}{\bar{n}^3}\alpha^5 + \cdots\Big), \tag{1}$$

$$f_{\bar{n}l} \equiv -\frac{6}{2l+1} + \frac{2}{\bar{n}}, \tag{2}$$

$$h_l \equiv \frac{16}{2l(2l+1)(2l+2)}, \tag{3}$$

where μ represents the mass of the scalar particle, $\alpha \equiv M\mu$ is the mass coupling and $a_* \equiv J/M^2$ denotes the BH spin. Here, M and J refer to the mass and angular momentum of the BH, respectively. The expressions for the imaginary parts of the eigenvalues are detailed in Refs. [2, 3].

Superradiance is activated when the condition $\omega < m\Omega_{\rm H}$ is satisfied, where $\Omega_{\rm H}$ is the angular velocity of the event horizon. This condition allows for the derivation of the critical black hole (BH) spin for each mode, given by:

$$a_{*\rm c}^{(nlm)} = \frac{4mM\omega_{nlm}}{m^2 + (2M\omega_{nlm})^2}. \tag{4}$$

Given that the BH spin has an upper limit of $a_{*\rm lim} \simeq 0.998$ [4], the superradiance condition $a_* > a_{*\rm c}^{(nlm)}$ implies that $M\mu \lesssim 0.5$.

Non-axisymmetric modes can induce disturbances in spacetime through rotation, leading to gravitational wave (GW) emission. Moreover, the $0, l, l$ mode and the $1, l, l$ mode can coexist, accompanied by GW emission, and are referred to as the dominant and subdominant modes, respectively. This coexistence results in GW beats due to differences in energy levels. Detailed calculations of GW beats are presented in Ref. [5]. In this discussion, we highlight some key properties and specifically focus on the case where $l = m = 1$, which represents the fastest modes.

In this scenario, three distinct GW frequencies are identified: $\widetilde{\omega}_1 \equiv 2\omega_{011}$, $\widetilde{\omega}_2 \equiv 2\omega_{111}$ and $\widetilde{\omega}_3 \equiv \omega_{011} + \omega_{111}$. From a particle perspective, the GW associated with $\widetilde{\omega}_1$ results from the annihilation of two $\{0, 1, 1\}$ scalar particles into a single graviton, and the mechanisms for the other frequencies follow similarly. In addition, the corresponding angular momentum have $\widetilde{m} = \pm 2$. However, the frequency $\widetilde{\omega}_4 \equiv \omega^{(111)} - \omega^{(011)}$, resulting from the transition between these states, does not lead to GW emission, as it corresponds to $\widetilde{m} = 0$.

For estimation purposes, the GWs emitted from this system exhibit the quasi-monochromatic frequency:

$$f_{\rm GW} \approx \mu/\pi \approx 48.4\,{\rm Hz} \cdot \frac{\mu}{10^{-13}\,{\rm eV}}. \tag{5}$$

The Laser Interferometer Gravitational-Wave Observatory (LIGO) is designed to detect gravitational waves (GWs) in the frequency range of 10 Hz to 1 kHz. Consequently, the mass of the boson, which generates GWs through annihilation, is estimated to be between 2.07×10^{-14} eV and 2.07×10^{-12} eV. The frequency of the GW beat is given by:

$$f_{\rm beat} \approx 0.831\,{\rm Hz} \left(\frac{M}{10 M_\odot}\right)^2 \left(\frac{f_{\rm GW}}{10^3\,{\rm Hz}}\right)^3. \tag{6}$$

II. THE WAVEFROM OF THE GW BEAT

When the $\{0, 1, 1\}$ mode and the $\{1, 1, 1\}$ mode coexist, one of GW strains at infinity is given by [5],

$$\begin{aligned} \lim_{r\to\infty} h_+ = -\frac{1}{\sqrt{2\pi} r} \Bigg\{ & \frac{N_{011}^2}{\omega_{011}^2} \frac{|U_1|}{\widetilde{\omega}_1^2} f_{+,1}(\theta) \cos\left[\widetilde{\omega}_1 (t - r_*) - 2\varphi - \widetilde{\phi}_1\right] \\ & + \frac{N_{111}^2}{\omega_{111}^2} \frac{|U_2|}{\widetilde{\omega}_2^2} f_{+,2}(\theta) \cos\left[\widetilde{\omega}_2 (t - r_*) - 2\varphi - \widetilde{\phi}_2\right] \\ & + 4\frac{N_{011} N_{111}}{\omega_{011}\omega_{111}} \frac{|U_3|}{\widetilde{\omega}_3^2} f_{+,3}(\theta) \cos\left[\widetilde{\omega}_3 (t - r_*) - 2\varphi - \widetilde{\phi}_3\right] \Bigg\}. \end{aligned} \tag{7}$$

The expression for $h_\times$ has a similar form, with the substitution of cosine terms for sine, and replacing $f_{+,i}(\theta)$ with $f_{\times,i}(\theta)$. Here, $\widetilde{\phi}_i = \arg(U_i)$ and $N_{nlm} \equiv M_{\rm s}^{(nlm)}/\omega_{nlm}$ denotes the occupation number, where $M_{\rm s}^{(nlm)}$ is the condensate mass of the $\{n, l, m\}$ mode. The expressions for U_i can be derived using the method outlined in Ref. [6], and are given by:

$$U_1 = \frac{1}{2}\sqrt{\frac{\pi}{5}}(22+3i\pi)M^6\mu^{10}, \tag{8}$$

$$U_2 = \frac{64}{2187}\sqrt{\frac{\pi}{5}}(22+3i\pi)M^6\mu^{10}, \tag{9}$$

$$U_3 = \frac{4}{81}\sqrt{\frac{\pi}{5}}(22+3i\pi)M^6\mu^{10}. \tag{10}$$

In addition, the tortoise coordinate r_* is given by

$$r_* = r - \frac{r_+ + r_-}{r_+ - r_-}\left[r_- \log\left(\frac{r-r_-}{2M}\right) - r_+ \log\left(\frac{r-r_+}{2M}\right)\right], \tag{11}$$

where $r_\pm = M\left(1 \pm \sqrt{1-a_*^2}\right)$ represent the inner and outer horizons. The angular functions $f_{+,i}(\theta)$ and $f_{\times,i}(\theta)$ are defined as:

$$f_{+,i}(\theta) = {}_{-2}S_{2,2,\widetilde{\omega}_i}(\theta) + {}_{-2}S_{2,-2,-\widetilde{\omega}_i}(\theta), \tag{12}$$

$$f_{\times,i}(\theta) = {}_{-2}S_{2,2,\widetilde{\omega}_i}(\theta) - {}_{-2}S_{2,-2,-\widetilde{\omega}_i}(\theta), \tag{13}$$

where ${}_sS_{\widetilde{l},\widetilde{m},\widetilde{\omega}}(\theta)$ is the spin-weighted spheroidal harmonics [7], which can be calculated using the Black Hole Perturbation Toolkit [8] based on Leaver's continued fraction method [9]. As $M\omega \to 0$, it reduces to the spin-weighted spherical harmonics [10, 11], and the angular functions become

$$\lim_{M\omega\to 0} f_{+,i}(\theta) = \frac{1}{8}\sqrt{\frac{5}{\pi}}(\cos(2\theta)+3), \tag{14}$$

$$\lim_{M\omega\to 0} f_{\times,i}(\theta) = \frac{1}{2}\sqrt{\frac{5}{\pi}}\cos(\theta). \tag{15}$$

For calculating these GW strains, we also need to determine the condensate mass for these modes. Assuming that $a_{*0} > a_{*\rm c}^{(011)}$, the mass of the $\{0, 1, 1\}$ mode can be obtained based on the conservation of energy and angular momentum equations:

$$a_{*\rm c}^{(011)} M_{\rm f}^2 - a_{*0} M_0^2 = \frac{M_{\rm s,f}^{(011)}}{\omega_{011,f}} - \frac{M_{\rm s,0}^{(011)}}{\omega_{011,0}}, \tag{16a}$$

$$M_0 + M_{\rm s,0}^{(011)} = M_{\rm f} + M_{\rm s,f}^{(011)}, \tag{16b}$$

where the subscript "0" represents the initial value while the subscript "f" denotes values when the $\{1,1,1\}$ mode reaches its mass maximum. By fitting the result of Eqs. (16), the mass of the $\{0,1,1\}$ mode becomes,

$$\frac{M_{\rm s,f}^{(011)}}{M_0} \approx M_0\mu\Big[a_{*0} - 4M_0\mu + 7.89a_{*0}(M_0\mu)^2 + (-16.55 + 6.45a_{*0}^2)(M_0\mu)^3\Big]. \tag{17}$$

For the $\{1,1,1\}$ mode, its mass can be estimated by,

$$\frac{M_{\rm s,f}^{(111)}}{M_0} \approx \frac{M_{\rm s,0}^{(111)}}{M_0}\left[\frac{M_{\rm s,f}^{(011)}}{M_{\rm s,0}^{(011)}}\right]^{\beta_{11}/\beta_{01}}, \tag{18}$$

where

$$\beta_{nl} \equiv \frac{(n+2l+1)!}{n!(n+l+1)^{2l+4}}. \tag{19}$$

For this specific case, the ratio β_{11}/β_{01} is approximately 0.35.

Finally, we turn h_+ and $h_\times$ into a detector output:

$$h = h_+F_+ + h_\times F_\times, \tag{20}$$

where the pattern functions F_+ and F_- depend on the orientation of the detector and the direction of the GW source [12–14]. For practical implementation, we utilize the LALSuite software suite [15]. The code for generating the waveform can be found in the file `Waveform.ipynb`. As an illustration, we demonstrate the GW beat phenomenon in Fig. 1. In addition, for a shorter time interval, the waveform appears to be monochrome. Note that the variation of frequency over time is not taken into account, i.e. $\dot{f} = 0$.

For this simulation, we configure the initial parameters as follows: the initial BH mass is set to $M_0 = 50M_\odot$, the initial BH spin is 0.99, the inclination angle is 0. We assume redshift of 0.00025. In addition, the scalar mass is set at 5.35×10^{-13} eV, corresponding to $M_0\mu = 0.2$. In this scenario, the GW frequency is 258.4 Hz while the beat frequency is 0.359 Hz, in the detector frame.

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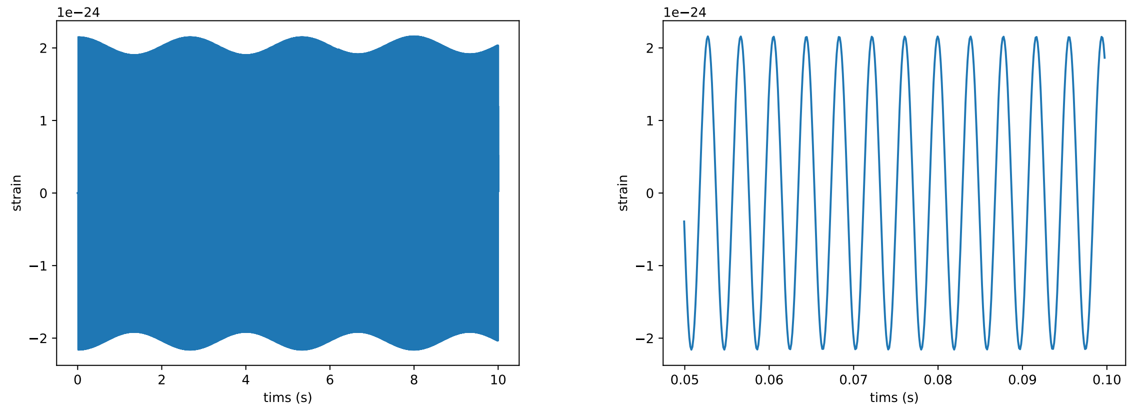

FIG. 1. **Left:** The dimensionless characteristic strain as a function of time. The initial BH mass is set to $M_0 = 50M_\odot$, the initial BH spin is 0.99, the inclination angle is 0. We assume a redshift of 0.00025. In addition, the scalar mass is set at 5.35×10^{-13} eV, corresponding to $M_0\mu = 0.2$. **Right:** Same to the left panel, but for a shorter time interval.

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