Education – Appendix

Options Arbitrage Relationships

In this section, we make similar assumptions to the conditions of the case – risk-free interest rate is zero, no dividends, only European options, single maturity, etc. A summary of the derived results is below. Let c be the price of a call with strike K, let p be the price of a put with the same strike, let S_0 be the current underlying price, and let S_T be the price of the underlying at expiry in the following discussion.

	Call	Put
Minimum	$\max\{S_0 - K, 0\}$	$\max\{K - S_0, 0\}$
Maximum	S_0	K

We consider a call such that $c > S_0$. We can buy the underlying at S_0 and sell a call with strike K at c, earning an immediate profit of $c - S_0$. We now consider what happens when the option expires. If the option expires out-of-the-money, the total profit is $c - 0 + (S_T - S_0)$. The smallest $(S_T - S_0)$ can be is $-S_0$ (if the price of the stock at expiry goes to zero), but $c > S_0$, so a positive profit results. If the option expires in-the-money, the total profit is

$$c - (S_T - K) + (S_T - S_0) = c + K - S_0.$$

Again, since $c > S_0$ and $K > 0$, a positive profit results. This violates the principle of no arbitrage and thus $c \leq S_0$. $p \leq K$can be shown in a similar manner by assuming $p > K$ and selling a put with strike K.

Now consider two portfolios, one consisting of a call with strike K and \$$K$ in cash (Portfolio A), and the other consisting of a single share of the underlying (Portfolio B). If the call in portfolio A is in-the-money at expiration, then $S_T > K$, so the option is exercised, the \$$K$ in cash is used to buy the underlying, and the portfolio is worth S_T. If the call in portfolio A is out-of-the-money at expiration, we have $S_T \leq K$, so the option is worthless and the portfolio is worth \$$K$. Portfolio A is always worth at least as much as the underlying (Portfolio B) at expiry, so if there are no arbitrage opportunities A must be worth at least as much as B today. This implies $c + K \geq S_0$ and $c \geq S_0 - K$. We note in addition that an option cannot be negative-valued (you can always choose not to exercise), so $c \geq \max\{S_0 - K, 0\}$. A similar argument for puts can be made considering portfolio C, consisting of a put struck at K plus a share of the underlying, compared to portfolio D, consisting of \$$K$, to show $\max\{K - S_0, 0\} \leq p$.

Using portfolios A and C from above, we can illustrate the principle of **put-call parity**. Suppose the call and put are struck at the same price K. If $S_T \leq K$ at expiry, the call is worthless and A is worth K, while the put is worth $K - S_T$ and the underlying is worth S_T, so C is also worth K. If $S_T > K$ at expiry, the call is worth $S_T - K$ and the cash is worth K, so A is worth S_T, while the put is worthless and so C is also worth S_T. Since these two portfolios must be worth the same no matter how the underlying and strike relate at expiry, by a no-arbitrage argument they must be worth the same now, and so

$$c + K = p + S_0$$

or

$$c - p = S_0 - K,$$

as stated in the education document.

Additional discussion of these static arbitrage relationships, including the effects of interest rates and dividends on options pricing and the differences between American and European options, can be found in Chapter 3 ("Arbitrage Bounds for Option Prices") of Sinclair (2010) or Chapter 10 ("Properties of Stock Options") of Hull (2010).

The Binomial Tree and Black-Scholes-Merton (BSM) Option Pricing Models

We will follow the discussion in Chapter 4 ("Pricing Models") of Sinclair (2010). A more mathematically advanced discussion can be found at several sources, including here, "Twelve Ways to Derive Black-Scholes" in Wilmott (2010), and Chapter 12 ("Binomial Trees") through 14 ("The Black-Scholes-Merton Model") of Hull (2010).

The BSM model is a limiting continuous-time extension of the **binomial pricing model**. The intuition behind the binomial model can be seen in a simple example, where we suppose a stock trades at S_0 and we own an option on this stock valued at V that has time to expiry T. We assume there are two possibilities at expiry – either the stock has moved up by a factor u or down by a factor d, and we suppose our option is worth V_u if the stock moves up and V_d if the stock moves down. Suppose we take a long position in one option and a short position in Δ shares of the underlying stock. We want to calculate the value of Δ so that our portfolio has the same value regardless of whether the underlying moves up or down. Thus,

$$V_u - \Delta \cdot S_0 \cdot u = V_d - \Delta \cdot S_0 \cdot d.$$

Solving for Δ,

$$\Delta = (V_u - V_d)/(S_0 \cdot u - S_0 \cdot d)$$

Since the portfolio we have created is now risk-free, it must grow at the risk-free rate r, and the value of the portfolio today $V - \Delta \cdot S_0$ must equal the present value of the riskless portfolio $(V_u - \Delta \cdot S_0 \cdot u)e^{-r}$ by a no-arbitrage argument. (We refer to this portfolio as **delta-hedged**.) We can solve for V and introduce the variable

$$p = (e^{rT} - d)/(u - d)$$

to get

$$V = e^{-r}\ (pV_u + (1-p)V_d)$$

This sheds light on a principle of derivatives pricing known as **risk-neutral valuation** – we are allowed to make the assumption that investors are **risk-neutral** when valuing derivatives, and so we can value derivatives in a **risk-neutral world**. Notice that we have not mentioned anything about the probability of an up or down move in the underlying, and the call is valued as the weighted average of something that looks like a probability of an up or down move discounted at the risk-free rate. This value p is in fact the probability of an upward move in the stock <u>in the risk-neutral world</u>, which does not always reflect the probability of an upward move in the real world accurately. We can make the binomial tree model more advanced by generalizing with more steps and matching u and d to the volatility of the underlying – see the reference texts for more details. Notably, as the binomial model increases in number of steps, the derived price converges to the result given by the BSM model.

We now derive the BSM model by again considering a delta-hedged option position $V - \Delta \cdot S_t$. Suppose that the underlying changes to S_{t+1} in the next time period. We note that the option value changes by $V(S_{t+1}) - V(S_t)$, the hedge changes by $\Delta \cdot S_{t+1} - \Delta \cdot S_t$, and we pay $r(V - \Delta \cdot S_t)$ to finance this position. The total change is

$$V(S_{t+1}) - V(S_t) + \Delta \cdot (S_{t+1} - S_t) - r(V - \Delta \cdot S_t).$$

We can approximate the change in option value due to the change in the underlying price with a second order Taylor expansion – let $\partial^2 V/\partial S^2 = \Gamma$. In addition, we observe a time effect – the option price decreases due to the passage of time at a rate $\partial V/\partial t = \theta$:

$$\Delta \cdot (S_{t+1} - S_t) + \frac{1}{2}(S_{t+1} - S_t)^2(\partial^2 V/\partial S^2) + \theta - \Delta \cdot (S_{t+1} - S_t) - r(V - \Delta \cdot S_t)$$

$$\frac{1}{2}(S_{t+1} - S_t)^2 \Gamma + \theta - r(V - \Delta \cdot S_t)$$

This equation gives the profit when the underlying price changes: $\frac{1}{2}(S_{t+1} - S_t)^2 \Gamma$ gives the effect of gamma, θ the effect of theta, and $-r(V - \Delta \cdot S_t)$ the effect of financing (which is 0 for the competition). On average, we observe $(S_{t+1} - S_t)^2 \approx \sigma^2 {S_t}^2$, where σ is the standard deviation of the underlying instrument's returns (volatility), so we get

$$\frac{1}{2}\sigma^2 {S_t}^2 \Gamma + \theta - r(V - \Delta \cdot S_t)$$

Since this is a riskless position and financed with borrowed money, we assume that the excess returns are zero, so we set the returns next period to zero:

$$\frac{1}{2}\sigma^2 {S_t}^2 \Gamma + \theta - r(V - \Delta \cdot S_t) = 0$$

This gives us the **Black-Scholes-Merton differential equation**. We note similarly to the binomial pricing model that we do not care about drift in the underlying price – it can be delta-hedged out, as the price of the derivative and underlying are perfectly correlated over very short time scales. However, unlike the binomial model, this riskless property is only preserved for a very short period of time, and **dynamic hedging** (rebalancing of the underlying) is required to maintain a **delta-neutral** position.

Considering the intrinsic value boundary conditions of $c_T = \max\{S_T - K, 0\}$ and $p_T = \max\{K - S_T, 0\}$ at expiry gives the **Black-Scholes-Merton pricing formulae** as seen before for European calls and puts:

$$c = S_0 N(d_1) - K e^{-rT} N(d_2)$$

$$p = K e^{-rT} N(-d_2) - S_0 N(-d_1)$$

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

where $N(\cdot)$ denotes the normal(0,1) cumulative density function, S_0 is the current underlying price, K is the strike price, r is the risk-free rate, T is the time to expiry (in years), and σ is the volatility of the underlying.

The Greeks and Volatility

We have reproduced this section from the education document with slightly more detail.

The Greeks describe different risk exposures of options positions and are calculated by taking derivatives of the price given by the BSM model with respect to different parameters. Portfolio Greeks can generally be calculated by summing the Greeks for each position in the portfolio. Some of the Greeks are described below:

- **Delta (Δ)** measures the rate of change of the theoretical option value with respect to changes in the underlying asset's price. This can be mathematically expressed as $\frac{\partial V}{\partial S}$. Using Black-Scholes assumptions in our case environment, $\Delta_c = N(d_1)$ and $\Delta_p = -N(-d_1)$. We note the delta of owning the long underlying is 1, the delta of calls is strictly positive ($0 < \Delta_c < 1$), and the delta of puts is strictly negative ($-1 < \Delta_p < 0$). Notably, if we differentiate the put-call parity equation $c - p = S_0 - K$ with respect to the underlying price S_0, we observe the relationship $\Delta_c - \Delta_p = 1$. We have seen the delta of an option before as the **hedge ratio** to create a riskless portfolio. Calculating the delta of your portfolio and choosing when to hedge and rebalance with the underlying is essential to success in this case.

- **Gamma (Γ)** measures the rate of change of delta with respect to changes in the underlying asset's price. This can be mathematically expressed as $\frac{\partial \Delta}{\partial S} = \frac{\partial^2 V}{\partial S^2}$. Using Black-Scholes assumptions in our case environment, $\Gamma_c = \Gamma_p = \frac{n(d_1)}{S_0 \sigma \sqrt{T}} > 0$, where $n(\cdot)$ denotes the normal(0,1) probability density function. The absolute value of Δ is increasing as S_0 increases for calls and is decreasing as S_0 increases for puts. Gamma is high for at-the-money options and is roughly normally distributed. Gamma can be thought of as a measure of the nonlinearity of an option or as how quickly a delta hedge must be adjusted. Note that the underlying does not have any gamma, so a **gamma hedge** requires taking positions in options contracts. An example of a strategy with high gamma is the **straddle**.

- **Theta (θ)** measures the sensitivity of the value of the option to the change in time (in years). Theta is mathematically expressed as $\frac{\partial V}{\partial t}$. Theta is often known as "time decay" because it reflects the fact that the time value of the option decreases as the option nears expiration. Using Black-Scholes assumptions in our case environment, $\theta_c = \theta_p = -\frac{S_0 n(d_1)\sigma}{2\sqrt{T}} < 0$. (Note if $r \neq 0$ that θ_p exceeds θ_c by rKe^{-rT} in general.) Theta is negative for both puts and calls and is commonly normalized to reflect per day option value decay (divide raw theta by number of days per year). Theta is most negative for at-the-money options, and in a delta-neutral portfolio, theta is a proxy for gamma.

- **Vega (ν)** measures the option's sensitivity to implied volatility (see below). Mathematically, this is $\frac{\partial V}{\partial \sigma}$. Using Black-Scholes assumptions in our case environment, $\nu_c = \nu_p = Sn(d_1)\sqrt{T} > 0$. Vega is notable because it involves taking a derivative with respect to a parameter of the model as opposed to a variable – we assumed volatility was constant when deriving the BSM model. This turns out to be a decent estimator of vega calculated from a more theoretically sound model, such as a stochastic volatility model. Vega is commonly scaled so that it represents the dollar change in option value for a one-point change in implied volatility (raw vega is divided by 100). Vega is highest for at-the-money options and increases as a function of time. Note that the underlying does not have any vega, so a **vega hedge** requires taking positions in options contracts. Portfolios that are gamma neutral are not generally vega neutral, and vice versa, but can be made so by trading two different options contracts on the same underlying.

If we ignore terms of higher order than Δt for a delta-neutral portfolio, we observe the relationships

$$\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma = r\Pi$$

$$\Delta\Pi = \Theta \cdot \Delta t + \frac{1}{2}\Gamma(\Delta S)^2$$

where ΔS is the change in the underlying asset price during a small interval Δt, Π is the value of the portfolio, $\Delta\Pi$ is the change in the value of the portfolio, σ is the volatility of the portfolio, and Θ is the theta of the portfolio. This shows that gamma and theta tend to have opposite sign and illustrates how theta can be regarded as a proxy for gamma in a delta-neutral portfolio.

Volatility is a measure of uncertainty about the returns provided by a stock, usually mathematically defined as the standard deviation of the return provided by the stock in one year when the return is expressed using continuous compounding. **Realized/historical volatility** can be measured by observing the stock price S_i, $i = 0,1,\ldots,n$, over some period T (measured in years), calculating the log return $u_i = \ln\left(\frac{S_i}{S_{i-1}}\right)$, and taking the standard deviation of the u_i

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(u_i - \bar{u})^2}$$

Then s is an estimator for the standard deviation of the daily returns $\sigma\sqrt{T}$, so $\hat{\sigma} = s/\sqrt{T}$. (Commonly for a one-year period the number of trading days per year, 252, is used instead of the number of calendar days per year.)

Implied volatility (IV) is the value of σ that, when fed into the BSM model with the other observable parameters, will output the current observed option premium in the market. Since all other parameters (strike, interest rate, underlying price, time to expiry) are observable and relatively deterministic, estimates of volatility are the main driver of options pricing fluctuations for a given instrument, and options prices are often quoted in terms of implied volatilities for this reason. (Note that changes in the underlying price are typically accounted for by proper hedging and their risk is thus not rewarded.) Implied volatility estimates affect the time value of the option, not the intrinsic value. The implied volatility of a European call is the same as the implied volatility of a European put option when they have the same strike and time to expiry, which can be shown by appealing to put-call parity for both the theoretical and market prices of calls and puts. IV can be numerically found by inverting the Black-Scholes model with the current market price and iteratively searching for the correct σ (for example, with Newton's method) – no closed-form/analytic solution exists to determine σ. IV reflects current market sentiment with regards to future volatility.

A plot of the implied volatility of an option against its strike price (holding time to expiry constant) is referred to as a **volatility smile**. (Volatility smiles often plot implied volatility against the ratio of the strike and underlying, K/S_0, to stabilize the smile against movements in the underlying.) The "smile" name comes from the fact that deep out-of-the-money strikes tend to have higher implied volatilities than at-the-money strikes. A volatility smile generates an **implied distribution** of the underlying asset price that is more fat-tailed than the standard lognormal distribution – this is known as **kurtosis**. Thus, using the volatility smile to model options prices assigns a higher probability of extreme moves than the BSM model predicts and thus values options struck at extreme values more than they would be worth using the BSM model. The volatility smile effect arises from real-world violations of the BSM assumptions, namely constant volatility and smooth price changes, which both increase the likelihood of extreme outcomes. Equity options often display a **volatility skew**, where implied volatility decreases as strike increases - the implied distribution has a heavier left tail and a less heavy right tail, implying the probability of extreme downward moves is much higher than predicted by BSM, while extreme upward moves are less likely. Potential explanations for this effect include leverage or fear of stock market crashes. More generally, the relationship of implied volatility and maturity holding strike constant is known as the **volatility term structure**, and we can plot implied volatility against both strike and maturity to develop the **volatility surface**, a tool that can be used to price options of any strike and maturity. See Hull (2010), Chapter 19 ("Volatility Smiles") for more details.

Successful competitors will be able to create their own models based off of BSM for how volatility should evolve over time and will compare their predictions to current implied volatilities to see if there are any profitable trading opportunities in the market.

References and Further Reading

- Hull, John. *Options, Futures, and Other Derivatives. Eighth Edition.* Pearson, 2010.
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- Wilmott, Paul. *Frequently Asked Questions in Quantitative Finance. Second Edition.* Wiley, 2010.