Proofs - Suppression–Operator Approximation

Dustyn Stanley

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1 Definition of the suppression operator L_α and its kernel

For each $\alpha > 0$, define the *suppression operator*

$$L_\alpha \;=\; e^{\alpha\Delta}$$

acting on mean-zero functions on ^{3}. Equivalently, in Fourier series,

$$\widehat{L_\alpha f}(k) \;=\; e^{-\alpha|k|^2}\,\widehat{f}(k), \qquad k \in^3 \setminus\{0\}, \quad \widehat{f}(0) = 0.$$

Its integral kernel $K_\alpha(x)$ is the mean-zero, 2π-periodic heat kernel:

$$L_\alpha f(x) = \int_{3} K_\alpha(x-y)\,f(y)\,dy, \quad K_\alpha(x) = \sum_{k\in^3\setminus\{0\}} e^{-\alpha|k|^2}\,e^{ik\cdot x}.$$

[Properties of K_α] For each $\alpha > 0$,

1. $\displaystyle\int_{3} K_\alpha(x)\,dx = 0$, so L_α preserves mean-zero.
2. There is $C > 0$ such that for all $x \in^3 \setminus\{0\}$,
$$|K_\alpha(x)| \;\le\; C\,\alpha^{-3/2}\,\exp\!\Big(-\frac{|x|^2}{4\alpha}\Big), \quad |\nabla K_\alpha(x)| \;\le\; C\,\alpha^{-2}\,\exp\!\Big(-\frac{|x|^2}{8\alpha}\Big).$$
3. Consequently, for any $1 \le p \le \infty$, L_α is bounded $L^p(^3) \to L^p(^3)$, with $\|L_\alpha f\|_{L^p} \le \|f\|_{L^p}$.

(1) By construction $\widehat{K_\alpha}(0) = 0$, so $\int K_\alpha = 0$.

(2) Write the kernel as a Poisson-summed Gaussian on ^{3}:

$$K_\alpha(x) = \sum_{m\in^3} \frac{1}{(4\pi\alpha)^{3/2}}\,\exp\!\Big(-\frac{|x-2\pi m|^2}{4\alpha}\Big) \;-\; \frac{1}{(2\pi)^3},$$

where the constant term enforces zero mean. Classical Gaussian bounds on ^{3} give the stated pointwise decay for $x \not\equiv 0 \pmod{2\pi}$. Differentiating under the sum yields the gradient bound.

(3) Since $K_\alpha \in L^1(^3)$ uniformly in α, Young's convolution inequality implies for $1 \le p \le \infty$,

$$\|L_\alpha f\|_{L^p} = \|K_\alpha * f\|_{L^p} \le \|K_\alpha\|_{L^1}\, \|f\|_{L^p} \le C\, \|f\|_{L^p}.$$

Moreover, in Fourier variables $|e^{-\alpha|k|^2}| \le 1$ shows $\|L_\alpha\|_{L^2 \to L^2} = 1$, so in fact $C = 1$.

2 Dyadic–multiplier decay for L_α (Littlewood–Paley argument)

[Exponential decay on frequency blocks] Let $L_\alpha = e^{\alpha\Delta}$ and Δ_j the Littlewood–Paley projections on ^{3}. Then for each $j \ge -1$ and all $1 \le p \le \infty$,

$$\|\Delta_j L_\alpha f\|_{L^p} \ \le\ e^{-c\,\alpha\, 2^{2j}}\, \|\Delta_j f\|_{L^p},$$

where $c > 0$ is an absolute constant (e.g. $c = 14$). In particular, the multiplier symbol $e^{-\alpha|k|^2}\varphi(2^{-j}k)$ on the support $\{|k| \sim 2^j\}$ decays like $e^{-c\alpha 2^{2j}}$.

Recall

$$\Delta_j L_\alpha f(x) = \sum_{k \in ^3} \varphi(2^{-j}k)\, e^{-\alpha|k|^2}\, \widehat{f}(k)\, e^{ik\cdot x}.$$

On the support of $\varphi(2^{-j}k)$ we have $12\,2^j \le |k| \le 2\,2^j$. Hence

$$e^{-\alpha|k|^2} \ \le\ \exp\bigl(-\alpha 14\, 2^{2j}\bigr) \ =\ e^{-c\,\alpha\, 2^{2j}}, \quad c = 14.$$

Therefore the Fourier multiplier defining $\Delta_j L_\alpha$ satisfies $\bigl|\varphi(2^{-j}k)\, e^{-\alpha|k|^2}\bigr| \le e^{-c\,\alpha\, 2^{2j}}$. By the periodic Mikhlin (or simply by taking the sup-norm of the multiplier on each dyadic shell), it follows that for each $1 \le p \le \infty$,

$$\|\Delta_j L_\alpha f\|_{L^p} \ \le\ \sup_k \bigl|\varphi(2^{-j}k)\, e^{-\alpha|k|^2}\bigr|\, \|\Delta_j f\|_{L^p} \ \le\ e^{-c\,\alpha\, 2^{2j}}\, \|\Delta_j f\|_{L^p}.$$

This establishes the stated dyadic decay estimate.

3 Uniform H^s– and $W^{s,p}$–bounds for L_α

[Uniform smoothness of L_α] Let $\alpha > 0$. Then for all $s \ge 0$ and $1 \le p \le \infty$,

$$\|L_\alpha f\|_{H^s} \ \le\ \|f\|_{H^s}, \qquad \|L_\alpha f\|_{W^{s,p}} \ \le\ \|f\|_{W^{s,p}}.$$

In fact, $L_\alpha = e^{\alpha\Delta}$ is a contraction on each Sobolev or Bessel potential space.

We treat the two cases in turn.

1. H^s**–bound.** By definition of the H^s norm on ^{3},

$$\|L_\alpha f\|_{H^s}^2 = \sum_{k\in^3}(1+|k|^2)^s\,|e^{-\alpha|k|^2}\widehat{f}(k)|^2 = \sum_k(1+|k|^2)^s\,e^{-2\alpha|k|^2}|\widehat{f}(k)|^2.$$

Since $e^{-2\alpha|k|^2}\le 1$ for all $\alpha>0$ and k, it follows immediately that $\|L_\alpha f\|_{H^s}\le \|f\|_{H^s}$.

2. $W^{s,p}$**–bound.** Recall $W^{s,p}(^3)$ may be defined by

$$\|f\|_{W^{s,p}} = \|(1-\Delta)^{s/2}f\|_{L^p} = \|T_m f\|_{L^p},$$

where T_m is the Fourier multiplier with symbol $m(k)=(1+|k|^2)^{s/2}$. Then

$$L_\alpha f = e^{\alpha\Delta}f\ \ (1-\Delta)^{s/2}L_\alpha f = T_m\big(e^{\alpha\Delta}f\big) = T_{m\,e^{-\alpha|k|^2}}f.$$

The combined symbol $m(k)e^{-\alpha|k|^2}=(1+|k|^2)^{s/2}e^{-\alpha|k|^2}$ is smooth and satisfies the periodic Mikhlin conditions:

$$\big|\partial^\beta\big((1+|\xi|^2)^{s/2}e^{-\alpha|\xi|^2}\big)\big| \ \le\ C_{\alpha,s,\beta}\,(1+|\xi|^2)^{s/2-|\beta|},$$

for all multi-indices β. Hence the operator $T_{m\,e^{-\alpha|k|^2}}$ is bounded on L^p with norm depending only on α,s,p. But since $|m(k)e^{-\alpha|k|^2}|\le(1+|k|^2)^{s/2}$, one sees the L^p–operator norm of $T_{m\,e^{-\alpha|k|^2}}$ is in fact ≤ 1. Therefore

$$\|L_\alpha f\|_{W^{s,p}} = \|(1-\Delta)^{s/2}L_\alpha f\|_{L^p}\le\|f\|_{W^{s,p}}.$$

This completes the proof of the uniform bounds.

4 Commutator $[L_\alpha,\,\nabla]$ estimate

[Estimate for $[L_\alpha,\,\nabla]$] Let $\alpha>0$ and $L_\alpha=e^{\alpha\Delta}$. For any $1\le p\le\infty$ and any sufficiently smooth $f{:}^3\to$,

$$\big\|\,[L_\alpha,\,\nabla]f\,\big\|_{L^p}\ \le\ C\,\alpha^{1/2}\,\|\nabla f\|_{L^p},$$

where C is an absolute constant.

Since L_α is a Fourier multiplier with symbol $m(k)=e^{-\alpha|k|^2}$, we have

$$\widehat{[L_\alpha,\partial_j]}f(k) = \big(-ik_je^{-\alpha|k|^2}-e^{-\alpha|k|^2}(-ik_j)\big)\,\widehat{f}(k) = \big(e^{-\alpha|k|^2}-1\big)\,(ik_j)\,\widehat{f}(k).$$

Hence

$$[L_\alpha,\nabla]f = \mathcal{F}^{-1}\Big((e^{-\alpha|k|^2}-1)(ik)\,\widehat{f}(k)\Big).$$

Observe for all $k\in^3$,

$$|\,e^{-\alpha|k|^2}-1|\ =\ \int_0^\alpha|\,dd\beta e^{-\beta|k|^2}\,|\,d\beta = \int_0^\alpha|k|^2\,e^{-\beta|k|^2}\,d\beta\ \le\ \alpha\,|k|^2.$$

Therefore

$$\left|(e^{-\alpha|k|^2}-1)(ik)\right| \;\le\; \alpha\,|k|^3.$$

Define the multiplier symbol

$$\sigma(k) \;=\; \alpha\,|k|^3.$$

Since $\sigma(k) \le C\,\alpha^{1/2}|k|$ on the support of the fractional multiplier (for $|k| \ge 1$), and the symbol $\alpha^{1/2}|k|$ satisfies the standard Mikhlin bounds, it follows by the periodic Mikhlin theorem that

$$\|[L_\alpha,\nabla]f\|_{L^p} \;\le\; C\,\alpha^{1/2}\,\|\nabla f\|_{L^p}.$$

This completes the proof.

5 Strong convergence $L_\alpha f \to f$ in H^s

[Convergence of suppression operator] Let $s \ge 0$ and $f \in H^s(^3)$. Then

$$\lim_{\alpha\to 0^+} \|L_\alpha f - f\|_{H^s} = 0,$$

where $L_\alpha = e^{\alpha\Delta}$.

Write f in Fourier series,

$$f(x) = \sum_{k\in^3} \widehat{f}(k)\,e^{ik\cdot x}, \qquad L_\alpha f(x) = \sum_{k\in^3} e^{-\alpha|k|^2}\,\widehat{f}(k)\,e^{ik\cdot x}.$$

Then

$$\|L_\alpha f - f\|_{H^s}^2 = \sum_{k\in^3}(1+|k|^2)^s\left|e^{-\alpha|k|^2}-1\right|^2|\widehat{f}(k)|^2.$$

For each fixed k, $e^{-\alpha|k|^2}-1 \to 0$ as $\alpha \to 0$. Moreover, since $\left|e^{-\alpha|k|^2}-1\right| \le 2$ and $\sum(1+|k|^2)^s|\widehat{f}(k)|^2 < \infty$, the dominated convergence theorem implies

$$\lim_{\alpha\to 0}\sum_k(1+|k|^2)^s\left|e^{-\alpha|k|^2}-1\right|^2|\widehat{f}(k)|^2 = 0.$$

Hence $\|L_\alpha f - f\|_{H^s} \to 0$ as claimed.

6 Construction of approximate solutions u_α

[Existence of suppressed-Navier–Stokes solutions] Let $u_0 \in H^2_\sigma(^3)$ be divergence-free and mean-zero, and fix $\alpha > 0$. Then there exists a unique global solution

$$u_\alpha \;\in\; C\big([0,\infty);H^2_\sigma(^3)\big) \;\cap\; L^2\big([0,\infty);H^3_\sigma(^3)\big)$$

of the *suppressed Navier–Stokes* system

$$\{\,\partial_t\,u_\alpha + L_\alpha\big(u_\alpha\cdot\nabla u_\alpha\big) - \nu\Delta u_\alpha + \nabla p_\alpha = 0, \nabla\cdot u_\alpha = 0, u_\alpha|_{t=0} = u_0,$$

where $L_\alpha = e^{\alpha\Delta}$. Moreover, u_α satisfies the global energy inequality

$$\|u_\alpha(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla u_\alpha(s)\|_{L^2}^2 \, ds \;\le\; \|u_0\|_{L^2}^2, \quad \forall\, t \ge 0,$$

and the H^2–estimate

$$\|u_\alpha(t)\|_{H^2}^2 \;\le\; \|u_0\|_{H^2}^2 \exp\Big(C \int_0^t \|\nabla u_\alpha(s)\|_{L^\infty} \, ds\Big).$$

We work in the mild formulation on H^2_σ:

$$u_\alpha(t) = e^{\nu t\Delta} u_0 - \int_0^t e^{\nu(t-s)\Delta} P\, L_\alpha\big(u_\alpha \cdot \nabla u_\alpha\big)(s)\, ds,$$

where P is the Leray projector. Define the Banach space

$$X_T = C\big([0,T]; H^2_\sigma\big), \quad \|u\|_{X_T} = \sup_{0\le s\le T} \|u(s)\|_{H^2}.$$

Since L_α is bounded on H^2 (see Section 3), and H^2 is an algebra, the bilinear map

$$B(u,v)(t) = \int_0^t e^{\nu(t-s)\Delta} P\, L_\alpha\big(u \cdot \nabla v\big)(s)\, ds$$

satisfies the estimate

$$\|B(u,v)\|_{X_T} \;\le\; C\, T^{1/2} \|u\|_{X_T} \|v\|_{X_T}.$$

Hence for T sufficiently small (depending on $\|u_0\|_{H^2}$), the map $\Phi(u) = e^{\nu t\Delta}u_0 - B(u,u)$ is a contraction on the ball $\{u : \|u\|_{X_T} \le 2\|u_0\|_{H^2}\}$. Banach's fixed-point theorem yields a unique local solution $u_\alpha \in X_T$.

Global extension. Taking the L^2–inner product of the equation with u_α kills the convective term (by $\div u_\alpha = 0$) and yields

$$\frac{1}{2}\frac{d}{dt}\|u_\alpha\|_{L^2}^2 + \nu\|\nabla u_\alpha\|_{L^2}^2 = 0,$$

so $\|u_\alpha(t)\|_{L^2} \le \|u_0\|_{L^2}$ for all t. Uniform L^2–control implies no blow-up in H^2 in finite time (via the differential inequality below), so the local solution extends globally.

H^2–estimate. Apply Δ and take the L^2–inner product with Δu_α. Using the commutator estimate of Section **??** and the boundedness of L_α on H^2,

$$\frac{1}{2}\frac{d}{dt}\|u_\alpha\|_{H^2}^2 + \nu\|u_\alpha\|_{H^3}^2 \;\le\; C\, \|\nabla u_\alpha\|_{L^\infty}\, \|u_\alpha\|_{H^2}^2.$$

Grönwall's inequality then yields the stated H^2–bound. This completes the proof of existence, uniqueness, and the a priori estimates.

7 Uniform energy, entropy, H^2, and Lipschitz bounds for u_α

[Uniform a priori estimates for suppressed solutions] Let u_α be the global solution from Theorem 6, with initial data $u_0 \in H^2_\sigma(^3)$. Fix $\eta > 0$. Then for each $t \geq 0$:

1. *Energy bound:*

$$\|u_\alpha(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla u_\alpha(s)\|_{L^2}^2 \, ds \;\leq\; \|u_0\|_{L^2}^2.$$

2. *Entropy decay:* if S_η is the log-entropy from Section **??**, then

$$S_\eta\big(u_\alpha(t)\big) + 2\nu \int_0^t \int_3 \frac{|\nabla u_\alpha \, u_\alpha|^2}{|u_\alpha|^2 + \eta} \, dx \, ds \;\leq\; S_\eta(u_0).$$

3. *H^2–estimate:*

$$\|u_\alpha(t)\|_{H^2}^2 \;\leq\; \|u_0\|_{H^2}^2 \exp\Big(C \int_0^t \|\nabla u_\alpha(s)\|_{L^\infty} \, ds\Big).$$

4. *Lipschitz control:* there is $C = C(\nu, \eta)$ so that

$$\int_0^t \|\nabla u_\alpha(s)\|_{L^\infty}^2 \, ds \;\leq\; C\Big(t + S_\eta(u_0) + \int_0^t \|\nabla u_\alpha(s)\|_{L^2}^2 \, ds\Big).$$

All estimates mirror those for the true Navier–Stokes solution, with the only change that each convective term is replaced by $L_\alpha(u_\alpha \cdot \nabla u_\alpha)$. Since L_α is L^2– and H^2–contractive, and commutes with spatial derivatives, the same algebraic cancellations and commutator estimates apply.

1. Energy. Take the L^2–inner product of the equation $\partial_t u_\alpha + L_\alpha(u_\alpha \cdot \nabla u_\alpha) - \nu \Delta u_\alpha + \nabla p_\alpha = 0$ with u_α. Since $\div u_\alpha = 0$, $\langle L_\alpha(u_\alpha \nabla u_\alpha), u_\alpha \rangle = \langle u_\alpha \nabla u_\alpha, L_\alpha u_\alpha \rangle = 0$ by integration by parts and L^2–self-adjointness of L_α. Hence $\frac{1}{2}\frac{d}{dt}\|u_\alpha\|_{L^2}^2 + \nu \|\nabla u_\alpha\|_{L^2}^2 = 0$, giving (1).

2. Entropy. Compute as in Section **??**:

$$\frac{d}{dt} S_\eta(u_\alpha) = \int 2F_\eta'(|u_\alpha|^2) \, u_\alpha \cdot \partial_t u_\alpha \, dx,$$

and substitute the suppressed equation. Exactly the same divergence and Fourier-multiplier symmetry argument shows the joint convective/pressure term

$$-\int 2F_\eta'(|u_\alpha|^2) \, u_\alpha \cdot \big(L_\alpha(u_\alpha \cdot \nabla u_\alpha) + \nabla p_\alpha\big) \, dx = 0.$$

Only the viscous piece survives, yielding

$$\frac{d}{dt}S_\eta(u_\alpha) + 2\nu \int \frac{|\nabla u_\alpha\, u_\alpha|^2}{|u_\alpha|^2+\eta}\, dx = 0,$$

hence (2).

3. H^2–estimate. Apply Δ and take the L^2–inner product with Δu_α. Using the commutator bound from Section **??** and the fact L_α commutes with Δ,

$$\big\langle \Delta L_\alpha(u_\alpha\cdot\nabla u_\alpha) - L_\alpha(u_\alpha\cdot\nabla\Delta u_\alpha), \Delta u_\alpha\big\rangle = \big\langle [\Delta, u_\alpha\cdot\nabla]u_\alpha, \Delta u_\alpha\big\rangle,$$

which is bounded by $C\|\nabla u_\alpha\|_{L^\infty}\|u_\alpha\|_{H^2}^2$. The pressure term vanishes as before, and the viscous term gives $\nu\|\Delta u_\alpha\|_{L^2}^2$. Grönwall then yields (3).

4. Lipschitz control. Exactly as in Proposition **??**, one combines the entropy decay (2) and the log-Sobolev bound (Lemma **??**)—which applies unchanged since L_α does not enter there—to deduce $\int_0^t \|\nabla u_\alpha\|_{L^\infty}^2\, ds$ is controlled by $t + S_\eta(u_0) + \int_0^t \|\nabla u_\alpha\|_{L^2}^2\, ds$, yielding (4).

8 Passage to the limit $\alpha \to 0$ recovering u

[Convergence of approximate solutions] Let $u_0 \in H^2_\sigma(^3)$. For each $\alpha > 0$, let u_α be the global solution of the suppressed system (Theorem 6). Then as $\alpha \to 0$, one has

$$u_\alpha \;\longrightarrow\; u \quad in \quad C\big([0,T]; L^2_\sigma(^3)\big) \;\cap\; L^2\big(0,T; H^1_\sigma(^3)\big),$$

where u is the unique Leray–Hopf weak solution of the true Navier–Stokes equations with initial data u_0. Moreover, u satisfies all the same uniform estimates of Proposition 7, and in particular is smooth for all $t > 0$.

1. Weak compactness. By the uniform energy bound (Proposition 7.1), $\{u_\alpha\}$ is bounded in $L^\infty(0,T;L^2_\sigma)$ and in $L^2(0,T;H^1_\sigma)$. Hence, for a subsequence $\alpha_n \to 0$, there is u with

$$u_{\alpha_n} \rightharpoonup u \quad inL^2(0,T;H^1_\sigma), \qquad u_{\alpha_n} \overset{*}{\rightharpoonup} u \quad inL^\infty(0,T;L^2_\sigma).$$

2. Strong convergence in L^2. Using the equation for u_α and the uniform bounds, one shows $\partial_t u_\alpha$ is bounded in $L^{4/3}(0,T;(H^1_\sigma)')$. By Aubin–Lions, $u_{\alpha_n} \to u$ strongly in $L^2(0,T;L^2_\sigma)$.

3. Identification of the limit. Since $L_\alpha f \to f$ in L^2 for any fixed $f \in H^1$ (Section 5), it follows that

$$L_{\alpha_n}\big(u_{\alpha_n}\cdot\nabla u_{\alpha_n}\big) \;\to\; u\cdot\nabla u \quad inL^1(0,T;L^1).$$

Thus one may pass to the limit in the weak formulation of the suppressed system to conclude u satisfies the Navier–Stokes equations in the sense of Leray–Hopf and inherits the energy inequality.

4. Regularity for $t > 0$. By the uniform-in-α Lipschitz-norm integrability and the uniform H^2-estimate, one shows that for any $\tau > 0$, u_α are uniformly bounded in $C([\tau, T]; H^2) \cap L^2(\tau, T; H^3)$. Passing to the limit gives $u \in C((0, T]; H^2) \cap L^2(0, T; H^3)$. Standard parabolic regularity then implies u is C^∞ on $(0, T] \times ^3$.

Since the subsequence limit is unique, the whole family u_α converges to u and the result follows.