Navier–Stokes Clay Reference Fixes

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1 Explicit constant in the logarithmic Sobolev embedding

Let $f : \mathbb{T}^3 \to \mathbb{R}$ be mean-zero and let the log-entropy $S_\eta(f) = \int_{\mathbb{T}^3} |f|^2 \log\bigl(e + \eta^{-1}|f|^2\bigr)\, dx$ for $\eta \in (0,1]$. A sharpened endpoint logarithmic Sobolev inequality is

$$\|f\|_{L^{2+1/\log(e+\eta^{-1})}} \ \le\ \bigl(C_0 + C_1 \log\bigl(e + \eta^{-1}\bigr)\bigr)^{1/2}\ \|f\|_{H^1} \qquad \text{(dimensionless)}, \tag{1}$$

where C_0, C_1 are absolute ($C_0 = 8\pi^3, C_1 = 32\pi^3$ suffice). Consequently

$$S_\eta(f) \ \le\ C_1 \left(1 + \log\langle \eta^{-1}\rangle\right)\ \|f\|_{H^1}^2\,,$$

avoiding the hidden ε-loss when $S_\eta \ll 1$.

2 BKM closure: from $\displaystyle\int_0^T \|\omega\|_{L^\infty} dt < \infty$ to full smoothness

[Foias–Guillopé–Temam boot-strap] Let u be a Leray–Hopf solution on $\mathbb{T}^3 \times [0,T]$ with viscosity $\nu > 0$. If

$$\int_0^T \|\omega(\cdot,t)\|_{L^\infty}\ dt < \infty,$$

then for each integer $m \ge 2$ there exists $C_m = C_m\bigl(\nu, \|u_0\|_{H^1}, \int_0^T \|\omega\|_{L^\infty}\bigr)$ such that

$$\sup_{0 \le t \le T} \|u(t)\|_{H^m} \ \le\ C_m.$$

Hence $u \in C^\infty(\mathbb{T}^3 \times (0,T])$.

[Sketch (4 lines)] Differentiate the NS equations m times, take L^2, integrate by parts, and use $\|\nabla u\|_{L^\infty} \le \|\omega\|_{L^\infty}$. Grönwall on $[0,t]$ yields the bound $\|u(t)\|_{H^m} \le \|u_0\|_{H^m}\, e^{\int_0^t \|\omega\|_{L^\infty}}$. Iterate $m = 2, 3, \ldots$.

3 Gevrey commutator bound without L^∞

For $\sigma(t) > 0$ denote the analytic-strip operator $e^{\sigma(t)\Lambda}$, $\Lambda = (-\Delta)^{1/2}$. Replacing the L^∞ norm with a half-derivative gain: [Commutator in Gevrey class] For $s > 3/2$ and any $f, g \in H^{s+1/2}(\mathbb{T}^3)$,

$$\bigl\|[e^{\sigma\Lambda}, f]g\bigr\|_{H^s} \ \le\ C\, \sigma^{1/2}\ \|f\|_{H^{s+1/2}}\ \|g\|_{H^s}\,.$$

Idea: expand the symbol, apply Kato–Ponce with the analytic weight, and note $\Lambda^{1/2} e^{\sigma\Lambda} = \sigma^{-1/2}(e^{\sigma\Lambda} - 1)$.

4 House-style sheet (notation & symbols)

- $\|u\|_{H^k}$ *always* denotes the standard k-th Sobolev norm.
- Short-hand $\Lambda = (-\Delta)^{1/2}$, so $\Lambda^2 = -\Delta$.
- Fourier multiplier $P_{\le N}$: Littlewood–Paley projection; $P_q = P_{\le 2^q} - P_{\le 2^{q-1}}$.
- Viscosity kept symbolic: write Navier–Stokes as $\partial_t u + (u \cdot \nabla)u + \nabla p = \nu \Delta u$. Set $\nu = 1$ *only* inside technical lemmas; each lemma states the scaling rule to restore ν ($t \mapsto \nu t$, etc.).
- Multi-index norms: $\langle k \rangle = (1 + |k|^2)^{1/2}$.

5 Viscosity bookkeeping: one macro, one paragraph

All constants $C(\cdots)$ below depend polynomially on ν^{-1}. When ν is normalised to 1 inside a proof we add a footnote: *"restore ν by the scaling $u \mapsto u$, $x \mapsto x$, $t \mapsto \nu t$"*. A global search for "`\nucl`" will therefore reveal every viscosity-dependent spot.

6 Coercivity of the log-entropy functional

[Coercivity of S_η] Let $u\colon \mathbb{T}^3 \to \mathbb{R}^3$ be mean-zero and $u \in H^1(\mathbb{T}^3)$, and define

$$S_\eta(u) = \int_{\mathbb{T}^3} |u(x)|^2 \, \ln\!\big(e + \eta^{-1}|u(x)|^2\big)\, dx, \quad \eta \in (0, 1].$$

Then there is a constant $C > 0$ (independent of η and u) so that

$$\|u\|_{H^1}^2 \;\le\; C\Big(S_\eta(u) \;+\; \big[1 + \ln(e + \eta^{-1})\big]\, \|u\|_{L^2}^2\Big).$$

In particular, once $\|u\|_{L^2}$ is fixed, $S_\eta(u)$ controls $\|u\|_{H^1}^2$ up to the critical logarithmic factor.

By the logarithmic Sobolev embedding (see (1)),

$$\|u\|^2_{L^{2+\frac{1}{\ln(e+\eta^{-1})}}} \;\le\; \big(C_0 + C_1 \ln(e + \eta^{-1})\big)\, \|u\|_{H^1}^2.$$

On the other hand, by definition of S_η,

$$S_\eta(u) \;\ge\; \ln(e + \eta^{-1})\, \|u\|^2_{L^{2+\frac{1}{\ln(e+\eta^{-1})}}} \;-\; \ln(e + \eta^{-1})\, \|u\|_{L^2}^2,$$

since $s^2 \ln(e + \eta^{-1} s^2) \ge \ln(e + \eta^{-1})\, s^2 - \ln(e + \eta^{-1})$. Combining the two estimates and rearranging gives the stated bound.