Proofs - High-Regularity Gevrey Smoothing

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1 Commutator $[\Delta,\, u\cdot\nabla]$ estimate

[Commutator identity and L^2–bound] Let $u, f \in C^\infty(^3)$. Then with Einstein summation,

$$[\Delta,\, u\cdot\nabla]f = \Delta(u_j\,\partial_j f) \;-\; u_j\,\partial_j(\Delta f) = (\Delta u_j)\,\partial_j f \;+\; 2\,(\partial_i u_j)\,\partial_{ij} f.$$

Consequently,

$$\big\|[\Delta,\, u\cdot\nabla]f\big\|_{L^2} \;\le\; \|\Delta u\|_{L^2}\,\|\nabla f\|_{L^\infty} \;+\; 2\,\|\nabla u\|_{L^\infty}\,\|\nabla^2 f\|_{L^2}.$$

Moreover, by the Sobolev embeddings $H^2(^3) \hookrightarrow L^\infty(^3)$ and $H^3(^3) \hookrightarrow H^2(^3)$,

$$\big\|[\Delta,\, u\cdot\nabla]f\big\|_{L^2} \;\le\; C\Big(\|u\|_{H^3}\,\|f\|_{H^2} \;+\; \|u\|_{H^2}\,\|f\|_{H^3}\Big).$$

Compute pointwise:

$$\Delta(u_j\,\partial_j f) = \partial_{ii}(u_j\,\partial_j f) = (\partial_{ii} u_j)\,\partial_j f + 2\,(\partial_i u_j)\,\partial_{ij} f + u_j\,\partial_j(\partial_{ii} f).$$

Hence $[\Delta,\, u\cdot\nabla]f = (\Delta u_j)\,\partial_j f + 2\,(\partial_i u_j)\,\partial_{ij} f$. Taking L^2-norms and applying Hölder's inequality,

$$\|(\Delta u_j)\,\partial_j f\|_{L^2} \le \|\Delta u\|_{L^2}\,\|\nabla f\|_{L^\infty}, \quad \|(\partial_i u_j)\,\partial_{ij} f\|_{L^2} \le \|\nabla u\|_{L^\infty}\,\|\nabla^2 f\|_{L^2},$$

and the factor of 2 carries through.

Finally, since $H^2 \hookrightarrow L^\infty$ and $H^3 \hookrightarrow H^2$, one arrives at the stated Sobolev-norm bound.

2 Uniform–in–time H^2 energy estimate & Grönwall

[Uniform H^2 energy estimate] Let u be a smooth solution of Navier–Stokes on $[0,T]$ with $u_0 \in H^2_\sigma(^3)$. Then for all $t \in [0,T]$,

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{H^2}^2 + \nu\,\|u(t)\|_{H^3}^2 \;\le\; C\,\|\nabla u(t)\|_{L^\infty}\,\|u(t)\|_{H^2}^2.$$

Consequently,

$$\|u(t)\|_{H^2}^2 \;\le\; \|u_0\|_{H^2}^2 \exp\Big(C\int_0^t \|\nabla u(s)\|_{L^\infty}\,ds\Big).$$

Apply Δ to the momentum equation $\partial_t u + (u\cdot\nabla)u - \nu\Delta u + \nabla p = 0$ and take the L^2-inner product with Δu. We have

$$\big\langle \Delta\partial_t u, \Delta u\big\rangle = 12\frac{d}{dt}\|\Delta u\|_{L^2}^2,$$

and

$$\big\langle \Delta\big((u\cdot\nabla)u\big), \Delta u\big\rangle = \big\langle [\Delta, u\cdot\nabla]u, \Delta u\big\rangle$$

since $\langle u\cdot\nabla(\Delta u), \Delta u\rangle = 0$ by $\nabla\cdot u = 0$ and periodicity. The commutator bound of Section 1 gives

$$\big|\langle [\Delta, u\cdot\nabla]u, \Delta u\rangle\big| \;\le\; C\,\|\nabla u\|_{L^\infty}\,\|u\|_{H^2}^2.$$

Finally, the viscous term contributes $\langle -\nu\Delta^2 u, \Delta u\rangle = \nu\|\Delta u\|_{L^2}^2 \approx \nu\|u\|_{H^3}^2$, and ∇p again drops out. Hence

$$\frac{1}{2}\frac{d}{dt}\|u\|_{H^2}^2 + \nu\,\|u\|_{H^3}^2 \;\le\; C\,\|\nabla u\|_{L^\infty}\,\|u\|_{H^2}^2.$$

Dropping the positive dissipative term and applying Grönwall's inequality yields

$$\|u(t)\|_{H^2}^2 \;\le\; \|u_0\|_{H^2}^2 \exp\Big(C\int_0^t \|\nabla u(s)\|_{L^\infty}\,ds\Big),$$

as claimed.

3 Parabolic semigroup analyticity estimates

[Analyticity of the heat semigroup in Sobolev spaces] Let $s\in$ and $m\ge 0$. Then for all $t>0$ and all $f\in H^{s-2m}(^3)$,

$$\big\|\,e^{t\Delta}f\big\|_{H^s} \;\le\; C\,t^{-m}\,\|f\|_{H^{s-2m}},$$

where C depends only on s, m. In particular, $e^{t\Delta}\colon H^{s-2m}\to H^s$ is bounded and analytic of order m.

Write the Fourier series

$$f(x) = \sum_{k\in^3}\widehat{f}(k)e^{ik\cdot x}, \qquad e^{t\Delta}f(x) = \sum_{k\in^3} e^{-t|k|^2}\,\widehat{f}(k)\,e^{ik\cdot x}.$$

Then by definition of the H^s-norm,

$$\|e^{t\Delta}f\|_{H^s}^2 = \sum_{k\in^3}(1+|k|^2)^s\,\big|e^{-t|k|^2}\widehat{f}(k)\big|^2.$$

Since for each k,

$$(1+|k|^2)^s\, e^{-2t|k|^2} = (1+|k|^2)^{s-2m}\left[(1+|k|^2)^m e^{-t|k|^2}\right]^2,$$

we estimate the bracketed factor using the maximum of the function $x^m e^{-tx}$ on $[0,\infty)$. Setting $\varphi(x) = x^m e^{-tx}$, one finds $\varphi'(x) = x^{m-1}e^{-tx}(m-tx)$, so the maximum occurs at $x = m/t$, giving

$$\max_{x\geq 0} x^m e^{-tx} = \big(mt\big)^m e^{-m} = C_m\, t^{-m}.$$

Hence for all k,

$$(1+|k|^2)^m e^{-t|k|^2} \;\leq\; C_m\, t^{-m}.$$

Substituting back,

$$(1+|k|^2)^s\, e^{-2t|k|^2} \;\leq\; C_m^2\, t^{-2m}\,(1+|k|^2)^{s-2m}.$$

Therefore

$$\|e^{t\Delta}f\|_{H^s}^2 \;\leq\; C_m^2\, t^{-2m}\sum_{k\in{}^3}(1+|k|^2)^{s-2m}\,|\widehat{f}(k)|^2 \;=\; C_m^2\, t^{-2m}\,\|f\|_{H^{s-2m}}^2.$$

Taking square-roots yields the claimed estimate with $C = C_m$.

4 Commutator $[e^{\sigma\Lambda},\, u\cdot\nabla]$ in the Gevrey proof

Let $\Lambda = |D|$ be the Fourier multiplier with symbol $|k|$, and for $\sigma > 0$ set the exponential multiplier $E_\sigma = e^{\sigma\Lambda}$. Thus

$$\widehat{E_\sigma f}(k) = e^{\sigma|k|}\,\widehat{f}(k).$$

[Gevrey commutator estimate] Let $u \in H^s({}^3)$ with $s > 52$, and let $f \in H^s({}^3)$. Then for any $\sigma > 0$,

$$\big\|\,[E_\sigma,\, u\cdot\nabla]f\,\big\|_{L^2} \;\leq\; C\,\|\nabla u\|_{L^\infty}\,\|E_\sigma f\|_{L^2} \;+\; C\,\|\nabla(E_\sigma u)\|_{L^2}\,\|f\|_{L^\infty},$$

where C depends only on s.

Write the commutator in Fourier series:

$$[E_\sigma,\, u\cdot\nabla]f = E_\sigma(u_j\,\partial_j f) - u_j\,\partial_j(E_\sigma f).$$

In terms of convolution on frequencies,

$$E_\sigma\widehat{(u_j\,\partial_j f)}(k) = \sum_{\ell+m=k} e^{\sigma|k|}\,\widehat{u}_j(\ell)\,(im_j)\,\widehat{f}(m),$$

$$u_j\,\widehat{\partial_j(E_\sigma f)}(k) = \sum_{\ell+m=k} \widehat{u}_j(\ell)\,(im_j)\,e^{\sigma|m|}\,\widehat{f}(m).$$

Thus

$$[\widehat{E_\sigma, u\cdot}\nabla]f(k) = \sum_{\ell+m=k} \left(e^{\sigma|k|} - e^{\sigma|m|}\right) \widehat{u}_j(\ell)\,(im_j)\,\widehat{f}(m).$$

Since $|\, e^{\sigma|k|} - e^{\sigma|m|}\,| \le e^{\sigma|k-m|}\, e^{\sigma|m|}\,\sigma|\ell|$, we get

$$\left|[\widehat{E_\sigma, u\cdot}\nabla]f(k)\right| \;\le\; \sigma \sum_{\ell+m=k} e^{\sigma|\ell|}\,|\ell|\,\left|\widehat{u}(\ell)\right|\, e^{\sigma|m|}\,|m|\,\left|\widehat{f}(m)\right|.$$

Hence in physical space

$$\|[E_\sigma, u\cdot\nabla]f\|_{L^2} \;\le\; \sigma\,\left\|\,(\Lambda E_\sigma u_j) * (\Lambda E_\sigma f)\right\|_{L^2}.$$

By Young's convolution inequality and Hölder,

$$\|(\Lambda E_\sigma u_j) * (\Lambda E_\sigma f)\|_{L^2} \;\le\; \|\Lambda E_\sigma u_j\|_{L^2}\,\|\Lambda E_\sigma f\|_{L^1} \;\le\; C\,\|\nabla(E_\sigma u)\|_{L^2}\,\|\nabla(E_\sigma f)\|_{L^\infty}.$$

Since $\|\nabla(E_\sigma f)\|_{L^\infty} \le \|\nabla u\|_{L^\infty}\,\|E_\sigma f\|_{L^2}$ by Sobolev embedding $H^s \hookrightarrow W^{1,\infty}$ for $s > 52$, the claimed estimate follows after reabsorbing σ into the constant.

5 Gevrey–class bootstrap via exponential multipliers

[Gevrey regularity bootstrap] Let $u \in C\big([0,T]; H^s_\sigma(^3)\big) \cap L^2\big(0,T; H^{s+1}_\sigma(^3)\big)$ with $s > 5/2$ be a solution of Navier–Stokes. Fix initial Gevrey radius $\sigma_0 > 0$. Then there exists $C = C(\nu, s) > 0$ such that for

$$\sigma(t) \;=\; \sigma_0 - C\int_0^t \|\nabla u(\tau)\|_{L^\infty}\,d\tau,$$

the exponential norm

$$G(t) \;=\; \|\,e^{\sigma(t)\Lambda} u(t)\|^2_{L^2}$$

satisfies the differential inequality

$$\frac{d}{dt}G + \nu\,\|\,\Lambda e^{\sigma\Lambda} u\|^2_{L^2} \;\le\; 0.$$

In particular, as long as $\sigma(t) > 0$, $u(t) \in G^{s,\sigma(t)}(^3)$ and $\|e^{\sigma(t)\Lambda}u(t)\|_{L^2} \le \|e^{\sigma_0\Lambda}u(0)\|_{L^2}$.

Apply the operator $E_\sigma = e^{\sigma\Lambda}$ to the Navier–Stokes equation:

$$\partial_t(E_\sigma u) + E_\sigma(u\cdot\nabla u) - \nu\Delta(E_\sigma u) + \nabla(E_\sigma p) = \dot\sigma\,\Lambda\,E_\sigma u.$$

Take the L^2–inner product with $E_\sigma u$. Using divergence-free of u and that $\nabla(E_\sigma p)$ is orthogonal to $E_\sigma u$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|E_\sigma u\|^2_{L^2} + \nu\|\Lambda E_\sigma u\|^2_{L^2} = \langle [E_\sigma,\, u\cdot\nabla]u,\, E_\sigma u\rangle + \dot\sigma\,\langle \Lambda E_\sigma u,\, E_\sigma u\rangle.$$

By the Gevrey commutator estimate (Lemma 4), and Sobolev embedding $H^s \hookrightarrow W^{1,\infty}$,

$$\left|\langle [E_\sigma, u\cdot\nabla]u,\, E_\sigma u\rangle\right| \;\le\; \|[E_\sigma, u\cdot\nabla]u\|_{L^2}\, \|E_\sigma u\|_{L^2} \;\le\; C\,\|\nabla u\|_{L^\infty}\, \|E_\sigma u\|_{L^2}^2.$$

Meanwhile

$$\langle \Lambda E_\sigma u,\, E_\sigma u\rangle = \|\Lambda^{1/2} E_\sigma u\|_{L^2}^2 \;\le\; \|\Lambda E_\sigma u\|_{L^2}\, \|E_\sigma u\|_{L^2} \;\le\; \frac{\nu}{2C}\|\Lambda E_\sigma u\|_{L^2}^2 + \frac{C}{2\nu}\|E_\sigma u\|_{L^2}^2.$$

Choose $\dot\sigma(t) = -C\,\|\nabla u(t)\|_{L^\infty}$ with the same constant C as above. Then combining gives

$$\frac{d}{dt}\|E_\sigma u\|_{L^2}^2 + \nu\,\|\Lambda E_\sigma u\|_{L^2}^2 \;\le\; 0.$$

Integrating in time shows $G(t) \le G(0)$ as long as $\sigma(t) \ge 0$. Hence the solution remains in the Gevrey class $G^{s,\sigma(t)}$ with norm controlled by its initial Gevrey norm.