Proofs - Vorticity Blow-up Criterion

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1 Vorticity formulation & Beale–Kato–Majda (BKM) criterion

Let $u\colon [0,T)\times^3 \to^3$ be a classical solution of Navier–Stokes with

$$u(\cdot,t) \in H^s_\sigma(^3), \qquad s > 52,$$

on its maximal interval of existence $[0,T^*)$. Denote the vorticity $\omega = \nabla \times u$.

[Vorticity equation] The vorticity ω satisfies

$$\partial_t \omega + (u \cdot \nabla)\,\omega = (\omega \cdot \nabla)\,u + \nu\,\Delta\omega,$$

with $\nabla \cdot \omega = 0$ and $\omega(\cdot,0) = \nabla \times u_0$.

Take the curl of $\partial_t u + (u \cdot \nabla)u - \nu\Delta u + \nabla p = 0$. Using $\nabla \times \nabla p = 0$ and $\nabla \times \Delta u = \Delta\omega$, and the identity $\nabla \times ((u \cdot \nabla)u) = (u \cdot \nabla)\omega - (\omega \cdot \nabla)u$, yields the stated equation.

[Beale–Kato–Majda blow-up criterion] Under the above hypotheses, if

$$\int_0^{T^*} \|\omega(\cdot,t)\|_{L^\infty}\,dt < \infty,$$

then the solution u can be continued smoothly beyond T^*. Equivalently, if $T^* < \infty$, then $\displaystyle\int_0^{T^*} \|\omega(t)\|_{L^\infty}\,dt = \infty$.

Fix $s > 52$. Apply the H^s-energy estimate to the Navier–Stokes equations:

$$\frac{1}{2}\frac{d}{dt}\|u\|_{H^s}^2 + \nu\|u\|_{H^{s+1}}^2 \;=\; -\int_3 \big[D^s(u \cdot \nabla u)\big] \cdot D^s u\,dx.$$

Since $H^s(^3)$ is an algebra and $s > 52$, commutator estimates (Kato–Ponce) give

$$\big|\langle D^s(u \cdot \nabla u),\, D^s u\rangle\big| \;\le\; C\|\nabla u\|_{L^\infty}\,\|u\|_{H^s}^2.$$

Hence

$$\frac{d}{dt}\|u\|_{H^s}^2 + 2\nu\|u\|_{H^{s+1}}^2 \;\le\; C\,\|\nabla u\|_{L^\infty}\,\|u\|_{H^s}^2.$$

By the Biot–Savart law on ^{3},

$$\|\nabla u\|_{L^\infty} \;\le\; C\,\|\omega\|_{L^\infty}.$$

Dropping the positive $\|u\|_{H^{s+1}}^2$ term and setting $Y(t) = \|u(t)\|_{H^s}^2$, we have

$$\frac{dY}{dt} \;\le\; C\,\|\omega(t)\|_{L^\infty}\,Y(t).$$

Gronwall's inequality then gives, for all $0 \le t < T^*$,

$$\|u(t)\|_{H^s}^2 \;\le\; \exp\Bigl(C\int_0^t \|\omega(\tau)\|_{L^\infty}\,d\tau\Bigr)\,\|u_0\|_{H^s}^2.$$

If $\int_0^{T^*} \|\omega\|_{L^\infty} < \infty$, the right-hand side remains finite as $t \to T^*$. Hence $\|u(t)\|_{H^s}$ stays bounded up to T^*, allowing the local-existence theorem in H^s to extend the solution beyond T^*. This contradiction shows that blow-up at $T^* < \infty$ can only occur if $\int_0^{T^*} \|\omega\|_{L^\infty}\,dt = \infty$.

2 Biot–Savart law on ^{3}

[Biot–Savart representation and gradient bound] Let $\omega \in C^0(^3;^3)$ be divergence-free with zero spatial mean. Define

$$u(x) \;=\; \int_{^3} K(x-y)\times\omega(y)\,dy,$$

where K is the periodic Biot–Savart kernel

$$K(x) \;=\; \nabla G(x), \qquad G(x) \;=\; -\sum_{m\in^3}\frac{1}{4\pi\,|x-2\pi m|},$$

the fundamental solution of $-\Delta$ on ^{3}. Then:

1. u is divergence-free, mean-zero, and satisfies $\nabla\times u = \omega$.
2. There is a constant C (independent of ω) such that

$$\|\nabla u\|_{L^\infty(^3)} \;\le\; C\,\|\omega\|_{L^\infty(^3)}.$$

1. **Representation and divergence-free property.** Since G satisfies $-\Delta G = \delta - (2\pi)^{-3}$ and has zero mean, one checks by differentiation under the integral that

$$\nabla\cdot u(x) = \int_{^3} \nabla\cdot\bigl(K(x-y)\times\omega(y)\bigr)\,dy = 0,$$

and

$$\nabla\times u(x) = \int_{^3} \nabla_x\times\bigl(K(x-y)\times\omega(y)\bigr)\,dy = \int_{^3} \omega(y)\,\delta(x-y)\,dy = \omega(x),$$

using $\nabla \times \nabla G = 0$ and the identity $\nabla_x \times \big(\nabla_x G \times \omega\big) = \omega\,\Delta G$.

2. ** $L^\infty \to L^\infty$ bound on ∇u.** Observe that

$$\partial_m u_i(x) = \int_3 \partial_m K_i^j(x-y)\,\omega_j(y)\,dy,$$

where $\partial_m K_i^j(z)$ is the mth derivative of the ith component of K paired with the jth entry of ω. One checks that the kernel $\partial_m K_i^j(z)$ is a periodic Calderón–Zygmund kernel satisfying

$$\big|\partial_m K_i^j(z)\big| \;\; |z|^{-3}, \qquad \big|\nabla_z \partial_m K_i^j(z)\big| \;\; |z|^{-4},$$

and has mean zero over each sphere. Hence by the periodic Calderón–Zygmund theorem, this convolution operator is bounded on $L^p(^3)$ for $1 < p < \infty$ and maps L^∞ into BMO :contentReference[oaicite:0]index=0:contentReference[oaicite:1]index=1.

Finally, since in our context ω is continuous (classical solution), BMO- and continuity-norms coincide up to a constant, giving

$$\|\partial_m u_i\|_{L^\infty} \;\le\; C\,\|\omega\|_{L^\infty}.$$

Taking the supremum over i, m yields the claimed Lipschitz bound.