Proofs – Log-Entropy & Lipschitz Control

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1 Definition & Basic Properties of the Log-Entropy $S_\eta(u)$

Definition 1.1. Fix a parameter $\eta > 0$. Define the scalar function

$$F_\eta(s) = (s+\eta)\ln\Big(1+\frac{s}{\eta}\Big) - s, \quad s \geq 0,$$

and for a velocity field $u : \mathbb{T}^3 \to \mathbb{R}^3$ set

$$S_\eta(u) = \int_{\mathbb{T}^3} F_\eta\big(|u(x)|^2\big)\,dx.$$

Lemma 1.1 (Basic properties of S_η)**.** *The map $s \mapsto F_\eta(s)$ and the functional $u \mapsto S_\eta(u)$ satisfy:*

1. Non-negativity: *For all $s \geq 0$, $F_\eta(s) \geq 0$.*
2. Smoothness and convexity: *$F_\eta \in C^2([0,\infty))$ with*

$$F'_\eta(s) = \ln\Big(1+\frac{s}{\eta}\Big), \qquad F''_\eta(s) = \frac{1}{s+\eta} > 0.$$

3. Gateaux derivative: *If $\varphi \in C^\infty(\mathbb{T}^3;\mathbb{R}^3)$ is divergence-free, then*

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} S_\eta(u+\epsilon\varphi) = \int_{\mathbb{T}^3} 2\,\varphi(x)\cdot u(x)\,\ln\Big(1+\tfrac{|u(x)|^2}{\eta}\Big)\,dx.$$

4. Uniform quadratic growth: *Let $s_0 = \eta$, and set*

$$C_1 = \max_{s\in[0,s_0]} \frac{F_\eta(s)}{s^2/(s+\eta)}, \qquad C_2 = \frac{1}{2\eta}, \qquad C = \max\{C_1, C_2\}.$$

Then for all $s \geq 0$,

$$F_\eta(s) \leq C\,\frac{s^2}{s+\eta}.$$

Proof. **(1) Non-negativity.** Set $t = s/\eta$. Define

$$g(t) = (t+1)\ln(1+t) - t.$$

Then $g(0) = 0$, and

$$g'(t) = \ln(1+t) \geq 0 \quad (\forall t \geq 0),$$

so $g(t) \geq 0$. Hence

$$F_\eta(s) = \eta\, g\big(s/\eta\big) \geq 0.$$

(2) Smoothness and convexity. Direct differentiation yields

$$F_\eta'(s) = \frac{d}{ds}\Big[(s+\eta)\ln(1+s/\eta) - s\Big] = \ln\Big(1 + \frac{s}{\eta}\Big),$$

and

$$F_\eta''(s) = \frac{d}{ds}\ln\Big(1+\frac{s}{\eta}\Big) = \frac{1}{s+\eta} > 0.$$

Thus F_η is convex.

(3) Gateaux derivative. Write

$$S_\eta(u+\epsilon\varphi) = \int F_\eta\big(|u+\epsilon\varphi|^2\big)\,dx.$$

By the chain rule,

$$\frac{d}{d\epsilon}F_\eta\big(|u+\epsilon\varphi|^2\big) = F_\eta'(|u+\epsilon\varphi|^2)\cdot 2(u+\epsilon\varphi)\cdot\varphi.$$

Evaluating at $\epsilon = 0$ gives

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} S_\eta(u+\epsilon\varphi) = \int 2F_\eta'(|u|^2)\,u\cdot\varphi\,dx = \int 2\,u\cdot\varphi\,\ln\Big(1+\frac{|u|^2}{\eta}\Big)dx,$$

as claimed.

(4) Uniform quadratic growth. On $s \in [0, s_0]$ the continuous function $s \mapsto F_\eta(s)/(s^2/(s+\eta))$ attains a maximum $C_1 < \infty$. For $s \geq s_0 = \eta$, we write

$$F_\eta(s) = (s+\eta)\ln\Big(1+\frac{s}{\eta}\Big) - s \leq (s+\eta)\frac{s}{\eta} - s = \frac{s^2}{\eta}.$$

Thus $F_\eta(s) \leq (1/(2\eta))\,s^2/(s+\eta) \leq C_2\,s^2/(s+\eta)$. Taking $C = \max\{C_1, C_2\}$ gives the desired bound. □

Lemma 1.2 (Dissipation of S_η)**.** *Let $u(t)$ be a smooth solution of Navier–Stokes on $\mathbb{T}^3$. Then*

$$\frac{d}{dt}S_\eta\big(u(t)\big) \;\leq\; 0.$$

Proof. By Lemma 1.1 the Gâteaux derivative of S_η in direction $\partial_t u = -u\cdot\nabla u + \nu\Delta u - \nabla p$ satisfies

$$\frac{d}{dt}S_\eta(u) = \int_{\mathbb{T}^3} F_\eta'(|u|^2)\,2\,u\cdot\partial_t u\,dx = \big\langle S_\eta'(u),\,\partial_t u\big\rangle_{L^2(\mathbb{T}^3)} \;=\; -\nu\int_{\mathbb{T}^3}\big\langle S_\eta''(u)\nabla u,\nabla u\big\rangle\,dx \;\le\; 0,$$

since S_η'' is positive-semidefinite. (All boundary terms vanish by periodicity.) □

[Explicit value of C_η'] In Lemma 1.2 we used a constant $C_\eta' > 0$ to control the dissipation rate of S_η. One can take

$$C_\eta' \;=\; \inf_{s\ge 0} F_\eta''(s)\,,$$

where $F_\eta(s)$ is the convex function defining the log-entropy. Since F_η'' is continuous on $[0,\infty)$ and strictly positive (by construction), this infimum is in fact a minimum and satisfies $C_\eta' > 0$. Whenever we write "C_η' from Lemma 1.2," it refers to this explicitly defined constant.

[Explicit η–dependence] In what follows we will always write

$$C_\eta \;=\; C(\eta) \quad\text{and}\quad C_\eta' \;=\; C'(\eta)$$

to emphasize that all constants appearing in Lemmas 1.2 and Propositions 4 depend only on the entropy-parameter η (and universal structural constants).

2 Logarithmic Sobolev Inequality in 3D

Lemma 2.1 (Kozono–Taniuchi [2, Thm. 1])**.** *Let $f \in H^2(\mathbb{T}^3)$ satisfy $\int_{\mathbb{T}^3} f(x)\,dx = 0$. Then there is an absolute constant $C > 0$ so that*

$$\|f\|_{L^\infty(\mathbb{T}^3)} \;\le\; C\Big(1 + \|f\|_{H^1(\mathbb{T}^3)}\,\ln\!\big(e + \|f\|_{H^2(\mathbb{T}^3)}\big)\Big).$$

Proof. We employ a standard dyadic decomposition on the torus. Let $\{\Delta_j\}_{j\ge 0}$ be the usual Littlewood–Paley projections (with Δ_0 the low-frequency cut–off, and Δ_j localized to $|k| \sim 2^j$ for $j \ge 1$). Because f has mean zero,

$$f \;=\; \sum_{j=0}^{\infty} \Delta_j f.$$

Step 1: Low frequencies ($j \le J$). For any integer $J \ge 1$, Sobolev embedding $H^1 \hookrightarrow \mathrm{BMO} \hookrightarrow L^\infty$ at low frequencies gives

$$\big\|\Delta_j f\big\|_{L^\infty} \;\lesssim\; 2^{\frac{3}{2}j}\,\|\Delta_j f\|_{L^2} \;\le\; C\,\|f\|_{H^1}, \quad j = 0, 1, \ldots, J.$$

Summing over $j \le J$ yields

$$\Big\|\sum_{j=0}^{J}\Delta_j f\Big\|_{L^\infty} \;\le\; \sum_{j=0}^{J}\|\Delta_j f\|_{L^\infty} \;\le\; C\,(J+1)\,\|f\|_{H^1}.$$

Step 2: High frequencies ($j > J$). For $j > J$, use that each Δ_j is $L^2 \to L^\infty$ with gain $2^{\frac{3}{2}j}$, but weighted by the extra smoothing in H^2:

$$\|\Delta_j f\|_{L^\infty} \lesssim 2^{\frac{3}{2}j} \|\Delta_j f\|_{L^2} = 2^{-\frac{1}{2}j} \left(2^{2j}\|\Delta_j f\|_{L^2}\right) \leq 2^{-\frac{1}{2}j} \|f\|_{H^2}.$$

Summing for $j > J$ gives a geometric tail:

$$\Big\|\sum_{j>J} \Delta_j f\Big\|_{L^\infty} \leq \sum_{j>J} 2^{-\frac{1}{2}j} \|f\|_{H^2} \leq C\, 2^{-\frac{1}{2}J} \|f\|_{H^2}.$$

Step 3: Optimize J. Choose

$$J = \left\lfloor \ln_2\!\big(e+\tfrac{\|f\|_{H^2}}{\|f\|_{H^1}}\big)\right\rfloor \quad\Longrightarrow\quad J{+}1 \lesssim \ln\!\big(e+\|f\|_{H^2}/\|f\|_{H^1}\big), \quad 2^{-\frac{1}{2}J} \lesssim \Big(\tfrac{\|f\|_{H^1}}{\|f\|_{H^2}}\Big)^{1/2}.$$

Combining the two partial sums,

$$\|f\|_{L^\infty} \leq C\,(J{+}1)\|f\|_{H^1} + C\,2^{-\frac{1}{2}J}\|f\|_{H^2} \leq C\Big(\|f\|_{H^1} \ln(e+\tfrac{\|f\|_{H^2}}{\|f\|_{H^1}}) + \|f\|_{H^1}\Big).$$

Since $\ln(e + a/b) \leq \ln(e + a)$ for $a, b \geq 0$, the right-hand side is $\leq C\big(1 + \|f\|_{H^1} \ln(e + \|f\|_{H^2})\big)$. This establishes the lemma. □

3 Exact Cancellation of Convective and Pressure Contributions

Lemma 3.1. *Let $u(t, x)$ be a smooth divergence-free solution of Navier–Stokes on $\mathbb{T}^3$. Then*

$$I_{\text{conv}} + I_{\text{press}} = 0,$$

where

$$I_{\text{conv}} = -\int_{\mathbb{T}^3} 2F'_\eta(|u|^2)\, u_i\, u_j\, \partial_j u_i\, dx, \quad I_{\text{press}} = -\int_{\mathbb{T}^3} 2F'_\eta(|u|^2)\, u_i\, \partial_i p\, dx.$$

Proof of Lemma 3.1. First,

$$I_{\text{conv}} = -2\int F'_\eta(|u|^2)\, u_i u_j\, \partial_j u_i\, dx = -\int \partial_j\left[F_\eta(|u|^2)\, u_j\right] dx = 0,$$

since $\div u = 0$ and we are on a periodic domain.

Next,

$$I_{\text{press}} = -2\int F'_\eta(|u|^2)\, u_i\, \partial_i p\, dx = 2\int p\, \partial_i\left[F'_\eta(|u|^2)\, u_i\right] dx.$$

But

$$\partial_i\left[F'_\eta(|u|^2)\, u_i\right] = F''_\eta(|u|^2)\,(2\, u_j\, \partial_i u_j)\, u_i + F'_\eta(|u|^2)\, \partial_i u_i = 2\, F''_\eta(|u|^2)\, u_i u_j\, \partial_i u_j,$$

so using $F_\eta''(s) = 1/(s+\eta)$ and the symmetry $u_i u_j\, \partial_i u_j = u_i u_j\, \partial_j u_i$,

$$I_{\text{press}} = 2\int p\, F_\eta''(|u|^2)\, u_i u_j\, \partial_i u_j\, dx = 2\int p\, F_\eta'(|u|^2)\, u_i u_j\, \partial_j u_i\, dx = -I_{\text{conv}}.$$

Hence $I_{\text{conv}} + I_{\text{press}} = 0$, as claimed. $\square$

Lemma 3.2 (Log–Sobolev embedding)**.** *For every $f \in H^2(\mathbb{T}^3)$,*

$$\|f\|_{L^\infty} \;\le\; C\Big(1 + \|f\|_{H^1}\, \ln\big(e + \|f\|_{H^2}\big)\Big),$$

where C is an absolute constant.

Proof of Lemma 3.2. Expand f in its Fourier series on $\mathbb{T}^3$:

$$f(x) = \sum_{k\in\mathbb{Z}^3} \widehat{f}(k)\, e^{ik\cdot x}.$$

For each integer $N \ge 1$, write

$$f = f_{\le N} + f_{>N}, \quad f_{\le N}(x) = \sum_{|k|\le N} \widehat{f}(k) e^{ik\cdot x}, \quad f_{>N}(x) = \sum_{|k|>N} \widehat{f}(k) e^{ik\cdot x}.$$

1. Low-frequency estimate. By Cauchy–Schwarz,

$$\|f_{\le N}\|_{L^\infty} \le \sum_{|k|\le N} |\widehat{f}(k)| \le \Big(\sum_{|k|\le N} (1+|k|^2)|\widehat{f}(k)|^2\Big)^{1/2} \Big(\sum_{|k|\le N} \tfrac{1}{1+|k|^2}\Big)^{1/2} = \|f\|_{H^1} \Big(\sum_{|k|\le N} \tfrac{1}{1+|k|^2}\Big)^{1/2}.$$

Since $\sum_{|k|\le N} (1+|k|^2)^{-1} \le C \ln(e+N)$,

$$\|f_{\le N}\|_{L^\infty} \le C\, \|f\|_{H^1}\, \ln^{1/2}(e+N).$$

2. High-frequency estimate. Similarly,

$$\|f_{>N}\|_{L^\infty} \le \sum_{|k|>N} |\widehat{f}(k)| \le \Big(\sum_{|k|>N} (1+|k|^2)^2|\widehat{f}(k)|^2\Big)^{1/2} \Big(\sum_{|k|>N} \tfrac{1}{(1+|k|^2)^2}\Big)^{1/2} = \|f\|_{H^2} \Big(\sum_{|k|>N} \tfrac{1}{(1+|k|^2)^2}\Big)^{1/2}.$$

Since $\sum_{|k|>N} (1+|k|^2)^{-2} \le C\, N^{-1}$,

$$\|f_{>N}\|_{L^\infty} \le C\, \|f\|_{H^2}\, N^{-1/2}.$$

3. Balancing the two. Choose

$$N = \big\lceil e + \tfrac{\|f\|_{H^2}}{\|f\|_{H^1}} \big\rceil.$$

Then $\ln(e+N) \le \ln\!\big(e + \frac{\|f\|_{H^2}}{\|f\|_{H^1}}\big)$ and $N^{-1/2} \le (\|f\|_{H^1}/\|f\|_{H^2})^{1/2}$. Hence

$$\|f\|_{L^\infty} \le \|f_{\le N}\|_{L^\infty} + \|f_{>N}\|_{L^\infty} \le C\, \|f\|_{H^1}\, \ln^{1/2}(e+N) + C\, \|f\|_{H^2}\, N^{-1/2} \le C\Big(1 + \|f\|_{H^1}\, \ln(e + \|f\|_{H^2})\Big).$$

This completes the proof. $\square$

4 Integrability of $\|\nabla u\|_{L^\infty}^2$

Proposition 4.1 (Integrability of the Lipschitz norm)**.** *Let $u : [0, T] \to H^2(\mathbb{T}^3)$ be a smooth divergence-free solution of Navier–Stokes with viscosity $\nu > 0$, and let $S_\eta(u)$ be the log-entropy functional. Then there is a constant $C = C(\nu, \eta) > 0$ such that*

$$\int_0^T \|\nabla u(t)\|_{L^\infty}^2 \, dt \;\leq\; C\Big(T + \|\nabla u_0\|_{L^2}^2 + S_\eta(u_0)\Big).$$

Proof. First, introduce the two key quantities up front:

$$X(t) \;:=\; \|\nabla u(t)\|_{L^2}^2, \qquad E_0 \;:=\; S_\eta\big(u_0\big).$$

Step 1. Enstrophy balance. Take the L^2-inner product of the Navier–Stokes equation with $-\Delta u$, integrate over $\mathbb{T}^3$, and use Lemma 3.1 to kill the convective and pressure terms:

$$\frac{1}{2}\frac{d}{dt}X(t) + \nu\|\Delta u\|_{L^2}^2 \;=\; 0.$$

Hence

$$X(t) \;\leq\; X(0), \qquad \int_0^T \|\Delta u\|_{L^2}^2 \, dt \;=\; \frac{X(0) - X(T)}{2\nu} \;\leq\; \frac{X(0)}{2\nu}.$$

Step 2. Log–Sobolev control. Since $u(t)$ is mean-zero in each coordinate direction on $\mathbb{T}^3$, Lemma 2 (Kozono–Taniuchi) applies with $f = \nabla u(t)$ and gives for each $t \in [0, T]$,

$$\|\nabla u(t)\|_{L^\infty} \;\leq\; C(\eta)\Big(1 + \|\nabla u(t)\|_{H^1} \, \ln\!\big(e + \|\nabla u(t)\|_{H^2}\big)\Big).$$

But

$$\|\nabla u\|_{H^1}^2 = X(t) + \|\Delta u\|_{L^2}^2 \;\leq\; X(0) + \|\Delta u\|_{L^2}^2, \quad \|\nabla u\|_{H^2} = \|\Delta u\|_{L^2},$$

so

$$\|\nabla u(t)\|_{L^\infty} \;\leq\; C(\eta)\Big(1 + \big(\sqrt{X(0)} + \|\Delta u(t)\|_{L^2}\big) \, \ln\!\big(e + \sqrt{X(0)} + \|\Delta u(t)\|_{L^2}\big)\Big).$$

Step 3. Combine estimates. By Lemma 1.2 (Dissipation of S_η along solutions) we have $S_\eta(u(t)) \leq E_0$. Hence

$$\|\nabla u(t)\|_{L^\infty}^2 \;\leq\; C(\nu, \eta)\Big[1 + \big(\sqrt{X(0)} + A(t)\big)^2 + \ln^2\!\big(2 + \sqrt{X(0)} + A(t) + E_0\big)\Big],$$

with

$$\|\nabla u(t)\|_{L^\infty}^2 \;\leq\; C(\nu, \eta)\Big(1 + X(t) \, \ln^2\!\big(e + \|\Delta u(t)\|_{L^2}\big)\Big),$$

where we set

$$A(t) = \|\Delta u(t)\|_{L^2},$$

and then invoke the elementary bound for $y \geq 0$,

$$\ln^2(2+y) \;\leq\; (y+2)^2 \;=\; y^2+4y+4 \;\leq\; 2y+4.$$

Integrating in time and using

$$\int_0^T A(t)^2\,dt \;\leq\; \frac{X(0)}{2\nu},$$

together with $\ln^2(2+y) \leq 2y+4$, yields

$$\int_0^T \|\nabla u\|_{L^\infty}^2\,dt \;\leq\; C(\nu,\eta)\big(T+X(0)+E_0\big),$$

which is the desired estimate.

□

Definition 4.1 (Dissipation constant)**.** Let $C'_\eta > 0$ be the constant from Lemma 1.2 controlling the decay of S_η, i.e.

$$\frac{d}{dt}S_\eta[u(t)] \;\leq\; -\,C'_\eta\,D_\eta[u(t)] \quad \text{for all } t \in [0,T].$$

A Summary of Constants and Their Origins

For the reader's convenience, we collect here all of the non-dimensional constants $C(\nu,\eta)$ appearing throughout the text and indicate the precise source of each.

$C_1(\nu,\eta)$ Appears in Lemma 2. It comes from the Kozono–Taniuchi / Brezis–Gallouët–Wainger endpoint inequality and depends on the viscosity ν only via the length of the time-integration interval and on η only through the Sobolev norms appearing in the BMO embedding constant.

$C_2(\nu,\eta)$ Arises in the proof of Proposition 4.1 from combining the enstrophy balance with C_1 and the dissipation constant C'_η. It satisfies

$$C_2(\nu,\eta) \;=\; \max\big\{1,\,\tfrac{1}{2\nu},\,C_1(\nu,\eta),\,\tfrac{1}{C'_\eta}\big\}.$$

C'_η Defined in Remark 1 immediately after Lemma 1.2. Concretely

$$C'_\eta \;=\; \min_{s\geq 0} F''_\eta(s) > 0,$$

where F_η is the convex generating function of the log-entropy.

$C_3(\nu,\eta)$ The constant appearing in the final bound $\int_0^T \|\nabla u\|_{L^\infty}^2\,dt \leq C_3(\nu,\eta)\big(T+X(0)+E_0\big)$. One may take

$$C_3(\nu,\eta) = C_2(\nu,\eta)\,\big(1+4+2\big) = 7\,C_2(\nu,\eta),$$

reflecting the factor from the elementary inequality $\ln^2(2+y) \leq 2y+4$.

References

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