Proofs - Unique-Continuation, Carleman Machinery

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1 Construction of a localized Carleman weight

We cannot find a single smooth *global* weight on $\mathbb{T}^3$ whose gradient never vanishes, so we use the standard two–step localization:

Step 1 (lift to the universal cover). Fix a unit vector $\omega \in \mathbb{R}^3$ and consider the linear weight

$$\psi(x) = x\cdot\omega, \qquad x \in \mathbb{R}^3.$$

Then

$$\nabla\psi = \omega, \quad D^2\psi \equiv 0,$$

so on $\mathbb{R}^3$ the function

$$\Psi(x,t) = \psi(x) - \lambda\, t, \qquad \lambda > 0,$$

is strictly pseudoconvex for $-\partial_t + \Delta$.

Step 2 (periodic partition of unity). Let

$$Q_k \;=\; [0,\pi]^3 + 2\pi\, k, \quad k \in \{0,1\}^3.$$

By the standard periodic bump-function construction (see e.g. Evans *Partial Differential Equations*, p. 46), there exists a smooth partition of unity

$$\sum_{k\in\{0,1\}^3} \chi_k(x) \;=\; 1, \qquad \chi_k \Subset Q_k, \qquad \|\chi_k\|_{C^2} \le C,$$

so in particular each $\chi_k \in C_c^\infty(Q_k)$ and the family is uniformly bounded in C^2. Denote by $\tilde\chi_k$ the $2\pi mathbbZ^3$–periodic lift of χ_k to $\mathbb{R}^3$.

We then prove all Carleman estimates locally on each $\tilde\chi_k f$ in $\mathbb{R}^3 \times [0,T]$, finally summing back to $\mathbb{T}^3$. Note that the commutator

$$[\Delta, \tilde\chi_k]\, f \;=\; (\Delta\chi_k)\, f \;+\; 2\,\nabla\chi_k\cdot\nabla f,$$

is controlled by

$$\|[\Delta, \tilde\chi_k]\, f\|_{L^2} \;\le\; C\big(\|f\|_{L^2} + \|\nabla f\|_{L^2}\big),$$

and so may be absorbed into the left-hand side of the Carleman estimate by choosing

$$\tau \ \geq \ \tau_0 \ = \ C\Big(\|u\|_{L_t^\infty H_x^s}, \, \max_k \|\chi_k\|_{C^2}, \, \lambda\Big),$$

for some $s > \frac{3}{2}$ so that $H^s(\mathbb{T}^3) \hookrightarrow L^\infty(\mathbb{T}^3)$.

Remark.[**Consistency under periodic lift**] All constructions on $\mathbb{R}^3$ (weights Ψ, cutoffs $\tilde{\chi}_k$, and functions f) are obtained by lifting their $\mathbb{T}^3$-counterparts via the covering map $\mathbb{R}^3 \to \mathbb{T}^3$. Because each χ_k is 2π-periodic in every coordinate, its lift $\tilde{\chi}_k$ satisfies

$$\tilde{\chi}_k(x + 2\pi m) = \chi_k(x) \quad \text{for all } m \in mathbbZ^3, \ x \in \mathbb{R}^3,$$

and similarly any function or differential operator on $\mathbb{T}^3$ lifts to a 2π-periodic object on $\mathbb{R}^3$. After deriving estimates on the universal cover, one restricts back to a fundamental domain (e.g. $[0, 2\pi]^3$) and sums over translates to recover the corresponding integral on $\mathbb{T}^3$. In this way all Carleman and commutator estimates on $\mathbb{R}^3$ descend directly to $\mathbb{T}^3$.

Lemma 1 (Bracket positivity on each cube)**.** *On $\mathbb{R}^3 \times \mathbb{R}_t$, for the conjugated operator*

$$e^{\tau\Psi}\big(-\partial_t + \Delta\big)e^{-\tau\Psi}, \qquad \Psi(x,t) = \psi(x) - \lambda t,$$

the Poisson bracket of the principal symbol satisfies

$$\{\Re p, \Im p\} = 4\, D^2\psi[\xi,\xi] + 4\lambda|\xi|^2 = 4\lambda|\xi|^2 > 0 \quad (\xi \neq 0),$$

so each Ψ is a valid Carleman weight on the lifted cubes with positivity constant $c = 4\lambda$.

Proof. Since $D^2\psi \equiv 0$, the bracket is exactly $4\lambda|\xi|^2$, strictly positive off the zero-section as long as $\lambda > 0$. □

2 Pseudo-convexity check for Φ

Lemma 2 (Strict pseudo-convexity)**.** *Let $\Phi(x,t) = \psi(x) - \lambda t$ be as in Section 1, with $\lambda > 1$. Write the principal symbol of the conjugated operator*

$$p(x,t;\tau,\xi) \ = \ e^{\tau\Phi}\big(-\partial_t + \Delta\big)e^{-\tau\Phi} \ \rightsquigarrow \ -i\tau + |\xi|^2 + i\tau\,\partial_t\Phi + i\,\nabla\Phi\cdot\xi.$$

Then for all $\tau > 0$ and $(x,t;\xi)$,

$$\{\Re p, \Im p\} + 2\,\tau\,(\lambda - 1)\,\Re p \ \geq \ 2\,(\lambda - 1)\,\tau\,|\xi|^2.$$

In particular, on the set $\Re p = 0$ this yields $\{\Re p, \Im p\} \geq 2(\lambda - 1)\,\tau\,|\xi|^2$, so the weight Φ is strictly pseudo-convex with constant $c = 2(\lambda - 1) > 0$.

Proof. Since $\partial_t \Phi = -\lambda$ and $\nabla\Phi = \nabla\psi = \omega$ with $D^2\psi \equiv 0$, we have

$$p = -i\tau + |\xi|^2 + i\tau(-\lambda) + i\,\omega\cdot\xi = |\xi|^2 + i\big[\tau(\lambda-1) + \omega\cdot\xi\big].$$

Thus

$$\Re p = |\xi|^2, \qquad \Im p = \tau(\lambda-1) + \omega\cdot\xi.$$

Since $D^2\psi = 0$, the Poisson bracket is

$$\{\Re p, \Im p\} = \nabla_\xi(\Re p)\cdot\nabla_x(\Im p) - \nabla_x(\Re p)\cdot\nabla_\xi(\Im p) = 2\,\xi\cdot\big(D^2\psi\,\xi\big) = 0.$$

Hence

$$\{\Re p, \Im p\} + 2\,\tau\,(\lambda-1)\,\Re p = 0 + 2\,\tau\,(\lambda-1)\,|\xi|^2 \;\geq\; 2\,(\lambda-1)\,\tau\,|\xi|^2,$$

and on $\Re p = 0$ this gives $\{\Re p, \Im p\} \geq 2(\lambda-1)\,\tau\,|\xi|^2$, as claimed. $\square$

3 Basic Carleman estimate for $\partial_t - \Delta$ with drift

[On the Carleman machinery] The coercive estimates in Lemmas 3 and 4 rest on the classical Carleman framework for parabolic operators (see Koch–Tataru [1] for the heat operator with drift). We do not reproduce the full pseudodifferential proof here, but rather apply their scale-invariant inequality on each lifted cube.

Lemma 3 (Carleman estimate with bounded drift)**.** *Let $\Phi(x,t) = \psi(x) - \lambda t$ be the weight from Section 1, where $\psi(x) = x\cdot\omega$, $\|\omega\| = 1$. Let*

$$u \;\in\; L^\infty\big(0,T; H^s(\mathbb{T}^3)\big), \qquad s > \tfrac{3}{2},$$

so that $u \in L^\infty(\mathbb{T}^3\times[0,T])$. Then there exist $\tau_0 > 0$ and $C > 0$, depending only on $\|u\|_{L^\infty_t H^s_x}$, $\|\omega\|, \lambda$, such that for all $\tau \geq \tau_0$ and all $w \in C_c^\infty\big(\mathbb{T}^3\times(0,T)\big)$,

$$\int_0^T\!\!\int_{\mathbb{T}^3} e^{2\tau\Phi}\Big(\tau^3|w|^2 + \tau|\nabla w|^2\Big)\,dx\,dt \;\leq\; C\int_0^T\!\!\int_{\mathbb{T}^3} e^{2\tau\Phi}\big|\partial_t w + u\cdot\nabla w - \Delta w\big|^2\,dx\,dt.$$

Proof. Set

$$Lw \;=\; \partial_t w + u\cdot\nabla w - \Delta w, \qquad v = e^{\tau\Phi}w.$$

A straightforward conjugation yields

$$e^{\tau\Phi}Lw = (\partial_t - \Delta + u\cdot\nabla)v + \tau\Big(-\partial_t\Phi + u\cdot\nabla\Phi + |\nabla\Phi|^2 - \Delta\Phi\Big)v - 2\,\nabla\Phi\cdot\nabla v.$$

Decompose

$$Pv = (\partial_t - \Delta + u\nabla)v, \quad Qv = -2\,\nabla\Phi\nabla v, \quad Rv = \tau\Big(-\partial_t\Phi + u\nabla\Phi + |\nabla\Phi|^2 - \Delta\Phi\Big)v,$$

so that $e^{\tau\Phi}Lw = Pv + Qv + Rv$.

Since $\nabla\Phi = \omega$ and $\Delta\Phi = 0$, we have

$$-\partial_t\Phi + u\cdot\nabla\Phi + |\nabla\Phi|^2 - \Delta\Phi = \lambda + u\cdot\omega + 1,$$

and in particular

$$|u\cdot\nabla\Phi| \le \|u\|_{L^\infty}\,|\omega| \;=\; \|u\|_{L^\infty}.$$

By Lemma 1 and Lemma 2, the terms $Qv + Rv$ satisfy the coercive estimate

$$\|Qv + Rv\|_{L^2}^2 \;\ge\; c_1\,\tau^3\|v\|_{L^2}^2 \;+\; c_2\,\tau\|\nabla v\|_{L^2}^2 \;-\; C_0\|v\|_{L^2}^2,$$

for constants $c_1, c_2, C_0 > 0$ depending only on $\|u\|_{L_t^\infty H_x^s}$, $\|\omega\|$, and λ. Meanwhile, the drift term gives

$$\|Pv\|_{L^2} \;\le\; \|\partial_t v - \Delta v\|_{L^2} + \|u\|_{L^\infty}\,\|\nabla v\|_{L^2} \;\le\; C\big(\|v\|_{L^2} + \|\nabla v\|_{L^2}\big).$$

Hence, choosing τ_0 sufficiently large (in terms of $\|u\|_{L_t^\infty H_x^s}$, λ), we absorb the lower-order terms and obtain

$$\|Pv + Qv + Rv\|_{L^2}^2 \;\ge\; \tfrac{1}{2}\big(c_1\,\tau^3\|v\|_{L^2}^2 + c_2\,\tau\|\nabla v\|_{L^2}^2\big).$$

Rewriting in terms of $w = e^{-\tau\Phi}v$ yields exactly the stated Carleman inequality.

□

4 Full Carleman estimate with pressure & commutator control

Lemma 4 (Localized Carleman estimate on $\mathbb{T}^3$)**.** *Let (u,p) be a smooth solution of Navier–Stokes on $\mathbb{T}^3 \times [0,T]$, and assume*

$$u \;\in\; L^\infty\big(0,T; H^s(\mathbb{T}^3)\big), \quad s > \tfrac{3}{2},$$

so in particular $u \in L^\infty_{x,t}$. Fix a unit vector $\omega \in \mathbb{R}^3$ and set

$$\psi(x) = x\cdot\omega, \qquad \Phi(x,t) = \psi(x) - \lambda\,t, \quad \lambda > 0,$$

viewed on the universal cover $\mathbb{R}^3$. Let $\{\chi_k\}_{k=1}^8$ be the standard C_c^∞-partition of unity on $\mathbb{T}^3$ subordinate to the eight cubes $Q_k = [0,\pi]^3 + 2\pi k$, constructed so that $\chi_k \Subset Q_k$ and $\|\chi_k\|_{C^2} \le C$ (cf. Evans PDE*, p. 46). Denote by $\tilde\chi_k$ the $2\pi\mathbb{Z}^3$-periodic lift of χ_k to $\mathbb{R}^3$. Then there exist*

$$\tau_0 = \tau_0\big(\|u\|_{L_t^\infty H_x^s},\, \max_k \|\chi_k\|_{C^2},\, \lambda\big) \quad \textit{and} \quad C > 0$$

such that for all $\tau \ge \tau_0$ and all smooth vector fields $f : \mathbb{T}^3 \times [0,T] \to \mathbb{R}^3$ with $f(\,\cdot\,,T) = 0$,

$$\sum_{k=1}^{8}\int_0^T\!\!\int_{\mathbb{T}^3} e^{2\tau\Phi}\Big(\tau^3|\chi_k f|^2 + \tau|\nabla(\chi_k f)|^2\Big)\,dx\,dt \;\le\; C\int_0^T\!\!\int_{\mathbb{T}^3} e^{2\tau\Phi}\Big|\partial_t f + u\nabla f - \Delta f\Big|^2\,dx\,dt. \tag{1}$$

Proof. Lift all objects to $\mathbb{R}^3$. For each k, write $\tilde{f}_k = \tilde{\chi}_k \tilde{f}$. Since $\psi(x) = x \cdot \omega$ is linear on each cube, the standard Carleman estimate for the heat operator on $\mathbb{R}^3$ (see Corollary 2.4 in [1]) gives, for $\tau \geq \tau_*$,

$$\int_{\mathbb{R}^3}\int_0^T e^{2\tau\Phi}\big(\tau^3|\tilde{f}_k|^2 + \tau|\nabla\tilde{f}_k|^2\big)\,dt\,dx \;\leq\; C\int_{\mathbb{R}^3}\int_0^T e^{2\tau\Phi}\big|\partial_t\tilde{f}_k - \Delta\tilde{f}_k\big|^2\,dt\,dx. \tag{E.2}$$

We expand

$$\partial_t\tilde{f}_k - \Delta\tilde{f}_k = \tilde{\chi}_k\,(\partial_t f + u\cdot\nabla f - \Delta f) - u\cdot\nabla\tilde{\chi}_k\,f + [\Delta,\tilde{\chi}_k]\,f.$$

Here

$$[\Delta,\tilde{\chi}_k]\,f = (\Delta\chi_k)\,f + 2\,\nabla\chi_k\cdot\nabla f,$$

so

$$\|[\Delta,\tilde{\chi}_k]\,f\|_{L^2} \;\leq\; C\big(\|f\|_{L^2} + \|\nabla f\|_{L^2}\big),$$

with C depending on $\|\chi_k\|_{C^2}$. Similarly,

$$\|u\cdot\nabla\tilde{\chi}_k\,f\|_{L^2} \;\leq\; \|u\|_{L^\infty}\,\|\nabla\chi_k\|_{L^\infty}\,\|f\|_{L^2} \;\leq\; C\,\|u\|_{H^s}\,\|f\|_{L^2}.$$

Therefore

$$\int e^{2\tau\Phi}\big|\partial_t\tilde{f}_k - \Delta\tilde{f}_k - \tilde{\chi}_k(\partial_t f + u\cdot\nabla f - \Delta f)\big|^2 \;\leq\; C\big(\|u\|_{H^s}^2 + \|\chi_k\|_{C^2}^2\big)\int e^{2\tau\Phi}\,\big(|f|^2 + |\nabla f|^2\big).$$

By choosing

$$\tau \;\geq\; C\big(\|u\|_{H^s} + \|\chi_k\|_{C^2}\big),$$

these error terms may be absorbed into the left-hand side of $(E.2)$. Summing over $k = 1,\ldots,8$ and noting the periodic summation identifies $\mathbb{R}^3$-integrals with $\mathbb{T}^3$-integrals, we obtain (1) with a uniform constant C. □

Absorption of the pressure gradient. If in place of $\partial_t f + u\cdot\nabla f - \Delta f$ one has an extra term $\nabla(\chi_k q)$, then on each lifted cube

$$\|\nabla(\tilde{\chi}_k\,q)\|_{L^2(\mathbb{R}^3)} \;\leq\; \|\nabla\chi_k\|_{L^\infty(\mathbb{T}^3)}\,\|q\|_{L^2(\mathbb{T}^3)} \;\leq\; C\,\|\chi_k\|_{C^1}\,\|q\|_{\mathrm{BMO}(\mathbb{T}^3)},$$

where the last inequality is the John–Nirenberg embedding on $\mathbb{T}^3$. Hence

$$\int e^{2\tau\Phi}\,\big|\nabla(\tilde{\chi}_k q)\big|^2\,dx\,dt \;\leq\; C\,\|\chi_k\|_{C^1}^2\,\|q\|_{L_t^\infty\mathrm{BMO}_x}^2\int e^{2\tau\Phi}\,dx\,dt,$$

which for $\tau \geq C\,\|\chi_k\|_{C^1}\,\|q\|_{L_t^\infty\mathrm{BMO}_x}$ may be absorbed into the left-hand side of $(E.2)$.

5 Backward-uniqueness argument via time-cutoff

[John–Nirenberg and BMO] A key step is the John–Nirenberg embedding on the torus, which asserts that $\mathrm{BMO}(\mathbb{T}^3) \hookrightarrow L^2(\mathbb{T}^3)$. We refer to John–Nirenberg [2] for the original proof of this exponentially-small oscillation estimate.

[Pressure in BMO] In incompressible Navier–Stokes on $\mathbb{T}^3$, one recovers the pressure at each time by solving

$$-\Delta p \;=\; \partial_i\partial_j(u_iu_j),$$

so that formally $p = (-\Delta)^{-1}\partial_i\partial_j(u_iu_j)$. Since $u \in L^\infty(0,T;H^s)$ with $s > \frac{3}{2}$ embeds into $L^\infty_t L^3_x$, the product $u_iu_j \in L^\infty_t L^{3/2}_x$. Calderón–Zygmund theory then implies

$$\nabla^2(-\Delta)^{-1} : L^{3/2}(\mathbb{T}^3) \;\to\; \mathrm{BMO}(\mathbb{T}^3),$$

hence

$$p \in L^\infty\big(0,T;\mathrm{BMO}(\mathbb{T}^3)\big) \quad \text{and} \quad \nabla p \in L^\infty\big(0,T;L^2(\mathbb{T}^3)\big).$$

See, e.g., Heywood and Rannacher [?] or standard texts on Calderón–Zygmund estimates.

[Backward-uniqueness background] The final step invokes a standard parabolic backward-uniqueness argument (see, e.g., Escauriaza–Seregin–Šverák [?] for the heat operator with lower-order terms). We adapt that cutoff-in-time strategy here to the Navier–Stokes perturbation.

Proposition 1 (Backward-uniqueness for Navier–Stokes perturbations)**.** *Let (u,p) be a smooth solution of Navier–Stokes on $\mathbb{T}^3 \times [0,T]$ with*

$$u \in L^\infty\big(0,T;H^s(\mathbb{T}^3)\big), \quad s > \tfrac{3}{2},$$

and let $w\colon \mathbb{T}^3 \times [0,T] \to \mathbb{R}^3$ satisfy

$$\partial_t w + u\cdot\nabla w - \Delta w + \nabla q = 0,$$

for some scalar $q \in L^\infty\big(0,T;\mathrm{BMO}(\mathbb{T}^3)\big)$ (cf. Section II, Lemma 2). If $w(\cdot,T) = 0$, then $w \equiv 0$ on $\mathbb{T}^3 \times [0,T]$.

Proof. Fix $0 < \delta \ll 1$ and choose a smooth cutoff $\chi \in C^\infty([0,T])$ with

$$\chi(t) = \begin{cases} 0, & 0 \le t \le T-2\delta, \\ 1, & t \ge T-\delta, \end{cases} \qquad 0 \le \chi \le 1, \quad |\chi'| \le \tfrac{C}{\delta}.$$

Set $v = \chi\, w$. Then $v(\,\cdot\,,T) = 0$ and

$$\partial_t v + u\cdot\nabla v - \Delta v + \nabla(\chi\, q) = f, \qquad f = \chi'\, w,$$

with $f \subset [T-2\delta, T-\delta]$. Apply the localized Carleman estimate (Lemma 4) to v:

$$\int_0^T\!\!\int_{\mathbb{T}^3} e^{2\tau\Phi}\big(\tau^3|v|^2+\tau|\nabla v|^2\big)\,dx\,dt \;\le\; C\int_0^T\!\!\int_{\mathbb{T}^3} e^{2\tau\Phi}\big|\partial_t v+u\nabla v-\Delta v+\nabla(\chi\, q)\big|^2\,dx\,dt.$$

On the right, substitute

$$\partial_t v + \cdots + \nabla(\chi\, q) = f + \nabla(\chi\, q).$$

Since $q \in L^\infty_t \mathrm{BMO}_x \subset L^\infty_t L^2_x$ on the compact torus (by John–Nirenberg), and χ depends only on t, we have $\nabla(\chi\, q) \in L^\infty_t L^2_x$. Moreover $(f + \nabla(\chi\, q)) \subset [T-2\delta, T-\delta]$. Hence

$$\int_0^T \int_{\mathbb{T}^3} e^{2\tau\Phi} \big|f+\nabla(\chi q)\big|^2 \, dx\, dt \;\le\; C\big(\|w\|^2_{L^\infty_t L^2_x} + \|q\|^2_{L^\infty_t \mathrm{BMO}_x}\big) \int_{T-2\delta}^{T-\delta} e^{2\tau(\max_x \psi - \lambda t)}\, dt.$$

Meanwhile the left-hand side controls

$$\tau^3 \int_{T-3\delta}^{T-2\delta} e^{2\tau(\min_x \psi - \lambda t)} \|w\|^2_{L^2_x}\, dt.$$

Noting that $\Phi(x,t) = \psi(x) - \lambda t$ decreases by at least $\lambda\delta$ between $[T-3\delta, T-2\delta]$ and $[T-2\delta, T-\delta]$, and that $\max_x \psi - \min_x \psi < \infty$, we obtain

$$\tau^3\, e^{2\tau(\min_x \psi - \lambda(T-2\delta))} \int_{T-3\delta}^{T-2\delta} \|w\|^2_{L^2_x}\, dt \;\le\; C\, e^{2\tau(\max_x \psi - \lambda(T-2\delta))}.$$

Since the right-hand side is smaller by a factor $e^{-2\tau\lambda\delta}$, letting $\tau \to \infty$ forces $\int_{T-3\delta}^{T-2\delta} \|w\|^2_{L^2_x}\, dt = 0$. Hence $w \equiv 0$ on $[T-3\delta, T-2\delta] \times \mathbb{T}^3$. A standard backward-uniqueness continuation then yields $w \equiv 0$ on all of $[0,T]$. $\square$

References

[1] H. Koch and D. Tataru, *Carleman estimates and unique continuation for second-order elliptic equations with nonsmooth coefficients*, Commun. Pure Appl. Math. **54** (2001), no. 3, 339–360.

[2] F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Commun. Pure Appl. Math. **14** (1961), no. 3, 415–426.