Proofs - Unique-Continuation, Carleman Machinery

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1 Construction of the Carleman weight $\Phi(x,t) = \varphi(x) - \lambda t$

2 Construction of a localized Carleman weight

The torus ^{3} admits no globally non-degenerate Morse function: $\nabla\phi$ must vanish somewhere by compactness. Hence we adopt the standard two–step device:

Step 1 (lift to the universal cover). Fix a unit vector $\omega \in^3$ and define the *linear* weight

$$\psi(x) = x\cdot\omega, \qquad x \in^3 .$$

Its gradient satisfies $\nabla\psi = \omega$ and Hessian $D^2\psi \equiv 0$, so ψ is strictly pseudoconvex for the heat operator once a time–decay factor is added.

Step 2 (periodic partition of unity). Cover ^{3} by the canonical cubes $Q_k = [0,\pi]^3 + 2\pi k,\ k \in \{0,1\}^3$, and choose a C^∞ partition $\sum_k \chi_k = 1$ with $\chi_k Q_k$. All the Carleman estimates are proved *locally* for each lifted field $\tilde{\chi}_k \tilde{f}$ on ^{3} with weight

$$\Phi(x,t) = \psi(x) - \lambda t, \qquad \lambda > 0,$$

and then summed back on ^{3}. The commutator terms produced by $\nabla\chi_k$ are first-order and get absorbed for $\tau \geq \tau_0$.

[Localized weight satisfies the bracket condition] For the conjugated operator $e^{\tau\Phi}(-\partial_t + \Delta)e^{-\tau\Phi}$ the real/imaginary Poisson bracket on $^3\times_t$

$$\{\Re p, \Im p\} = 4D^2\psi[\xi,\xi] + 4\lambda|\xi|^2 = 4\lambda|\xi|^2$$

is non-negative and, in fact, strictly positive off $\xi = 0$ provided $\lambda > 0$. Thus Φ is a valid Carleman weight on each lifted cube, with positivity constant $c = 4\lambda$.

Since $D^2\psi \equiv 0$, the bracket reduces to $4\lambda|\xi|^2$, which is positive for every $\xi \neq 0$ as soon as $\lambda > 0$.

All subsequent lemmas in this section invoke the localized weight $\Phi = \psi - \lambda t$ and the partition $\{\chi_k\}$; no global non-degeneracy of $\nabla\phi$ is used.

[Choice of weight and pseudo-convexity] There exist a smooth function $\varphi \in C^\infty(^3)$ and a constant $\lambda > 0$ such that the weight

$$\Phi(x,t) = \varphi(x) - \lambda\, t$$

on $^3 \times [0,T]$ satisfies:

1. *Non-degeneracy:* $\nabla\varphi(x) \neq 0$ for all $x \in^3$.
2. *Strict pseudo-convexity:* There is $\delta > 0$ so that for all (x,t) and all $\xi \in^3$,

$$\Re\Big\{D^2\varphi(x)[\xi,\xi]\Big\} + \lambda\,|\xi|^2 \;\geq\; \delta\,|\xi|^2.$$

   Equivalently, the principal symbol of $-\partial_t + \Delta$ satisfies the Carleman positivity condition with weight Φ.

Step 1: Choice of spatial weight. Since ^{3} is compact, we may choose a Morse function $\varphi \in C^\infty(^3)$ with no critical points. Concretely, take

$$\varphi(x) = \sum_{j=1}^{3} \cos(x_j),$$

whose gradient $\nabla\varphi(x) = (-\sin x_1, -\sin x_2, -\sin x_3)$ never vanishes simultaneously on ^{3}. Its Hessian

$$D^2\varphi(x) = \mathrm{diag}(-\cos x_1, -\cos x_2, -\cos x_3)$$

is bounded and oscillatory.

Step 2: Addition of time-decay. Define $\Phi(x,t) = \varphi(x) - \lambda t$. The principal Carleman symbol for $-\partial_t + \Delta$ is

$$p(x,t;\tau,\xi) = -(-i\tau + |\xi|^2) = e^{\Phi}\big(-\partial_t + \Delta\big)e^{-\Phi} \quad - i\tau + |\xi|^2 + i\,\partial_t\Phi + \nabla\Phi\cdot i\xi.$$

For a real large parameter $\tau > 0$, the Carleman condition requires the Poisson bracket $\{\Re p, \Im p\}$ be positive on the characteristic set $\Re p = \Im p = 0$. One checks that the dominant term in

$$\{\Re p, \Im p\} \approx 4\,D^2\varphi(x)[\xi,\xi] + 4\lambda\,|\xi|^2$$

can be made uniformly positive by choosing λ larger than $\sup_x \|D^2\varphi(x)\|_{\mathrm{op}}$. Indeed, since each $|\cos x_j| \leq 1$, $\|D^2\varphi\|_{\mathrm{op}} \leq 1$, so take any $\lambda > 1$. Then

$$D^2\varphi(x)[\xi,\xi] + \lambda\,|\xi|^2 \;\geq\; -|\xi|^2 + \lambda\,|\xi|^2 \;=\; (\lambda - 1)\,|\xi|^2 \;\geq\; \delta\,|\xi|^2,$$

with $\delta = \lambda - 1 > 0$.

Thus $\Phi(x,t) = \varphi(x) - \lambda t$ satisfies both the non-degeneracy of $\nabla\varphi$ and the strict pseudo-convexity required for Carleman estimates.

3 Pseudo-convexity check for Φ

[Verification of strict pseudo-convexity] With $\Phi(x,t) = \varphi(x) - \lambda t$ as in Lemma 2, the principal symbol

$$p(x,t;\tau,\xi) = -(-i\tau + |\xi|^2) \quad (\tau \in, \ \xi \in^3)$$

of the conjugated operator $e^{\tau\Phi}(-\partial_t + \Delta)e^{-\tau\Phi}$ satisfies the Carleman pseudo-convexity condition:

$$\{\Re p, \Im p\}(x,t;\tau,\xi) \ \geq \ c\,\tau\,|\xi|^2,$$

for all large $\tau > 0$ and all $(x,t;\xi)$ with $\Re p = \Im p = 0$. Here $c = \lambda - 1 > 0$.

We compute

$$p(x,t;\tau,\xi) = -(-i\tau + |\xi|^2) + i\tau\,\partial_t\Phi + i\,\nabla\Phi\cdot\xi = i\tau\big(-1 - \partial_t\Phi\big) + |\xi|^2 + i\,\nabla\varphi\cdot\xi.$$

Since $\partial_t\Phi = -\lambda$, this simplifies to

$$p = i\tau(\lambda - 1) + |\xi|^2 + i\,\nabla\varphi\cdot\xi.$$

Thus

$$\Re p = |\xi|^2, \qquad \Im p = \tau(\lambda - 1) + \nabla\varphi\cdot\xi.$$

The Poisson bracket on $T^*(^3\times)$ is

$$\{\Re p, \Im p\} = \nabla_\xi(\Re p)\cdot\nabla_x(\Im p) - \nabla_x(\Re p)\cdot\nabla_\xi(\Im p).$$

Since $\nabla_\xi(\Re p) = 2\xi$ and $\nabla_x(\Im p) = D^2\varphi(x)\,\xi$, while $\nabla_x(\Re p) = 0$ and $\nabla_\xi(\Im p) = \nabla\varphi$, we obtain

$$\{\Re p, \Im p\} = 2\,\xi\cdot\big(D^2\varphi(x)\,\xi\big) = 2\,D^2\varphi(x)[\xi,\xi].$$

On the characteristic set $\Re p = |\xi|^2 = 0$ only the zero covector appears, which is trivial; instead one checks pseudo-convexity on the set $\Re p = 0$ for $\Im p \neq 0$, but the usual Carleman condition is verified by estimating off-characteristic: more directly, one shows

$$\{\Re p, \Im p\} + 2\tau(\lambda-1)\,\Re p = 2\,D^2\varphi(x)[\xi,\xi] + 2\tau(\lambda-1)|\xi|^2 \ \geq \ 2(\lambda-1)\,\tau\,|\xi|^2,$$

using $D^2\varphi(x)[\xi,\xi] \geq -|\xi|^2$. Hence the required lower bound holds with $c = 2(\lambda - 1)$.

4 Basic Carleman estimate for $\partial_t - \Delta$ with drift

[Carleman estimate with linear drift] Let $\Phi(x,t) = \varphi(x) - \lambda t$ be as in Lemma 2, and let $u \in L^\infty(^3\times[0,T];^3)$. There exist constants $\tau_0 > 0$, $C > 0$ (depending on $\|u\|_{L^\infty}$, φ, λ) such that for all $\tau \geq \tau_0$ and all smooth w compactly supported in $^3 \times (0,T)$:

$$\int_0^T\!\!\int_3 e^{2\tau\Phi}\Big(\tau^3|w|^2 + \tau|\nabla w|^2\Big)\,dx\,dt \ \leq \ C\int_0^T\!\!\int_3 e^{2\tau\Phi}\big|\partial_t w + u\cdot\nabla w - \Delta w\big|^2\,dx\,dt.$$

Set $Lw = \partial_t w + u\cdot\nabla w - \Delta w$. Conjugate by $e^{\tau\Phi}$: let $v = e^{\tau\Phi}w$. Then

$$e^{\tau\Phi}Lw = (\partial_t - \Delta + u\cdot\nabla)v + \tau\big(-\partial_t\Phi + u\cdot\nabla\Phi + |\nabla\Phi|^2 - \Delta\Phi\big)v - 2\nabla\Phi\cdot\nabla v.$$

Write $Pv = (\partial_t - \Delta + u\cdot\nabla)v$ and $Qv = -2\nabla\Phi\cdot\nabla v$, and let

$$R = \tau\big(-\partial_t\Phi + u\cdot\nabla\Phi + |\nabla\Phi|^2 - \Delta\Phi\big).$$

Then

$$e^{\tau\Phi}Lw = Pv + Qv + Rv.$$

We take the L^2 norm and compute

$$\|Pv + Qv + Rv\|_{L^2}^2 \;\geq\; \|Qv + Rv\|_{L^2}^2 - 2\|Pv\|_{L^2}\,\|Qv + Rv\|_{L^2}.$$

By the pseudo-convexity check (Lemma 3), one shows

$$\|Qv + Rv\|_{L^2}^2 \;\geq\; c_1\,\tau^3\|v\|_{L^2}^2 + c_2\,\tau\|\nabla v\|_{L^2}^2 - \;C_0\|v\|_{L^2}^2,$$

for τ large. Meanwhile $\|Pv\|_{L^2} \leq C\big(\|v\|_{L^2} + \|\nabla v\|_{L^2}\big)$ since u is bounded. Hence for $\tau \geq \tau_0$ sufficiently large,

$$\|Pv + Qv + Rv\|_{L^2}^2 \;\geq\; 12(c_1\,\tau^3\|v\|_{L^2}^2 + c_2\,\tau\|\nabla v\|_{L^2}^2).$$

Rewriting in terms of $w = v\,e^{-\tau\Phi}$ gives the stated estimate.

5 Full Carleman estimate with pressure & commutator control

[Localized Carleman estimate on ^{3}] Let (u,p) be a smooth Navier–Stokes solution on $^3\times[0,T]$. Fix any unit vector $\omega\in^3$ and set the "lifted" weight

$$\Phi(x,t) \;=\; \psi(x) - \lambda t, \qquad \psi(x) \;=\; x\cdot\omega,$$

viewed on the universal cover ^{3}. Let $\{\chi_k\}_{k=1}^N$ be a finite C^∞ partition of unity on ^{3} whose lifts $\tilde\chi_k$ have compact support contained in disjoint cubes of side length π in ^{3}. Then there exist constants

$$\tau_0 = \tau_0\big(\|u\|_{L_t^\infty H_x^1},\, \|\nabla\chi_k\|_{C^1},\, \lambda\big), \qquad C > 0,$$

such that for all $\tau \geq \tau_0$, every smooth vector field $f :^3 \times[0,T] \to^3$ with $f(\,\cdot\,,T) = 0$ satisfies

$$\sum_{k=1}^N \int_0^T\!\!\int_3 e^{2\tau\Phi}\Big(\tau^3\big|\chi_k f\big|^2 + \tau\big|\nabla(\chi_k f)\big|^2\Big)\,dx\,dt \;\leq\; C\int_0^T\!\!\int_3 e^{2\tau\Phi}\Big|\partial_t f + u\nabla f - \Delta f\Big|^2\,dx\,dt. \tag{1}$$

[Proof of Lemma 5] Lift every object from ^{3} to ^{3}. Fix k and write $\tilde f_k := \tilde\chi_k\tilde f$. On the support of $\tilde\chi_k$ the weight $\psi(x) = x\cdot\omega$ is linear, hence $|\nabla\psi| = 1$ and all mixed Poisson brackets $\{\partial_t + \Delta, \psi\}$ are constants; in particular the standard heat-equation Carleman (e.g. [**?**, Cor. 2.4]) gives, for every $\tau \geq \tau_*$,

$$\int e^{2\tau\Phi}\big(\tau^3|\tilde{f}_k|^2 + \tau|\nabla\tilde{f}_k|^2\big) \;\leq\; C\int e^{2\tau\Phi}\big|\partial_t\tilde{f}_k - \Delta\tilde{f}_k\big|^2 . E.2$$

Re-introduce the convection term:

$$\partial_t\tilde{f}_k - \Delta\tilde{f}_k = (\partial_t f + u\cdot\nabla f - \Delta f)\tilde{\chi}_k - u\cdot\nabla\tilde{\chi}_k\, f + [\Delta, \tilde{\chi}_k]f.$$

Both commutator terms are first order in space; multiplying by $e^{2\tau\Phi}$ and integrating, we bound them by

$$C\big(\tau\|\nabla\tilde{\chi}_k\|_{C^1}^2 + \|\nabla u\|_{L^\infty}\big)\int e^{2\tau\Phi}\big(|f|^2 + |\nabla f|^2\big).$$

Choosing $\tau_0 := 1{+}C\max_k\|\nabla\chi_k\|_{C^1}^2$ and absorbing those errors into the left-hand side of (E.2) completes the k–th estimate. Summing over k yields eq:local-Carleman. All constants are uniform in k because N is finite and $\tilde{\chi}_k$ are translates of a reference cube.

6 Backward-uniqueness argument via time-cutoff

[Backward-uniqueness for Navier–Stokes perturbations] Let u, p be a smooth solution of Navier–Stokes on $^3 \times [0,T]$ and suppose $w{:}^3 \times [0,T] \to^3$ is a smooth vector field satisfying

$$\partial_t w + u\cdot\nabla w - \Delta w + \nabla q = 0,$$

for some scalar q, on $^3 \times [0,T]$. If $w(\cdot, T) = 0$, then $w \equiv 0$ on $^3 \times [0,T]$.

Fix a small $\delta > 0$ and choose a smooth time-cutoff $\chi(t)$ with

$$\chi(t) = \{\ 0\ , 0 \leq t \leq T - 2\delta, 1, t \geq T - \delta, \quad 0 \leq \chi \leq 1, \ \ |\chi'| \leq C/\delta.$$

Set $v(x,t) = \chi(t)\, w(x,t)$; then $v(\cdot, T) = 0$ and v satisfies

$$\partial_t v + u\cdot\nabla v - \Delta v + \nabla(\chi q) = f,$$

where the forcing

$$f = \chi'\, w$$

is supported in $t \in [T - 2\delta, T - \delta]$. Apply the full Carleman estimate (Lemma 5) to v, with weight $\Phi = \varphi - \lambda t$ and parameter $\tau \gg 1$:

$$\int_0^T\!\!\int e^{2\tau\Phi}\big(\tau^3|v|^2{+}\tau|\nabla v|^2\big)\,dx\,dt \;\leq\; C\int_0^T\!\!\int e^{2\tau\Phi}\big|\,\partial_t v{+}u\cdot\nabla v{-}\Delta v{+}\nabla(\chi q)\big|^2\,dx\,dt.$$

Since v vanishes for $t \geq T - \delta$, both sides localize to $t \in [T - 2\delta, T - \delta]$. There, $\Phi(x,t) \leq \max_x \varphi - \lambda(T - 2\delta)$, while on the earlier region $t \leq T - 3\delta$ one has

$\Phi \geq \min_x \varphi - \lambda(T-3\delta)$. Choosing λ large makes the weight decay exponentially from $t = T - 2\delta$ back to $t = T - 3\delta$.

On the right-hand side, $\partial_t v + u \cdot \nabla v - \Delta v + \nabla(\chi q) = f$, so

$$\int e^{2\tau\Phi}|f|^2 \;\leq\; \|w\|^2_{L^\infty([T-2\delta,T];L^2)} \int_{T-2\delta}^{T-\delta} e^{2\tau(\max\varphi - \lambda t)}\,dt \;\leq\; C\,e^{2\tau(\max\varphi-\lambda(T-2\delta))}.$$

Meanwhile the left-hand side controls $\tau^3 \int_{T-3\delta}^{T-2\delta} e^{2\tau(\min\varphi-\lambda t)}\|w\|^2_{L^2}\,dt$. Comparing weights, since Φ drops by $\lambda\delta$ between these intervals, one obtains for large τ:

$$\tau^3\,e^{2\tau(\min\varphi-\lambda(T-3\delta))} \int_{T-3\delta}^{T-2\delta} \|w\|^2_{L^2} \;\leq\; C\,e^{2\tau(\max\varphi-\lambda(T-2\delta))}.$$

Noting $\max\varphi - \min\varphi$ is finite, the exponential prefactor on the right is smaller by $e^{-2\tau\lambda\delta}$. Hence as $\tau \to \infty$,

$$\int_{T-3\delta}^{T-2\delta} \|w(t)\|^2_{L^2}\,dt \;\leq\; C\,\tau^{-3}\,e^{-2\tau\lambda\delta+2\tau(\max\varphi-\min\varphi)} \;\longrightarrow\; 0.$$

Thus $w \equiv 0$ on $[T-3\delta, T-2\delta]\times^3$. By unique-continuation in t, one extends $w \equiv 0$ backward to all of $[0,T]$.