# Proofs - Extension to R3

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## 1 Smooth partition of unity into large cubes

**Lemma 1** (Scale-free Bogovskiĭ on periodic blocks). *Let* $Q_L = [0, L]^3$ *and denote by* $\bar{f}_L(x) = f(Lx)$ *the pull-back of* $f : Q_L \to \mathbb{R}$ *to the reference cube* $Q_1 = [0, 1]^3$. *There is a linear Bogovskiĭ operator* $B_1 : \{g \in C^\infty(Q_1) : \int_{Q_1} g = 0\} \to C^\infty(Q_1; \mathbb{R}^3)$ *satisfying*

$$\nabla \cdot B_1[g] = g, \quad \|B_1[g]\|_{H^{s+1}(Q_1)} \; \le \; C \, \|g\|_{H^s(Q_1)} \quad \text{for all } s \ge 0,$$

*with a constant* $C$ *independent of* $g$. *Defining*

$$B_L[f](x) \; = \; L \left( B_1[\bar{f}_L] \right)(x/L),$$

*we get a Bogovskiĭ operator on* $Q_L$ *satisfying*

$$\nabla \cdot B_L[f] = f, \quad \|B_L[f]\|_{H^{s+1}(Q_L)} \; \le \; C \, \|f\|_{H^s(Q_L)}, \quad \int_{Q_L} B_L[f] \, dx = 0,$$

*where* $C$ *is* the same *constant as above, and in particular* independent of $L$.

*Proof.* By construction,

$$\bar{f}_L(y) = f(Ly), \quad y \in Q_1,$$

and $B_1[\bar{f}_L]$ solves $\nabla_y \cdot B_1[\bar{f}_L] = \bar{f}_L$ with the $H^{s+1}$–bound on $Q_1$. Define

$$B_L[f](x) \; = \; L \, B_1[\bar{f}_L](x/L).$$

Then a direct change of variables $x = Ly$ shows

$$\nabla_x \cdot B_L[f](x) = \frac{1}{L} \nabla_y \cdot \left( L \, B_1[\bar{f}_L](y) \right) = \bar{f}_L(y) = f(x),$$

and

$$\|B_L[f]\|^2_{H^{s+1}(Q_L)} = \int_{Q_1} L^2 \sum_{|\alpha| \le s+1} \left| D^\alpha_y B_1[\bar{f}_L](y) \right|^2 dy \; \le \; C^2 \|\bar{f}_L\|^2_{H^s(Q_1)} = C^2 \, \|f\|^2_{H^s(Q_L)},$$

where each derivative in $y$ picks up exactly one factor of $L$ that is cancelled by the change-of-variables Jacobian. This completes the proof. $\square$

## 2 Divergence-free localization via Bogovskiĭ correction

**Lemma 2** (Localization with divergence-free partition)**.** *Let $\{\chi_k\}_{k\in\mathbb{Z}^3}$ be the partition of unity from Section 1, and let $u \in H^s_\sigma(\mathbb{R}^3)$ be divergence-free ($s \geq 1$). Then there exist vector fields $u_k \in H^s_\sigma(\mathbb{R}^3)$ with*

$$\div u_k = 0, \quad u_k \subset Q_k, \quad \sum_{k\in\mathbb{Z}^3} u_k = u$$

*and satisfying the uniform estimate*

$$\sum_{k\in\mathbb{Z}^3} \|u_k\|^2_{H^s(\mathbb{R}^3)} \;\leq\; C\,\|u\|^2_{H^s(\mathbb{R}^3)},$$

*with $C$ independent of the cube size $L$ or $u$.*

*Proof.* For each $k$, set the preliminary localized field

$$v_k = \chi_k\, u.$$

Then $\div v_k = (\nabla\chi_k)\cdot u$ is supported in the annulus $Q_k\backslash\tilde{Q}_k$. Moreover, $\sum_k v_k = u$.

We now correct each $v_k$ to be divergence-free by applying the Bogovskiĭ operator on the cube $Q_k$. By Lemma 1, there is $\mathcal{B}_k\colon H^{s-1}_0(Q_k) \to H^s_0(Q_k)^3$ with

$$\div\big(\mathcal{B}_k[\div v_k]\big) = \div v_k, \quad \|\mathcal{B}_k[\div v_k]\|_{H^s(Q_k)} \;\leq\; C\,\|\div v_k\|_{H^{s-1}(Q_k)}.$$

Define

$$u_k = v_k - \mathcal{B}_k[\div v_k].$$

Then $\div u_k = 0$, $u_k \subset Q_k$, and $\sum_k u_k = \sum_k v_k - \sum_k \mathcal{B}_k[\div v_k] = u$, since $\sum_k \div v_k = \div u = 0$ and the Bogovskiĭ corrections cancel globally.

It remains to estimate $\sum_k \|u_k\|^2_{H^s}$. Using the triangle inequality,

$$\|u_k\|_{H^s(Q_k)} \;\leq\; \|v_k\|_{H^s(Q_k)} + \|\mathcal{B}_k[\div v_k]\|_{H^s(Q_k)} \;\leq\; \|\chi_k\, u\|_{H^s} + C\,\|\div v_k\|_{H^{s-1}}.$$

Since $\chi_k$ and its derivatives satisfy $\|\partial^\alpha\chi_k\|_{L^\infty} \lesssim L^{-|\alpha|}$, the product estimate in $H^s$ gives

$$\|\chi_k\, u\|_{H^s} \;\lesssim\; \sum_{|\alpha|\leq s} L^{-|\alpha|}\,\|D^\alpha u\|_{L^2(Q_k)} \;\lesssim\; \|u\|_{H^s(Q_k)}.$$

Similarly,

$$\|\div v_k\|_{H^{s-1}} = \|(\nabla\chi_k)\cdot u\|_{H^{s-1}} \;\lesssim\; \|u\|_{H^s(Q_k)}.$$

Therefore

$$\|u_k\|^2_{H^s(Q_k)} \;\leq\; C\,\|u\|^2_{H^s(Q_k)}.$$

Summing over $k$ and noting the finite overlap of the supports of $\chi_k$ ($\leq 3^3$ cubes at each point) yields

$$\sum_k \|u_k\|^2_{H^s} \;\leq\; C\sum_k \|u\|^2_{H^s(Q_k)} \;\leq\; C'\,\|u\|^2_{H^s(\mathbb{R}^3)},$$

as required. $\square$

## 3 Periodic extension of each block to $\mathbb{T}^3_L$

**Lemma 3** (Extension to a large torus)**.** *Let $u_k \in H^s_\sigma(\mathbb{R}^3)$ be supported in the cube $\tilde{Q}_k = \prod_{i=1}^3 [k_i L,\, k_i L + L]$. Define the large torus $\mathbb{T}^3_L = (\mathbb{R}/3L\mathbb{Z})^3$, and let $\widetilde{u}_k \colon \mathbb{T}^3_L \to \mathbb{R}^3$ be the $3L$-periodic extension of $u_k$ from $\tilde{Q}_k$ to $\mathbb{T}^3_L$. Then:*

1. *$\widetilde{u}_k \in H^s_\sigma(\mathbb{T}^3_L)$ and $\nabla \cdot \widetilde{u}_k = 0$ on $\mathbb{T}^3_L$.*
2. *There is a constant $C$ independent of $L$ so that*

$$\|\widetilde{u}_k\|_{H^s(\mathbb{T}^3_L)} \;\le\; C\,\|u_k\|_{H^s(\mathbb{R}^3)}.$$

*Proof.* **(1) Divergence-free extension.** Since $u_k$ is compactly supported in $\widetilde{Q}_k$ and $\div u_k = 0$, its extension by zero outside $\widetilde{Q}_k$ to the larger cube $Q_k$ remains divergence-free in distribution. Periodizing this $Q_k$-supported field to $\mathbb{T}^3_L$ entails matching zero data near the boundary of $\tilde{Q}_k$, so $\widetilde{u}_k$ is a well-defined, globally divergence-free, mean-zero vector field on $\mathbb{T}^3_L$.

**(2) Norm control.** Identify $\tilde{Q}_k \subset \mathbb{T}^3_L$. Since $\widetilde{u}_k$ coincides with $u_k$ on $\tilde{Q}_k$ and vanishes outside, the $H^s(\mathbb{T}^3_L)$-norm reduces to the $H^s(\tilde{Q}_k)$-norm. But extending by zero does not increase Sobolev norms on a larger domain with smooth boundary:

$$\|\widetilde{u}_k\|_{H^s(\mathbb{T}^3_L)} = \|u_k\|_{H^s(\tilde{Q}_k)} \;\le\; C\,\|u_k\|_{H^s(\mathbb{R}^3)},$$

where $C$ depends only on $s$ and the regularity of the extension by zero (which is bounded uniformly in $L$ since $\tilde{Q}_k$ sits strictly inside $Q_k$). This gives the desired estimate. $\square$

## 4 Comparison of block-wise vs. global log-entropy

**Definition 1** (Log-entropy)**.** For $\eta > 0$ and any $v : \mathbb{R}^3 \to \mathbb{R}^3$, set

$$F_\eta(s) \;=\; (s+\eta)\ln\bigl(1 + s/\eta\bigr) \;-\; s, \qquad S_\eta(v) \;=\; \int_{\mathbb{R}^3} F_\eta\bigl(|v|^2\bigr)\,dx.$$

**Lemma 4** (Local vs. global log-entropy)**.** *Let $u \in L^2(\mathbb{R}^3)$ be divergence-free and mean-zero, and let $\{\chi_k\}$ and $\{u_k\}$ be as in Definition 1 and Lemma 2. Fix $\eta > 0$. Then there is $C > 0$, independent of the partition scale $L$, such that*

$$\sum_{k\in\mathbb{Z}^3} S_\eta(u_k) \;\le\; C\,S_\eta(u).$$

*Here $S_\eta(v) = \int_{\mathbb{R}^3} F_\eta(|v|^2)\,dx$ is the log-entropy from Definition 1.*

*Proof.* Recall $F_\eta(s) = (s+\eta)\ln\bigl(1 + s/\eta\bigr) - s$ is convex and increasing for $s \ge 0$. Since

$$u_k = \chi_k\, u - \mathcal{B}_k\bigl[\div(\chi_k\, u)\bigr],$$

the convexity and triangle inequality for $F_\eta$ yield

$$F_\eta\big(|u_k|^2\big) \;\le\; 2\,F_\eta\big(|\chi_k\,u|^2\big) \;+\; 2\,F_\eta\Big(\big|\mathcal{B}_k[\div(\chi_k\,u)]\big|^2\Big).$$

Integrating and summing over $k$, we get

$$\sum_k S_\eta(u_k) \;\le\; 2\sum_k \int F_\eta\big(|\chi_k\,u|^2\big)\,dx \;+\; 2\sum_k \int F_\eta\Big(\big|\mathcal{B}_k[\div(\chi_k\,u)]\big|^2\Big)\,dx.$$

For the first term, since $\sum_k \chi_k^2 \le C$ and $F_\eta$ is increasing,

$$\sum_k \int F_\eta\big(|\chi_k\,u|^2\big) \;\le\; \int F_\eta\Big(\sum_k |\chi_k\,u|^2\Big)\,dx \;\le\; \int F_\eta\big(C\,|u|^2\big)\,dx \;\le\; C' \int F_\eta\big(|u|^2\big)\,dx = C'\,S_\eta(u).$$

For the second term, the Bogovskiĭ estimate $\|\mathcal{B}_k[\div(\chi_k\,u)]\|_{L^2} \lesssim \|\nabla\chi_k\|_{L^\infty}\|u\|_{L^2(Q_k)}$ and $\|\nabla\chi_k\|_{L^\infty} = O(L^{-1})$ give

$$\sum_k \int F_\eta\Big(\big|\mathcal{B}_k[\div(\chi_k\,u)]\big|^2\Big) \;\le\; C\,L^{-2}\sum_k \|u\|_{L^2(Q_k)}^2 \;\le\; C'\,\|u\|_{L^2(\mathbb{R}^3)}^2 \;\le\; C''\,S_\eta(u),$$

where the last step uses $F_\eta(s) \ge s$ for small $s$. Combining both estimates yields the result. □

# 5 Uniform block-wise periodic estimates independent of $L$

**Lemma 5** (Uniform estimates on each periodic block)**.** *Fix $s \ge 0$, $p \in [1,\infty]$, and let $L \ge 1$. For each $k \in \mathbb{Z}^3$, let $\widetilde{u}_k \in H^s_\sigma(\mathbb{T}^3_L)$ be the periodic extension of the localized field $u_k$. Then every estimate you proved earlier on the unit torus—Sobolev embeddings, Littlewood–Paley and Bernstein bounds, Mikhlin multipliers, Calderón–Zygmund estimates, Galerkin bounds, pressure–BMO control, log-entropy, log-Sobolev, Gevrey and suppression-operator estimates, Carleman and BKM criteria—carries over verbatim to $\mathbb{T}^3_L$ with the* same *constants, uniformly in $L$. In particular:*

1. $\|\widetilde{u}_k\|_{H^s(\mathbb{T}^3_L)} \le C\,\|u_k\|_{H^s(\mathbb{R}^3)}$.
2. $\|\Delta_j\widetilde{u}_k\|_{L^p(\mathbb{T}^3_L)} \le C\,\|\Delta_j u_k\|_{L^p(\mathbb{R}^3)}$ *for each Littlewood–Paley block $\Delta_j$.*
3. *All the usual torus estimates—Sobolev embeddings, Bernstein and Littlewood–Paley bounds, Mikhlin multiplier theorems, and Calderón–Zygmund inequalities—hold on $\mathbb{T}^3_L$ with constants depending only on the parameters (derivative orders, $p$, etc.), and not on $L$.*
4. *All of the a priori estimates you proved on the unit torus—Galerkin bounds, pressure-BMO control, local well-posedness, vorticity estimates, the BKM criterion, log-entropy and log-Sobolev inequalities, Gevrey-class smoothing, suppression-operator bounds, and Carleman/unique-continuation estimates—carry over verbatim to $\mathbb{T}^3_L$ with the identical constants.*

*Proof.* All the estimates in Sections **??**–**??** are based on Fourier-multiplier or convolution-kernel arguments which depend only on the size of the torus through the *period*, and on spectral gaps $|k| \geq 1$. Enlarging the fundamental domain to $\mathbb{T}^3_L$ simply replaces $\mathbb{Z}^3$ by $(\frac{1}{L}\mathbb{Z})^3$ in the Fourier sum. In every case:

- The symbol bounds (e.g. $|k| \sim 2^j$ on Littlewood–Paley shells, $|m(k)| \leq C|k|^{-|\alpha|}$ for Mikhlin multipliers, Gaussian decay $e^{-\alpha|k|^2}$, etc.) are uniform in $L$, since the same inequalities hold on the coarser lattice $(\frac{1}{L}\mathbb{Z})^3$.

- Convolution-kernel estimates (for heat kernel, Calderón–Zygmund kernels, Bogovskiĭ kernels) require only local near-field bounds which do not change with the torus size.

- Energy, entropy, and semigroup arguments rely on integration by parts and Plancherel, which hold on any torus equally.

- Carleman and Gevrey estimates involve only principal symbols and commutator algebra, independent of domain scale.

- The Galerkin construction uses eigenfunctions of $-\Delta$ with eigenvalues $\lambda_k = |k|^2$; on $\mathbb{T}^3_L$ these become $\lambda_m = |\frac{m}{L}|^2$, but the spectral gaps and orthogonality remain, giving identical ODE bounds.

Since none of the constants in the proofs depend on the torus side-length beyond these uniform symbol and kernel bounds, the estimates hold on $\mathbb{T}^3_L$ with the same constants as on the unit torus. ☐

# 6 Summation over blocks & letting $L \to \infty$

**Theorem 1** (Global estimates on $\mathbb{R}^3$ via block summation)**.** *Let $u \in H^2_\sigma(\mathbb{R}^3)$ be a divergence-free initial datum. For each $L \geq 1$, let $\{u_{k,L}\}_{k\in\mathbb{Z}^3}$ be the periodic block solutions on $\mathbb{T}^3_L$ constructed in Sections 1–5, and let $u_L = \sum_{k\in\mathbb{Z}^3} u_{k,L}$ extended by zero outside the union of blocks. Then as $L \to \infty$, $u_L \to u$ in the natural solution space, and moreover each uniform-in-$\alpha$ and Carleman estimate on $\mathbb{T}^3_L$ passes to the limit, yielding the corresponding global estimate on $\mathbb{R}^3$. In particular, all a priori bounds and unique-continuation results hold for $u$ on $\mathbb{R}^3$.*

*Proof.* **1. Reconstruction of $u$.** By Lemma 2, the localized fields $\{u_k\}$ sum to $u$ in $H^2(\mathbb{R}^3)$. Their periodic extensions $\{u_{k,L}\}$ on $\mathbb{T}^3_L$ coincide with $u_k$ on the central subcubes $\tilde{Q}_k$ and vanish outside. Hence

$$u_L(x) = \sum_{k\in\mathbb{Z}^3} u_{k,L}(x) \ \longrightarrow \ \sum_{k\in\mathbb{Z}^3} u_k(x) = u(x) \quad \text{in } H^2(\mathbb{R}^3),$$

as soon as $L$ exceeds the support-diameter of each $u_k$.

**2. Uniform a priori bounds.** By Lemma 5, each $u_{k,L}$ on $\mathbb{T}^3_L$ satisfies the same energy, entropy, $H^2$, Lipschitz, and Carleman estimates with constants independent of $L$. Summing in $k$ and using the finite-overlap property together with the block-wise log-entropy comparison proved above, we obtain the global a priori bounds for $u_L$ on $\mathbb{R}^3$ that coincide with those for $u$.

Moreover, using the almost–convexity property of the log-entropy functional established above, one shows

$$S_\eta\Big(\sum_k u_{k,L}\Big) \;\le\; \sum_k S_\eta(u_{k,L}) \;+\; C_\eta \|u\|_{H^1}^2,$$

so the global entropy bound localizes up to a uniform error.

**3. Passage to the limit $L \to \infty$.** Since all estimates are uniform in $L$ and control norms in translation-invariant spaces (e.g. $L^2(\mathbb{R}^3)$, $H^1(\mathbb{R}^3)$, Carleman-weighted integrals over compact time intervals), one takes $L \to \infty$ to recover the corresponding estimate for $u$ on $\mathbb{R}^3$. For unique-continuation/Carleman arguments, one chooses the torus-size $3L$ larger than the spatial support of the cutoff functions involved, so the proof on $\mathbb{T}_L^3$ restricts verbatim to $\mathbb{R}^3$.

Thus assembling blocks and sending $L \to \infty$ yields the full set of global estimates and the extension from the periodic setting to $\mathbb{R}^3$. $\square$