Proofs – Functional-Analytic Preliminaries

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1 Function-space setup & notation

Throughout, we work on the three-dimensional torus $\mathbb{T}^3 = (\mathbb{R}/2\pi\mathbb{Z})^3$ with spatial coordinate $x \in \mathbb{T}^3$ and Fourier modes $k \in \mathbb{Z}^3$. For a (mean-zero) function $f\colon \mathbb{T}^3 \to \mathbb{R}$ we write its Fourier series

$$f(x) \;=\; \sum_{k\in\mathbb{Z}^3} \widehat{f}(k)\, e^{ik\cdot x}, \qquad \widehat{f}(k) = \frac{1}{(2\pi)^3}\int_{\mathbb{T}^3} f(x)\, e^{-ik\cdot x}\, dx.$$

1.1 Sobolev and Bessel-potential spaces

For $s \in \mathbb{R}$ the (periodic) L^2-based Sobolev space $H^s(\mathbb{T}^3)$ is

$$H^s(\mathbb{T}^3) = \Big\{ f : \mathbb{T}^3 \to \mathbb{R} \;\Big|\; \|f\|_{H^s}^2 \;=\; \sum_{k\in\mathbb{Z}^3} (1+|k|^2)^s\, |\widehat{f}(k)|^2 \;<\; \infty \Big\}.$$

More generally, for $s \geq 0$ and $1 \leq p < \infty$, the Bessel-potential (or Sobolev–Slobodeckiĭ) space $W^{s,p}(\mathbb{T}^3)$ is defined by

$$W^{s,p}(\mathbb{T}^3) = \big\{ f : \mathbb{T}^3 \to \mathbb{R} :\; (1-\Delta)^{s/2} f \in L^p(\mathbb{T}^3) \big\}, \quad \|f\|_{W^{s,p}} = \|(1-\Delta)^{s/2} f\|_{L^p}.$$

When $s \in (0,1)$ one may also use the equivalent Gagliardo seminorm:

$$[f]_{W^{s,p}}^p = \int_{\mathbb{T}^3}\int_{\mathbb{T}^3} \frac{|f(x)-f(y)|^p}{|x-y|^{3+sp}}\, dx\, dy, \quad \|f\|_{W^{s,p}}^p = \|f\|_{L^p}^p + [f]_{W^{s,p}}^p.$$

1.2 BMO space

A locally integrable function $f\colon \mathbb{T}^3 \to \mathbb{R}$ is in $\mathrm{BMO}(\mathbb{T}^3)$ if

$$\|f\|_{\mathrm{BMO}} = \sup_{Q\subset\mathbb{T}^3} \frac{1}{|Q|}\int_Q \big|f(x) - f_Q\big|\, dx \;<\; \infty, \quad f_Q = \frac{1}{|Q|}\int_Q f(x)\, dx,$$

where the supremum is taken over all cubes $Q \subset \mathbb{T}^3$.

1.3 Gevrey classes

For $s \geq 1$ and $\sigma > 0$, the Gevrey class $G^{s,\sigma}(\mathbb{T}^3)$ consists of all $f \in C^\infty(\mathbb{T}^3)$ such that

$$\|f\|_{G^{s,\sigma}} = \left\| e^{\sigma |D|^{1/s}} f \right\|_{L^2(\mathbb{T}^3)} = \Big(\sum_{k \in \mathbb{Z}^3} e^{2\sigma |k|^{1/s}} \, |\widehat{f}(k)|^2 \Big)^{1/2} < \infty.$$

Equivalently, $f \in G^{s,\sigma}$ if there exist $C, M > 0$ so that

$$\|D^\alpha f\|_{L^2} \;\leq\; C \, M^{|\alpha|} \, (\alpha!)^s \quad \forall \text{ multi-indices } \alpha.$$

All constants C appearing below may depend on fixed parameters such as s, p, or σ, which will be indicated explicitly when needed.

2 Poincaré inequality on $\mathbb{T}^3$

Lemma 1 (Poincaré inequality on $\mathbb{T}^3$)**.** *Let $f \colon \mathbb{T}^3 \to \mathbb{R}$ be a smooth function with zero mean,*

$$\int_{\mathbb{T}^3} f(x) \, dx = 0.$$

Then

$$\|f\|_{L^2(\mathbb{T}^3)} \;\leq\; C_P \, \|\nabla f\|_{L^2(\mathbb{T}^3)},$$

where one may take $C_P = 1$ when $\mathbb{T}^3 = (\mathbb{R}/2\pi\mathbb{Z})^3$.

Proof. Expand f in its Fourier series,

$$f(x) \;=\; \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \widehat{f}(k) \, e^{ik \cdot x}, \qquad \widehat{f}(k) = \frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} f(x) e^{-ik \cdot x} \, dx.$$

The zero-mean condition means $\widehat{f}(0) = 0$. Then

$$\|f\|_{L^2}^2 = \sum_{k \neq 0} |\widehat{f}(k)|^2, \qquad \|\nabla f\|_{L^2}^2 = \sum_{k \neq 0} |k|^2 \, |\widehat{f}(k)|^2.$$

Since for every $k \neq 0$ we have $|k|^2 \geq 1$, it follows that

$$\|f\|_{L^2}^2 = \sum_{k \neq 0} |\widehat{f}(k)|^2 \;\leq\; \sum_{k \neq 0} |k|^2 \, |\widehat{f}(k)|^2 = \|\nabla f\|_{L^2}^2.$$

Taking square-roots gives the claimed inequality with $C_P = 1$. $\square$

3 Sobolev embeddings on $\mathbb{T}^3$

Lemma 2 (Sobolev embedding on $\mathbb{T}^3$). *Let $1 \le p \le \infty$ and suppose*

$$s > \frac{3}{2} - \frac{3}{p}.$$

Then there exists a constant $C = C(s,p)$ so that for every $f \in H^s(\mathbb{T}^3)$,

$$\|f\|_{L^p(\mathbb{T}^3)} \le C\, \|f\|_{H^s(\mathbb{T}^3)}.$$

In particular, if $s > \frac{3}{2}$ then $H^s(\mathbb{T}^3) \hookrightarrow L^\infty(\mathbb{T}^3)$.

Proof. Fix $N \ge 1$. Decompose f into low and high Fourier modes:

$$f = P_{\le N} f + P_{>N} f, \quad P_{\le N} f(x) = \sum_{|k| \le N} \widehat{f}(k) e^{ik\cdot x}, \quad P_{>N} = I - P_{\le N}.$$

1. *Low-frequency part.* By Cauchy–Schwarz and the fact there are $O(N^3)$ modes with $|k| \le N$,

$$\|P_{\le N} f\|_{L^p} \le \|P_{\le N} f\|_{L^2}\, |\mathbb{T}^3|^{\frac{1}{p}-\frac{1}{2}} \lesssim N^{\frac{3}{2}-\frac{3}{p}}\, \|f\|_{L^2}.$$

2. *High-frequency part.* Since $(1+|k|^2)^{s/2} \ge (1+N^2)^{s/2}$ for $|k| > N$,

$$\|P_{>N} f\|_{L^2}^2 = \sum_{|k|>N} |\widehat{f}(k)|^2 \le (1+N^2)^{-s} \sum_{|k|>N} (1+|k|^2)^s |\widehat{f}(k)|^2 = (1+N^2)^{-s}\, \|f\|_{H^s}^2.$$

Hence for any $p \ge 2$,

$$\|P_{>N} f\|_{L^p} \le \|P_{>N} f\|_{L^2} \le (1+N^2)^{-s/2}\, \|f\|_{H^s}.$$

Combining these,

$$\|f\|_{L^p} \le C\Big(N^{\frac{3}{2}-\frac{3}{p}} + (1+N^2)^{-s/2}\Big)\|f\|_{H^s}.$$

Since $s > \frac{3}{2} - \frac{3}{p}$, we may choose N sufficiently large so that

$$N^{\frac{3}{2}-\frac{3}{p}} \approx (1+N^2)^{-s/2},$$

whence

$$N^{\frac{3}{2}-\frac{3}{p}} + (1+N^2)^{-s/2} \le 2\, N^{\frac{3}{2}-\frac{3}{p}} \le C(s,p).$$

Thus $\|f\|_{L^p} \le C(s,p) \|f\|_{H^s}$, as claimed. $\square$

4 Littlewood–Paley decomposition on $\mathbb{T}^3$

Fix nonnegative radial cutoffs $\chi, \varphi \in C_c^\infty(\mathbb{R}^3)$ such that

$$\chi(\xi) = \begin{cases} 1, & |\xi| \le 1, \\ 0, & |\xi| \ge 2, \end{cases} \qquad \varphi(\xi) = \chi(\xi/2) - \chi(\xi),$$

so that

$$\varphi \subset \{\tfrac{1}{2} \le |\xi| \le 2\}, \quad \chi(\xi) + \sum_{j \ge 0} \varphi(2^{-j}\xi) \equiv 1, \quad \forall\, \xi \in \mathbb{R}^3.$$

View these as functions on the integer lattice $\mathbb{Z}^3$.

Definition 1. For each $j \ge 0$, define the Fourier projections on $\mathbb{T}^3$ by

$$\widehat{\Delta_j f}(k) = \varphi(2^{-j}k)\, \widehat{f}(k), \qquad \widehat{S_j f}(k) = \chi(2^{-j}k)\, \widehat{f}(k), \quad k \in \mathbb{Z}^3.$$

Also set $\Delta_{-1} = S_0$. Then

$$S_j f = \sum_{m=-1}^{j-1} \Delta_m f, \qquad f = \sum_{j=-1}^{\infty} \Delta_j f,$$

with convergence in $\mathcal{D}'(\mathbb{T}^3)$.

Proposition 1 (Properties of Δ_j, S_j)**.**

1. Frequency-support orthogonality: $\Delta_j \Delta_{j'} f \equiv 0$ *if* $|j - j'| \ge 2$.
2. L^p–boundedness: *For every* $1 \le p \le \infty$ *there is* C_p *with*
$$\|\Delta_j f\|_{L^p} \le C_p \|f\|_{L^p}, \quad \|S_j f\|_{L^p} \le C_p \|f\|_{L^p}.$$
3. Bernstein-type derivative bound: *For every multi–index* α*,*
$$\|D^\alpha \Delta_j f\|_{L^p} \;\le\; C_{\alpha,p}\, 2^{j|\alpha|}\, \|\Delta_j f\|_{L^p}.$$
4. Sobolev-norm equivalence: *For any* $s \in \mathbb{R}$*,*
$$c \sum_{j=-1}^{\infty} 2^{2js} \|\Delta_j f\|_{L^2}^2 \;\le\; \|f\|_{H^s}^2 \;\le\; C \sum_{j=-1}^{\infty} 2^{2js} \|\Delta_j f\|_{L^2}^2,$$
where $c, C > 0$ *depend only on* s*.*

Proof. **(1) Orthogonality.** If $|j - j'| \ge 2$ then the supports of $\varphi(2^{-j}\cdot)$ and $\varphi(2^{-j'}\cdot)$ in $\mathbb{R}^3$ are disjoint. Hence for each k, at least one factor in $\varphi(2^{-j}k)\, \varphi(2^{-j'}k)$ vanishes, so $\Delta_j \Delta_{j'} f \equiv 0$.

(2) L^p–boundedness. Each Δ_j is a Fourier multiplier with symbol $m_j(k) = \varphi(2^{-j}k)$, which is C^∞ and compactly supported away from the origin. Likewise S_j has symbol $\chi(2^{-j}k)$. By the periodic Mikhlin theorem, these multipliers satisfy $\|T_{m_j}f\|_{L^p} \le C_p\|f\|_{L^p}$. Hence $\|\Delta_j f\|_{L^p} \le C_p\|f\|_{L^p}$ and similarly $\|S_j f\|_{L^p} \le C_p\|f\|_{L^p}$.

(3) Derivative bounds. Note

$$D^\alpha \Delta_j f = T_{m_{j,\alpha}} f, \qquad m_{j,\alpha}(k) = (ik)^\alpha \, \varphi(2^{-j}k).$$

On the support of $\varphi(2^{-j}\cdot)$ we have $|k| \sim 2^j$, so $|\partial^\beta m_{j,\alpha}(k)| \lesssim 2^{j|\alpha|}\, |k|^{-|\beta|}$ for all β. Again by Mikhlin,

$$\|D^\alpha \Delta_j f\|_{L^p} = \|T_{m_{j,\alpha}} f\|_{L^p} \ \le\ C_{p,\alpha}\, 2^{j|\alpha|} \|f\|_{L^p}.$$

Restricting f to $\Delta_j f$ on the right gives the stated form.

(4) Sobolev-norm equivalence. Write

$$\|f\|_{H^s}^2 = \sum_{k\in\mathbb{Z}^3} (1+|k|^2)^s |\widehat{f}(k)|^2 = \sum_{j=-1}^{\infty} \sum_{k\in\mathbb{Z}^3} (1+|k|^2)^s \, |\widehat{\Delta_j f}(k)|^2.$$

On the support of $\widehat{\Delta_j f}$, one has

$$2^{2j} \le 1 + |k|^2 \le C(1 + 2^{2j}),$$

so

$$2^{2js} \le (1+|k|^2)^s \le C\, 2^{2js},$$

with C depending only on s. Therefore

$$c\, 2^{2js} \|\Delta_j f\|_{L^2}^2 \ \le\ \sum_k (1+|k|^2)^s |\widehat{\Delta_j f}(k)|^2 \ \le\ C\, 2^{2js} \|\Delta_j f\|_{L^2}^2.$$

Summing over j gives the claimed two-sided bound. □

5 Bernstein and periodic Mikhlin multiplier theorems

Lemma 3 (Bernstein inequalities)**.** *Let $1 \le p \le q \le \infty$. For each Littlewood–Paley block Δ_j on $\mathbb{T}^3$,*

$$\|\Delta_j f\|_{L^q(\mathbb{T}^3)} \ \le\ C\, 2^{3j\left(\frac{1}{p}-\frac{1}{q}\right)} \|\Delta_j f\|_{L^p(\mathbb{T}^3)}, \quad \|D^\alpha \Delta_j f\|_{L^p(\mathbb{T}^3)} \ \le\ C\, 2^{j|\alpha|} \|\Delta_j f\|_{L^p(\mathbb{T}^3)},$$

and for the low-frequency cutoff $S_j = \sum_{m\le j-1} \Delta_m$,

$$\|S_j f\|_{L^q(\mathbb{T}^3)} \ \le\ C\, 2^{3j\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{L^p(\mathbb{T}^3)}.$$

Proof. Recall that

$$\Delta_j f(x) = \sum_{k\in\mathbb{Z}^3} \varphi(2^{-j}k)\,\widehat{f}(k)\,e^{ik\cdot x} = (K_j * f)(x),$$

where the convolution kernel

$$K_j(x) = \sum_{k\in\mathbb{Z}^3} \varphi(2^{-j}k)\,e^{ik\cdot x} = 2^{3j}\sum_{m\in\mathbb{Z}^3} \check{\varphi}\big(2^j(x-2\pi m)\big),$$

and $\check{\varphi}\in\mathcal{S}(\mathbb{R}^3)$ is the inverse-Fourier transform of φ. In particular,

$$|K_j(x)| \;\le\; 2^{3j}\sum_{m\in\mathbb{Z}^3} \frac{C_N}{\big(1+2^j|x-2\pi m|\big)^N} \quad \forall N,$$

so for any $1\le r\le\infty$,

$$\|K_j\|_{L^r(\mathbb{T}^3)} \;\le\; 2^{3j}\Big(\int_{\mathbb{T}^3}\big|\sum_m \tfrac{C_N}{(1+2^j|x-2\pi m|)^N}\big|^r dx\Big)^{1/r} \;\le\; C\,2^{3j\left(1-\frac{1}{r}\right)},$$

provided $N>3$. Now Young's convolution inequality on the compact group $\mathbb{T}^3$ gives, with $\frac{1}{p}+\frac{1}{r}=1+\frac{1}{q}$,

$$\|\Delta_j f\|_{L^q} = \|K_j * f\|_{L^q} \;\le\; \|K_j\|_{L^r}\,\|f\|_{L^p} \;\le\; C\,2^{3j\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_{L^p}.$$

To obtain the stated form, replace f by $\Delta_j f$.

For derivatives, observe

$$D^\alpha\Delta_j f(x) = \sum_k \varphi(2^{-j}k)\,(ik)^\alpha\,\widehat{f}(k)\,e^{ik\cdot x} = (D^\alpha K_j)*f,$$

and since $|(ik)^\alpha\varphi(2^{-j}k)|\lesssim 2^{j|\alpha|}$, the same kernel argument yields $\|D^\alpha\Delta_j f\|_{L^p}\le C\,2^{j|\alpha|}\|f\|_{L^p}$.

Finally, for $S_j=\sum_{m\le j-1}\Delta_m$, sum the above in m and use $\sum_{m\le j-1}2^{3m(\frac{1}{p}-\frac{1}{q})}\le C\,2^{3j(\frac{1}{p}-\frac{1}{q})}$. □

Theorem 1 (Periodic Mikhlin multiplier theorem)**.** *Let $m\colon\mathbb{R}^3\setminus\{0\}\to$ be smooth with*

$$|\partial^\alpha m(\xi)| \;\le\; C_\alpha\,|\xi|^{-|\alpha|}, \quad |\alpha|\le 4,\ \xi\ne 0.$$

Define

$$T_m f(x) = \sum_{k\in\mathbb{Z}^3\setminus\{0\}} m(k)\,\widehat{f}(k)\,e^{ik\cdot x}, \quad \widehat{f}(0)=0.$$

Then for every $1<p<\infty$, there is C_p such that $\|T_m f\|_{L^p(\mathbb{T}^3)}\le C_p\,\|f\|_{L^p(\mathbb{T}^3)}$.

Proof. We show T_m is a Calderón–Zygmund operator on $\mathbb{T}^3$. Its convolution kernel is

$$K(x) = \sum_{k \neq 0} m(k)\, e^{ik\cdot x}, \quad x \in \mathbb{T}^3 \setminus \{0\}.$$

By the smoothness and decay of m, one may decompose

$$K(x) = K_{\mathrm{loc}}(x) + K_{\mathrm{rem}}(x),$$

where K_{loc} is supported in $|x| \leq \delta$ and satisfies the standard singular-integral estimates:

$$|K_{\mathrm{loc}}(x)| \;\leq\; \frac{C}{|x|^3}, \quad |\nabla K_{\mathrm{loc}}(x)| \;\leq\; \frac{C}{|x|^4},$$

and $K_{\mathrm{rem}} \in L^1(\mathbb{T}^3)$. Concretely, partition the sum over $\mathbb{Z}^3$ into dyadic shells and use Poisson summation paired with the derivative bounds on m to verify these pointwise kernel estimates.

Since $K_{\mathrm{rem}} \in L^1$, convolution with it is bounded on all L^p. For K_{loc}, the size and smoothness estimates imply that $T_{\mathrm{loc}} f = K_{\mathrm{loc}} * f$ extends to a bounded operator on $L^p(\mathbb{T}^3)$ for $1 < p < \infty$ by the standard Calderón–Zygmund theorem (adapted to the compact setting).

Hence $T_m = T_{\mathrm{loc}} + T_{\mathrm{rem}}$ is bounded on $L^p(\mathbb{T}^3)$ for $1 < p < \infty$, as required. $\square$

6 Calderón–Zygmund theorem on $\mathbb{T}^3$

Theorem 2 (Singular integrals on $\mathbb{T}^3$)**.** *Let $K_0 \in C^1(\mathbb{R}^3 \setminus \{0\})$ satisfy the size, smoothness, and cancellation conditions*

$$|K_0(x)| \leq \frac{A}{|x|^3}, \quad |\nabla K_0(x)| \leq \frac{B}{|x|^4}, \quad \int_{|x|=r} K_0(x)\, dS(x) = 0 \quad (\forall r > 0).$$

Define the periodic kernel

$$K(x) = \sum_{m \in \mathbb{Z}^3} K_0(x - 2\pi m),$$

and the principal-value operator

$$Tf(x) = \mathrm{p.v.} \int_{\mathbb{T}^3} K(x-y)\, f(y)\, dy.$$

Then:

1. *T extends to a bounded operator $L^p(\mathbb{T}^3) \to L^p(\mathbb{T}^3)$ for every $1 < p < \infty$.*
2. *T extends to a bounded operator $L^\infty(\mathbb{T}^3) \to \mathrm{BMO}(\mathbb{T}^3)$.*

Proof. We break the proof into four steps.

1. Weak-$(1,1)$ estimate. Let $f \in L^1(\mathbb{T}^3)$ and $\alpha > 0$. By the classical Calderón–Zygmund decomposition, there exist disjoint cubes $\{Q_i\}$ and functions $g, b = \sum_i b_i$ with

$$f = g + b, \quad \|g\|_{L^\infty} \le \alpha, \quad \int_{Q_i} b_i = 0, \quad b_i \subset Q_i,$$

and $\sum_i |Q_i| \le \frac{\|f\|_{L^1}}{\alpha}$, $\|g\|_{L^1} \le \|f\|_{L^1}$, $\|b_i\|_{L^1} \le 2\alpha|Q_i|$. Then

$$\{x : |Tf(x)| > 2\alpha\} \;\subset\; \{x : |Tg(x)| > \alpha\} \;\cup\; \{x : |Tb(x)| > \alpha\}.$$

For the "good" part g, using the $L^2 \to L^2$ boundedness of T (Step 2 below) and Chebyshev:

$$|\{x : |Tg| > \alpha\}| \;\le\; \frac{1}{\alpha^2}\|Tg\|_{L^2}^2 \;\le\; \frac{C}{\alpha^2}\|g\|_{L^2}^2 \;\le\; \frac{C}{\alpha}\|g\|_{L^1} \;\le\; \frac{C\|f\|_{L^1}}{\alpha}.$$

For the "bad" part $b = \sum b_i$, set $5Q_i$ the cube with same center and side-length 5 times that of Q_i. One shows for $x \notin \bigcup_i 5Q_i$:

$$|Tb_i(x)| = \left|\int_{Q_i} \big[K(x-y) - K(x-c_i)\big] b_i(y)\,dy\right| \;\le\; C\,\frac{\ell(Q_i)}{|x-c_i|^4}\,\|b_i\|_{L^1},$$

using the mean-zero of b_i and the smoothness of K_0. Hence $\{x : |Tb| > \alpha\} \subset \bigcup_i 5Q_i$ and

$$\big|\{x : |Tb| > \alpha\}\big| \;\le\; \sum_i |5Q_i| \;\le\; 5^3 \sum_i |Q_i| \;\le\; \frac{125\,\|f\|_{L^1}}{\alpha}.$$

Combining gives the weak-$(1,1)$ bound:

$$|\{x : |Tf| > \lambda\}| \;\le\; \frac{C\,\|f\|_{L^1}}{\lambda}.$$

2. Strong-$(2,2)$ estimate. Since K has mean-zero on the torus and its Fourier transform (the multiplier $m(k) = \widehat{K}(k)$) is bounded, Plancherel's theorem yields

$$\|Tf\|_{L^2} = \|m(D)f\|_{L^2} \;\le\; \|m\|_{L^\infty(\mathbb{Z}^3)}\,\|f\|_{L^2} \;\le\; C\,\|f\|_{L^2}.$$

3. Interpolation to L^p. By Marcinkiewicz interpolation between weak-$(1,1)$ and strong-$(2,2)$, T is bounded $L^p \to L^p$ for all $1 < p \le 2$. Duality then covers $2 < p < \infty$.

4. $L^\infty \to \mathrm{BMO}$**.** Let $f \in L^\infty$ and Q any cube. Write $f = f\,\mathbf{1}_{3Q} + f\,\mathbf{1}_{\mathbb{T}^3\setminus 3Q}$, and correspondingly $Tf = I + II$.

(i) Local part $I = T(f\,\mathbf{1}_{3Q})$. Since $\|f\,\mathbf{1}_{3Q}\|_{L^2} \le |3Q|^{1/2}\|f\|_\infty$,

$$\frac{1}{|Q|}\int_Q \big|I - (I)_Q\big| \;\le\; \frac{2}{|Q|}\int_Q |I| \;\le\; \frac{2}{|Q|^{1/2}}\|I\|_{L^2} \;\le\; \frac{2C}{|Q|^{1/2}}\|f\,\mathbf{1}_{3Q}\|_{L^2} \;\le\; C'\|f\|_\infty.$$

(ii) Tail part $II = T(f\,\mathbf{1}_{\mathbb{T}^3\setminus 3Q})$. For any $x, y \in Q$,

$$|II(x) - II(y)| \ \le\ \int_{\mathbb{T}^3\setminus 3Q} \big|K(x-z) - K(y-z)\big|\,\big|f(z)\big|\,dz \ \le\ \|f\|_\infty \int_{|z-c_Q|\ge\frac{3}{2}\ell(Q)} \frac{C\,|x-y|}{|z-c_Q|^4}\,dz \ \le\ C''\|f\|_\infty,$$

where c_Q and $\ell(Q)$ are the center and side-length of Q. Hence $\frac{1}{|Q|}\int_Q |II - (II)_Q| \le \sup_{x,y\in Q} |II(x) - II(y)| \le C''\|f\|_\infty$.

Combining (i) and (ii) shows $\|Tf\|_{\mathrm{BMO}} \le C\,\|f\|_\infty$.

Thus T is bounded on L^p for $1 < p < \infty$ and maps L^∞ into BMO, as claimed. $\square$

7 John–Nirenberg inequality

Lemma 4 (John–Nirenberg inequality)**.** *There exist universal constants* $c_1, c_2 > 0$ *such that for any* $f \in \mathrm{BMO}(\mathbb{T}^3)$*, any cube* $Q \subset \mathbb{T}^3$*, and any* $\lambda > 0$*,*

$$\big|\{x \in Q : |f(x) - f_Q| > \lambda\}\big| \ \le\ c_1\,|Q|\,\exp\!\Big(-\tfrac{c_2\,\lambda}{\|f\|_{\mathrm{BMO}}}\Big).$$

As a consequence, for each $1 \le p < \infty$ *there is* $C_p > 0$ *so that*

$$\|f - f_Q\|_{L^p(Q)} \ \le\ C_p\,\|f\|_{\mathrm{BMO}}\,|Q|^{1/p}.$$

Proof. Set $\alpha = \|f\|_{\mathrm{BMO}}$. Fix a cube Q and $\lambda > 0$. We first handle the case $\lambda = n\alpha$ for an integer $n \ge 1$.

Step 1: Decomposition for $\lambda = \alpha$. Consider the collection $\{Q^1_j\}$ of maximal dyadic subcubes of Q satisfying

$$\frac{1}{|Q^1_j|}\int_{Q^1_j} |f - f_Q|\,dx \ >\ \alpha.$$

By maximality these cubes are pairwise disjoint, and on the remaining set $\Omega^0 = Q \setminus \bigcup_j Q^1_j$ we have $|f(x) - f_Q| \le \alpha$ almost everywhere. Moreover, by the BMO bound,

$$\sum_j |Q^1_j| = \sum_j \frac{1}{\alpha}\int_{Q^1_j} |f - f_Q| \ \le\ \frac{1}{\alpha}\int_Q |f - f_Q| \ \le\ \frac{|Q|\,\alpha}{\alpha} \ =\ |Q|.$$

But each Q^1_j satisfies $\int_{Q^1_j} |f - f_Q| > \alpha\,|Q^1_j|$, so $\sum_j |Q^1_j| \le \frac{1}{2}\,|Q|$. Hence

$$\big|\{x \in Q : |f - f_Q| > \alpha\}\big| \ \le\ \sum_j |Q^1_j| \ \le\ \tfrac{1}{2}\,|Q|.$$

Step 2: Iteration for $\lambda = n\alpha$. On each bad cube Q_j^1, apply the same argument with average $f_{Q_j^1}$. We obtain subcubes $\{Q_{j,k}^2\} \subset Q_j^1$ covering at most half of $\bigcup_j Q_j^1$, on whose complement $|f - f_{Q_j^1}| \le \alpha$. By the triangle inequality,

$$\{x \in Q : |f - f_Q| > 2\alpha\} \ \subset \ \bigcup_{j,k} Q_{j,k}^2,$$

and $\left|\bigcup_{j,k} Q_{j,k}^2\right| \le \frac{1}{2} \left|\bigcup_j Q_j^1\right| \le \frac{1}{4} |Q|$. Iterating n times yields

$$\left|\{x \in Q : |f - f_Q| > n\alpha\}\right| \ \le \ 2^{-n} \, |Q|.$$

Step 3: Exponential bound for general λ. Given arbitrary $\lambda > 0$, choose $n = \lfloor \lambda/\alpha \rfloor$. Then

$$2^{-n} \ = \ \exp\bigl(-n \ln 2\bigr) \ \le \ \exp\Bigl(-\tfrac{\lambda-\alpha}{\alpha} \ln 2\Bigr) \ \le \ c_1 \exp\Bigl(-\tfrac{c_2\,\lambda}{\alpha}\Bigr),$$

with $c_1 = e^{\ln 2}$ and $c_2 = \ln 2$. Therefore

$$\left|\{x \in Q : |f - f_Q| > \lambda\}\right| \ \le \ 2^{-n} |Q| \ \le \ c_1 \, |Q| \, e^{-c_2 \lambda/\alpha},$$

which is the desired measure estimate.

Step 4: L^p–estimate. Finally, for $1 \le p < \infty$,

$$\int_Q |f - f_Q|^p = \int_0^\infty p\lambda^{p-1} \left|\{x : |f - f_Q| > \lambda\}\right| d\lambda \ \le \ c_1 |Q| \int_0^\infty p\lambda^{p-1} e^{-c_2\lambda/\alpha} \, d\lambda.$$

The integral converges and equals $C_p \, \alpha^p$. Taking the pth root gives $\|f - f_Q\|_{L^p(Q)} \le C_p \, \|f\|_{\mathrm{BMO}} \, |Q|^{1/p}$. □

8 Bogovskiĭ operator on cubes

Lemma 5 (Bogovskiĭ operator on a cube)**.** *Let $Q \subset \mathbb{R}^3$ be a cube and, for $s \ge 1$, set*

$$H_0^{s-1}(Q) = \bigl\{ f \in H^{s-1}(Q) : \int_Q f \, dx = 0 \bigr\}.$$

Then there is a bounded linear operator $\mathcal{B} : H_0^{s-1}(Q) \to H_0^s(Q)^3$ (where H_0^s is the closure of $C_c^\infty(Q)$ in H^s) such that for every $f \in H_0^{s-1}(Q)$,

$$\operatorname{div} \mathcal{B}[f] = f, \qquad \|\mathcal{B}[f]\|_{H^s(Q)} \ \le \ C \, \|f\|_{H^{s-1}(Q)},$$

with C depending only on s and the shape of Q (hence uniform over all cubes of comparable aspect-ratio).

Proof. We construct $\mathcal{B}[f]$ as the unique $u \in H_0^1(Q)^3$ solving the mixed (Stokes) problem

$$\begin{cases} -\Delta u + u + \nabla p = 0 & \text{in } Q, \\ \operatorname{div} u = f & \text{in } Q, \\ u = 0 & \text{on } \partial Q, \end{cases}$$

with the pressure $p \in L_0^2(Q)$. Define the spaces

$$V = H_0^1(Q)^3, \quad W = \{q \in L^2(Q) : \int_Q q\,dx = 0\},$$

and the bilinear forms

$$a(u,v) = \int_Q \big(\nabla u : \nabla v + u \cdot v\big)\,dx, \qquad b(v,q) = \int_Q q \operatorname{div} v\,dx.$$

The mixed formulation is: find $(u,p) \in V \times W$ such that

$$\begin{aligned} a(u,v) + b(v,p) &= 0 && \forall\, v \in V, \\ b(u,q) &= \int_Q f\,q\,dx && \forall\, q \in W. \end{aligned}$$

By standard Babuska–Brezzi theory on the convex domain Q, a is coercive on $\ker b = \{v \in V : \operatorname{div} v = 0\}$, and b satisfies the inf–sup condition. Hence there is a unique solution (u,p), with

$$\|u\|_{H^1(Q)} + \|p\|_{L^2(Q)} \;\le\; C\,\|f\|_{L^2(Q)}.$$

Elliptic regularity for the Stokes system on the convex cube then gives, for each integer $s \ge 1$,

$$\|u\|_{H^s(Q)} + \|p\|_{H^{s-1}(Q)} \;\le\; C_s\,\|f\|_{H^{s-1}(Q)}.$$

Setting $\mathcal{B}[f] = u$ yields $\operatorname{div} \mathcal{B}[f] = f$, $\mathcal{B}[f]\big|_{\partial Q} = 0$, and the claimed H^s-estimate. $\square$