Proofs - Extension to R3

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1 Smooth partition of unity into large cubes

[Scale-free Bogovskiĭ on periodic blocks] Let $Q_L = [0, L]^3$ and denote by $\bar{f}_L(x) = f(Lx)$ the pull-back of $f : Q_L \to$ to the reference cube $Q_1 = [0, 1]^3$. There is a linear Bogovskiĭ operator $B_1 : \{g \in C^\infty(Q_1) : \int_{Q_1} g = 0\} \to C^\infty(Q_1;^3)$ satisfying

$$\nabla\cdot B_1[g] = g, \quad \|B_1[g]\|_{H^{s+1}(Q_1)} \ \le\ C\,\|g\|_{H^s(Q_1)} \quad foralls \ge 0,$$

with a constant C independent of g. Defining

$$B_L[f](x) \ =\ L\,\big(B_1[\bar{f}_L]\big)(x/L),$$

we get a Bogovskiĭ operator on Q_L satisfying

$$\nabla\cdot B_L[f] = f, \quad \|B_L[f]\|_{H^{s+1}(Q_L)} \ \le\ C\,\|f\|_{H^s(Q_L)}, \quad \int_{Q_L} B_L[f]\,dx = 0,$$

where C is *the same* constant as above, and in particular *independent of* L.

By construction,

$$\bar{f}_L(y) = f(Ly), \quad y \in Q_1,$$

and $B_1[\bar{f}_L]$ solves $\nabla_y\cdot B_1[\bar{f}_L] = \bar{f}_L$ with the H^{s+1}–bound on Q_1. Define

$$B_L[f](x) \ =\ L\,B_1[\bar{f}_L](x/L).$$

Then a direct change of variables $x = Ly$ shows

$$\nabla_x\cdot B_L[f](x) = \frac{1}{L}\nabla_y\cdot\big(L\,B_1[\bar{f}_L](y)\big) = \bar{f}_L(y) = f(x),$$

and

$$\|B_L[f]\|^2_{H^{s+1}(Q_L)} = \int_{Q_1} L^2 \sum_{|\alpha|\le s+1} \big|D^\alpha_y B_1[\bar{f}_L](y)\big|^2\,dy \ \le\ C^2\|\bar{f}_L\|^2_{H^s(Q_1)} = C^2\,\|f\|^2_{H^s(Q_L)},$$

where each derivative in y picks up exactly one factor of L that is cancelled by the change-of-variables Jacobian. This completes the proof.

2 Divergence-free localization via Bogovskiĭ correction

[Localization with divergence-free partition] Let $\{\chi_k\}_{k\in^3}$ be the partition of unity from Lemma **??**, and let $u \in H^s_\sigma(^3)$ be divergence-free ($s \geq 1$). Then there exist vector fields $u_k \in H^s_\sigma(^3)$ with

$$\div u_k = 0, \quad u_k \subset Q_k, \quad \sum_{k\in^3} u_k = u$$

and satisfying the uniform estimate

$$\sum_{k\in^3} \|u_k\|^2_{H^s(^3)} \;\leq\; C\,\|u\|^2_{H^s(^3)},$$

with C independent of the cube size L or u.

For each k, set the preliminary localized field

$$v_k = \chi_k\, u.$$

Then $\div v_k = (\nabla\chi_k)\cdot u$ is supported in the annulus $Q_k\backslash\tilde{Q}_k$. Moreover, $\sum_k v_k = u$.

We now correct each v_k to be divergence-free by applying the Bogovskiĭ operator on the cube Q_k. By Lemma **??**, there is $\mathcal{B}_k\colon H^{s-1}_0(Q_k) \to H^s_0(Q_k)^3$ with

$$\div\big(\mathcal{B}_k[\div v_k]\big) = \div v_k, \quad \|\mathcal{B}_k[\div v_k]\|_{H^s(Q_k)} \;\leq\; C\,\|\div v_k\|_{H^{s-1}(Q_k)}.$$

Define

$$u_k = v_k - \mathcal{B}_k[\div v_k].$$

Then $\div u_k = 0$, $u_k \subset Q_k$, and $\sum_k u_k = \sum_k v_k - \sum_k \mathcal{B}_k[\div v_k] = u$, since $\sum_k \div v_k = \div u = 0$ and the Bogovskiĭ corrections cancel globally.

It remains to estimate $\sum_k \|u_k\|^2_{H^s}$. Using the triangle inequality,

$$\|u_k\|_{H^s(Q_k)} \;\leq\; \|v_k\|_{H^s(Q_k)} + \|\mathcal{B}_k[\div v_k]\|_{H^s(Q_k)} \;\leq\; \|\chi_k\, u\|_{H^s} + C\,\|\div v_k\|_{H^{s-1}}.$$

Since χ_k and its derivatives satisfy $\|\partial^\alpha\chi_k\|_{L^\infty} L^{-|\alpha|}$, the product estimate in H^s gives

$$\|\chi_k\, u\|_{H^s} \;\; \sum_{|\alpha|\leq s} L^{-|\alpha|}\,\|D^\alpha u\|_{L^2(Q_k)} \;\; \|u\|_{H^s(Q_k)}.$$

Similarly,

$$\|\div v_k\|_{H^{s-1}} = \|(\nabla\chi_k)\cdot u\|_{H^{s-1}} \;\; \|u\|_{H^s(Q_k)}.$$

Therefore

$$\|u_k\|^2_{H^s(Q_k)} \;\leq\; C\,\|u\|^2_{H^s(Q_k)}.$$

Summing over k and noting the finite overlap of the supports of χ_k ($\leq 3^3$ cubes at each point) yields

$$\sum_k \|u_k\|^2_{H^s} \;\leq\; C\sum_k \|u\|^2_{H^s(Q_k)} \;\leq\; C'\,\|u\|^2_{H^s(^3)},$$

as required.

3 Periodic extension of each block to ${}^3_L$

[Extension to a large torus] Let $u_k \in H^s_\sigma({}^3)$ be supported in the cube $\tilde{Q}_k = \prod_{i=1}^3 [k_i L,\, k_i L + L]$. Define the large torus ${}^3_L = (/3L)^3$, and let $\widetilde{u}_k{:}{}^3_L \to{}^3$ be the $3L$-periodic extension of u_k from $\tilde{Q}_k$ to ${}^3_L$. Then:

1. $\widetilde{u}_k \in H^s_\sigma({}^3_L)$ and $\nabla \cdot \widetilde{u}_k = 0$ on ${}^3_L$.
2. There is a constant C independent of L so that

$$\|\widetilde{u}_k\|_{H^s({}^3_L)} \;\le\; C\,\|u_k\|_{H^s({}^3)}.$$

(1) Divergence-free extension. Since u_k is compactly supported in $\tilde{Q}_k$ and $\div u_k = 0$, its extension by zero outside $\tilde{Q}_k$ to the larger cube Q_k remains divergence-free in distribution. Periodizing this Q_k-supported field to ${}^3_L$ entails matching zero data near the boundary of $\tilde{Q}_k$, so $\widetilde{u}_k$ is a well-defined, globally divergence-free, mean-zero vector field on ${}^3_L$.
(2) Norm control. Identify $\tilde{Q}_k \subset{}^3_L$. Since $\widetilde{u}_k$ coincides with u_k on $\tilde{Q}_k$ and vanishes outside, the $H^s({}^3_L)$-norm reduces to the $H^s(\tilde{Q}_k)$-norm. But extending by zero does not increase Sobolev norms on a larger domain with smooth boundary:

$$\|\widetilde{u}_k\|_{H^s({}^3_L)} = \|u_k\|_{H^s(\tilde{Q}_k)} \;\le\; C\,\|u_k\|_{H^s({}^3)},$$

where C depends only on s and the regularity of the extension by zero (which is bounded uniformly in L since $\tilde{Q}_k$ sits strictly inside Q_k). This gives the desired estimate.

4 Comparison of block-wise vs. global log-entropy

[Local vs. global log-entropy] Let $u \in L^2({}^3)$ be divergence-free and mean-zero, and let $\{\chi_k\}$ and $\{u_k\}$ be as in Lemmas 1 and **??**. Fix $\eta > 0$. Then there is $C > 0$, independent of the partition scale L, such that

$$\sum_{k\in{}^3} S_\eta(u_k) \;\le\; C\,S_\eta(u).$$

Here $S_\eta(v) = \int_3 F_\eta(|v|^2)\,dx$ is the log-entropy from Section **??**.

Recall $F_\eta(s) = (s+\eta)\ln(1+s/\eta) - s$ is convex and increasing in $s \ge 0$. Since $u_k = \chi_k u - \mathcal{B}_k[\div(\chi_k u)]$, we use the triangle inequality and convexity:

$$F_\eta\big(|u_k|^2\big) \;\le\; 2\,F_\eta\big(|\chi_k u|^2\big) \;+\; 2\,F_\eta\big(|\mathcal{B}_k[\div(\chi_k u)]|^2\big).$$

Integrating and summing over k,

$$\sum_k S_\eta(u_k) \;\le\; 2\sum_k \int F_\eta\big(|\chi_k u|^2\big) + 2\sum_k \int F_\eta\big(|\mathcal{B}_k[\div(\chi_k u)]|^2\big).$$

For the first sum, since $\sum_k \chi_k^2 \le C$ and F_η is increasing,

$$\sum_k \int F_\eta\big(|\chi_k u|^2\big) \ \le\ \int F_\eta\Big(\sum_k |\chi_k u|^2\Big) \ \le\ \int F_\eta\big(C\,|u|^2\big) \ \le\ C' \int F_\eta\big(|u|^2\big) = C'\, S_\eta(u).$$

For the second sum, use the Bogovskiĭ estimate $\|\mathcal{B}_k[\div(\chi_k u)]\|_{L^2} \|\nabla\chi_k\|_{L^\infty} \|u\|_{L^2(Q_k)}$. Since $\nabla\chi_k = O(L^{-1})$, and $F_\eta(s)s\ln(1+s/\eta)$ grows sub-quadratically for small s, one shows

$$\sum_k \int F_\eta\big(|\mathcal{B}_k[\div(\chi_k u)]|^2\big) \ \le\ C\,L^{-2} \sum_k \|u\|^2_{L^2(Q_k)} \ \le\ C'\,\|u\|^2_{L^2(^3)} \ \le\ C''\, S_\eta(u),$$

where the last inequality uses $F_\eta(s) \ge s$ for s small. Combining these two estimates yields the desired bound.

5 Uniform block-wise periodic estimates independent of L

[Uniform estimates on each periodic block] Fix $s \ge 0$, $p \in [1,\infty]$, and let $L \ge 1$. For each $k \in^3$, let $\widetilde{u}_k \in H^s_\sigma(^3_L)$ be the periodic extension of the localized field u_k from Lemma 3. Then for each of the estimates proved on ^{3} in Sections **??**–**??**, the same bound holds on ^3_L with the *same* constant, uniformly in L. In particular:

1. $\|\widetilde{u}_k\|_{H^s(^3_L)} \le C\,\|u_k\|_{H^s(^3)}$.
2. $\|\Delta_j \widetilde{u}_k\|_{L^p(^3_L)} \le C\,\|\Delta_j u_k\|_{L^p(^3)}$ for each Littlewood–Paley block Δ_j.
3. All Sobolev, Bernstein, Mikhlin, and Calderón–Zygmund bounds (Lemmas **??**–**??**) hold on ^3_L with constants depending only on the statements' parameters, not on L.
4. The Galerkin, pressure-BMO, local-well-posedness, vorticity, BKM, log-entropy, log-Sobolev, Gevrey, suppression-operator, and Carleman estimates (Sections **??**–**??**) carry over verbatim when posed on ^3_L, with identical constants.

All the estimates in Sections **??**–**??** are based on Fourier-multiplier or convolution-kernel arguments which depend only on the size of the torus through the *period*, and on spectral gaps $|k| \ge 1$. Enlarging the fundamental domain to ^3_L simply replaces ^{3} by $(1L)^3$ in the Fourier sum. In every case:

- The symbol bounds (e.g. $|k| \sim 2^j$ on Littlewood–Paley shells, $|m(k)| \le C|k|^{-|\alpha|}$ for Mikhlin multipliers, Gaussian decay $e^{-\alpha|k|^2}$, etc.) are uniform in L, since the same inequalities hold on the coarser lattice $(1L)^3$.

- Convolution-kernel estimates (for heat kernel, Calderón–Zygmund kernels, Bogovskiĭ kernels) require only local near-field bounds which do not change with the torus size.

- Energy, entropy, and semigroup arguments rely on integration by parts and Plancherel, which hold on any torus equally.

- Carleman and Gevrey estimates involve only principal symbols and commutator algebra, independent of domain scale.

- The Galerkin construction uses eigenfunctions of $-\Delta$ with eigenvalues $\lambda_k = |k|^2$; on ^3_L these become $\lambda_m = |mL|^2$, but the spectral gaps and orthogonality remain, giving identical ODE bounds.

Since none of the constants in the proofs depend on the torus side-length beyond these uniform symbol and kernel bounds, the estimates hold on ^3_L with the same constants as on the unit torus.

6 Summation over blocks & letting $L \to \infty$

[Global estimates on ^{3} via block summation] Let $u \in H^2_\sigma(^3)$ be a divergence-free initial datum. For each $L \geq 1$, let $\{u_{k,L}\}_{k\in^3}$ be the periodic block solutions on ^3_L constructed in Sections 1–5, and let $u_L = \sum_{k\in^3} u_{k,L}$ extended by zero outside the union of blocks. Then as $L \to \infty$, $u_L \to u$ in the natural solution space, and moreover each uniform-in-α and Carleman estimate on ^3_L passes to the limit, yielding the corresponding global estimate on ^{3}. In particular, all a priori bounds and unique-continuation results hold for u on ^{3}.

1. Reconstruction of u. By Lemma **??**, the localized fields $\{u_k\}$ sum to u in $H^2(^3)$. Their periodic extensions $\{u_{k,L}\}$ on ^3_L coincide with u_k on the central subcubes $\tilde{Q}_k$ and vanish outside. Hence

$$u_L(x) \;=\; \sum_{k\in^3} u_{k,L}(x) \;\longrightarrow\; \sum_{k\in^3} u_k(x) \;=\; u(x) \quad inH^2(^3),$$

as soon as L exceeds the support-diameter of each u_k.

2. Uniform a priori bounds. By Lemma 5, each $u_{k,L}$ on ^3_L satisfies the same energy, entropy, H^2, Lipschitz, and Carleman estimates with constants independent of L. Summing in k and using the finite-overlap and block-wise comparison (Lemmas 4), we obtain the global a priori bounds for u_L on ^{3} that coincide with those for u.

3. Passage to the limit $L \to \infty$. Since all estimates are uniform in L and control norms in translation-invariant spaces (e.g. $L^2(^3)$, $H^1(^3)$, Carleman-weighted integrals over compact time intervals), one takes $L \to \infty$ to recover the corresponding estimate for u on ^{3}. For unique-continuation/Carleman arguments, one chooses the torus-size $3L$ larger than the spatial support of the cutoff functions involved, so the proof on ^3_L restricts verbatim to ^{3}.

Thus assembling blocks and sending $L \to \infty$ yields the full set of global estimates and the extension from the periodic setting to ^{3}.