Proofs - Auxiliary ODE Grönwall Tools

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May 2025

1 Generalized Grönwall inequalities

Lemma 1 (Integral and differential Grönwall inequalities)**.** *Let $T > 0$, and suppose $y : [0,T] \to [0,\infty)$ is continuous.*

1. *(Integral form) If there are nonnegative functions $a, b \in L^1(0,T)$ such that*

$$y(t) \;\leq\; a(t) + \int_0^t b(s)\, y(s)\, ds, \quad \forall\, t \in [0,T],$$

then

$$y(t) \;\leq\; a(t) + \int_0^t a(s)\, b(s)\, \exp\Big(\int_s^t b(\sigma)\, d\sigma\Big)\, ds.$$

In particular, if a is nondecreasing then

$$y(t) \;\leq\; a(t)\, \exp\Big(\int_0^t b(s)\, ds\Big).$$

2. *(Differential form) If $y \in C^1([0,T])$ satisfies*

$$\frac{dy}{dt} \;\leq\; \alpha(t)\, y(t) + \beta(t), \quad \forall\, t \in [0,T],$$

for integrable functions α, β, then

$$y(t) \;\leq\; y(0)\, \exp\Big(\int_0^t \alpha(s)\, ds\Big) + \int_0^t \beta(s)\, \exp\Big(\int_s^t \alpha(\sigma)\, d\sigma\Big)\, ds.$$

Proof. **1. Integral form.** Define $z(t) = \displaystyle\int_0^t b(s)\, y(s)\, ds$. Then the hypothesis gives

$$y(t) \leq a(t) + z(t) \quad \Longrightarrow \quad z'(t) = b(t)\, y(t) \leq b(t)\, a(t) + b(t)\, z(t).$$

Apply the differential-form Grönwall inequality (part 2) to z:

$$z(t) \leq \int_0^t b(s)\, a(s)\, \exp\Big(\int_s^t b(\sigma)\, d\sigma\Big)\, ds.$$

Hence

$$y(t) \le a(t) + z(t) \le a(t) + \int_0^t a(s)\, b(s)\, \exp\Big(\int_s^t b(\sigma)\, d\sigma\Big)\, ds.$$

If a is nondecreasing then $a(s) \le a(t)$ for $0 \le s \le t$, so

$$y(t) \le a(t)\Big[1 + \int_0^t b(s)\, \exp\Big(\int_s^t b(\sigma)\, d\sigma\Big)\, ds\Big] = a(t)\, \exp\Big(\int_0^t b(s)\, ds\Big).$$

2. Differential form. Rearrange the inequality as

$$\frac{d}{dt}\big(e^{-\int_0^t \alpha}\, y(t)\big) = e^{-\int_0^t \alpha}\Big(y'(t) - \alpha(t)\, y(t)\Big) \le e^{-\int_0^t \alpha}\, \beta(t).$$

Integrate from 0 to t:

$$e^{-\int_0^t \alpha}\, y(t) - y(0) \le \int_0^t e^{-\int_0^s \alpha}\, \beta(s)\, ds.$$

Multiplying by $e^{\int_0^t \alpha}$ yields

$$y(t) \le y(0)\, e^{\int_0^t \alpha} + \int_0^t \beta(s)\, e^{\int_s^t \alpha}\, ds,$$

which is the claimed result. $\square$

2 ODE–comparison lemma for $\dot y \le A\, y + B\, y^2$

Lemma 2 (Comparison for quadratic growth ODE)**.** *Let $A, B \ge 0$. Suppose $y \in C^1([0,T])$ satisfies*

$$\frac{dy}{dt} \;\le\; A\, y + B\, y^2, \quad y(0) = y_0 \ge 0.$$

Then for all $t \in [0,T]$ with $1 - \frac{B}{A} y_0 \big(e^{At} - 1\big) > 0$,

$$y(t) \;\le\; \frac{y_0\, e^{At}}{1 - \frac{B}{A}\, y_0\, (e^{At} - 1)}.$$

Proof. Consider the Riccati equation

$$\frac{dz}{dt} = A\, z + B\, z^2, \quad z(0) = y_0,$$

whose explicit solution is

$$z(t) = \frac{y_0\, e^{At}}{1 - \frac{B}{A}\, y_0\, (e^{At} - 1)},$$

valid as long as the denominator stays positive. Since y satisfies $\dot y \le A\, y + B\, y^2$ and $y(0) = z(0)$, the standard comparison principle for ODEs implies $y(t) \le z(t)$ for all t up to the blow-up time of z. This yields the stated bound. $\square$

3 Continuity-in-time lemmas

Lemma 3 (Weak-continuity in L^2)**.** *Let $u \in L^\infty(0,T;L^2_\sigma(\mathbb{T}^3))$ satisfy*

$$\partial_t u \ \in \ L^{4/3}\big(0,T;(H^1_\sigma(\mathbb{T}^3))'\big).$$

Then $u \in C_w\big([0,T];L^2_\sigma(\mathbb{T}^3)\big)$, i.e. $t \mapsto \langle u(t),\varphi\rangle$ is continuous for every $\varphi \in L^2_\sigma(\mathbb{T}^3)$.

Proof. Fix $\varphi \in H^1_\sigma(\mathbb{T}^3) \subset L^2_\sigma(\mathbb{T}^3)$. For a.e. $s,t \in [0,T]$,

$$\langle u(t)-u(s),\varphi\rangle = \int_s^t \langle \partial_\tau u(\tau),\varphi\rangle \, d\tau.$$

Since $\partial_t u \in L^{4/3}(0,T;(H^1)')$ and $\varphi \in H^1$, Hölder gives

$$|\langle u(t)-u(s),\varphi\rangle| \le \|\partial_t u\|_{L^{4/3}(s,t;(H^1)')} \, \|\varphi\|_{L^4(s,t;H^1)} \le C\,|t-s|^{1/4}\, \|\partial_t u\|_{L^{4/3}(0,T;(H^1)')}\, \|\varphi\|_{H^1}.$$

Thus $t \mapsto \langle u(t),\varphi\rangle$ is Hölder-continuous of exponent $1/4$, hence continuous. Density of H^1_σ in L^2_σ and the uniform $L^\infty_t L^2_x$–bound then upgrade this to continuity against all $\varphi \in L^2_\sigma$, proving $u \in C_w([0,T];L^2_\sigma)$. □

Lemma 4 (Strong-continuity in L^2)**.** *Let*

$$u \in L^2\big(0,T;H^1_\sigma(\mathbb{T}^3)\big), \quad \partial_t u \in L^2\big(0,T;(H^1_\sigma(\mathbb{T}^3))'\big).$$

Then $u \in C\big([0,T];L^2_\sigma(\mathbb{T}^3)\big)$ and, moreover,

$$\|u(t)\|^2_{L^2} = \|u(0)\|^2_{L^2} + 2\int_0^t \langle \partial_s u(s), u(s)\rangle_{(H^1)',H^1}\, ds.$$

Proof. This is the classical Lions–Magenes lemma (see *e.g.* [**?**]), specialized to the Gelfand triplet

$$H^1_\sigma(\mathbb{T}^3) \ \hookrightarrow \ L^2_\sigma(\mathbb{T}^3) \ \hookrightarrow \ (H^1_\sigma(\mathbb{T}^3))'.$$

Precisely, since $u \in L^2(0,T;H^1_\sigma)$ and $\partial_t u \in L^2(0,T;(H^1)')$, one has $u \in C([0,T];L^2_\sigma)$ and the above identity holds by differentiating $\|u(t)\|^2_{L^2}$ in time in the weak sense. A direct proof uses a mollification in time and passage to the limit, which recovers both continuity and the energy-identity formula. □

4 Interpolation inequalities in time–space norms

Lemma 5 (Interpolation in mixed Lebesgue spaces)**.** *Let $0 \le \theta \le 1$, and let $1 \le p_0,p_1,q_0,q_1 \le \infty$. Define*

$$\frac{1}{p} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}, \qquad \frac{1}{q} = \frac{\theta}{q_0} + \frac{1-\theta}{q_1}.$$

If $f\colon [0,T]\times\mathbb{T}^3\to\mathbb{R}$ *satisfies*

$$f \in L^{p_0}\big(0,T;L^{q_0}(\mathbb{T}^3)\big) \quad \textit{and} \quad f \in L^{p_1}\big(0,T;L^{q_1}(\mathbb{T}^3)\big),$$

then $f \in L^p(0,T;L^q(\mathbb{T}^3))$ *and there is a constant* C*, depending only on the indices, such that*

$$\|f\|_{L^p_t L^q_x} \;\le\; \|f\|^{\theta}_{L^{p_0}_t L^{q_0}_x}\;\|f\|^{1-\theta}_{L^{p_1}_t L^{q_1}_x}.$$

In particular, for any $r \in [p_0,p_1]$ *and* $s \in [q_0,q_1]$ *with* $\frac{1}{r} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}$*,* $\frac{1}{s} = \frac{\theta}{q_0} + \frac{1-\theta}{q_1}$*, one has* $\|f\|_{L^r_t L^s_x} \le \|f\|^{\theta}_{L^{p_0}_t L^{q_0}_x}\|f\|^{1-\theta}_{L^{p_1}_t L^{q_1}_x}$*.*

Proof. Apply the Riesz–Thorin interpolation theorem first in x for each fixed t, to deduce

$$\|f(t,\cdot)\|_{L^q_x} \;\le\; \|f(t,\cdot)\|^{\theta}_{L^{q_0}_x}\,\|f(t,\cdot)\|^{1-\theta}_{L^{q_1}_x}.$$

Raising to the pth power and integrating in t gives

$$\|f\|^p_{L^p_t L^q_x} = \int_0^T \|f(t)\|^p_{L^q_x}\,dt \le \int_0^T \|f(t)\|^{\theta p}_{L^{q_0}_x}\,\|f(t)\|^{(1-\theta)p}_{L^{q_1}_x}\,dt.$$

Now apply Hölder's inequality in time with exponents $\frac{p_0}{\theta p}$ and $\frac{p_1}{(1-\theta)p}$, which satisfy $\frac{\theta p}{p_0} + \frac{(1-\theta)p}{p_1} = 1$. One obtains

$$\|f\|^p_{L^p_t L^q_x} \;\le\; \Big(\int_0^T \|f(t)\|^{p_0}_{L^{q_0}_x}\,dt\Big)^{\theta p/p_0}\;\Big(\int_0^T \|f(t)\|^{p_1}_{L^{q_1}_x}\,dt\Big)^{(1-\theta)p/p_1}.$$

Taking pth roots yields the stated interpolation inequality. $\square$