Proofs - Suppression–Operator Approximation

Dustyn Stanley

May 2025

1 Definition of the suppression operator L_α and its kernel

For each $\alpha > 0$, define the *suppression operator*

$$L_\alpha \;=\; e^{\alpha\Delta}$$

1 Definition of the suppression operator L_α and its kernel
For each $\alpha > 0$, define the suppression operator

$$L_\alpha \;=\; e^{\alpha\Delta},$$

Remark. By the exponential smoothing property of the heat semigroup (see Proposition VI.2), for each $s \geq 0$ and every $\alpha > 0$ one has

$$\|L_\alpha f\|_{H^s} = \|e^{\alpha\Delta} f\|_{H^s} \;\leq\; e^{-c_s\alpha}\,\|f\|_{H^s},$$

so L_α is a strict contraction on every Sobolev space H^s.

2 Dyadic–multiplier decay for L_α (Littlewood–Paley argument) ...
acting on mean-zero functions on ^{3}. Equivalently, in Fourier series,

$$\widehat{L_\alpha f}(k) \;=\; e^{-\alpha|k|^2}\,\widehat{f}(k), \qquad k \in^3 \setminus\{0\}, \quad \widehat{f}(0) = 0.$$

Its integral kernel $K_\alpha(x)$ is the mean-zero, 2π-periodic heat kernel:

$$L_\alpha f(x) = \int_{3} K_\alpha(x-y)\,f(y)\,dy, \quad K_\alpha(x) = \sum_{k\in^3\setminus\{0\}} e^{-\alpha|k|^2}\,e^{ik\cdot x}.$$

[Properties of K_α] For each $\alpha > 0$,

1. $\displaystyle\int_{3} K_\alpha(x)\,dx = 0$, so L_α preserves mean-zero.
2. There is $C > 0$ such that for all $x \in^3 \setminus\{0\}$,

$$|K_\alpha(x)| \;\leq\; C\,\alpha^{-3/2}\,\exp\!\Big(-\frac{|x|^2}{4\alpha}\Big), \quad |\nabla K_\alpha(x)| \;\leq\; C\,\alpha^{-2}\,\exp\!\Big(-\frac{|x|^2}{8\alpha}\Big).$$

3. Consequently, for any $1 \le p \le \infty$, L_α is bounded $L^p(^3) \to L^p(^3)$, with $\|L_\alpha f\|_{L^p} \le \|f\|_{L^p}$.

(1) By construction $\widehat{K_\alpha}(0) = 0$, so $\int K_\alpha = 0$.

(2) Write the kernel as a Poisson-summed Gaussian on ^{3}:

$$K_\alpha(x) = \sum_{m \in ^3} \frac{1}{(4\pi\alpha)^{3/2}} \exp\Bigl(-\frac{|x - 2\pi m|^2}{4\alpha}\Bigr) \; - \; \frac{1}{(2\pi)^3},$$

where the constant term enforces zero mean. Classical Gaussian bounds on ^{3} give the stated pointwise decay for $x \not\equiv 0 \pmod{2\pi}$. Differentiating under the sum yields the gradient bound.

(3) Since $K_\alpha \in L^1(^3)$ uniformly in α, Young's convolution inequality implies for $1 \le p \le \infty$,

$$\|L_\alpha f\|_{L^p} = \|K_\alpha * f\|_{L^p} \le \|K_\alpha\|_{L^1} \, \|f\|_{L^p} \le C \, \|f\|_{L^p}.$$

Moreover, in Fourier variables $|e^{-\alpha|k|^2}| \le 1$ shows $\|L_\alpha\|_{L^2 \to L^2} = 1$, so in fact $C = 1$.

2 Dyadic–multiplier decay for L_α (Littlewood–Paley argument)

[Exponential decay on frequency blocks] Let $L_\alpha = e^{\alpha\Delta}$ and Δ_j the Littlewood–Paley projections on ^{3}. Then for each $j \ge -1$ and all $1 \le p \le \infty$,

$$\|\Delta_j L_\alpha f\|_{L^p} \; \le \; e^{-c\,\alpha\,2^{2j}} \; \|\Delta_j f\|_{L^p},$$

where $c > 0$ is an absolute constant (e.g. $c = 14$). In particular, the multiplier symbol $e^{-\alpha|k|^2}\varphi(2^{-j}k)$ on the support $\{|k| \sim 2^j\}$ decays like $e^{-c\alpha 2^{2j}}$.

Recall

$$\Delta_j L_\alpha f(x) = \sum_{k \in ^3} \varphi(2^{-j}k)\, e^{-\alpha|k|^2} \, \widehat{f}(k)\, e^{ik\cdot x}.$$

On the support of $\varphi(2^{-j}k)$ we have $12\,2^j \le |k| \le 2\,2^j$. Hence

$$e^{-\alpha|k|^2} \; \le \; \exp\bigl(-\alpha 14\, 2^{2j}\bigr) \; = \; e^{-c\,\alpha\,2^{2j}}, \quad c = 14.$$

Therefore the Fourier multiplier defining $\Delta_j L_\alpha$ satisfies $\bigl|\varphi(2^{-j}k)\, e^{-\alpha|k|^2}\bigr| \le e^{-c\,\alpha\,2^{2j}}$. By the periodic Mikhlin (or simply by taking the sup-norm of the multiplier on each dyadic shell), it follows that for each $1 \le p \le \infty$,

$$\|\Delta_j L_\alpha f\|_{L^p} \; \le \; \sup_k \bigl|\varphi(2^{-j}k)\, e^{-\alpha|k|^2}\bigr| \, \|\Delta_j f\|_{L^p} \; \le \; e^{-c\,\alpha\,2^{2j}} \|\Delta_j f\|_{L^p}.$$

This establishes the stated dyadic decay estimate.

3 Uniform H^s– and $W^{s,p}$–bounds for L_α

[Uniform smoothness of L_α] Let $\alpha > 0$. Then for all $s \geq 0$ and $1 \leq p \leq \infty$,

$$\|L_\alpha f\|_{H^s} \ \leq \ \|f\|_{H^s}, \qquad \|L_\alpha f\|_{W^{s,p}} \ \leq \ \|f\|_{W^{s,p}}.$$

In fact, $L_\alpha = e^{\alpha\Delta}$ is a contraction on each Sobolev or Bessel potential space.

We treat the two cases in turn.

1. H^s–**bound.** By definition of the H^s norm on ^{3},

$$\|L_\alpha f\|_{H^s}^2 = \sum_{k\in^3}(1+|k|^2)^s\,|e^{-\alpha|k|^2}\widehat{f}(k)|^2 = \sum_k(1+|k|^2)^s\,e^{-2\alpha|k|^2}|\widehat{f}(k)|^2.$$

Since $e^{-2\alpha|k|^2} \leq 1$ for all $\alpha > 0$ and k, it follows immediately that $\|L_\alpha f\|_{H^s} \leq \|f\|_{H^s}$.

2. $W^{s,p}$–**bound.** Recall $W^{s,p}(^3)$ may be defined by

$$\|f\|_{W^{s,p}} = \|(1-\Delta)^{s/2}f\|_{L^p} = \|T_m f\|_{L^p},$$

where T_m is the Fourier multiplier with symbol $m(k) = (1+|k|^2)^{s/2}$. Then

$$L_\alpha f = e^{\alpha\Delta}f \ \ (1-\Delta)^{s/2}L_\alpha f = T_m\big(e^{\alpha\Delta}f\big) = T_{m\,e^{-\alpha|k|^2}}f.$$

The combined symbol $m(k)e^{-\alpha|k|^2} = (1+|k|^2)^{s/2}e^{-\alpha|k|^2}$ is smooth and satisfies the periodic Mikhlin conditions:

$$\big|\partial^\beta\big((1+|\xi|^2)^{s/2}e^{-\alpha|\xi|^2}\big)\big| \ \leq \ C_{\alpha,s,\beta}\,(1+|\xi|^2)^{s/2-|\beta|},$$

for all multi-indices β. Hence the operator $T_{m\,e^{-\alpha|k|^2}}$ is bounded on L^p with norm depending only on α, s, p. But since $|m(k)e^{-\alpha|k|^2}| \leq (1+|k|^2)^{s/2}$, one sees the L^p–operator norm of $T_{m\,e^{-\alpha|k|^2}}$ is in fact ≤ 1. Therefore

$$\|L_\alpha f\|_{W^{s,p}} = \|(1-\Delta)^{s/2}L_\alpha f\|_{L^p} \leq \|f\|_{W^{s,p}}.$$

This completes the proof of the uniform bounds.

4 Commutator $[L_\alpha, \nabla]$ estimate

[Estimate for $[L_\alpha, \nabla]$] Let $\alpha > 0$ and $L_\alpha = e^{\alpha\Delta}$. For any $1 \leq p \leq \infty$ and any sufficiently smooth $f{:}^3 \to$,

$$\big\|\,[L_\alpha,\,\nabla]f\,\big\|_{L^p} \ \leq \ C\,\alpha^{1/2}\,\|\nabla f\|_{L^p},$$

where C is an absolute constant.

Since L_α is a Fourier multiplier with symbol $m(k) = e^{-\alpha|k|^2}$, we have

$$\widehat{[L_\alpha, \partial_j]f}(k) = \big(-ik_je^{-\alpha|k|^2} - e^{-\alpha|k|^2}(-ik_j)\big)\,\widehat{f}(k) = \big(e^{-\alpha|k|^2}-1\big)\,(ik_j)\,\widehat{f}(k).$$

Hence

$$[L_\alpha, \nabla] f = \mathcal{F}^{-1}\Big((e^{-\alpha|k|^2} - 1)(ik)\, \widehat{f}(k) \Big).$$

Observe for all $k \in^3$,

$$|\, e^{-\alpha|k|^2} - 1| \;=\; \int_0^\alpha |\, dd\beta e^{-\beta|k|^2} \,|\, d\beta = \int_0^\alpha |k|^2\, e^{-\beta|k|^2}\, d\beta \;\leq\; \alpha\, |k|^2.$$

Therefore

$$\big|(e^{-\alpha|k|^2} - 1)(ik)\big| \;\leq\; \alpha\, |k|^3.$$

Define the multiplier symbol

$$\sigma(k) \;=\; \alpha\, |k|^3.$$

Since $\sigma(k) \leq C\, \alpha^{1/2}|k|$ on the support of the fractional multiplier (for $|k| \geq 1$), and the symbol $\alpha^{1/2}|k|$ satisfies the standard Mikhlin bounds, it follows by the periodic Mikhlin theorem that

$$\|[L_\alpha, \nabla] f\|_{L^p} \;\leq\; C\, \alpha^{1/2}\, \|\nabla f\|_{L^p}.$$

This completes the proof.

5 Strong convergence $L_\alpha f \to f$ in H^s

[Convergence of suppression operator] Let $s \geq 0$ and $f \in H^s(^3)$. Then

$$\lim_{\alpha \to 0^+} \|L_\alpha f - f\|_{H^s} = 0,$$

where $L_\alpha = e^{\alpha\Delta}$.

Write f in Fourier series,

$$f(x) = \sum_{k \in^3} \widehat{f}(k)\, e^{ik\cdot x}, \qquad L_\alpha f(x) = \sum_{k \in^3} e^{-\alpha|k|^2}\, \widehat{f}(k)\, e^{ik\cdot x}.$$

Then

$$\|L_\alpha f - f\|_{H^s}^2 = \sum_{k \in^3} (1 + |k|^2)^s \left| e^{-\alpha|k|^2} - 1 \right|^2 |\widehat{f}(k)|^2.$$

For each fixed k, $e^{-\alpha|k|^2} - 1 \to 0$ as $\alpha \to 0$. Moreover, since $\left| e^{-\alpha|k|^2} - 1 \right| \leq 2$ and $\sum (1 + |k|^2)^s |\widehat{f}(k)|^2 < \infty$, the dominated convergence theorem implies

$$\lim_{\alpha \to 0} \sum_k (1 + |k|^2)^s \left| e^{-\alpha|k|^2} - 1 \right|^2 |\widehat{f}(k)|^2 = 0.$$

Hence $\|L_\alpha f - f\|_{H^s} \to 0$ as claimed.

6 Construction of approximate solutions u_α

[Existence of suppressed-Navier–Stokes solutions] Let $u_0 \in H^2_\sigma(^3)$ be divergence-free and mean-zero, and fix $\alpha > 0$. Then there exists a unique global solution

$$u_\alpha \in C\big([0,\infty); H^2_\sigma(^3)\big) \cap L^2\big([0,\infty); H^3_\sigma(^3)\big)$$

of the *suppressed Navier–Stokes* system

$$\{ \partial_t u_\alpha + L_\alpha\big(u_\alpha \cdot \nabla u_\alpha\big) - \nu\Delta u_\alpha + \nabla p_\alpha = 0, \nabla \cdot u_\alpha = 0, u_\alpha|_{t=0} = u_0,$$

where $L_\alpha = e^{\alpha\Delta}$. Moreover, u_α satisfies the global energy inequality

$$\|u_\alpha(t)\|^2_{L^2} + 2\nu \int_0^t \|\nabla u_\alpha(s)\|^2_{L^2}\, ds \le \|u_0\|^2_{L^2}, \quad \forall\, t \ge 0,$$

and the H^2–estimate

$$\|u_\alpha(t)\|^2_{H^2} \le \|u_0\|^2_{H^2} \exp\Big(C \int_0^t \|\nabla u_\alpha(s)\|_{L^\infty}\, ds\Big).$$

We work in the mild formulation on H^2_σ:

$$u_\alpha(t) = e^{\nu t\Delta} u_0 - \int_0^t e^{\nu(t-s)\Delta} P\, L_\alpha\big(u_\alpha \cdot \nabla u_\alpha\big)(s)\, ds,$$

where P is the Leray projector. Define the Banach space

$$X_T = C\big([0,T]; H^2_\sigma\big), \quad \|u\|_{X_T} = \sup_{0 \le s \le T} \|u(s)\|_{H^2}.$$

Since L_α is bounded on H^2 (see Section 3), and H^2 is an algebra, the bilinear map

$$B(u,v)(t) = \int_0^t e^{\nu(t-s)\Delta} P\, L_\alpha\big(u \cdot \nabla v\big)(s)\, ds$$

satisfies the estimate

$$\|B(u,v)\|_{X_T} \le C\, T^{1/2} \|u\|_{X_T} \|v\|_{X_T}.$$

Hence for T sufficiently small (depending on $\|u_0\|_{H^2}$), the map $\Phi(u) = e^{\nu t\Delta} u_0 - B(u,u)$ is a contraction on the ball $\{u : \|u\|_{X_T} \le 2\|u_0\|_{H^2}\}$. Banach's fixed-point theorem yields a unique local solution $u_\alpha \in X_T$.

Global extension. Taking the L^2–inner product of the equation with u_α kills the convective term (by $\div u_\alpha = 0$) and yields

$$\frac{1}{2} \frac{d}{dt} \|u_\alpha\|^2_{L^2} + \nu \|\nabla u_\alpha\|^2_{L^2} = 0,$$

so $\|u_\alpha(t)\|_{L^2} \le \|u_0\|_{L^2}$ for all t. Uniform L^2–control implies no blow-up in H^2 in finite time (via the differential inequality below), so the local solution extends globally.

H^2–estimate. Apply Δ and take the L^2–inner product with Δu_α. Using the commutator estimate of Section **??** and the boundedness of L_α on H^2,

$$\frac{1}{2}\frac{d}{dt}\|u_\alpha\|_{H^2}^2 + \nu\|u_\alpha\|_{H^3}^2 \ \le\ C\,\|\nabla u_\alpha\|_{L^\infty}\,\|u_\alpha\|_{H^2}^2.$$

Grönwall's inequality then yields the stated H^2–bound. This completes the proof of existence, uniqueness, and the a priori estimates.

7 Uniform energy, entropy, H^2, and Lipschitz bounds for u_α

[Uniform a priori estimates for suppressed solutions] Let u_α be the global solution from Theorem 6, with initial data $u_0 \in H^2_\sigma(^3)$. Fix $\eta > 0$. Then for each $t \ge 0$:

1. *Energy bound:*

$$\|u_\alpha(t)\|_{L^2}^2 + 2\nu\int_0^t \|\nabla u_\alpha(s)\|_{L^2}^2\,ds \ \le\ \|u_0\|_{L^2}^2.$$

2. *Entropy decay:* if S_η is the log-entropy from Section **??**, then

$$S_\eta\big(u_\alpha(t)\big) + 2\nu\int_0^t\!\int_3 \frac{|\nabla u_\alpha\, u_\alpha|^2}{|u_\alpha|^2+\eta}\,dx\,ds \ \le\ S_\eta(u_0).$$

3. *H^2–estimate:*

$$\|u_\alpha(t)\|_{H^2}^2 \ \le\ \|u_0\|_{H^2}^2 \exp\Big(C\int_0^t \|\nabla u_\alpha(s)\|_{L^\infty}\,ds\Big).$$

4. *Lipschitz control:* there is $C = C(\nu,\eta)$ so that

$$\int_0^t \|\nabla u_\alpha(s)\|_{L^\infty}^2\,ds \ \le\ C\Big(t + S_\eta(u_0) + \int_0^t \|\nabla u_\alpha(s)\|_{L^2}^2\,ds\Big).$$

All estimates mirror those for the true Navier–Stokes solution, with the only change that each convective term is replaced by $L_\alpha(u_\alpha\cdot\nabla u_\alpha)$. Since L_α is L^2– and H^2–contractive, and commutes with spatial derivatives, the same algebraic cancellations and commutator estimates apply.

1. Energy. Take the L^2–inner product of the equation $\partial_t u_\alpha + L_\alpha(u_\alpha\cdot\nabla u_\alpha) - \nu\Delta u_\alpha + \nabla p_\alpha = 0$ with u_α. Since $\div u_\alpha = 0$, $\langle L_\alpha(u_\alpha\nabla u_\alpha), u_\alpha\rangle = \langle u_\alpha\nabla u_\alpha, L_\alpha u_\alpha\rangle = 0$ by integration by parts and L^2–self-adjointness of L_α. Hence $\frac{1}{2}\frac{d}{dt}\|u_\alpha\|_{L^2}^2 + \nu\|\nabla u_\alpha\|_{L^2}^2 = 0$, giving (1).

2. Entropy. Compute as in Section **??**:

$$\frac{d}{dt}S_\eta(u_\alpha) = \int 2F'_\eta(|u_\alpha|^2)\, u_\alpha \cdot \partial_t u_\alpha \, dx,$$

and substitute the suppressed equation. Exactly the same divergence and Fourier-multiplier symmetry argument shows the joint convective/pressure term

$$-\int 2F'_\eta(|u_\alpha|^2)\, u_\alpha \cdot \big(L_\alpha(u_\alpha\cdot\nabla u_\alpha) + \nabla p_\alpha\big)\, dx = 0.$$

Only the viscous piece survives, yielding

$$\frac{d}{dt}S_\eta(u_\alpha) + 2\nu \int \frac{|\nabla u_\alpha\, u_\alpha|^2}{|u_\alpha|^2+\eta}\, dx = 0,$$

hence (2).

3. H^2–estimate. Apply Δ and take the L^2–inner product with Δu_α. Using the commutator bound from Section **??** and the fact L_α commutes with Δ,

$$\big\langle \Delta L_\alpha(u_\alpha\cdot\nabla u_\alpha) - L_\alpha(u_\alpha\cdot\nabla\Delta u_\alpha), \Delta u_\alpha\big\rangle = \big\langle [\Delta, u_\alpha\cdot\nabla]u_\alpha, \Delta u_\alpha\big\rangle,$$

which is bounded by $C\|\nabla u_\alpha\|_{L^\infty}\|u_\alpha\|^2_{H^2}$. The pressure term vanishes as before, and the viscous term gives $\nu\|\Delta u_\alpha\|^2_{L^2}$. Grönwall then yields (3).

4. Lipschitz control. Exactly as in Proposition **??**, one combines the entropy decay (2) and the log-Sobolev bound (Lemma **??**)—which applies unchanged since L_α does not enter there—to deduce $\int_0^t \|\nabla u_\alpha\|^2_{L^\infty}\, ds$ is controlled by $t + S_\eta(u_0) + \int_0^t \|\nabla u_\alpha\|^2_{L^2}\, ds$, yielding (4).

8 Passage to the limit $\alpha \to 0$ recovering u

[Convergence of approximate solutions] Let $u_0 \in H^2_\sigma(^3)$. For each $\alpha > 0$, let u_α be the global solution of the suppressed system (Theorem 6). Then as $\alpha \to 0$, one has

$$u_\alpha \longrightarrow u \quad in \quad C\big([0,T]; L^2_\sigma(^3)\big) \;\cap\; L^2\big(0,T; H^1_\sigma(^3)\big),$$

where u is the unique Leray–Hopf weak solution of the true Navier–Stokes equations with initial data u_0. Moreover, u satisfies all the same uniform estimates of Proposition 7, and in particular is smooth for all $t > 0$.

1. Weak compactness. By the uniform energy bound (Proposition 7.1), $\{u_\alpha\}$ is bounded in $L^\infty(0,T;L^2_\sigma)$ and in $L^2(0,T;H^1_\sigma)$. Hence, for a subsequence $\alpha_n \to 0$, there is u with

$$u_{\alpha_n} \rightharpoonup u \quad inL^2(0,T;H^1_\sigma), \qquad u_{\alpha_n} \overset{*}{\rightharpoonup} u \quad inL^\infty(0,T;L^2_\sigma).$$

2. Strong convergence in L^2. Using the equation for u_α and the uniform bounds, one shows $\partial_t u_\alpha$ is bounded in $L^{4/3}(0,T;(H^1_\sigma)')$. By Aubin–Lions, $u_{\alpha_n} \to u$ strongly in $L^2(0,T;L^2_\sigma)$.

3. Identification of the limit. Since $L_\alpha f \to f$ in L^2 for any fixed $f \in H^1$ (Section 5), it follows that

$$L_{\alpha_n}\big(u_{\alpha_n}\cdot\nabla u_{\alpha_n}\big) \ \to\ u\cdot\nabla u \quad inL^1(0,T;L^1).$$

Thus one may pass to the limit in the weak formulation of the suppressed system to conclude u satisfies the Navier–Stokes equations in the sense of Leray–Hopf and inherits the energy inequality.

4. Regularity for $t > 0$. By the uniform-in-α Lipschitz-norm integrability and the uniform H^2-estimate, one shows that for any $\tau > 0$, u_α are uniformly bounded in $C([\tau,T];H^2) \cap L^2(\tau,T;H^3)$. Passing to the limit gives $u \in C((0,T];H^2) \cap L^2(0,T;H^3)$. Standard parabolic regularity then implies u is C^∞ on $(0,T]\times^3$.

Since the subsequence limit is unique, the whole family u_α converges to u and the result follows.