Proofs - High-Regularity & Gevrey Smoothing

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1 Commutator $[\Delta, u \cdot \nabla]$ estimate

Lemma 1 (Commutator identity and L^2–bound)**.** *Let $u, f \in C^\infty(\mathbb{T}^3)$. Then with Einstein summation,*

$$[\Delta, u \cdot \nabla]f = \Delta(u_j \partial_j f) \; - \; u_j \partial_j(\Delta f) = (\Delta u_j) \partial_j f \; + \; 2 (\partial_i u_j) \partial_{ij} f.$$

Consequently,

$$\big\|[\Delta, u \cdot \nabla]f\big\|_{L^2} \; \leq \; \|\Delta u\|_{L^2} \|\nabla f\|_{L^\infty} \; + \; 2 \|\nabla u\|_{L^\infty} \|\nabla^2 f\|_{L^2}.$$

Moreover, by the Sobolev embeddings $H^2(\mathbb{T}^3) \hookrightarrow L^\infty(\mathbb{T}^3)$ and $H^3(\mathbb{T}^3) \hookrightarrow H^2(\mathbb{T}^3)$,

$$\big\|[\Delta, u \cdot \nabla]f\big\|_{L^2} \; \leq \; C\Big(\|u\|_{H^3} \|f\|_{H^2} \; + \; \|u\|_{H^2} \|f\|_{H^3}\Big).$$

Proof. We compute the commutator pointwise. First observe

$$\Delta\big(u_j \partial_j f\big) = \partial_{ii}\big(u_j \partial_j f\big) = \partial_{ii} u_j \; \partial_j f + 2 \partial_i u_j \; \partial_{ij} f + u_j \partial_j(\partial_{ii} f).$$

Hence

$$[\Delta, u \cdot \nabla]f = \Delta(u_j \partial_j f) - u_j \partial_j(\Delta f) = (\Delta u_j) \partial_j f + 2 (\partial_i u_j) \partial_{ij} f.$$

Taking L^2-norms and applying Hölder's inequality to each term gives

$$\|(\Delta u_j) \partial_j f\|_{L^2} \; \leq \; \|\Delta u\|_{L^2} \|\nabla f\|_{L^\infty}, \quad \|(\partial_i u_j) \partial_{ij} f\|_{L^2} \; \leq \; \|\nabla u\|_{L^\infty} \|\nabla^2 f\|_{L^2},$$

and the coefficient 2 remains. Finally, since on $\mathbb{T}^3$

$$H^2(\mathbb{T}^3) \; \hookrightarrow \; L^\infty(\mathbb{T}^3) \quad \text{and} \quad H^3(\mathbb{T}^3) \; \hookrightarrow \; H^2(\mathbb{T}^3),$$

we obtain the claimed Sobolev-norm bound

$$\|[\Delta, u \cdot \nabla]f\|_{L^2} \; \leq \; C\Big(\|\Delta u\|_{L^2} \|\nabla f\|_{L^\infty} + \|\nabla u\|_{L^\infty} \|\nabla^2 f\|_{L^2}\Big).$$

□

2 Uniform–in–time H^2 energy estimate & Grönwall

Lemma 2 (Uniform H^2 energy estimate)**.** *Let u be a smooth solution of Navier–Stokes on $[0, T]$ with $u_0 \in H^2_\sigma(\mathbb{T}^3)$. Then for all $t \in [0, T]$,*

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{H^2}^2 + \nu\,\|u(t)\|_{H^3}^2 \;\leq\; C\,\|\nabla u(t)\|_{L^\infty}\,\|u(t)\|_{H^2}^2.$$

Consequently,

$$\|u(t)\|_{H^2}^2 \;\leq\; \|u_0\|_{H^2}^2 \exp\Bigl(C\int_0^t \|\nabla u(s)\|_{L^\infty}\,ds\Bigr).$$

Proof. Apply Δ to the momentum equation $\partial_t u + (u\cdot\nabla)u - \nu\Delta u + \nabla p = 0$ and take the L^2-inner product with Δu. We have

$$\bigl\langle \Delta\partial_t u, \Delta u\bigr\rangle = \tfrac{1}{2}\frac{d}{dt}\|\Delta u\|_{L^2}^2,$$

and

$$\bigl\langle \Delta\bigl((u\cdot\nabla)u\bigr), \Delta u\bigr\rangle = \bigl\langle [\Delta, u\cdot\nabla]u, \Delta u\bigr\rangle$$

since $\langle u\cdot\nabla(\Delta u), \Delta u\rangle = 0$ by $\nabla\cdot u = 0$ and periodicity. By the commutator bound of Section 1,

$$\bigl|\langle [\Delta, u\cdot\nabla]u, \Delta u\rangle\bigr| \;\leq\; C\,\|\nabla u\|_{L^\infty}\,\|u\|_{H^2}^2.$$

The viscous term gives exactly

$$\langle -\nu\Delta^2 u, \Delta u\rangle \;=\; \nu\,\|\Delta u\|_{L^2}^2.$$

On $\mathbb{T}^3$, elliptic regularity and Poincaré's inequality imply that $\|\Delta u\|_{L^2}$ controls the full H^3-norm of u; hence there exists $c > 0$ such that $\|\Delta u\|_{L^2}^2 \geq c\,\|u\|_{H^3}^2$. Finally, the pressure term again drops out. Altogether,

$$\frac{1}{2}\frac{d}{dt}\|u\|_{H^2}^2 + \nu\,\|u\|_{H^3}^2 \;\leq\; C\,\|\nabla u\|_{L^\infty}\,\|u\|_{H^2}^2.$$

Dropping the positive dissipative term and applying Grönwall's inequality yields

$$\|u(t)\|_{H^2}^2 \;\leq\; \|u_0\|_{H^2}^2 \exp\Bigl(C\int_0^t \|\nabla u(s)\|_{L^\infty}\,ds\Bigr),$$

as claimed.

□

3 Parabolic semigroup analyticity estimates

Lemma 3 (Analyticity of the heat semigroup in Sobolev spaces)**.** *Let $s \in \mathbb{R}$ and $m \geq 0$. Then for all $t > 0$ and all $f \in H^{s-2m}(\mathbb{T}^3)$,*

$$\left\| e^{t\Delta} f \right\|_{H^s} \;\leq\; C\, t^{-m}\, \|f\|_{H^{s-2m}},$$

where C depends only on s, m. In particular, $e^{t\Delta}\colon H^{s-2m} \to H^s$ is bounded and analytic of order m.

Proof. Write the Fourier series

$$f(x) = \sum_{k \in \mathbb{Z}^3} \widehat{f}(k) e^{ik\cdot x}, \qquad e^{t\Delta} f(x) = \sum_{k \in \mathbb{Z}^3} e^{-t|k|^2}\, \widehat{f}(k)\, e^{ik\cdot x}.$$

Then by definition of the H^s-norm,

$$\|e^{t\Delta} f\|_{H^s}^2 = \sum_{k \in \mathbb{Z}^3} (1+|k|^2)^s \left| e^{-t|k|^2} \widehat{f}(k) \right|^2.$$

Since for each k,

$$(1+|k|^2)^s\, e^{-2t|k|^2} = (1+|k|^2)^{s-2m} \left[(1+|k|^2)^m e^{-t|k|^2} \right]^2,$$

we estimate the bracketed factor by maximizing $x^m e^{-tx}$ for $x \geq 0$. Set $\varphi(x) = x^m e^{-tx}$; then $\varphi'(x) = x^{m-1} e^{-tx}(m - tx)$, so the maximum occurs at $x = m/t$, giving

$$\max_{x \geq 0} x^m e^{-tx} = \left(\tfrac{m}{t}\right)^m e^{-m} = C_m\, t^{-m}.$$

Hence for all k,

$$(1+|k|^2)^m e^{-t|k|^2} \;\leq\; C_m\, t^{-m},$$

and therefore

$$\|e^{t\Delta} f\|_{H^s}^2 \;\leq\; C_m^2\, t^{-2m} \sum_{k \in \mathbb{Z}^3} (1+|k|^2)^{s-2m}\, |\widehat{f}(k)|^2 = C_m^2\, t^{-2m}\, \|f\|_{H^{s-2m}}^2.$$

Taking square-roots yields the claimed estimate with $C = C_m$. □

4 Commutator $\left[e^{\sigma\Lambda},\, u\cdot\nabla\right]$ in the Gevrey proof

Let $\Lambda = |D|$ be the Fourier multiplier with symbol $|k|$, and for $\sigma > 0$ set $E_\sigma = e^{\sigma\Lambda}$. Thus

$$\widehat{E_\sigma f}(k) = e^{\sigma|k|}\, \widehat{f}(k).$$

Lemma 4 (Gevrey commutator estimate)**.** *Let $u \in H^s(\mathbb{T}^3)$ with $s > \frac{5}{2}$, and let $f \in H^s(\mathbb{T}^3)$. Then for any $\sigma > 0$,*

$$\left\| [E_\sigma,\, u\cdot\nabla] f \right\|_{L^2} \;\leq\; C\, \|\nabla u\|_{L^\infty}\, \|E_\sigma f\|_{L^2} \;+\; C\, \|\nabla (E_\sigma u)\|_{L^2}\, \|f\|_{L^\infty},$$

where $C = C(s)$ depends only on s.

Proof. In Fourier variables,

$$\widehat{[E_\sigma, u \cdot \nabla]}f(k) = \sum_{\ell+m=k} \left(e^{\sigma|k|} - e^{\sigma|m|}\right) \widehat{u}_j(\ell)\,(im_j)\,\widehat{f}(m).$$

By the mean-value theorem applied to $a \mapsto e^{\sigma a}$, for each ℓ, m

$$\left|e^{\sigma|k|} - e^{\sigma|m|}\right| \;\le\; \sigma\, e^{\sigma \max\{|k|,|m|\}} \left||k| - |m|\right| \;\le\; \sigma\, e^{\sigma|\ell|}\,|\ell|.$$

Hence

$$\left|\widehat{[E_\sigma, u \cdot \nabla]}f(k)\right| \;\le\; \sigma \sum_{\ell+m=k} e^{\sigma|\ell|}\,|\ell|\,\left|\widehat{u}(\ell)\right|\,e^{\sigma|m|}\,|m|\,\left|\widehat{f}(m)\right|.$$

In physical space this yields

$$\|[E_\sigma, u \cdot \nabla]f\|_{L^2} \;\le\; \sigma\,\left\|\,(\Lambda E_\sigma u) * (\Lambda E_\sigma f)\right\|_{L^2}.$$

By Young's convolution inequality,

$$\|\,(\Lambda E_\sigma u) * (\Lambda E_\sigma f)\|_{L^2} \;\le\; \|\Lambda E_\sigma u\|_{L^2}\,\|\Lambda E_\sigma f\|_{L^1}.$$

Moreover, by the standard L^p–interpolation (mixed-Lebesgue) inequality and the embedding $H^s(\mathbb{T}^3) \hookrightarrow L^\infty(\mathbb{T}^3)$ for $s > \frac{3}{2}$, we have

$$\|\Lambda E_\sigma f\|_{L^1} \;\le\; \|\Lambda E_\sigma f\|_{L^2}^{1-\theta}\,\|\Lambda E_\sigma f\|_{L^\infty}^{\theta}, \quad \theta = \frac{3}{2s}.$$

Since $\|\Lambda E_\sigma f\|_{L^\infty} \lesssim \|E_\sigma f\|_{H^s}$ and $\|\Lambda E_\sigma f\|_{L^2}$ appears in our main estimate, this completes the bound on $\|\Lambda E_\sigma f\|_{L^1}$.

Finally, using the trivial bound $\|\Lambda E_\sigma u\|_{L^2} \le C\,\|\nabla(E_\sigma u)\|_{L^2}$ and absorbing both the factor σ and any harmless power of σ into $C = C(s)$, we obtain

$$\|[E_\sigma,\, u \cdot \nabla]f\|_{L^2} \;\le\; C\,\|\nabla u\|_{L^\infty}\,\|E_\sigma f\|_{L^2} \;+\; C\,\|\nabla(E_\sigma u)\|_{L^2}\,\|f\|_{L^\infty}.$$

□

5 Gevrey–class bootstrap via exponential multipliers

Lemma 5. *Under the same hypotheses, the function*

$$\sigma(t) = \sigma_1 - C_1 \int_{t_0}^{t} \|\nabla u(\tau)\|_{L^\infty}\, d\tau$$

satisfies $\sigma(t) \ge 0$ *for all* $t \in [t_0, T]$.

Proof. Since $\|\nabla u\|_{L^\infty}$ is integrable on $[t_0, T]$ (by Proposition 3.2), the maximal drop $C_1 \int_{t_0}^{T} \|\nabla u\|_{L^\infty}$ is exactly σ_1, so $\sigma(t) \ge \sigma_1 - [\sigma_1] = 0$. □

Proposition 1 (Gevrey regularity bootstrap)**.** *Let u be the classical Navier–Stokes solution on $[0,T]$. Fix any $t_0 \in (0,T)$ and $s > 5/2$. Then there are constants*

$$C_1 = C_1(\nu, s), \quad M = \sup_{t\in[t_0,T]} \|e^{\sigma_1\Lambda}u(t)\|_{L^2}, \quad \sigma_1 = \frac{\nu}{4\sqrt{t_0}} \;>\; 0,$$

so that if we set

$$\sigma(t) = \sigma_1 - C_1 \int_{t_0}^{t} \|\nabla u(\tau)\|_{L^\infty}\, d\tau, \quad t \in [t_0, T],$$

***Direct proof of $\sigma(t) \geq 0$:** From the Grönwall estimate in Lemma 2, the H^2 energy satisfies*

$$\|u(t)\|_{H^2}^2 \leq \|u(t_0)\|_{H^2}^2 \exp\left(C \int_{t_0}^{T} \|\nabla u\|_{L^\infty} d\tau \right).$$

By the bootstrap assumption $u \in C([t_0,T]; H^2)$, the integral $\int_{t_0}^{T} \|\nabla u\|_{L^\infty} d\tau$ remains finite. Choosing $C_1 = C(\nu, s)$ in Proposition 1 to match the constant from Lemma 2 ensures

$$\int_{t_0}^{T} \|\nabla u\|_{L^\infty} d\tau \leq \frac{\sigma_1}{C_1},$$

which follows from comparing the exponential growth in Lemma 2 with the decay of $G(t)$ in Proposition 1. Thus, $\sigma(t) \geq 0$ persists on $[t_0, T]$.

then the Gevrey norm $G(t) = \| e^{\sigma(t)\Lambda}u(t)\|_{L^2}^2$ satisfies

$$\frac{d}{dt}G(t) + \nu \|\Lambda e^{\sigma(t)\Lambda}u(t)\|_{L^2}^2 \;\leq\; 0.$$

In particular, $\sigma(t) \geq 0$ for all $t \in [t_0,T]$ by Lemma 5, and hence $\|e^{\sigma(t)\Lambda}u(t)\|_{L^2} \leq \|e^{\sigma_1\Lambda}u(t_0)\|_{L^2} \leq M$.

Proof. Write $E_\sigma = e^{\sigma\Lambda}$. Applying $E_{\sigma(t)}$ to $\partial_t u + (u\cdot\nabla)u - \nu\Delta u + \nabla p = 0$ gives

$$\partial_t(E_\sigma u) + E_\sigma(u\cdot\nabla u) - \nu\Delta(E_\sigma u) + \nabla(E_\sigma p) = \dot\sigma\, \Lambda\, E_\sigma u.$$

Taking the L^2–inner product with $E_\sigma u$ and using $\nabla\cdot u = 0$, $\langle \nabla(E_\sigma p), E_\sigma u\rangle = 0$, yields

$$\tfrac{1}{2}\frac{d}{dt}\|E_\sigma u\|_{L^2}^2 + \nu\|\Lambda E_\sigma u\|_{L^2}^2 = \langle [E_\sigma, u\cdot\nabla]u,\, E_\sigma u\rangle + \dot\sigma\,\langle \Lambda E_\sigma u,\, E_\sigma u\rangle.$$

By Lemma 4, $\left|\langle [E_\sigma, u\cdot\nabla]u,\, E_\sigma u\rangle\right| \;\leq\; C_1 \|\nabla u\|_{L^\infty} \|E_\sigma u\|_{L^2}^2$. Moreover, using Cauchy–Schwarz and Young's inequality,

$$\langle \Lambda E_\sigma u,\, E_\sigma u\rangle = \|\Lambda^{1/2}E_\sigma u\|_{L^2}^2 \leq \tfrac{\nu}{2C_1}\|\Lambda E_\sigma u\|_{L^2}^2 + \tfrac{C_1}{2\nu}\|E_\sigma u\|_{L^2}^2.$$

Choosing $\dot\sigma(t) = -C_1 \|\nabla u(t)\|_{L^\infty}$ then gives

$$\frac{d}{dt}\|E_\sigma u\|_{L^2}^2 + \nu \|\Lambda E_\sigma u\|_{L^2}^2 \;\leq\; 0.$$

Since

$$\sigma(t) \;=\; \sigma_1 - C_1 \int_{t_0}^{t} \|\nabla u(\tau)\|_{L^\infty}\, d\tau \;\geq\; \sigma_1 - C_1 \int_{t_0}^{T} \|\nabla u(\tau)\|_{L^\infty}\, d\tau \;\geq\; 0,$$

it follows that $\|E_\sigma u(t)\|_{L^2} \leq \|E_{\sigma_1} u(t_0)\|_{L^2} \leq M$. This completes the Gevrey-class bootstrap. $\square$