Proofs – Vorticity & Blow-up Criterion

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1 Vorticity formulation & Beale–Kato–Majda criterion

Let

$$u \colon [0, T^*) \times \mathbb{T}^3 \longrightarrow \mathbb{R}^3$$

be a classical solution of the Navier–Stokes equations with viscosity $\nu > 0$, satisfying

$$u(\cdot, t) \in H^s_\sigma(\mathbb{T}^3), \qquad s > \tfrac{5}{2},$$

on its maximal interval of existence $[0, T^*)$. Denote the vorticity by

$$\omega = \nabla \times u.$$

Lemma 1.1 (Vorticity equation)**.** *The vorticity ω satisfies*

$$\partial_t \omega + (u \cdot \nabla)\, \omega = (\omega \cdot \nabla)\, u + \nu\, \Delta\omega,$$

together with $\nabla \cdot \omega = 0$ and initial data $\omega(\cdot, 0) = \nabla \times u_0$.

Proof. Start from the momentum equation

$$\partial_t u + (u \cdot \nabla) u - \nu \Delta u + \nabla p = 0.$$

Take the curl of both sides. Since $\nabla \times \nabla p = 0$ and $\nabla \times \Delta u = \Delta(\nabla \times u) = \Delta\omega$, we get

$$\partial_t(\nabla \times u) + \nabla \times \big((u \cdot \nabla) u\big) - \nu\, \Delta\omega = 0.$$

The vector identity

$$\nabla \times \big((u \cdot \nabla) u\big) = (u \cdot \nabla)\, \omega - (\omega \cdot \nabla)\, u$$

gives the stated vorticity equation. Finally, divergence and initial condition follow by commuting derivatives. □

Lemma 1.2 (Vorticity–gradient comparison)**.** *At each point (x, t),*

$$|\omega| = \big|\nabla \times u\big| = \big|\partial_i u_j - \partial_j u_i\big| \ \le\ 2\,\big|\nabla u\big|.$$

Hence $\|\omega(t)\|_{L^\infty} \le 2\, \|\nabla u(t)\|_{L^\infty}$.

Proof. By definition,

$$\omega_i = \epsilon_{ijk}\,\partial_j u_k,$$

so each component of ω is a difference of two partials of u. Therefore

$$|\omega|^2 = \sum_i (\epsilon_{ijk}\partial_j u_k)^2 \;\le\; 4\sum_{j,k}(\partial_j u_k)^2 = 4\,|\nabla u|^2,$$

which yields the claimed bound. $\square$

Theorem 1.1 (Beale–Kato–Majda blow-up criterion)**.** *Under the above hypotheses, if*

$$\int_0^{T^*} \|\omega(\cdot,t)\|_{L^\infty}\,dt < \infty,$$

then the solution u extends smoothly past T^*. Equivalently, if $T^* < \infty$, then

$$\int_0^{T^*} \|\omega(t)\|_{L^\infty}\,dt = \infty.$$

Proof. Fix $s > \frac{5}{2}$. Apply the H^s-energy method. Let D^s be the Fourier multiplier with symbol $|k|^s$. Taking the L^2 inner product of D^s applied to the momentum equation with $D^s u$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|_{H^s}^2 + \nu\,\|u\|_{H^{s+1}}^2 = -\int_{\mathbb{T}^3} D^s\big((u\cdot\nabla)u\big)\cdot D^s u\,dx.$$

Since $s > \frac{3}{2}$, the Sobolev embedding $H^s(\mathbb{T}^3) \hookrightarrow W^{1,\infty}(\mathbb{T}^3)$ holds and $H^s(\mathbb{T}^3)$ is an algebra. We write

$$D^s\big((u\cdot\nabla)u\big) = u\cdot\nabla(D^s u) + \big[D^s,\, u\cdot\nabla\big]u,$$

where the commutator satisfies (by standard Fourier-multiplier estimates)

$$\|[D^s,\, u\cdot\nabla]u\|_{L^2} \;\le\; C\,\|\nabla u\|_{L^\infty}\,\|u\|_{H^s}.$$

Thus

$$\Big|\langle D^s((u\cdot\nabla)u),\, D^s u\rangle\Big| \le \|[D^s,\, u\cdot\nabla]u\|_{L^2}\,\|u\|_{H^s} \le C\,\|\nabla u\|_{L^\infty}\,\|u\|_{H^s}^2.$$

Combining yields

$$\frac{d}{dt}\|u\|_{H^s}^2 + 2\nu\,\|u\|_{H^{s+1}}^2 \le C\,\|\nabla u\|_{L^\infty}\,\|u\|_{H^s}^2.$$

By Lemma 1.2, $\|\nabla u\|_{L^\infty} \le 2\|\omega\|_{L^\infty}$, so

$$\frac{d}{dt}\|u\|_{H^s}^2 \le C\,\|\omega\|_{L^\infty}\,\|u\|_{H^s}^2.$$

Gronwall's inequality on $[0,t]$ gives

$$\|u(t)\|_{H^s}^2 \le \exp\left(C\int_0^t \|\omega(\tau)\|_{L^\infty}\,d\tau\right)\|u_0\|_{H^s}^2.$$

If $\int_0^{T^*}\|\omega\|_{L^\infty} < \infty$, the right–hand side remains finite as $t \to T^*$, so $\|u(t)\|_{H^s}$ stays bounded up to T^*. By the local existence theory in H^s, u extends past T^*, contradicting maximality. Therefore blow–up forces

$$\int_0^{T^*}\|\omega(t)\|_{L^\infty}\,dt = \infty.$$

□

2 Biot–Savart law on $\mathbb{T}^3$

Lemma 2.1 (Biot–Savart representation and Lipschitz bound)**.** *Let $\omega \in C^0(\mathbb{T}^3;\mathbb{R}^3)$ be divergence-free with zero mean. Define*

$$u(x) \;=\; \int_{\mathbb{T}^3} K(x-y)\times\omega(y)\,dy,$$

where $K = \nabla G$ and

$$G(x) = -\sum_{m\in\mathbb{Z}^3}\frac{1}{4\pi\,|x-2\pi m|}$$

is the periodic Green's function of $-\Delta$. Then:

1. *u is divergence-free, mean-zero, and $\nabla\times u = \omega$.*
2. *There exists $C>0$ such that*

$$\|\nabla u\|_{L^\infty(\mathbb{T}^3)} \;\le\; C\,\|\omega\|_{L^\infty(\mathbb{T}^3)}.$$

Proof. **(1) Divergence-free and curl.** Since $-\Delta G = \delta - (2\pi)^{-3}$ on $\mathbb{T}^3$ and G has zero mean, differentiating under the integral shows

$$\nabla\cdot u = \int_{\mathbb{T}^3}\nabla_x\cdot\big(K(x-y)\times\omega(y)\big)\,dy = 0,$$

while the vector identity

$$\nabla_x\times\big(K(x-y)\times\omega(y)\big) = \omega(y)\,(-\Delta_x G)(x-y)$$

yields

$$\nabla\times u = \int_{\mathbb{T}^3}\omega(y)\,\delta(x-y)\,dy = \omega(x).$$

(2) Lipschitz bound. Writing components,

$$\partial_m u_i(x) = \int_{\mathbb{T}^3} \partial_m K_i^j(x-y)\,\omega_j(y)\,dy.$$

One checks that each scalar kernel $\partial_m K_i^j(z)$ satisfies the standard Calderón–Zygmund estimates on $\mathbb{T}^3$:

$$\left|\partial_m K_i^j(z)\right| \le C|z|^{-3}, \quad \left|\nabla_z \partial_m K_i^j(z)\right| \le C|z|^{-4},$$

and has zero average over each fundamental cell. By the periodic Calderón–Zygmund theory, the operator

$$\omega \;\mapsto\; \partial_m u_i$$

is bounded $L^p(\mathbb{T}^3) \to L^p(\mathbb{T}^3)$ for $1 < p < \infty$, and since $\omega \in C^0$, continuity plus the L^p–bounds imply

$$\|\partial_m u_i\|_{L^\infty} \le C\,\|\omega\|_{L^\infty}.$$

Taking supremum over i, m gives the desired Lipschitz estimate. □