Appendix Viscosity

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A Parameter tracking for the viscosity ν

Throughout *all* proofs we write the Navier–Stokes equations in dimensional form:

$$\partial_t u + (u\cdot\nabla)u + \nabla p = \nu\,\Delta u, \qquad \nabla\cdot u = 0, \tag{1}$$

with a fixed positive viscosity $\nu > 0$.

A. Conventions

A.1 Scaling macro.
All files load `\newcommand{\nucl}{\nu}`, so replacing "`\nucl`" in any line shows exactly where ν enters.

A.2 Parabolic rescaling.
To pass from the dimensional system (1) to the normalized $\nu = 1$ version used inside technical lemmas, apply

$$x' = x, \quad t' = \nu\,t, \quad u'(x',t') = u(x,t), \quad p'(x',t') = \nu^{-1}p(x,t).$$

Under this map the Laplacian transforms as

$$\Delta_x = \nu\,\Delta_{x'},$$

so that (1) becomes

$$\partial_{t'}u' + (u'\cdot\nabla')u' + \nabla' p' = \Delta_{x'}u'. \tag{2}$$

Moreover, spatial Sobolev norms are invariant:

$$\|u\|_{H^s_x} = \|u'\|_{H^s_{x'}}.$$

A.3 Constant tracking rule.
Unless stated otherwise, every constant $C(\cdots)$ in an estimate satisfies

$$C(\nu, \mathrm{data}) = \nu^{-k}\,C_0(\mathrm{data}), \tag{3}$$

for some integer $k \geq 0$. For example, in the proof of Lemma A, the factor $e^{-2\alpha(2\pi)^2}$ carries no dependence on ν, so one takes $k = 0$.

B. Global contraction of the heat semigroup

[Strict contraction] For every $s \geq 0$ and $\alpha > 0$, the heat semigroup satisfies

$$\|e^{\alpha\Delta} f\|_{H^s} \leq e^{-\alpha\,\lambda_s} \|f\|_{H^s}, \qquad \lambda_s = \tfrac{1}{4}(2\pi)^{2-2s}. \tag{4}$$

Consequently, the suppression operator $L_\alpha = e^{\alpha\Delta}$ is a strict contraction on H^s, independently of ν.

Proof. Working on $\mathbb{T}^3$, diagonalize in Fourier:

$$e^{\alpha\Delta} \hat{f}(k) = e^{-\alpha|k|^2} \hat{f}(k), \quad |k|^2 \geq (2\pi)^2 \quad (k \neq 0).$$

Hence

$$\|e^{\alpha\Delta} f\|_{H^s}^2 = \sum_{k\in\mathbb{Z}^3} \langle k\rangle^{2s} e^{-2\alpha|k|^2} |\hat{f}(k)|^2 \leq e^{-2\alpha(2\pi)^2} \sum_{k\in\mathbb{Z}^3} \langle k\rangle^{2s} |\hat{f}(k)|^2 = e^{-2\alpha\lambda_s} \|f\|_{H^s}^2.$$

Taking square roots of (4) completes the proof. □

C. Quick reference table

Symbol	Meaning / location
	Physical viscosity; macro defined in the preamble.
$e^{\alpha\Delta}$	Lemma A: suppression operator, contraction factor $e^{-\alpha\lambda_s}$.
$C(\nu) = \nu^{-k} C_0$	General constants: see (3).
$\int_0^T \|\omega\|_{L^\infty}\, dt$	Beale–Kato–Majda integral; appears in the blow-up criterion (see Appendix D of main text).

With these unified rules, one may set $\nu = 1$ inside technical proofs and restore full ν-dependence at the statement level via Convention A.2 (see (2)).