Proofs - Existence Basic Navier–Stokes Facts

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1 Galerkin construction of Leray–Hopf solutions

Let

$$L^2_\sigma(\mathbb{T}^3) = \overline{\{\,\varphi \in C^\infty(\mathbb{T}^3;\mathbb{R}^3)\colon \nabla\cdot\varphi = 0\}}^{\|\cdot\|_{L^2}}, \quad H^1_\sigma(\mathbb{T}^3) = L^2_\sigma \cap H^1(\mathbb{T}^3;\mathbb{R}^3).$$

Theorem 1 (Existence of Leray–Hopf weak solutions)**.** *Let $u_0 \in L^2_\sigma(\mathbb{T}^3)$. Then for each $T > 0$ there exists*

$$u \in C_w\big([0,T]; L^2_\sigma(\mathbb{T}^3)\big) \;\cap\; L^2\big(0,T; H^1_\sigma(\mathbb{T}^3)\big)$$

such that

$$\int_0^T\!\!\int_{\mathbb{T}^3} \big[u\cdot\partial_t\varphi + (u\otimes u) : \nabla\varphi + \nabla u : \nabla\varphi\big]\,dx\,dt + \int_{\mathbb{T}^3} u_0(x)\cdot\varphi(x,0)\,dx = 0$$

for all divergence-free $\varphi \in C^\infty_c([0,T)\times\mathbb{T}^3;\mathbb{R}^3)$, and the energy inequality

$$\|u(t)\|^2_{L^2} + 2\int_0^t \|\nabla u(s)\|^2_{L^2}\,ds \;\le\; \|u_0\|^2_{L^2} \quad \textit{for a.e. } t \in [0,T].$$

Proof. **Step 1: Choice of basis and Galerkin system.** Let $\{w_k\}_{k=1}^\infty$ be an orthonormal basis of $L^2_\sigma(\mathbb{T}^3)$ consisting of eigenfunctions of the Stokes operator $-P\Delta$, with eigenvalues $0 < \lambda_1 \le \lambda_2 \le \cdots$. For each n set

$$H_n = \operatorname{span}\{w_1,\dots,w_n\}, \quad u^n(t,x) = \sum_{i=1}^n g_i^n(t)\,w_i(x).$$

Projecting the Navier–Stokes equations onto H_n yields the finite-dimensional ODE system: for $i = 1,\dots,n$,

$$\frac{d}{dt}g_i^n + \sum_{j=1}^n\sum_{k=1}^n c_{ijk}\,g_j^n\,g_k^n + \nu\,\lambda_i\,g_i^n = 0, \qquad g_i^n(0) = \big\langle u_0,\, w_i\big\rangle_{L^2},$$

where $c_{ijk} = \big\langle (w_j\cdot\nabla)w_k,\, w_i\big\rangle$. By classical ODE theory, there is a unique $g^n \in C^1([0,T])$ solving this system.

Step 2: Uniform energy estimates. Multiply the ith equation by g_i^n and sum over i:

$$\frac{1}{2}\frac{d}{dt}\|u^n\|_{L^2}^2 + \nu\|\nabla u^n\|_{L^2}^2 + \sum_{i,j,k} c_{ijk}\, g_i^n g_j^n g_k^n = 0.$$

But skew-symmetry of c_{ijk} (coming from $\div w_j = 0$) gives $\sum_{i,j,k} c_{ijk} g_i^n g_j^n g_k^n = 0$. Hence

$$\frac{d}{dt}\|u^n\|_{L^2}^2 + 2\nu\|\nabla u^n\|_{L^2}^2 = 0.$$

Integrating in time yields the uniform bounds

$$\sup_{t\in[0,T]} \|u^n(t)\|_{L^2}^2 \;\le\; \|u_0\|_{L^2}^2, \qquad \int_0^T \|\nabla u^n(s)\|_{L^2}^2\, ds \;\le\; \tfrac{1}{2\nu}\|u_0\|_{L^2}^2.$$

Step 3: Compactness and limit passage. By the uniform bounds,

$$u^n \rightharpoonup u \quad \text{in } L^2(0,T;H^1_\sigma), \qquad u^n \rightharpoonup^* u \quad \text{in } L^\infty(0,T;L^2_\sigma).$$

Moreover, $\partial_t u^n = P_n\big(-P(u^n\!\cdot\!\nabla u^n)+\nu\Delta u^n\big)$ is uniformly bounded in $L^{4/3}(0,T;(H^1_\sigma)')$, so by the Aubin–Lions lemma $u^n \to u$ strongly in $L^2(0,T;L^2_\sigma)$. Hence the non-linear term converges: $\int_0^T\!\int (u^n\otimes u^n) : \nabla\varphi \to \int (u\otimes u) : \nabla\varphi$.

Passing to the limit in the projected weak formulation gives

$$\int_0^T\!\!\int \big[u\cdot\partial_t\varphi + (u\otimes u) : \nabla\varphi + \nu\nabla u : \nabla\varphi\big]\, dx\, dt + \int u_0\cdot\varphi(0)\, dx = 0$$

for all divergence-free φ. Lower-semicontinuity of the norm then yields the energy inequality for u.

Step 4: Time-continuity. Weak continuity in L^2 follows from the uniform energy bound and the weak form. Thus $u \in C_w([0,T];L^2_\sigma)$.

This completes the construction of a global Leray–Hopf weak solution. □

2 Pressure reconstruction & BMO regularity

Lemma 1 (Pressure in BMO for H^2 solutions)**.** *Let $u\colon [0,T]\times\mathbb{T}^3 \to \mathbb{R}^3$ be a divergence-free vector field with*

$$u(\cdot,t) \in H^2(\mathbb{T}^3), \qquad \int_{\mathbb{T}^3} u(x,t)\, dx = 0, \qquad \sup_{t\in[0,T]} \|u(\cdot,t)\|_{H^2} < \infty.$$

For each t, define the mean-zero pressure

$$-\Delta p(\cdot,t) = \partial_i\partial_j\big(u_i(\cdot,t)\, u_j(\cdot,t)\big), \qquad \int_{\mathbb{T}^3} p(x,t)\, dx = 0.$$

Lemma 2 (Uniform BMO_x control of the pressure)**.** *Let (u,p) be a smooth solution of Navier–Stokes on $\mathbb{T}^3 \times [0,T]$ with*

$$u \in L_t^\infty H_x^2, \quad \nabla \cdot u = 0, \quad \int_{\mathbb{T}^3} p\,dx = 0.$$

Then for almost every $t \in [0,T]$ the pressure solves $-\Delta p = \partial_i \partial_j (u_i\, u_j)$ and satisfies

$$p \;\in\; L_t^\infty BMO_x, \qquad \|p(t)\|_{BMO_x} \;\le\; C\,\|u(t)\|_{H_x^2}^2,$$

where C is an absolute Calderón–Zygmund constant.

Proof. Fix t. On the torus $p = R_{ij}[u_i\, u_j]$, where $R_{ij} = -\partial_i \partial_j \Delta^{-1}$ is mean-zero and bounded $L^\infty \to BMO$. By Sobolev embedding, $\|u_i\, u_j\|_{L^\infty} \le \|u\|_{L^\infty}^2 \le C\|u\|_{H^2}^2$. Hence $\|p(t)\|_{BMO} \le C_{CZ}\,\|u_i\, u_j\|_{L^\infty} \le C\,\|u(t)\|_{H^2}^2$. Uniformity in t follows from $u \in L_t^\infty H_x^2$. □

Then $p(\cdot,t) \in \mathrm{BMO}(\mathbb{T}^3)$ for all t, and there is $C > 0$ so that

$$\|p(\cdot,t)\|_{\mathrm{BMO}} \;\le\; C\,\big\|\,u(\cdot,t)\big\|_{H^2}^2, \qquad \forall\, t \in [0,T].$$

In particular, $p \in L^\infty\big([0,T];\mathrm{BMO}(\mathbb{T}^3)\big)$.

Proof. Fix t. Set $f(x) = u_i(x,t)\,u_j(x,t)$. Since $u(\cdot,t) \in H^2(\mathbb{T}^3)$, the Sobolev embedding $H^2 \hookrightarrow L^\infty$ on $\mathbb{T}^3$ gives

$$\|f\|_{L^\infty} \;\le\; \|u(\cdot,t)\|_{L^\infty}^2 \;\le\; C\,\|u(\cdot,t)\|_{H^2}^2.$$

On the periodic domain $\mathbb{T}^3$, the unique mean-zero solution of $-\Delta p = \partial_i \partial_j f$ is given by the Fourier series

$$p(x,t) = \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \frac{k_i k_j}{|k|^2}\,\widehat{f}(k)\,e^{ik\cdot x},$$

where $\widehat{f}(k) = \frac{1}{(2\pi)^3}\int_{\mathbb{T}^3} f(y)e^{-ik\cdot y}\,dy$. Define the multiplier

$$T_{ij}[f](x) = \sum_{k \neq 0} \frac{k_i k_j}{|k|^2}\,\widehat{f}(k)\,e^{ik\cdot x},$$

so that $p = T_{ij}[f]$. The symbol $m_{ij}(k) = k_i k_j/|k|^2$ is smooth on $\mathbb{R}^3 \setminus \{0\}$ and satisfies the Mikhlin conditions $\left|\partial^\alpha m_{ij}(\xi)\right| \lesssim |\xi|^{-|\alpha|}$ for $|\alpha| \le 4$. By the periodic Mikhlin (Calderón–Zygmund) theorem (Theorem **??**), T_{ij} extends to a bounded operator

$$T_{ij} : L^\infty(\mathbb{T}^3) \;\longrightarrow\; \mathrm{BMO}(\mathbb{T}^3),$$

with $\|T_{ij}[f]\|_{\mathrm{BMO}} \le C\,\|f\|_{L^\infty}$. Combining these estimates,

$$\|p(\cdot,t)\|_{\mathrm{BMO}} = \|T_{ij}[f]\|_{\mathrm{BMO}} \;\le\; C\,\|f\|_{L^\infty} \;\le\; C'\,\|u(\cdot,t)\|_{H^2}^2,$$

uniformly in t. This shows $p \in L_t^\infty \mathrm{BMO}_x$, as claimed. □

3 Local well-posedness in H^2

Theorem 2 (Local existence and uniqueness in H^2)**.** *Let $u_0 \in H^2_\sigma(\mathbb{T}^3)$. Then there exists $T > 0$, depending only on $\|u_0\|_{H^2}$, and a unique solution*

$$u \ \in \ C\big([0,T]; H^2_\sigma(\mathbb{T}^3)\big) \ \cap \ L^2\big(0,T; H^3_\sigma(\mathbb{T}^3)\big)$$

of the Navier–Stokes equations

$$\begin{cases} \partial_t u + (u\cdot\nabla)u - \nu\Delta u + \nabla p = 0, \\ \nabla\cdot u = 0, \\ u|_{t=0} = u_0, \end{cases}$$

satisfying $u(0) = u_0$. Moreover, the map $u_0 \mapsto u$ is Lipschitz continuous from bounded sets in H^2_σ into $C([0,T]; H^2_\sigma)$.

Proof. We work in the mild (Duhamel) formulation on the divergence-free subspace:

$$u(t) = e^{\nu t\Delta}u_0 - \int_0^t e^{\nu(t-s)\Delta} P \div \big(u\otimes u\big)(s)\, ds,$$

where P is the Leray projection onto L^2_σ. Set

$$X_T = C\big([0,T]; H^2_\sigma(\mathbb{T}^3)\big), \quad \|u\|_{X_T} = \sup_{0\le t\le T} \|u(t)\|_{H^2}.$$

Semigroup and bilinear estimates. The heat semigroup satisfies for all $t \ge 0$,

$$\|e^{\nu t\Delta} f\|_{H^2} \le \|f\|_{H^2}, \qquad \big\| e^{\nu(t-s)\Delta} P \div F\big\|_{H^2} \ \le \ C\,(t-s)^{-1/2}\,\|F\|_{H^2}.$$

Moreover, $H^2(\mathbb{T}^3)$ is an algebra and

$$\|u\otimes v\|_{H^2} \ \le \ C\,\|u\|_{H^2}\,\|v\|_{H^2}.$$

Fixed-point argument. Define the map $\Phi\colon X_T \to X_T$ by

$$\Phi(u)(t) = e^{\nu t\Delta}u_0 - \int_0^t e^{\nu(t-s)\Delta} P \div \big(u\otimes u\big)(s)\, ds.$$

Then for $u \in X_T$,

$$\big\|\Phi(u)(t)\big\|_{H^2} \ \le \ \|u_0\|_{H^2} + C\int_0^t (t-s)^{-1/2}\,\|u(s)\|^2_{H^2}\, ds.$$

Taking the supremum over $t \in [0,T]$ gives

$$\|\Phi(u)\|_{X_T} \ \le \ \|u_0\|_{H^2} + C\,T^{1/2}\,\|u\|^2_{X_T}.$$

Similarly, for $u, v \in X_T$,

$$\|\Phi(u) - \Phi(v)\|_{X_T} \ \le \ C\,T^{1/2}\,\big(\|u\|_{X_T} + \|v\|_{X_T}\big)\,\|u-v\|_{X_T}.$$

Choice of T and conclusion. Choose $R = 2\|u_0\|_{H^2}$ and then $T > 0$ so small that

$$C\,T^{1/2}\,R \;\le\; \tfrac{1}{4}, \quad C\,T^{1/2}\,R^2 \;\le\; \tfrac{1}{2}.$$

Then Φ maps the closed ball $\{u \in X_T : \|u\|_{X_T} \le R\}$ into itself and is a contraction of Lipschitz constant $< \frac{1}{2}$. Banach's fixed-point theorem yields a unique $u \in X_T$ solving the mild equation. Standard parabolic regularity upgrades the solution to $\partial_t u \in L^2(0,T;L^2)$ and $u \in L^2(0,T;H^3)$. Continuity of the data-to-solution map follows from the contraction argument.

Thus the Navier–Stokes equations admit a unique local solution in H^2, as claimed. $\square$