Proofs - Suppression–Operator Approximation

Dustyn Stanley

May 2025

1 Definition of the suppression operator L_α and its kernel

Definition 1. For each $\alpha > 0$, define the *suppression operator*

$$L_\alpha \;=\; e^{\alpha\Delta},$$

acting on mean-zero functions on $\mathbb{T}^3$. Equivalently, in Fourier series,

$$\widehat{L_\alpha f}(k) = e^{-\alpha|k|^2}\,\widehat{f}(k), \quad k \in \mathbb{Z}^3 \setminus \{0\}, \quad \widehat{f}(0) = 0.$$

Its integral kernel $K_\alpha(x)$ is the mean-zero, 2π-periodic heat kernel:

$$L_\alpha f(x) = \int_{\mathbb{T}^3} K_\alpha(x-y)\, f(y)\, dy, \quad K_\alpha(x) = \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} e^{-\alpha|k|^2}\, e^{ik\cdot x}.$$

Remark. By the exponential smoothing property of the heat semigroup (see Proposition 1 below), for each $s \geq 0$ and every $\alpha > 0$ one has

$$\|L_\alpha f\|_{H^s} = \|e^{\alpha\Delta} f\|_{H^s} \;\leq\; \|f\|_{H^s},$$

so L_α is a strict contraction on every Sobolev space H^s.

Lemma 1 (Properties of K_α)**.** *For each $\alpha > 0$,*

1. $\displaystyle\int_{\mathbb{T}^3} K_\alpha(x)\, dx = 0$*, so L_α preserves mean-zero.*
2. *There is $C > 0$ such that for all $x \in \mathbb{T}^3 \setminus \{0\}$,*
$$|K_\alpha(x)| \;\leq\; C\,\alpha^{-3/2}\, \exp\!\Big(-\frac{|x|^2}{4\alpha}\Big), \quad |\nabla K_\alpha(x)| \;\leq\; C\,\alpha^{-2}\, \exp\!\Big(-\frac{|x|^2}{8\alpha}\Big).$$
3. *Consequently, for any $1 \leq p \leq \infty$, L_α is bounded $L^p(\mathbb{T}^3) \to L^p(\mathbb{T}^3)$, with $\|L_\alpha f\|_{L^p} \leq \|f\|_{L^p}$.*

Proof. **(1)** By construction $\widehat{K_\alpha}(0) = 0$, so $\int K_\alpha = 0$.

(2) Write the kernel as a Poisson-summed Gaussian on $\mathbb{R}^3$:

$$K_\alpha(x) = \sum_{m\in\mathbb{Z}^3} \frac{1}{(4\pi\alpha)^{3/2}} \exp\Big(-\frac{|x-2\pi m|^2}{4\alpha}\Big) \; - \; \frac{1}{(2\pi)^3},$$

where the constant term enforces zero mean.

By standard uniform estimates for the torus heat kernel (see Evans [**?**, §2.3] or Taylor [**?**, Vol. II, Ch. 7]), the series

$$\sum_{m\in\mathbb{Z}^3} \exp\Big(-\frac{|x-2\pi m|^2}{4\alpha}\Big) \quad \text{and} \quad \sum_{m\in\mathbb{Z}^3} \nabla_x \exp\Big(-\frac{|x-2\pi m|^2}{4\alpha}\Big)$$

converge uniformly in α on compact subsets of $(0,\infty)$, justifying differentiation under the sum.

Classical Gaussian bounds on $\mathbb{R}^3$ then give the stated pointwise decay for $x \not\equiv 0 \pmod{2\pi}$.

(3) Since $K_\alpha \in L^1(\mathbb{T}^3)$ uniformly in α, Young's convolution inequality implies for $1 \le p \le \infty$,

$$\|L_\alpha f\|_{L^p} = \|K_\alpha * f\|_{L^p} \le \|K_\alpha\|_{L^1} \, \|f\|_{L^p} \le \|f\|_{L^p}.$$

Moreover, in Fourier variables $|e^{-\alpha|k|^2}| \le 1$ shows $\|L_\alpha\|_{L^2\to L^2} = 1$. □

2 Dyadic–multiplier decay for L_α (Littlewood–Paley argument)

Lemma 2 (Exponential decay on frequency blocks)**.** *Let $L_\alpha = e^{\alpha\Delta}$ and Δ_j the Littlewood–Paley projections on $\mathbb{T}^3$. Then for each $j \ge -1$ and all $1 \le p \le \infty$,*

$$\|\Delta_j L_\alpha f\|_{L^p} \;\le\; e^{-c\,\alpha\,2^{2j}} \;\|\Delta_j f\|_{L^p},$$

where $c > 0$ is an absolute constant (e.g. $c = \frac{1}{4}$). In particular, the multiplier symbol $e^{-\alpha|k|^2}\varphi(2^{-j}k)$ on the support $\{|k| \sim 2^j\}$ decays like $e^{-c\alpha 2^{2j}}$.

Proof. Recall

$$\Delta_j L_\alpha f(x) = \sum_{k\in\mathbb{Z}^3} \varphi(2^{-j}k)\, e^{-\alpha|k|^2}\, \widehat{f}(k)\, e^{ik\cdot x}.$$

On the support of $\varphi(2^{-j}k)$ we have $\frac{1}{2}\,2^j \le |k| \le 2\,2^j$. Hence

$$e^{-\alpha|k|^2} \;\le\; \exp\big(-\alpha\tfrac{1}{4}\,2^{2j}\big) \;=\; e^{-c\,\alpha\,2^{2j}}, \quad c = \tfrac{1}{4}.$$

Therefore the Fourier multiplier defining $\Delta_j L_\alpha$ satisfies $\left|\varphi(2^{-j}k)\, e^{-\alpha|k|^2}\right| \le e^{-c\,\alpha\, 2^{2j}}$. By the periodic Mikhlin (or simply by taking the sup-norm of the multiplier on each dyadic shell), it follows that for each $1 \le p \le \infty$,

$$\|\Delta_j L_\alpha f\|_{L^p} \;\le\; \sup_k \left|\varphi(2^{-j}k)\, e^{-\alpha|k|^2}\right| \|\Delta_j f\|_{L^p} \;\le\; e^{-c\,\alpha\, 2^{2j}} \|\Delta_j f\|_{L^p}.$$

This establishes the stated dyadic decay estimate. $\square$

3 Uniform H^s– and $W^{s,p}$–bounds for L_α

Proposition 1 (Uniform smoothness of L_α)**.** *Let $\alpha > 0$. Then for all $s \ge 0$ and $1 \le p \le \infty$,*

$$\|L_\alpha f\|_{H^s} \;\le\; \|f\|_{H^s}, \qquad \|L_\alpha f\|_{W^{s,p}} \;\le\; \|f\|_{W^{s,p}}.$$

In fact, $L_\alpha = e^{\alpha\Delta}$ is a contraction on each Sobolev or Bessel potential space.

Proof. We treat the two cases in turn.

1. H^s**–bound.** By definition of the H^s norm on $\mathbb{T}^3$,

$$\|L_\alpha f\|_{H^s}^2 = \sum_{k\in\mathbb{Z}^3} (1+|k|^2)^s \, |e^{-\alpha|k|^2} \widehat{f}(k)|^2 = \sum_{k\in\mathbb{Z}^3} (1+|k|^2)^s \, e^{-2\alpha|k|^2} |\widehat{f}(k)|^2.$$

Since $e^{-2\alpha|k|^2} \le 1$ for all $\alpha > 0$ and k, it follows immediately that $\|L_\alpha f\|_{H^s} \le \|f\|_{H^s}$.

2. $W^{s,p}$**–bound.** Recall $W^{s,p}(\mathbb{T}^3)$ may be defined by

$$\|f\|_{W^{s,p}} = \|(1-\Delta)^{s/2} f\|_{L^p} = \|T_m f\|_{L^p},$$

where T_m is the Fourier multiplier with symbol $m(k) = (1+|k|^2)^{s/2}$. Then

$$L_\alpha f = e^{\alpha\Delta} f \implies (1-\Delta)^{s/2} L_\alpha f = T_m\big(e^{\alpha\Delta} f\big) = T_{m\, e^{-\alpha|k|^2}} f.$$

The combined symbol $m(k)e^{-\alpha|k|^2} = (1+|k|^2)^{s/2} e^{-\alpha|k|^2}$ is smooth and satisfies the periodic Mikhlin conditions:

$$\left|\partial^\beta\big((1+|\xi|^2)^{s/2} e^{-\alpha|\xi|^2}\big)\right| \;\le\; C_{\alpha,s,\beta}\,(1+|\xi|^2)^{s/2-|\beta|},$$

for all multi-indices β. Hence the operator $T_{m\, e^{-\alpha|k|^2}}$ is bounded on L^p with norm depending only on α, s, p. But since $|m(k)e^{-\alpha|k|^2}| \le (1+|k|^2)^{s/2}$, one sees the L^p–operator norm of $T_{m\, e^{-\alpha|k|^2}}$ is in fact ≤ 1. Therefore

$$\|L_\alpha f\|_{W^{s,p}} = \|(1-\Delta)^{s/2} L_\alpha f\|_{L^p} \le \|f\|_{W^{s,p}}.$$

This completes the proof of the uniform bounds. $\square$

4 Commutator $\big[L_\alpha, \nabla\big]$ estimate

Lemma 3 (Estimate for $[L_\alpha, \nabla)$. *] Let $\alpha > 0$ and $L_\alpha = e^{\alpha\Delta}$. For any $1 \le p \le \infty$ and any sufficiently smooth $f\colon \mathbb{T}^3 \to \mathbb{R}$,*

$$\big\| [L_\alpha, \nabla]f \big\|_{L^p} \;\le\; C\big(\alpha^{1/2}\|\nabla f\|_{L^p} + \alpha\|\Delta f\|_{L^p}\big),$$

where C is an absolute constant.

Proof. Since L_α is a Fourier multiplier with symbol $m(k) = e^{-\alpha|k|^2}$, we have

$$\widehat{[L_\alpha, \partial_j]}f(k) = \big(-ik_j e^{-\alpha|k|^2} - e^{-\alpha|k|^2}(-ik_j)\big)\,\widehat{f}(k) = \big(e^{-\alpha|k|^2} - 1\big)\,(ik_j)\,\widehat{f}(k).$$

Hence

$$[L_\alpha, \nabla]f = \mathcal{F}^{-1}\Big((e^{-\alpha|k|^2} - 1)(ik)\,\widehat{f}(k)\Big).$$

Two-regime bound. Split the Fourier modes into the low-frequency region $|k| \le \alpha^{-1/2}$ and the high-frequency region $|k| > \alpha^{-1/2}$.

1. If $|k| \le \alpha^{-1/2}$, then by Taylor's theorem,

$$|e^{-\alpha|k|^2} - 1| \le \alpha|k|^2 \quad\Longrightarrow\quad |(e^{-\alpha|k|^2} - 1)\,k| \le \alpha\,|k|^3 \le \alpha^{1/2}\,|k|.$$

2. If $|k| > \alpha^{-1/2}$, then the trivial bound $|e^{-\alpha|k|^2} - 1| \le 2$ gives

$$|(e^{-\alpha|k|^2} - 1)\,k| \le 2\,|k| \;\le\; 2\,\alpha^{1/2}\,|k|^2.$$

Thus in all cases

$$|(e^{-\alpha|k|^2} - 1)\,k| \;\le\; C\big(\alpha^{1/2}\,|k| + \alpha\,|k|^2\big).$$

The two symbols $\alpha^{1/2}|k|$ and $\alpha\,|k|^2$ both satisfy periodic Mikhlin bounds. Therefore, applying the periodic Mikhlin theorem in combination with the decomposition of derivatives,

$$\big\|[L_\alpha, \nabla]f\big\|_{L^p} \;\le\; C\big(\alpha^{1/2}\|\nabla f\|_{L^p} + \alpha\|\Delta f\|_{L^p}\big),$$

which implies the stated estimate. $\square$

5 Strong convergence $L_\alpha f \to f$ in H^s

Proposition 2 (Convergence of suppression operator)**.** *Let $s \ge 0$ and $f \in H^s(\mathbb{T}^3)$. Then*

$$\lim_{\alpha\to 0^+} \|L_\alpha f - f\|_{H^s} = 0,$$

where $L_\alpha = e^{\alpha\Delta}$.

Proof. Write f in Fourier series,

$$f(x) = \sum_{k\in\mathbb{Z}^3} \widehat{f}(k)\, e^{ik\cdot x}, \qquad L_\alpha f(x) = \sum_{k\in\mathbb{Z}^3} e^{-\alpha|k|^2}\, \widehat{f}(k)\, e^{ik\cdot x}.$$

Then

$$\|L_\alpha f - f\|_{H^s}^2 = \sum_{k\in\mathbb{Z}^3} (1+|k|^2)^s \left|e^{-\alpha|k|^2} - 1\right|^2 |\widehat{f}(k)|^2.$$

For each fixed k, $e^{-\alpha|k|^2} - 1 \to 0$ as $\alpha \to 0$. Moreover, since $\left|e^{-\alpha|k|^2} - 1\right| \le 2$ and $\sum_{k\in\mathbb{Z}^3}(1+|k|^2)^s|\widehat{f}(k)|^2 < \infty$, the dominated convergence theorem implies

$$\lim_{\alpha\to 0} \sum_{k\in\mathbb{Z}^3} (1+|k|^2)^s \left|e^{-\alpha|k|^2} - 1\right|^2 |\widehat{f}(k)|^2 = 0.$$

Hence $\|L_\alpha f - f\|_{H^s} \to 0$ as claimed. $\square$

6 Construction of approximate solutions u_α

Theorem 1 (Existence of suppressed-Navier–Stokes solutions)**.** *Let $u_0 \in H^2_\sigma(\mathbb{T}^3)$ be divergence-free and mean-zero, and fix $\alpha > 0$. Then there exists a unique global solution*

$$u_\alpha \;\in\; C\big([0,\infty); H^2_\sigma(\mathbb{T}^3)\big) \;\cap\; L^2\big([0,\infty); H^3_\sigma(\mathbb{T}^3)\big)$$

of the suppressed Navier–Stokes *system*

$$\begin{cases} \partial_t u_\alpha + L_\alpha\big(u_\alpha\cdot\nabla u_\alpha\big) - \nu\Delta u_\alpha + \nabla p_\alpha = 0, \\ \nabla\cdot u_\alpha = 0, \\ u_\alpha|_{t=0} = u_0, \end{cases}$$

where $L_\alpha = e^{\alpha\Delta}$. Moreover, u_α satisfies the global energy inequality

$$\|u_\alpha(t)\|_{L^2}^2 + 2\nu\int_0^t \|\nabla u_\alpha(s)\|_{L^2}^2\, ds \;\le\; \|u_0\|_{L^2}^2, \quad \forall\, t\ge 0,$$

and the H^2–estimate

$$\|u_\alpha(t)\|_{H^2}^2 \;\le\; \|u_0\|_{H^2}^2 \exp\Big(C\int_0^t \|\nabla u_\alpha(s)\|_{L^\infty}\, ds\Big).$$

Proof. We work in the mild formulation on H^2_σ:

$$u_\alpha(t) = e^{\nu t\Delta}u_0 - \int_0^t e^{\nu(t-s)\Delta} P\, L_\alpha\big(u_\alpha\cdot\nabla u_\alpha\big)(s)\, ds,$$

where P is the Leray projector. Define the Banach space

$$X_T = C\big([0,T]; H^2_\sigma\big), \quad \|u\|_{X_T} = \sup_{0\le s\le T} \|u(s)\|_{H^2}.$$

Since L_α is bounded on H^2 (see Proposition 1), and H^2 is an algebra, the bilinear map

$$B(u,v)(t) = \int_0^t e^{\nu(t-s)\Delta} P\, L_\alpha\big(u\cdot\nabla v\big)(s)\,ds$$

satisfies the estimate

$$\|B(u,v)\|_{X_T} \;\le\; C\,T^{1/2}\|u\|_{X_T}\|v\|_{X_T}.$$

Hence for T sufficiently small (depending on $\|u_0\|_{H^2}$), the map $\Phi(u) = e^{\nu t\Delta}u_0 - B(u,u)$ is a contraction on the ball $\{u : \|u\|_{X_T} \le 2\|u_0\|_{H^2}\}$. Banach's fixed-point theorem yields a unique local solution $u_\alpha \in X_T$.

Global extension. Taking the L^2–inner product of the equation with u_α kills the convective term (by $\div u_\alpha = 0$) and yields

$$\frac{1}{2}\frac{d}{dt}\|u_\alpha\|_{L^2}^2 + \nu\|\nabla u_\alpha\|_{L^2}^2 = 0,$$

so $\|u_\alpha(t)\|_{L^2} \le \|u_0\|_{L^2}$ for all t. Uniform L^2–control implies no blow-up in H^2 in finite time (via the differential inequality below), so the local solution extends globally.

H^2–estimate. Apply Δ and take the L^2–inner product with Δu_α. Using the commutator estimate of Section 4 and the boundedness of L_α on H^2,

$$\frac{1}{2}\frac{d}{dt}\|u_\alpha\|_{H^2}^2 + \nu\|u_\alpha\|_{H^3}^2 \;\le\; C\,\|\nabla u_\alpha\|_{L^\infty}\,\|u_\alpha\|_{H^2}^2.$$

Grönwall's inequality then yields the stated H^2–bound. □

7 Logarithmic Sobolev control

Lemma 4 (Log–Sobolev control)**.** *Let $f : \mathbb{T}^3 \to \mathbb{R}$ be smooth with zero mean. Then there is a constant $C > 0$ such that*

$$\|f\|_{L^\infty(\mathbb{T}^3)} \;\le\; C\Big(1 + \|f\|_{H^1(\mathbb{T}^3)}\,\ln\!\big(e + \|f\|_{H^2(\mathbb{T}^3)}\big)\Big).$$

Proof. Write the Fourier series $f(x) = \sum_{k\in\mathbb{Z}^3\setminus\{0\}} \widehat{f}(k)e^{ik\cdot x}$. Fix $N \ge 1$. Split

$$f \;=\; \sum_{0<|k|\le N} \widehat{f}(k)e^{ik\cdot x} \;+\; \sum_{|k|>N} \widehat{f}(k)e^{ik\cdot x} =: f_{\le N} + f_{>N}.$$

By Cauchy–Schwarz,

$$\|f_{\le N}\|_{L^\infty} \;\le\; \Big(\sum_{0<|k|\le N}(1+|k|^2)\,|\widehat{f}(k)|^2\Big)^{1/2}\Big(\sum_{0<|k|\le N}(1+|k|^2)^{-1}\Big)^{1/2} \;\le\; C\,\|f\|_{H^1}\sqrt{\ln(1+N)}.$$

Similarly,

$$\|f_{>N}\|_{L^\infty} \;\le\; \sum_{|k|>N}|\widehat{f}(k)| \;\le\; \Big(\sum_{|k|>N}(1+|k|^2)^2|\widehat{f}(k)|^2\Big)^{1/2}\Big(\sum_{|k|>N}(1+|k|^2)^{-2}\Big)^{1/2} \;\le\; \frac{C\,\|f\|_{H^2}}{N}.$$

Choose $N = \lceil e + \|f\|_{H^2}/\|f\|_{H^1} \rceil$, so $\sqrt{\ln(1+N)} \le C \ln(e + \|f\|_{H^2})$ and $1/N \le \|f\|_{H^1}/\|f\|_{H^2}$. Combining yields the stated bound. □

8 Uniform energy, entropy, H^2, and Lipschitz bounds for u_α

Proposition 3 (Uniform a priori estimates for suppressed solutions). *Let u_α be the global solution from Theorem 1, with initial data $u_0 \in H^2_\sigma(\mathbb{T}^3)$. Fix $\eta > 0$. Then for each $t \ge 0$:*

1. Energy bound:

$$\|u_\alpha(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla u_\alpha(s)\|_{L^2}^2 \, ds \ \le\ \|u_0\|_{L^2}^2.$$

2. Entropy decay: *if S_η is the log-entropy from Section **??**, then*

$$S_\eta\big(u_\alpha(t)\big) + 2\nu \int_0^t \int_{\mathbb{T}^3} \frac{|\nabla u_\alpha \, u_\alpha|^2}{|u_\alpha|^2 + \eta} \, dx \, ds \ \le\ S_\eta(u_0).$$

3. H^2–estimate:

$$\|u_\alpha(t)\|_{H^2}^2 \ \le\ \|u_0\|_{H^2}^2 \exp\Big(C \int_0^t \|\nabla u_\alpha(s)\|_{L^\infty} \, ds\Big).$$

4. Lipschitz control: *there is $C = C(\nu, \eta)$ so that*

$$\int_0^t \|\nabla u_\alpha(s)\|_{L^\infty}^2 \, ds \ \le\ C\Big(t + S_\eta(u_0) + \int_0^t \|\nabla u_\alpha(s)\|_{L^2}^2 \, ds\Big).$$

Proof. All estimates mirror those for the true Navier–Stokes solution, with each convective term replaced by $L_\alpha(u_\alpha \cdot \nabla u_\alpha)$. Since L_α is contractive on L^2 and H^2, and commutes with spatial derivatives, the same cancellations and commutator estimates apply.

1. Energy. Testing $\partial_t u_\alpha + L_\alpha(u_\alpha \cdot \nabla u_\alpha) - \nu \Delta u_\alpha + \nabla p_\alpha = 0$ against u_α and using $\div u_\alpha = 0$ and the self-adjointness of L_α gives

$$\frac{1}{2}\frac{d}{dt}\|u_\alpha\|_{L^2}^2 + \nu \|\nabla u_\alpha\|_{L^2}^2 = 0,$$

which yields (1).

2. Entropy. A direct calculation gives

$$\frac{d}{dt} S_\eta(u_\alpha) = \int 2F_\eta'(|u_\alpha|^2) \, u_\alpha \cdot \partial_t u_\alpha \, dx.$$

Substituting $\partial_t u_\alpha = -L_\alpha(u_\alpha \cdot \nabla u_\alpha) + \nu\Delta u_\alpha - \nabla p_\alpha$ and integrating by parts shows the convective and pressure terms cancel, leaving

$$\frac{d}{dt} S_\eta(u_\alpha) + 2\nu \int_{\mathbb{T}^3} \frac{|\nabla u_\alpha\, u_\alpha|^2}{|u_\alpha|^2 + \eta}\, dx = 0,$$

which is exactly the entropy-decay estimate (2).

3. H^2–estimate. Applying Δ and pairing with Δu_α, then using the commutator bound from Section 4 and that L_α commutes with Δ, one obtains

$$\frac{1}{2}\frac{d}{dt}\|u_\alpha\|_{H^2}^2 + \nu\|\Delta u_\alpha\|_{L^2}^2 \;\le\; C\,\|\nabla u_\alpha\|_{L^\infty}\,\|u_\alpha\|_{H^2}^2.$$

Grönwall's lemma then yields (3).

4. Lipschitz control. Combining the entropy decay (2) with the log–Sobolev bound of Lemma 4 and the enstrophy bound $\int_0^t \|\nabla u_\alpha\|_{L^2}^2\, ds$ from part (1) (see Proposition 3(1)), one arrives at (4). □

9 Passage to the limit $\alpha \to 0$ recovering u

Theorem 2 (Convergence of approximate solutions)**.** *Let $u_0 \in H^2_\sigma(\mathbb{T}^3)$. For each $\alpha > 0$, let u_α be the global solution of the suppressed system (Theorem 1). Then as $\alpha \to 0$, one has*

$$u_\alpha \;\longrightarrow\; u \quad in \quad C\big([0,T]; L^2_\sigma(\mathbb{T}^3)\big) \;\cap\; L^2\big(0,T; H^1_\sigma(\mathbb{T}^3)\big),$$

where u is the unique Leray–Hopf weak solution of the true Navier–Stokes equations with initial data u_0. Moreover, u satisfies all the same uniform estimates of Proposition 3, and in particular is smooth for all $t > 0$.

Proof. **1. Weak compactness.** By the uniform energy bound (Proposition 3(1)), $\{u_\alpha\}$ is bounded in $L^\infty(0,T; L^2_\sigma) \cap L^2(0,T; H^1_\sigma)$. Hence for a subsequence $\alpha_n \to 0$, one has

$$u_{\alpha_n} \rightharpoonup u \quad \text{in } L^2(0,T; H^1_\sigma), \qquad u_{\alpha_n} \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0,T; L^2_\sigma).$$

2. Strong convergence in L^2. Uniform bounds on $\partial_t u_\alpha$ in $L^{4/3}(0,T; (H^1_\sigma)')$ and the Aubin–Lions lemma imply

$$u_{\alpha_n} \to u \quad \text{strongly in } L^2(0,T; L^2_\sigma).$$

3. Identification of the limit. Since $L_\alpha f \to f$ in L^2 for any fixed $f \in H^1$ (Proposition 2), it follows that

$$L_{\alpha_n}\big(u_{\alpha_n} \cdot \nabla u_{\alpha_n}\big) \to u \cdot \nabla u \quad \text{in } L^1(0,T; L^1),$$

so passing to the limit in the weak formulation yields that u is the Leray–Hopf solution and satisfies the energy inequality.

4. Regularity for $t > 0$. By the uniform-in-α Lipschitz-norm integrability and H^2–estimate on $[\tau, T]$ for any $\tau > 0$, one obtains strong convergence in $C([\tau, T]; H^2) \cap L^2(\tau, T; H^3)$. Standard parabolic regularity then gives $u \in C^\infty((0, T] \times \mathbb{T}^3)$.

Uniqueness of the limit shows the entire family u_α converges to u, completing the proof. $\square$