Proofs - Auxiliary ODE Grönwall Tools

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1 Generalized Grönwall inequalities

[Integral and differential Grönwall inequalities] Let $T > 0$, and suppose $y : [0,T] \to [0,\infty)$ is continuous.

1. (Integral form) If there are nonnegative functions $a, b \in L^1(0,T)$ such that

$$y(t) \ \le \ a(t) + \int_0^t b(s)\, y(s)\, ds, \quad \forall\, t \in [0,T],$$

then

$$y(t) \ \le \ a(t) + \int_0^t a(s)\, b(s)\, \exp\Big(\int_s^t b(\sigma)\, d\sigma\Big)\, ds.$$

In particular, if a is nondecreasing then

$$y(t) \ \le \ a(t)\, \exp\Big(\int_0^t b(s)\, ds\Big).$$

2. (Differential form) If $y \in C^1([0,T])$ satisfies

$$\frac{dy}{dt} \ \le \ \alpha(t)\, y(t) + \beta(t), \quad \forall\, t \in [0,T],$$

for integrable functions α, β, then

$$y(t) \ \le \ y(0)\, \exp\Big(\int_0^t \alpha(s)\, ds\Big) + \int_0^t \beta(s)\, \exp\Big(\int_s^t \alpha(\sigma)\, d\sigma\Big)\, ds.$$

1. Integral form. Define $z(t) = \displaystyle\int_0^t b(s)\, y(s)\, ds$. Then the hypothesis gives

$$y(t) \le a(t) + z(t) \quad \Longrightarrow \quad z'(t) = b(t)\, y(t) \le b(t)\, a(t) + b(t)\, z(t).$$

Apply the differential-form Grönwall inequality (part 2) to z:

$$z(t) \le \int_0^t b(s)\, a(s)\, \exp\Big(\int_s^t b(\sigma)\, d\sigma\Big)\, ds.$$

Hence

$$y(t) \le a(t) + z(t) \le a(t) + \int_0^t a(s)\, b(s)\, \exp\Bigl(\int_s^t b(\sigma)\, d\sigma\Bigr)\, ds.$$

If a is nondecreasing then $a(s) \le a(t)$ for $0 \le s \le t$, so

$$y(t) \le a(t)\Bigl[1 + \int_0^t b(s)\, \exp\Bigl(\int_s^t b(\sigma)\, d\sigma\Bigr)\, ds\Bigr] = a(t)\, \exp\Bigl(\int_0^t b(s)\, ds\Bigr).$$

2. Differential form. Rearrange the inequality as

$$\frac{d}{dt}\bigl(e^{-\int_0^t \alpha}\, y(t)\bigr) = e^{-\int_0^t \alpha}\Bigl(y'(t) - \alpha(t)\, y(t)\Bigr) \le e^{-\int_0^t \alpha}\, \beta(t).$$

Integrate from 0 to t:

$$e^{-\int_0^t \alpha}\, y(t) - y(0) \le \int_0^t e^{-\int_0^s \alpha}\, \beta(s)\, ds.$$

Multiplying by $e^{\int_0^t \alpha}$ yields

$$y(t) \le y(0)\, e^{\int_0^t \alpha} + \int_0^t \beta(s)\, e^{\int_s^t \alpha}\, ds,$$

which is the claimed result.

2 ODE–comparison lemma for $\dot{y} \le A\, y + B\, y^2$

[Comparison for quadratic growth ODE] Let $A, B \ge 0$. Suppose $y \in C^1([0,T])$ satisfies

$$\frac{dy}{dt} \;\le\; A\, y + B\, y^2, \quad y(0) = y_0 \ge 0.$$

Then for all $t \in [0,T]$ with $1 - BAy_0\bigl(e^{At} - 1\bigr) > 0$,

$$y(t) \;\le\; \frac{y_0\, e^{At}}{1 - BA\, y_0\, (e^{At} - 1)}.$$

Consider the Riccati equation

$$\frac{dz}{dt} = A\, z + B\, z^2, \quad z(0) = y_0,$$

whose explicit solution is

$$z(t) = \frac{y_0\, e^{At}}{1 - BA\, y_0\, (e^{At} - 1)},$$

valid as long as the denominator stays positive. Since y satisfies $\dot{y} \le A\, y + B\, y^2$ and $y(0) = z(0)$, the standard comparison principle for ODEs implies $y(t) \le z(t)$ for all t up to the blow-up time of z. This yields the stated bound.

3 Continuity-in-time lemmas

[Weak-continuity in L^2] Let $u \in L^\infty(0,T;L^2_\sigma(^3))$ satisfy

$$\partial_t u \ \in \ L^{4/3}\big(0,T;(H^1_\sigma(^3))'\big).$$

Then $u \in C_w\big([0,T];L^2_\sigma(^3)\big)$, i.e. $t \mapsto \langle u(t),\varphi\rangle$ is continuous for every $\varphi \in L^2_\sigma(^3)$.

Fix $\varphi \in H^1_\sigma(^3) \subset L^2_\sigma(^3)$. For a.e. $s,t \in [0,T]$,

$$\langle u(t)-u(s),\varphi\rangle = \int_s^t \langle \partial_\tau u(\tau),\varphi\rangle \, d\tau.$$

Since $\partial_t u \in L^{4/3}(0,T;(H^1)')$ and $\varphi \in H^1$, Hölder gives

$$|\langle u(t){-}u(s),\varphi\rangle| \le \|\partial_t u\|_{L^{4/3}(s,t;(H^1)')}\,\|\varphi\|_{L^4(s,t;H^1)} \le C\,|t{-}s|^{1/4}\,\|\partial_t u\|_{L^{4/3}(0,T;(H^1)')}\,\|\varphi\|_{H^1}.$$

Thus $t \mapsto \langle u(t),\varphi\rangle$ is Hölder-continuous of exponent $1/4$, hence continuous. Density of H^1_σ in L^2_σ and the uniform $L^\infty_t L^2_x$–bound then upgrade this to continuity against all $\varphi \in L^2_\sigma$, proving $u \in C_w([0,T];L^2_\sigma)$.

[Strong-continuity in L^2] Let

$$u \in L^2\big(0,T;H^1_\sigma(^3)\big), \quad \partial_t u \in L^2\big(0,T;(H^1_\sigma(^3))'\big).$$

Then $u \in C\big([0,T];L^2_\sigma(^3)\big)$ and, moreover,

$$\|u(t)\|^2_{L^2} = \|u(0)\|^2_{L^2} + 2\int_0^t \langle \partial_s u(s),u(s)\rangle_{(H^1)',H^1}\, ds.$$

This is the classical Lions–Magenes lemma (see *e.g.* [**?**]), specialized to the Gelfand triplet

$$H^1_\sigma(^3) \ \hookrightarrow \ L^2_\sigma(^3) \ \hookrightarrow \ (H^1_\sigma(^3))'.$$

Precisely, since $u \in L^2(0,T;H^1_\sigma)$ and $\partial_t u \in L^2(0,T;(H^1)')$, one has $u \in C([0,T];L^2_\sigma)$ and the above identity holds by differentiating $\|u(t)\|^2_{L^2}$ in time in the weak sense. A direct proof uses a mollification in time and passage to the limit, which recovers both continuity and the energy-identity formula.

4 Interpolation inequalities in time–space norms

[Interpolation in mixed Lebesgue spaces] Let $0 \le \theta \le 1$, and let $1 \le p_0,p_1,q_0,q_1 \le \infty$. Define

$$\frac{1}{p} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}, \qquad \frac{1}{q} = \frac{\theta}{q_0} + \frac{1-\theta}{q_1}.$$

If $f\colon [0,T]\times^3 \to$ satisfies

$$f \in L^{p_0}\big(0,T;L^{q_0}(^3)\big) \quad \textit{and} \quad f \in L^{p_1}\big(0,T;L^{q_1}(^3)\big),$$

then $f \in L^p(0,T;L^q(^3))$ and there is a constant C, depending only on the indices, such that

$$\|f\|_{L^p_t L^q_x} \ \le\ \|f\|^{\theta}_{L^{p_0}_t L^{q_0}_x}\ \|f\|^{1-\theta}_{L^{p_1}_t L^{q_1}_x}.$$

In particular, for any $r \in [p_0, p_1]$ and $s \in [q_0, q_1]$ with $1r = \theta p_0 + 1 - \theta p_1$, $1s = \theta q_0 + 1 - \theta q_1$, one has $\|f\|_{L^r_t L^s_x} \le \|f\|^{\theta}_{L^{p_0}_t L^{q_0}_x} \|f\|^{1-\theta}_{L^{p_1}_t L^{q_1}_x}$.

Apply the Riesz–Thorin interpolation theorem first in x for each fixed t, to deduce

$$\|f(t,\cdot)\|_{L^q_x} \ \le\ \|f(t,\cdot)\|^{\theta}_{L^{q_0}_x}\ \|f(t,\cdot)\|^{1-\theta}_{L^{q_1}_x}.$$

Raising to the pth power and integrating in t gives

$$\|f\|^p_{L^p_t L^q_x} = \int_0^T \|f(t)\|^p_{L^q_x}\, dt \le \int_0^T \|f(t)\|^{\theta p}_{L^{q_0}_x}\, \|f(t)\|^{(1-\theta)p}_{L^{q_1}_x}\, dt.$$

Now apply Hölder's inequality in time with exponents $p_0 \theta p$ and $p_1(1-\theta)p$, which satisfy $\theta p p_0 + (1-\theta) p p_1 = 1$. One obtains

$$\|f\|^p_{L^p_t L^q_x} \ \le\ \Big(\int_0^T \|f(t)\|^{p_0}_{L^{q_0}_x}\, dt\Big)^{\theta p/p_0}\ \Big(\int_0^T \|f(t)\|^{p_1}_{L^{q_1}_x}\, dt\Big)^{(1-\theta)p/p_1}.$$

Taking pth roots yields the stated interpolation inequality.