Divergence for n=2 Case

$$\text{Let } f(z) = z^2 + C \quad |C| \leq 2 \;\; C = c + di \qquad z = a + bi$$

Need to find z such that $f(z)$ increases or decreases without bound

$$\frac{|f(z)|}{|z|} = \frac{|z^2 + C|}{|z|}$$

If the ratio of $|z_{k+1}|$ to $|z_k|$ is greater than 1, the distance to the origin increases without bound

$$|z^2 + C| = \sqrt{(z^2)^2 + C^2}$$

$$\sqrt{(z^2)^2 + C^2} \geq |z^2| - |C| \qquad \text{via triangle theorem}$$

This is because the sum of the two sides of a triangle is always greater than the length of the third side, and the length of the third side can never be less than the difference between the two sides (straight lines)

$$|z^2 + C| \geq |z|^2 - |C|$$

Therefore,

$$\frac{|f(z)|}{|z|} \geq \frac{|z|^2 - |C|}{|z|}$$

$$\frac{|f(z)|}{|z|} \geq |z| - \frac{|C|}{|z|}$$

$$|z| - \frac{|C|}{|z|} \geq |z| - 1 \quad \text{when} \quad |z| \geq |C|$$

$$|z| - 1 \geq 1 \quad \text{when} \quad |z| \geq |2|$$

$$\therefore \frac{|f(z)|}{|z|} \geq 1 \quad \text{when} \quad |z| \geq 2 \geq |C|$$

As a result, the point grows rapidly when $|z| \geq 2$, so we know all points that lead to an iteration that has a magnitude above 2, the sequence diverges.

Divergence for General Case

$$\text{Let } f(z) = z^n + C \quad C = c + di \qquad z = a + bi$$

$$\frac{|f(z)|}{|z|} = \frac{|z^n + C|}{|z|}$$

$$|z^n + C| \geq |z^n| - |C| \qquad \text{via triangle theorem}$$

Therefore,

$$\frac{|f(z)|}{|z|} \geq \frac{|z^n| - |C|}{|z|}$$

$$\frac{|f(z)|}{|z|} \geq \frac{|z|^n - |C|}{|z|} \qquad \text{via De Moivre's Theorem}$$

$$\frac{|f(z)|}{|z|} \geq |z|^{n-1} - \frac{|C|}{|z|}$$

$$|z|^{n-1} - \frac{|C|}{|z|} \geq |z|^{n-1} - 1 \quad \text{when} \quad |z| \geq |C|$$

$$|z|^{n-1} - 1 \geq 1 \quad \text{when} \quad |z|^{n-1} \geq |2|$$

$$\therefore \frac{|f(z)|}{|z|} \geq 1 \quad \text{when} \quad |z| \geq \sqrt[n-1]{2} \geq |C|$$

As a result, the point grows rapidly when $|z| \geq \sqrt[n-1]{2}$, so we know all points that lead to an iteration that has a magnitude above $\sqrt[n-1]{2}$, the sequence diverges.

Magnitude of Complex Numbers (n=2)

$$z = a + bi$$

$$|z| = \sqrt{a^2 + b^2} = \sqrt{a^2 + b^2}$$

$$z^2 = a^2 + 2abi - b^2 = (a^2 - b^2) + 2abi$$

$$|z^2| = \sqrt{(a^2 - b^2)^2 + (2ab)^2} = \sqrt{(a^4 + b^4 - 2a^2b^2) + 4a^2b^2} = \sqrt{(a^4 + b^4 + 2a^2b^2}$$

$$= \sqrt{(a^2 + b^2)^2} = a^2 + b^2$$

$$\therefore \quad |z^2| = |z|^2$$

Powers of Complex Numbers (general)

$$\text{Let } z = a + bi \qquad |z| = r$$

Since z is complex, it can be written as $z = r(\cos\theta + i\sin\theta)$ for some θ

$$e^{ix} = \cos x + i\sin x$$

$$(e^{ix})^n = (\cos x + i\sin x)^n = e^{nix} = \cos nx + i\sin nx$$

$$z^n = r^n(\cos\theta + i\sin\theta)^n$$

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

$$|z^n| = |r^n(\cos n\theta + i\sin n\theta)|$$

$$= r^n|\cos n\theta + i\sin n\theta|$$

$$= r^n \qquad \text{via Pythagorean Theorem}$$

$$\therefore \quad |z^n| = |z|^n$$