Stability Analysis for a Class of Systems: From Model-Based Methods to Data-Driven Methods

Zhuo Wang and Derong Liu, *Fellow, IEEE*

***Abstract*—This paper presents some new methods to analyze the stability of the equilibrium point, and the bounded-input/bounded-state (BIBS) and the bounded-input/bounded-output (BIBO) stabilities for linear systems and a class of nonlinear systems. We study these systems in both continuous-time and discrete-time forms, using approaches from model-based methods to data-driven methods. For linear systems, our work about the BIBS and the BIBO stabilities does not concern the state controllability or the state observability. The data-driven methods can analyze the stabilities by doing some simple experiments and observations, while not identifying the corresponding parameter matrices. Our methods are easier to understand and more convenient to use in engineering practice.**

***Index Terms*—Bounded-input/bounded-state (BIBS) and bounded-input/bounded-output (BIBO) stabilities, continuous-time and discrete-time systems, data-driven methods, model-based methods, stability of the equilibrium point.**

I. Introduction

IN SYSTEMS theory and control engineering, stability theory plays an important role. If a control system is not stable, it cannot work well and will have serious issues in performance.

The study of control system stabilities has a long history. From the 1860s to the 1930s, research was focused on the stability of the equilibrium point, while not considering the effect of the control input. From the 1930s to the 1950s, both linear and nonlinear SISO systems, which can be described by differential equations, became popular in the field of control theory and control engineering. In those days, in addition to studying the stability of the equilibrium point, people also studied the influence of control input on the system stabilities. The studied control systems were relatively simple, whose output solutions corresponding to the typical inputs (such as the step function, the velocity function and the periodic function, etc.) are easy to obtain. People may confirm the bounded-input/bounded-output (BIBO) stability simply by observing the solution of the system.

After R. E. Kalman developed the state-space model in 1960, especially since the 1990s, the industrial systems have become more and more complex. Accordingly, the state space models are widely applied for system description, and MIMO systems rather than SISO systems are widely adopted. In addition, people found that it is not enough to only analyze the stability of the equilibrium point. Since there always exist control inputs in practical systems, and if the system is not bounded-input/bounded-state (BIBS)/BIBO stable, there will be bad performances or great damages, even though the equilibrium point is stable. In this context, the study of the BIBS and the BIBO stabilities has both theoretical and practical significances, and some results have been proposed on this topic [1], [2]. But, most of them have specific requirements. For instance, when analyzing the linear time-invariant (time-varying) systems, it is often required that the system is (uniformly) completely state controllable/observable [3]–[6]; when analyzing the nonlinear systems, people usually need to use some complicated mathematical tools such as the Dini derivatives and the $\mathcal{K}$-function [7], [8]. This causes these methods not easy to understand and inconvenient to use in engineering practice.

At present, most stability analysis approaches are based on the system mathematical models [9]–[11]. However, sometimes the system mathematical model may be unknown. To analyze the system stabilities, model-based methods usually need to determine the structure of the system model and identify the corresponding parameters, or construct the system model based on mechanisms first, and then perform the stability analysis. Up until now, there have been many methods for system identification and model construction [12]–[19]. However, these model-based methods spend much time on system identification and model construction, but cannot avoid the identification errors and the modeling errors. In this case, data-driven methods for direct system analysis, which only use measured/observed online and/or offline data, might be a good choice and meaningful.

In this paper, we develop some model-based and data-driven methods to analyze the stability of the equilibrium point, the BIBS and the BIBO stabilities for linear systems and a class of nonlinear systems, including their continuous-time and discrete-time forms. Our work has no specific requirements on the state controllability/observability when analyzing the linear systems; and no need of the Dini derivatives or the $\mathcal{K}$-function when analyzing the nonlinear systems. These data-driven methods have the advantage that they can confirm the stabilities by doing some simple experiments and observations, with no need to know the system mathematical models. This may avoid the identification errors and the modeling errors. Our results are easier to understand and more convenient to use in practical applications, and thus are meaningful.

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Z. Wang is with the Department of Electrical and Computer Engineering, University of Alberta, Edmonton, AB T6G 2V4, Canada (e-mail: wangzhuo12300@gmail.com).

D. Liu is with the State Key Laboratory of Management and Control for Complex Systems, Institute of Automation, Chinese Academy of Sciences, Beijing 100190, China, and also with the Department of Electrical and Computer Engineering, University of Illinois at Chicago, Chicago, IL 60607 USA (e-mail: derong.liu@ia.ac.cn; derongliu@gmail.com).

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II. Stability Analysis of Continuous-Time Systems: Model-Based Methods

In this paper, we first study the stabilities of continuous-time systems. Before the introduction of our results, we introduce the following definitions.

Definition 1 ([20, Chapter 6]): If every bounded input of a system results in a bounded state, then that system is called bounded-input/bounded-state (BIBS) stable. More specifically, the system is BIBS stable if there exist two constants $0 < L_u, L_x < \infty$, such that for all t_0 the conditions

$$x(t_0) = 0, \quad \|u(t)\| \leq L_u, \quad t \geq t_0$$

imply that $\|x(t)\| \leq L_x$ for all $t \geq t_0$. (In this paper, the symbol $\|\cdot\|$ denotes the 2—norm).

Definition 2 ([20, Chapter 6]): If every bounded input of a system results in a bounded output, then that system is called bounded-input/bounded-output (BIBO) stable. More specifically, the system is BIBO stable if there exist two constants $0 < L_u, L_y < \infty$, such that for all t_0 the conditions

$$x(t_0) = 0, \quad \|u(t)\| \leq L_u, \quad t \geq t_0$$

imply that $\|y(t)\| \leq L_y$ for all $t \geq t_0$.

Definitions 1 and 2 are applicable to discrete-time systems.

A. Stability Analysis of Continuous-Time Linear Systems

This section starts with the continuous-time linear time-varying systems described by

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ y(t) = C(t)x(t) \quad (t \geq 0) \end{cases} \tag{1}$$

where $x(t) \in R^n$, $u(t) \in R^m$, and $y(t) \in R^q$ are the state, the input and the output of system (1), respectively. $A(t)$, $B(t)$, and $C(t)$ are the corresponding time-varying parameter matrices.

Lemma 1 ([21, Section 4.6]): The equilibrium point $x = 0$ of system (1) is uniformly asymptotically stable if and only if the state transition matrix $\Phi(t, t_0)$ satisfies the inequality

$$\|\Phi(t, t_0)\| \leq \alpha \exp\{-\beta(t - t_0)\} \qquad \forall t \geq t_0 \geq 0 \tag{2}$$

for some positive constants α and β.

Theorem 1: For system (1), if for all $t \geq t_0 \geq 0$,

1) the equilibrium point $x = 0$ is uniformly asymptotically stable;
2) $\sup_{t\geq 0}\{\|A(t)\|\} = M_A < \infty$ and $\sup_{t\geq 0}\{\|B(t)\|\} = M_B < \infty$;

then system (1) is BIBS stable. In addition, if $\sup_{t\geq 0}\{\|C(t)\|\} = M_C < \infty$, then system (1) is also BIBO stable.

Proof: The solution of system (1) is

$$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau.$$

Since the equilibrium point $x = 0$ is uniformly asymptotically stable, then by Lemma 1, there exist two constants $\alpha, \beta > 0$, such that $\|\Phi(t, t_0)\| \leq \alpha \exp\{-\beta(t - t_0)\}(t \geq t_0 \geq 0)$.

Set $x(t_0) = 0$. If $\|u(t)\| \leq L_u < \infty$ for all $t \geq t_0 \geq 0$, then by condition 2

$$\begin{aligned} \|x(t)\| &\leq \int_{t_0}^{t} \|\Phi(t, \tau)\| \, \|B(\tau)\| \, \|u(\tau)\| \, d\tau \\ &\leq \alpha M_B L_u \int_{t_0}^{t} \exp\{-\beta(t - \tau)\} \, d\tau \leq \frac{\alpha}{\beta} M_B L_u. \end{aligned} \tag{3}$$

Let $L_x = (\alpha/\beta)M_B L_u$. Then, for any $\|u(t)\| \leq L_u (t \geq t_0 \geq 0)$, $\|x(t)\| \leq L_x < \infty$. By Definition 1, system (1) is BIBS stable. Moreover, if $\sup_{t\geq 0}\{\|C(t)\|\} = M_C < \infty$, then

$$\|y(t)\| = \|C(t)x(t)\| \leq \|C(t)\| \, \|x(t)\| \leq M_C L_x. \tag{4}$$

Let $L_y = M_C L_x$. Then, $\forall t \geq t_0 \geq 0$, $\|y(t)\| \leq L_y < \infty$. By Definition 2, system (1) is BIBO stable. ■

For the case that $u(t) = u_d(t) + u_r(t)$, where $u_d(t)$ is the deterministic component and $u_r(t)$ is the random component, Theorem 1 still holds as long as $\|u_d(t)\| \leq L_d < \infty$ and $\|u_r(t)\| \leq L_r < \infty$. The rest of results in this paper also apply to this case, and we will not repeat the discussion.

Corollary 1: In system (1), if $A(t) \equiv A$, $\sup_{t\geq 0}\{\|B(t)\|\} = M_B < \infty$ (or, $B(t) \equiv B$), $\sup_{t\geq 0}\{\|C(t)\|\} = M_C < \infty$ (or, $C(t) \equiv C$), and all the eigenvalues of A satisfy that $\mathrm{Re}\{\lambda_i(A)\} < 0 (1 \leq i \leq n)$, then this system is both BIBS and BIBO stable.

In this case, $\Phi(t, t_0) = e^{A(t-t_0)}$ and the equilibrium point $x = 0$ is uniformly asymptotically stable. By Lemma 1, there exist two constants $\alpha', \beta' > 0$, such that $\|e^{A(t-t_0)}\| \leq \alpha' \exp\{-\beta'(t - t_0)\}$. The rest of the proof are similar to those of Theorem 1, and thus will not be repeated.

Corollary 2: For system (1), if for all $t \geq t_0 \geq 0$,

1) $\sup_{t\geq 0}\{\|A(t)\|\} = M_A < \infty$ and $\sup_{t\geq 0}\{\|B(t)\|\} = M_B < \infty$;
2) there exists a positive definite matrix $Q(t) \in R^{n\times n}$, such that $A^T(t) + A(t) = -Q(t)$, and the eigenvalues $\lambda_i(Q(t))$ satisfy $\min_{1\leq i\leq n}\{\inf_{t\geq t_0\geq 0}[\lambda_i(Q(t))]\} = \lambda > 0$;

then the equilibrium point $x = 0$ of system (1) is uniformly asymptotically stable, and the system is BIBS stable. Moreover, if $\sup_{t\geq 0}\{\|C(t)\|\} = M_C < \infty$, then system (1) is also BIBO stable.

Proof: First, set the input $u(t) \equiv 0$. Define a Lyapunov function $V(x(t)) = x^T(t)x(t)$, then

$$\dot{V}(x(t)) = x^T(t)\left[A^T(t) + A(t)\right]x(t) = -x^T(t)Q(t)x(t).$$

By condition 2, $\dot{V}(x(t)) \leq -\lambda x^T(t)x(t)$. According to Lyapunov stability theory, the equilibrium point $x = 0$ of system (1) is uniformly asymptotically stable.

Similar to Theorem 1, it can easily be proved that system (1) is both BIBS and BIBO stable. ■

Theorem 1 and Corollaries 1 and 2 show that if the equilibrium point $x = 0$ of the continuous-time linear time-varying (time-invariant) system is uniformly asymptotically (asymptotically) stable, and the corresponding parameter matrices are bounded, then the system is certainly both BIBS and BIBO stable.

When studying the BIBS/BIBO stabilities of system (1), earlier researchers [3]–[6] provided the necessary and sufficient conditions that require the system to be uniformly completely state controllable and uniformly completely state observable. The contribution of Theorem 1 and Corollary 2 is that they present the sufficient conditions of the BIBS/BIBO stabilities, which do not have these requirements. By contrast, our methods are more convenient in practice. Next, we give an example to illustrate Theorem 1 and Corollary 2.

Example 1: Consider the following system:

$$\begin{cases} \begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix} -\frac{2+t}{1+t} & (2+e^{-t}) \\ -(2+e^{-t}) & -1 \end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}3\\ -\sin t\end{bmatrix}u(t) \\ \begin{bmatrix}y_1(t)\\ y_2(t)\end{bmatrix} = \begin{bmatrix}\cos t & -1\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}, \quad t \geq 0. \end{cases} \tag{5}$$

Here, the eigenvalues of $A(t)$ are

$$\lambda_{1,2}\left[A(t)\right] = -\frac{3+2t}{2(1+t)} \pm j\frac{1}{2}\sqrt{4(2+e^{-t})^2 - \frac{1}{(1+t)^2}}$$

and $\mathrm{Re}\{\lambda_{1,2}[A(t)]\} = -((3+2t)/2(1+t)) < -1 (t \geq 0)$. Thus, the equilibrium point $x=0$ is uniformly asymptotically stable, and then condition 1 of Theorem 1 is satisfied. Because $\sup_{t\geq 0}\{\|A(t)\|\} = (1/2)\sqrt{46+6\sqrt{5}}$, $\sup_{t\geq 0}\{\|B(t)\|\} = \sqrt{10}$ and $\sup_{t\geq 0}\{\|C(t)\|\} = (1/2)\sqrt{6+2\sqrt{5}}$, condition 2 of Theorem 1 and condition 1 of Corollary 2 are satisfied, respectively. On the other hand, for $Q(t) = -[A^T(t) + A(t)]$, we have $\min_{i=1,2}\{\inf_{t\geq 0} \lambda_i[Q(t)]\} = 2 > 0$. Condition 2 of Corollary 2 is also satisfied.

By Theorem 1 or Corollary 2, system (5) is both BIBS and BIBO stable.

B. Stability Analysis of a Class of Continuous-Time Nonlinear Systems

In this subsection, we first study the following continuous-time nonlinear time-varying system:

$$\begin{cases} \dot{x}(t) = f\left(t, x(t), u(t)\right) \\ y(t) = h(t, x(t), u(t)), \quad t \geq 0 \end{cases} \tag{6}$$

where f and h are both continuous in x and u, and bounded with respect to t, while $f(t,0,0) \equiv 0$ and $h(t,0,0) \equiv 0$.

Theorem 2: For system (6), if for all $x(t) \in R^n$, $u(t) \in R^m$ and $t \geq t_0 \geq 0$,

1) $\partial f(t,x(t),u(t))/\partial x(t)$ is continuous in $x(t)$;
2) there exist two symmetric positive definite matrices P, $Q(t,x(t)) \in R^{n\times n}$, such that

$$\frac{\partial f^T\left(t, x(t), 0\right)}{\partial x(t)}P + P\frac{\partial f\left(t, x(t), 0\right)}{\partial x(t)} = -Q\left(t, x(t)\right)$$

and the eigenvalues of $Q(t,x(t))$ satisfy

$$\min_{1\leq i\leq n}\left\{\inf_{t\geq t_0\geq 0, x(t)\in R^n}\left\{\lambda_i\left[Q\left(t,x(t)\right)\right]\right\}\right\} = \lambda > 0;$$

3) $\|f(t,x(t),u(t)) - f(t,x(t),0)\| \leq L_f\|u(t)\|$, where $L_f < \infty$ is a positive constant;

then the system is both BIBS and BIBO stable.

Proof: Let $V(t) = x^T(t)Px(t)$, where P is introduced in condition 2. By (6), we have

$$\begin{aligned} \dot{V}(t) &= x^T(t)Pf(t,x(t),u(t)) + f^T\,(t,x(t),u(t))Px(t) \\ &= x^T(t)P[f(t,x(t),0) + f(t,x(t),u(t)) - f(t,x(t),0)] \\ &\quad + [f(t,x(t),0) + f(t,x(t),u(t)) - f(t,x(t),0)]^T Px(t) \\ &= x^T(t)Pf\,(t,x(t),0) + f^T\,(t,x(t),0)\,Px(t) + 2x^T(t)P \\ &\quad \times \left[f\,(t,x(t),u(t)) - f\,(t,x(t),0)\right]. \end{aligned} \tag{7}$$

Since $f(t,0,0) \equiv 0$, by condition 1 and Lagrange's mean value theorem, $f(t,x(t),0) = (\partial f(t,\bar{x}(t),0)/\partial x(t))x(t)$, where $\bar{x}(t) = \theta(t)x(t)$ and $0 < \theta(t) < 1$ for all $t \geq t_0 \geq 0$. According to conditions 2 and 3, we can obtain

$$\begin{aligned} \dot{V}(t) &= x^T(t)\left[\frac{\partial f^T\left(t,\bar{x}(t),0\right)}{\partial x(t)}P + P\frac{\partial f\left(t,\bar{x}(t),0\right)}{\partial x(t)}\right]x(t) \\ &\quad + 2x^T(t)P\left[f\left(t,x(t),u(t)\right) - f\left(t,x(t),0\right)\right] \\ &= -x^T(t)Q\left(t,\bar{x}(t)\right)x(t) + 2x^T(t) \\ &\quad \times P\left[f\left(t,x(t),u(t)\right) - f\left(t,x(t),0\right)\right] \\ &\leq -\lambda\left\|x(t)\right\|^2 + 2\left\|x(t)\right\|\left\|P\right\| \\ &\quad \times \left\|f\left(t,x(t),u(t)\right) - f\left(t,x(t),0\right)\right\| \\ &\leq -\lambda\left\|x(t)\right\|^2 + 2L_f\|P\|\left\|u(t)\right\|\left\|x(t)\right\|. \end{aligned}$$

If $\|u(t)\| \leq L_u < \infty$ for all $t \geq t_0 \geq 0$, then

$$\dot{V}(t) \leq \left(2L_fL_u\|P\| - \lambda\left\|x(t)\right\|\right)\left\|x(t)\right\|. \tag{8}$$

Obviously, for any $\|x(t)\| > (2/\lambda)L_fL_u\|P\|$, $\dot{V}(t) < 0$, and $V(t)$ will monotonically decrease until $\|x(t)\| \leq (2/\lambda)L_fL_u\|P\|$. Considering $x(t_0)$, let $L_x = \max\{\|x(t_0)\|, (2/\lambda)L_fL_u\|P\|\}$. Then, for any $\|u(t)\| \leq L_u$, $\|x(t)\| \leq L_x < \infty$, and system (6) is BIBS stable.

Since h is continuous in both x and u, and bounded with respect to t, the output $y(t)$ will be bounded with bounded input $u(t)$ and bounded state $x(t)$. Therefore, the system is also BIBO stable. ■

Conditions 1 and 2 of Theorem 2 can ensure the global exponential stability of the equilibrium point $x=0$ of system (6), and condition 3 is a global Lipschitz condition. Theorem 2 demonstrates a fact that as long as $x=0$ is globally exponentially stable and the global Lipschitz condition 3 is satisfied, system (6) is both BIBS and BIBO stable.

Next, we consider a special case of system (6)

$$\begin{cases} \dot{x}(t) = f\left(t, x(t)\right) + G\left(t, x(t)\right)u(t) \\ y(t) = h(t, x(t), u(t)), \quad t \geq 0 \end{cases} \tag{9}$$

where f and G are both continuous in x, and bounded with respect to t, while $f(t,0) \equiv 0$ and $G(t,0) \equiv 0$; h is continuous in both x and u, and bounded with respect to t.

Corollary 3: For system (9), if for all $t \geq t_0 \geq 0$ and $x(t) \in R^n$,

1) $\partial f(t,x(t))/\partial x(t)$ is continuous in $x(t)$;

2) there exist two symmetric positive definite matrices $P, Q(t,x(t)) \in R^{n\times n}$, such that

$$\frac{\partial f^T(t,x(t))}{\partial x(t)}P + P\frac{\partial f(t,x(t))}{\partial x(t)} = -Q(t,x(t))$$

and the eigenvalues of $Q(t,x(t))$ satisfy

$$\min_{1\le i\le n}\left\{\inf_{t\ge t_0\ge 0, x(t)\in R^n}\{\lambda_i[Q(t,x(t))]\}\right\} = \lambda > 0;$$

3) $\|G(t,x(t))\| \le L_G < \infty$;

then the system is both BIBS and BIBO stable.

The proof is similar to that of Theorem 2, and thus will be omitted here.

There are few results about the BIBS and the BIBO stabilities of systems (6) and (9). The contribution of Theorem 2 and Corollary 3 is that they provide some new sufficient conditions of the BIBS and the BIBO stabilities. They do not use the mathematical tools such as the Dini derivatives and the $\mathcal{K}$-function, and thus are easier to understand and more convenient for applications. Next, we present an example to illustrate Theorem 2 and Corollary 3.

Example 2:

$$\begin{cases} \dot{x}_1(t) = -x_1(t) - x_1^3(t) + x_2(t) + 3u_1(t) \\ \dot{x}_2(t) = -x_1(t) - 7x_2(t) - 2x_2^5(t) + \sin(t)u_2(t) \\ y(t) = 5x_1(t) + 6u_2(t) - e^{-t}, \quad t \ge 0. \end{cases} \tag{10}$$

Choose $P = I$, which is the identity matrix. Then, we can easily see that conditions 1–3 of Theorem 2 and Corollary 3 are all satisfied, respectively. Then, system (10) is both BIBS and BIBO stable.

Next, we present another example, which shows that for nonlinear systems, the uniform global asymptotic stability of the equilibrium point $x = 0$ does not guarantee the BIBS and the BIBO stabilities. This is much different from linear systems.

Example 3:

$$\begin{cases} \dot{x}(t) = -\frac{x(t)}{1+x^2(t)} + u(t) \\ y(t) = 3x(t) + \cos(t), \quad t \ge 0. \end{cases} \tag{11}$$

The internal dynamics of system (11) is $\dot{x}(t) = -(x(t)/(1 + x^2(t)))$. Let $V(x(t)) = x^2(t)$. Then, $\dot{V}(x(t)) = (-2x^2(t)/((1 + x^2(t)))$. According to Lyapunov stability theory, the equilibrium point $x = 0$ is uniformly globally asymptotically stable. But, it is not globally exponentially stable. Let $x(0) = 0$, $u(t) \equiv 1$, we have

$$x(t) = \int_0^t \left[\frac{-x(t)}{1+x^2(t)} + 1\right] d\tau \ge \int_0^t \left(1 - \frac{1}{2}\right) d\tau = \frac{1}{2}t.$$

Obviously, system (11) is neither BIBS stable nor BIBO stable.

III. Stability Analysis of Discrete-Time Systems: Model-Based Methods

In this section, we study the stabilities of discrete-time systems, which include linear systems and a class of nonlinear systems.

A. Stability Analysis of Discrete-Time Linear Systems

Consider the discrete-time linear time-varying systems as follows:

$$\begin{cases} x(k+1) = A(k)x(k) + B(k)u(k) \\ y(k) = C(k)x(k), \quad k \ge 0 \end{cases} \tag{12}$$

where $x(k) \in R^n$, $u(k) \in R^m$, and $y(k) \in R^q$ are the state, the input and the output of system (12), respectively. $A(k)$, $B(k)$, and $C(k)$ are the time-varying parameter matrices.

Theorem 3: For system (12), if $\sup_{k\ge 0}\{\|A(k)\|\} = M_A < 1$ and $\sup_{k\ge 0}\{\|B(k)\|\} = M_B < \infty$, then the equilibrium point $x = 0$ of system (12) is uniformly asymptotically stable, and the system is BIBS stable. In addition, if $\sup_{k\ge 0}\{\|C(k)\|\} = M_C < \infty$, then system (12) is also BIBO stable.

Proof: For system (12), let $u(k) \equiv 0$. Then, $\forall x(0) \ne 0$

$$x(k+1) = A(k)x(k) = \prod_{j=0}^{k} A(j)x(0), \quad k \ge 0$$

$$\|x(k+1)\| \le \prod_{j=0}^{k} \|A(j)\|\,\|x(0)\| \le M_A^{k+1}\,\|x(0)\|. \tag{13}$$

Since $M_A < 1$, $x(k) \to 0$ as $k \to \infty$. Then, the equilibrium point $x = 0$ of system (12) is uniformly asymptotically stable.

Set $x(0) = 0$. Then $\forall k \ge 1$, the solution of system (12) is

$$x(k+1) = B(k)u(k) + \sum_{j=1}^{k}\prod_{i=j}^{k} A(i)B(j-1)u(j-1)$$

while $x(1) = B(0)u(0)$. With given conditions, $\forall k \ge 1$

$$\begin{aligned} \|x(k+1)\| \le{}& \|B(k)\|\,\|u(k)\| \\ &+ \sum_{j=1}^{k}\prod_{i=j}^{k}\|A(i)\|\,\|B(j-1)\| \times \|u(j-1)\| \\ \le{}& M_B\,\|u(k)\| + M_B\sum_{j=1}^{k} M_A^{k-j+1}\,\|u(j-1)\|. \end{aligned} \tag{14}$$

If $\|u(k)\| \le L_u < \infty (k \ge 0)$, then

$$\|x(k+1)\| \le M_B L_u + M_B L_u \sum_{j=1}^{\infty} M_A^j = \frac{M_B L_u}{1 - M_A}.$$

Let $L_x = (M_B L_u/(1 - M_A))$. Then, for any $\|u(k)\| \le L_u (k \ge 0)$, $\|x(k)\| \le L_x < \infty$. By Definition 1, system (12) is BIBS stable.

If $\sup_{k\ge 0}\{\|C(k)\|\} = M_C < \infty$, then $\|y(k)\| \le M_C L_x$. Let $L_y = M_C L_x$. For any $\|u(k)\| \le L_u$, $\|y(k)\| \le L_y < \infty$. Therefore, by Definition 2, system (12) is BIBO stable. ■

Next, we study the time-invariant form of system (12)

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k), \quad k \ge 0. \end{cases} \tag{15}$$

Theorem 4: Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ denote the eigenvalues of A in (15). If $|\lambda_j| < 1 (1 \leq j \leq n)$, then system (15) is both BIBS and BIBO stable. (Here, $|\cdot|$ denotes the absolute value).

Proof: Let $b_i = [0, \ldots, 1, \ldots, 0]^T \in R^m (1 \leq i \leq m)$, whose ith element is 1, and all other elements are zeros. The input can be expressed as $u(k) = \sum_{i=1}^m \alpha_i(k) b_i$, where $\alpha_i(k)(1 \leq i \leq m)$ are real numbers. Set $x(0) = 0$, then

$$
\begin{aligned}
x(k+1) &= \sum_{j=0}^{k} A^j B u(k-j) \\
&= \sum_{j=0}^{k} A^j B \sum_{i=1}^{m} \alpha_i(k-j) b_i \\
&= \sum_{i=1}^{m} \sum_{j=0}^{k} \alpha_i(k-j) A^j B b_i \\
\|x(k+1)\| &\leq \sum_{i=1}^{m} \left\| \sum_{j=0}^{k} \alpha_i(k-j) A^j \right\|_F \|B b_i\| \\
&\leq \sum_{i=1}^{m} \left\| \sum_{j=0}^{k} \alpha_i(k-j) A^j \right\|_F \|B\|_F \qquad (16)
\end{aligned}
$$

where $\|\cdot\|_F$ represents the Frobenius norm.

Suppose that A has s $(1 \leq s \leq n)$ different eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_s$, and the multiplicity of λ_h $(1 \leq h \leq s)$ is m_h. Then, $\sum_{h=1}^s m_h = n$. Assume that $J \in R^{n \times n}$ is the Jordan canonical form of A. There is a nonsingular matrix $P \in R^{n \times n}$, such that $A = PJP^{-1}$, $A^j = PJ^jP^{-1}$ $(0 \leq j \leq k)$. Notice that $J^j = \mathrm{diag}[J_1^j(\lambda_1), J_2^j(\lambda_2), \ldots, J_s^j(\lambda_s)]$, where

$$
J_h^j(\lambda_h) = \begin{bmatrix} \lambda_h^j & \mathcal{C}_j^1 \lambda_h^{j-1} & \cdots & \mathcal{C}_j^{m_h-1} \lambda_h^{j-m_h+1} \\ 0 & \lambda_h^j & \cdots & \vdots \\ \vdots & \vdots & \ddots & \mathcal{C}_j^1 \lambda_h^{j-1} \\ 0 & 0 & \cdots & \lambda_h^j \end{bmatrix}. \qquad (17)
$$

Then, any nonzero element of $J_h^j(\lambda_h)$ can be represented by $\mathcal{C}_j^l \lambda_h^{j-l}$, where $\mathcal{C}_j^l = j!/l!(j-l)!$ $(0 \leq l \leq m_h - 1$ and $l \leq j)$.

If $\|u(k)\| \leq L_u < \infty$ for all $k \geq 0$, then $|\alpha_i(k)| \leq L_u$ $(1 \leq i \leq m)$, and

$$
\begin{aligned}
\left| \sum_{j=0}^{k} \alpha_i(k-j) \mathcal{C}_j^l \lambda_h^{j-l} \right| &\leq \sum_{j=0}^{k} |\alpha_i(k-j)| \, \mathcal{C}_j^l \, |\lambda_h|^{j-l} \\
&\leq L_u \sum_{j=0}^{k} \mathcal{C}_j^l |\lambda_h|^{j-l}. \qquad (18)
\end{aligned}
$$

In addition,

$$
\lim_{j \to \infty} \frac{\mathcal{C}_j^l |\lambda_h|^{j-l}}{\mathcal{C}_{j-1}^l |\lambda_h|^{j-l-1}} = \lim_{j \to \infty} \frac{j}{j-l} |\lambda_h| = |\lambda_h|. \qquad (19)
$$

Because $|\lambda_h| < 1$ $(1 \leq h \leq s)$, by D'Alembert's criterion, the series $\sum_{j=0}^k \mathcal{C}_j^l |\lambda_h|^{j-l}$ converges as $k \to \infty$. From the above discussions, we can infer that

$$
\begin{aligned}
&\left\| \sum_{j=0}^{k} \alpha_i(k-j) J^j \right\|_F \\
&\quad = \left\| \sum_{j=0}^{k} \alpha_i(k-j) \mathrm{diag}\left[J_1^j(\lambda_1), \ldots, J_s^j(\lambda_s) \right] \right\|_F \\
&\quad = \left[\sum_{h=1}^{s} \sum_{t=1}^{m_h} \sum_{\tau=t}^{m_h} \left| \sum_{j=0}^{k} \alpha_i(k-j) \mathcal{C}_j^{\tau-t} \lambda_h^{j-\tau+t} \right|^2 \right]^{\frac{1}{2}} \\
&\quad \leq \left[\sum_{h=1}^{s} \sum_{t=1}^{m_h} \sum_{\tau=t}^{m_h} \left(L_u \sum_{j=0}^{k} \mathcal{C}_j^{\tau-t} |\lambda_h|^{j-\tau+t} \right)^2 \right]^{\frac{1}{2}} \\
&\quad \leq L_u \left[\sum_{h=1}^{s} \sum_{t=1}^{m_h} \sum_{\tau=t}^{m_h} \left(\sum_{j=0}^{\infty} \mathcal{C}_j^{\tau-t} |\lambda_h|^{j-\tau+t} \right)^2 \right]^{\frac{1}{2}}. \qquad (20)
\end{aligned}
$$

Let $M = [\sum_{h=1}^s \sum_{t=1}^{m_h} \sum_{\tau=t}^{m_h} (\sum_{j=0}^\infty \mathcal{C}_j^{\tau-t} |\lambda_h|^{j-\tau+t})^2]^{1/2}$. Since the series $\sum_{j=0}^\infty \mathcal{C}_j^{\tau-t} |\lambda_h|^{j-\tau+t}$ converges, $M < \infty$. Therefore, from (16) and (20), we can obtain

$$
\begin{aligned}
\|x(k+1)\| &\leq \sum_{i=1}^{m} \left\| \sum_{j=0}^{k} \alpha_i(k-j) A^j \right\|_F \|B\|_F \\
&= \sum_{i=1}^{m} \left\| \sum_{j=0}^{k} \alpha_i(k-j) P J^j P^{-1} \right\|_F \|B\|_F \\
&\leq \sum_{i=1}^{m} \|P\|_F \|P^{-1}\|_F L_u M \|B\|_F \\
&= mM \|P\|_F \|P^{-1}\|_F \|B\|_F L_u. \qquad (21)
\end{aligned}
$$

Let $L_x = mM\|P\|_F\|P^{-1}\|_F\|B\|_F L_u$. For any $\|u(k)\| \leq L_u$, $\|x(k)\| \leq L_x < \infty$. By Definition 1, system (15) is BIBS stable. In addition, $\|y(k)\| \leq \|C\| L_x$. Let $L_y = \|C\| L_x$. Then, for any $\|u(k)\| \leq L_u$, $\|y(k)\| \leq L_y < \infty$. By Definition 2, system (15) is also BIBO stable. ■

Theorems 3 and 4 show that if the equilibrium point $x = 0$ of the discrete-time linear system is uniformly asymptotically (asymptotically) stable, and the corresponding parameter matrices are bounded, then the system is both BIBS and BIBO stable. The contribution of Theorems 3 and 4 is that, they have no concern with the state controllability/observability. Because $\|A\| < 1$ implies $|\lambda_j| < 1$ $(1 \leq j \leq n)$ instead of vice versa, Theorem 4 is less conservative than Theorem 3.

B. Stability Analysis of a Class of Discrete-Time Nonlinear Systems

In the following, we study a class of discrete-time nonlinear systems:

$$\begin{cases} x(k+1) = f\left(k, x(k)\right) + G\left(k, x(k)\right) u(k) \\ y(k) = h\left(k, x(k), u(k)\right), \quad k \geq 0 \end{cases} \tag{22}$$

where f and G are both continuous in x, and bounded with respect to k, while $f(k,0) \equiv 0$ and $G(k,0) \equiv 0$; h is continuous in both x and u, and bounded with respect to k.

Theorem 5: For system (22), if

1) $\partial f(k,x(k))/\partial x(k)$ is continuous with respect to $x(k)$, and $\sup\limits_{k\geq 0, x(k)\in R^n} \{\|\partial f(k,x(k))/\partial x(k)\|\} = L_f < 1$;
2) $\sup\limits_{k\geq 0, x(k)\in R^n} \{\|G(k,x(k))\|\} = L_G < \infty$;

then the system is both BIBS and BIBO stable.

Proof: Let $A(k,x(k)) = \int_0^1 (\partial f(k,\theta x(k))/\partial x(k)) d\theta$. Since $f(k,0) \equiv 0$, by condition 1, $f(k,x(k)) = A(k,x(k))x(k)$ and $\|A(k,x(k))\| \leq \int_0^1 \|\partial f(k,\theta x(k))/\partial x(k)\| d\theta \leq L_f$. Then

$$\begin{aligned} x(k+1) &= A\left(k,x(k)\right)x(k) + G\left(k,x(k)\right)u(k) \\ &= G\left(k,x(k)\right)u(k) + \prod_{i=0}^{k} A\left(i,x(i)\right)x(0) \\ &\quad + \sum_{j=1}^{k}\prod_{i=j}^{k} A\left(i,x(i)\right)G\left((j-1),x(j-1)\right)u(j-1). \end{aligned} \tag{23}$$

Without loss of generality, set $x(0) = 0$. If $\|u(k)\| \leq L_u < \infty$ $(k \geq 0)$, then by conditions 1 and 2 and (23), we shall have

$$\begin{aligned} &\|x(k+1)\| \\ &\leq \|G\left(k,x(k)\right)\| \, \|u(k)\| + \sum_{j=1}^{k}\prod_{i=j}^{k} \|A\left(i,x(i)\right)\| \\ &\quad \times \|G\left((j-1),x(j-1)\right)\| \, \|u(j-1)\| \\ &\leq L_G L_u + L_G L_u \sum_{j=1}^{k} L_f^{k-j+1} \\ &\leq L_G L_u \sum_{j=0}^{\infty} L_f^j = \frac{L_G L_u}{1-L_f}. \end{aligned} \tag{24}$$

Let $L_x = L_G L_u/(1-L_f)$. For all $\|u(k)\| \leq L_u$, $\|x(k)\| \leq L_x < \infty$. By Definition 1, system (22) is BIBS stable.

Since h is continuous in both x and u, and bounded with respect to k, then the system is also BIBO stable. ■

IV. Stability Analysis of Discrete-Time Linear Time-Invariant Systems: Data-Driven Methods

In the previous sections, the stability analysis is based on the system models. However, sometimes the system model is unknown. To analyze the stabilities, model-based methods need to know the structure of the system model and identify the corresponding parameters, or establish the system model based on mechanisms. These model-based approaches usually cannot avoid the identification errors and the modeling errors. In this situation, data-driven methods, which can directly analyze the system stabilities using measured/observed state and/or I/O data, should have practical significance.

In this section, we will develop some data-driven methods to analyze the stabilities of discrete-time linear time-invariant systems in the following form:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k), \quad k \geq 0. \end{cases} \tag{25}$$

Lemma 2 ([20, Section 6.10]): The equilibrium point $x = 0$ of system (25) is asymptotically stable if and only if all eigenvalues of A are within the unit circle of the complex plane, i.e., if $\lambda_1, \lambda_2, \ldots, \lambda_n$ denote the eigenvalues of A, then $|\lambda_j| < 1$ $(1 \leq j \leq n)$.

Theorem 6: For system (25), select n linearly independent initial states $x^{[i]}(0) \neq 0$ $(1 \leq i \leq n)$, and set $u^{[i]}(k) \equiv 0$. Observe the trajectories $x^{[i]}(k)$ $(k \geq 0)$ starting from these initial states. If $x^{[i]}(k) \to 0$ as $k \to \infty$, then

1) the equilibrium point $x = 0$ of system (25) is asymptotically stable;
2) system (25) is both BIBS and BIBO stable.

Proof: Since the initial states $x^{[i]}(0) \neq 0$ $(1 \leq i \leq n)$ are linearly independent, then for any other $x(0) \neq 0$, there exist n real numbers a_i, which are not all zeros, such that $x(0) = \sum_{i=1}^n a_i x^{[i]}(0)$. Let $u(k) \equiv 0$, then we have

$$\begin{aligned} x(1) &= Ax(0) = \sum_{i=1}^{n} a_i A x^{[i]}(0) = \sum_{i=1}^{n} a_i x^{[i]}(1) \\ x(2) &= Ax(1) = \sum_{i=1}^{n} a_i A x^{[i]}(1) = \sum_{i=1}^{n} a_i x^{[i]}(2) \\ &\vdots \\ x(k) &= \sum_{i=1}^{n} a_i A x^{[i]}(k-1) = \sum_{i=1}^{n} a_i x^{[i]}(k). \end{aligned} \tag{26}$$

Since it is observed that $x^{[i]}(k) \to 0$ as $k \to \infty$, then $x(k) \to 0$ as $k \to \infty$. Considering the arbitrariness of $x(0)$, the equilibrium point $x = 0$ is asymptotically stable.

By Lemma 2 and Theorem 4, $|\lambda_i(A)| < 1$ $(1 \leq i \leq n)$ and system (25) is both BIBS and BIBO stable. ■

For system (25), it is either completely state controllable, or not. When system (25) is not completely state controllable, there exists a nonsingular linear transformation $T \in R^{n\times n}$, such that $x(k) = Tz(k)$, and (25) can be transformed into

$$\begin{bmatrix} z_1(k+1) \\ z_2(k+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(k) \tag{27}$$

where $A_{11} \in R^{l\times l}$ $(1 \leq l < n)$, $z_1(k)$ is controllable, and $z_2(k)$ is uncontrollable.

Lemma 3 ([16, Section 2.2]): Assume that system (25) is completely state controllable. Then, the system is BIBS stable if and only if the equilibrium point $x = 0$ of system (25) is asymptotically stable.

Lemma 4 ([16, Section 2.2]): System (27) is BIBS stable if and only if

1) $|\lambda_s(A_{11})| < 1$ $(1 \le s \le l)$;
2) $|\lambda_h(A_{22})| \le 1$ $(1 \le h \le n-l)$, and for any $|\lambda_h(A_{22})| = 1$, it only has the elementary divisor of order one.

With the above preparations, we now introduce the next data-driven stability analysis method.

Do m experiments on system (25). In each experiment, set

$$u^{[j]}(k) \equiv u^{[j]} = [0, \ldots, 0, 1, 0, \ldots, 0]^T \quad (1 \le j \le m) \quad (28)$$

whose jth element is 1 and all other elements are zeros. Set the initial states $x^{[j]}(0) = x_0 = 0$. Then, observe the trajectories $x^{[j]}(k) (k \ge 0)$ starting from x_0.

Theorem 7: For system (25), if $\|x^{[j]}(\infty)\| = M_j < \infty$ $(1 \le j \le m)$, then

1) the system is both BIBS and BIBO stable;
2) the equilibrium point $x = 0$ of system (25) is stable.

Proof: With $x_0 = 0$, $x^{[j]}(k+1) = \sum_{i=0}^{k} A^{k-i} B u^{[j]}$.

Suppose that the different eigenvalues of A are $\lambda_1, \ldots, \lambda_s$ $(1 \le s \le n)$, and the multiplicity of λ_h $(1 \le h \le s)$ is m_h. Then, $\sum_{h=1}^{s} m_h = n$. Assume that $J \in R^{n \times n}$ is the Jordan canonical form of A. There is a nonsingular matrix $P \in R^{n \times n}$, such that $A = PJP^{-1}$, $A^i = PJ^iP^{-1} (0 \le i \le k)$. Notice that $J^i = \mathrm{diag}[J_1^i(\lambda_1), J_2^i(\lambda_2), \ldots, J_s^i(\lambda_s)]$, where

$$J_h^i(\lambda_h) = \begin{bmatrix} \lambda_h^i & \mathcal{C}_i^1 \lambda_h^{i-1} & \cdots & \mathcal{C}_i^{m_h-1} \lambda_h^{i-m_h+1} \\ 0 & \lambda_h^i & \cdots & \vdots \\ \vdots & \vdots & \ddots & \mathcal{C}_i^1 \lambda_h^{i-1} \\ 0 & 0 & \cdots & \lambda_h^i \end{bmatrix}.$$

Any nonzero element of $J_h^i(\lambda_h)$ can be denoted by $\mathcal{C}_i^l \lambda_h^{i-l}$, where $\mathcal{C}_i^l = i!/l!(i-l)! (0 \le l \le m_h - 1$ and $l \le i)$.

Let $H^{[j]}(k-i) = J^{k-i} P^{-1} B u^{[j]}$. Then, $x^{[j]}(k+1) = P \sum_{i=0}^{k} H^{[j]}(k-i)$. Since $P^{-1}Bu^{[j]}$ is a constant vector, each element of $H^{[j]}(k-i)$ is in the form of $\beta_{k-i-l}^h \lambda_h^{k-i-l}$ $(0 \le l \le m_h - 1$ and $l \le k-i)$, where β_{k-i-l}^h is a real number.

Since $\|x^{[j]}(\infty)\| = M_j < \infty$ $(1 \le j \le m)$, by Abel's Theorem, the power series $\lim_{k\to\infty} \sum_{i=0}^{k} \beta_{k-i-l}^h \lambda_h^{k-i-l}$ absolutely converges, and $\lim_{k\to\infty} \sum_{i=0}^{k} H^{[j]}(k-i)$ absolutely converges too. This means $\lim_{k\to\infty} \sum_{i=0}^{k} H_p^{[j]}(k-i) L_u$ also absolutely converges, where $H_p^{[j]}(k-i)$ is the pth $(1 \le p \le n)$ element of $H^{[j]}(k-i)$, and $\|u(k)\| \le L_u < \infty$ $(k \ge 0)$.

Next, we study the case that the input is $u_\alpha^{[j]}(k) = \alpha_j(k) u^{[j]}$, where $\alpha_j(k)$ $(1 \le j \le m)$ are real numbers. The corresponding state is $x_\alpha^{[j]}(k+1) = P \sum_{i=0}^{k} H^{[j]}(k-i)\alpha_j(i)$. Let $\eta_\alpha^{[j]}(k+1) = \sum_{i=0}^{k} H^{[j]}(k-i)\alpha_j(i)$, we shall have $x_\alpha^{[j]}(k+1) = P\eta_\alpha^{[j]}(k+1)$. Let $x_{\alpha,p}^{[j]}(k)$, $\eta_{\alpha,p}^{[j]}(k)$ denote the pth $(1 \le p \le n)$ elements of $x_\alpha^{[j]}(k)$ and $\eta_\alpha^{[j]}(k)$, respectively. Then, $\eta_{\alpha,p}^{[j]}(k+1) = \sum_{i=0}^{k} H_p^{[j]}(k-i)\alpha_j(i)$.

Because $\lim_{k\to\infty} \sum_{i=0}^{k} H_p^{[j]}(k-i) L_u = \sum_{i=0}^{\infty} H_p^{[j]}(i) L_u$ absolutely converges,

$$\sum_{i=0}^{\infty} H_p^{[j]}(i) L_u = \sum_{i_1=0}^{\infty} H_p^{[j]}(i_1) L_u - \sum_{i_2=0}^{\infty} \left| H_p^{[j]}(i_2) \right| L_u$$

where $H_p^{[j]}(i_1) \ge 0$ and $H_p^{[j]}(i_2) < 0$, $\sum_{i_1=0}^{\infty} H_p^{[j]}(i_1) L_u$ and $\sum_{i_2=0}^{\infty} |H_p^{[j]}(i_2)| L_u$ both converge. Since $|\alpha_j(k)| \le L_u$

$$\left| H_p^{[j]}(i_1)\alpha_j(i_1) \right| \le H_p^{[j]}(i_1) L_u,$$
$$\left| H_p^{[j]}(i_2)\alpha_j(i_2) \right| \le \left| H_p^{[j]}(i_2) \right| L_u.$$

According to Weierstrass Theorem, $\sum_{i_1=0}^{\infty} H_p^{[j]}(i_1)\alpha_j(i_1)$ and $\sum_{i_2=0}^{\infty} H_p^{[j]}(i_2)\alpha_j(i_2)$ both converge. In summary, $\lim_{k\to\infty} \sum_{i=0}^{k} H_p^{[j]}(k-i)\alpha_j(i) = \lim_{k\to\infty} \eta_{\alpha,p}^{[j]}(k+1)$ $(1 \le p \le n)$ converges. Such that $\lim_{k\to\infty} \eta_\alpha^{[j]}(k)$ converges, which means $\lim_{k\to\infty} x_\alpha^{[j]}(k) = \lim_{k\to\infty} P\eta_\alpha^{[j]}(k)$ converges. This indicates that any bounded input $u_\alpha^{[j]}(k) = \alpha_j(k) u^{[j]}$ will result in a bounded state $x_\alpha^{[j]}(k)$.

More generally, any input $u(k)$ can be expressed as $u(k) = \sum_{j=1}^{m} \alpha_j(k) u^{[j]}$. Let $x(0) = 0$, then the solution of (25) is

$$x(k+1) = \sum_{i=0}^{k} A^{k-i} B u(i) = \sum_{j=1}^{m} \left[\sum_{i=0}^{k} A^{k-i} B u^{[j]} \alpha_j(i) \right] = \sum_{j=1}^{m} x_\alpha^{[j]}(k+1).$$

Since $x_\alpha^{[j]}(k) (1 \le j \le m)$ are bounded with $\|u(k)\| \le L_u (k \ge 0)$, any bounded input $u(k)$ will result in a bounded state $x(k)$. By Definition 1, the system is BIBS stable. Moreover, with constant matrix C, the system is also BIBO stable.

Because system (25) is either completely state controllable or not completely state controllable, we may discuss it in two cases. If system (25) is completely state controllable, by Lemma 3, the equilibrium point $x = 0$ of system (25) is asymptotically stable, and naturally it is stable.

If system (25) is not completely state controllable, it can be transformed into system (27). Because nonsingular linear transformation does not change the eigenvalues of a matrix, nor the BIBS stability of the system, according to Lemma 4, $|\lambda_p(A)| \le 1 (1 \le p \le n)$, and for any $|\lambda_p(A)| = 1$, it only has the elementary divisor of order one. In this case, it is easy to infer that the equilibrium point $x = 0$ is stable. In summary of the above two cases, the equilibrium point $x = 0$ of system (25) is stable, and so the proof is completed. ■

The contribution of Theorems 6 and 7 is that they do not need to know the state controllability as Lemmas 3 and 4 do. Another advantage is that, to analyze the system stabilities, our data-driven methods will do some simple experiments and observations. They do not identify A, B and C, so that identification errors can be avoided.

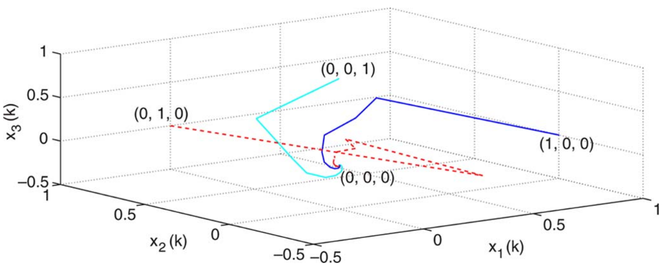

Fig. 1. State trajectories of $x^{[1]}(k)$, $x^{[2]}(k)$, and $x^{[3]}(k)$.

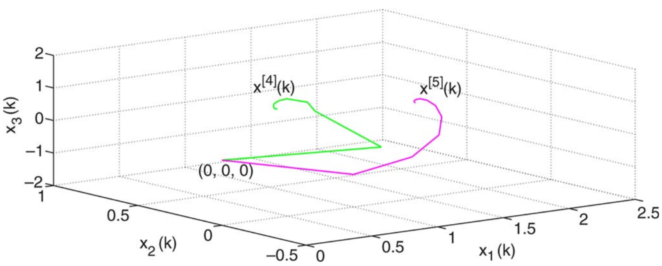

Fig. 2. State trajectories of $x^{[4]}(k)$ and $x^{[5]}(k)$.

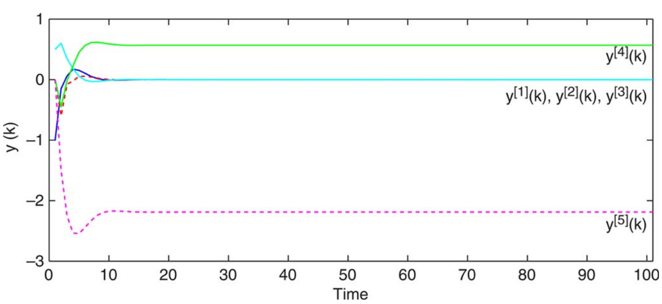

Fig. 3. Output trajectories of $y^{[1]}(k)$, $y^{[2]}(k)$, $y^{[3]}(k)$, $y^{[4]}(k)$, and $y^{[5]}(k)$.

Example 4: Here, we use the following system as an example to illustrate our data-driven stability analysis methods. Consider

$$\begin{cases} x(k+1)=\begin{bmatrix}0.4 & 0.5 & -0.3\\ 0.3 & -0.2 & 0.1\\ 0.5 & -0.2 & 0.6\end{bmatrix} x(k)+\begin{bmatrix}0.7 & 1\\ -0.4 & 0\\ 0.5 & -1\end{bmatrix} u(k),\\ y(k)=[-1,0,0.5]x(k), \quad k\geq 0. \end{cases} \tag{29}$$

Set $x^{[1]}(0)=[1,0,0]^T$, $x^{[2]}(0)=[0,1,0]^T$, and $x^{[3]}(0)=[0,0,1]^T$, which are linearly independent; and set the corresponding inputs as $u^{[i]}(k)\equiv[0,0]^T$ $(i=1,2,3)$. We then use Matlab programs to simulate the state trajectories starting from $x^{[1]}(0)$, $x^{[2]}(0)$ and $x^{[3]}(0)$, which are shown in Fig. 1. We can see that $x^{[1]}(k), x^{[2]}(k), x^{[3]}(k)\to 0$ as $k\to\infty$. By Theorem 6, the equilibrium point $x=0$ of system (29) is asymptotically stable.

Next, set $x^{[4]}(0)=x^{[5]}(0)=[0,0,0]^T$, and set the inputs as $u^{[4]}(k)\equiv[1,0]^T$, $u^{[5]}(k)\equiv[0,1]^T$. The state trajectories of $x^{[4]}(k)$ and $x^{[5]}(k)$ are shown in Fig. 2. We can see that $\|x^{[4]}(\infty)\|=1.6444$ and $\|x^{[5]}(\infty)\|=2.2145$. By Theorem 7, the equilibrium point $x=0$ is stable. By Theorems 6 and 7, system (29) is both BIBS and BIBO stable.

In this example, the BIBS stability can be confirmed by observing the state trajectories shown in Figs. 1 and 2, and the BIBO stability can be confirmed by observing the output trajectories shown in Fig. 3.

V. Conclusion

In this paper, we developed some model-based and data-driven methods to analyze the stability of the equilibrium point, the BIBS and the BIBO stabilities for linear systems and a class of nonlinear systems. In Section II, we studied continuous-time systems using model-based approaches; in Section III, we expanded the research of model-based stability analysis to discrete-time systems; then in Section IV, we presented some data-driven methods for the stability analysis of discrete-time linear time-invariant systems. Our results have the advantage that they do not have specific requirements on the state controllability or the state observability, nor use complicated mathematical tools such as the Dini derivatives and the $\mathcal{K}$-function. Thus, they are easy to understand and convenient to use in practice. These data-driven methods showed another advantage that they can analyze the stabilities by doing some simple experiments and observations, while not identifying the parameter matrices. They may definitely avoid the identification errors, and be more convenient in engineering applications. For the case that the input has random component, as long as the random component is bounded, our methods are still applicable.

For the future work, we will study the BIBS/BIBO stability problems of linear/nonlinear time-delay systems and linear/nonlinear periodic systems.

References

[1] Z. Li, Z. Fei, and H. Gao, "Stability and stabilisation of Markovian jump systems with time-varying delay: An input-output approach," *IET Control Theory Appl.*, vol. 6, no. 17, pp. 2601–2610, Nov. 2012.

[2] M. Lazar, W. P. M. H. Heemels, and A. R. Teel, "Further input-to-state stability subtleties for discrete-time systems," *IEEE Trans. Autom. Control*, vol. 58, no. 6, pp. 1609–1613, Jun. 2013.

[3] L. M. Silverman and B. D. O. Anderson, "Controllability, observability and stability of linear systems," *SIAM J. Control*, vol. 6, no. 1, pp. 121–130, 1968.

[4] Y. Yasuda, H. Ito, and H. Ishizu, "On the input-output stability of linear time-varying systems," *SIAM J. Appl. Math.*, vol. 38, no. 1, pp. 175–188, Feb. 1980.

[5] E. D. Sontag, "On the input-to-state stability property," *Eur. J. Control*, vol. 1, no. 1, pp. 24–36, 1995.

[6] L. Huang, *The Theoretical Basis of the Stability and Robustness*. Beijing, China: Science Press, 2003.

[7] P. P. Varaiya and R. Liu, "Bounded-input bounded-output stability of nonlinear time-varying differential systems," *SIAM J. Control*, vol. 4, no. 4, pp. 698–704, 1966.

[8] E. D. Sontag and Y. Wang, "On characterizations of the input-to-state stability property," *Syst. Control Lett.*, vol. 24, no. 5, pp. 351–359, Apr. 1995.

[9] N. Milivojevic, M. Krishnamurthy, Y. Gurkaynak, A. Sathyan, Y. J. Lee, and A. Emadi, "Stability analysis of FPGA-based control of brushless DC motors and generators using digital PWM technique," *IEEE Trans. Ind. Electron.*, vol. 59, no. 1, pp. 343–351, Jan. 2012.

[10] R. Turner, S. Walton, and R. Duke, "A case study on the application of the Nyquist stability criterion as applied to interconnected loads and sources on grids," *IEEE Trans. Ind. Electron.*, vol. 60, no. 7, pp. 2740–2749, Jul. 2013.

[11] S. Anand and B. G. Fernandes, "Reduced-order model and stability analysis of low-voltage DC microgrid," *IEEE Trans. Ind. Electron.*, vol. 60, no. 11, pp. 5040–5049, Nov. 2013.

[12] S. Yin, S. Ding, A. Haghani, H. Hao, and P. Zhang, "A comparison study of basic datadriven fault diagnosis and process monitoring methods on the benchmark Tennessee Eastman process," *J. Process Control*, vol. 22, no. 9, pp. 1567–1581, Oct. 2012.

[13] H. Gao, W. Zhan, H. R. Karimi, X. Yang, and S. Yin, “Allocation of actuators and sensors for coupled-adjacent-building vibration attenuation,” *IEEE Trans. Ind. Electron.*, vol. 60, no. 12, pp. 5792–5801, Dec. 2013.
[14] I. Sadinezhad and V. G. Agelidis, “Frequency adaptive least-squares-Kalman technique for real-time voltage envelope and flicker estimation,” *IEEE Trans. Ind. Electron.*, vol. 59, no. 8, pp. 3330–3341, Aug. 2012.
[15] I. Sadinezhad and V. G. Agelidis, “Real-time power system phasors and harmonics estimation using a new decoupled recursive-least-squares technique for DSP implementation,” *IEEE Trans. Ind. Electron.*, vol. 60, no. 6, pp. 2295–2308, Jun. 2013.
[16] C.-H. Lu, “Wavelet fuzzy neural networks for identification and predictive control of dynamic systems,” *IEEE Trans. Ind. Electron.*, vol. 58, no. 7, pp. 3046–3058, Jul. 2011.
[17] S.-N. Huang and K.-K. Tan, “Intelligent friction modeling and compensation using neural network approximations,” *IEEE Trans. Ind. Electron.*, vol. 59, no. 8, pp. 3342–3349, Aug. 2012.
[18] C.-H. Gao, L. Jian, and S.-H. Luo, “Modeling of the thermal state change of blast furnace hearth with support vector machines,” *IEEE Trans. Ind. Electron.*, vol. 59, no. 2, pp. 1134–1145, Feb. 2012.
[19] C.-F. Juang and G.-C. Chen, “A TS fuzzy system learned through a support vector machine in principal component space for real-time object detection,” *IEEE Trans. Ind. Electron.*, vol. 59, no. 8, pp. 3309–3320, Aug. 2012.
[20] P. J. Antsaklis and A. N. Michel, *Linear Systems*. Berlin, Germany: Birkhäuser, 2006.
[21] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Englewood Cliffs, NJ, USA: Prentice-Hall, 2001.

Zhuo Wang received the B.E. degree in automation from Beihang University, Beijing, China, and the Ph.D. degree in electrical and computer engineering from the University of Illinois at Chicago, Chicago, IL, USA.

He is currently a Post Doctoral Fellow in the Department of Electrical and Computer Engineering at the University of Alberta, Edmonton, AB, Canada. His research interests include data-driven control and system analysis, artificial neural networks, and event-triggered control.

Derong Liu (S'91-M'94-SM'96-F'05) received the Ph.D. degree in electrical engineering from the University of Notre Dame, Notre Dame, IN, USA, in 1994.

He was a Staff Fellow with the General Motors Research and Development Center, Warren, MI, USA, from 1993 to 1995. He was an Assistant Professor in the Department of Electrical and Computer Engineering, Stevens Institute of Technology, Hoboken, NJ, USA, from 1995 to 1999. He joined the University of Illinois at Chicago, Chicago, IL, USA in 1999, and became a Full Professor of electrical and computer engineering and of computer science in 2006. He has published 14 books (six research monographs and eight edited volumes).

Dr. Liu received the Faculty Early Career Development (CAREER) Award from the National Science Foundation (1999), the University Scholar Award from the University of Illinois (2006–2009), and the Overseas Outstanding Young Scholar Award from the National Natural Science Foundation of China (2008). He was selected for the “100 Talents Program” by the Chinese Academy of Sciences in 2008. Currently, he is the Editor-in-Chief of the IEEE TRANSACTIONS ON NEURAL NETWORKS AND LEARNING SYSTEMS.