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# Problems in Classical Electromagnetism

157 Exercises with Solutions

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# Preface

This book comprises 157 problems in classical electromagnetism, originating from the second-year course given by the authors to the undergraduate students of physics at the University of Pisa in the years from 2002 to 2017. Our course covers the basics of classical electromagnetism in a fairly complete way. In the first part, we present electrostatics and magnetostatics, electric currents, and magnetic induction, introducing the complete set of Maxwell's equations. The second part is devoted to the conservation properties of Maxwell's equations, the classical theory of radiation, the relativistic transformation of the fields, and the propagation of electromagnetic waves in matter or along transmission lines and waveguides. Typically, the total amount of lectures and exercise classes is about 90 and 45 hours, respectively. Most of the problems of this book were prepared for the intermediate and final examinations. In an examination test, a student is requested to solve two or three problems in 3 hours. The more complex problems are presented and discussed in detail during the classes.

The prerequisite for tackling these problems is having successfully passed the first year of undergraduate studies in physics, mathematics, or engineering, acquiring a good knowledge of elementary classical mechanics, linear algebra, differential calculus for functions of one variable. Obviously, classical electromagnetism requires differential calculus involving functions of more than one variable. This, in our undergraduate programme, is taught in parallel courses of the second year. Typically, however, the basic concepts needed to write down the Maxwell equations in differential form are introduced and discussed in our electromagnetism course, in the simplest possible way. Actually, while we do not require higher mathematical methods as a prerequisite, the electromagnetism course is probably the place where the students will encounter for the first time topics such as Fourier series and transform, at least in a heuristic way.

In our approach to teaching, we are convinced that checking the ability to solve a problem is the best way, or perhaps the only way, to verify the understanding of the theory. At the same time, the problems offer examples of the application of the theory to the real world. For this reason, we present each problem with a title that often highlights its connection to different areas of physics or technology,

so that the book is also a survey of historical discoveries and applications of classical electromagnetism. We tried in particular to pick examples from different contexts, such as, e.g., astrophysics or geophysics, and to include topics that, for some reason, seem not to be considered in several important textbooks, such as, e.g., radiation pressure or homopolar/unipolar motors and generators. We also included a few examples inspired by recent and modern research areas, including, e.g., optical metamaterials, plasmonics, superintense lasers. These latter topics show that nowadays, more than 150 years after Maxwell's equations, classical electromagnetism is still a vital area, which continuously needs to be understood and revisited in its deeper aspects. These certainly cannot be covered in detail in a second-year course, but a selection of examples (with the removal of unnecessary mathematical complexity) can serve as a useful introduction to them. In our problems, the students can have a first glance at "advanced" topics such as, e.g., the angular momentum of light, longitudinal waves and surface plasmons, the principles of laser cooling and of optomechanics, or the longstanding issue of radiation friction. At the same time, they can find the essential notions on, e.g., how an optical fiber works, where a plasma display gets its name from, or the principles of funny homemade electrical motors seen on YouTube.

The organization of our book is inspired by at least two sources, the book *Selected Problems in Theoretical Physics* (ETS Pisa, 1992, in Italian; World Scientific, 1994, in English) by our former teachers and colleagues A. Di Giacomo, G. Paffuti and P. Rossi, and the great archive of *Physics Examples and other Pedagogic Diversions* by Prof. K. McDonald (<http://puhep1.princeton.edu/%7Emcdonald/examples/>) which includes probably the widest source of advanced problems and examples in classical electromagnetism. Both these collections are aimed at graduate and postgraduate students, while our aim is to present a set of problems and examples with valuable physical contents, but accessible at the undergraduate level, although hopefully also a useful reference for the graduate student as well.

Because of our scientific background, our inspirations mostly come from the physics of condensed matter, materials and plasmas as well as from optics, atomic physics and laser-matter interactions. It can be argued that most of these subjects essentially require the knowledge of quantum mechanics. However, many phenomena and applications can be introduced within a classical framework, at least in a phenomenological way. In addition, since classical electromagnetism is the first field theory met by the students, the detailed study of its properties (with particular regard to conservation laws, symmetry relations and relativistic covariance) provides an important training for the study of wave mechanics and quantum field theories, that the students will encounter in their further years of physics study.

In our book (and in the preparation of tests and examinations as well), we tried to introduce as many original problems as possible, so that we believe that we have reached a substantial degree of novelty with respect to previous textbooks. Of course, the book also contains problems and examples which can be found in existing literature: this is unavoidable since many classical electromagnetism problems are, indeed, classics! In any case, the solutions constitute the most

important part of the book. We did our best to make the solutions as complete and detailed as possible, taking typical questions, doubts and possible mistakes by the students into account. When appropriate, alternative paths to the solutions are presented. To some extent, we tried not to bypass tricky concepts and ostensible ambiguities or “paradoxes” which, in classical electromagnetism, may appear more often than one would expect.

The sequence of Chapters 1–12 follows the typical order in which the contents are presented during the course, each chapter focusing on a well-defined topic. Chapter 13 contains a set of problems where concepts from different chapters are used, and may serve for a general review. To our knowledge, in some undergraduate programs the second-year physics may be “lighter” than at our department, i.e., mostly limited to the contents presented in the first six chapters of our book (i.e., up to Maxwell’s equations) plus some preliminary coverage of radiation (Chapter 10) and wave propagation (Chapter 11). Probably this would be the choice also for physics courses in the mathematics or engineering programs. In a physics program, most of the contents of our Chapters 7–12 might be possibly presented in a more advanced course at the third year, for which we believe our book can still be an appropriate tool.

Of course, this book of problems must be accompanied by a good textbook explaining the theory of the electromagnetic field in detail. In our course, in addition to lecture notes (unpublished so far), we mostly recommend the volume II of the celebrated *Feynman Lectures on Physics* and the volume 2 of the *Berkeley Physics Course* by E. M. Purcell. For some advanced topics, the famous *Classical Electrodynamics* by J. D. Jackson is also recommended, although most of this book is adequate for a higher course. The formulas and brief descriptions given at the beginning of the chapter are not meant at all to provide a complete survey of theoretical concepts, and should serve mostly as a quick reference for most important equations and to clarify the notation we use as well.

In the first Chapters 1–6, we use both the SI and Gaussian c.g.s. system of units. This choice was made because, while we are aware of the wide use of SI units, still we believe the Gaussian system to be the most appropriate for electromagnetism because of fundamental reasons, such as the appearance of a single fundamental constant (the speed of light  $c$ ) or the same physical dimensions for the electric and magnetic fields, which seems very appropriate when one realizes that such fields are parts of the same object, the electromagnetic field. As a compromise we used both units in that part of the book which would serve for a “lighter” and more general course as defined above, and switched definitely (except for a few problems) to Gaussian units in the “advanced” part of the book, i.e., Chapters 7–13. This choice is similar to what made in the 3rd Edition of the above-mentioned book by Jackson.

Problem-solving can be one of the most difficult tasks for the young physicist, but also one of the most rewarding and entertaining ones. This is even truer for the older physicist who tries to *create* a new problem, and admittedly we learned a lot from this activity which we pursued for 15 years (some say that the only person who certainly learns something in a course is the teacher!). Over this long time, occasionally we shared this effort and amusement with colleagues including in

particular Francesco Ceccherini, Fulvio Cornolti, Vanni Ghimenti, and Pietro Menotti, whom we wish to warmly acknowledge. We also thank Giuseppe Bertin for a critical reading of the manuscript. Our final thanks go to the students who did their best to solve these problems, contributing to an essential extent to improve them.

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# Chapter 1

## Basics of Electrostatics

**Topics.** The electric charge. The electric field. The superposition principle. Gauss's law. Symmetry considerations. The electric field of simple charge distributions (plane layer, straight wire, sphere). Point charges and Coulomb's law. The equations of electrostatics. Potential energy and electric potential. The equations of Poisson and Laplace. Electrostatic energy. Multipole expansions. The field of an electric dipole.

**Units.** An aim of this book is to provide formulas compatible with both SI (French: *Système International d'Unités*) units and Gaussian units in Chapters 1–6, while only Gaussian units will be used in Chapters 7–13. This is achieved by introducing some system-of-units-dependent constants.

The first constant we need is *Coulomb's constant*,  $k_e$ , which for instance appears in the expression for the force between two electric point charges  $q_1$  and  $q_2$  in vacuum, with position vectors  $\mathbf{r}_1$  and  $\mathbf{r}_2$ , respectively. The Coulomb force acting, for instance, on  $q_1$  is

$$\mathbf{f}_1 = k_e \frac{q_1 q_2}{|\mathbf{r}_1 - \mathbf{r}_2|^2} \hat{\mathbf{r}}_{12}, \quad (1.1)$$

where  $k_e$  is Coulomb's constant, dependent on the units used for force, electric charge, and length. The vector  $\mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2$  is the distance from  $q_2$  to  $q_1$ , pointing towards  $q_1$ , and  $\hat{\mathbf{r}}_{12}$  the corresponding unit vector. Coulomb's constant is

$$k_e = \begin{cases} \frac{1}{4\pi\epsilon_0} 8.987 \dots \times 10^9 \text{ N} \cdot \text{m}^2 \cdot \text{C}^{-2} \simeq 9 \times 10^9 \text{ m/F} & \text{SI} \\ 1 & \text{Gaussian.} \end{cases} \quad (1.2)$$

Constant  $\epsilon_0 \simeq 8.854\,187\,817\,620 \dots \times 10^{-12}$  F/m is the so-called “dielectric permittivity of free space”, and is defined by the formula

$$\varepsilon_0 = \frac{1}{\mu_0 c^2}, \quad (1.3)$$

where  $\mu_0 = 4\pi \times 10^{-7}$  H/m (by definition) is the vacuum magnetic permeability, and  $c$  is the speed of light in vacuum,  $c = 299\,792\,458$  m/s (this is a precise value, since the length of the meter is defined from this constant and the international standard for time).

**Basic equations** The two basic equations of this Chapter are, in differential and integral form,

$$\nabla \cdot \mathbf{E} = 4\pi k_e \varrho, \quad \oint_S \mathbf{E} \cdot d\mathbf{S} = 4\pi k_e \int_V \varrho d^3 r \quad (1.4)$$

$$\nabla \times \mathbf{E} = 0, \quad \oint_C \mathbf{E} \cdot d\ell = 0. \quad (1.5)$$

where  $\mathbf{E}(\mathbf{r}, t)$  is the electric field, and  $\varrho(\mathbf{r}, t)$  is the volume charge density, at a point of location vector  $\mathbf{r}$  at time  $t$ . The infinitesimal volume element is  $d^3 r = dx dy dz$ . In (1.4) the functions to be integrated are evaluated over an arbitrary volume  $V$ , or over the surface  $S$  enclosing the volume  $V$ . The function to be integrated in (1.5) is evaluated over an arbitrary closed path  $C$ . Since  $\nabla \times \mathbf{E} = 0$ , it is possible to define an electric potential  $\varphi = \varphi(\mathbf{r})$  such that

$$\mathbf{E} = -\nabla \varphi. \quad (1.6)$$

The general expression of the potential generated by a given charge distribution  $\varrho(\mathbf{r})$  is

$$\varphi(\mathbf{r}) = k_e \int_V \frac{\varrho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3 r'. \quad (1.7)$$

The force acting on a volume charge distribution  $\varrho(\mathbf{r})$  is

$$\mathbf{f} = \int_V \varrho(\mathbf{r}') \mathbf{E}(\mathbf{r}') d^3 r'. \quad (1.8)$$

As a consequence, the force acting on a point charge  $q$  located at  $\mathbf{r}$  (which corresponds to a charge distribution  $\varrho(\mathbf{r}') = q\delta(\mathbf{r} - \mathbf{r}')$ , with  $\delta(\mathbf{r})$  the Dirac-delta function) is

$$\mathbf{f} = q \mathbf{E}(\mathbf{r}). \quad (1.9)$$

The electrostatic energy  $U_{es}$  associated with a given distribution of electric charges and fields is given by the following expressions

$$U_{es} = \int_V \frac{\mathbf{E}^2}{8\pi k_e} d^3 r. \quad (1.10)$$

$$U_{\text{es}} = \frac{1}{2} \int_V \varrho \varphi d^3 r, \quad (1.11)$$

Equations (1.10–1.11) are valid provided that the volume integrals are finite and that all involved quantities are well defined.

The multipole expansion allows us to obtain simple expressions for the leading terms of the potential and field generated by a charge distribution at a distance much larger than its extension. In the following we will need only the expansion up to the dipole term,

$$\varphi(\mathbf{r}) \approx k_e \left( \frac{Q}{r} + \frac{\mathbf{p} \cdot \mathbf{r}}{r^3} + \dots \right), \quad (1.12)$$

where  $Q$  is the total charge of the distribution and the electric dipole moment is

$$\mathbf{p} \equiv \int_V \mathbf{r}' \rho(\mathbf{r}') d^3 \mathbf{r}'. \quad (1.13)$$

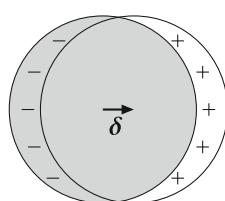
If  $Q = 0$ , then  $\mathbf{p}$  is independent on the choice of the origin of the reference frame. The field generated by a dipolar distribution centered at  $\mathbf{r} = 0$  is

$$\mathbf{E} = k_e \frac{3\hat{\mathbf{r}}(\mathbf{p} \cdot \hat{\mathbf{r}}) - \mathbf{p}}{r^3}. \quad (1.14)$$

We will briefly refer to a localized charge distribution having a dipole moment as “an electric dipole” (the simplest case being two opposite point charges  $\pm q$  with a spatial separation  $\delta$ , so that  $\mathbf{p} = q\delta$ ). A dipole placed in an external field  $\mathbf{E}_{\text{ext}}$  has a potential energy

$$U_p = -\mathbf{p} \cdot \mathbf{E}_{\text{ext}}. \quad (1.15)$$

## 1.1 Overlapping Charged Spheres



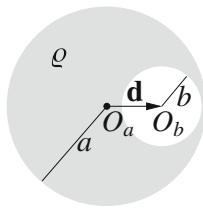
**Fig. 1.1**

We assume that a neutral sphere of radius  $R$  can be regarded as the superposition of two “rigid” spheres: one of uniform positive charge density  $+q_0$ , comprising the nuclei of the atoms, and a second sphere of the same radius, but of negative uniform charge density  $-q_0$ , comprising the electrons. We further assume that it is possible to shift the two spheres relative to each other by a quantity  $\delta$ , as shown in Fig. 1.1, without perturbing the internal structure of either sphere.

Find the electrostatic field generated by the global charge distribution

- a) in the “inner” region, where the two spheres overlap,
- b) in the “outer” region, i.e., outside both spheres, discussing the limit of small displacements  $\delta \ll R$ .

## 1.2 Charged Sphere with Internal Spherical Cavity



**Fig. 1.2**

A sphere of radius  $a$  has uniform charge density  $\varrho$  over all its volume, excluding a spherical cavity of radius  $b < a$ , where  $\varrho = 0$ . The center of the cavity,  $O_b$  is located at a distance  $\mathbf{d}$ , with  $|\mathbf{d}| < (a - b)$ , from the center of the sphere,  $O_a$ . The mass distribution of the sphere is proportional to its charge distribution.

- a) Find the electric field inside the cavity.  
Now we apply an external, uniform electric field  $\mathbf{E}_0$ .  
Find
- b) the force on the sphere,
- c) the torque with respect to the center of the sphere, and the torque with respect to the center of mass.

## 1.3 Energy of a Charged Sphere

A total charge  $Q$  is distributed uniformly over the volume of a sphere of radius  $R$ . Evaluate the electrostatic energy of this charge configuration in the following three alternative ways:

- a) Evaluate the work needed to assemble the charged sphere by moving successive infinitesimals shells of charge from infinity to their final location.
- b) Evaluate the volume integral of  $u_E = |\mathbf{E}|^2 / (8\pi k_e)$  where  $\mathbf{E}$  is the electric field [Eq. (1.10)].
- c) Evaluate the volume integral of  $\varrho\phi/2$  where  $\varrho$  is the charge density and  $\phi$  is the electrostatic potential [Eq. (1.11)]. Discuss the differences with the calculation made in b).

## 1.4 Plasma Oscillations

A square metal slab of side  $L$  has thickness  $h$ , with  $h \ll L$ . The conduction-electron and ion densities in the slab are  $n_e$  and  $n_i = n_e/Z$ , respectively,  $Z$  being the ion charge.

An external electric field shifts all conduction electrons by the same amount  $\delta$ , such that  $|\delta| \ll h$ , perpendicularly to the base of the slab. We assume that both  $n_e$  and  $n_i$  are constant, that the ion lattice is unperturbed by the external field, and that boundary effects are negligible.

a) Evaluate the electrostatic field generated by the displacement of the electrons.

b) Evaluate the electrostatic energy of the system.

Now the external field is removed, and the “electron slab” starts oscillating around its equilibrium position.

c) Find the oscillation frequency, at the small displacement limit ( $\delta \ll h$ ).

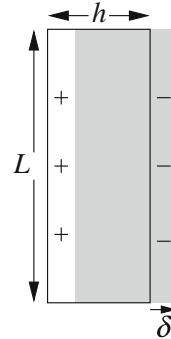


Fig. 1.3

## 1.5 Mie Oscillations

Now, instead of a the metal slab of Problem 1.4, consider a metal sphere of radius  $R$ . Initially, all the conduction electrons ( $n_e$  per unit volume) are displaced by  $-\delta$  (with  $\delta \ll R$ ) by an external electric field, analogously to Problem 1.1.

a) At time  $t = 0$  the external field is suddenly removed. Describe the subsequent motion of the conduction electrons under the action of the self-consistent electrostatic field, neglecting the boundary effects on the electrons close to the surface of the sphere.

b) At the limit  $\delta \rightarrow 0$  (but assuming  $en_e\delta = \sigma_0$  to remain finite, i.e., the charge distribution is a surface density), find the electrostatic energy of the sphere as a function of  $\delta$  and use the result to discuss the electron motion as in point a).

## 1.6 Coulomb explosions

At  $t = 0$  we have a spherical cloud of radius  $R$  and total charge  $Q$ , comprising  $N$  point-like particles. Each particle has charge  $q = Q/N$  and mass  $m$ . The particle density is uniform, and all particles are at rest.

- a)** Evaluate the electrostatic potential energy of a charge located at a distance  $r < R$  from the center at  $t = 0$ .

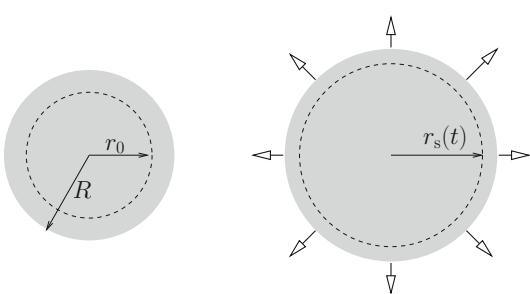


Fig. 1.4

$r_0 < r_s < r_0 + dr$ , with  $r_0 + dr < R$ , at  $t = 0$ . Show that the equation of motion of the layer is

$$m \frac{d^2 r_s}{dt^2} = k_e \frac{qQ}{r_s^2} \left( \frac{r_0}{R} \right)^3 \quad (1.16)$$

- c)** Find the initial position of the particles that acquire the maximum kinetic energy during the cloud expansion, and determinate the value of such maximum energy.
- d)** Find the energy spectrum, i.e., the distribution of the particles as a function of their final kinetic energy. Compare the total kinetic energy with the potential energy initially stored in the electrostatic field.
- e)** Show that the particle density remains spatially uniform during the expansion.

## 1.7 Plane and Cylindrical Coulomb Explosions

Particles of identical mass  $m$  and charge  $q$  are distributed with zero initial velocity and uniform density  $n_0$  in the infinite slab  $|x| < a/2$  at  $t = 0$ . For  $t > 0$  the slab expands because of the electrostatic repulsion between the pairs of particles.

- a)** Find the equation of motion for the particles, its solution, and the kinetic energy acquired by the particles.
- b)** Consider the analogous problem of the explosion of a uniform distribution having cylindrical symmetry.

- b)** Due to the Coulomb repulsion, the cloud begins to expand radially, keeping its spherical symmetry. Assume that the particles do not overtake one another, i.e., that if two particles were initially located at  $r_1(0)$  and  $r_2(0)$ , with  $r_2(0) > r_1(0)$ , then  $r_2(t) > r_1(t)$  at any subsequent time  $t > 0$ . Consider the particles located in the infinitesimal spherical shell

## 1.8 Collision of two Charged Spheres

Two rigid spheres have the same radius  $R$  and the same mass  $M$ , and opposite charges  $\pm Q$ . Both charges are uniformly and rigidly distributed over the volumes of the two spheres. The two spheres are initially at rest, at a distance  $x_0 \gg R$  between their centers, such that their interaction energy is negligible compared to the sum of their “internal” (construction) energies.

- a)** Evaluate the initial energy of the system.

The two spheres, having opposite charges, attract each other, and start moving at  $t = 0$ .

- b)** Evaluate the velocity of the spheres when they touch each other (i.e. when the distance between their centers is  $x = 2R$ ).

- c)** Assume that, after touching, the two spheres penetrate each other without friction. Evaluate the velocity of the spheres when the two centers overlap ( $x = 0$ ).

## 1.9 Oscillations in a Positively Charged Conducting Sphere

An electrically neutral metal sphere of radius  $a$  contains  $N$  conduction electrons. A fraction  $f$  of the conduction electrons ( $0 < f < 1$ ) is removed from the sphere, and the remaining  $(1 - f)N$  conduction electrons redistribute themselves to an equilibrium configurations, while the  $N$  lattice ions remain fixed.

- a)** Evaluate the conduction-electron density and the radius of their distribution in the sphere.

Now the conduction-electron sphere is rigidly displaced by  $\delta$  relatively to the ion lattice, with  $|\delta|$  small enough for the conduction-electron sphere to remain inside the ion sphere.

- b)** Evaluate the electric field inside the conduction-electron sphere.

- c)** Evaluate the oscillation frequency of the conduction-electron sphere when it is released.

## 1.10 Interaction between a Point Charge and an Electric Dipole

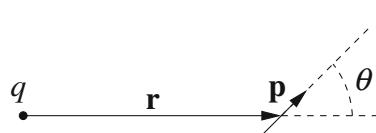


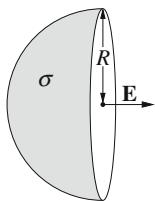
Fig. 1.5

An electric dipole  $\mathbf{p}$  is located at a distance  $\mathbf{r}$  from a point charge  $q$ , as in Fig. 1.5. The angle between  $\mathbf{p}$  and  $\mathbf{r}$  is  $\theta$ .

- a)** Evaluate the electrostatic force on the dipole.

- b)** Evaluate the torque acting on the dipole.

## 1.11 Electric Field of a Charged Hemispherical Surface



A hemispherical surface of radius  $R$  is uniformly charged with surface charge density  $\sigma$ . Evaluate the electric field and potential at the center of curvature (hint: start from the electric field of a uniformly charged ring along its axis).

Fig. 1.6

# Chapter 2

## Electrostatics of Conductors

**Topics.** The electrostatic potential in vacuum. The uniqueness theorem for Poisson's equation. Laplace's equation, harmonic functions and their properties. Boundary conditions at the surfaces of conductors: Dirichlet, Neumann and mixed boundary conditions. The capacity of a conductor. Plane, cylindrical and spherical capacitors. Electrostatic field and electrostatic pressure at the surface of a conductor. The method of image charges: point charges in front of plane and spherical conductors.

**Basic equations** Poisson's equation is

$$\nabla^2 \varphi(\mathbf{r}) = -4\pi k_e \varrho(\mathbf{r}), \quad (2.1)$$

where  $\varphi(\mathbf{r})$  is the electrostatic potential, and  $\varrho(\mathbf{r})$  is the electric charge density, at the point of vector position  $\mathbf{r}$ . The solution of Poisson's equation is unique if one of the following boundary conditions is true

1. **Dirichlet boundary condition:**  $\varphi$  is known and well defined on all of the boundary surfaces.
2. **Neumann boundary condition:**  $\mathbf{E} = -\nabla \varphi$  is known and well defined on all of the boundary surfaces.
3. **Modified Neumann boundary condition** (also called Robin boundary condition): conditions where boundaries are specified as conductors with known charges.
4. **Mixed boundary conditions:** a combination of Dirichlet, Neumann, and modified Neumann boundary conditions:

Laplace's equation is the special case of Poisson's equation

$$\nabla^2 \varphi(\mathbf{r}) = 0, \quad (2.2)$$

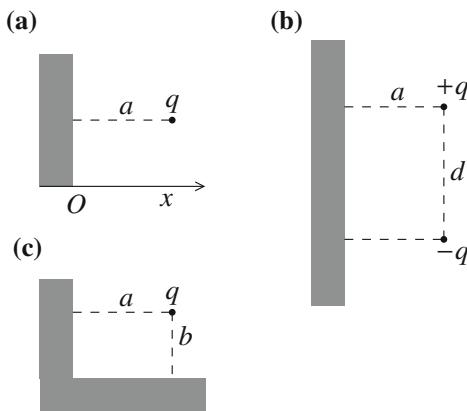
which is valid in vacuum.

## 2.1 Metal Sphere in an External Field

A metal sphere of radius  $R$  consists of a “rigid” lattice of ions, each of charge  $+Ze$ , and valence electrons each of charge  $-e$ . We denote by  $n_i$  the ion density, and by  $n_e$  the electron density. The net charge of the sphere is zero, therefore  $n_e = Zn_i$ . The sphere is located in an external, constant, and uniform electric field  $\mathbf{E}_0$ . The field causes a displacement  $\delta$  of the “electron sea” with respect to the ion lattice, so that the total field inside the sphere,  $\mathbf{E}$ , is zero. Using Problem 1.1 as a model, evaluate

- the displacement  $\delta$ , giving a numerical estimate for  $E_0 = 10^3$  V/m;
- the field generated by the sphere at its exterior, as a function of  $\mathbf{E}_0$ ;
- the surface charge density on the sphere.

## 2.2 Electrostatic Energy with Image Charges



**Fig. 2.1**

distance  $a$  from an infinite conducting plane.

- A charge  $q$  is at distances  $a$  and  $b$ , respectively, from two infinite conducting half planes forming a right dihedral angle.

Consider the configurations of point charges in the presence of conducting planes shown in Fig. 2.1. For each case, find the solution for the electrostatic potential over the whole space and evaluate the electrostatic energy of the system. Use the method of image charges.

- A charge  $q$  is located at a distance  $a$  from an infinite conducting plane.
- Two opposite charges  $+q$  and  $-q$  are at a distance  $d$  from each other, both at the same

## 2.3 Fields Generated by Surface Charge Densities

Consider the case a) of Problem 2.2: we have a point charge  $q$  at a distance  $a$  from an infinite conducting plane.

- Evaluate the surface charge density  $\sigma$ , and the total induced charge  $q_{\text{ind}}$ , on the plane.

**b)** Now assume to have a *nonconducting* plane with the same surface charge distribution as in point **a**). Find the electric field in the whole space.

**c)** A non conducting spherical surface of radius  $a$  has the same charge distribution as the conducting sphere of Problem 2.4. Evaluate the electric field in the whole space.

## 2.4 A Point Charge in Front of a Conducting Sphere

A point charge  $q$  is located at a distance  $d$  from the center of a conducting grounded sphere of radius  $a < d$ . Evaluate

- a)** the electric potential  $\varphi$  over the whole space;
- b)** the force on the point charge;
- c)** the electrostatic energy of the system.

Answer the above questions also in the case of an isolated, uncharged conducting sphere.

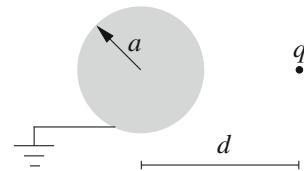


Fig. 2.2

## 2.5 Dipoles and Spheres

An electric dipole  $\mathbf{p}$  is located at a distance  $d$  from the center of a conducting sphere of radius  $a$ . Evaluate the electrostatic potential  $\varphi$  over the whole space assuming that

- a)**  $\mathbf{p}$  is perpendicular to the direction from  $\mathbf{p}$  to the center of the sphere,
- b)**  $\mathbf{p}$  is directed towards the center of the sphere.
- c)**  $\mathbf{p}$  forms an arbitrary angle  $\theta$  with respect to the straight line passing through the center of the sphere and the dipole location.

In all three cases consider the two possibilities of i) a grounded sphere, and ii) an electrically uncharged isolated sphere.

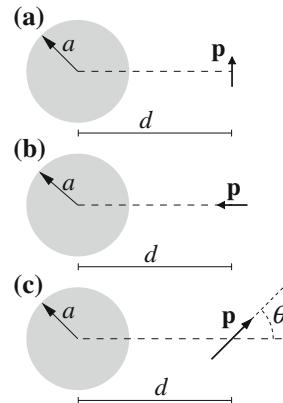


Fig. 2.3

## 2.6 Coulomb's Experiment

Coulomb, in his original experiment, measured the force between two charged metal spheres, rather than the force between two “point charges”. We know that the field of a sphere whose surface is uniformly charged equals the field of a point charge,

and that the force between two charge distributions, each of spherical symmetry, equals the force between two point charges

$$\mathbf{F} = k_e \frac{q_1 q_2}{r^2} \hat{\mathbf{r}}, \quad (2.3)$$

where  $q_1$  and  $q_2$  are the charges on the spheres, and  $\mathbf{r} = r \hat{\mathbf{r}}$  is the distance between the two centers of symmetry. But we also know that electric induction modifies the surface charge densities of conductors, so that a correction to (2.3) is needed. We expect the induction effects to be important if the radius  $a$  of the spheres is not negligibly small with respect to  $r$ .



Fig. 2.4

**a)** Using the method of image charges, find the solution for the electrical potential outside the spheres as a series expansion, and identify the expansion parameter. For simplicity, assume the spheres to be identical and to have the same charge  $Q$ , as in the figure.

**b)** Evaluate the lowest order correction to the force between the spheres with respect to Coulomb's law (2.3).

## 2.7 A Solution Looking for a Problem

An electric dipole  $\mathbf{p}$  is located at the origin of a Cartesian frame, parallel to the  $z$  axis, in the presence of a uniform electric field  $\mathbf{E}$ , also parallel to the  $z$  axis.

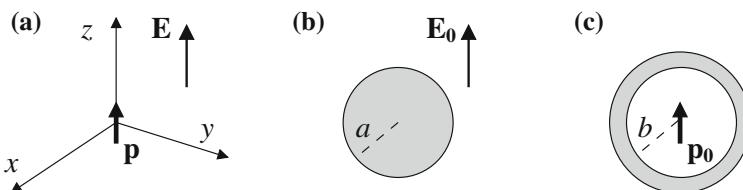


Fig. 2.5

**a)** Find the total electrostatic potential  $\varphi = \varphi(\mathbf{r})$ , with the condition  $\varphi = 0$  on the  $xy$  plane. Show that, in addition to the  $xy$  plane, there is another equipotential surface with  $\varphi = 0$ , that this surface is spherical, and calculate its radius  $R$ .

Now use the result from point **a)** to find the electric potential in the whole space for the following problems:

- b) A conducting sphere of radius  $a$  is placed in a uniform electric field  $\mathbf{E}_0$ ;
- c) a dipole  $\mathbf{p}_0$  is placed in the center of a conducting spherical shell of radius  $b$ .
- d) Find the solution to problem c) using the method of image charges.

## 2.8 Electrically Connected Spheres

Two conducting spheres of radii  $a$  and  $b < a$ , respectively, are connected by a thin metal wire of negligible capacitance. The centers of the two spheres are at a distance  $d \gg a > b$  from each other. A total net charge  $Q$  is located on the system.

Evaluate to zeroth order approximation, neglecting the induction effects on the surfaces of the two spheres,

- a) how the charge  $Q$  is partitioned between the two spheres,

- b) the value  $V$  of the electrostatic potential of the system (assuming zero potential at infinity) and the capacitance  $C = Q/V$ ,

- c) the electric field at the surface of each sphere, comparing the intensities and discussing the limit  $b \rightarrow 0$ .

- d) Now take the electrostatic induction effects into account and improve the preceding results to the first order in  $a/d$  and  $b/d$ .

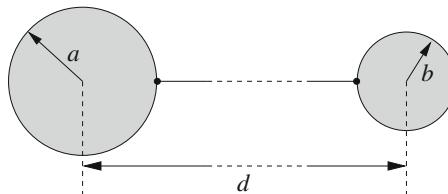


Fig. 2.6

## 2.9 A Charge Inside a Conducting Shell

A point charge  $q$  is located at a distance  $d$  from the center of a spherical conducting shell of internal radius  $R > d$ , and external radius  $R' > R$ . The shell is grounded, so that its electric potential is zero.

- a) Find the electric potential and the electric field in the whole space.
- b) Evaluate the force acting on the charge.
- c) Show that the total charge induced on the surface of the internal sphere is  $-q$ .
- d) How does the answer to a) change if the shell is not grounded, but electrically isolated with a total charge equal to zero?

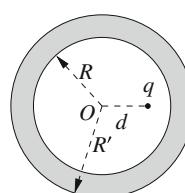


Fig. 2.7

## 2.10 A Charged Wire in Front of a Cylindrical Conductor

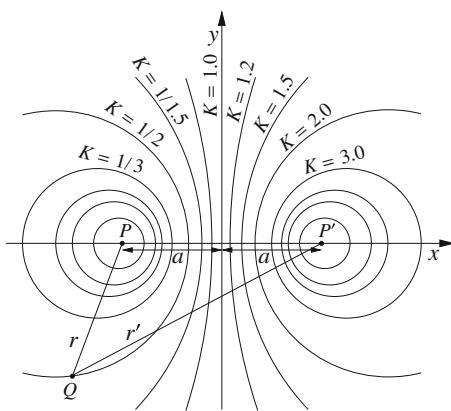


Fig. 2.8

$(-a, 0)$  and  $(+a, 0)$ , respectively. Use the geometrical result of point **a**) to show that the equipotential surfaces of the electrostatic field generated by the two wires are infinite cylindrical surfaces whose intersections with the  $xy$  plane are the circumferences shown in Fig. 2.8.

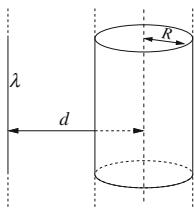


Fig. 2.9

We have two fixed points  $P \equiv (-a, 0)$  and  $P' \equiv (+a, 0)$  on the  $xy$  plane, and a third, generic point  $Q \equiv (x, y)$ . Let  $r = \overline{QP}$  and  $r' = \overline{QP'}$  be the distances of  $Q$  from  $P$  and  $P'$ , respectively.

**a)** Show that the family of curves defined by the equation  $r/r' = K$ , with  $K > 0$  a constant, is the family of circumferences drawn in Fig. 2.8.

**b)** Now consider the electrostatic field generated by two straight infinite, parallel wires of linear charge densities  $\lambda$  and  $-\lambda$ , respectively. We choose a Cartesian reference frame such that the  $z$  axis is parallel to the wires, and the two wires intersect the  $xy$  plane at

**c)** Use the results of points **a)** and **b)** to solve the following problem by the method of image charges. An infinite straight wire of linear charge density  $\lambda$  is located in front of an infinite conducting cylindrical surface of radius  $R$ . The wire is parallel to the axis of the cylinder, and the distance between the wire and axis of the cylinder is  $d$ , with  $d > R$ , as shown in Fig. 2.9. Find the electrostatic potential in the whole space.

## 2.11 Hemispherical Conducting Surfaces

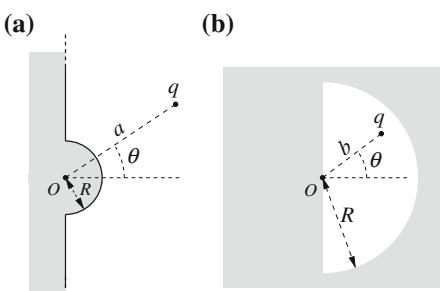


Fig. 2.10

Find the configurations of image charges that solve the problems represented in Fig. 2.10a, 2.10b, and the corresponding induced-charge distributions, remembering that the electric potential of an infinite conductor is zero.

**a)** The plane infinite surface of a conductor has a hemispherical boss of

radius  $R$ , with curvature center in  $O$ . A point charge  $q$  is located at a distance  $a > R$  from  $O$ , the line segment from  $O$  to  $q$  forms an angle  $\theta$  with the symmetry axis of the problem.

- b)** An infinite conductor has a hemispherical cavity of radius  $R$ . A point charge  $q$  is located inside the cavity, at a distance  $b < R$  from  $O$ . Again, the line segment from  $O$  to  $q$  forms an angle  $\theta$  with the symmetry axis of the problem.

## 2.12 The Force Between the Plates of a Capacitor

The plates of a flat, parallel-plate capacitor have surface  $S$  and separation  $h \ll \sqrt{S}$ . Find the force between the plates, both for an isolated capacitor (as a function of the constant charge  $Q$ ), and for a capacitor connected to an ideal voltage source (as a function of the constant voltage  $V$ ). In both cases, use *two* different methods, i.e., calculate the force

- a)** from the electrostatic pressure on the surface of the plates,
- b)** from the expression of the energy as a function of the distance between the plates.

## 2.13 Electrostatic Pressure on a Conducting Sphere

A conducting sphere of radius  $a$  has a net charge  $Q$  and it is electrically isolated. Find the electrostatic pressure at the surface of the sphere

- a)** directly, from the surface charge density and the electric field on the sphere,
- b)** by evaluating variation of the electrostatic energy with respect to  $a$ .
- c)** Now calculate again the pressure on the sphere, assuming that the sphere is not isolated, but connected to an ideal voltage source, keeping the sphere at the constant potential  $V$  with respect to infinity.

## 2.14 Conducting Prolate Ellipsoid

- a)** Show that the equipotential surfaces generated by a uniformly charged line segment are prolate ellipsoids of revolution, with the focal points coinciding with the end points of the segment.
- b)** Evaluate the electric field generated by a conducting prolate ellipsoid of revolution of major axis  $2a$  and minor axis  $2b$ , carrying a charge  $Q$ . Evaluate the electric capacity of the ellipsoid, and the capacity of a confocal ellipsoidal capacitor.
- c)** Use the above results to evaluate an approximation for the capacity of a straight conducting cylindrical wire of length  $h$ , and diameter  $2b$ .

# Chapter 3

## Electrostatics of Dielectric Media

**Topics.** Polarization charges. Dielectrics. Permanent and induced polarization. The auxiliary vector  $\mathbf{D}$ . Boundary conditions at the surface of dielectrics. Relative dielectric permittivity  $\epsilon_r$ .

**Basic equations** We  $\mathbf{P}$  denote the *electric polarization* (electric dipole moment per unit volume) of a material. Some special materials have a permanent non-zero electric polarization, but in most cases a polarization appears only in the presence of an electric field  $\mathbf{E}$ . We consider linear dielectric materials, for which  $\mathbf{P}$  is parallel and proportional to  $\mathbf{E}$ , thus

$$\mathbf{P} = \begin{cases} \epsilon_0 \chi \mathbf{E}, & \text{where } \chi = \epsilon_r - 1, \quad \text{SI} \\ \chi \mathbf{E}, & \text{where } \chi = \frac{\epsilon_r - 1}{4\pi}, \quad \text{Gaussian,} \end{cases} \quad (3.1)$$

where  $\chi$  is called the *electric susceptibility* and  $\epsilon_r$  the *relative permittivity* of the material.<sup>1</sup> Notice that  $\epsilon_r$  is a dimensionless quantity with the same numerical value both in SI and Gaussian units.

We shall denote by  $\varrho_b$  and  $\varrho_f$  the volume densities of bound electric charge and of free electric charge, respectively, and by  $\sigma_b$  and  $\sigma_f$  the surface densities of bound charge. Quantities  $\varrho_b$  and  $\sigma_b$  are related to the electric polarization  $\mathbf{P}$  by

$$\varrho_b = -\nabla \cdot \mathbf{P}, \quad \text{and} \quad \sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}, \quad (3.2)$$

---

<sup>1</sup>In anisotropic media (such as non-cubic crystals)  $\mathbf{P}$  and  $\mathbf{E}$  may be not parallel to each other, in this case  $\chi$  and  $\epsilon_r$  are actually second rank tensors. Here, however, we are interested only in isotropic and homogeneous media, for which  $\chi$  and  $\epsilon_r$  are scalar quantities.

where  $\hat{\mathbf{n}}$  is the unit vector pointing outwards from the boundary surface of the polarized material. We may thus rewrite (1.4) as

$$\nabla \cdot \mathbf{E} = \begin{cases} \frac{\varrho_f + \varrho_b}{\varepsilon_0} = \frac{\varrho_f}{\varepsilon_0} - \frac{1}{\varepsilon_0} \nabla \cdot \mathbf{P}, & \text{SI} \\ 4\pi(\varrho_f + \varrho_b) = 4\pi\varrho_f - 4\pi \nabla \cdot \mathbf{P}, & \text{Gaussian.} \end{cases} \quad (3.3)$$

We can also introduce the auxiliary vector  $\mathbf{D}$  (also called *electrical displacement*) defined as

$$\mathbf{D} = \begin{cases} \varepsilon_0 \mathbf{E} + \mathbf{P}, & \text{SI,} \\ \mathbf{E} + 4\pi \mathbf{P}, & \text{Gaussian,} \end{cases} \quad (3.4)$$

so that

$$\nabla \cdot \mathbf{D} = \begin{cases} \varrho_f, & \text{SI,} \\ 4\pi\varrho_f, & \text{Gaussian.} \end{cases} \quad (3.5)$$

In addition,  $\nabla \times \mathbf{E} = 0$  holds in static conditions. Thus, at the interface between two different dielectric materials, the component of  $\mathbf{E}$  parallel to the interface surface, and the perpendicular component of  $\mathbf{D}$  are continuous. In a material of electric permittivity  $\varepsilon_r$

$$\mathbf{D} = \begin{cases} \varepsilon_0 \varepsilon_r \mathbf{E}, & \text{SI} \\ \varepsilon_r \mathbf{E}, & \text{Gaussian.} \end{cases} \quad (3.6)$$

To facilitate the use of the basic equations in this chapter also with the system independent units, we summarize some of them in the following table:

**Table 3.1** Basic equations for electrostatics in dielectrics

Quantity	SI	Gaussian	System independent
Polarization $\mathbf{P}$ of an isotropic dielectric medium of relative permittivity $\varepsilon_r$	$\varepsilon_0(\varepsilon_r - 1) \mathbf{E}$	$\frac{\varepsilon_r - 1}{4\pi} \mathbf{E}$	$\frac{\varepsilon_r - 1}{4\pi k_e} \mathbf{E}$
$\nabla \cdot \mathbf{E}$	$\frac{\varrho_f + \varrho_b}{\varepsilon_0}$	$4\pi(\varrho_f + \varrho_b)$	$4\pi k_e (\varrho_f + \varrho_b)$
$\nabla \cdot (\varepsilon_r \mathbf{E})$	$\frac{\varrho_f}{\varepsilon_0}$	$4\pi\varrho_f$	$4\pi k_e \varrho_f$
$\nabla \times \mathbf{E}$	0	0	0

### 3.1 An Artificial Dielectric

We have a tenuous suspension of conducting spheres, each of radius  $a$ , in a liquid dielectric material of relative dielectric permittivity  $\epsilon_r = 1$ . The number of spheres per unit volume is  $n$ .

- a)** Evaluate the dielectric susceptibility  $\chi$  of the system as a function of the fraction of the volume filled by the conducting spheres. Use the mean field approximation (MFA), according to which the electric field may be assumed to be uniform throughout the medium.
- b)** The MFA requires the field generated by a single sphere on its nearest neighbor to be much smaller than the mean field due to the collective contribution of all the spheres. Derive a condition on  $n$  and  $a$  for the validity of the MFA.

### 3.2 Charge in Front of a Dielectric Half-Space

A plane divides the whole space into two halves, one of which is empty and the other filled by a dielectric medium of relative permittivity  $\epsilon_r$ . A point charge  $q$  is located in vacuum at a distance  $d$  from the medium as shown in Fig. 3.1.

- a)** Find the electric potential and electric field in the whole space, using the method of image charges.
- b)** Evaluate the surface polarization charge density on the interface plane, and the total polarization charge of the plane.
- c)** Find the field generated by the polarization charge in the whole space.

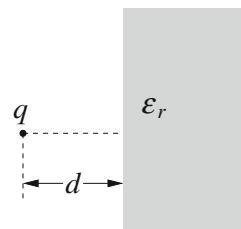


Fig. 3.1

### 3.3 An Electrically Polarized Sphere

Ferroelectricity is the property of some materials like Rochelle salt, carnauba wax, barium titanate, lead titanate, ..., that possess a spontaneous electric polarization in the absence of external fields.

- a)** Consider a ferroelectric sphere of radius  $a$  and uniform polarization  $\mathbf{P}$ , in the absence of external fields, and evaluate the electric field in the whole space (hint: see Problem 1.1).
- b)** Now consider again a ferroelectric sphere of radius  $a$  and uniform polarization  $\mathbf{P}$ , but with a concentrical spherical hole of radius  $b < a$ . Evaluate the electric field and the displacement field in the whole space.

### 3.4 Dielectric Sphere in an External Field

A dielectric sphere of relative permittivity  $\epsilon_r$  and radius  $a$  is placed in vacuum, in an initially uniform external electric field  $\mathbf{E}_0$ , as shown in Fig. 3.2.

- a)** Find the electric field in the whole space (hint: use the results of Problem 3.3 and the superposition principle).

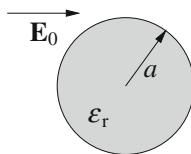


Fig. 3.2

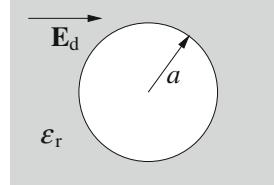


Fig. 3.3

A spherical cavity of radius  $a$  is located inside an infinite dielectric medium of relative permittivity  $\epsilon_r$ , as in Fig. 3.3. The system is in the presence of an external electric field which, far from the cavity (i.e., at a distance  $\gg a$ ), is uniform and equal to  $\mathbf{E}_d$ .

- b)** Find the electric field in the whole space.

### 3.5 Refraction of the Electric Field at a Dielectric Boundary

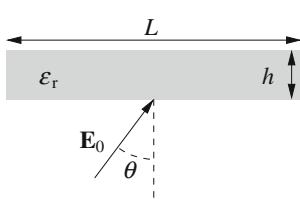


Fig. 3.4

A dielectric slab of thickness  $h$ , length  $L \gg h$ , and dielectric permittivity  $\epsilon_r$ , is placed in an external uniform electric field  $\mathbf{E}_0$ . The angle between  $\mathbf{E}_0$  and the normal to the slab surface is  $\theta$ , as in Fig. 3.4.

- a)** Find the electric field  $\mathbf{E}'$  inside the slab and the angle  $\theta'$  between  $\mathbf{E}'$  and the normal to the slab surface.  
**b)** Find the polarization charge densities in the dielectric medium.

- c)** Evaluate the torque exerted by the external field on the slab, if any. Neglect all boundary effects.

### 3.6 Contact Force between a Conducting Slab and a Dielectric Half-Space

A conducting square slab of surface  $S = a^2$  and thickness  $h \ll a$  is in contact with a dielectric medium of relative permittivity  $\epsilon_r$ . The dielectric medium is much larger than the slab, thus, we can consider it as a hemisphere of radius  $R \gg a$ , with the slab in contact with its base, as shown in Fig. 3.5.a. Part b) of Fig. 3.5 is an enlargement of the area enclosed in the dashed rectangle of part a). With this assumption, we can assume the slab to be in contact with a semi-infinite medium filling the half-space  $x > 0$ , while we have vacuum in the half space  $x < 0$ . The conducting slab carries a total charge  $Q$ , and we assume that the boundary effects at its edges are negligible.

- a) Considering both the cases in which the slab is in contact with the dielectric, and in which it is displaced by an amount  $\xi \ll a$  to the left, find the free charge densities on the left ( $\sigma_1$ ) and right ( $\sigma_2$ ) surfaces of the slab, the polarization charge density ( $\sigma_b$ ) at the surface of the dielectric, and the electric field in the whole space.
- b) Calculate the electrostatic force acting on the slab.
- c) How do these results change if the dielectric medium is assumed to be an infinite (in the  $y$  and  $z$  directions) layer of *finite* thickness  $w$  in the  $x$  direction?

### 3.7 A Conducting Sphere between two Dielectrics

A conducting sphere of mass density  $\varrho$  and radius  $R$  floats in a liquid of density  $\varrho_1 > 2\varrho$  and relative dielectric permittivity  $\epsilon_{r1}$  in the presence of the gravitational field. Above the liquid there is a gaseous medium of mass density  $\varrho_2 \ll \varrho$  and relative dielectric permittivity  $\epsilon_{r2} < \epsilon_{r1}$ . The sphere is given a charge  $Q$  such that exactly one half of its volume is submerged. Evaluate

- a) the electric field in the whole space, the surface free charge densities on the sphere, and the surface polarization charge densities of the two dielectrics, as functions of  $R$ ,  $\epsilon_{r1}$ ,  $\epsilon_{r2}$  and  $Q$ ;
- b) the value of  $Q$ .

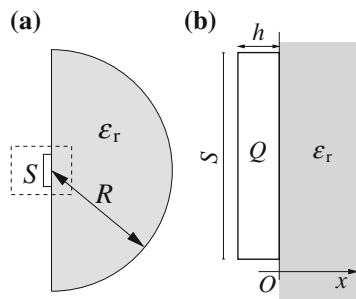


Fig. 3.5

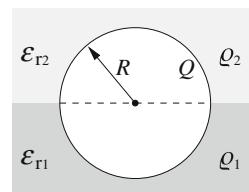


Fig. 3.6

### 3.8 Measuring the Dielectric Constant of a Liquid

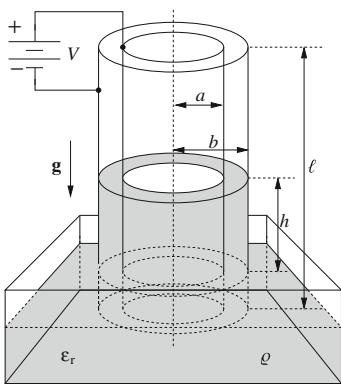


Fig. 3.7

A cylindrical capacitor has internal radius  $a$ , external radius  $b > a$ , and length  $\ell \gg b$ , so that the boundary effects are negligible. The axis of the capacitor is vertical, and the bottom of the capacitor is immersed in a vessel containing a liquid of mass density  $\varrho$  and dielectric permittivity  $\varepsilon_r$ , in the presence of the gravitational field. If a voltage source maintains a potential difference  $V$  between the two cylindrical plates, the liquid rises for a height  $h$  in the cylindrical shell between the plates. Show how one can evaluate the value of  $\varepsilon_r$  from the measurement of  $h$ .

(This is a problem from Ref. [1]).

### 3.9 A Conducting Cylinder in a Dielectric Liquid

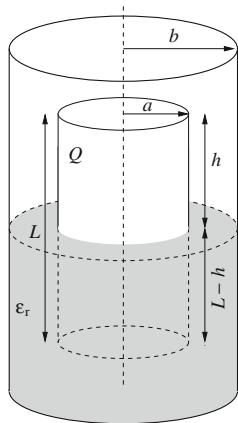


Fig. 3.8

A conducting cylinder of mass  $M$ , radius  $a$  and height  $L \gg a$  is immersed for a depth  $L - h$  (with  $h \gg a$ ) in a dielectric liquid having relative permittivity  $\varepsilon_r$ . The liquid is contained in a cylindrical vessel of radius  $b > a$ , with conducting lateral surface. A free charge  $Q$  is located on the internal cylinder. Boundary effects are assumed to be negligible. The cylinder is free to move vertically preserving its axis. Find

- the electric field  $\mathbf{E}(a)$  at the surface of the internal cylinder, and the surface charge densities;
- the electric field in the region between the lateral surface of the internal cylinder and the container of the liquid ( $a < r < b$ );
- the electrostatic force on the internal cylinder.
- Assume that the internal cylinder has mass  $M$ , and the liquid has mass density  $\varrho > M/(\pi a^2 L)$ . Discuss the equilibrium conditions.

### 3.10 A Dielectric Slab in Contact with a Charged Conductor

A dielectric slab of relative permeability  $\epsilon_r$ , thickness  $h$  and surface  $S \gg h$  is in contact with a plane conducting surface, carrying a uniform surface charge density  $\sigma$ , as in Fig. 3.9. Boundary effects are negligible.

- a) Evaluate the electric field in the whole space.
- b) Evaluate the polarization surface-charge densities on the dielectric surfaces.

c) How do the answers to points a)

and b) change if the slab is moved

at a distance  $s < h$  from the conducting plane? How does the electrostatic energy of the system depend on  $s$ ? Is there an interaction force between slab and conductor?

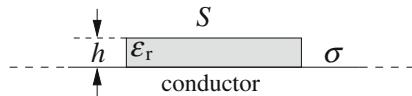


Fig. 3.9

### 3.11 A Transversally Polarized Cylinder

An infinite cylinder of radius  $a$  has an internal uniform electric polarization  $\mathbf{P}$ , perpendicular to its axis, as shown in Fig. 3.10. Evaluate the electric charge density on the lateral surface of the cylinder, the electric potential and the electric field in the whole space.

Hint: see Problem 1.1.

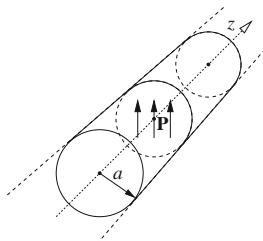


Fig. 3.10

## Reference

1. J. D. Jackson, *Classical Electrodynamics*, John Wiley & Sons, New York, 1975, Problem 4.13

# Chapter 4

## Electric Currents

**Topics.** Electric current density. Continuity equation. Stationary electric currents. Drude model for a conductor. Ohm's law. Joule heating.

**Basic equations** The electric current density  $\mathbf{J} = \mathbf{J}(\mathbf{r}, t)$  is the local flow of charge per unit area and surface, which appears in the continuity equation

$$\partial_t \rho + \nabla \cdot \mathbf{J} = 0, \quad (4.1)$$

that states the conservation of the total electric charge. In integral form

$$\frac{dQ}{dt} \equiv \int_V \partial_t \rho d^3 r = \int_S \mathbf{J} \cdot d\mathbf{S} \equiv I. \quad (4.2)$$

where  $Q$  is the total charge contained into the volume  $V$  bounded by the closed surface  $S$ . Usually the flux (or electric current)  $I$  is defined also for an open surface, as the total charge crossing the surface per unit time.

The quantity

$$w = \mathbf{J} \cdot \mathbf{E} \quad (4.3)$$

is interpreted as the work per unit time and volume done by the EM fields on a distribution of currents.

In a model of matter where there are several species of charged particles (labeled with the index  $s$ ) each having a charge  $q_s$ , a density of particles  $n_s = n_s(\mathbf{r}, t)$  and flowing with velocity  $\mathbf{v}_s = \mathbf{v}_s(\mathbf{r}, t)$ , the current density is given by

$$\mathbf{J} = \sum_s q_s n_s \mathbf{v}_s. \quad (4.4)$$

In a metal where electrons are the only charge carrier,  $\mathbf{J} = -en_e \mathbf{v}_e$ .

Drude's model for electrons in a metal assumes the classical equation of motion

$$m_e \frac{d\mathbf{v}_e}{dt} = -e\mathbf{E} - m_e v \mathbf{v}_e , \quad (4.5)$$

where  $v$  is a phenomenological friction coefficient. In a steady state ( $d\mathbf{v}_e/dt = 0$ ) this leads to Ohm's law for a conductor

$$\mathbf{J} = \frac{n_e e^2}{m_e v} \mathbf{E} \equiv \sigma \mathbf{E} \equiv \frac{\mathbf{E}}{\rho} , \quad (4.6)$$

where  $\sigma$  is the conductivity and  $\rho = 1/\sigma$  the resistivity of the material.<sup>1</sup>

In a material satisfying (4.6), the latter implies that the current  $I$  flowing between two points (or layers) at different values of the electric potential, the potential drop  $V$  is proportional to  $I$ , leading to the definition of the resistance  $R$ :

$$V = RI . \quad (4.7)$$

In the common (but particular) example of a straight conductor of length  $\ell$  and cross-section area  $A$ , such that the electric field is uniform inside the conductor, one obtains  $R = \ell/(\sigma A) = \rho \ell/A$ . The equations (Kirchoff's laws) describing DC electric circuits, i.e. networks of interconnected conductors each satisfying (4.7), can be found in any textbook and will not be repeated here.

Equation (4.7) is known as Ohm's law, but it is appropriate to use this name also for the underlying and more general Equation (4.6) due to G. Kirchoff. An Ohmic conductor is defined as any material which satisfies (4.6). For such materials, Equation (4.3) gives the power per unit volume dissipated into the material as a consequence of the friction term,

$$\mathbf{J} \cdot \mathbf{E} = \sigma E^2 = \frac{E^2}{\rho} , \quad (4.8)$$

which causes the heating of the material (Joule effect). For the above mentioned example, this is equivalent to state that the power dissipated into the whole conductor is  $W = RI^2$ .

Notice that all the above equations have the same form both in the SI and in the Gaussian system. However, the units of measure are different. For example, the current  $I$  is measured in C/s or Ampère (1 A = 1 C/S) in SI, and in statCoulomb/s or "statAmpère" in Gaussian units, while the resistance is measured in Ohms ( $\Omega$ ) in SI and in s/cm in Gaussian units. For the latter,  $\sigma$  has the dimensions of the inverse of a time, and is thus measured in  $s^{-1}$ , while  $\rho$  can be measured in s.

<sup>1</sup>Unfortunately the lower-case Greek letters commonly used as symbols for resistivity and conductivity are the same used for volume and surface charge densities, respectively. However, the meaning of the symbols used in the formulas throughout the book should be clear from the context.

## 4.1 The Tolman-Stewart Experiment

The experiment of Tolman and Stewart [1] was conceived in order to show that conduction in metals is due to electrons. A metallic torus (ring) of major radius  $a$  and minor radius  $b$  is spun at a very high angular velocity  $\omega$  around its axis. We assume that  $b \ll a$ , so that the radial motion of the charge carriers can be neglected. The cross section of the ring is  $S = \pi b^2$ .

At time  $t = 0$  the rotation of the ring is suddenly stopped. A current  $I = I(t)$  flowing in the ring and decaying in time is observed for  $t > 0$ .

- Using the Drude model for conduction in metals, find  $I = I(t)$  and its characteristic decay time  $\tau$  for a ring of copper (electrical conductivity  $\sigma \simeq 10^7 \Omega^{-1}m^{-1}$  and electron density  $n_e = 8.5 \times 10^{28} m^{-3}$ ).
- Evaluate the charge that flows in the ring from  $t = 0$  to  $t = \infty$  as a function of  $\sigma$ .

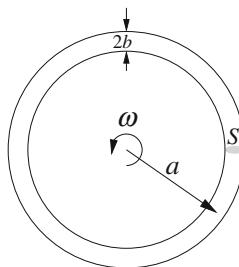


Fig. 4.1

## 4.2 Charge Relaxation in a Conducting Sphere

A conducting sphere of radius  $a$  and conductivity  $\sigma$  has a net charge  $Q$ . At time  $t = 0$  the charge is uniformly distributed over the volume of the sphere, with a volume charge density  $\rho_0 = Q/(3/4\pi a^3)$ . Since in static conditions the charge in an isolated conductor can only be located on the conductor's surface, for  $t > 0$  the charge progressively migrates to the surface of the sphere.

- Evaluate the time evolution of the charge distribution on the sphere, and of the electric field everywhere in space. Give a numerical value for the time constant  $\tau$  in the case of a good conductor (e.g., copper).
- Evaluate the time evolution of the electrostatic energy of the sphere during the charge redistribution.
- Show that the energy dissipated into Joule heat equals the loss of electrostatic energy.

## 4.3 A Coaxial Resistor

Two coaxial cylindrical plates of very low resistivity  $\rho_0$  have radii  $a$  and  $b$ , respectively, with  $a < b$ . The space between the cylindrical plates is filled up to a height  $h$  with a medium of resistivity  $\rho \gg \rho_0$ , as in Fig. 4.2. A voltage source maintains a constant potential difference  $V$  between the plates.

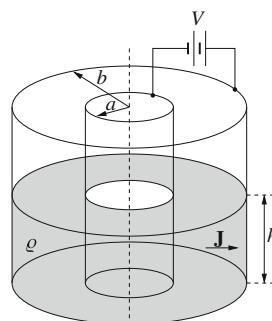


Fig. 4.2

- a)** Evaluate the resistance  $R$  of the system.  
**b)** Discuss the relation between  $R$  and the capacitance of a cylindrical capacitor of radii  $a$  and  $b$ .

#### 4.4 Electrical Resistance between two Submerged Spheres (1)

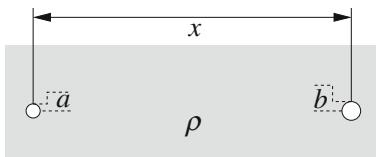


Fig. 4.3

**a)** Two highly conducting spheres of radii  $a$  and  $b$ , respectively, are deeply submerged in the water of a lake, at a distance  $x$  from each other, with  $x \gg a$  and  $x \gg b$ . The water of the lake has resistivity  $\rho$ . Evaluate the approximate resistance between the two spheres, using the results of the answer to point **b**) of problem 4.3.

**b)** Now suppose that the two spheres are not completely submerged, but just sunk so that their centers are exactly at the level of the surface of the lake, as shown in the figure. Evaluate the resistance between them.

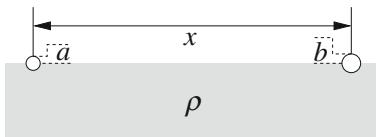


Fig. 4.4

#### 4.5 Electrical Resistance between two Submerged Spheres (2)

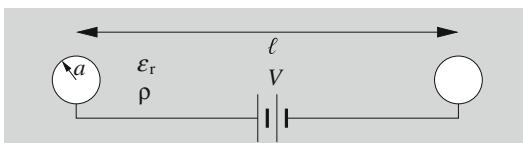


Fig. 4.5

Two identical, perfectly conducting spheres of radius  $a$  are immersed in a fluid of resistivity  $\rho$  and relative electric permittivity  $\epsilon_r$ . The distance between the centers of the two spheres is  $\ell \gg a$ . A constant potential difference difference

$V$  is maintained between the spheres by a suitable voltage source.

As a first approximation, assume the charge to be uniformly distributed over the surface of each sphere, neglecting electrostatic induction effects. Evaluate

- a)** the charge on each sphere,
- b)** the resistance  $R$  and the current  $I$  flowing between the spheres.
- c)** Find the temporal law and the decay time for the discharge of the spheres when the voltage source is disconnected.
- d)** Discuss how electrostatic induction modifies the previous answers, to the lowest order in  $a/\ell$ .

## 4.6 Effects of non-uniform resistivity

Two geometrically identical cylindrical conductors have both height  $h$  and radius  $a$ , but different resistivities  $\rho_1$  and  $\rho_2$ . The two cylinders are connected in series as in Fig. 4.6, forming a single conducting cylinder of height  $2h$  and cross section  $S = \pi a^2$ . The two opposite bases are connected to a voltage source maintaining a potential difference  $V$  through the system, as shown in the figure.

- a)** Evaluate the electric fields, the electric current and the current densities flowing in the two cylinders in stationary conditions.

- b)** Evaluate the surface charge densities at the surface separating the two materials, and at the base surfaces connected to the voltage source.

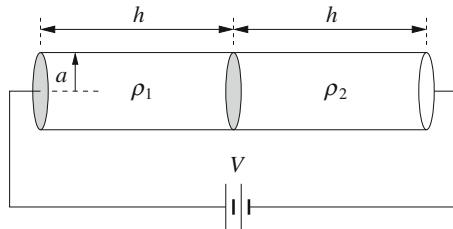


Fig. 4.6

## 4.7 Charge Decay in a Lossy Spherical Capacitor

A spherical capacitor has internal radius  $a$  and external radius  $b$ . The spherical shell  $a < r < b$  is filled by a lossy dielectric medium of relative dielectric permittivity  $\epsilon_r$  and conductivity  $\sigma$ . At time  $t = 0$ , the charge of the capacitor is  $Q_0$ .

- a)** Evaluate the time constant for the discharge of the capacitor.

- b)** Evaluate the power dissipated by Joule heating inside the capacitor, and compare it with the temporal variation of the electrostatic energy.

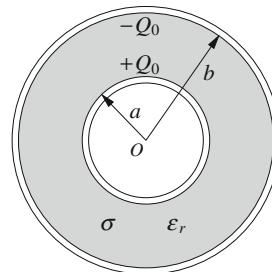


Fig. 4.7

## 4.8 Dielectric-Barrier Discharge

The plates of a parallel-plate capacitor have surface  $S$  and separation  $d$ . The space between the plates is divided into two layers, parallel to the plates, of thickness  $d_1$  and  $d_2$ , respectively, with  $d_1 + d_2 = d$ , as in Fig. 4.8. The layer of thickness  $d_1$  is filled with a gas of negligible dielectric susceptibility ( $\chi = 0, \epsilon_r \approx 1$ ), while the layer of thickness  $d_2$  is filled with a dielectric material of dielectric permittivity  $\epsilon_r > 1$ . The electric potential difference between the plates,  $V$ , is kept constant by a voltage source. Boundary effects can be neglected.

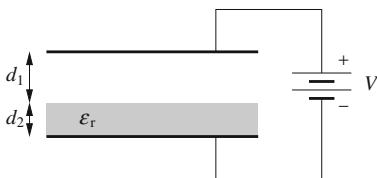


Fig. 4.8

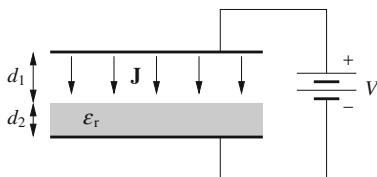


Fig. 4.9

a) Find the electric field inside the capacitor. An ionization discharge is started in the gaseous layer at  $t = 0$ , and the gas instantaneously becomes conducting. We assume that, for  $t > 0$ , the ionized gas can be considered as an Ohmic conductor of constant and uniform resistivity  $\rho$ .

b) After a sufficiently long time we observe that the current stops flowing in the gas, and the system reaches a steady state (i.e., all physical quantities are constant). Find the electric field in the capacitor in these conditions, and the surface free charge density between the two layers.

c) Find the time dependence of the electric field during the transient phase ( $t > 0$ ), and the relaxation time needed by the system to reach the steady state condition.

## 4.9 Charge Distribution in a Long Cylindrical Conductor

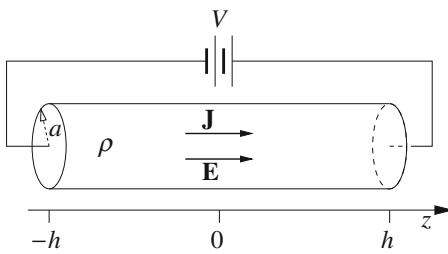


Fig. 4.10

Consider a conducting homogeneous cylindrical wire of radius  $a$  and length  $2h$ , with  $a \ll h$ , and resistivity  $\rho$ . The wire is connected to a voltage source that keeps a constant potential difference  $V$  across its ends. We know that the electric field  $\mathbf{E}$  and, consequently, the current density  $\mathbf{J} = \mathbf{E}/\rho$  must be uniform inside the wire, see Problem 4.6. This implies the presence of charge distributions generating the uniform field. Only surface charge distributions are allowed in a conductor in steady conditions. The charge distributions on the bases of the cylinder are not sufficient for generating an even approximately uniform field in our case of  $a \ll 2h$ . Thus, a charge density  $\sigma_L$  must be present also on the lateral surface. Verify that a surface charge density  $\sigma_L = \gamma z$ , where  $\gamma$  is a constant and  $z$  is the coordinate along the cylinder axis, leads to a good approximation for the field inside the conductor far from the ends [2].

## 4.10 An Infinite Resistor Ladder

An infinite resistor ladder consists of an infinite number of resistors, all of resistance  $R$ , arranged as in Fig. 4.11. Evaluate the resistance measured between the terminals  $A$  and  $B$ . Hint: use an invariance property of the ladder.

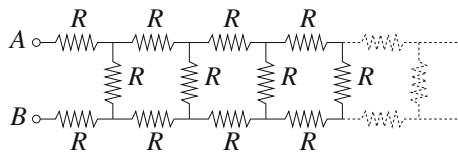


Fig. 4.11

## References

1. R.C. Tolman, T.D. Stewart, The electromotive force produced by the acceleration of metals. *Phys. Rev.* **8**, 97–116 (1916)
2. C.A. Coombes, H. Laue, Electric fields and charge distributions associated with steady currents. *Am. J. Phys.* **49**, 450–451 (1981)

# Chapter 5

## Magnetostatics

**Topics.** Stationary magnetic field in vacuum. Lorentz force. Motion of an electric point charge in a magnetic field. The magnetic force on a current. The magnetic field of steady currents. “Mechanical” energy of a circuit in a magnetic field. Biot-Savart law. Ampères’ circuital law. The magnetism of matter. Volume and surface magnetization current densities (bound currents). Magnetic susceptibility. The “auxiliary” vector  $\mathbf{H}$ . Magnetic field boundary conditions. Equivalent magnetic charge method.

**Units.** In order to write formulas compatible with both SI and Gaussian units, we introduce two new “system dependent” constants,  $k_m$  and  $b_m$ , defined as

$$k_m = \begin{cases} \frac{\mu_0}{4\pi}, & \text{SI,} \\ \frac{1}{c}, & \text{Gaussian,} \end{cases} \quad b_m = \begin{cases} 1, & \text{SI,} \\ \frac{1}{c}, & \text{Gaussian,} \end{cases} \quad (5.1)$$

where, again,  $\mu_0 = 4\pi \times 10^{-7} \text{ T}\cdot\text{m}/\text{A}$  is the “magnetic permeability of vacuum”, and  $c = 29979\,245\,800 \text{ cm/s}$  is the light speed in vacuum.

**Basic equations** The two Maxwell equations for the magnetic field  $\mathbf{B}$  relevant to this chapter are

$$\nabla \cdot \mathbf{B} = 0, \quad (5.2)$$

$$\nabla \times \mathbf{B} = 4\pi k_m \mathbf{J}. \quad (5.3)$$

Equation (5.2) is always valid (in the absence of *magnetic monopoles*), while (5.3) is valid in the absence of time-dependent electric fields. It is thus possible to introduce a *vector potential*  $\mathbf{A}$ , such that

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}), \quad (5.4)$$

Imposing the gauge condition  $\nabla \cdot \mathbf{A} = 0$ , the vector potential satisfies

$$\nabla^2 \mathbf{A}(\mathbf{r}) = -4\pi k_m \mathbf{J}(\mathbf{r}), \quad (5.5)$$

which is the vector analogous of Poisson's equation (2.1). Thus,

$$\mathbf{A}(\mathbf{r}) = k_m \int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3 r'. \quad (5.6)$$

A particular and typical case is that of closed “line” currents, e.g. flowing in a circuit having wires of negligible thickness. In such case one may replace  $\mathbf{J}(\mathbf{r}') d^3 r'$  by  $I(\mathbf{r}') d\ell$  and calculate the field via the Biot-Savart formula

$$\mathbf{B}(\mathbf{r}) = k_m \oint \frac{I(\mathbf{r} - \mathbf{r}') d\ell \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}, \quad (5.7)$$

where the integral is extended to the closed path of the current.

The force exerted by a magnetic field over a distribution of currents is

$$\mathbf{f} = b_m \int_v \mathbf{J}(\mathbf{r}') \times \mathbf{B}(\mathbf{r}') d^3 r'. \quad (5.8)$$

A single point charge  $q$  located at  $\mathbf{r}$  and moving with velocity  $\mathbf{v}$  is equivalent to a current density  $\mathbf{j}(\mathbf{r}') = q\delta(\mathbf{r} - \mathbf{r}')\mathbf{v}$ , so that the magnetic force on the point charge is

$$\mathbf{f} = b_m q \mathbf{v} \times \mathbf{B}. \quad (5.9)$$

The energy associated to a magnetic field distribution is given by the expression

$$U_m = \int_V \frac{b_m B^2}{8\pi k_m \mu_r} d^3 r. \quad (5.10)$$

In the absence of magnetic monopoles, the first non-vanishing term of the multipole expansion is the magnetic dipole  $\mathbf{m}$

$$\mathbf{m} = \frac{1}{2} \int_V \mathbf{r}' \times \mathbf{J}(\mathbf{r}') d^3 r'. \quad (5.11)$$

In the simple case of a small plane coil of area  $A$  and electric current  $I$  this reduces to the line integral over the coil path  $C$

$$\mathbf{m} = \frac{I}{2} \oint_C \mathbf{r}' \times d\ell = AI\hat{\mathbf{n}}, \quad (5.12)$$

where  $\hat{\mathbf{n}}$  is perpendicular to the coil surface. A magnetic dipole term located at  $\mathbf{r} = 0$  generates a magnetic field

$$\mathbf{B}(\mathbf{r}) = k_m \frac{3\hat{\mathbf{r}}(\mathbf{m} \cdot \hat{\mathbf{r}}) - \mathbf{m}}{r^3}. \quad (5.13)$$

In an external magnetic field  $\mathbf{B}_{\text{ext}}$ , the magnetic force on a magnetic dipole is

$$\mathbf{f} = (\mathbf{m} \cdot \nabla) \mathbf{B}_{\text{ext}}. \quad (5.14)$$

The magnetization density  $\mathbf{M}$  of a material is defined as the dipole moment per unit volume,

$$\mathbf{M} = \frac{d\mathbf{m}}{dV}. \quad (5.15)$$

Ampère equivalence theorem states that a magnetization density  $\mathbf{M} = \mathbf{M}(\mathbf{r})$  is always equivalent to a distribution of volume current density  $\mathbf{J}_m$  and surface current density  $\mathbf{K}_m$  bound to the material, and given by

$$\mathbf{J}_m = \frac{1}{b_m} \nabla \times \mathbf{M}, \quad (5.16)$$

$$\mathbf{K}_m = \frac{1}{b_m} \mathbf{M} \cdot \hat{\mathbf{n}}, \quad (5.17)$$

where  $\hat{\mathbf{n}}$  is the unit normal vector pointing outwards from the boundary surface of the material. The total volume and surface current densities are thus

$$\mathbf{J} = \mathbf{J}_f + \mathbf{J}_m, \quad \mathbf{K} = \mathbf{K}_f + \mathbf{K}_m, \quad (5.18)$$

the subscript f denoting the *free* (e.g., conduction) current densities.

The auxiliary field  $\mathbf{H}$  is defined as

$$\mathbf{H} = \begin{cases} \frac{\mathbf{B}}{\mu_0} - \mathbf{M}, & \text{SI,} \\ \mathbf{B} - 4\pi\mathbf{M}, & \text{Gaussian,} \end{cases} \quad (5.19)$$

so that Equation (5.3) becomes

$$\nabla \times \mathbf{H} = 4\pi k_m \mathbf{J}_f, \quad (5.20)$$

A material may have either a permanent magnetization, or a magnetization induced by a magnetic field. In linear, isotropic diamagnetic and paramagnetic materials  $\mathbf{M}$  is parallel and proportional to  $\mathbf{H}$ ,

$$\mathbf{M} = \chi_m \mathbf{H}, \quad (5.21)$$

where  $\chi_m$  is the *magnetic susceptibility* of the material, with  $\chi_m < 0$  for diamagnetic materials and  $\chi_m > 0$  for paramagnetic materials.<sup>1</sup> The (*relative*) *magnetic permeability*  $\mu_r$  is defined as

$$\mu_r = \begin{cases} 1 + \chi_m, & \text{SI,} \\ 1 + 4\pi\chi_m, & \text{Gaussian.} \end{cases} \quad (5.22)$$

We have  $\mu_r < 1$  for diamagnetic materials and  $\mu_r > 1$  for paramagnetic materials. Inserting (5.21) and (5.22) into (5.19) we obtain

$$\mathbf{B} = \begin{cases} \mu_0\mu_r \mathbf{H}, & \text{SI,} \\ \mu_r \mathbf{H}, & \text{Gaussian,} \end{cases} \quad (5.23)$$

valid for isotropic, non-ferromagnetic, materials.

To facilitate the use of the basic equations in this chapter also with the system independent units, we summarize some of them in the following table (Table 5.1):

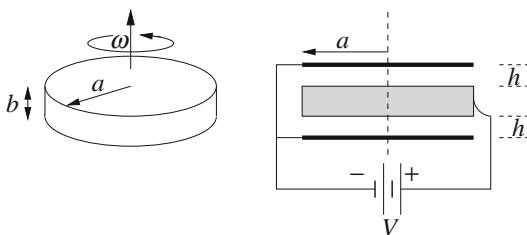
**Table 5.1** Basic equations for magnetostatics

Quantity	SI	Gaussian	System independent
$\nabla \times \mathbf{B}$	$\mu_0 \mathbf{J}$	$\frac{4\pi}{c} \frac{\mathbf{v}}{q} \mathbf{J}$	$4\pi k_m \mathbf{J}$
Magnetic force on a point charge $q$ moving with velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$	$q \mathbf{v} \times \mathbf{B}$	$q \frac{\mathbf{v}}{c} \times \mathbf{B}$	$b_m q \mathbf{v} \times \mathbf{B}$
Magnetic field $d\mathbf{B}$ generated by a wire element $d\ell$ carrying a current $I$ at a distance $r$ (Biot-Savart's law)	$\frac{\mu_0}{4\pi} \frac{Id\ell \times \hat{\mathbf{r}}}{r^2}$	$\frac{1}{c} \frac{Id\ell \times \hat{\mathbf{r}}}{r^2}$	$k_m \frac{Id\ell \times \hat{\mathbf{r}}}{r^2}$
Magnetic moment $\mathbf{m}$ of a ring circuit carrying an electric current $I$ , and enclosing a surface $\mathbf{S}$	$IS$	$\frac{1}{c} IS$	$b_m IS$
Volumetric magnetic energy density $u_m$	$\frac{B^2}{2\mu_0\mu_r}$	$\frac{B^2}{8\pi\mu_r}$	$\frac{b_m B^2}{8\pi k_m \mu_r}$

<sup>1</sup>The magnetization is expressed in terms of the auxiliary field  $\mathbf{H}$ , rather than in terms of the magnetic field  $\mathbf{B}$ , for historical reasons. In ferromagnetic materials there is no one-to-one correspondence between  $\mathbf{M}$  and  $\mathbf{H}$  (between  $\mathbf{M}$  and  $\mathbf{B}$ ) because of *magnetic hysteresis*

## 5.1 The Rowland Experiment

This experiment by Henry A. Rowland (1876) aimed at showing that moving charges generate magnetic fields. A metallic disk of radius  $a$  and thickness  $b \ll a$  is electrically charged and kept in rotation with a constant angular velocity  $\omega$ .

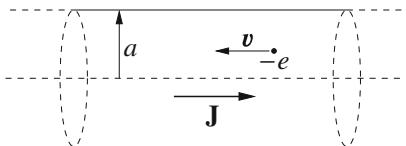


**Fig. 5.1**

- a) The disk rotates between two conducting plates, one at a distance  $h \approx 0.5$  cm above its upper surface, and the other at  $h$  below its lower surface, as in Fig. 5.1. The two plates are connected to the same terminal of a voltage source maintaining a potential difference  $V_0 = 10^4$  V, while the other terminal is connected to the disk by a sliding contact. Evaluate the surface charge density on the disk surfaces.
- b) Calculate the magnetic field  $\mathbf{B}_c$  near the center of the disk and the magnetic field component  $B_r$ , parallel and close to the disk surfaces, as a function of the distance  $r$  from the axis. Typical experimental values were  $a = 10$  cm, and  $\omega \approx 2\pi \times 10^2$  rad/s (period  $T = 2\pi/\omega = 10^{-2}$  s).
- c) The field component  $B_r$  generated by the disk at  $r = a$  can be measured by orienting the apparatus so that  $\hat{\mathbf{r}}$  is perpendicular to the Earth's magnetic field  $\mathbf{B}_\oplus$ , of strength  $B_\oplus \approx 5 \times 10^{-5}$  T, and measuring the deviation of a magnetic needle when the disk rotates. Find the deviation angle of the needle.

## 5.2 Pinch Effect in a Cylindrical Wire

A uniform current density  $\mathbf{J}$  flows in an infinite cylindrical conductor of radius  $a$ . The current carriers are electrons (charge  $-e$ ) of number volume density  $n_e$  and drift velocity  $\mathbf{v}$ , parallel to the axis of the cylinder. Ions can be considered as fixed in space, with uniform number density  $n_i$  and charge  $Ze$ . The system is globally neutral.



**Fig. 5.2**

- a) Evaluate the magnetic field generated by the current, and the resulting magnetic force on the electrons.

The magnetic force modifies the volume distribution of the electrons and this, in turn, gives origin to a static electric field. At equilibrium the magnetic force on the electrons is compensated by the electrostatic force.

- b)** Evaluate the electric field that compensates the magnetic force on the electrons, and the corresponding charge distribution.
- c)** Evaluate the effect in “standard” conditions for a good Ohmic conductor.

### 5.3 A Magnetic Dipole in Front of a Magnetic Half-Space

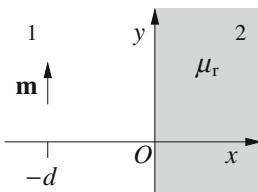


Fig. 5.3

The plane  $x = 0$  divides the space into two half-spaces, labeled 1 and 2, respectively. We have vacuum in half-space 1, while half-space 2 is filled by a medium of magnetic permittivity  $\mu_r$ . A magnetic dipole  $\mathbf{m}$ , parallel to the  $y$  axis, is located in vacuum at position  $x = -d$ . Find

- a)** the magnetic field  $\mathbf{B}$  in the whole space,
- b)** the force acting on the magnetic dipole.

### 5.4 Magnetic Levitation

In a given region of space we have a static magnetic field, which, in a cylindrical reference frame  $(r, \phi, z)$ , is symmetric around the  $z$  axis, i.e., is independent of  $\phi$ , and can be written  $\mathbf{B} = \mathbf{B}(r, z)$ . The field component along  $z$  is  $B_z(z) = B_0 z / L$ , where  $B_0$  and  $L$  are constant parameters.

- a)** Find the radial component  $B_r$  close to the  $z$  axis.

A particle of magnetic polarizability  $\alpha$  (such that it acquires an induced magnetic dipole moment  $\mathbf{m} = \alpha \mathbf{B}$  in a magnetic field  $\mathbf{B}$ ), is located close to the  $z$  axis.

- b)** Find the potential energy of the particle in the magnetic field.
- c)** Discuss the existence of equilibrium positions for the particle, and find the frequency of oscillations for small displacements from equilibrium either along  $z$  or  $r$  (let  $M$  be the mass of the particle).

### 5.5 Uniformly Magnetized Cylinder

A magnetically “hard” cylinder of radius  $R$  and height  $h$ , with  $R \ll h$ , carries a uniform magnetization  $\mathbf{M}$  parallel to its axis.

- a)** Show that the volume magnetization current density  $\mathbf{J}_m$  is zero inside the cylinder, while the lateral surface of the cylinder carries a surface magnetization current density  $\mathbf{K}_m$ , with  $|\mathbf{K}_m| = |\mathbf{M}|$ .
- b)** Find the magnetic field  $\mathbf{B}$  inside and outside the cylinder, at the limit  $h \rightarrow \infty$ .
- c)** Now consider the opposite case of a “flat” cylinder, i.e.,  $h \ll R$ , and evaluate the magnetic field  $\mathbf{B}_0$  at the center of the cylinder.
- d)** According to the result of **c)**,  $\lim_{R/h \rightarrow \infty} \mathbf{B}_0 = 0$ . Obtain the same result using the equivalent magnetic charge method.

## 5.6 Charged Particle in Crossed Electric and Magnetic Fields

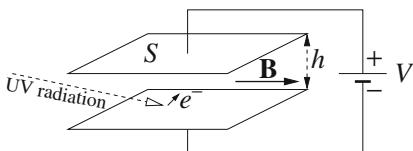


Fig. 5.4

following problem. We have a parallel-plate capacitor with surface  $S$ , plate separation  $h$  and voltage  $V$ , as in Fig. 5.4. A uniform magnetic field  $\mathbf{B}$  is applied to the capacitor, perpendicular to the capacitor electric field, i.e., parallel to the plates. Ultraviolet radiation causes the negative plate to emit electrons with zero initial velocity. Evaluate the minimum value of  $\mathbf{B}$  for which the electrons cannot reach the positive plate.

A particle of electric charge  $q$  and mass  $m$  is initially at rest in the presence of a uniform electric field  $\mathbf{E}$  and a uniform magnetic field  $\mathbf{B}$ , perpendicular to  $\mathbf{E}$ .

- a)** Describe the subsequent motion of the particle.
- b)** Use the above result to discuss the

## 5.7 Cylindrical Conductor with an Off-Center Cavity

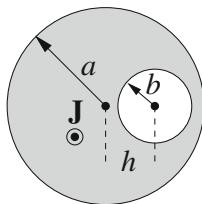


Fig. 5.5

An infinite cylindrical conductor of radius  $a$  has a cylindrical cavity of radius  $b$  bored parallel to, and centered at a distance  $h < a - b$  from the cylinder axis as in Fig. 5.5, which shows a section of the conductor. The current density  $\mathbf{J}$  is perpendicular to, and uniform over the section of the conductor (i.e., excluding the cavity!). The figure shows a section of the conductor. Evaluate the magnetic field  $\mathbf{B}$ , showing that it is *uniform* inside the cavity.

## 5.8 Conducting Cylinder in a Magnetic Field

A conducting cylinder of radius  $a$  and height  $h \gg a$  rotates around its axis at constant angular velocity  $\omega$  in a uniform magnetic field  $\mathbf{B}_0$ , parallel to the cylinder axis.

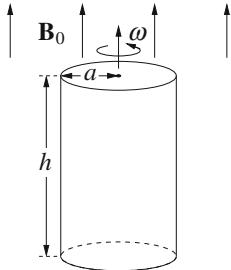


Fig. 5.6

- a) Evaluate the magnetic force acting on the conduction electrons, assuming  $\omega = 2\pi \times 10^2 \text{ s}^{-1}$  and  $B = 5 \times 10^{-5} \text{ T}$  (the Earth's magnetic field), and the ratio of the magnetic force to the centrifugal force. Assume that the cylinder is rotating in stationary conditions. Evaluate
- b) the electric field inside the cylinder, and the volume and surface charge densities;
- c) the magnetic field  $\mathbf{B}_1$  generated by the rotation currents inside the cylinder, and the order of magnitude of  $B_1/B_0$  (assume  $a \approx 0.1 \text{ m}$ ).

## 5.9 Rotating Cylindrical Capacitor

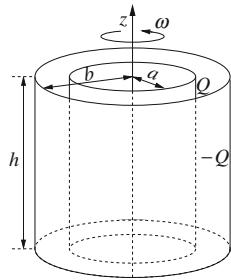


Fig. 5.7

the electrostatic forces.

The concentric cylindrical shells of a cylindrical capacitor have radii  $a$  and  $b > a$ , respectively, and height  $h \gg b$ . The capacitor charge is  $Q$ , with  $+Q$  on the inner shell of radius  $a$ , and  $-Q$  on the outer shell of radius  $b$ , as in Fig. 5.7. The whole capacitor rotates about its axis with angular velocity  $\omega = 2\pi/T$ . Boundary effects are negligible.

- a) Evaluate the magnetic field  $\mathbf{B}$  generated by the rotating capacitor over the whole space.
- b) Evaluate the magnetic forces on the charges of the two rotating cylindrical shells, and compare them to

## 5.10 Magnetized Spheres

- a) A sphere of radius  $R$  has a uniform and permanent magnetization  $\mathbf{M}$ . Calculate the magnetic field inside and outside the sphere. (Hint: see Problem 3.3.)
- b) A sphere of radius  $R$  has a total charge  $Q$  uniformly distributed on its surface. The sphere rotates with angular velocity  $\omega$ . Calculate the magnetic field inside and outside the sphere.

- c) A sphere of radius  $R$  has a magnetic permeability  $\mu_r$  and is located in an external, uniform magnetic field  $\mathbf{B}_0$ . Calculate the total magnetic field inside and outside the sphere, discussing the limit of a perfectly diamagnetic material ( $\mu_r = 0$ ), as a superconductor.

# Chapter 6

## Magnetic Induction and Time-Varying Fields

**Topics.** Magnetic induction. Faraday's law. Electromotive force. The slowly varying current approximation. Mutual inductance and self-inductance. Energy stored in an inductor. Magnetically coupled circuits. Magnetic energy. Displacement current and the complete Maxwell's equations.

**Basic equations** In the presence of a time-varying magnetic field, Equation (1.5) is modified into the exact equation

$$\nabla \times \mathbf{E} = -b_m \partial_t \mathbf{B}, \quad (6.1)$$

so that the line integral of  $\nabla \times \mathbf{E}$  around a closed path  $C$  is

$$\oint_C \mathbf{E} \cdot d\ell = -b_m \int_S \partial_t \mathbf{B} \cdot d\mathbf{S} \quad (6.2)$$

Thus, for a *fixed* path, the line integral of  $\mathbf{E}$  equals the time derivative of the flux of the time-varying field  $\mathbf{B}$  through a surface delimited by the contour  $C$ .

The *electromotive force* (emf)  $\mathcal{E}$  in a *real* circuit having *moving* parts is the work done by the Lorentz force on a unit charge over the circuit path,

$$\mathcal{E} = \oint_{\text{circ}} (\mathbf{E} + b_m \mathbf{V} \times \mathbf{B}) \cdot d\ell \equiv -b_m \frac{d}{dt} \Phi_{\text{circ}}(\mathbf{B}), \quad (6.3)$$

where  $\mathbf{V}$  is the velocity of the circuit element; now in (6.3) the flux  $\Phi_{\text{circ}}(\mathbf{B})$  of  $\mathbf{B}$  through the circuit may vary because of *both* the temporal variation of  $\mathbf{B}$  and of the circuit geometry. Equation (6.3) is the general Faraday's law of induction.

For a system of two electric circuits, the magnetic flux through each circuit can be written as a function of the currents flowing in each circuit,

$$\Phi_2 = L_1 I_1 + M_{21} I_2, \quad \Phi_1 = L_2 I_2 + M_{12} I_1, \quad (6.4)$$

where the terms containing the (self-)inductance coefficients  $L_i$  are the contribution to flux generated by the circuit itself, and the terms containing the mutual inductance coefficients  $M_{21} = M_{12}$  give the flux generated by one circuit over the other.

Finally, for time-varying fields the complete Maxwell's equation replacing (5.3) is

$$\nabla \times \mathbf{B} = 4\pi k_m \mathbf{J} + \frac{k_m}{k_e} \partial_t \mathbf{E} = \begin{cases} \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \partial_t \mathbf{E} & \text{(Gaussian),} \\ \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \partial_t \mathbf{E} & \text{(SI.)} \end{cases} \quad (6.5)$$

## 6.1 A Square Wave Generator

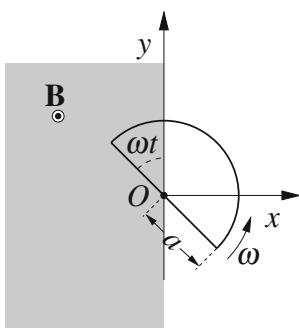


Fig. 6.1

We have a uniform magnetic field  $\mathbf{B} = B\hat{\mathbf{z}}$  in the half space  $x < 0$  of a Cartesian coordinate system, while the field is zero for  $x > 0$ . A semicircular loop of radius  $a$  and resistance  $R$  lies in the  $xy$  plane, with the center of the full circumference at the origin  $O$  of our coordinate system, as in Fig. 6.1. The loop rotates around the  $z$  axis at constant angular velocity  $\omega$ .

First, assume that the self-inductance of the coil is negligible and evaluate

- the current circulating in the coil;
- the torque exerted by the magnetic forces on the coil, and the mechanical power needed to keep the coil in rotation. Compare this to the electric power dissipated in the coil.

- Now consider the presence of the self-inductance of the coil, and discuss how it affects the answer to point a).

## 6.2 A Coil Moving in an Inhomogeneous Magnetic Field

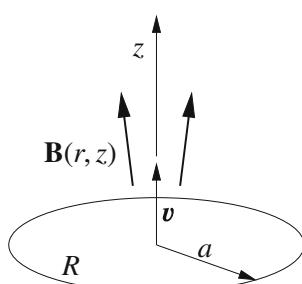


Fig. 6.2

A magnetic field has rotational symmetry around a straight line, that we choose as the longitudinal axis,  $z$ , of a cylindrical reference frame  $(r, \phi, z)$ . The  $z$  component of the field on the  $z$  axis,  $B_z(0, z)$ , is known and equals  $B_z(0, z) = B_0 z/L$ , where  $L$  is a constant. A circular coil has radius  $a$ , resistance  $R$ , and axis coinciding with the  $z$  axis of our reference frame. The coil performs a translational motion at constant velocity  $v = v\hat{\mathbf{z}}$ , and its radius  $a$  is assumed to be small enough that the magnetic field is always approximately uniform over the surface limited by the coil.

- Find the current  $\mathbf{I}$  flowing in the coil.
- Find the power  $P$  dissipated by the coil due to Joule heating, and the corresponding frictional force  $\mathbf{f}$  on the coil.
- Calculate  $\mathbf{f}$  as the resultant magnetic force on the loop carrying the current  $I$ .

### 6.3 A Circuit with “Free-Falling” Parts

In the presence of the Earth’s gravitational field  $\mathbf{g}$ , two high-conducting bars are located vertically, at a distance  $a$  from each other. A uniform, horizontal magnetic field  $\mathbf{B}$  is perpendicular to the plane defined by the vertical bars. Two horizontal bars, both of mass  $m$ , resistance  $R/2$  and length  $a$ , are constrained to move, without friction, with their ends steadily in contact with the two vertical bars. The resistance of the two fixed vertical bars is assumed to be much smaller than  $R/2$ , so that the net resistance of the resulting rectangular circuit is, with very good approximation, always  $R$ , independently of the positions of the two horizontal bars.

First, assume that the upper horizontal bar is fixed, while the lower bar starts a “free” fall at  $t = 0$ . Let’s denote by  $v = v(t)$  the velocity of the falling bar at time  $t$ , with  $v(0) = 0$ .

- Write the equation of motion for the falling bar, find the solution for  $v(t)$  and show that, asymptotically, the bar approaches a terminal velocity  $v_t$ .
- Evaluate the power dissipated in the circuit by Joule heating when  $v(t) = v_t$ , and the mechanical work done per unit time by gravity in these conditions.

Now consider the case in which, at  $t = 0$ , the upper bar already has a velocity  $v_0 \neq 0$  directed downwards, while the lower bar starts a “free” fall.

- Write the equations of motion for both falling bars, and discuss the asymptotic behavior of their velocities  $v_1(t)$  and  $v_2(t)$ , and of the current in the circuit  $I(t)$ .

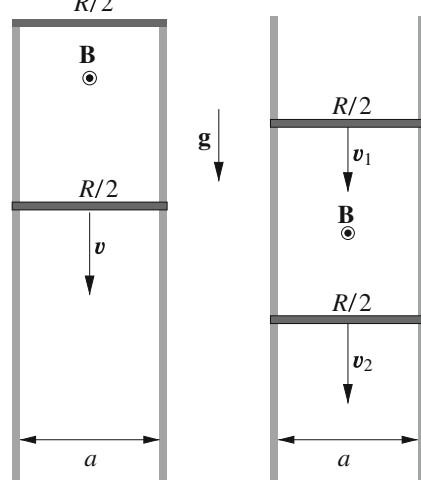
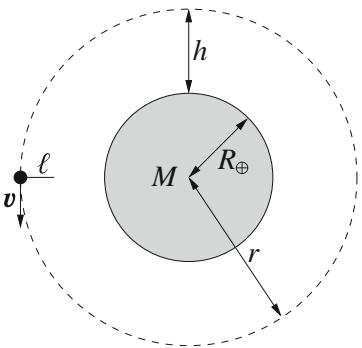


Fig. 6.3

## 6.4 The Tethered Satellite



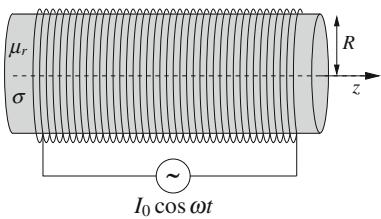
**Fig. 6.4**

(leash, or lead line), consisting in a metal cable of length  $\ell = 1$  km, hangs from the satellite, pointing to the Earth's center.

- a) Find the electromotive force on the wire.
- b) The satellite is traveling through the ionosphere, where charge carriers in outer space are available to close the circuit, thus a current can flow along the wire. Assume that the ionosphere is rigidly rotating at the same angular velocity as the Earth. Find the power dissipated by Joule heating in the wire and the mechanical force on the wire as a function of its resistance  $R$ .

The Earth's magnetic field at the Earth's surface roughly approximates the field of a magnetic dipole placed at the Earth's center. Its magnitude ranges from  $2.5 \times 10^{-5}$  to  $6.5 \times 10^{-5}$  T (0.25 to 0.65 G in Gaussian units), with a value  $B_{\text{eq}} \approx 3.2 \times 10^{-5}$  T at the equator. A satellite moves on the magnetic equatorial plane with a velocity  $v \approx 8$  km/s at a constant height  $h \approx 100$  km over the Earth's surface, as shown in the figure (not to scale!).

## 6.5 Eddy Currents in a Solenoid



**Fig. 6.5**

A long solenoid consists of a helical coil of  $n$  turns per unit length wound around a soft ferromagnetic cylinder of radius  $R$  and length  $\ell \gg R$ . The ferromagnetic material has a relative magnetic permeability  $\mu_r$ , and an electrical conductivity  $\sigma$ . An AC current  $I = I_0 \cos \omega t$  flows in the coil.

- a) Find the electric field induced in the solenoid.
- b) Explain why the cylinder warms up and evaluate the dissipated power.
- c) Evaluate how the induced currents affect the magnetic field in the solenoid. (Boundary effects and the displacement current are assumed to be negligible).

## 6.6 Feynman's "Paradox"

A non-conducting ring of radius  $R$  is at rest on the  $xy$  plane, with its center at the origin of the coordinate system. The ring has mass  $m$ , negligible thickness, and an electric charge  $Q$  distributed uniformly on it, so that the ring has a linear charge density  $\lambda = Q/(2\pi a)$ . The ring is free to rotate around its axis without friction.

A superconducting circular ring of radius  $a \ll R$ , coaxial to the charged ring and carrying an electric current  $I_0$ , also lies on the  $xy$  plane, as in Fig. 6.6. At time  $t = 0$  the superconducting loop is heated above its critical temperature, and switches to normal conductivity. Consequently, its current decays to zero according to a law  $I = I(t)$ .

- a)** Neglecting self-induction effects, evaluate the angular velocity  $\omega = \omega(t)$  of the charged ring as a function of the current  $I(t)$  in the smaller ring. Evaluate the final angular velocity  $\omega_f$ , and the final angular momentum  $L_f$ , of the charged ring.
- b)** Evaluate the magnetic field at the ring center,  $\mathbf{B}_c$ , generated by the rotation of the ring.
- c)** Discuss how the results of **a)** are modified by taking the “self-inductance”  $\mathcal{L}$  of the charged ring into account.

This is one of the possible versions of the so-called *Feynman's disc paradox* [2], presented in Vol. II, Section 17-4, of *The Feynman's Lectures on Physics*. The apparent paradox arises because the initial total *mechanical* angular momentum of the system is zero, no external torque is applied, and one could (wrongly) expect the final total angular momentum to be zero, i.e., no rotation of the ring. This conclusion is wrong, of course, for reasons further discussed in Prob. 8.8.

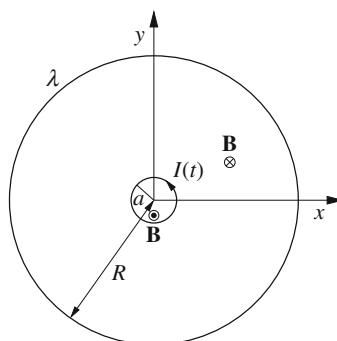


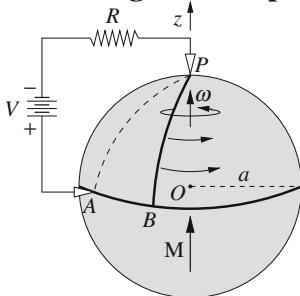
Fig. 6.6

## 6.7 Induced Electric Currents in the Ocean

A fluid flows with uniform velocity  $\mathbf{v}$  in the presence of a constant and uniform magnetic field  $\mathbf{B}$  perpendicular to  $\mathbf{v}$ . The fluid has an electrical conductivity  $\sigma$  and volumetric mass density  $\varrho$ .

- a)** Evaluate the electric current density  $\mathbf{J}$  induced in the fluid.
- b)** Give a numerical estimate of  $|\mathbf{J}|$  for the terrestrial oceans, knowing that the Earth's magnetic field has an average value  $B \approx 0.5 \text{ G} = 5 \times 10^{-5} \text{ T}$ , the conductivity of sea water is  $\sigma \approx 4 \Omega^{-1} \text{ m}^{-1}$  ( $\sigma \approx 3.6 \times 10^{10} \text{ s}^{-1} \text{ cm}^{-1}$  in Gaussian units), and a typical value of the flow velocity is  $v = 1 \text{ m/s}$ .
- c)** Due to the presence of the induced current, the magnetic force tends to slow down the fluid. Estimate the order of magnitude for the time constant of this effect.

## 6.8 A Magnetized Sphere as Unipolar Motor



**Fig. 6.7**

A magnetized, non-conducting sphere has radius  $a$ , mass  $m$  and permanent, uniform magnetization  $\mathbf{M}$  throughout its volume. An electric circuit is formed by pasting a conducting wire along a half meridian, from the pole  $P$  to the equator, and another conducting wire around the whole equator of the sphere, as shown in Fig. 6.7. The circuit is closed by two brush contacts (the white arrows in Fig. 6.7) connecting the pole  $P$ , and a point  $A$  of the wire on the equator of the sphere, to a voltage source of electromotive force  $V$ . The resulting circuit has resistance  $R$ .

- a)** Evaluate the torque on the sphere when a current  $I$  flows in the circuit.
- b)** If the sphere is free to rotate without friction around the  $z$  axis of a cylindrical coordinate system, parallel to  $\mathbf{M}$  and passing through the center  $O$  of the sphere, it reaches asymptotically a terminal angular velocity  $\omega_t$ . Evaluated  $\omega_t$  and the characteristic time of the system.

## 6.9 Induction Heating

Consider a homogeneous material of electrical conductivity  $\sigma$  and relative magnetic permeability  $\mu_r$ , both real, positive and independent of frequency. The electric permittivity is  $\epsilon_r = 1$ .

- a)** Show that, if the displacement current density  $\partial_t \mathbf{E} / (4\pi k_e)$  can be neglected, the magnetic field  $\mathbf{B}$  inside the material obeys the equation

$$\partial_t \mathbf{B} = \alpha \nabla^2 \mathbf{B}, \quad (6.6)$$

and determine the value of the real constant  $\alpha$ .

The material fills the half-space  $x > 0$  in the presence of a uniform oscillating magnetic field  $\mathbf{B}_0 = \hat{\mathbf{y}} B_0 \cos(\omega t) = \hat{\mathbf{y}} \operatorname{Re}(B_0 e^{-i\omega t})$  in the half-space  $x < 0$ .

- b)** Evaluate the magnetic field  $\mathbf{B}(x, t)$  for  $x > 0$ , assuming that the displacement current is negligible. Discuss under what conditions the result is a good approximation for the case of a finite slab of the material.
- c)** Evaluate the power dissipated in the medium by Joule heating.

## 6.10 A Magnetized Cylinder as DC Generator

A long hard-iron cylinder has height  $h$ , radius  $a \ll h$ , and permanent, uniform magnetization  $\mathbf{M}$  throughout its volume. The magnetization is parallel to the cylinder axis, which we choose as the  $z$  axis of a cylindrical coordinate system  $(r, \phi, z)$ .

**a)** Show that the magnetic field inside the cylinder, far from the two bases, is  $\mathbf{B}_0 \simeq 4\pi(k_m/b_m)\mathbf{M}$ , or  $\mathbf{B}_0 \simeq \mu_0\mathbf{M}$  in SI units,  $\mathbf{B}_0 = 4\pi\mathbf{M}$  in Gaussian units. Show that the magnitude of the  $z$  component of the field at the two bases is  $B_z \simeq B_0/2$ .

**b)** Two brush contacts (the white arrows in Fig. 6.8) connect the center of the upper base of the cylinder,  $A$ , and a point on the equator of the cylinder,  $B$ , to a voltmeter. The cylinder is kept in rotation around the  $z$  axis with constant angular velocity  $\omega$ . Evaluate the electromotive force measured by the voltmeter.

This problem is taken from an example of [1], Section 88, page 379.

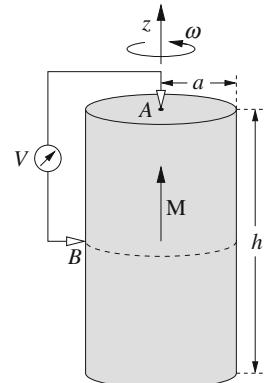


Fig. 6.8

## 6.11 The Faraday Disk and a Self-Sustained Dynamo

A perfectly conducting disk, of radius  $a$  and thickness  $h \ll a$ , rotates at constant angular velocity  $\omega$  (parallel to the disk axis), in the presence of a uniform and constant magnetic field  $\mathbf{B}$  parallel to  $\omega$ .

**a)** Evaluate the electric field  $\mathbf{E}$  in the disk in steady state conditions, and the corresponding potential drop between the center and the boundary of the disk (hint: the total force on charge carriers must be zero at equilibrium).

**b)** We now form a closed circuit by connecting the center of the disk to a point of the circumference by brush contacts (white arrows in the figure), as in Fig. 6.9. Let  $R$  be the total resistance of the resulting circuit. Calculate the external torque needed to keep the disk in rotation at constant angular speed.

**c)** Finally, we place the rotating disk at the center of a long solenoid of radius  $b > a$  and  $n$  turns per unit length. The disk and the solenoid are coaxial, as shown in Fig. 6.10.

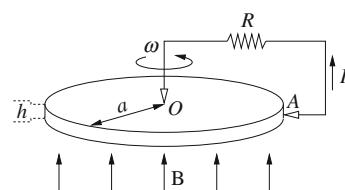


Fig. 6.9

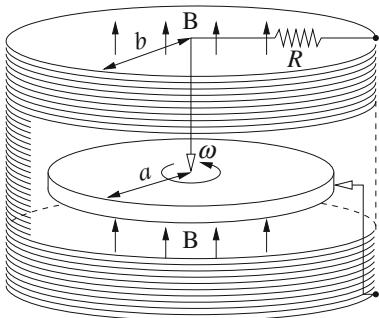
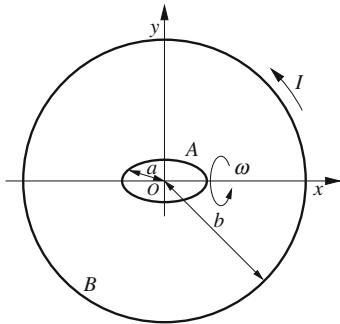
**Fig. 6.10**

Fig. 6.10 illustrates an elementary model for a dynamo self-sustained by rotation, such as the generation mechanism of the Earth's magnetic field [3].

The two brush contacts of point **b**) are now connected to the ends of the solenoid coil, so that the rotating disk provides the current circulating in the turns. The total resistance of the disk-solenoid circuit is  $R$ . The circulating current is thus due to the disk rotation and to the presence of the magnetic field  $\mathbf{B}$ , that the current itself generates by circulating in the solenoid (self-sustained dynamo). Find the value of  $\omega$  for steady-state conditions.

## 6.12 Mutual Induction between Circular Loops

**Fig. 6.11**

The centers of two circular conducting loops  $A$  and  $B$ , of radii  $a$  and  $b \gg a$ , respectively, are located at the origin  $O$  of a Cartesian reference frame. At time  $t = 0$  both loops lie on the  $xy$  plane. While the larger loop remains at rest, the smaller loop, of resistance  $R$ , rotates about one of its diameters, lying on the  $x$  axis, with angular velocity  $\omega$ , as shown in Fig. 6.11. A constant current  $I$  circulates in the larger loop.

- a)** Evaluate the current  $I_A$  induced in loop  $A$ , neglecting self-inductance effects.
- b)** Evaluate the power dissipated in loop  $A$  due to Joule heating.
- c)** Evaluate the torque needed to keep loop  $A$  in rotation, and the associated mechanical power.
- d)** Now consider the case when loop  $A$  is at rest on the  $xy$  plane, with a constant current  $I$  circulating in it, while loop  $B$  rotates around the  $x$  axis with constant angular velocity  $\omega$ . Evaluate the electromotive force induced in  $B$ , neglecting self-inductance effects.

## 6.13 Mutual Induction between a Solenoid and a Loop

A conducting loop of radius  $a$  and resistance  $R$  is located with its center at the center of solenoid of radius  $b > a$  and  $n$  turns per unit length, as in Fig. 6.12. The loop rotates at constant angular velocity  $\omega$  around a diameter perpendicular to the solenoid axis, while a steady current  $I$  flows in the solenoid.

- a)** Evaluate the flux of the magnetic field through the rotating coil as a function of time.
- b)** Evaluate the torque exerted by the external forces on the loop in order to keep it rotating at constant angular velocity.

Now assume that the solenoid is disconnected from the current source, and that the rotating loop is replaced by a magnetic dipole  $\mathbf{m}$ , still rotating at constant angular velocity  $\omega$ , as in Fig. 6.13.

- c)** Evaluate the electromotive force induced in the solenoid.

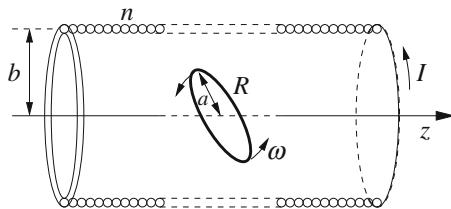


Fig. 6.12

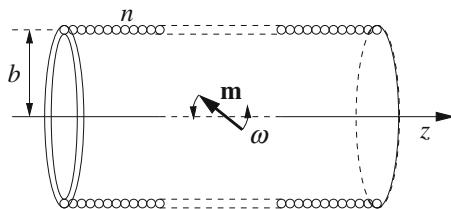
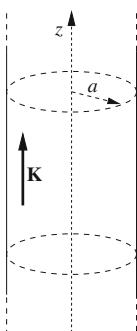


Fig. 6.13

## 6.14 Skin Effect and Eddy Inductance in an Ohmic Wire

A long, straight cylindrical wire of radius  $r_0$  and conductivity  $\sigma$  (which we assume to be real and constant in the frequency range considered) carries an alternating current of angular frequency  $\omega$ . The impedance per unit length of the wire,  $Z_\ell$ , can be defined as the ratio of the electric field at the wire surface to the total current through the wire cross section. Evaluate  $Z_\ell$  as a function of  $\omega$ .

## 6.15 Magnetic Pressure and Pinch effect for a Surface Current



**Fig. 6.14**

A current  $I$  flows on the surface of a cylinder of radius  $a$  and infinite length, in the direction parallel to the axis  $\hat{z}$ . The current layer has negligible thickness, so that we can write  $I = 2\pi a K$ , with  $\mathbf{K} = K \hat{z}$  the surface current density. Calculate

- the magnetic field  $\mathbf{B}$  in the whole space,
- the force per unit surface  $P$  on the cylinder surface
- the variation of magnetic field energy (per unit length)  $dU_m$  associated to an infinitesimal variation of the radius  $da$ . Explain why  $P \neq -(2\pi a)^{-1} dU_m / da$  and how to calculate  $P$  correctly from the energy variation.

Notice: for point b) it might be useful to show first that for a magnetostatic field we have

$$\mathbf{J} \times \mathbf{B} = \frac{1}{4\pi k_m} \left[ (\mathbf{B} \cdot \nabla) \mathbf{B} - \frac{1}{2} \nabla B^2 \right]. \quad (6.7)$$

## 6.16 Magnetic Pressure on a Solenoid

A current source supplies a constant current  $I$  to a solenoid of radius  $a$ , length  $h \gg a$ , so that boundary effects are negligible, and  $n$  coils per unit length.

- Evaluate the magnetic pressure on the solenoid surface directly, by computing the magnetic force on the coils.
- Now evaluate the magnetic pressure on the solenoid surface by evaluating the variation of the magnetic energy of the system for an infinitesimal increase  $da$  of the radius of the solenoid, and the corresponding work done by the current source in order to keep  $I$  constant.

## 6.17 A Homopolar Motor

A *homopolar motor* is a direct current electric motor consisting of a circuit carrying a direct current  $I$  in the presence of a static magnetic field. The circuit is free to rotate around a fixed axis, so that the angle between the current and the magnetic field remains constant in time in each part of the circuit. The resulting electromotive force is continuous, and the homopolar motor needs no device, like a commutator, to switch the current flow. But it still requires slip rings (or brush contacts) to operate. “Homopolar” means that the electrical polarity of the conductor (the direction of the current flow at each point of the circuit) and the magnetic field do not change in time, and the motor does not require commutation. A simple practical realization of a homopolar motor is shown in Figs. 6.15 and 6.16, based on a Wikipedia entry.

The idea is the following: an electrochemical cell drives a DC current into the double circuit shown in the figures, while a magnetic field is generated generated by the permanent magnet cylindrical located at the bottom of the cell, in electrical contact with its negative pole, as shown in Fig. 6.16. The magnetic field has rotational symmetry around the  $z$  axis and is constant in time, in spite of the magnet rotation, and the circuit is free to rotate around the  $z$  axis. The magnetic forces on the current-carrying circuit exert a torque, and the circuit starts to rotate.

The dimensions, mass and resistance of the circuit (the mass includes battery and magnet), the voltage of the battery and the magnetic field strength generated by the magnet at each point of the circuit are known. Find the torque acting on the circuit, and the angular velocity of the system as a function of time,

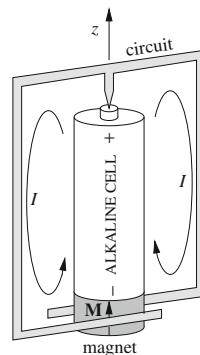


Fig. 6.15

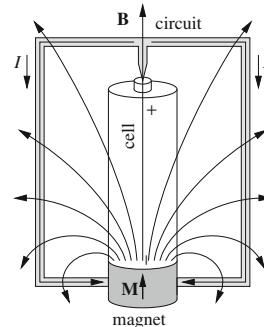


Fig. 6.16

## References

1. R. Becker, *Electromagnetic Fields and Interactions*, vol. I (Electromagnetic Theory and Relativity, Blackie, London and Glasgow, 1964)
2. R. P. Feynman, R. B. Leighton, M. Sands, *The Feynman Lectures on Physics*, Addison-Wesley Publishing Company, Reading, MA, 2006. Volume II, Section 17-4
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# Chapter 7

## Electromagnetic Oscillators and Wave Propagation

**Topics** Harmonic oscillators. Resonances. Coupled oscillators, normal modes and eigenfrequencies. Basics of the Fourier transform. Electric circuits: impedances, simple LC and RLC circuits. Waves. The wave equation. Monochromatic waves. Dispersion. Wavepackets. Phase velocity and group velocity. Transmission lines.

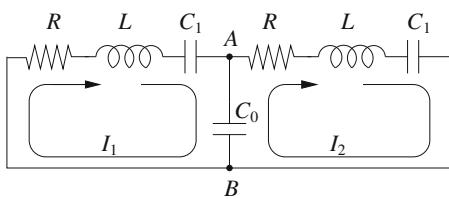
Useful formulas for this chapter:

Fourier transform of the Gaussian function

$$\int_{-\infty}^{+\infty} e^{-(\alpha k)^2} e^{ikx} dk = \frac{\sqrt{\pi}}{\alpha} e^{-x^2/4\alpha^2}, \quad (7.1)$$

where in general  $\alpha$  is a complex number with  $\operatorname{Re}(\alpha) > 0$ .

## 7.1 Coupled RLC Oscillators (1)



**Fig. 7.1**

$I_1$  and  $I_2$  are flowing in the absence of voltage sources, and assume  $R = 0$ .

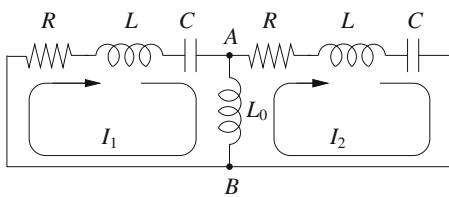
- a) Find the equations for the time evolution of  $I_1$  and  $I_2$ . Describe the normal modes of the system, i.e., look for steady-state solutions of the form

$$I_1(t) = A_1 e^{-i\omega t}, \quad I_2(t) = A_2 e^{-i\omega t}, \quad (7.2)$$

determining the possible values for  $\omega$ . Find a mechanical equivalent of the circuit.

- b) Now consider the effect of the nonzero resistances  $R$  in series with each of the two inductances  $L$ . Find the solutions for  $I_1$  and  $I_2$  in this case.  
c), Evaluate  $I_1$  and  $I_2$  as functions of  $\omega$  if a voltage source  $V = V_0 e^{-i\omega t}$  is inserted into the left mesh of the circuit.

## 7.2 Coupled RLC Oscillators (2)



**Fig. 7.2**

are flowing in the absence of sources, and assume  $R = 0$ . Find the equations for the time evolution of  $I_1 = I_1(t)$  and  $I_2 = I_2(t)$ . Determine the normal modes of the circuit.

- b) Now assume  $R \neq 0$ . Show that now the modes of the system are damped, and determine the damping rates.

Consider an electrical circuit consisting of two identical resistors  $R$ , two identical inductors  $L$ , two identical capacitors  $C_1$ , and a capacitor  $C_0$ , all arranged in two meshes as in Fig. 7.1. Let  $I_1$  and  $I_2$  be the current intensities flowing in the left and right mesh of the circuit, respectively, as shown in the figure. Initially, assume that  $I_1$  and

An electrical circuit consists of two identical resistors  $R$ , two identical inductors  $L$ , two identical capacitors  $C$ , and an inductor  $L_0$ , all arranged in two meshes as in Fig. 7.2. Let  $I_1$  and  $I_2$  be the currents flowing in the left and right mesh of the circuit, respectively, as shown in the figure.

- a) Initially, assume that the currents

### 7.3 Coupled RLC Oscillators (3)

An electrical circuit consists of three identical resistors  $R$ , three identical inductors  $L$ , and two identical capacitors  $C$ , arranged in three meshes as in Fig. 7.3. Let  $I_1$ ,  $I_2$ , and  $I_3$  be the currents flowing in the three meshes, as in the figure. Initially, assume  $R = 0$ .

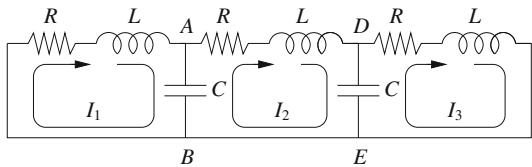


Fig. 7.3

- a) Write the equations for the time evolution of  $I_n(t)$ . Find a mechanical system with three degrees of freedom and the same equations of motion as those for  $I_n(t)$ .
- b) Determine the normal oscillation modes of the system and their frequencies.
- c) Now assume  $R \neq 0$ , and determine the decay rate of the normal modes.

### 7.4 The LC Ladder Network

An  $LC$  ladder network is formed by  $N$  inductors  $L$ , and  $N$  capacitors  $C$ , arranged as shown in Fig. 7.4. We denote by  $I_n = I_n(t)$  the current in the  $n$ th inductor. Resistance effects are assumed to be negligible. The distance between two neighboring nodes is  $a$ .

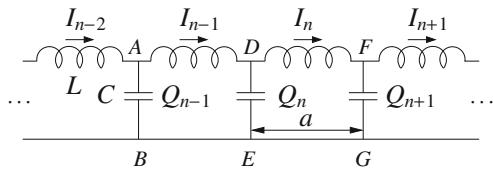


Fig. 7.4

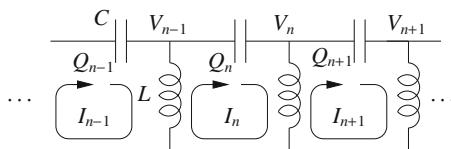
- a) Find the equations for the time evolution of  $I_n$ . Which is a mechanical equivalent of the system?
- b) Show that solutions exist in the form of propagating monochromatic waves

$$I_n = C e^{i(kna - \omega t)} \quad (7.3)$$

and find the dispersion relation between  $k$  and  $\omega$ .

- c) For a given value of  $\omega$ , find the allowed values of  $k$  with the boundary conditions  $I_0 = I_N = 0$ .
- d) Discuss the limit to a continuum system,  $N \rightarrow \infty$ ,  $n \rightarrow \infty$ ,  $a \rightarrow 0$ , with  $na \rightarrow x$ . In this case inductance and capacity are continuously distributed, i.e., defined per unit length.

## 7.5 The *CL* Ladder Network



**Fig. 7.5**

nth mesh, across the  $n$ th capacitor, i.e., between the network nodes at  $V_{n-1}$  and  $V_n$ .

a) Show that the currents  $I_n$  satisfy the coupled equations

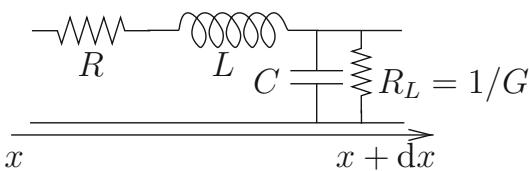
$$L \frac{d^2}{dt^2} (I_{n+1} - 2I_n + I_{n-1}) = \frac{I_n}{C}. \quad (7.4)$$

b) Show that the solutions of (7.4) have the form

$$I_n = A e^{i(kna - \omega t)}, \quad (7.5)$$

with  $a$  the distance between two adjacent network elements, and determine the dispersion relation  $\omega = \omega(k)$ .

## 7.6 Non-Dispersive Transmission Line



**Fig. 7.6**

For the non-dispersive transmission line, there is a resistance  $R$  in series with  $L$ , which accounts for the finite resistivity of the two conductors which form the line. In addition, we assume a finite leakage of current between the two conductors (i.e., in the direction “transverse” to the propagation) which is modeled by a second resistance  $R_L$  in parallel to  $C$ . The corresponding *conductance* is  $G = 1/R_L$ . In the limit of a continuous system with homogeneous, distributed properties, we define all quantities per unit length by replacing  $R$  with  $R_\ell dx$ ,  $L$  with  $L_\ell dx$ ,  $C$  with  $C_\ell dx$  and  $G$  with  $G_\ell dx$  (it is proper to use  $G$  as a quantity defined per unit length instead of  $R_L$  because the latter is proportional to the *inverse* of the length of the line).

The “elementary cell” scheme of a transmission line is sketched in the figure. In addition to the inductance  $L$  and capacitance  $C$  typical of the ideal “LC” trans-

**a)** Show that the current intensity  $I = I(x, t)$  satisfies the equation

$$(\partial_x^2 - L_\ell C_\ell \partial_t^2)I = (R_\ell C_\ell + L_\ell G_\ell)\partial_t I + R_\ell G_\ell I = 0. \quad (7.6)$$

**b)** Study the propagation of a monochromatic current signal of frequency  $\omega$ , i.e., search for solutions

$$I = I_0 e^{ikx-i\omega t}, \quad (7.7)$$

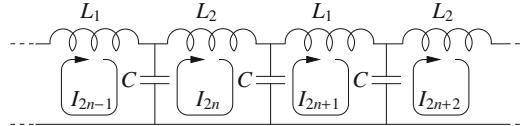
for  $x > 0$  with the boundary condition  $I(0, t) = I_0 e^{-i\omega t}$ , and determine the dispersion relation  $k = k(\omega)$ .

**c)** Find the condition on the line parameters for which a wavepacket traveling along the lines undergoes attenuation of the amplitude but *no* dispersion. This condition corresponds to solutions having the general form

$$I(x, t) = e^{-\kappa x} f(x - vt), \quad (7.8)$$

where  $f(x)$  is an arbitrary differentiable function. Find the expression for  $v$  and  $\kappa$ .

## 7.7 An “Alternate” LC Ladder Network



**Fig. 7.7**

Consider an “alternate” LC ladder network comprising identical capacitors  $C$  and inductors of value alternatively  $L_1$  and  $L_2$ , as shown in Fig. 7.7. Let  $I_{2n}$  be the current flowing in the mesh of the  $n$ th inductor of value  $L_2$ , and  $I_{2n+1}$  the current flowing in the  $n$ -th inductor of value  $L_1$ .

**a)** Show that the currents satisfy the equations

$$L_2 \frac{d^2 I_{2n}}{dt^2} = \frac{1}{C} (I_{2n-1} - 2I_{2n} + I_{2n+1}), \quad L_1 \frac{d^2 I_{2n+1}}{dt^2} = \frac{1}{C} (I_{2n} - 2I_{2n+1} + I_{2n+2}). \quad (7.9)$$

What is a mechanical equivalent of this network?

**b)** Search for solutions of (7.9) of the form

$$I_{2n} = I_e e^{i[2nka-\omega t]}, \quad I_{2n+1} = I_o e^{i[(2n+1)ka-\omega t]}, \quad (7.10)$$

where  $I_e$  and  $I_o$  (the subscripts “e” and “o” stand for *even* and *odd*, respectively) are two constants, and determine the dispersion relation  $\omega = \omega(k)$ . Determine the allowed frequency range for wave propagation (for simplicity, assume  $L_2 \ll L_1$ ).

## 7.8 Resonances in an LC Ladder Network

Consider the semi-infinite *LC* ladder network shown in Fig. 7.8. Let  $I_n = I_n(t)$  be the current flowing in the  $n$ -th mesh of the circuit. An ideal current source provides the input current

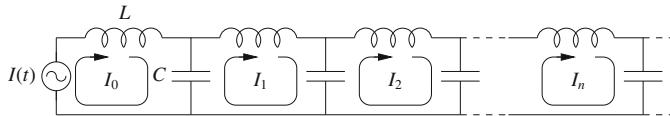
$$I(t) = I_s e^{-i\omega t}, \quad (7.11)$$

where

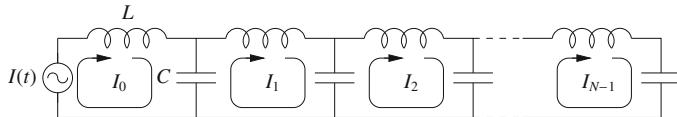
- a) Assuming  $\omega < 2\omega_0$ , evaluate  $I_n(t)$  as a function of  $I_s$  and  $\omega$ .
- b) Now find  $I_n(t)$  assuming  $\omega > 2\omega_0$ . Hint: search for a solution of the form

$$I_n(t) = A\alpha^n e^{-i\omega t}, \quad (7.12)$$

determining the dependence of  $\alpha$  on  $\omega$  and  $\omega_0$ .



**Fig. 7.8**



**Fig. 7.9**

Now assume that our *LC* ladder is finite, comprising  $N$  meshes numbered from 0 to  $N - 1$ , as in Fig. 7.9. Evaluate  $I_n(t)$  both for the case  $\omega > 2\omega_0$  and for the case  $\omega < 2\omega_0$ , determining for which values of  $\omega$  resonances are observed.

## 7.9 Cyclotron Resonances (1)

Consider a particle of charge  $q$  and mass  $m$  in the presence of a constant, uniform magnetic field  $\mathbf{B} = B_0 \hat{\mathbf{z}}$ , and of a uniform electric field of amplitude  $E_0$ , *rotating* with frequency  $\omega$  in the  $(x, y)$  plane, either in clockwise or in counterclockwise direction (Fig. 7.10 shows the counterclockwise case).

- a)** Describe the motion of the particle as a function of  $B$ ,  $E$ , and  $\omega$ , and show that, given  $B$ , a resonance is observed for the appropriate sign and value of  $\omega$ .
- b)** Evaluate the solution of the equations of motion at resonance in the absence of friction.
- c)** Now assume the presence of a frictional force  $\mathbf{f} = -m\gamma\mathbf{v}$ , where  $\mathbf{v}$  is the velocity of the particle. Find the steady-state solution of the equations of motion, and calculate the power dissipated by friction as a function of  $\omega$ .

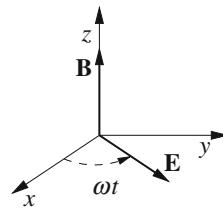


Fig. 7.10

## 7.10 Cyclotron Resonances (2)

Consider a particle of charge  $q$  and mass  $m$  in the presence of a constant uniform magnetic field  $\mathbf{B} = B_0\hat{\mathbf{z}}$ , and of an oscillating uniform electric field  $\mathbf{E} = E_0\hat{\mathbf{x}}\cos\omega t$ .

- a)** Write the equations of motion (assuming no friction) and determine the resonance frequency of the system (hint: show that the equations for the velocity components  $v_x$  and  $v_y$  can be separated into two uncoupled equations of the forced harmonic oscillator type)
- b)** Now assume the presence of a frictional force  $\mathbf{f} = -m\gamma\mathbf{v}$  where  $\gamma \ll \omega$  and  $\gamma \ll qB_0/m$ . Find the steady state solution of the equations of motion and the spectrum of the absorbed power (hint: the equations for  $v_x$  and  $v_y$  cannot be separated in this case, but seeking a solution in the form  $\mathbf{v} = \mathbf{v}_0e^{-i\omega t}$ , with  $\mathbf{v}_0$  a complex vector, will work).

## 7.11 A Quasi-Gaussian Wave Packet

Let us consider a wave packet of Gaussian profile propagating with velocity  $v$  along the  $x$  axis in a non-dispersive medium, with dispersion relation  $\omega(k) = kv$ . In these conditions, the wave packet's profile remains constant, and the packet is described by the function  $g(x - vt)$  (Fig. 7.11)

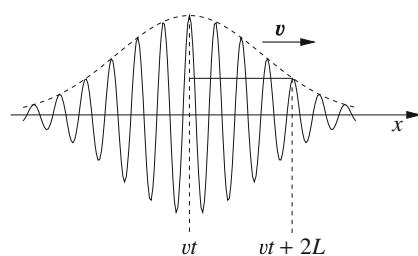


Fig. 7.11

$$\begin{aligned} g(x - vt) &= \sqrt{\pi} \frac{A}{L} e^{ik_0(x-vt)} e^{-(x-vt)^2/4L^2} \\ &= \int_{-\infty}^{+\infty} \tilde{g}(k) e^{ik(x-vt)} dk, \end{aligned} \quad (7.13)$$

where  $L$ ,  $A$  and  $k_0$  are constant parameters, and  $\tilde{g}(k) = A e^{-(k-k_0)^2 L^2}$  is the Fourier transform of  $g$ . Now consider a second wave packet described by a function  $f$ , whose Fourier transform is

$$\tilde{f}(k) = \tilde{g}(k) e^{i\phi(k)} = A e^{-(k-k_0)^2 L^2} e^{i\phi(k)}, \quad (7.14)$$

where the “phase perturbation”  $\phi(k)$  is a smooth function, that can be approximated by its Taylor polynomial expansion of degree 2 around  $k = k_0$ ,

$$\phi(k) \simeq \phi(k_0) + \phi'(k_0)(k - k_0) + \frac{1}{2}\phi''(k_0)(k - k_0)^2, \quad (7.15)$$

where  $\phi'$  and  $\phi''$  are the first and second derivatives of  $\phi$ . The second wave packet can be considered as an “attempt” to build up a Gaussian wave packet from its spectral components, but with some error on the relative phases of the components themselves. Find the width of the wave packet and discuss its shape in order to show its deviations from the Gaussian profile.

## 7.12 A Wave Packet along a Weakly Dispersive Line

A transmission line extends from  $x = 0$  to  $x = +\infty$ . A generator at  $x = 0$  inputs a signal

$$f(t) = A e^{-i\omega_0 t} e^{-t^2/\tau^2}, \quad (7.16)$$

where  $A$  and  $\tau$  are constant and  $\omega_0 \tau \gg 1$ , i.e., the signal is “quasi-monochromatic”. The dispersion relation of the transmission line can be written

$$\omega = \omega(k) = kv(1 + bk), \quad (7.17)$$

where  $v$  and  $b$  are known constants, and we assume  $k > 0$ .

**a)** Find the expression  $f(x, t)$  for the propagating signal, i.e., for the wave packet traveling along the line, assuming  $b = 0$ .

From now on, assume dispersive effects to be small but not negligible, i.e., assume  $b k_0 \ll 1$ , where  $k_0 = k(\omega_0)$  according to (7.17).

**b)** Within the above approximation, write the phase and group velocities as functions of  $\omega_0$  to the lowest order at which dispersive effects are present.

- c)** Give an estimate of the instant  $t_x$  when the “peak” of the signal reaches the position  $x$ , and of the corresponding length of the wave packet.
- d)** Now find the expression of the wave-packet shape as a function of  $(x, t)$ , by calculating the integral

$$f(x, t) = \int e^{ik(\omega)x - i\omega t} \tilde{f}(\omega) d\omega, \quad (7.18)$$

where  $\tilde{f}(\omega)$  is the Fourier transform of the wave packet. As a reasonable approximation, keep only factors up to the second order in  $(k - k_0)^2$ , for instance use

$$k(\omega) \approx k(\omega_0) + k'(\omega_0)(\omega - \omega_0) + \frac{1}{2}k''(\omega_0)(\omega - \omega_0)^2. \quad (7.19)$$

# Chapter 8

## Maxwell Equations and Conservation Laws

**Topics** Maxwell's equations. Conservation laws: energy, momentum and angular momentum of the electromagnetic field. Poynting's theorem. Radiation pressure.

Basic equations of this chapter:

(Note: Gaussian cgs units are used in this chapter unless otherwise specified.)

Maxwell's equations

$$\nabla \cdot \mathbf{E} = 4\pi\rho , \quad (8.1)$$

$$\nabla \cdot \mathbf{B} = 0 , \quad (8.2)$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\partial_t \mathbf{B} , \quad (8.3)$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\partial_t \mathbf{E} . \quad (8.4)$$

Energy conservation (Poynting's) theorem

$$\partial_t u + \nabla \cdot \mathbf{S} = -\mathbf{J} \cdot \mathbf{E} , \quad (8.5)$$

where

$$u = \frac{1}{8\pi} (\mathbf{E}^2 + \mathbf{B}^2) \quad (8.6)$$

is the energy density of the EM field, and

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} \quad (8.7)$$

is the Poynting (also named Poynting-Umov) vector.

Momentum conservation theorem:

$$\partial_t \mathbf{g} + \nabla \cdot \mathbf{T} = -\left( \rho \mathbf{E} + \frac{1}{c} \mathbf{J} \times \mathbf{B} \right), \quad (8.8)$$

where

$$\mathbf{g} = \frac{1}{4\pi c} (\mathbf{E} \times \mathbf{B}) = \frac{\mathbf{S}}{c^2} \quad (8.9)$$

is the momentum density of the EM field, and  $\mathbf{T}$  is Maxwell's stress tensor with components

$$T_{ij} = \frac{1}{4\pi} \left[ \frac{1}{2} (\mathbf{E}^2 + \mathbf{B}^2) \delta_{ij} - E_i E_j - B_i B_j \right]. \quad (8.10)$$

Thus,  $\nabla \cdot \mathbf{T}$  is a vector with components

$$(\nabla \cdot \mathbf{T})_i = \sum_{j=1}^{j=3} \partial_j T_{ij}. \quad (8.11)$$

Angular momentum density of an EM field

$$\boldsymbol{\ell} = \mathbf{r} \times \mathbf{g} = \mathbf{r} \times \frac{\mathbf{S}}{c}. \quad (8.12)$$

## 8.1 Poynting Vector(s) in an Ohmic Wire

A constant and uniformly distributed current density  $\mathbf{J} = \sigma \mathbf{E}$  flows inside an infinite straight wire of radius  $a$  and conductivity  $\sigma$ .

- a)** Calculate the Poynting vector  $\mathbf{S} = (c/4\pi)\mathbf{E} \times \mathbf{B}$  and discuss the energy conservation in the wire.
- b)** The Poynting vector occurs in Poynting's theorem only through its divergence, since the theorem only requires that the flux of the Poynting vector through any a closed surface describes the net flow of electromagnetic energy. Show that, consequently,  $\mathbf{S}' = \varphi \mathbf{J}$ , where  $\varphi$  is the electrostatic potential, is also a suitable choice for  $\mathbf{S}$  (hint: substitute  $\mathbf{E} = -\nabla\varphi$  into (8.7) and manipulate the result).

## 8.2 Poynting Vector(s) in a Capacitor

A plane capacitor consists of two parallel circular plates of radius  $a$ , at a distance  $h \ll a$  from each other. The electric field inside the capacitor is slowly varying in time,  $\mathbf{E} = E(t)\hat{\mathbf{z}}$ , for instance, assume  $E = E_0 t/\tau$ . Boundary effects are negligible (Fig. 8.1).

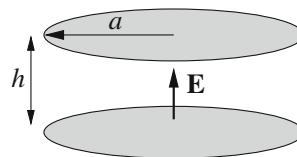


Fig. 8.1

- a)** Evaluate the magnetic field  $\mathbf{B}$  inside the capacitor.

**b)** Calculate the Poynting vector  $\mathbf{S} = (c/4\pi)\mathbf{E} \times \mathbf{B}$ , and show that the flux of  $\mathbf{S}$  though any surface enclosing the capacitor equals the time variation of the energy associated to the electromagnetic field.

**c)** Show that an alternative Poynting vector is

$$\mathbf{S}' = \frac{1}{4\pi} \varphi \partial_t \mathbf{E}, \quad (8.13)$$

where  $\varphi$  is the electric potential ( $\mathbf{E} = -\nabla\varphi$ ). Verify that also the flux of  $\mathbf{S}'$  through the closed surface of point **b**) equals the variation of the energy in the volume inside the surface [hint: proceed as in point **b**) of Problem 8.1].

## 8.3 Poynting's Theorem in a Solenoid

A time-dependent current,  $I = I(t) = I_0 t/\tau$ , flows through the coils of an infinitely long, cylindrical solenoid. The solenoid has radius  $a$  and  $n$  turns per unit length.

- a)** Find the magnetic and electric fields,  $\mathbf{B}$  and  $\mathbf{E}$ , inside the solenoid.

- b)** Verify the law of energy conservation (Poynting's theorem), for a closed internal cylindrical surface, coaxial to the solenoid.
- c)** Now verify Poynting's theorem for an *external*, coaxial cylindrical surface (remember that  $\mathbf{B} = 0$  outside an infinite solenoid).

## 8.4 Poynting Vector in a Capacitor with Moving Plates

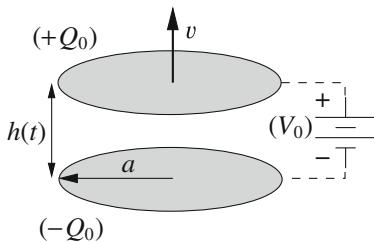


Fig. 8.2

A plane capacitor consists of two circular metallic plates of radius  $a$ , parallel to each other. One plate is kept at rest while the other moves at constant velocity  $v$ , so that the distance between the plates is  $h = h(t) = h_0 + vt$ . In the following we consider only the case in which  $h \ll a$  at any time  $t$ , so that boundary effects are negligible. We also assume that  $v$  is small enough to ensure the validity of the slowly varying current approximation 7.11 A Quasi-Gaussian Wave Packet (Fig. 8.2).

Considering both the case of electrically isolated plates having opposite charges  $\pm Q_0$ , and the case of plates connected through a voltage source keeping a constant electric potential drop  $V_0$  between them, calculate

- the force  $F$  needed to keep  $v$  constant,
- the rate of change of the electrostatic energy  $U$ ,
- the magnetic field between the plates,
- the Poynting vector  $\mathbf{S}$  and its flux through a cylindrical surface enclosing, and coaxial with, the capacitor; use this last result to discuss energy conservation in the system.

## 8.5 Radiation Pressure on a Perfect Mirror

A perfect mirror is defined as a medium inside which  $\mathbf{E} = 0$  and  $\mathbf{B} = 0$ . Thus, an EM wave cannot penetrate the mirror surface and will be reflected by it.

Find the radiation pressure  $P_{\text{rad}}$ , i.e., the cycle-averaged force per unit surface exerted by a plane wave incident on the surface of a perfect plane mirror, as a function of the intensity  $I$  of the wave by each of the following three methods:

- Consider the reflection of a square wave packet of arbitrary, but finite, duration. Determine  $P_{\text{rad}}$  from the difference between the total momentum of the incident wave packet and the momentum of the reflected wave packet.

- b)** Calculate the force on the mirror directly, from the knowledge of the EM fields and of the charge and current densities on the mirror surface.  
**c)** Determine  $P_{\text{rad}}$  from Maxwell's stress tensor.

## 8.6 A Gaussian Beam

In optics, a Gaussian beam is a beam of monochromatic electromagnetic radiation whose transverse magnetic and electric field amplitude profiles are given by the Gaussian function. Gaussian beams are important because they are a very good approximation of the radiation emitted by most laser sources. Here we consider a linearly-polarized Gaussian beam propagating along the  $z$ -axis and whose transverse profile is symmetrical around such axis. The origin of the coordinate systems is chosen so that the beam has minimum width on the  $z = 0$  plane. We assume that, *close to the  $z = 0$  plane*, the *transverse* components of the EM fields can be written as

$$\begin{aligned} E_x &= E_0(r) \cos(kz - \omega t) = E_0 e^{-r^2/r_0^2} \cos(kz - \omega t), \\ B_y &= B_0(r) \cos(kz - \omega t) = B_0 e^{-r^2/r_0^2} \cos(kz - \omega t), \end{aligned} \quad (8.14)$$

where  $r = \sqrt{x^2 + y^2} < r_0$  and  $k = \omega/c$ . The parameter  $r_0$  is called the *waist* of the beam.

- a)** Show that, in addition to the transverse components (8.14), *longitudinal* components  $E_z$  and  $B_z$  must exist, and give their expression.  
**b)** Compute the Poynting vector of the beam  $\mathbf{S}$ , and its average over a period  $\langle \mathbf{S} \rangle$ , showing which components are vanishing.  
**c)** Verify that the fields (8.14) do *not* satisfy the wave equation in vacuum, hence they are only an approximate expression, as mentioned above. Explain in which range of  $z$ , depending on the value of  $kr_0$ , the approximate expressions are accurate.

## 8.7 Intensity and Angular Momentum of a Light Beam

A circularly polarized monochromatic light beam of frequency  $\omega$  propagates along the  $z$  direction. The beam has a finite width in the plane perpendicular to  $z$ . We assume that in a region of space, close to the “waist” (i.e., to the plane where the beam has minimal width), the *transverse* components of the EM fields can be written approximately as

$$\begin{aligned} E_x &= +E_0(r) \cos(kz - \omega t), & E_y &= -E_0(r) \sin(kz - \omega t), \\ B_x &= E_0(r) \sin(kz - \omega t), & B_y &= E_0(r) \cos(kz - \omega t), \end{aligned} \quad (8.15)$$

where  $r = \sqrt{x^2 + y^2}$ ,  $k = \omega/c$ , and  $E_0(r)$  is a known real function.

- a)** Write the intensity  $I = I(r)$ , defined as the “energy flow along  $z$ ”, i.e.,  $I(r) = \mathbf{S} \cdot \hat{\mathbf{z}}$  where  $\mathbf{S}$  is the Poynting vector.
- b)** Show that, in addition to the transverse components of the fields, also *longitudinal* components ( $E_z$ ,  $B_z$ ) must exist, and give their expression.
- c)** Evaluate the  $S_x$  and  $S_y$  component of  $\mathbf{S}$ , and discuss the result.
- d)** Show that the density of angular momentum (8.12) of the beam can be written as

$$\ell_z = \ell_z(r) = -\frac{r}{2c\omega} dr I, \quad (8.16)$$

and compute the quantity

$$L_z = \int_0^\infty \ell_z(r) 2\pi r dr \quad (8.17)$$

as a function of the total power of the beam  $W = \int_0^\infty I(r) 2\pi r dr$ .

## 8.8 Feynman's Paradox solved

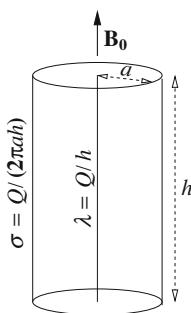


Fig. 8.3

uniform magnetic field  $\mathbf{B}_{\text{ext}}$ , parallel to the system axis. Assume that boundary effects can be neglected.

The system in Fig. 8.3 is composed by a non-conducting cylindrical surface of height  $h$  and radius  $a$ , over which there is a net charge  $Q$  uniformly distributed with surface density  $\sigma = Q/(2\pi ah)$ , and a wire of same length oriented along the cylinder axis and having charge  $-Q$  distributed with uniform linear density  $\lambda = -Q/h$ , so that the system is globally neutral. The cylindrical surface is free to rotate around its axis without friction, and has moment of inertia  $I$  per unit length. The system is at rest in the presence of an external

Starting at time  $t = 0$ , the external magnetic field is reduced from its initial value  $\mathbf{B}_{\text{ext}} = \mathbf{B}_0$  to zero at a time  $t_f \gg a/c$ , according to some temporal law  $\mathbf{B}_{\text{ext}} = \mathbf{B}_{\text{ext}}(t)$ .

- a)** Initially assuming that the field generated by the motion of the charges on the cylinder is negligible, evaluate the angular velocity  $\omega = \omega(t)$  of the cylinder as a function of time during the decay of  $\mathbf{B}_{\text{ext}}$ , and the corresponding mechanical angular momentum  $\mathbf{L}_c$  of the cylinder.
- b)** Now take the field generated by rotating charges into account, and evaluate how the results of **a)** change.
- c)** Consistently with Eqs. (8.8–8.9), we introduce the *angular momentum* of a given distribution of electromagnetic fields as

$$\mathbf{L}_{\text{EM}} = \int \mathbf{r} \times \mathbf{g} d^3r, \quad (8.18)$$

where  $\mathbf{g} = \mathbf{E} \times \mathbf{B}/4\pi$  is the electromagnetic momentum density. Use Eq. (8.18) to check the conservation of the *total* angular momentum for the system (thus solving the “paradox” as outlined in Problem 6.6).

## 8.9 Magnetic Monopoles

Assume that an experiment gives evidence of the existence of “magnetic monopoles”, i.e., of point-like particles with a *net magnetic charge*  $q_m$ , such that the magnetic field  $\mathbf{B}_m$  generated by such charge is

$$\mathbf{B}_m = \alpha \frac{q_m}{r^2} \hat{\mathbf{r}}, \quad (8.19)$$

while in the presence of an “external” magnetic field  $\mathbf{B}_{\text{ext}}$  the force on the particle is  $\mathbf{f} = q_m \mathbf{B}_{\text{ext}}$ . Thus, for example, the interaction force between two particles with magnetic charges  $q_{m1}$  and  $q_{m2}$  is given by

$$\mathbf{f}_{1 \rightarrow 2} = \alpha \frac{q_{m1} q_{m2}}{r_{12}^2} \hat{\mathbf{r}}_{12}, \quad \mathbf{f}_{2 \rightarrow 1} = -\mathbf{f}_{1 \rightarrow 2}. \quad (8.20)$$

where  $\mathbf{r}_{12}$  is the distance vector directed from charge 1 to charge 2. We also assume that conservation of the total magnetic charge holds.

- a)** Determine, both in SI and Gaussian units, the expressions for the coefficient  $\alpha$  and the dimensions of the magnetic charge  $q_m$  with respect to the electric charge  $q_e$ . (Hint: we may assume that the field generated by two magnetic charges  $+q_m$  and  $-q_m$ , separated by a distance  $\mathbf{h}$ , is equivalent to the field of a magnetic dipole  $\mathbf{m} = q_m \mathbf{h}$  at distances  $r \gg |\mathbf{h}|$ .)
- b)** Complete Maxwell’s equations in order to take the presence of magnetic monopoles into account.
- c)** Now consider a beam of magnetic monopoles of radius  $a$ , of uniform density and infinite length. The number density of the particles of the beam is  $n$ , and all particles have the same magnetic charge  $q_m$  and the same velocity  $v$ . Find the electric and magnetic fields generated by the beam.

# Chapter 9

## Relativistic Transformations of the Fields

**Topics** Relativistic covariance of Maxwell's equations. Four-vectors in electromagnetism: four-current, four-potential. The electromagnetic four-tensor. Lorentz transformations of the fields.

Basic equations of this chapter:

Relation of four-current and four-potential to densities and potentials in three-dimensional space

$$J_\mu \equiv (\rho c, \mathbf{J}), \quad A_\mu = (\phi, \mathbf{A}). \quad (9.1)$$

Lorentz transformations of a four-vector  $K_\mu = (K_0, \mathbf{K})$  from the frame  $S$  to the frame  $S'$  moving with relative velocity  $v = \beta c$  with respect to  $S$ :

$$K'_0 = \gamma(K_0 - \beta \cdot \mathbf{K}), \quad K'_{||} = \gamma(K_{||} - \beta K_0), \quad \mathbf{K}'_{\perp} = \mathbf{K}_{\perp}, \quad (9.2)$$

where the subscripts “ $||$ ” and “ $\perp$ ” denote the directions parallel and perpendicular to  $\beta$ , respectively, and  $\gamma = 1/\sqrt{1-\beta^2}$ .

Compact three-dimensional formulas for the transformation of the EM fields are

$$E'_{||} = E_{||}, \quad \mathbf{E}'_{\perp} = \gamma(\mathbf{E}_{\perp} + \beta \times \mathbf{B}), \quad B'_{||} = B_{||}, \quad \mathbf{B}'_{\perp} = \gamma(\mathbf{B}_{\perp} - \beta \times \mathbf{E}), \quad (9.3)$$

or, equivalently

$$\mathbf{E}' = \gamma(\mathbf{E} + \beta \times \mathbf{B}) - \frac{\gamma^2}{\gamma+1} \beta(\beta \cdot \mathbf{E}), \quad (9.4)$$

$$\mathbf{B}' = \gamma(\mathbf{B} - \beta \times \mathbf{E}) - \frac{\gamma^2}{\gamma+1} \beta(\beta \cdot \mathbf{B}). \quad (9.5)$$

The three-dimensional “Newtonian” force transforms as

$$F'_{\parallel} = F_{\parallel} - \frac{V\mathbf{F} \cdot \mathbf{v}_{\perp}/c^2}{1 - Vv_{\parallel}/c^2}, \quad F'_{\perp} = \frac{\mathbf{F}_{\perp}}{\gamma(1 - Vv_{\parallel}/c^2)}, \quad (9.6)$$

where  $V$  is the boost velocity, and  $v_{\parallel}$  and  $v_{\perp}$  are the components of the particle velocity parallel and perpendicular to  $\mathbf{V}$ , respectively. We have  $\mathbf{F} = d\mathbf{p}/dt$  in the  $S$  frame and  $\mathbf{F}' = d\mathbf{p}'/dt'$  in the  $S'$  frame.

## 9.1 The Fields of a Current-Carrying Wire

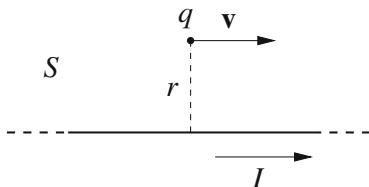


Fig. 9.1

ate Lorentz transformation. What can be inferred on the EM fields in  $S'$  from the expression of  $\mathbf{F}'$ ?

- b)** Use the Lorentz transformations to obtain the charge and current densities of the wire in  $S'$ , and the related EM fields. Evaluate the scalar and vector potentials in  $S'$ . Compare the results to what obtained from the direct transformation rules for the EM field.
- c)** The answers to points **a)** and **b)** imply that in  $S'$  there is a net charge density on the wire. Recover this result by calculating the linear densities of electrons (flowing with velocity  $v_e$  in  $S$ ) and ions (at rest in  $S$ ) in  $S'$  via the Lorentz transformations for velocity and length. (this last point corresponds to the one presented by E.M. Purcell in Ref. [1].)

## 9.2 The Fields of a Plane Capacitor

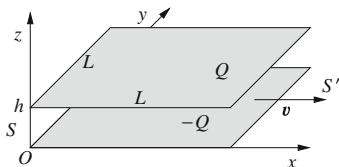


Fig. 9.2

Evaluate, in a reference frame  $S'$  moving with respect to  $S$  with velocity  $v = \beta c$  parallel to the capacitor plates,

In the laboratory frame  $S$ , a plane capacitor has parallel square plates of area  $A = L^2$ , located at a distance  $h \ll L$  from each other, so that the boundary effects can be assumed to be negligible. The plates have electric charges  $\pm Q$ , uniformly distributed over their surfaces, with surface charge density  $\pm\sigma = \pm Q/A$ , respectively (Fig. 9.2).

- a) the electric and magnetic fields in the region between the plates;
- b) the sources of the fields;
- c) the force per unit surface and the total force on each plate, comparing the results to the corresponding values in  $S$ .

### 9.3 The Fields of a Solenoid

In the laboratory frame  $S$  a constant current  $I$  flows in an infinite solenoid of radius  $R$  and  $n$  turns per unit length. At a given instant  $t = 0$  a test particle of charge  $q$  is located inside the solenoid, with a velocity  $v$  perpendicular to the axis of the solenoid, as shown in Fig. 9.3.

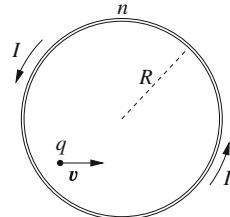


Fig. 9.3

- a) Find the electromagnetic fields and the force on the particle both in  $S$ , and in the frame  $S'$  where the particle is instantaneously at rest ( $v' = 0$ ).
- b) Assuming  $v/c \ll 1$ , evaluate the sources of the fields in  $S'$  up to the first order in  $v/c$ .

### 9.4 The Four-Potential of a Plane Wave

Consider a monochromatic plane wave, propagating in vacuum along the  $x$  axis of the Cartesian laboratory frame  $S$ , linearly polarized along  $\hat{y}$ , and of frequency  $\omega$ .

- a) Show that the electric field  $\mathbf{E} = \mathbf{E}(x, t)$  and the magnetic field  $\mathbf{B} = \mathbf{B}(x, t)$  of the wave can be obtained from a suitable four-potential  $A_\mu = (\Phi, \mathbf{A}) = (0, 0, A_y, 0)$ .

Now consider the same wave observed in a frame  $S'$ , moving with velocity  $v = v\hat{y}$  with respect to  $S$ .

- b) Evaluate the frequency  $\omega'$  and the wave vector  $\mathbf{k}'$  of the wave in  $S'$ . Calculate the electric field  $\mathbf{E}' = \mathbf{E}'(\mathbf{r}', t')$  and the magnetic field  $\mathbf{B}' = \mathbf{B}'(\mathbf{r}', t')$  in  $S'$  as functions of  $\mathbf{E}$  in the  $S$  frame.

- c) Verify that the wave is linearly polarized in  $S'$  and show that  $\mathbf{E}'$  and  $\mathbf{B}'$  can be obtained from a four-potential  $A'_\mu = (0, \mathbf{A}')$ , where  $\mathbf{A}' = \mathbf{A}'(\mathbf{r}', t')$ .

- d) Find the four-potential  $\tilde{A}'_\mu$  obtained from  $A_\mu$  through a Lorentz transformation. Verify that  $\mathbf{E}'$  and  $\mathbf{B}'$  can be obtained also from  $\tilde{A}'_\mu$ .

- e) Show that  $A'_\mu$  and  $\tilde{A}'_\mu$  are related by a gauge transformation.

### 9.5 The Force on a Magnetic Monopole

Assume that an experiment has given evidence for the existence of magnetic monopoles, i.e., point-like particles which, in the presence of a magnetic field  $\mathbf{B}$ , are subject to a force

$$\mathbf{F}_m = q_m \mathbf{B}, \quad (9.7)$$

where  $q_m$  is the magnetic charge of the monopole. We assume that these particles have no electric charge.

- a) Show that the force exerted by an electric field  $\mathbf{E}$  on a monopole moving in the laboratory frame with velocity  $v$  is

$$\mathbf{F}_e = -q_m \frac{\mathbf{v}}{c} \times \mathbf{E}. \quad (9.8)$$

- b) The “Lorentz force” on a magnetic monopole is the sum of (9.7) and (9.8). Use this expression to study the motion of a magnetic monopole of mass  $m$  in either an electric field  $\mathbf{E}$  or in perpendicular  $\mathbf{E}$  and  $\mathbf{B}$  fields, where the fields are both constant and uniform, and  $E > B$ . For simplicity assume a non-relativistic motion. Compare the results to those of Problem 5.6, point a).

## 9.6 Reflection from a Moving Mirror

An electromagnetic wave of frequency  $\omega$  and electric field amplitude  $E_i$ , linearly polarized along the  $y$  axis, is perpendicularly incident on a perfect conductor whose bounding surface lies on the  $yz$  plane. The perfect conductor behaves as a perfect mirror, i.e., we have  $\mathbf{E} = 0$  and  $\mathbf{B} = 0$  inside the material ( $x > 0$ ).

- a) Evaluate the field of the reflected wave and the total electromagnetic field. The mirror is now set in motion with respect to the laboratory frame  $S$ , with a constant velocity  $\mathbf{v} = \hat{x}v$  parallel to the  $x$  axis.  
b) Find the frequencies and the fields of the incident and reflected waves in the  $S'$  frame, where the mirror is at rest.  
c) Find the frequency and the fields of the reflected wave in the  $S$  frame.  
d) Discuss the continuity of the fields at the moving mirror surface.

## 9.7 Oblique Incidence on a Moving Mirror

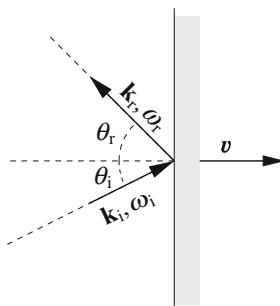


Fig. 9.4

In the laboratory frame  $S$ , a perfectly reflecting mirror moves with constant velocity  $v$ , perpendicular to its surface. In  $S$ , the wave vector  $\mathbf{k}_i$  of an incident EM wave makes an angle  $\theta_i$  with the normal to the mirror surface, as in Fig. 9.4. The incident wave has frequency  $\omega_i$ . Find

- a) the frequency  $\omega'_i$  of the incident wave, the incidence angle  $\theta'_i$ , and the reflection angle  $\theta'_r$  in the  $S'$  frame, where the mirror is at rest;  
b) the frequency  $\omega_r$  of the reflected wave, and the reflection angle  $\theta_r$ , in the  $S$  frame. What happens if  $\cos\theta_i \geq v/c$ ?

## 9.8 Pulse Modification by a Moving Mirror

In the laboratory frame  $S$  we have an EM square wave packet of amplitude  $E_i$ , comprising  $N$  complete oscillations of frequency  $\omega_i$ , therefore of duration  $\tau_i = 2\pi N/\omega_i$ . Assume that  $N \gg 1$ , so that the packet is “quasi-monochromatic”. The wave packet impinges perpendicularly on a perfect mirror. In the laboratory frame, the mirror itself is moving with constant velocity  $v$  perpendicularly to its surface (Fig. 9.5).

- Determine the form, duration and amplitude of the reflected wave packet.
- Compare the total energies of the incident and reflected wave packets, and determine the amount of mechanical work  $W$  done by the mirror during the reflection stage (consider all quantities per unit surface).
- Show that  $W$  is equal to the integral over time and volume of  $\mathbf{J} \cdot \mathbf{E}$ , in agreement with Poynting’s theorem.

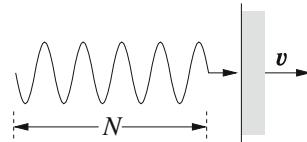


Fig. 9.5

## 9.9 Boundary Conditions on a Moving Mirror

The reflecting surface of a perfect mirror is parallel to the  $yz$  plane of a laboratory Cartesian reference frame  $S$ . The mirror is translating with constant velocity  $v$  parallel to the  $x$  axis, as in Fig. 9.6. A plane monochromatic wave of frequency  $\omega_i$ , amplitude  $E_i$ , and wave vector  $\mathbf{k}_i = \hat{x}\omega_i/c$ , linearly polarized along the  $y$  axis, impinges onto the mirror.

- Show that in the laboratory frame  $S$  both *total* (i.e., incident + reflected) fields  $\mathbf{E}$  and  $\mathbf{B}$  are *discontinuous* at the mirror surface, lying on the plane  $x = x(t)$ , with  $dx/dt = v$ .
- The EM fields can be derived from a vector potential  $\mathbf{A}(x, t)$ . Show that the boundary conditions for the EM fields at the mirror surface are equivalent to the condition

$$\frac{d}{dt} \mathbf{A}[x(t), t] = 0 , \quad (9.9)$$

which states that the value of the vector potential at the surface is constant (i.e., time-independent) in the *laboratory* frame.

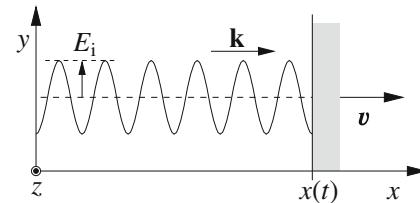


Fig. 9.6

- c) As a consequence of (9.9), we can assume  $\mathbf{A}[x(t), t] \equiv 0$ . Use this boundary condition to obtain the frequency ( $\omega_r$ ) and amplitude ( $E_r$ ) of the reflected wave in the laboratory frame.

## Reference

1. E.M. Purcell, *Electricity and Magnetism* (Berkeley Physics Course—Vol. 2, Section 5.9), 2nd edn. (McGraw-Hill Book Company, New York, 1984)

# Chapter 10

## Radiation Emission and Scattering

**Topics** The radiation field. Multipole expansion. Electric dipole radiation. Magnetic dipole radiation.

Basic equations of this chapter:

Fields in the radiation zone of a point-like source at  $r = 0$  having an electric dipole moment  $\mathbf{p}(t)$ :

$$\mathbf{E}(\mathbf{r}, t) = \frac{[\ddot{\mathbf{p}}(t_{\text{ret}}) \times \hat{\mathbf{r}}] \times \hat{\mathbf{r}}}{rc^2}, \quad \mathbf{B}(\mathbf{r}, t) = \hat{\mathbf{r}} \times \mathbf{E} \quad (10.1)$$

where  $t_{\text{ret}} = t - r/c$ .

Instantaneous radiation power from the electric dipole source and its angular distribution

$$P_{\text{rad}} = \frac{2}{3c^3} |\ddot{\mathbf{p}}|^2, \quad \frac{dP_{\text{rad}}}{d\Omega} = \frac{3P_{\text{rad}}}{4\pi} \sin^2 \theta, \quad (10.2)$$

where  $\theta$  is the angle between  $\mathbf{p}$  and  $\mathbf{r}$ , and the infinitesimal solid angle  $d\Omega = 2\pi \sin \theta d\theta$ .

Analogous formulas for the fields and the power of a magnetic dipole  $\mathbf{m}(t)$ :

$$\mathbf{E}(\mathbf{r}, t) = -\frac{[\dot{\mathbf{m}}(t_{\text{ret}}) \times \hat{\mathbf{f}}]}{rc^2}, \quad \mathbf{B}(\mathbf{r}, t) = \hat{\mathbf{f}} \times \mathbf{E}, \quad (10.3)$$

$$P_{\text{rad}} = \frac{2}{3c^3} |\dot{\mathbf{m}}|^2, \quad \frac{dP_{\text{rad}}}{d\Omega} = \frac{3P_{\text{rad}}}{4\pi} \sin^2 \theta. \quad (10.4)$$

### 10.1 Cyclotron Radiation

An electron moves in the  $xy$  plane in the presence of a constant and uniform magnetic field  $\mathbf{B} = B_0 \hat{\mathbf{z}}$ . The initial velocity is  $v_0 \ll c$ , so that the motion is non-

relativistic and the electron moves on a circular orbit of radius  $r_L = v_0/\omega_L$  and frequency  $\omega_L = eB_0/m_ec$  (Larmor frequency).

- a)** Describe the radiation emitted by the electron in the dipole approximation specifying its frequency, its polarization for radiation observed along the  $z$  axis, and along a direction lying in the  $xy$  plane, and the total irradiated power  $P_{\text{rad}}$ . Discuss the validity of the dipole approximation.
- b)** The electron gradually loses energy because of the emitted radiation. Use the equation  $P_{\text{rad}} = -dU/dt$ , where  $U$  is the total energy of the electron, to show that the electron actually spirals toward the “center” of its orbit. Evaluate the time constant  $\tau$  of the energy loss, assuming  $\tau \gg \omega_L^{-1}$ , and provide a numerical estimate.
- c)** The spiral motion cannot occur if we consider the Lorentz force  $\mathbf{f}_L = -(e/c)\mathbf{v} \times \mathbf{B}$  as the only force acting on the electron. Show that a spiral motion can be obtained by adding a friction force  $\mathbf{f}_{\text{fr}}$  proportional to the electron velocity.

## 10.2 Atomic Collapse

In the classical model for the hydrogen atom, an electron travels in a circular orbit of radius  $a_0$  around the proton.

- a)** Evaluate the frequency  $\omega$  of the radiation emitted by the orbiting electron, and the emitted radiation power, both as functions of  $a_0$ .
- b)** Use the results of point **a)** to show that, classically, the electron would collapse on the nucleus, and find the decay time assuming  $a_0 = 0.53 \times 10^{-8}$  cm (Bohr radius, actually obtained from quantum considerations) .

## 10.3 Radiative Damping of the Elastically Bound Electron

The motion of a classical, elastically bound electron in the absence of external fields is described by the equation

$$\frac{d^2\mathbf{r}}{dt^2} + \eta \frac{d\mathbf{r}}{dt} + \omega_0^2 \mathbf{r} = 0 , \quad (10.5)$$

where the vector  $\mathbf{r}$  is the distance of the electron from its equilibrium position,  $\eta$  is a friction coefficient, and  $\omega_0$  is the undamped angular frequency. We assume that at time  $t = 0$  the electron is located at  $\mathbf{r}(0) = \mathbf{s}_0$ , with zero initial velocity.

- a)** As a first step, find the solution of (10.5) assuming  $\eta = 0$ , and evaluate the cycle-averaged emitted radiation power  $P_{\text{rad}}$  due to the electron acceleration.
- b)** Assuming the oscillation amplitude to decay due to the radiative energy loss, estimate the decay time  $\tau$  using the result of point **a)** for the emitted power  $P_{\text{rad}}$ . Determine under which conditions  $\tau$  is much longer than one oscillation period.

Now assume  $\eta \neq 0$ , with  $\eta \ll \omega_0$ , in Eq.(10.5). In the following, neglect quantities of the order  $(\eta/\omega_0)^2$  or higher.

- c) Describe the motion of the electron and determine, *a posteriori*, the value of  $\eta$  that reproduces the radiative damping.

## 10.4 Radiation Emitted by Orbiting Charges

Two identical point charges  $q$  rotate with constant angular velocity  $\omega$  on the circular orbit  $x^2 + y^2 = R^2$  on the  $z = 0$  plane of a Cartesian reference frame.

- a) Write the most general trajectory for the charges both in polar coordinates  $r_i = r_i(t)$ ,  $\phi_i = \phi_i(t)$  and in Cartesian coordinates  $x_i = x_i(t)$ ,  $y_i = y_i(t)$  (where  $i = 1, 2$  labels the charge) and calculate the electric dipole moment of the system.
- b) Characterize the dipole radiation emitted by the two-charge system, discussing how the power depends on the initial conditions, and finding the polarization of the radiation emitted along the  $\hat{\mathbf{x}}$ ,  $\hat{\mathbf{y}}$  and  $\hat{\mathbf{z}}$  directions.
- c) Answer questions a) and b) in the case where the charges are orbiting with *opposite* angular velocity.
- d) Now consider a system of *three* identical charges on the circular orbit with the same angular velocity. Find the initial conditions for which the radiation power is either zero or has its maximum.
- e) Determine whether the magnetic dipole moment gives some contribution to the radiation, for each of the above specified cases.

## 10.5 Spin-Down Rate and Magnetic Field of a Pulsar

A pulsar is a neutron star with mass  $M \approx 1.4M_{\odot} \approx 2.8 \times 10^{33}$  g (where  $M_{\odot}$  is the Sun mass), and radius  $R \approx 10$  km =  $10^6$  cm. The star rotates with angular velocity  $\omega$  and has a magnetic moment  $\mathbf{m}$ , which is, in general, *not* parallel to the rotation axis. [1]

- a) Describe the radiation emitted by the pulsar, and find the total radiated power, assuming that the angle between the magnetic moment and the rotation axis is  $\alpha$ , as in Fig. 10.1.
- b) Find the “spindown rate” (decay constant of the rotation) of the pulsar, assuming that energy loss is due to radiation only.

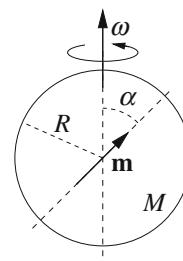
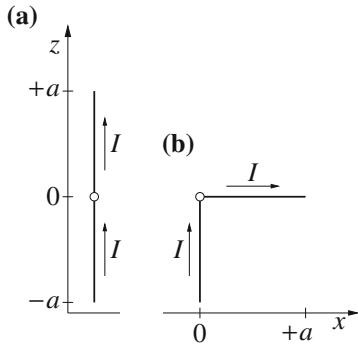


Fig. 10.1

- c) Explain how, from the knowledge of mass, radius, rotation period  $T$ , and time derivative  $dT/dt$  of the pulsar one can estimate the magnetic field at the pulsar surface. Give a numerical approximation based on the results of observations [3] which give  $T = 7.476551 \pm 3$  s and  $\dot{T} = (2.8 \pm 1.4) \times 10^{-11} \approx 10^{-3}$  s/year (for simplicity assume that  $\mathbf{m}$  is perpendicular).

## 10.6 A Bent Dipole Antenna



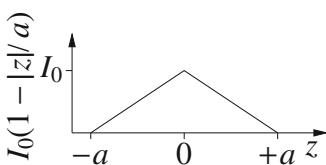
**Fig. 10.2**

A dipole antenna consists of two identical conductive elements, usually two metal rods, each of length  $a$  and resistance  $R$ . The driving current is applied between the two halves of the antenna, so that the current flows as shown in Fig. 10.2a). For a “short” antenna ( $a \ll \lambda = 2\pi c/\omega$ ) the current can be approximately specified as [2]

$$I = I(z, t) = \operatorname{Re} \left[ I_0 \left( 1 - \frac{|z|}{a} \right) e^{-i\omega t} \right]. \quad (10.6)$$

The dependence of the current oscillation amplitude on  $z$  is shown in Fig. 10.3. Calculate

- a) the cycle-averaged the dissipated power  $P_{\text{diss}}$ ;  
 b) the linear charge density  $q_\ell$  on the rods of the antenna, and the antenna electric dipole moment  $\mathbf{p}$ ;  
 c) the cycle-averaged radiated power  $P_{\text{rad}}$  and the ratio  $P_{\text{rad}}/P_{\text{diss}}$ .



**Fig. 10.3**

- d) Find the directions along which there no radiation is observed.

Now assume that the upper rod of the dipole antenna is bent by  $90^\circ$ , so that it is parallel to the  $x$  axis, as shown in Fig. 10.2b), without perturbing either the current or the charge density anywhere in the two rods.

- e) Answer questions a), b) and c) again for the bent antenna, pointing out the differences with the straight antenna.

## 10.7 A Receiving Circular Antenna

A receiving circular antenna is a circular coil of radius  $a$  and resistance  $R$ . The amplitude of the received signal is proportional to the current induced in the antenna by an incoming EM wave (Fig. 10.4).

- a)** Assume that the incoming signal is a monochromatic, linearly polarized wave of wavelength  $\lambda \gg a$ , and electric field amplitude  $E_0$ . Find how the antenna must be oriented with respect to the wave vector  $\mathbf{k}$  and to the polarization in order to detect the maximum signal, and evaluate the signal amplitude.
- b)** In a receiving linear antenna the signal is approximately proportional to  $E_{\parallel}\ell$ , where  $E_{\parallel}$  is the component of the electric field of the wave parallel to the antenna, and  $\ell$  is the length of the antenna. Old portable TV sets were provided with both a linear and a circular antenna, typical dimensions were  $\ell \approx 50$  cm and  $a \approx \ell/2$ . Which antenna is best suited to detect EM waves with  $\lambda$  in the  $10^2$ – $10^3$  cm range?
- c)** Calculate the power  $P_{\text{rad}}$  scattered by the antenna, and the ratio  $P_{\text{rad}}/P_{\text{diss}}$ , where  $P_{\text{diss}}$  is the power dissipated in the antenna by Joule heating.

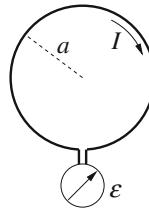


Fig. 10.4

## 10.8 Polarization of Scattered Radiation

An EM wave impinges on a particle that acquires an electric dipole moment  $\mathbf{p} = \alpha \mathbf{E}$ , where  $\mathbf{E}$  is the electric field of the wave at the position of the particle. Assume that the size of the particle is much smaller than the wavelength of the incoming wave.

- a)** Find the polarization of the scattered radiation as a function of the polarization of the incoming wave, and of the angle between the directions of observation and propagation.
- b)** If the incoming radiation is *unpolarized*, what can be said about the polarization of the scattered radiation?

## 10.9 Polarization Effects on Thomson Scattering

An electron is in the field of an elliptically polarized plane wave of frequency  $\omega$  propagating along the  $z$  axis of a Cartesian reference frame. The electric field of the wave can be written as

$$\mathbf{E} = E_0 [\hat{\mathbf{x}} \cos \theta \cos(kz - \omega t) + \hat{\mathbf{y}} \sin \theta \sin(kz - \omega t)], \quad (10.7)$$

where  $\theta$  is a constant real number with  $0 \leq \theta \leq \pi/2$ , such that we have linear polarization along the  $x$  axis for  $\theta = 0$ , linear polarization along the  $y$  axis for  $\theta = \pi/2$ , and circular polarization for  $\theta = \pi/4$ .

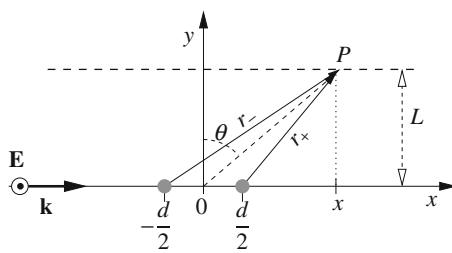
First, neglecting the effects of the magnetic force  $-ev \times \mathbf{B}/c$ ,

- a) characterize the radiation scattered by the electron by determining the frequency and the polarization observed along each axis ( $x, y, z$ ), and find a direction along which the radiation is circularly polarized;
- b) calculate the total (cycle-averaged) scattered power and discuss its dependence on  $\theta$ ;

Now consider the effect of the magnetic force on the scattering process.

- c) Evaluate the  $-ev \times \mathbf{B}/c$  term by calculating the  $\mathbf{B}$  field from (10.7) and using the result of point a) for  $v$ . Discuss the direction and frequency of the magnetic force and its dependence on  $\theta$  as well.
- d) Discuss how the scattering of the incident wave is modified by the magnetic force by specifying which new frequencies are observed, in which direction and with which polarization, and the modification of the scattered power.

## 10.10 Scattering and Interference



**Fig. 10.5**

the scatterer position. The intensity  $I_s$  of the scattered radiation is measured on the  $y = L$  plane, with both  $L \gg d$  and  $L \gg \lambda$ .

- a) Evaluate the phase difference  $\Delta\phi$  between the two scattered waves in a generic point  $P \equiv (x, L, 0)$ , with  $L$  a constant, as a function of the observation angle  $\theta = \arctan(x/L)$ , as shown in Fig. 10.5.
- b) Study the scattered intensity distribution  $I_s = I_s(\theta)$  as a function of  $kd$ , where  $\mathbf{k}$  is the wave vector of the incoming wave. Determine for which values of  $kd$  interference fringes appear.

A monochromatic plane wave propagates along the  $x$  axis of a Cartesian coordinate system. The wave is linearly polarized in the  $\hat{\mathbf{z}}$  direction, and has wavelength  $\lambda$ . Two identical, point-like scatterers are placed on the  $x$  axis at  $x = \pm d/2$ , respectively, as in Fig. 10.5. The dipole moment of each scatterer is  $\mathbf{p} = \alpha \mathbf{E}$ , where  $\mathbf{E}$  is the electric field of the incoming wave at

## 10.11 Optical Beats Generating a “Lighthouse Effect”

Two oscillating dipoles,  $\mathbf{p}_-$  and  $\mathbf{p}_+$ , are located at  $(0, -d/2, 0)$  and  $(0, +d/2, 0)$ , respectively, in a Cartesian reference frame. The two dipoles are parallel to the  $z$  axis and oscillate, with equal amplitude, at slightly different frequencies  $\omega_{\pm} = \omega_0 \pm \delta\omega/2$ , with  $\delta\omega \ll \omega_0$ . In complex representation we have  $\mathbf{p}_{\pm} = \mathbf{p}_0 e^{-i\omega_{\pm}t}$ . The distance between the two dipoles is  $d = \lambda_0/2 = \pi c/\omega_0$ . The radiation emitted by the dipoles is observed at a point  $P$  at a distance  $\mathbf{r}$  from the origin, with  $r \gg \lambda_0$ , on the  $z = 0$  plane. Let  $\phi$  be the angle between  $\mathbf{r}$  and the  $x$  axis, as shown in Fig. 10.6.

- a) Determine the direction of the electric field in  $P$  and its dependence on  $\phi$  and  $\omega_{\pm}$ , up to the first order in  $\delta\omega/\omega_0$ .

The wave intensity in  $P$  is measured by two detectors with different temporal resolutions: the first detector measures the “instantaneous” flux averaged over an interval  $\Delta t$  such that  $2\pi/\omega_0 \ll \Delta t \ll 2\pi/\delta\omega$ , while the second detector averages over  $\Delta t' \gg 2\pi/\delta\omega$ .

- b) Determine the dependence on the angle  $\phi$  and the time  $t$  of the fluxes measured with the two detectors.  
c) How do the above results change if the observation point is located in the  $x = 0$  plane?

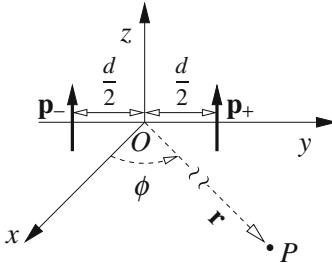


Fig. 10.6

## 10.12 Radiation Friction Force

An accelerated point charge emits radiation. Considering for definiteness an electron performing a periodic (non-relativistic for simplicity) motion in an oscillating external field, there is a finite amount of energy leaving the electron as radiation, but on the average the external field produces no work. Thus, to account self-consistently for the energy lost as radiation, it is necessary to modify the Newton-Lorentz force by adding a new “friction” term  $\mathbf{F}_{\text{rad}}$  so that the mechanical work done by  $\mathbf{F}_{\text{rad}}$  equals the radiated energy.<sup>1</sup>

We thus write for the electron

$$m_e \frac{d\mathbf{v}}{dt} = -e \left( \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) + \mathbf{F}_{\text{rad}}, \quad (10.8)$$

<sup>1</sup>From another viewpoint,  $\mathbf{F}_{\text{rad}}$  aims to describe the back-action or *reaction* of the self-generated EM fields on the accelerated charge.

and look for a suitable expression for  $\mathbf{F}_{\text{rad}}$  starting from the condition

$$\int_t^{t+T} \mathbf{F}_{\text{rad}}(t) \cdot \mathbf{v}(t) dt = - \int_t^{t+T} P_{\text{rad}}(t) dt , \quad (10.9)$$

where  $T$  is the period of the electron motion and  $P_{\text{rad}}(t)$  is the instantaneous radiated power, which is given by the Larmor formula

$$P_{\text{rad}}(t) = \frac{2e^2}{3c^2} \left| \frac{d\mathbf{v}}{dt} \right|^2 . \quad (10.10)$$

**a)** Show by direct substitution of the expression for  $F_{\text{rad}}$

$$\mathbf{F}_{\text{rad}} = m_e \tau \frac{d^2 \mathbf{v}}{dt^2} \quad (10.11)$$

into (10.9), that the equation is verified, and find the expression of the constant  $\tau$ , estimating its numerical value.

**b)** Determine the steady state solution of (10.8), where  $\mathbf{F}_{\text{rad}}$  is given by (10.11), for an electron in a uniform, oscillating electric field

$$\mathbf{E}(t) = \text{Re}(-e \mathbf{E}_0 e^{-i\omega t}) . \quad (10.12)$$

Compare the result with what obtained using the simple classical model an electron subject to a frictional force

$$m_e \frac{d\mathbf{v}}{dt} = \mathbf{F}_{\text{ext}} - m_e \eta \mathbf{v} . \quad (10.13)$$

## References

1. C. Bernardini, C. Guaraldo, *Fisica del Nucleo* (Editori Riuniti, Roma, 1982)
2. J.D. Jackson, *Classical Electrodynamics*, §9.2 and 9.4, 3rd edn. (Wiley, New York, London, Sydney, 1998)
3. C. Kouveliotou et al., An X-ray pulsar with a superstrong magnetic field in the soft  $\gamma$ -ray repeater SGR1806-20. Nature **393**, 235–237 (1998)

# Chapter 11

## Electromagnetic Waves in Matter

**Topics** Wave equation in continuous media. Classical model of the electron, bound and free electrons. Frequency-dependent conductivity  $\sigma(\omega)$  and dielectric permittivity  $\varepsilon(\omega)$  for harmonic fields. Relation between  $\sigma(\omega)$  and  $\varepsilon(\omega)$ . Transverse and longitudinal waves. The refraction index. Propagation of monochromatic waves in matter. Dispersion relations. Reflection and transmission at a plane interface: Snell's law, Fresnel's formulas, total reflection, Brewster's angle. Anisotropic media.

Basic equations of this chapter:

Wave equation for the electric field:

$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \partial_t^2 \mathbf{E} - \nabla(\nabla \cdot \mathbf{E}) = \frac{4\pi}{c^2} \mathbf{J} = \frac{4\pi}{c^2} \partial_t^2 \mathbf{P}. \quad (11.1)$$

(Notice that  $\mathbf{J} = \partial_t \mathbf{P}$ .)

Definition of  $\sigma(\omega)$ ,  $\chi(\omega)$  and  $\varepsilon(\omega)$  for harmonic fields  $\mathbf{E}(\mathbf{r}, t) = \text{Re}[\tilde{\mathbf{E}}(\mathbf{r})e^{-i\omega t}]$ ,  $\mathbf{J}(\mathbf{r}, t) = \text{Re}[\tilde{\mathbf{J}}(\mathbf{r})e^{-i\omega t}]$ ,  $\mathbf{P}(\mathbf{r}, t) = \text{Re}[\tilde{\mathbf{P}}(\mathbf{r})e^{-i\omega t}]$ :

$$\tilde{\mathbf{J}} = \sigma(\omega) \tilde{\mathbf{E}}, \quad \tilde{\mathbf{P}} = \chi(\omega) \tilde{\mathbf{E}}, \quad (11.2)$$

$$\varepsilon(\omega) = 1 + 4\pi\chi(\omega), \quad \chi(\omega) = \frac{i\sigma(\omega)}{\omega} \quad (\omega \neq 0). \quad (11.3)$$

Dispersion relation in a medium and refraction index  $n(\omega)$ :

$$\frac{k^2 c^2}{\omega^2} = \varepsilon(\omega) = n^2(\omega). \quad (11.4)$$

## 11.1 Wave Propagation in a Conductor at High and Low Frequencies

In a classical treatment, a metal has  $n_e$  conduction electrons per unit volume, whose equations of motion in the presence of an external electric field  $\mathbf{E}(\mathbf{r}, t)$  are

$$m_e \frac{d\mathbf{v}}{dt} = -e\mathbf{E}(\mathbf{r}, t) - m_e \eta \mathbf{v}, \quad \mathbf{v} = \frac{d\mathbf{r}}{dt}, \quad (11.5)$$

where  $-e$  and  $m_e$  are electron charge and mass, respectively, and  $\eta$  is a constant describing friction.

- a)** Determine the complex conductivity of the metal,  $\sigma = \sigma(\omega)$ , as a function of the angular frequency  $\omega$  of the electric field, and the values of  $\omega$  for which  $\sigma$  is either purely real or purely imaginary. Discuss these limits for a good conductor, whose DC conductivity (i.e., its conductivity for static fields) has values of the order of  $\sigma_{DC} \sim 5 \times 10^{17} \text{ s}^{-1}$ .

Now consider a monochromatic, plane EM wave, linearly polarized along the  $y$  axis and traveling in the positive direction along the  $x$  axis of a Cartesian coordinate system. The wave is incident on a conductor filling the  $x > 0$  half-space, while we have vacuum in the  $x < 0$  half-space.

- b)** Consider both cases of  $\sigma$  purely real and purely imaginary, and determine the frequency ranges in which the wave is evanescent inside the metal.  
**c)** Find the time-averaged EM energy flux through the metal surface and show that it is equal to the amount of energy dissipated inside the metal.

## 11.2 Energy Densities in a Free Electron Gas

A plane, monochromatic, transverse electromagnetic wave propagates in a medium containing  $n_e$  free electrons per unit volume. The electrons move with negligible friction. Calculate

- a)** the dispersion relation of the wave, the phase ( $v_\varphi$ ) and group ( $v_g$ ) velocities, and the relation between the amplitudes of the electric ( $E_0$ ) and magnetic ( $B_0$ ) fields;  
**b)** the EM energy density  $u_{EM}$  (averaged over an oscillation period) as a function of  $E_0$ ;  
**c)** the kinetic energy density  $u_K$  (averaged over an oscillation period), defined as  $u_K = n_e m_e \langle v^2 \rangle / 2$ , where  $v$  is the electron oscillation velocity, and the total energy density  $u = u_{EM} + u_K$ .  
**d)** Assume that the medium fills the half-space  $x > 0$ , while we have vacuum in the half-space  $x < 0$ . An EM wave, propagating along the  $x$  axis, enters the medium. Assume that both  $v_g$  and  $v_\varphi$  are real quantities. Use the above results to verify the conservation of the energy flux, expressed by the relation

$$c(u_i - u_r) = v_g u_t , \quad (11.6)$$

where  $u_i$ ,  $u_r$  and  $u_t$  are the total energy densities for the incident, reflected and transmitted waves, respectively.

## 11.3 Longitudinal Waves

Consider a *longitudinal* monochromatic plane wave, propagating in a medium along the  $x$  axis of a Cartesian reference frame. “Longitudinal” means that the electric field  $\mathbf{E}$  of the wave is parallel to the wavevector  $\mathbf{k}$ . Assume that the electric and magnetic fields of the wave are

$$\mathbf{E} = \mathbf{E}(x, t) = \hat{\mathbf{x}} E_0 e^{ikx - i\omega t}, \quad \mathbf{B} \equiv 0 , \quad (11.7)$$

respectively, and that the optical properties of the medium are described by a given frequency-dependent dielectric permittivity  $\varepsilon_r(\omega)$ .

- a) Show that the possible frequencies for the wave (11.7) correspond to zeros of the dielectric permittivity,  $\varepsilon_r(\omega) = 0$ .
- b) Find the charge and current densities in the medium associated to the presence of the wave fields (11.7).
- c) Assuming that the optical properties of the medium are determined by  $n_e$  classical electrons per unit volume, bound to atoms by an elastic force  $-m_e \omega_0^2 \mathbf{r}$ , determine  $\varepsilon_r(\omega)$  and the dispersion relation for the longitudinal wave.

## 11.4 Transmission and Reflection by a Thin Conducting Foil

A plane wave of frequency  $\omega = 2\pi c/\lambda$  strikes at normal incidence a thin metal foil of thickness  $d \ll \lambda$ . At the limit of an infinitely thin foil, the volume electron density in space can be approximated as  $n_v(x) = n_e d \delta(x)$ , where  $n_e$  is the volume electron density in the conductor, so that  $n_e d$  is the surface electron density on the foil, and  $\delta(x)$  is the Dirac delta function. Analogously, the volume current density in space can be approximated as  $\mathbf{J}(x, t) = \mathbf{K}(t) \delta(x)$ , where  $\mathbf{K}(t)$  is the surface current density on the foil.

- a) Prove the following relations for the field components parallel to the foil surface

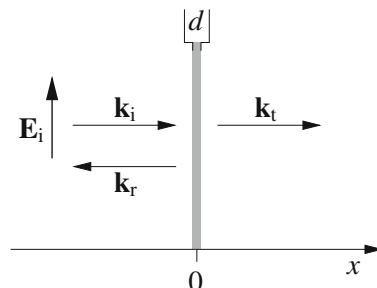


Fig. 11.1

$$E_{\parallel}(0^+) - E_{\parallel}(0^-) = 0, \quad B_{\parallel}(0^+) - B_{\parallel}(0^-) = \frac{4\pi}{c} K. \quad (11.8)$$

**b)** Evaluate the EM field in the whole space as a function of the foil conductivity  $\sigma$ , with  $\sigma$ , in general, a complex scalar quantity. Assume a linear dependence of the current density  $\mathbf{J}$  on the electric field  $\mathbf{E}$ , using the complex notation  $\mathbf{J} = \text{Re}(\tilde{\mathbf{J}} e^{-i\omega t})$ .

**c)** Now use the classical equation of motion for the electrons in the metal

$$m_e \frac{d\mathbf{v}}{dt} = -e\mathbf{E} - m_e \eta \mathbf{v}, \quad (11.9)$$

where  $\eta$  is a damping constant, to obtain an expression for  $\sigma$ , and evaluate the cycle-averaged absorbed power at the limits  $v \gg \omega$  and  $v \ll \omega$ , respectively.

**d)** Verify the conservation of energy for the system by showing that the flux of EM energy into the foil equals the absorbed power.

## 11.5 Anti-reflection Coating

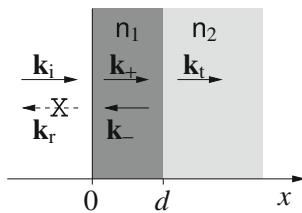


Fig. 11.2

A monochromatic plane EM wave of angular frequency  $\omega$  travels in vacuum ( $x < 0$ ) along the  $x$  direction of a Cartesian coordinate system. On the plane  $x = 0$  the wave strikes normally a semi-infinite composite medium. The medium comprises a first layer, between the planes  $x = 0$  and  $x = d$ , of real refractive index  $n_1$ , followed by a semi-infinite layer filling the half-space  $x > d$ , of real refractive index  $n_2$ , as shown in Fig. 11.2.

We want to determine the conditions on  $n_1$  and  $d$  in order to have a *total* transmission of the incident wave, so that there is *no* reflected wave in the vacuum region. Proceed as follows:

- a)** write the general solution for the EM wave in each region of space;
- b)** write the relations between the amplitudes of the EM fields in each region due to matching conditions at the two interfaces;
- c)** having determined from point **b**) the relation between  $n_1$ ,  $n_2$  and  $d$  necessary to the absence of reflection, find the values of  $n_1$  and  $d$  for which a solution exists in the  $n_2 = 1$  case.
- d)** How does the answer to point **c**) change if  $n_2 \neq 1$ ?

## 11.6 Birefringence and Waveplates

The refractive index of *anisotropic* crystals depends on both the propagation direction and the polarization of the incoming EM wave. We choose a Cartesian reference frame such that the  $x = 0$  plane separates the investigated medium from vacuum. The wave vector of the incident wave,  $\mathbf{k}_i$ , lies in the  $xy$  plane and forms an angle  $\theta_i$  with the  $x$  axis, as shown in Fig. 11.3. In this context we consider a material whose refractive index has the values  $n_s$  for a wave polarized perpendicularly to the incidence  $xy$  plane ( $S$  polarization, from German *senkrecht*, perpendicular), and  $n_p$  for waves whose electric field lies in the  $xy$  plane ( $P$  polarization, from *parallel*). Here, both  $n_s$  and  $n_p$  are assumed to be real and positive, with  $n_p > n_s$ . The treatment of the opposite case,  $n_s > n_p$ , is straightforward.

a) Assume that the incoming wave is linearly polarized, and that its electric field forms an angle  $\psi = \pi/4$  with the  $z$  axis, so that its polarization is a mixture of  $S$  and  $P$  polarizations. The incident ray splits into two refracted rays at different angles,  $\theta_{t\pm} = \theta_t \pm \alpha$ , as shown in Fig. 11.3, where  $\mathbf{k}_{t+}$  corresponds to  $S$ , and  $\mathbf{k}_{t-}$  to  $P$  polarization. Show how the values of  $n_s$  and  $n_p$  can be obtained from the measurements of  $\theta_t$  and  $\alpha$ . Assume that  $n_p = \bar{n} + \delta n$ , and  $n_- = \bar{n} - \delta n$ , with  $\delta n/\bar{n} \ll 1$ , and keep only first-order terms in  $\delta n/\bar{n}$ .

b) Now assume normal incidence ( $\theta_i = 0$ ), and that the electric field of the linearly polarized incoming wave,  $\mathbf{E}_i$ , still forms an angle  $\psi = \pi/4$  with the  $\hat{z}$  axis, as in Fig. 11.4. The crystal has a thickness  $d \gg \lambda$ . Find the values of  $d$  such that the light exiting the crystal is either *circularly* polarized, or linearly polarized, but rotated by  $\pi/2$  with respect to the polarization of the incident light. Neglect the difference between the reflection coefficients for  $S$  and  $P$  polarizations.

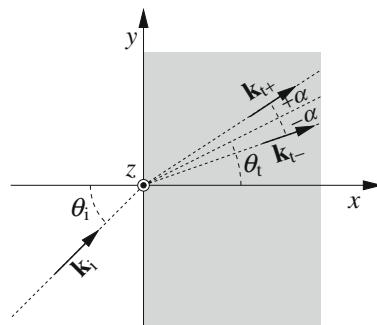


Fig. 11.3

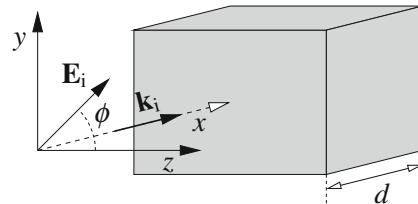


Fig. 11.4

## 11.7 Magnetic Birefringence and Faraday Effect

An EM plane wave of frequency  $\omega$  travels in a medium in the presence of a static uniform magnetic field  $\mathbf{B}_0 = B_0 \hat{z}$ , where  $\hat{z}$  is the  $z$  unit vector of a Cartesian reference

frame.  $B_0$  is much stronger than the magnetic field of the wave. The direction of the wave propagation is also parallel to  $\hat{\mathbf{z}}$ . The medium contains  $n_e$  bound electrons per unit volume, obeying the classical equations of motion

$$m_e \frac{d^2 \mathbf{r}}{dt^2} = -e \left( \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) - m_e \omega_0^2 \mathbf{r}, \quad \mathbf{v} = \frac{d\mathbf{r}}{dt}. \quad (11.10)$$

where  $m_e$  and  $-e$  are the electron mass and charge, respectively.

- a)** Show that the propagation of the wave depends on its polarization by evaluating the refractive index for *circular* polarization, either left-handed or right-handed.
- b)** Now consider the propagation of a *linearly* polarized wave. Assume the electric field at  $z = 0$  to be given by  $\mathbf{E}_i(z = 0, t) = \hat{\mathbf{x}} E_i e^{-i\omega t}$ , and a relatively weak magnetic field so that  $\omega \gg \omega_c$  and terms of order higher than  $\omega_c/\omega$  may be neglected. Find the electric field at the position  $z = \ell$ , showing that the polarization has *rotated* (Faraday effect).

## 11.8 Whistler Waves

Lightnings excite transverse EM signals which propagate in the ionosphere, mostly in the direction parallel to the Earth's magnetic field lines.

- a)** Show that, in a frequency range to be determined, and depending on the wave polarization, the dispersion relation for such signals has the form

$$\omega = \alpha k^2, \quad (11.11)$$

with  $\alpha$  a constant depending on the free electron density  $n_e$  and the magnetic field  $B_0$  (both assumed to be uniform for simplicity). Give a numerical estimate for the frequency range, knowing that typical values are  $n_e \approx 10^5 \text{ cm}^{-3}$ , and  $B_0 \approx 0.5 \text{ G}$ .

- b)** Determine the group and phase velocities following from (11.11) as functions of  $\omega$ , and compare them to  $c$ .
- c)** Suppose that a lightning locally excites a pulse having a frequency spectrum extending from a value  $\omega_1$  to  $\omega_2 = 2\omega_1$ , within the frequency range determined at point **a**). Assuming the pulse to be “short” (in a sense to be clarified *a posteriori*), estimate the pulse length after propagation over a distance  $L \approx 10^4 \text{ km}$ . Try to explain why these signals are called *whistlers*.

(Refer to [1], Sect. 7.6, and to Problem 11.7 for the propagation of EM waves along a magnetic field).

## 11.9 Wave Propagation in a “Pair” Plasma

A “pair” plasma is composed by electrons and positrons with equal density  $n_0$  (pair annihilation is neglected).

- a)** In the absence of external fields, find the dispersion relation for transverse EM waves, determining cut-off and/or resonance frequencies, if any.
- b)** Find and discuss the dispersion relation as in **a)**, but for waves propagating along the direction of an external, static magnetic field  $\mathbf{B}_0$  (see also Problem 11.7).

## 11.10 Surface Waves

A homogeneous medium fills the  $x > 0$  half-space of a Cartesian reference frame, while we have vacuum for  $x < 0$ . The dielectric permittivity of the medium,  $\varepsilon = \varepsilon(\omega)$ , assumes real values in the frequency range of interest. A monochromatic EM wave propagates along the  $y$ -direction, parallel to the interface between the medium and vacuum. Inside the medium, the magnetic field of the wave has the  $z$ -component only, given by

$$B_z = B_0 e^{-qx} \cos(ky - \omega t) = \operatorname{Re}(B_0 e^{-qx} e^{iky-i\omega t}) \quad (x > 0), \quad (11.12)$$

where  $q$  is a real and positive quantity.

- a)** Using the wave equation for  $\mathbf{B}$  inside the dielectric medium, find a relation between  $q$ ,  $k$  and  $\omega$ .
- b)** Write the expression for the electric field  $\mathbf{E}$  inside the medium.
- c)** Calculate the Poynting vector  $\mathbf{S}$  and specify the direction of the time-averaged EM energy flow.

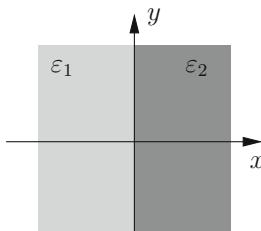


Fig. 11.5

where

$$B_z(x) = \begin{cases} B_1 e^{+q_1 x}, & x < 0 \\ B_2 e^{-q_2 x}, & x > 0 \end{cases}. \quad (11.14)$$

Now consider two different homogeneous media of dielectric permittivities  $\varepsilon_1$  and  $\varepsilon_2$ , respectively, filling the  $x < 0$  and  $x > 0$  half-spaces. A linearly-polarized EM wave propagates along the  $y$ -axis on the  $x = 0$  interface, with the magnetic field given by

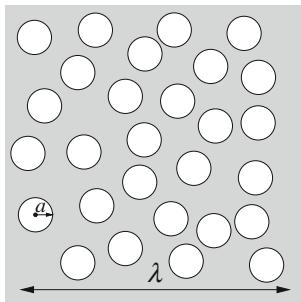
$$\mathbf{B} = \operatorname{Re}[\hat{\mathbf{z}} B_z(x) e^{iky-i\omega t}], \quad (11.13)$$

- d)** Using the boundary conditions for  $B_z$  at the  $x = 0$  surface, find the relation between  $B_1$  and  $B_2$ .

- e)** Using the continuity of  $E_y$  at the  $x = 0$  surface, find the relation between  $q_1$  and  $q_2$ . Show that  $\varepsilon_1$  and  $\varepsilon_2$  must have *opposite* sign in order to have  $q_{1,2} > 0$ , i.e., vanishing fields for  $|x| \rightarrow \infty$ .
- f)** From the results of points **a)** and **e)** find the dispersion relation  $\omega = \omega(k)$  as a function of  $\varepsilon_1$  and  $\varepsilon_2$ , showing that wave propagation requires  $\varepsilon_1 + \varepsilon_2 < 0$ .
- g)** If medium 1 is vacuum ( $\varepsilon_1 = 1$ ), how should medium 2 and the wave frequency be chosen in order to fulfill the condition found at point **f)**?

## 11.11 Mie Resonance and a “Plasmonic Metamaterial”

A plane, monochromatic wave of frequency  $\omega$  impinges on a small sphere of radius  $a \ll \lambda = 2\pi c/\omega$ . The sphere is made of a material whose dielectric function  $\varepsilon = \varepsilon(\omega)$  can be written as



**Fig. 11.6**

$$\varepsilon = 1 - \frac{\omega_p^2}{\omega^2 - \omega_0^2 + i\omega\eta} \quad (11.15)$$

where, according to the model of the elastically bound electron,  $\omega_p$  is the plasma frequency,  $\omega_0$  is the resonance frequency of bound electrons, and  $\eta$  is a damping constant.

- a)** Find the induced field and polarization inside the sphere, and discuss any resonant behavior. (Hint: have a look back at Problem 3.4)
- b)** Assume that the EM wave is propagating inside a material where there are  $n_s$  metallic ( $\omega_0 = 0$ ) nanospheres per unit volume, with  $n_s \lambda^3 \gg 1 \gg \lambda/a$ . Find the *macroscopic* polarization of the material and discuss the propagation of the wave as a function of the frequency  $\omega$ .

## Reference

1. J.D. Jackson, *Classical Electrodynamics*, §9.2 and 9.4, 3rd edn. (Wiley, New York, London, Sydney, 1998)

# Chapter 12

## Transmission Lines, Waveguides, Resonant Cavities

**Topics.** Guided propagation of EM waves. Transmission lines, TEM mode. Waveguides, TE and TM modes. Resonant cavities and discretization of frequencies.

## 12.1 The Coaxial Cable

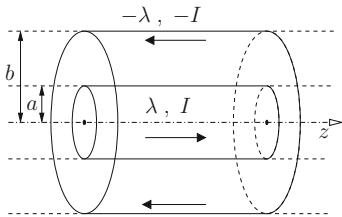


Fig. 12.1

A coaxial cable consists of two coaxial, infinitely long conductors: an inner cylinder of radius  $a$ , and an outer cylindrical shell of internal radius  $b > a$ . In general, if there is a charge per unit length  $\lambda$  on the inner conductor, there is an opposite charge  $-\lambda$  on the outer conductor. Similarly, if a total current  $I$  flows through the inner conductor, an opposite “return” current  $-I$  flows in the outer one.

We use a cylindrical coordinate system  $(r, \phi, z)$  with the cable axis as  $z$  axis, and, at first, we assume that the region  $a < r < b$  is filled by an insulating medium of dielectric permittivity  $\epsilon = 1$  and magnetic permeability  $\mu = 1$ .

- a)** Evaluate the capacitance and inductance per unit length of the cable.
- b)** Describe the propagation of a current signal  $I(z, t)$  and of an associated linear charge signal  $\lambda(z, t)$  along the cable, remembering the results of Problem 7.4. How are  $I(z, t)$  and  $\lambda(z, t)$  related to each other?
- c)** For given  $I(z, t)$  and  $\lambda(z, t)$ , find the electric field  $\mathbf{E}$  and the magnetic field  $\mathbf{B}$  in the space between the conductors, assuming that both  $\mathbf{E}$  and  $\mathbf{B}$  are *transverse*, i.e. perpendicular to the direction of propagation (such configuration is called TEM mode).
- d)** Now consider a semi-infinite cable with an ideal source imposing the voltage  $V(t)$  between the inner and outer conductors at the end of the cable. Show that the work done by the generator equals the flux of the Poynting vector through the cable (far enough from the end, so that we may neglect boundary effects).
- e)** How do the preceding answers change if the medium between the internal and external conductors has real and positive values for  $\epsilon$  and  $\mu$ , but different from unity?

## 12.2 Electric Power Transmission Line

Consider a thin, infinite straight wire along the  $z$  axis of a cylindrical coordinate system  $(r, \phi, z)$ . The wire is located in a medium of relative electric permittivity  $\epsilon_r = 1$  and relative magnetic permeability  $\mu_r = 1$ . Assume a current  $I = I(z, t)$  to flow in the wire, with

$$I = I(z, t) = I_0 e^{ikz-i\omega t}. \quad (12.1)$$

- a)** Calculate the linear charge density  $\lambda = \lambda(z, t)$  on the wire.
- b)** Assume that the electric and magnetic fields have only their radial and azimuthal components, respectively,

$$E_\phi = E_z = 0, \quad E_r = E_r(r) e^{ikz-i\omega t}, \quad B_r = B_z = 0, \quad B_\varphi = B_\varphi(r) e^{ikz-i\omega t}. \quad (12.2)$$

Calculate  $E_r$  and  $B_\varphi$  as functions of  $I_0$  and  $\omega$ , and use Maxwell's equations to evaluate the phase velocity of the signal  $v_\varphi = \omega/k$ .

- c) A high voltage transmission line comprises two straight parallel wires, at a constant distance  $d = 5$  m and typical height over the ground  $h = 30$  m. The two wires have opposite current intensities  $\pm I(z, t)$  given by (12.1), where typically  $I_0 = 10^3$  A and  $\omega = 2\pi \times 50$  s<sup>-1</sup>. Calculate the electric and magnetic fields on the symmetry plane between the two wires, and evaluate their magnitude on the ground.

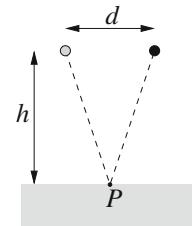


Fig. 12.2

## 12.3 TEM and TM Modes in an “Open” Waveguide

An “open” waveguide comprises two parallel, perfectly conducting planes, between which the waves propagate. Let us choose a Cartesian coordinate system  $(x, y, z)$  such that the two conducting planes are at  $y = \pm a/2$ , respectively, as in Fig. 12.3. An EM wave of frequency  $\omega$  propagates in the waveguide along  $\hat{x}$ . The magnetic field of the wave is directed along  $\hat{z}$  and has the form

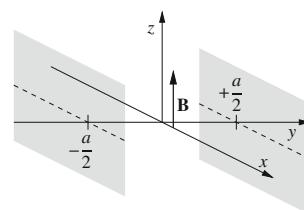


Fig. 12.3

$$B_z(x, y, t) = B_0 \cos(k_y y) e^{ik_x x - i\omega t}. \quad (12.3)$$

- a) Find the relations between  $\omega$ ,  $k_x$  and  $k_y$ .
- b) Find the expression for the electric field  $\mathbf{E} = \mathbf{E}(x, y, t)$  of the EM wave.
- c) Find how the possible values for  $k_y$  are determined by the boundary conditions on  $\mathbf{E}$ , and discuss the existence of cut-off frequencies.
- d) Find the flux of energy along the direction of propagation  $\hat{x}$ , showing that it is proportional to the group velocity of the wave.

## 12.4 Square and Triangular Waveguides

A waveguide has perfectly conducting walls and a square section of side  $a$ , as shown in Fig. 12.4. We choose a Cartesian coordinate system  $(x, y, z)$  where the interior of the waveguide is delimited by the four planes  $x = 0$ ,  $x = a$ ,  $y = 0$  and  $y = a$ . Consider the

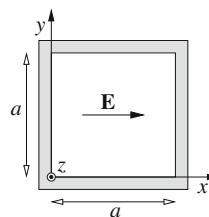


Fig. 12.4

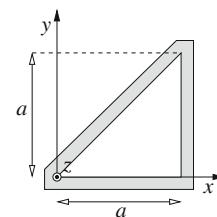


Fig. 12.5

propagation along  $\hat{\mathbf{z}}$  of a wave of frequency  $\omega$ , whose electric field  $\mathbf{E}(x, y, z, t)$  is perpendicular to  $\hat{\mathbf{z}}$  (a TE mode). Assume that the electric field can be written as

$$\mathbf{E}(x, y, z, t) = \tilde{\mathbf{E}}(x, y) e^{ik_{zz} - i\omega t}, \quad (12.4)$$

where  $\tilde{\mathbf{E}}(x, y)$  on  $x$  and  $y$  only.

- a)** Assume that  $\mathbf{E}$  is parallel to  $\hat{\mathbf{x}}$ , i.e.  $\mathbf{E} = \hat{\mathbf{x}} E_x$ , and determine the lowest value of  $\omega$  for which the TE mode can propagate in the waveguide, and the corresponding expressions for the electric and magnetic fields.
- b)** Determine the lowest frequency and the EM fields for a waveguide delimited by the conducting planes  $x = 0$ ,  $y = 0$ , and  $y = x$ , whose cross section is the right isosceles triangle shown in Fig. 12.5.

## 12.5 Waveguide Modes as an Interference Effect

An electric dipole  $\mathbf{p} = p \hat{\mathbf{y}}$  is located at the origin of a Cartesian coordinate system  $(x, y, z)$ , between two infinite, perfectly conducting planes located at  $y = \pm a$ , respectively, as shown in Fig. 12.6.

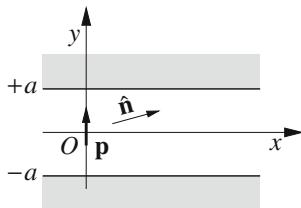


Fig. 12.6

the corresponding constraints on the possible values of the oscillation frequency  $\omega$ . Now consider two types of waves, labeled “0” and “1”, respectively, propagating between the two conducting planes with their wavevectors  $\mathbf{k}_{0,1}$  lying in the  $z = 0$  plane. Assume that the only nonzero component of the magnetic field of both waves is parallel to  $\hat{\mathbf{z}}$  (TM waves), and that the magnetic fields have the form

$$\mathbf{B}_0 = \hat{\mathbf{z}} B_0 e^{ik_{0x}x - i\omega t}, \quad \mathbf{B}_1 = \hat{\mathbf{z}} B_1 \sin(k_{1y}y) e^{ik_{1x}x - i\omega t}. \quad (12.5)$$

- c)** Find the relation between the components of the wavevectors and  $\omega$  for both waves.
- d)** Find the expressions for the electric fields  $\mathbf{E}_{0,1}$  of the waves corresponding to the magnetic fields (12.5).

- a)** Find the electrostatic potential between the two conducting planes, using the method of images.

Now assume that the dipole is oscillating, in complex notation  $\mathbf{p} = p_0 e^{-i\omega t}$ , and consider the emitted radiation in the region between the two conducting planes, at large distances from the dipole, i.e., with both  $|x| \gg \lambda$  and  $|x| \gg a$ .

- b)** Find in which directions  $\hat{\mathbf{n}}$ , lying in the  $z = 0$  plane, we observe constructive interference between the waves emitted by the dipole and its images, and

- e) Verify (or impose when appropriate) that for the expressions found in d) the component of  $\mathbf{E}$  parallel to the planes vanishes at their surface, and the related constraints on  $\mathbf{k} = (k_x, k_y)$ . What is the relation with the orders of interference found at point b)?

## 12.6 Propagation in an Optical Fiber

Figure 12.7 represents a simple model for an optical fiber. In a Cartesian reference frame  $(x, y, z)$  the space between the planes  $y = \pm a/2$  is filled by a material of a real and positive refractive index  $n > 1$  (in the frequency range of interest), while we have vacuum ( $n = 1$ ) in the regions  $y > a/2$  and  $y < -a/2$ . A monochromatic electromagnetic wave of frequency  $\omega$  propagates parallel to  $\hat{x}$  inside the fiber. We

assume that the only nonzero component of the electric field  $\mathbf{E}$  of the wave is parallel to  $z$  (i.e. perpendicular to the plane of the figure). Further, we assume that the wave is the superposition of two plane waves with wavevectors  $\mathbf{k}_1 \equiv (k_x, k_y, 0) \equiv k(\sin\theta, \cos\theta, 0)$ , and  $\mathbf{k}_2 \equiv (k_x, -k_y, 0) \equiv k(\sin\theta, -\cos\theta, 0)$ , where  $\theta$  is the angle of incidence shown in the figure. We have, in complex notation,

$$\begin{aligned}\mathbf{E} &= \hat{\mathbf{y}} E_z(x, y, t) = \hat{\mathbf{y}} (E_1 e^{i\mathbf{k}_1 \cdot \mathbf{r} - i\omega t} + E_2 e^{i\mathbf{k}_2 \cdot \mathbf{r} - i\omega t}) \\ &= \hat{\mathbf{y}} (E_1 e^{ik_x x + ik_y y - i\omega t} + E_2 e^{ik_x x - ik_y y - i\omega t}).\end{aligned}\quad (12.6)$$

- a) Find the relation between  $k$  and  $\omega$ , and the range of  $\theta$  for which the wave propagates without energy loss through the boundary surfaces at  $y = \pm a/2$ .  
b) The *amplitude reflection coefficient*  $r = E_r/E_i$  is the ratio of the complex amplitude of the reflected wave to the amplitude of the incident wave, at the surface separating two media. In the case of total reflection we have  $r = e^{i\delta}$ , with  $\delta$  a real number. Show that, in our case, we have

$$k_y a + \delta = m\pi, \quad \text{with } m \in \mathbb{N}, \quad (12.7)$$

and write the equation for the cut-off frequencies of the fiber. Find the values of  $k_y$  explicitly at the  $n \sin\theta \gg 1$ ,  $\theta \rightarrow \pi/2$  limit.

- c) How do the results change if  $\mathbf{E}$  lies in the  $xy$  plane?

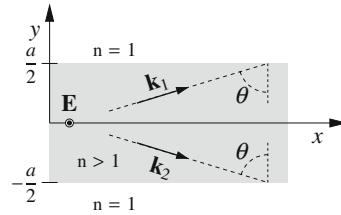


Fig. 12.7

## 12.7 Wave Propagation in a Filled Waveguide

A waveguide has rectangular cross section and perfectly conducting walls. We choose a Cartesian reference frame where the waves propagate parallel to the  $x$  axis, and the conducting walls lie on the  $y = \pm a/2$  and  $z = \pm a/2$  planes, as in Fig. 12.8. The waveguide is uniformly filled with a medium having refractive index  $n=n(\omega)$ .

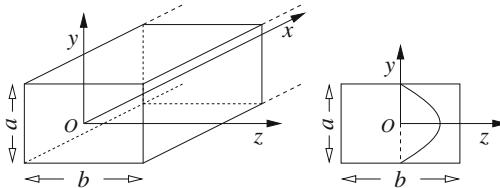


Fig. 12.8

electrons, i.e., a plasma, with plasma frequency  $\omega_p$ . In this case we have for the refractive index  $n^2(\omega) = 1 - \omega_p^2/\omega^2$ .

**b)** Now assume that the medium fills only the  $x > 0$  region of the waveguide. A monochromatic wave of the lowest frequency that can propagate in both regions ( $x < 0$  and  $x > 0$ ) travels in the guide from  $x = -\infty$ . Find the amplitudes of the reflected and transmitted waves at the  $x = 0$  interface.

**a)** Consider the propagation of a TE mode of frequency  $\omega$ , for which the electric field is  $\mathbf{E} = \hat{\mathbf{z}} E_z(y) e^{ikx-i\omega t}$ . Find the general expression for  $E_z(y)$  and the dispersion relation  $\omega = \omega(k)$ . Determine the cut-off frequencies for the particular case in which the filling medium is a gas of free

## 12.8 Schumann Resonances

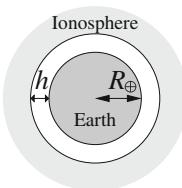


Fig. 12.9

The system formed by the Earth and the ionosphere can be considered as a resonant cavity. The cavity is delimited by two conducting, concentric spherical surfaces: the Earth's surface (radius  $R_\oplus \approx 6400$  km) and to the lower border of the ionosphere, located at an altitude  $h \approx 100$  km above, as shown in Fig. 12.9, obviously out of scale. Inside this “cavity” there are standing electromagnetic waves of particular frequencies, called *Schumann resonances*.

We want to estimate the typical frequency  $\omega$  of these resonances, assuming that both the Earth and the ionosphere are perfect conductors, and thus completely reflect the electromagnetic waves in the resonant frequency range.

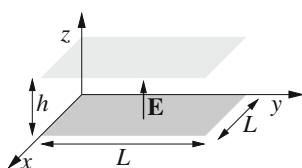


Fig. 12.10

In order to avoid mathematical complications due to the spherical geometry of the problem, we choose a simplified, flat model consisting in a rectangular parallelepiped with two square, conducting bases of side  $L$ , and height  $h$ . In a Cartesian reference frame, the base standing for the Earth surface lies on the  $z = 0$  plane, while the base standing for the surface

at the bottom of the ionosphere lies on the  $z = h$  plane, as shown in Fig. 12.10. We choose  $L = 2\pi R_\oplus$ , and, in order to reproduce somehow spherical geometry, we impose periodic boundary conditions on the lateral surface of the parallelepiped, namely

$$\mathbf{E}(0, y, z, t) = \mathbf{E}(L, y, z, t), \quad \mathbf{E}(x, 0, z, t) = \mathbf{E}(x, L, z, t), \quad (12.8)$$

where  $\mathbf{E}$  is the field of the wave, the same conditions are assumed for the magnetic field of the wave. We assume  $\epsilon_r = 1$  and  $\mu_r = 1$  in the interior of our parallelepiped. Further, we assume a TE mode with an electric field of the form

$$\mathbf{E} = \hat{\mathbf{z}} E_0 e^{ik_x x + ik_y y - i\omega t}. \quad (12.9)$$

- a)** Find the possible values of  $k_x$ ,  $k_y$ ,  $\omega$  and give a numerical estimate of  $\omega$  and the corresponding wavelength for the lowest frequency mode.
- b)** The low-frequency conductivity of sea water is  $\sigma \simeq 4.4 \Omega^{-1} m^{-1}$ . Discuss if approximating the surface of the oceans as a perfect conductor is reasonable at the frequency of the Schumann resonances .

# Chapter 13

## Additional Problems

### 13.1 Electrically and Magnetically Polarized Cylinders

Let us consider a cylinder of relative magnetic permeability  $\mu_r$ , located in a uniform magnetic field  $\mathbf{B}_0$  parallel to the cylinder axis, and the analogous problem of a cylinder of relative electric permittivity  $\varepsilon_r$  located in a uniform electric field  $\mathbf{E}_0$  parallel to the cylinder axis. In both cases the cylinder has radius  $a$  and height  $h$ .

- a)** First, consider “long” cylinders, with  $a \ll h$ . Evaluate the magnetic field  $\mathbf{B}_i$ , and, respectively, the electric field  $\mathbf{E}_i$ , inside the cylinders, neglecting boundary effects.
- b)** Now evaluate the internal magnetic and electric fields in the case of “flat” cylinders,  $a \gg h$ , again neglecting boundary effects.
- c)** Evaluate the fields of point **a)** at the next order of accuracy, taking the boundary effects at the lowest nonzero order in  $a/h$  into account.
- d)** Evaluate the fields of point **b)** at the lowest nonzero order in  $h/a$ .

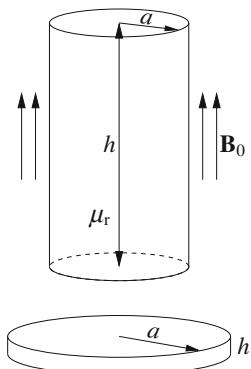


Fig. 13.1

### 13.2 Oscillations of a Triatomic Molecule

A triatomic symmetric linear molecule, like  $\text{CO}_2$ , can be schematized as a central point mass  $M$ , of charge  $2q$ , and two identical point masses  $m$ , each of charge  $-q$ , which, when the molecule is at rest, are located symmetri-

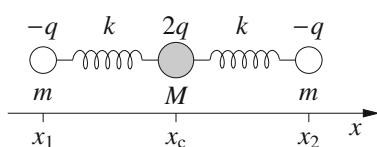


Fig. 13.2

cally around  $M$ , as shown in Fig. 13.2. In the case of longitudinal small-amplitude vibrations, the interactions between the three masses (including the electrostatic forces) can be described as two identical springs, each of rest length  $\ell$  and elastic constant  $k$ , located as in the figure.

Let  $x_1$  and  $x_2$  be the positions of the two lateral masses, and  $x_c$  the position of the central mass. We want to study the longitudinal vibrations of the molecule in its center-of-mass reference frame, defined by the condition

$$x_{\text{cm}} = \frac{mx_1 + mx_2 + Mx_c}{2m + M} = 0. \quad (13.1)$$

When the molecule is at rest we have thus

$$x_1 = -\ell, \quad x_c = 0, \quad x_2 = \ell. \quad (13.2)$$

- a)** Find the normal longitudinal oscillation modes of the molecule, and their frequencies.
- b)** The molecule emits radiation because the charged masses oscillate around their equilibrium positions. If the electric dipole term is dominant, the frequency of only one of the normal modes is observed in the spectrum of the emitted radiation. Explain why, and evaluate the observed frequency.
- c)** Assume that, initially, the molecule is “excited” by locating the masses at  $x_1 = -\ell + d_1$ ,  $x_2 = \ell + d_2$ , and  $x_c$  such that  $x_{\text{cm}} = 0$ . Then, at  $t = 0$ , the three masses are simultaneously released. Find the power radiated at  $t > 0$ .

### 13.3 Impedance of an Infinite Ladder Network

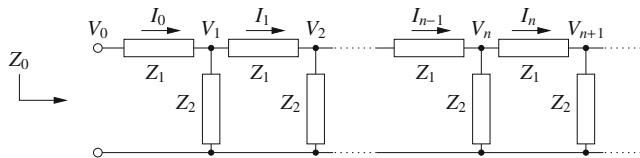


Fig. 13.3

Consider the (semi-)infinite ladder of Fig. 13.3, where each (identical) section contains a “horizontal” impedance  $Z_1 = Z_1(\omega)$ , and a “vertical” impedance  $Z_2 = Z_2(\omega)$ .

- a)** Calculate the input impedance  $Z_0$  of the semi-infinite network. How can a real, finite network be terminated after  $N$  sections, so that its impedance has also the value  $Z_0$ ?
- b)** Let  $V_n$  be the voltage at the  $n$ th node. Find the relation between  $V_n$  and  $V_{n+1}$  and, from this, the dependence of  $V_n$  on  $n$  and on the input voltage  $V_0$ . Discuss the result for the case of a purely resistive network ( $Z_1 = R_1$ ,  $Z_2 = R_2$ ).

Now consider the case of an  $LC$  network, with  $Z_1 = -i\omega L$  and  $Z_2 = -1/i\omega C$ , in the presence of an input signal  $V_0(t) = V_0 e^{-i\omega t}$ .

- c) Find the frequency range in which signals can propagate in the network. Show that there is a cut-off frequency  $\omega_{co}$ , such that the signal is damped if  $\omega > \omega_{co}$ , and evaluate the damping factor.
- d) Discuss the case of a  $CL$  network, with  $Z_1 = -1/i\omega C$  and  $Z_2 = -i\omega L$ .

## 13.4 Discharge of a Cylindrical Capacitor

A cylindrical capacitor has internal radius  $a$ , external radius  $b > a$ , and height  $h \gg b$ . For  $t < 0$ , the two cylindrical plates have charges  $\pm Q_0$ , respectively, and are disconnected. At  $t = 0$  the plates are connected through a resistor  $R$  as in Fig. 13.4. We assume that during the discharge i) the slowly-varying current approximation holds, ii) the surface charge density on the plates remains uniform, iii) we can neglect the effects of the external circuit and the resistance of the plates, iv) other boundary effects are negligible. We use a cylindrical coordinate system  $(r, \phi, z)$  with the capacitor axis as  $z$  axis, and the origin at the center of the capacitor.

- a) Calculate the current  $I = I(z, t)$  flowing on the plates, and the magnetic field  $\mathbf{B} = \hat{\phi} B_\phi(r, z, t)$  inside the capacitor, for  $|z| \ll (h/2)$ .
- b) Calculate the Poynting vector  $\mathbf{S}$  for  $|z| \ll (h/2)$ , and verify that its flux through a cylindrical surface coaxial to the capacitor equals the time variation of the electrostatic energy inside the surface.
- c) Discuss the validity of the slowly varying current approximation and of the assumption of uniform charge distribution over the plates.

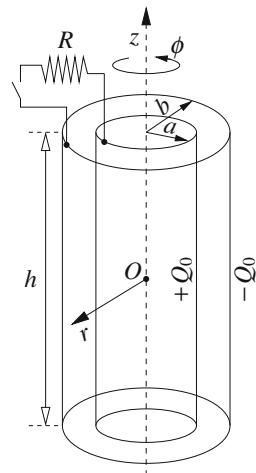


Fig. 13.4

## 13.5 Fields Generated by Spatially Periodic Surface Sources

Evaluate the electromagnetic fields and potentials generated by the following three surface charge and/or current densities, located on the  $y = 0$  plane of a Cartesian coordinate system,

- a) a static surface charge density  $\sigma = \sigma_0 \cos kx$ ;
- b) a static surface current density  $\mathbf{K} = \hat{z} K_0 \cos kx$ ;

c) a time-dependent surface current density  $\mathbf{K} = \hat{\mathbf{z}} K_0 e^{-i\omega t} \cos kx$ , discussing for which values of  $\omega$  (for fixed  $k$ ) the fields are propagating.

d) In case c), calculate the time- and space-averaged power dissipated per unit surface on the  $y = 0$  plane

$$W = \frac{k}{2\pi} \int_{-\pi/k}^{+\pi/k} \langle \mathbf{K}(x, t) \cdot \mathbf{E}(x, y = 0, t) \rangle dx, \quad (13.3)$$

and find for which values of  $\omega$  we have  $W = 0$ . Discuss the result with respect to the findings of point c).

## 13.6 Energy and Momentum Flow Close to a Perfect Mirror

Consider a plane EM wave, propagating along the  $x$  axis of a Cartesian coordinate system, of frequency  $\omega$  and *elliptical* polarization, with electric field

$$\mathbf{E}_i = \frac{E_0}{\sqrt{1 + \epsilon^2}} [\hat{\mathbf{y}} \cos(kx - \omega t) - \hat{\mathbf{z}} \epsilon \sin(kx - \omega t)], \quad (13.4)$$

where  $k = \omega/c$ , and  $\epsilon$  is a real parameter,  $0 < \epsilon < 1$ , characterizing the eccentricity of the polarization ellipse. The normalization factor  $1/\sqrt{1 + \epsilon^2}$  has been chosen so that the intensity of the wave is  $I = cE_0^2/8\pi$  for any value of  $\epsilon$ . The wave is incident on a plane, perfect mirror located at  $x = 0$ .

- a) Evaluate the Poynting vector  $\mathbf{S} = \mathbf{S}(x, t)$  in front of the mirror, including the contribution of the reflected wave. Find the value of  $\epsilon$  for which  $\mathbf{S} = 0$  everywhere, and the corresponding angle between the total electric ( $\mathbf{E}$ ) and magnetic ( $\mathbf{B}$ ) fields.  
 b) Find the force per unit surface on the mirror  $F_x = T_{xx}$ , where  $T_{xx} = T_{11}(x = 0^-)$  is the (1,1) component of the stress tensor at  $x = 0^-$ . Show that, in general,  $F_x$  has both a steady and an oscillating component, and find the frequency of the latter. For which value of  $\epsilon$  the oscillating component is missing?

## 13.7 Laser Cooling of a Mirror

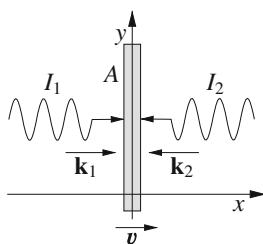


Fig. 13.5

A plane mirror has surface area  $A$ , finite thickness, mass  $M$ , and its two opposite surfaces are perfectly reflecting. At  $t = 0$  the mirror lies on the  $x = 0$  plane of a Cartesian coordinate system, as in Fig. 13.5. Two plane EM waves of intensities  $I_1$  and  $I_2$ , respectively, are impinging at normal incidence on the two surfaces.

- a)** Find the total force on the mirror and the direction of its acceleration if  $I_1 > I_2$ .  
 Now assume that the two waves have equal intensities,  $I_1 = I_2 = I$ , and that the mirror is moving with velocity  $\mathbf{v} = \hat{\mathbf{x}}v$ .
- b)** Evaluate the force on the mirror, in the system where the mirror is at rest.  
**c)** Discuss the motion of the mirror under the action of the force found at point **b**), at the limit  $v \ll c$ .

## 13.8 Radiation Pressure on a Thin Foil

An EM wave of frequency  $\omega$ , traveling along the  $x$  axis of a Cartesian coordinate system, is impinging normally on a very thin foil of thickness  $d$  and surface  $A$  (see Problem 11.4, in particular Fig. 11.1). The foil is perfectly conducting in a frequency range containing  $\omega$ . As shown in the solution of Problem 11.4, the (complex) transmission and reflection coefficients of the foil are

$$t = \frac{1}{1 + \eta}, \quad r = -\frac{\eta}{1 + \eta}, \quad \text{where} \quad \eta = i \frac{\omega_p^2 d}{2\omega c}, \quad (13.5)$$

and  $\omega_p$  is the plasma frequency of the foil.

- a)** Show that the radiation pressure on the thin foil is

$$P_{\text{rad}} = \frac{2RI}{c}, \quad (13.6)$$

where  $I$  is the intensity of the wave and  $R \equiv |\mathbf{r}|^2$ .

- b)** Now assume that the foil is moving with velocity  $\mathbf{v} = \beta c \hat{\mathbf{x}}$  in the laboratory frame. Assuming  $R = 1$  (a perfectly reflecting foil), evaluate the force on the foil.  
**c)** How does the answer to **b**) change if  $R = R(\omega) < 1$ ?

## 13.9 Thomson Scattering in the Presence of a Magnetic Field

In a Cartesian reference frame, an electron moves in the presence of a uniform and constant magnetic field  $\mathbf{B}_0 = \hat{\mathbf{z}}B_0$  and of a monochromatic plane EM wave, propagating along  $\hat{\mathbf{z}}$ , of electric field

$$\mathbf{E}(z, t) = \hat{\mathbf{y}} E_i e^{ikz - i\omega t}, \quad (13.7)$$

with  $E_i \ll B_0$ .

- a)** Describe the motion of the electron in steady state conditions, neglecting friction and the effect of the magnetic field of the wave.

- b)** Calculate the power radiated by the electron. Discuss the dependence of the emitted spectrum on  $\omega$ , and the angular distribution of the emitted radiation at the limits  $\omega \ll \omega_c$  and  $\omega \gg \omega_c$ , where  $\omega_c = eB_0/m_e$  is the cyclotron frequency of the electron in the presence of  $\mathbf{B}_0$ .

### 13.10 Undulator Radiation

In a Cartesian laboratory reference frame  $S \equiv (x, y, z)$ , we have a static magnetic field  $\mathbf{B}$ . In a certain region of space, free of charges and currents, the magnetic field is independent of  $z$ , and its  $y$  and  $z$  components can be written as

$$B_y = b(y) \cos(kx), \quad B_z \equiv 0, \quad (13.8)$$

where  $b(y)$  is an even function of  $y$ . The field is generated by sources located outside the region of interest, at finite values of  $|y|$ .

- a)** Show that, in the region of interest, we must have

$$B_y = B_0 \cos(kx) \cosh(ky), \quad (13.9)$$

and determine the expression for  $B_x$ .

Now assume that, in the laboratory frame  $S$ , an electron enters our magnetic field region with initial velocity  $\mathbf{v} = \hat{\mathbf{x}} v$ .

- b)** Describe the electron motion in the frame  $S'$ , moving at the velocity  $\mathbf{v}$  relative to the laboratory frame  $S$ , and discuss the emitted radiation. (Assume the electron motion to be non-relativistic and keep only linear terms in the equation of motion.)  
**c)** Determine the frequency of the radiation emitted in the directions both parallel and antiparallel to  $\mathbf{v}$ , as observed in  $S$ . In which directions the radiation intensity is zero in  $S'$ ?

### 13.11 Electromagnetic Torque on a Conducting Sphere

A plane, monochromatic, circularly polarized electromagnetic wave, of wavelength  $\lambda = 2\pi c/\omega$  and amplitude  $E_0$ , impinges on a small metallic sphere of radius  $a \ll \lambda$ . We assume that  $\omega$  is low enough so that the metal can be considered as an Ohmic conductor, of conductivity  $\sigma$  independent of frequency.

- a)** Evaluate the dipole moment induced on the sphere.  
**b)** Show that the EM wave exerts a torque on the sphere.

## 13.12 Surface Waves in a Thin Foil

A very thin conductive foil is located between the  $x = -\ell$  and  $x = +\ell$  planes of a Cartesian coordinate system. A *surface wave* propagates along both sides of the foil, in the  $y$ -direction. The fields of the surface wave have only the following non-zero components:  $E_x$ ,  $E_y$ , and  $B_z$ , all of them independent of  $z$ . We know that the electric field component *parallel* to the propagation direction is

$$E_y(x, y, t) = E_0 e^{-q|x|} e^{i(ky - \omega t)}, \quad (13.10)$$

where the frequency  $\omega$  is such that  $\lambda = 2\pi c/\omega \gg \ell$ . In these conditions, the foil can be treated, with good approximation, as the superposition of a surface charge  $\sigma(y, z, t)$ , and a surface current  $\mathbf{K}(y, z, t)$ , all lying on the  $x = 0$  plane.

Starting from (13.10) and Maxwell's equations in vacuum, evaluate

- a)** the field components  $E_x$  and  $B_z$ , specifying their parity with respect to  $\hat{\mathbf{x}}$ , and the surface current  $\mathbf{K}(y, z, t)$ ,
- b)** the Poynting vector and the time-averaged flux of electromagnetic energy associated to the surface wave,
- c)** the relations between  $q$ ,  $k$ , and  $\omega$ .

Now assume that, in the relevant frequency range, the relation between the current density  $\mathbf{J}$  and the electric field  $\mathbf{E}$  in the foil can be written (for harmonic fields) as

$$\mathbf{J} = 4\pi i \frac{\omega_p^2}{\omega} \mathbf{E}, \quad (13.11)$$

where  $\omega_p$  is the plasma frequency of the foil. Equation (13.11) characterizes an ideal conductor in the high-frequency regime (Problem 11.1)

- d)** Using (13.11) and the boundary conditions for the fields of a thin foil discussed in Problem 11.4, obtain an additional relation between  $q$ ,  $k$ , and  $\omega$ .
- e)** By combining the results of points **d**) and **e**) find the dispersion relation  $\omega = \omega(k)$  and discuss its limits of validity.

## 13.13 The Fizeau Effect

A plane electromagnetic wave of frequency  $\omega$  and wavevector  $\mathbf{k} = \hat{\mathbf{x}}k$  propagates in a homogenous medium, while the medium itself is moving with velocity  $\mathbf{u} = \hat{\mathbf{x}}u$  (thus parallel to  $\mathbf{k}$ ) in the laboratory frame. The refractive index of the medium is real and positive,  $n > 0$ , *in the rest frame of the medium*. Assume  $u \ll c$ , and answer the following questions evaluating all results up to the first order in  $\beta = u/c$ .

First, assume a non-dispersive medium, with  $n$  *independent* of frequency.

- a)** Evaluate the phase velocity of the wave,  $v_\varphi$ , *in the laboratory frame*.

Now assume that the medium is dispersive, with  $n$  depending on frequency according to a known law  $n = n(\omega)$ , defined *in the rest frame of the medium*.

**b)** Evaluate the phase velocity in the laboratory frame in this case, showing that now

$$v_\varphi \simeq \frac{c}{n(\omega)} + \beta c \left( 1 - \frac{1}{n^2(\omega)} + \frac{\omega}{n(\omega)} \partial_\omega n(\omega) \right) + O(\beta^2). \quad (13.12)$$

Hint: use the first-order Doppler effect for evaluating the relation between the frequency in the laboratory frame and the frequency observed in the rest frame of the medium.

**c)** Use (13.12) to show that, in a medium containing free electrons moving with negligible friction (a simple metal or an ideal plasma), the phase velocity does *not* depend on  $\beta$  up to to first order [1].

## 13.14 Lorentz Transformations for Longitudinal Waves

Consider a longitudinal wave with fields

$$\mathbf{E} = \mathbf{E}(x, t) = \hat{\mathbf{x}} E_0 e^{i(kx - \omega t)}, \quad \mathbf{B} \equiv 0, \quad (13.13)$$

in the (Cartesian) laboratory frame  $S$ . We have shown in Problem (11.3) that the phase velocity of this wave,  $v_\varphi = \omega_L/k_L$  is undetermined, and can have arbitrary values.

Find the frequency, wavevector and fields of the wave in a frame  $S'$ , moving with velocity  $\mathbf{v}$  with respect to  $S$ , for the three following cases:

- a)**  $\mathbf{v} = v_\varphi \hat{\mathbf{x}}$ , with  $v_\varphi < c$ ;
- b)**  $\mathbf{v} = (c^2/v_\varphi) \hat{\mathbf{x}}$ , with  $v_\varphi > c$ ;
- c)**  $\mathbf{v} = V \hat{\mathbf{y}}$  with  $V < c$ .

## 13.15 Lorentz Transformations for a Transmission Cable

A transmission cable can be schematized as an infinite straight conducting wire. We choose a cylindrical coordinate system  $(r, \phi, z)$  with the  $z$  axis along the wire. A monochromatic charge and current signal, of frequency  $\omega$ , propagates along the cable, with total current  $I$  and linear charge density  $\lambda$  given by, in complex notation,

$$I = I(z, t) = I_0 e^{ikz-i\omega t}, \quad \lambda = \lambda(z, t) = \lambda_0 e^{ikz-i\omega t}. \quad (13.14)$$

The cable is located in a uniform medium of real dielectric permittivity  $\epsilon > 1$ , and magnetic permeability  $\mu = 1$ , in the frequency region of interest.

- a) Find the relation between  $I_0$  and  $\lambda_0$ .  
 b) Evaluate the electric and magnetic fields in the medium,  $\mathbf{E}(r, z, t)$  and  $\mathbf{B}(r, z, t)$ , assuming that they are in a TEM mode, i.e.,

$$\mathbf{E}(r, z, t) = \mathbf{E}(r) e^{ikz - i\omega t}, \quad \mathbf{B}(r, z, t) = \mathbf{B}(r) e^{ikz - i\omega t}, \quad (13.15)$$

with  $\mathbf{E} \cdot \hat{\mathbf{z}} = 0$  and  $\mathbf{B} \cdot \hat{\mathbf{z}} = 0$ . Evaluate the dispersion relation between the frequency  $\omega$  and the wave vector  $k$ .

- c) Show that the fields and their sources are independent of time in a reference frame  $S'$ , moving at the phase velocity  $\hat{\mathbf{z}}v_\varphi = \hat{\mathbf{z}}(\omega/k)$  relative to the laboratory frame  $S$  where the wire is at rest. Show that, in  $S'$ , we have  $\mathbf{E}' = 0$  and  $I' = 0$ , while  $\lambda' \neq 0$  and  $\mathbf{B}' \neq 0$ . Explain this apparently surprising result.

## 13.16 A Waveguide with a Moving End

Two perfectly conducting plane surfaces located at  $y = \pm a/2$ , respectively, form a waveguide. The waveguide is terminated at  $x = 0$  by a perfectly conducting wall, as shown in Fig. 13.6. Consider the propagation of a monochromatic  $\text{TE}_{10}$  wave along the  $x$  axis. The electric field of the wave has only the  $z$  component. In complex notation we have  $E_z(x, y, t) = E_z(y) e^{ikx - i\omega t}$ , where  $\omega$  and  $k$  are related by the dispersion relation of the  $\text{TE}_{10}$  mode.

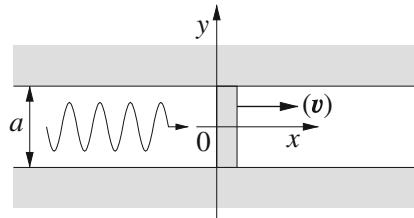


Fig. 13.6

- a) Find the total electric and magnetic fields inside the waveguide.

Now assume that the end of the waveguide moves with constant velocity  $\mathbf{v} = v \hat{\mathbf{x}}$ .

- b) Assuming  $v < kc^2/\omega$ , determine the frequency  $\omega_r$  and the wavevector  $k_r$  of the reflected wave. Verify that  $\omega_r$  and  $k_r$  are related by the dispersion relation of the  $\text{TE}_{10}$  mode.

- c) What happens in the  $v > kc^2/\omega$  case?

## 13.17 A “Relativistically” Strong Electromagnetic Wave

We consider a circularly polarized, plane electromagnetic wave propagating parallel to the  $z$  axis of a Cartesian reference frame. The wave fields are

$$\mathbf{E} = E_0 [\hat{\mathbf{x}} \cos(kz - \omega t) - \hat{\mathbf{y}} \sin(kz - \omega t)], \quad (13.16)$$

$$\mathbf{B} = E_0 [\hat{\mathbf{x}} \sin(kz - \omega t) + \hat{\mathbf{y}} \cos(kz - \omega t)]. \quad (13.17)$$

We assume that the field is strong enough that electrons oscillate at *relativistic* velocities *not* much smaller than  $c$ . The relativistically correct equation of motion for an electron in the presence of the intense wave is

$$\frac{d\mathbf{p}}{dt} = -e \left( \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right), \quad (13.18)$$

where  $\mathbf{p} = m_e \gamma \mathbf{v}$ , and  $\gamma = 1/\sqrt{1-v^2/c^2} = \sqrt{1+\mathbf{p}^2/(m_e c)^2}$ .

We want to study the propagation of such a “relativistic” wave in a medium with free electrons (ions are considered at rest).

- a) Show that it is self-consistent to assume that the electron motion occurs on the  $xy$  plane. Do this in two steps. First we assume that, at  $t = 0$ , the  $z$  component of the momentum of the electrons is zero,  $p_z = 0$ . Then solve the equations of motion in steady state conditions and verify the consistency of the assumption *a posteriori*.
- b) Show that the Lorentz factor  $\gamma$  is time-independent, and give its expression.
- c) Calculate the refraction index for a medium with free electron density  $n_e$ .
- d) Find the dispersion relation and the cut-off frequency for the electromagnetic wave, comparing the result with the “non-relativistic” case of low field amplitudes.

### 13.18 Electric Current in a Solenoid

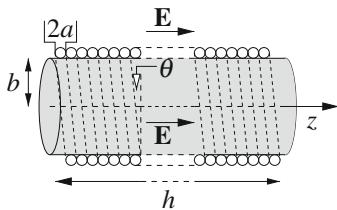


Fig. 13.7

A solenoid is made by winding a thin conducting wire of radius  $a$  and conductivity  $\sigma$  around a non-conducting cylinder of radius  $b \gg a$  and height  $h \gg b$ . Thus the solenoid coil has a pitch

$$\theta = \arctan \left( \frac{a}{\pi b} \right) \ll 1, \quad (13.19)$$

since the wire moves in the  $\hat{z}$  direction by a step of length  $2a$  at every turn of length  $2\pi b$ . The solenoid is located in an external uniform electric field  $\mathbf{E} = E\hat{z}$ .

- a) Evaluate the magnetic field  $\mathbf{B}$  both *inside* ( $r < b$ ) and *outside* ( $r > b$ ) the solenoid, neglecting boundary effects.
- b) Calculate the flux of the Poynting vector  $\mathbf{S} = c\mathbf{E} \times \mathbf{B}/4\pi$  through a cylindrical surface external and coaxial to the solenoid, and compare its value with the power dissipated by Joule heating.

### 13.19 An Optomechanical Cavity

We have a one-dimensional cavity limited by two perfectly conducting plane surfaces located at  $x = \pm d/2$ , respectively, as in Fig. 13.8. The electromagnetic field inside the cavity has frequency  $\omega$ , peak amplitude of the electric field  $E_0$ , and is linearly polarized parallel to the walls.

- a) Find the possible values for  $\omega$  and write the most general form of the electromagnetic field.
- b) Calculate the electromagnetic energy per unit surface  $U$  inside the cavity.
- c) Calculate the radiation pressure  $P$  on the walls as a function of  $E_0$ , and the ratio  $U/P$ .
- d) Now assume that the two cavity walls are finite squares, each of mass  $M$  and surface  $S \gg d^2$ . Each cavity wall is connected to an external fixed wall by a spring of Hooke's constant  $K$ , as shown in Fig. 13.9. Neglecting boundary effects, evaluate the relation between frequency and amplitude of the electromagnetic modes of the cavity.

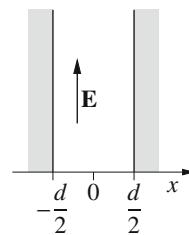


Fig. 13.8

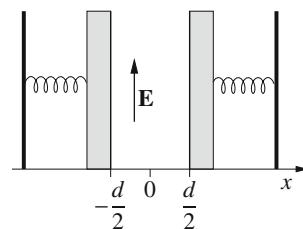


Fig. 13.9

### 13.20 Radiation Pressure on an Absorbing Medium

In an appropriate Cartesian reference frame we have vacuum in the  $x < 0$  half-space, while the  $x > 0$  half-space is filled with a medium of *complex* refractive index

$$\mathbf{n} = n_1 + i n_2 , \quad (13.20)$$

with  $n_1 > 1 \gg n_2$ . A monochromatic plane wave of frequency  $\omega$  and intensity  $I_i$ , propagating in the positive  $x$  direction, is incident on the  $x = 0$  plane. Calculate

- a) the power absorbed by the medium,  $W_{\text{abs}}$ , showing that  $W_{\text{abs}} = (1 - R) I_i$ , where  $R$  is the reflection coefficient ( $R = |r|^2$  where  $r$  is the usual amplitude coefficient for the reflected wave as defined in the Fresnel formulas);
- b) the pressure on the medium,  $P_{\text{rad}}$ , showing that  $P_{\text{rad}} = (1 + R) I_i / c$ .

### 13.21 Scattering from a Perfectly Conducting Sphere

In a Cartesian reference frame  $(x, y, z)$ , a linearly polarized, plane monochromatic electromagnetic wave has electric field

$$\mathbf{E}(x, t) = \hat{\mathbf{y}} E_0 \cos(kx - \omega t), \quad \text{where } k = \frac{\omega}{c}, \quad (13.21)$$

and impinges on a metallic sphere of radius  $a \ll \lambda = 2\pi/k$ , located at the origin of the reference frame. We further assume that the sphere is “perfectly conducting” at the frequency of the wave, so that the total electric field can be assumed to be zero over the whole volume of the sphere.

- a)** Find the power scattered by the sphere in the electric dipole approximation, and the corresponding scattering cross section.
- b)** Assuming that the sphere is also perfectly diamagnetic ( $\mathbf{B} = 0$  inside the sphere), find the contribution of the magnetic dipole term to the scattering cross section.

### 13.22 Radiation and Scattering from a Linear Molecule

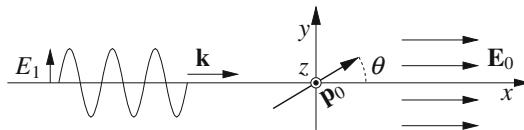


Fig. 13.10

A simple model for a polar linear molecule, neglecting vibrations, is a one-dimensional rigid rotor associated to an electric dipole moment  $\mathbf{p}_0$ . The molecule has moment of inertia  $\mathcal{I}$  about any rotational axis passing through its barycenter and perpendicular to the molecule. Let us consider a polar linear molecule located in a uniform and constant electric field  $\mathbf{E}_0$ , parallel to the  $x$  axis of a Cartesian coordinate system (right part of Fig. 13.10).

- a)** Find the equilibrium positions of the molecule, and discuss the motion when the molecule at time  $t = 0$  is slightly displaced from its stable equilibrium position.
- b)** Describe the radiation emitted by the molecule during small amplitude oscillations, and estimate the damping time of such oscillations.

Now assume that a monochromatic plane wave, linearly polarized along the  $y$  axis, of frequency  $\omega$  and electric field amplitude  $E_1$ , is propagating along the  $x$  axis. Also assume that the length of the molecule,  $d$ , is much smaller than the wavelength,  $d \ll \lambda = 2\pi c/\omega = 2\pi/k$ .

- c)** Describe the motion of the molecule in these conditions.
- d)** Calculate the power scattered by the molecule and its scattering cross section.

## 13.23 Radiation Drag Force

The classical motion of a particle of charge  $q$  and mass  $m$ , under the simultaneous action of an electric field  $\mathbf{E}$  and a magnetic field  $\mathbf{B}$ , is described by the equation

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{d\mathbf{v}}{dt} = \frac{q}{m} \left( \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) - \eta \mathbf{v}, \quad (13.22)$$

where  $\eta$  is a damping coefficient.

Given a Cartesian reference frame  $(x, y, z)$ , consider the motion of the particle in the presence of a plane, monochromatic electromagnetic wave propagating along the  $x$  axis. The wave is linearly polarized along  $y$  with electric field amplitude  $E_0$ , has frequency  $\omega$  and wave vector  $\mathbf{k} = \hat{x}\omega/c$ . Assume the velocity of the particle to be much smaller than  $c$ , so that, as a first order approximation, we can neglect the  $\mathbf{v} \times \mathbf{B}/c$  term.

- a) Solve (13.22) in steady-state conditions.
- b) Calculate the cycle-averaged power  $P_{\text{abs}}$  absorbed by the particle, i.e., the work made by the electromagnetic force over an oscillation period.
- c) Calculate the cycle-averaged power  $P_{\text{rad}}$  radiated by the particle, and obtain an expression for the damping coefficient  $\eta$  assuming that all the absorbed power is re-emitted as radiation,  $P_{\text{rad}} = P_{\text{abs}}$ .
- d) Now use the result of point b) to evaluate the effect of the term  $\mathbf{v} \times \mathbf{B}/c$ . The cycle-averaged force along  $x$ , which accelerates the particle in the wave propagation direction, is

$$F_x = \left\langle q \left( \frac{\mathbf{v}}{c} \times \mathbf{B} \right)_x \right\rangle. \quad (13.23)$$

Calculate  $F_x$  and the  $P_{\text{abs}}/F_x$  ratio.

- e) Assume that instead of a point particle we have a small sphere of radius  $a$ , such that  $ka \ll 1$ , containing  $N \gg 1$  particles (plus a neutralizing background). Find the force on the sphere and the related acceleration as a function of  $N$  (neglect any collective effect such as screening of the electromagnetic field inside the sphere).

## Reference

1. I. Lerche, On a curiosity arising from Fizeau's experiment. Am. J. Phys. **43**, 910 (1975)

# Chapter S-1

## Solutions for Chapter 1

### S-1.1 Overlapping Charged Spheres

**a)** The electrostatic field at any point in space is the sum of the fields generated by each charged sphere (superposition principle). The field generated by a single uniformly charged sphere at its interior is  $\mathbf{E}(\mathbf{r}) = 4\pi k_e \varrho_0 \mathbf{r}/3$ , where  $\varrho_0$  is the charge density and  $\mathbf{r}$  is the position vector relative to the center of the sphere. Thus, the two spheres generate at their interiors the fields  $\mathbf{E}_\pm = \pm 4\pi k_e \varrho_0 \mathbf{r}_\pm/3$ , respectively,  $\mathbf{r}_\pm$  being the position vectors relative to the two centers. We assume that the centers are located on the  $x$  axis at points  $O_+ \equiv (+\delta/2, 0, 0)$  and  $O_- \equiv (-\delta/2, 0, 0)$ . We thus have  $\mathbf{r}_\pm = \mathbf{r} \pm \boldsymbol{\delta}/2$ , where  $\mathbf{r}$  is the position vector relative to the origin  $O \equiv (0, 0, 0)$ . The total field in the overlap region is

$$\mathbf{E}_{in} = +\frac{4\pi k_e}{3} \varrho_0 \left( \mathbf{r} - \frac{\boldsymbol{\delta}}{2} \right) - \frac{4\pi k_e}{3} \varrho_0 \left( \mathbf{r} + \frac{\boldsymbol{\delta}}{2} \right) = -\frac{4\pi k_e}{3} \varrho_0 \boldsymbol{\delta}. \quad (S-1.1)$$

The internal field  $\mathbf{E}_{in}$  is thus *uniform* and proportional to  $-\boldsymbol{\delta}$ .

**b)** The electrostatic field generated by a uniformly charged sphere, with volume charge density  $\varrho_0$ , *outside* its volume equals is the field of a point charge  $Q = 4\pi R^3 \varrho_0/3$  located at its center. Thus, the electrostatic field in the outer region (outside both spheres) is the sum of the fields of two point charges  $\pm Q$  located at  $O_+$  and  $O_-$ , respectively. If  $R \gg \delta$ , this is equivalent to the field of an electric dipole of moment

$$\mathbf{p} = Q\boldsymbol{\delta} = \frac{4\pi R^3}{3} \varrho_0 \boldsymbol{\delta} \quad (S-1.2)$$

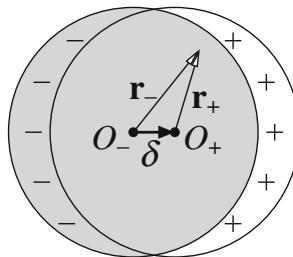


Fig. S-1.1

located at the origin and lying on the  $x$  axis. The external field is thus

$$\mathbf{E}_{\text{ext}}(\mathbf{r}) = k_e \frac{3\hat{\mathbf{r}}(\mathbf{p} \cdot \hat{\mathbf{r}}) - \mathbf{p}}{r^3}, \quad (\text{S-1.3})$$

where  $\mathbf{r} = r\hat{\mathbf{r}}$  is the vector position relative to the origin.

In the two transition “shell” regions, of net charge densities  $\pm\varrho_0$ , the field is the sum of the inner field of one sphere and of the outer field of the other. We omit to write down the expression for brevity (Fig. S-1.1).

## S-1.2 Charged Sphere with Internal Spherical Cavity

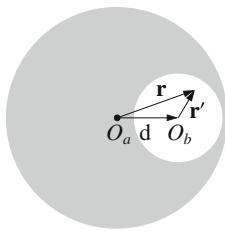


Fig. S-1.2

a) Once again we use the superposition principle. Our charged sphere with an internal spherical cavity can be thought of as the superposition of two uniformly charged spheres: a sphere of radius  $a$  centered in  $O_a$ , with charge density  $\varrho$ , and a smaller sphere of radius  $b$  centered in  $O_b$ , with charge density  $-\varrho$ . The electric field everywhere in space is the sum of the fields generated by the two spheres. The field generated by a uniformly charged sphere at its inside is  $\mathbf{E} = (4\pi k_e/3)\varrho \mathbf{r}$ , where  $\mathbf{r}$  is the distance from the center of the sphere. The total field inside the cavity at a point of vector position  $\mathbf{r}$  relative to  $O_a$ , and vector position  $\mathbf{r}'$  relative to  $O_b$ , is thus

$$\mathbf{E}_{\text{cav}} = \frac{4\pi k_e}{3} \varrho (\mathbf{r} - \mathbf{r}') = \frac{4\pi k_e}{3} \varrho \mathbf{d}, \quad (\text{S-1.4})$$

uniform and parallel to the straight line passing through  $O_a$  and  $O_b$ . If  $\mathbf{d} = 0$  we obtain  $\mathbf{E} = 0$ , as expected from Gauss’s law and symmetry considerations.

b) In an external field  $\mathbf{E}_0$  the total force on the system is the sum of the forces that  $\mathbf{E}_0$  would exert on the two point charges  $Q_a = 4\pi a^3 \varrho/3$  and  $Q_b = -4\pi b^3 \varrho/3$ , located in  $O_a$  and  $O_b$ , respectively, so that

$$\mathbf{F} = \frac{4\pi}{3} \varrho (a^3 - b^3) \mathbf{E}_0. \quad (\text{S-1.5})$$

c) Since the vector sum of the forces is different from zero, the torque depends on our choice of the origin. The torque about the center of the sphere  $O_a$  is

$$\tau = \mathbf{d} \times \mathbf{F} = -\frac{4\pi}{3} \varrho b^3 \mathbf{d} \times \mathbf{E}_0. \quad (\text{S-1.6})$$

Let us introduce a reference system with the  $x$  axis passing through  $O_a$  and  $O_b$ , and the origin in  $O_a$ . We denote by  $\varrho_m = \alpha\varrho$  the mass density, with  $\alpha$  some constant value. Our coordinate origin is thus the location of the center of mass of the cavity-less sphere of radius  $a$  and mass  $M_{tot} = 4\pi a^3 \varrho_m / 3$ , while  $x = d$  is the location of the center of mass of a sphere of radius  $b$  and mass  $M_b = 4\pi b^3 \varrho_m / 3$ . Let us denote by  $x_c$  the center of mass of the sphere with cavity, of mass  $M_c = 4\pi(a^3 - b^3)\varrho_m / 3$ . We have

$$0 = \frac{M_c x_c + M_b d}{M_{tot}}, \quad \text{thus} \quad x_c = -d \frac{M_b}{M_c} = -d \frac{b^3}{a^3 - b^3}. \quad (\text{S-1.7})$$

The torque about the center of mass  $x_c$  is thus

$$\tau_c = \frac{b^3 d}{a^3 - b^3} \frac{4\pi}{3} \varrho a^3 \hat{x} \times \mathbf{E}_0 - \left( d + \frac{b^3 d}{a^3 - b^3} \right) \frac{4\pi}{3} \varrho b^3 \hat{x} \times \mathbf{E}_0 = 0, \quad (\text{S-1.8})$$

as was to be expected, since each charged volume element  $d^3 r$  is subject to the force  $\varrho \mathbf{E}_0 d^3 r$ , and acquires an acceleration

$$\mathbf{a} = \frac{\varrho \mathbf{E}_0 d^3 r}{\varrho_m d^3 r} = \frac{\mathbf{E}_0}{\alpha}, \quad (\text{S-1.9})$$

equal for each charged volume element (Fig. S-1.2).

### S-1.3 Energy of a Charged Sphere

a) We can assemble the sphere by moving successive infinitesimal shells of charge from infinity to their final location. Let us assume that we have already assembled a sphere of charge density  $\varrho$  and radius  $r < R$ , and that we are adding a further shell of thickness  $dr$ . The assembled sphere has charge  $q(r) = \varrho(4\pi r^3 / 3)$ , and its potential  $\varphi(r, r')$  at any point at distance  $r' \geq r$  from the center of the sphere is

$$\varphi(r, r') = k_e \frac{q(r)}{r'} = k_e \varrho \frac{4\pi r^3}{3} \frac{1}{r'} . \quad (\text{S-1.10})$$

The work needed to move the new shell of charge  $dq = \varrho 4\pi r^2 dr$  from infinity to  $r$  is

$$dW = \varphi(r, r') dq = k_e \varrho \frac{4\pi r^3}{3} \frac{1}{r} \varrho 4\pi r^2 dr = k_e \frac{(4\pi \varrho)^2}{3} r^4 dr. \quad (\text{S-1.11})$$

The total work needed to assemble the sphere is obtained by integrating  $dW$  from  $r = 0$  (no sphere) up to the final radius  $R$

$$U_0 = k_e \frac{(4\pi\varrho)^2}{3} \int_0^R r^4 dr = k_e \frac{(4\pi\varrho)^2 R^5}{15} = \frac{3k_e}{5} \frac{Q^2}{R}, \quad (\text{S-1.12})$$

where  $Q = (\varrho 4\pi R^3)/3$  is the total charge of the sphere.

**b)** The electric field everywhere in space is, according to Gauss's law,

$$E(r) = k_e Q \times \begin{cases} \frac{r}{R^3}, & r \leq R \\ \frac{1}{r^2}, & r \geq R, \end{cases} \quad (\text{S-1.13})$$

and the integral of the corresponding energy density  $u_E = E^2/(8\pi k_e)$  over the whole space is

$$\begin{aligned} U_0 &= \int_0^\infty \frac{E^2(r)}{8\pi k_e} 4\pi r^2 dr = \frac{k_e^2 Q^2}{2k_e} \left[ \int_0^R \left( \frac{r}{R^3} \right)^2 r^2 dr + \int_R^\infty \left( \frac{1}{r^2} \right)^2 r^2 dr \right] \\ &= k_e \frac{Q^2}{2} \left( \frac{1}{5R} + \frac{1}{R} \right) = \frac{3k_e}{5} \frac{Q^2}{R}. \end{aligned} \quad (\text{S-1.14})$$

**c)** The electrostatic potential of the sphere everywhere in space is

$$\varphi(r) = k_e Q \times \begin{cases} -\frac{r^2}{2R^3} + \frac{3}{2R}, & r \leq R \\ \frac{1}{r}, & r > R \end{cases} \quad (\text{S-1.15})$$

where the constant  $3k_e Q/(2R)$  appearing for  $r \leq R$  is needed for  $\varphi(r)$  to be continuous at  $r = R$ . Since  $\varrho = 0$  for  $r > R$ , we need only the integral of  $\varrho\varphi/2$  inside the sphere

$$\begin{aligned} U_0 &= \frac{1}{2} \int_0^R \varrho k_e Q \left( -\frac{r^2}{2R^3} + \frac{3}{2R} \right) 4\pi r^2 dr = k_e \frac{Q}{4} \frac{Q}{R^3/3} \left( -\frac{R^2}{5} + R^2 \right) \\ &= \frac{3k_e}{4} \frac{Q^2}{R^3} \frac{4R^2}{5} = \frac{3k_e}{5} \frac{Q^2}{R}. \end{aligned} \quad (\text{S-1.16})$$

All methods, including **b)** and **c)**, lead to the correct result, as expected. However, a comparison between methods **b)** and **c)** shows that it is incorrect to interpret the “energy density” of the electric field as the “energy stored in a given region of space per unit volume”. If we give this meaning to quantity  $E^2/(8\pi k_e)$ , as in **b)**, we conclude that the energy is spread over the whole space. If, on the other hand, we assume the energy density to be  $\frac{1}{2}\varrho\varphi$ , as in **c)**, the energy is “stored” only inside the volume of the sphere, i.e., “where the charge is”. Thus, the concept of energy density is ambiguous, while the total electrostatic energy of the system is a well defined quantity, at least in the absence of point charges.

## S-1.4 Plasma Oscillations

a) Assuming  $\delta > 0$ , the collective rigid displacement of the conduction electrons due to the external field gives origin to the charge density

$$\varrho(x) = \begin{cases} 0, & x < 0, \\ +en, & 0 < x < \delta, \\ 0, & \delta < x < h, \\ -en, & h < x < h + \delta, \\ 0, & x > h + \delta. \end{cases} \quad (\text{S-1.17})$$

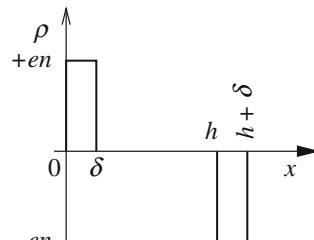


Fig. S-1.3

The electrostatic field  $\mathbf{E}(x)$  generated by this charge distribution is obtained by integrating the equation  $\nabla \cdot \mathbf{E} = \partial_x E_x = 4\pi k_e \varrho$  with the boundary condition  $\mathbf{E}(-\infty) = 0$ :

$$E_x(x) = 4\pi en k_e \begin{cases} 0, & x < 0, \\ x, & 0 < x < \delta, \\ \delta, & \delta < x < h, \\ h + \delta - x, & h < x < h + \delta, \\ 0, & x > h + \delta. \end{cases} \quad (\text{S-1.18})$$

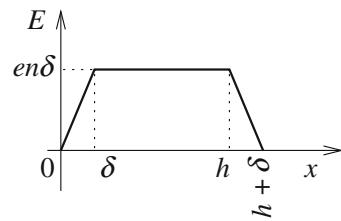


Fig. S-1.4

If we assume a negative displacement  $-\delta$  (with  $\delta > 0$ ) the charge density and the electric field are

$$\varrho(x) = \begin{cases} 0, & x < -\delta, \\ -en, & -\delta < x < 0, \\ 0, & 0 < x < h - \delta, \\ +en, & h - \delta < x < h, \\ 0, & x > h. \end{cases} \quad E_x(x) = 4\pi en k_e \begin{cases} 0, & x < -\delta, \\ -x - \delta, & -\delta < x < 0, \\ -\delta, & 0 < x < h - \delta, \\ x - h - \delta, & h - \delta < x < h, \\ 0, & x > h. \end{cases} \quad (\text{S-1.19})$$

The plots are obtained from Figs. S-1.3 and S-1.4, respectively, by flipping around the  $x$  axis and translating by  $\delta$  towards the negative  $x$  values.

b) The electrostatic energy of the system, in the case of a positive displacement, can be evaluated by integrating the “energy density”  $u = E_x^2/(8\pi k_e)$  over the whole space:

$$\begin{aligned} U_{\text{es}} &= \int \frac{E_x^2}{8\pi k_e} d^3r = \frac{L^2}{8\pi k_e} \int_0^{h+\delta} E_x^2 dx \\ &= \frac{L^2}{8\pi k_e} (4\pi en)^2 \left[ \int_0^\delta x^2 dx + \int_\delta^h \delta^2 dx + \int_h^{h+\delta} (h + \delta - x)^2 dx \right] \\ &= 2\pi k_e (enL)^2 \left[ \frac{\delta^3}{3} + \delta^2(h - \delta) + \frac{\delta^3}{3} \right] = 2\pi k_e (enL)^2 \left( h\delta^2 - \frac{\delta^3}{3} \right), \end{aligned} \quad (\text{S-1.20})$$

where, due to the symmetry of the problem, we used  $d^3r = L^2 dx$ . Exactly the same result is obtained for a negative displacement by  $-\delta$ . The  $\delta$  appearing in the last line of (S-1.20) is actually to be interpreted as the absolute value  $|\delta|$ .

c) At the limit  $\delta \ll h$  we can neglect the third-order term in  $\delta$  of (S-1.20), and approximate  $U_{\text{es}} \simeq 2\pi k_e(enL)^2 h \delta^2$ , which is the potential energy of a harmonic oscillator. The force on the “electron slab” is thus

$$F = -\frac{\partial U_{\text{es}}}{\partial \delta} = -4\pi k_e(enL)^2 h \delta, \quad (\text{S-1.21})$$

where  $\delta$  can be positive or negative. The equation of motion for the electrons is

$$M\ddot{\delta} = F \equiv -M\omega^2\delta, \quad (\text{S-1.22})$$

where  $M = m_e n L^2 h$  is the total mass of the conduction-electron slab. We thus have

$$\omega^2 = \frac{4\pi k_e n e^2}{m_e} \equiv \omega_p^2, \quad (\text{S-1.23})$$

where  $\omega_p$  is called the *plasma frequency*, and is an intrinsic property of the given conductor, dependent only on the density of free electrons.

## S-1.5 Mie Oscillations

a) In Problem 1.1 we showed that the electric field is uniform and equal to  $-4\pi k_e \varrho_0 \delta / 3$ , with  $\varrho_0 = en_e$ , in the region where the conduction-electrons sphere overlap,

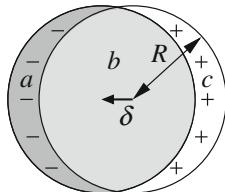


Fig. S-1.5

$$\mathbf{E}_{\text{int}} = -\frac{4\pi k_e}{3} en_e \delta. \quad (\text{S-1.24})$$

We assume that the displacement  $\delta$  is sufficiently small for the volumes  $a$  (only conduction electrons) and  $c$  (only ion lattice) of Fig. S-1.5 to be negligible compared to the overlap volume  $b$ , an order of magnitude for  $\delta$

is found in (S-2.4) of Solution S-2.1. Assuming further that conduction electrons behave like a “rigid” body, oscillating in phase with the same displacement  $\delta = \delta(t)$  from their rest positions, the equation of motion for the single electron is

$$m_e \frac{d^2 \delta}{dt^2} = -e \mathbf{E}_{\text{int}} = -e \frac{4\pi k_e}{3} en_e \delta = -m_e \frac{\omega_p^2}{3} \delta, \quad (\text{S-1.25})$$

where  $\omega_p$  is the plasma frequency (S-1.23). Thus, the displacement of each conduction electron from its rest position is

$$\delta(t) = \delta(0) \cos\left(\frac{\omega_p}{\sqrt{3}} t\right), \quad (\text{S-1.26})$$

where  $\delta(0)$  is a constant and  $\omega_p/\sqrt{3}$  is called the Mie frequency. This type of motion is known as Mie oscillation (or surface plasmon of the sphere).

**b)** The electrostatic energy of the system is given by the integral

$$U_{\text{es}} = \int \frac{E^2}{8\pi k_e} d^3r. \quad (\text{S-1.27})$$

For  $\delta$  approaching 0, the electric field is given by (S-1.24) inside the sphere ( $r < R$ ), and by (S-1.3), i.e., an electric dipole field, outside the sphere ( $r > R$ ). Thus we can split the integral of (S-1.27) into the sum of two terms, corresponding to the integration domains  $r < R$  and  $r > R$ , respectively

$$U_{\text{es}} = U_{\text{es}}^{\text{in}} + U_{\text{es}}^{\text{out}} = \frac{1}{8\pi k_e} \left( \int_{r < R} E^2 d^3r + \int_{r > R} E^2 d^3r \right). \quad (\text{S-1.28})$$

For  $r < R$  the field is uniform and we immediately find

$$U_{\text{es}}^{\text{in}} = \frac{1}{8\pi k_e} \left( k_e \frac{4\pi}{3} e n_e \delta \right)^2 \frac{4\pi}{3} R^3 = k_e \frac{8\pi^2}{27} (e n_e \delta)^2 R^3. \quad (\text{S-1.29})$$

For evaluating the contribution of the outer region, we substitute

$$d^3r = r^2 \sin\theta dr d\theta d\phi, \quad \text{and} \quad E^2 = \left( \frac{k_e p}{r^3} \right)^2 (3 \cos^2 \theta + 1), \quad (\text{S-1.30})$$

where  $\mathbf{p} = Q\boldsymbol{\delta} = \boldsymbol{\delta} e n_e 4\pi R^3 / 3$  (see Problem 1.1), into the second integral at the right-hand side of (S-1.28)

$$U_{\text{es}}^{\text{out}} = 2\pi \frac{1}{8\pi k_e} k_e^2 p^2 \int_R^{+\infty} r^2 dr \int_0^\pi \sin\theta d\theta \frac{3 \cos^2 \theta + 1}{r^6}, \quad (\text{S-1.31})$$

and obtain

$$U_{\text{es}}^{\text{out}} = k_e \frac{p^2}{3R^3} = k_e \frac{4\pi}{9} (e n_e \delta)^2 \frac{4\pi}{3} R^3 = k_e \frac{16\pi^2}{27} (e n_e \delta)^2 R^3, \quad (\text{S-1.32})$$

thus  $U_{\text{es}}^{\text{out}} = 2U_{\text{es}}^{\text{in}}$ . For the total energy we have finally

$$U_{\text{es}} = U_{\text{es}}^{\text{in}} + U_{\text{es}}^{\text{out}} = k_e \frac{8\pi^2}{9} (en_e \delta)^2 R^3. \quad (\text{S-1.33})$$

The derivative of  $U_{\text{es}}$  with respect to  $\delta$  gives the force associated to the displacement of the electron sphere:

$$F = -\frac{\partial U_{\text{es}}}{\partial \delta} = -k_e \frac{16\pi^2}{9} R^3 (en_e)^2 \delta. \quad (\text{S-1.34})$$

The equation of motion for the rigid sphere of electrons is  $M d^2\delta/(dt^2) = \mathbf{F}$ , where  $M = m_e n_e 4\pi R^3 / 3$  is the total mass of electrons. Thus

$$\frac{d^2\delta}{dt^2} = -k_e \frac{4\pi n_e e^2}{3m_e} \delta \equiv -\frac{\omega_p^2}{3} \delta, \quad (\text{S-1.35})$$

and we are back to the oscillations at the Mie frequency of (S-1.26).

## S-1.6 Coulomb Explosions

**a)** The electric field has radial symmetry,  $\mathbf{E} = E(r)\hat{\mathbf{r}}$ . According to Gauss's law we have  $4\pi r^2 E(r) = 4\pi k_e Q_{\text{int}}(r)$ , where  $Q_{\text{int}}(r)$  is the charge inside the sphere of radius  $r$ . At  $t = 0$  we have  $Q_{\text{int}} = Q(r/R)^3$  for  $r < R$ , and  $Q_{\text{int}} = Q$  for  $r > R$ , thus the electric field inside and outside the cloud is, respectively,

$$E(r) = k_e Q \times \begin{cases} \frac{r}{R^3}, & r \leq R, \\ \frac{1}{r^2}, & r \geq R. \end{cases} \quad (\text{S-1.36})$$

Due to the spherical symmetry of the problem, we have  $E = -\partial\varphi/\partial r$ , where  $\varphi$  is the electric potential, and the potential at  $t = 0$  can be obtained by a simple integration:

$$\varphi(r) = k_e Q \times \begin{cases} -\frac{r^2}{2R^3} + \frac{3}{2R}, & r \leq R, \\ \frac{1}{r}, & r \geq R. \end{cases} \quad (\text{S-1.37})$$

As in Equation (S-1.15) of Problem 1.3, the integration constants have been chosen so that  $\varphi(\infty) = 0$  and  $\varphi(r)$  is continuous at  $r = R$ . The potential energy of a *test* charge  $q_t$  located at distance  $r$  from the center is thus  $q_t \varphi(r)$ .

**b)** Under the action of the electric field, the test charge would move and convert all its potential energy into kinetic energy *if the field remained stationary during the charge motion*, i.e., *if all the source charges of the field remained fixed*. At  $t = 0$  the

electric field inside the spherical cloud increases with  $r$ . Thus, the “outer” particles, located at larger  $r$ , have a higher acceleration than the “inner” particles, located at smaller  $r$ . After an infinitesimal time interval the “outer” particles will acquire a higher velocity (we are assuming that all particles are at rest at  $t = 0$ ), and will not be overtaken by the “inner” ones. Moreover, also the acceleration has radial symmetry, and thus any spherical shell preserves its shape in time. These arguments can be iterated for any following time, proving the validity of our assumptions that the particles do not overtake one another, and that the spherical symmetry is preserved.

Let us denote by  $r_s(r_0, t)$  the position of a particle initially located at  $r_0$ . Since the particles do not overtake one another, the charge inside a sphere of radius  $r_s(r_0, t)$  is constant. The electric field intensity at  $r_s(r_0, t)$  can be evaluated by applying Gauss’s law: from

$$4\pi r_s^2(r_0, t) E[r_s(r_0, t)] = 4\pi k_e Q \left(\frac{r_0}{R}\right)^3 \quad (\text{S-1.38})$$

we obtain

$$E[r_s(r_0, t)] = k_e \frac{Q}{r_s^2(r_0, t)} \left(\frac{r_0}{R}\right)^3, \quad (\text{S-1.39})$$

from which Equation (1.16) can be derived. The forces on the particles, and thus their accelerations, increase with increasing  $r_0$ , in agreement with our “non-overtaking” result. Note that the electric field, and thus the force, at  $t = 0$  is proportional to  $r_0$ , not to  $r_0^3$ , because we have  $r_s^2(r_0, 0) = r_0^2$  at the denominator.

**c)** Each infinitesimal spherical shell expands from its initial radius  $r_0$  to its final radius  $r_s(r_0, \infty) = \infty$  under the action of the force (1.16). The final kinetic energy of a particle belonging to the shell,  $K_{\text{fin}}(r_0)$ , equals the work done by the force on the particle

$$K_{\text{fin}}(r_0) = \int_{r_0}^{+\infty} k_e \frac{qQ}{r_i^2} \left(\frac{r_0}{R}\right)^3 dr = k_e \frac{qQ}{r_0} \left(\frac{r_0}{R}\right)^3 = k_e q Q \frac{r_0^2}{R^3}. \quad (\text{S-1.40})$$

Quantity  $K_{\text{fin}}(r_0)$  is a monotonically increasing function of  $r_0$ , thus its maximum value  $K_{\text{max}}$  is observed for  $r_0 = R$

$$K_{\text{max}} = K_{\text{fin}}(R) = k_e \frac{qQ}{R}. \quad (\text{S-1.41})$$

This means that the particles initially located at  $r_0 = R$ , i.e., at the cloud surface, acquire the maximum final kinetic energy.

**d)** The energy distribution, or energy spectrum, function  $f(K)$  is defined so that the number  $dN$  of particles with kinetic energy in the interval  $(K, K + dK)$  equals  $f(K) dK$ , therefore  $f(K) = dN/dK$ . A particle belonging to the shell  $r_0 < r_s < r_0 + dr_0$  at  $t = 0$  has a kinetic energy in the interval  $(K_{\text{fin}}(r_0), K_{\text{fin}}(r_0) + dK_{\text{fin}})$  at  $t = \infty$ , where

$$dK_{\text{fin}} = \frac{2k_e q Q r_0}{R^3} dr_0. \quad (\text{S-1.42})$$

On the other hand, at  $t = 0$  the number of particles in the shell  $(r_0, r_0 + dr_0)$  is

$$dN = N \frac{3}{4\pi R^3} 4\pi r_0^2 dr_0 = N \frac{3r_0^2}{R^3} dr_0, \quad (\text{S-1.43})$$

and the number of particles in a given shell is constant during the motion, since the particles do not overtake one another. Thus, inserting (S-1.40) and (S-1.41), we obtain

$$f(K_{\text{fin}}) = \frac{dN}{dK_{\text{fin}}} = N \frac{3r_0^2}{R^3} \frac{R^3}{2k_e q Q r_0} = N \frac{3r_0}{2k_e q Q} = \frac{3N}{2K_{\text{max}}^{3/2}} \sqrt{K_{\text{fin}}}, \quad (\text{S-1.44})$$

valid for  $K_{\text{fin}} \leq K_{\text{max}}$ .

The final total kinetic energy is

$$\begin{aligned} K_{\text{tot}} &= \int_0^{K_{\text{max}}} K f(K) dK = \frac{3N}{2K_{\text{max}}^{3/2}} \int_0^{K_{\text{max}}} K^{3/2} dK = \frac{3N}{5} K_{\text{max}} \\ &= \frac{3k_e N q Q}{5} = \frac{3k_e}{5} \frac{Q^2}{R}, \end{aligned} \quad (\text{S-1.45})$$

which equals the total electrostatic energy stored in the charged sphere at  $t = 0$  (Problem 1.3). Here we have substituted  $Nq = Q$ . Thus, all the electrostatic energy stored in the initial configuration is eventually converted into kinetic energy.

It is a relatively common error to assume that the final kinetic energy of a particle initially in the shell  $r_0 < r_s < r_0 + dr$  is equal to the potential energy of the same particle at  $t = 0$ , i.e., that  $K_{\text{fin}} = q\varphi(r_0)$ , where  $\varphi$  is given by (S-1.37). This is obviously wrong, because a particle initially at  $r_0 = 0$  has the highest possible initial potential energy,  $\varphi(0) = 3k_e Q/(2R)$ , while it undergoes the lowest possible gain in kinetic energy (zero)! Moreover, this behavior would not preserve the total energy of the system, because the initial potential energy of the sphere is (see Problem 1.3)

$$U(0) = \frac{1}{2} \sum_i q\varphi[r_i(0)], \quad \text{not} \quad U(0) = \sum_i q\varphi[r_i(0)].$$

The point is that while the field is electrostatic ( $\nabla \times \mathbf{E} = 0$ ) at any time, it is *time dependent*. Thus,  $\varphi$  can be defined for any value of  $t$ , but it *cannot* be used to evaluate the final kinetic energy, because  $\varphi$  changes as the particles move.

The gain in kinetic energy equals the initial potential energy  $q\varphi(R, t = 0)$  for the particles initially at  $r_i = R$ , i.e., for the most external ones. Only these particles are accelerated by a field that can be treated as static, being simply equal to the field of a point charge  $Q$  located at  $r = 0$  at any time.

e) If we introduce the new variable  $x(t) = r_s(r_0, t)/r_0$ , (1.16) becomes

$$m \frac{d^2x}{dt^2} = k_e \frac{qQ}{R^3 x^2}, \quad (\text{S-1.46})$$

which is independent of  $r_0$ . The solution of (S-1.46),  $x = x(t)$ , with the initial condition  $x(0) = 1$ , is thus valid for all the particles of the cloud. Thus, if two shells, labeled 1 and 2, have initial radii  $r_{10}$  and  $r_{20}$ , with  $r_{20} > r_{10}$ , their subsequent radii will be  $r_1(t) = r_s(r_{10}, t) = r_{10}x(t)$  and  $r_2(t) = r_s(r_{20}, t) = r_{20}x(t)$ . It will always be  $r_2(t) > r_1(t)$ , and the internal shell cannot overtake the external one. The number of particles contained between the layers 1 and 2 is constant and equal to

$$\delta N_{12} = \frac{N}{R^3} (r_{20}^3 - r_{10}^3). \quad (\text{S-1.47})$$

Thus the particle density between the two layers at time  $t$  is

$$n(t) = \frac{3}{4\pi [r_2^3(t) - r_1^3(t)]} \delta N_{12} = \frac{3N}{4\pi R^3} \frac{r_{20}^3 - r_{10}^3}{(r_{20}^3 - r_{10}^3)x^3(t)} = \frac{3N}{4\pi R^3} x^{-3}(t). \quad (\text{S-1.48})$$

This result does not depend on the particular choice of the two layers, and the particle density is uniform at any time  $t$ , and decreases with increasing time as  $x^{-3}(t)$ .

## S-1.7 Plane and Cylindrical Coulomb Explosions

- a) The electric field is parallel to the  $x$  axis and independent of the  $y$  and  $z$  coordinates for symmetry reasons, thus we have  $\mathbf{E}(x, y, z) = E(x)\hat{x}$ . Again for symmetry reasons, the electric field is antisymmetric with respect to the  $x = 0$  plane, so that  $E(-x) = -E(x)$ . Thus it is sufficient to consider the field for  $x \geq 0$ . The charge density at  $t = 0$  is  $\varrho(x) = qn_0\Theta(a/2 - |x|)$ , where  $\Theta(x)$  is the Heaviside step function, defined as  $\Theta(x) = 1$  for  $x > 0$ , and  $\Theta(x) = 0$  for  $x < 0$ . The electric field at  $t = 0$  can be evaluated by integrating the equation  $\nabla \cdot \mathbf{E} = \partial_x E_x = 4\pi k_e \varrho(x)$ , with the boundary condition  $\mathbf{E}(0) = 0$ , obtaining

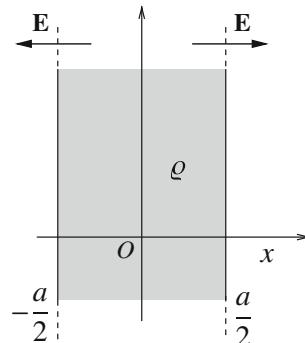


Fig. S-1.6

$$E(x) = 4\pi k_e \begin{cases} qn_0 x, & x < \frac{a}{2}, \\ qn_0 \frac{a}{2}, & x > \frac{a}{2}. \end{cases} \quad (\text{S-1.49})$$

Since the particles are at rest at  $t=0$ , and the electric field increases with increasing  $x$ , the particles cannot overtake one another. The motion of a particle initially at  $x_0$  is described by an equation  $x_s = x_s(x_0, t)$ . Let us consider a parallelepiped of base  $S$  lying on the  $x = 0$  plane and height  $x_s(x_0, t)$ . The charge inside our parallelepiped is constant in time since no particle can cross the moving base. We can apply Gauss's law to evaluate the electric field on the particle located at  $x_s(x_0, t)$

$$E[x_s(x_0, t), t] S = 4\pi k_e Q_{in}(t) = 4\pi k_e Q_{in}(0) = 4\pi k_e q n_0 x_0 S . \quad (\text{S-1.50})$$

Thus, the field accelerating each particle is constant in time, and equals

$$E(x_0) = 4\pi k_e q n_0 x_0 , \quad (\text{S-1.51})$$

where  $x_0$  is the particle position at  $t = 0$ . The equation of motion is thus

$$m \frac{d^2 x_s(x_0, t)}{dt^2} = q E(x_0) = 4\pi k_e q^2 n_0 x_0 , \quad (\text{S-1.52})$$

with the initial conditions  $x_s(x_0, 0) = x_0$  and  $\dot{x}_s(x_0, 0) = 0$ . The solution is

$$x_s(x_0, t) = x_0 + 2\pi k_e \frac{q^2 n_0 x_0}{m} t^2 = x_0 \left( 1 + \frac{\omega_p^2 t^2}{2} \right) , \quad (\text{S-1.53})$$

where  $\omega_p = \sqrt{4\pi k_e q^2 n_0 / m}$  is the “plasma frequency” of the infinite charged layer at  $t = 0$  (see Problem 1.4). Thus, the acceleration of an infinitesimal plane layer of thickness  $dx$  is proportional to its initial  $x$  coordinate, and more external layers are faster than more internal ones. The velocity, and the kinetic energy, of each layer grow indefinitely with time, which is not surprising since the system has an infinite initial potential energy (Fig. S-1.6).

If we introduce the dimensionless variable  $\xi = x_s/x_0$ , its equation of motion

$$m \frac{d^2 \xi}{dt^2} = q E(x_0) = 4\pi k_e q^2 n_0 , \quad \xi(0) = 1 , \quad \frac{d\xi(0)}{dt} = 0 , \quad (\text{S-1.54})$$

is independent of  $x_0$ . Thus the position of any particle can be written in the form

$$x_s(x_0, t) = x_0 \xi(t) , \quad (\text{S-1.55})$$

which shows that the particle density, and the charge density, remain uniform during the explosion.

**b)** The case of the Coulomb explosion of a system of charged particles initially confined, at rest, inside an infinite cylinder of radius  $a$ , is similar. We use cylindrical coordinates  $(r, \phi, z)$ , and assume that the initial particle density is uniform and equal to  $n_0$  for  $r < a$ , and zero for  $r > a$ . All particles have mass  $m$  and charge  $q$ . According to Gauss's law, at  $t = 0$  the field at position  $(r_0, \phi, z)$ , with  $r_0 < a$ , is

$$E(r_0) = 2\pi k_e n_0 q r_0 . \quad (\text{S-1.56})$$

Again, the particles cannot overtake one another because the electric field increases with increasing  $r_0$ . A particle initially at  $r_0$  will move along the  $r$  coordinate according to a law  $r_s = r_s(r_0, t)$ , with  $r_s(r_0, 0) = r_0$ . The field acting on the particle at time  $t$  is

$$E(r_0, t) = \frac{2\pi k_e n_0 q r_0^2}{r_s(r_0, t)}, \quad (\text{S-1.57})$$

and its equation of motion is

$$m \frac{d^2 r_s(r_0, t)}{dt^2} = q \frac{2\pi k_e n_0 q r_0^2}{r_s(r_0, t)}. \quad (\text{S-1.58})$$

It is not possible to solve (S-1.58) for  $r_s(r_0, t)$  in a simple way, however, we can multiply both sides by  $dr_s(r_0, t)/dt$ , obtaining

$$m \frac{dr_s(r_0, t)}{dt} \frac{d^2 r_s(r_0, t)}{dt^2} = 2\pi k_e n_0 q^2 r_0^2 \frac{1}{r_s(r_0, t)} \frac{dr_s(r_0, t)}{dt}. \quad (\text{S-1.59})$$

Equation (S-1.59) can be rewritten

$$\frac{m}{2} \frac{d}{dt} \left[ \frac{dr_s(r_0, t)}{dt} \right]^2 = 2\pi k_e n_0 q^2 r_0^2 \frac{d}{dt} \ln[r_s(r_0, t)], \quad (\text{S-1.60})$$

which can be integrated with respect to time, leading to

$$\begin{aligned} \frac{m}{2} \left[ \frac{dr_s(r_0, t)}{dt} \right]^2 &= 2\pi k_e n_0 q^2 r_0^2 \ln[r_s(r_0, t)] + C \\ &= 2\pi k_e n_0 q^2 r_0^2 \ln\left(\frac{r_s(r_0, t)}{r_0}\right), \end{aligned} \quad (\text{S-1.61})$$

where the integration constant  $C$  has been determined by the condition that the kinetic energy of the particle must be zero at  $t = 0$ , when  $r_s(r_0, t) = r_0$ . The first side of (S-1.61) is the kinetic energy  $K(r_s)$  at time  $t$ , when the particle is located at  $r_s(r_0, t)$ , which we can simply denote by  $r_s$ . Thus we have the following, seemingly time-independent equation for the kinetic energy of a particle initially located at  $r_0$

$$K(r_s) = 2\pi k_e n_0 q^2 r_0^2 \ln\left(\frac{r_s}{r_0}\right). \quad (\text{S-1.62})$$

At the limit  $r_s \rightarrow \infty, t \rightarrow \infty$ , the integral diverges logarithmically. Again, this is due to the infinite potential energy initially stored in the system.

## S-1.8 Collision of two Charged Spheres

- a)** The electrostatic energy of a uniformly charged sphere of radius  $R$  and total charge  $Q$  is, according to the result of Problem 1.3,

$$U_0 = \frac{3}{5} k_e \frac{Q^2}{R}, \quad (\text{S-1.63})$$

so that the initial energy of our system of two spheres is

$$U_{\text{tot}} = 2U_0 = \frac{6}{5} k_e \frac{Q^2}{R}. \quad (\text{S-1.64})$$

- b)** Let us denote by  $x$  the distance between the centers of the two spheres. When  $x$  is such that the interaction energy  $U_{\text{int}}(x)$  is no longer negligible with respect to  $U_0$ , but still larger than  $2R$ , the total potential energy  $U_{\text{pot}}(x)$  of the system is

$$U_{\text{pot}}(x) = 2U_0 + U_{\text{int}}(x). \quad (\text{S-1.65})$$

As long as  $x \geq 2R$  the force between the spheres is identical to the force between two point charges  $\pm Q$  located at the centers of the spheres, and

$$U_{\text{int}}(x) = -k_e \frac{Q^2}{x}. \quad (\text{S-1.66})$$

Both the total momentum and the total energy of the system are conserved. Thus, the velocities of the two spheres are always equal and opposite. As long as  $x \geq 2R$  the total energy of the system  $U_{\text{tot}} = 2U_0$  equals the sum of the potential and kinetic energies of the system

$$U_{\text{tot}} = 2 \frac{1}{2} M v^2(x) + 2U_0 + U_{\text{int}}(x), \quad (\text{S-1.67})$$

where  $M$  is the mass of each sphere, and  $\pm v(x)$  are the velocities of the two spheres. Thus

$$\frac{6}{5} k_e \frac{Q^2}{R} = M v^2(x) + \frac{6}{5} k_e \frac{Q^2}{R} - k_e \frac{Q^2}{x}, \quad \text{and} \quad v(x) = \sqrt{k_e \frac{Q^2}{Mx}}. \quad (\text{S-1.68})$$

When  $x = 2R$ , the velocity is

$$v(2R) = \sqrt{k_e \frac{Q^2}{2MR}}. \quad (\text{S-1.69})$$

- c)** When the two spheres overlap completely, the charge density and the electrostatic field are zero over the whole space, so that also the electrostatic energy is zero. This means that all the initial energy has been converted into kinetic energy, i.e.,

$$2 \frac{1}{2} M v^2(0) = 2 U_0, \quad (\text{S-1.70})$$

from which we obtain

$$v(0) = \sqrt{\frac{6}{5} k_e \frac{Q^2}{MR}}. \quad (\text{S-1.71})$$

## S-1.9 Oscillations in a Positively Charged Conducting Sphere

- a)** At equilibrium, the remaining  $(1 - f)N$  conduction electrons must be subject to zero electric field. For symmetry reasons, this is possible only if they occupy a spherical volume of radius  $b < a$  concentric with the conducting sphere, where the where  $e$  is the elementary electric charge, and  $n_e$  and  $n_i$  are the conduction-electron density and the ion density, respectively. Thus, we must have  $n_e = n_i$ , with total charge density  $\varrho$  is zero. We thus have

$$\varrho(r) = e(n_i - n_e) = 0 \quad \text{for } r < b, \quad (\text{S-1.72})$$

where

$$n_i = \frac{3N}{4\pi a^3} \quad \text{and} \quad n_e = \frac{3(1-f)N}{4\pi b^3}, \quad (\text{S-1.73})$$

and we get

$$b = a \sqrt[3]{1-f}, \quad (\text{S-1.74})$$

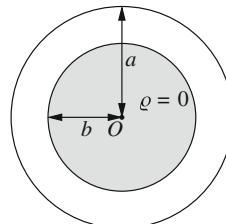


Fig. S-1.7

and

$$\varrho(r) = \frac{3Ne}{4\pi a^3} \quad \text{for } b < r < a. \quad (\text{S-1.75})$$

Note that the electric field is nonzero in the spherical shell  $b < r < a$ . However, this region is not conducting, since the conduction electrons are confined in the inner region  $r < b$  (Fig. S-1.7).

- b)** Now the conduction electron sphere is rigidly displaced by an amount  $\delta$  relative to the metal sphere centered in  $O$ , so that its center is in  $O'$ , as in Fig. S-1.8. The electric field

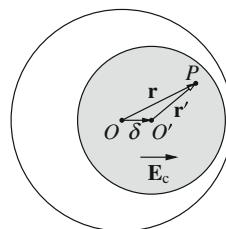


Fig. S-1.8

in any point of space can be evaluated by superposition, adding the field generated by the ion lattice, of charge density  $\varrho_i = en_i$  and the field generated by the conduction electrons of charge density  $\varrho_e = -en_e$ . The electric field  $\mathbf{E}_c$  in a point  $P$  inside the conduction-electron sphere, of vector position  $\mathbf{r}$  relative to  $O$ , and  $\mathbf{r}'$  relative to  $O'$ , is

$$\mathbf{E}_c = \frac{4\pi k_e}{3} (\varrho_i \mathbf{r} - \varrho_e \mathbf{r}') = \frac{4\pi k_e}{3} \frac{3Ne}{4\pi a^3} (\mathbf{r} - \mathbf{r}') = \frac{k_e Ne^2}{a^3} \delta, \quad (\text{S-1.76})$$

spatially uniform and parallel to  $\delta$ .

c) Each conduction electron is subject to the force

$$\mathbf{F}_c = -e \mathbf{E}_c = -\frac{k_e Ne^2}{a^3} \delta, \quad (\text{S-1.77})$$

proportional to the displacement from its equilibrium position. The equation of motion for each electron is thus

$$m_e \frac{d^2 \delta}{dt^2} = \mathbf{F}_c = -\frac{k_e Ne^2}{a^3} \delta = -m_e \omega_M^2 \delta, \quad (\text{S-1.78})$$

where  $m_e$  is the electron mass, and  $\omega_M$  the oscillation frequency for the resulting harmonic motion. The oscillation frequency is thus

$$\omega_M = \sqrt{\frac{k_e Ne^2}{m_e a^3}}, \quad (\text{S-1.79})$$

i.e., the Mie frequency of (S-1.26).

## S-1.10 Interaction between a Point Charge and an Electric Dipole

The potential energy of an electric dipole  $\mathbf{p}$  in the presence of an external electric field  $\mathbf{E}$  is

$$U = -\mathbf{p} \cdot \mathbf{E} = -pE \cos \theta, \quad (\text{S-1.80})$$

and, in our case, we have

$$\mathbf{E} = k_e \frac{q}{r^2} \hat{\mathbf{r}}, \quad \text{and} \quad U = -k_e \frac{qp \cos \theta}{r^2}. \quad (\text{S-1.81})$$

a) The force acting on the dipole is thus

$$\mathbf{F} = -\nabla U = -2k_e \frac{qp \cos \theta}{r^3} \hat{\mathbf{r}}. \quad (\text{S-1.82})$$

Assuming that the point charge  $q$  is positive, the force is attractive if  $\cos\theta > 0$ , repulsive if  $\cos\theta < 0$ .

**b)** The torque acting on the dipole is

$$\tau = -\partial_\theta U \hat{\mathbf{z}} = -k_e \frac{qp \sin\theta}{r^2} \hat{\mathbf{z}}, \quad (\text{S-1.83})$$

where  $\hat{\mathbf{z}}$  is the unit vector perpendicular to the plane determined by the vectors  $\mathbf{r}$  and  $\mathbf{p}$ , pointing out of paper in Fig. 1.5. If  $q > 0$ , the torque tends to align  $\mathbf{p}$  to  $\mathbf{r}$ .

Alternatively, we can think of the dipole  $\mathbf{p}$  as the limit approached by a system of two opposite charges  $q'$  and  $-q'$ , at a distance  $2h$  from each other (as in Fig. S-1.9), as  $h \rightarrow 0$  and  $q' \rightarrow \infty$ , the product  $2hq' = p$  being constant. We assume that the point charges  $q$  and  $q'$  are positive. The distances  $r_1$  of  $q'$  from  $q$ , and  $r_2$  of  $-q'$  from  $q$ , can be written, as functions of  $r$ ,  $h$ , and  $\theta$ ,

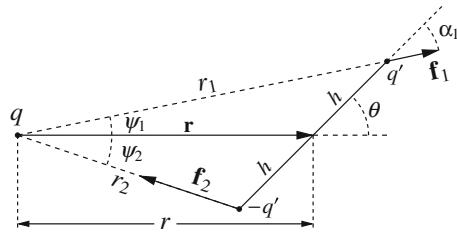


Fig. S-1.9

$$r_1 = \sqrt{r^2 + h^2 + 2rh \cos\theta}, \quad r_2 = \sqrt{r^2 + h^2 - 2rh \cos\theta}, \quad (\text{S-1.84})$$

where we have used the law of cosines. The force  $\mathbf{f}_1$  acting on the positive charge  $q'$  of the dipole is

$$\mathbf{f}_1 = k_e \frac{qq'}{r_1^2} \hat{\mathbf{r}}_1 = k_e \frac{qq'}{r^2 + h^2 + 2rh \cos\theta} \hat{\mathbf{r}}_1, \quad (\text{S-1.85})$$

and the force  $\mathbf{f}_2$  acting on  $-q'$  is

$$\mathbf{f}_2 = -k_e \frac{qq'}{r_1^2} \hat{\mathbf{r}}_2 = -k_e \frac{qq'}{r^2 + h^2 - 2rh \cos\theta} \hat{\mathbf{r}}_2, \quad (\text{S-1.86})$$

the minus sign meaning that the force is attractive for  $q > 0$ . Both angles  $\psi_1$  and  $\psi_2$  approach zero as  $h \rightarrow 0$ , and therefore

$$\lim_{h \rightarrow 0} \hat{\mathbf{r}}_1 = \hat{\mathbf{r}}, \quad \text{and} \quad \lim_{h \rightarrow 0} \hat{\mathbf{r}}_2 = \hat{\mathbf{r}}. \quad (\text{S-1.87})$$

Thus, the total force  $\mathbf{F}$  acting on the dipole can be written

$$\begin{aligned}
 \mathbf{F} &= \lim_{\substack{h \rightarrow 0 \\ q' \rightarrow \infty \\ 2hq' = p}} (\mathbf{f}_1 + \mathbf{f}_2) = \lim_{\substack{h \rightarrow 0 \\ q' \rightarrow \infty \\ 2hq' = p}} \hat{\mathbf{r}} k_e qq' \left( \frac{1}{r^2 + h^2 + 2rh \cos \theta} - \frac{1}{r^2 + h^2 - 2rh \cos \theta} \right) \\
 &= \lim_{\substack{h \rightarrow 0 \\ q' \rightarrow \infty \\ 2hq' = p}} \hat{\mathbf{r}} k_e qq' \frac{-4rh \cos \theta}{(r^2 + h^2 + 2rh \cos \theta)(r^2 + h^2 - 2rh \cos \theta)} \\
 &= -2k_e \frac{qp \cos \theta}{r^3} \hat{\mathbf{r}}, \tag{S-1.88}
 \end{aligned}$$

in agreement with (S-1.82).

As  $h \rightarrow 0$  and  $\psi_1 \rightarrow 0$ , the angle  $\alpha_1$  approaches  $\theta$ , and the limit of the torque of  $\mathbf{f}_1$  on the dipole is

$$\tau_1 = -\hat{\mathbf{z}} \lim_{\substack{h \rightarrow 0 \\ q' \rightarrow \infty \\ 2hq' = p}} h f_1 \sin \alpha_1 = -\hat{\mathbf{z}} \lim_{\substack{h \rightarrow 0 \\ q' \rightarrow \infty \\ 2hq' = p}} k_e \frac{qq' h \sin \alpha_1}{r^2 + h^2 + 2rh \cos \theta} = -k_e \frac{qp \sin \theta}{2r^2} \hat{\mathbf{z}}, \tag{S-1.89}$$

analogously, the limit of the torque of  $\mathbf{f}_2$  is

$$\tau_2 = -k_e \frac{qp \sin \theta}{2r^2} \hat{\mathbf{z}}, \tag{S-1.90}$$

and the total torque on the dipole is

$$\tau = \tau_1 + \tau_2 = -k_e \frac{qp \sin \theta}{r^2} \hat{\mathbf{z}}, \tag{S-1.91}$$

in agreement with (S-1.83).

## S-1.11 Electric Field of a Charged Hemispherical surface

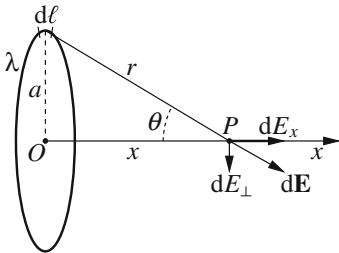


Fig. S-1.10

We start from the electric field generated by a ring of radius  $a$  and linear charge density  $\lambda$  in a generic point  $P$  on its axis, at a distance  $x$  from the center  $O$  of the ring. With reference to Fig. S-1.10, the infinitesimal ring arc  $d\ell$ , of charge  $\lambda d\ell$ , generates a field  $d\mathbf{E}$  in  $P$ . The magnitude of  $d\mathbf{E}$  is

$$dE = k_e \frac{\lambda d\ell}{a^2 + x^2}. \tag{S-1.92}$$

The field  $d\mathbf{E}$  has a component  $dE_x$  parallel to the ring axis, and a component  $dE_\perp$  perpendicular to the axis. We need only the parallel component

$$\begin{aligned} dE_x &= \cos\theta dE = k_e \frac{\lambda d\ell}{a^2 + x^2} \frac{x}{\sqrt{a^2 + x^2}} \\ &= k_e \frac{\lambda x d\ell}{(a^2 + x^2)^{3/2}}, \end{aligned} \quad (\text{S-1.93})$$

because the perpendicular component cancels out because of symmetry when we integrate over the whole ring. When we integrate,  $\theta$  and  $r$  do not depend on the position of  $d\ell$ , and the total field in  $P$  is

$$\begin{aligned} E_x &= k_e \frac{\lambda x}{(a^2 + x^2)^{3/2}} \int_0^{2\pi a} d\ell = k_e \frac{2\pi a \lambda x}{(a^2 + x^2)^{3/2}} \\ &= \begin{cases} \frac{1}{4\pi\epsilon_0} \frac{2\pi a \lambda x}{(a^2 + x^2)^{3/2}} & \text{SI} \\ \frac{2\pi a \lambda x}{(a^2 + x^2)^{3/2}} & \text{Gaussian,} \end{cases} \end{aligned} \quad (\text{S-1.94})$$

which can be rewritten

$$\mathbf{E} = \hat{\mathbf{x}} k_e \frac{Q x}{(a^2 + x^2)^{3/2}}, \quad (\text{S-1.95})$$

where  $Q = 2\pi a \lambda$  is the total charge of the ring.

The charged hemispherical surface can be divided into infinitesimal strips between “parallels” of colatitude  $\theta$  and  $\theta + d\theta$  with respect to the symmetry axis of the hemisphere, as in Fig. S-1.11. Each infinitesimal strip is equivalent to a charged ring of radius  $R \sin\theta$  and total charge  $dQ = \sigma 2\pi R^2 \sin\theta d\theta$ . The curvature center of the hemisphere is located on the axis of the rings, at a distance  $x = R \cos\theta$  from the center of each ring. Thus, the contribution of each strip to the field at the center is

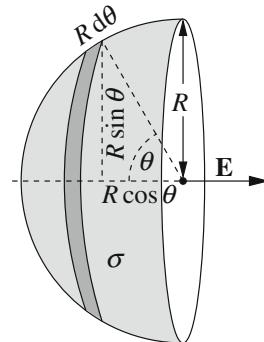


Fig. S-1.11

$$\begin{aligned} dE &= k_e \frac{\sigma 2\pi R^2 \sin\theta d\theta R \cos\theta}{(R^2 \sin^2\theta + R^2 \cos^2\theta)^{3/2}} \\ &= k_e \frac{\sigma 2\pi R^3 \cos\theta \sin\theta d\theta}{R^3} = k_e \sigma 2\pi \cos\theta \sin\theta d\theta, \end{aligned} \quad (\text{S-1.96})$$

and the total field is

$$E = k_e \sigma 2\pi \int_0^{\pi/2} \cos \theta \sin \theta d\theta = k_e \pi \sigma, \quad (\text{S-1.97})$$

independent of the radius  $R$ .

# Chapter S-2

## Solutions for Chapter 2

### S-2.1 Metal Sphere in an External Field

a) The total electric field inside a conductor must be zero in static conditions. Thus, in the presence of an external field  $\mathbf{E}_0$ , the surface charge distribution of our sphere must generate a field  $\mathbf{E}_{\text{in}} = -\mathbf{E}_0$  at its inside. As we found in Problem 1.1, a rigid displacement  $-\delta$  of the electron sphere (or “electron sea”) with respect to the ion lattice gives origin to the internal uniform field (S-1.1)

$$\mathbf{E}_{\text{in}} = -k_e \frac{4\pi}{3} \varrho_0 \delta, \quad (\text{S-2.1})$$

where  $\varrho_0 = en_e$  is the charge density of the “electron sphere”. The magnitude of the displacement  $\delta$  is thus

$$\delta = \frac{3E_0}{4\pi k_e \varrho_0}. \quad (\text{S-2.2})$$

For a rough numerical estimate for  $n_e$ , we can assume that each atom contributes a single conduction electron ( $Z = 1$ ). If  $M$  is the atomic mass of our atoms,  $M$  grams of metal contain  $N_A \simeq 6.0 \times 10^{23}$  atoms (Avogadro constant), and occupy a volume of  $M/\varrho_m$  cm<sup>3</sup>, where  $\varrho_m$  is the mass density. Typical values for a metal are  $M \sim 60$  and  $\varrho_m \sim 8$  g/cm<sup>3</sup>, leading to

$$n_e \sim \frac{N_A \varrho_m}{M} \sim 10^{22} \text{ cm}^{-3}, \quad \text{and} \quad \varrho_0 = en_e \sim 5 \times 10^{12} \text{ statC/cm}^3. \quad (\text{S-2.3})$$

In SI units we have  $n_e \sim 10^{29}$  m<sup>-3</sup> and  $\varrho_0 \sim 1.6 \times 10^{-10}$  C/m<sup>3</sup>. Substituting into (S-2.2) and assuming  $E_0 = 1000$  V/m (0.003 statV/cm), we finally obtain

$$\delta \sim 10^{-15} \text{ cm}. \quad (\text{S-2.4})$$

This value for  $\delta$  is smaller by orders of magnitude than the spacing between the atoms in a crystalline lattice ( $\sim 10^{-8}$  cm), therefore it makes sense to consider the

charge as distributed on the surface. Formally, this is equivalent to take the limits  $\delta \rightarrow 0$  and  $\varrho_0 \rightarrow \infty$ , keeping constant the product

$$\sigma_0 = \varrho_0 \delta = \frac{3E_0}{4\pi k_e} . \quad (\text{S-2.5})$$

**b)** According to Problem 1.1, the field generated by the charge distribution of the metal sphere outside its volume equals the field of an electric dipole  $\mathbf{p} = Q\delta$ , where  $Q = (4\pi/3)R^3en_e$ , located at the center of the sphere. Replacing  $\delta$  by its value of (S-2.2) we have for the dipole moment

$$\mathbf{p} = \frac{R^3}{k_e} \mathbf{E}_0 . \quad (\text{S-2.6})$$

The field outside the sphere ( $r > R$ ) is the sum of  $\mathbf{E}_0$  and the field generated by  $\mathbf{p}$

$$\mathbf{E} = \mathbf{E}_0 + [3(\mathbf{E}_0 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{E}_0] \left( \frac{R}{r} \right)^3 . \quad (\text{S-2.7})$$

**c)** The external field at the surface of the sphere is obtained by replacing  $r$  by  $R$  in (S-2.7)

$$\mathbf{E}_{\text{surf}} = \mathbf{E}_0 + 3(\mathbf{E}_0 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{E}_0 = 3(\mathbf{E}_0 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} , \quad (\text{S-2.8})$$

which is perpendicular to the surface, as expected. The surface charge density is

$$\sigma = \frac{1}{4\pi k_e} \mathbf{E}_{\text{surf}} \cdot \hat{\mathbf{r}} = k_e \frac{3}{4\pi} E_0 \cos \theta = \sigma_0 \cos \theta , \quad (\text{S-2.9})$$

where  $\sigma_0 = 3k_e E_0 / (4\pi)$ , and  $\theta$  is the angle between  $\hat{\mathbf{r}}$  and  $\mathbf{E}_0$ .

## S-2.2 Electrostatic Energy with Image Charges

In all cases, the conducting (half-)planes divide the whole space into two regions: one free of charges (A), and one containing electrical charges (B), as shown in Fig. S-2.1. Since the charge distribution is finite, the electric potential  $\varphi$  equals

zero at the boundaries of both regions, i.e., on the conducting surfaces and at infinity. We can thus use the uniqueness theorem for Poisson's equation. The potential  $\varphi$  (and therefore the electric field  $\mathbf{E}$ ) is uniformly equal to zero in region A. The potential problem in region B is solved if we find an *image* charge distribution, located in region A, that replicates the boundary conditions of region B. The potential and the electric field (and thus the forces on the real charges) in region B are the same as if the image charges were real.

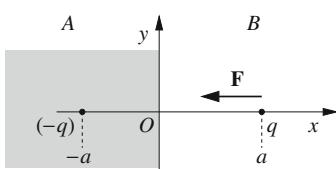


Fig. S-2.1

a) We introduce a reference frame with the  $x$  axis perpendicular to the conducting plane, and passing through the charge. The origin, and the  $y$  and  $z$  axes, lie on the plane. The charge is thus located at  $(x = a, y = 0, z = 0)$ , and the potential problem for  $x > 0$  is solved by placing an image charge  $q' = -q$  at  $(x = -a, y = 0, z = 0)$ . The force on the real charge is  $F = -k_e q^2/(4a^2)$ . The electrostatic energy  $U_{\text{es}}$  of the system equals the work  $W$  done by the field when the real charge  $q$  moves from  $x = a$  to  $x = +\infty$ . Simultaneously, the image charge will move from  $x = -a$  to  $x = -\infty$ , but no additional work is needed for this, since what actually moves is the surface charge on the conducting plane, which is constantly at zero potential. Thus we have

$$U_{\text{es}} = W = \int_a^\infty F dx = -k_e \frac{q^2}{4} \int_a^\infty \frac{dx}{x^2} = -k_e \frac{q^2}{4a}. \quad (\text{S-2.10})$$

This is half the electrostatic energy  $U_{\text{real}}$  of a system comprising two *real* charges,  $q$  and  $-q$ , at a distance  $2a$  from each other. The  $1/2$  factor is due to the fact that, if two *real* charges move to infinity in opposite directions, the work done by the field is

$$W_{\text{real}} = \int_{+a}^{+\infty} F dx - \int_{-a}^{-\infty} (-F) dx = 2 \int_{+a}^{+\infty} F_x dx = -k_e \frac{q^2}{2a}, \quad (\text{S-2.11})$$

since the force acting on  $-q$  is the opposite of the force acting on  $q$ , and  $U_{\text{real}} = W_{\text{real}}$ .

The  $1/2$  factor can also be explained by evaluating the electrostatic energies for our system, and for the system of the two real charges. In both cases, because of the cylindrical symmetry around the  $x$  axis, the electrostatic field is a function of the longitudinal coordinate  $x$  and of the radial distance  $r = \sqrt{y^2 + z^2}$  only, i.e.,  $\mathbf{E} = \mathbf{E}(x, r)$ . In the case of the two real charges we have

$$\begin{aligned} U_{\text{real}} &= \frac{1}{8\pi k_e} \int d^3 r \mathbf{E}^2 = \frac{1}{8\pi k_e} \int_{-\infty}^{\infty} dx \int_0^{\infty} 2\pi r dr \mathbf{E}^2(x, r) \\ &= 2 \frac{1}{8\pi k_e} \int_0^{\infty} dx \int_0^{\infty} 2\pi r dr \mathbf{E}^2(x, r), \end{aligned} \quad (\text{S-2.12})$$

since  $\mathbf{E}(x, r) = -\mathbf{E}(-x, r)$ , so that  $\mathbf{E}^2(x, r) = \mathbf{E}^2(-x, r)$ . In the case of the charge in front of a conducting plane we have

$$U_{\text{es}} = \frac{1}{8\pi k_e} \int_0^{\infty} dx \int_0^{\infty} 2\pi r dr \mathbf{E}^2(x, r), \quad (\text{S-2.13})$$

because  $\mathbf{E} = 0$  for  $x < 0$  (in region A), while the field is the same as in the “real” case for  $x > 0$ . Thus  $U_{\text{es}} = U_{\text{real}}/2$ . The electrostatic energy includes both the interaction energy between the charges,  $U_{\text{int}}$ , and the “self-energy”,  $U_{\text{self}}$ , of each charge. For the “real” system we have

$$U_{\text{real}} = U_{\text{self}}(q) + U_{\text{self}}(-q) + U_{\text{int}}(q, -q) = 2U_{\text{self}}(q) + U_{\text{int}}(q, -q), \quad (\text{S-2.14})$$

since  $U_{\text{self}}(-q) = U_{\text{self}}(q)$ . For the charge in front of the conducting plane we have

$$U_{\text{es}} = U_{\text{self}}(q) + U_{\text{int}}(q, \text{plane}), \quad (\text{S-2.15})$$

since there is only one real charge. Actually, the self-energy  $U_{\text{self}}$  approaches infinity if we let the charge radius approach 0, but this issue is not really relevant here. In any case, the divergence may be treated by assuming an arbitrarily small, but non-zero radius for the charge. Since  $U_{\text{es}} = U_{\text{real}}/2$ , we also have  $U_{\text{int}}(q, \text{plane}) = U_{\text{int}}(q, -q)/2$ .

**b)** Again, we choose a reference frame with the  $x$  axis perpendicular to the conducting plane, so that  $q$  has coordinates  $(a, d/2, 0)$  and  $-q$  has coordinates  $(a, -d/2, 0)$ . The potential problem for the  $x > 0$  half-space is solved by placing an image charge  $-q$  at  $(-a, d/2, 0)$ , and an image charge  $q$  at  $(-a, -d/2, 0)$ . According to the arguments at the end of point **a)**, the electrostatic energy  $U_{\text{es}}$  of our system is one half of the energy  $U_{\text{real}}$  of a system of four charges, all of them real, at the same locations. We can evaluate  $U_{\text{real}}$  by inserting the four charges one by one, each interacting only with the previously inserted charges.

$$U_{\text{real}} = k_e \left( -2 \frac{q^2}{d} - 2 \frac{q^2}{2a} + 2 \frac{q^2}{\sqrt{d^2 + 4a^2}} \right). \quad (\text{S-2.16})$$

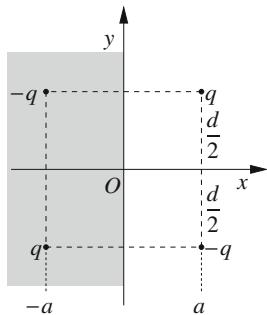


Fig. S-2.2

The same result is obtained by evaluating the work of the electric forces when the two real charges are moved to infinite distance from the plane, and infinite distance from each other. This can be done in two steps. First we move the charge at  $(a, d/2, 0)$ , then the charge at  $(a, -d/2, 0)$ . When we move the first charge, three forces are acting on it:  $\mathbf{F}_1$ , due to its own image, which is simultaneously moving to  $-\infty$ , and  $\mathbf{F}_2$  and  $\mathbf{F}_3$ , due to the second real charge and to its image, at distances  $r_2$  and  $r_3$ , respectively. The total work on the first charge is thus

$$\begin{aligned} W &= W_1 + W_2 + W_3 \\ &= \int_a^\infty \mathbf{F}_1 \cdot d\mathbf{r} + \int_a^\infty \mathbf{F}_2 \cdot d\mathbf{r} + \int_a^\infty \mathbf{F}_3 \cdot d\mathbf{r}, \end{aligned} \quad (\text{S-2.17})$$

where  $\mathbf{r}$  is the position vector of the first charge, and the first integral is the same as the integral of (S-2.10) and equals  $-k_e q^2/(4a)$ . The second integral can be rewritten, in terms of the angle  $\theta$  of Fig. S-2.3,

$$\begin{aligned}
 W_2 &= -k_e \int_a^\infty \frac{q^2}{r_2^2} \sin \theta dx \\
 &= -k_e q^2 \int_0^{\pi/2} \frac{\cos^2 \theta}{d^2} \sin \theta \frac{d}{\cos^2 \theta} d\theta \\
 &= -k_e \frac{q^2}{d} \int_0^{\pi/2} \sin \theta d\theta \\
 &= -k_e \frac{q^2}{d}, \tag{S-2.18}
 \end{aligned}$$

where we have used the facts that  $r_2 = d/\cos \theta$  and  $dx = (d/\cos^2 \theta) d\theta$ . The third integral of (S-2.17) can be treated analogously, in terms of the angle  $\psi$  of Fig. S-2.3,

$$\begin{aligned}
 W_3 &= k_e \int_a^\infty \frac{q^2}{r_3^2} \sin \psi dx \\
 &= k_e \frac{q^2}{d} \int_{\psi_0}^{\pi/2} \sin \psi d\psi \\
 &= k_e \frac{q^2}{\sqrt{4a^2 + d^2}}, \tag{S-2.19}
 \end{aligned}$$

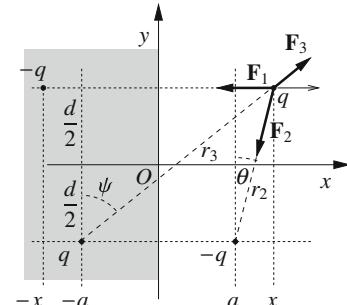


Fig. S-2.3

where  $\psi_0$  is the value of  $\psi$  when  $q$  is at  $x = a$ , i.e.,  $\psi_0 = \arccos(d/\sqrt{4a^2 + d^2})$ . Thus, the work done by the electric field when the first charge is moved to infinity is

$$W = k_e \left( -\frac{q^2}{4a} - \frac{q^2}{d} + \frac{q^2}{\sqrt{4a^2 + d^2}} \right). \tag{S-2.20}$$

We must still move the second real charge to infinity, this is done in the presence of its own image charge only, and the work is  $-k_e q^2/(4a)$ . We finally have

$$U_{\text{es}} = W - k_e \frac{q^2}{4a} = k_e \left( -\frac{q^2}{2a} - \frac{q^2}{d} + \frac{q^2}{\sqrt{4a^2 + d^2}} \right), \tag{S-2.21}$$

i.e., one half of the value of  $U_{\text{real}}$  of (S-2.16), as expected.

c) We choose a reference frame with the half planes ( $x = 0, y \geq 0$ ) and ( $y = 0, x \geq 0$ ) coinciding with the two conducting half-planes. Thus, the real charge  $q$  is located at  $(x = a, y = b, z = 0)$ . If we add two image charges  $q'_1 = q'_2 = -q$  at  $(-a, b, 0)$  and  $(a, -b, 0)$ , respectively, and an image charge  $q'_3 = q$  at  $(-a, -b, 0)$ , the potential is zero on the  $x = 0$  and  $y = 0$  planes, and at infinity. This

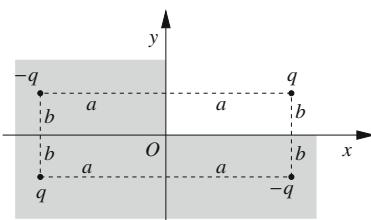


Fig. S-2.4

solves the potential problem in the dihedral angle where the real charge is located. Following the discussions of points **a)** and **b)**, the electrostatic energy of this system is one quarter of the energy of a system comprising four charges, all of them real, in the same locations, since the energy density is zero in three quarters of the whole space.

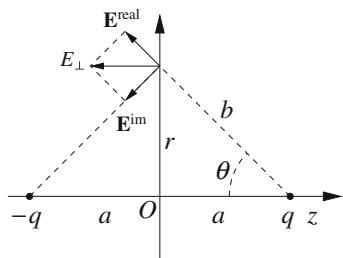
$$U_{\text{es}} = \frac{1}{4} k_e \left( \frac{-q^2}{a} - \frac{q^2}{b} + \frac{q^2}{\sqrt{b^2 + a^2}} \right). \quad (\text{S-2.22})$$

Alternatively, we can calculate the work done by the electric field when the real charge is moved from  $(a, b, 0)$  to  $(\infty, \infty, 0)$ .

### S-2.3 Fields Generated by Surface Charge Densities

**a)** We use cylindrical coordinates  $(r, \phi, z)$  with the origin  $O$  on the conducting plane, and the  $z$  axis perpendicular to the plane and passing through the real charge  $q$ . The real charge is located at  $(0, \phi, z)$ , and the image charge at  $(0, \phi, -a)$ ,  $\phi$  being irrelevant when  $r = 0$ . The electric field on the conducting plane is perpendicular to the plane,

and depends only on  $r$ . At a generic point  $P \equiv (r, \phi, 0)$  on the plane the magnitude of the field  $\mathbf{E}^{\text{real}}$  generated by the real charge is



$$E^{\text{real}} = k_e \frac{q}{b^2} = k_e \frac{q}{a^2 + r^2}. \quad (\text{S-2.23})$$

The field generated at  $P$  by the image charge,  $\mathbf{E}^{\text{im}}$ , has the same magnitude, the same  $z$  component, but opposite  $r$  component of  $\mathbf{E}^{\text{real}}$ , as in Fig. S-2.5. The total electric field in  $P$  is thus perpendicular to the plane and has magnitude

$$E(r) = 2E_z^{\text{real}}(r) = -2k_e \frac{q}{a^2 + r^2} \frac{a}{\sqrt{(a^2 + r^2)^3}} = -2k_e \frac{qa}{(a^2 + r^2)^{3/2}}. \quad (\text{S-2.24})$$

The surface charge density is thus

$$\sigma(r) = \frac{1}{4\pi k_e} E(r) = -\frac{1}{2\pi} \frac{qa}{(a^2 + r^2)^{3/2}}, \quad (\text{S-2.25})$$

and the annulus between  $r$  and  $r + dr$  on the conducting plain has a charge

$$\begin{aligned} dq_{\text{ind}} &= \sigma 2\pi r dr = \frac{qar dr}{(a^2 + r^2)^{3/2}} \\ &= -\frac{1}{2\pi} \frac{q}{a^2} \cos^3 \theta 2\pi a^2 \tan \theta \frac{d\theta}{\cos^2 \theta} = -q \sin \theta d\theta, \end{aligned} \quad (\text{S-2.26})$$

since  $r = a \tan \theta$ . The total induced charge on the conducting plane is

$$q_{\text{ind}} = \int_0^{\pi/2} dq_{\text{ind}} = -q \int_0^{\pi/2} \sin \theta d\theta = -q. \quad (\text{S-2.27})$$

- b)** In the problem of a real charge  $q$  located on the  $z$  axis, at  $z = a$ , in front of a conducting plane, the only real charges are  $q$  and the surface charge distribution  $\sigma$  on the plane. What we observe is no field in the half-space  $z < 0$ , while in the half-space  $z > 0$  we observe a field equivalent to the field of  $q$ , plus the field of an image charge  $-q$  located on the  $z$  axis at  $z = -a$ . The field generated by the surface charge distribution alone is thus equivalent to the field of a charge  $-q$  located at  $z = +a$  in the half-space  $z < 0$ , and to the field of a charge  $-q$  located at  $z = -a$  in the half-space  $z > 0$ . In the half space  $z < 0$ , the field of the surface charge distribution and the field of the real charge cancel each other. The discontinuity of the field at  $z = 0$  is due to the presence of a finite surface charge density on the conducting plane, which implies an infinite volume charge density.

- c)** Let us introduce a spherical coordinate system  $(r, \theta, \phi)$  into Problem 2.4, with the origin  $O$  at the center of the conducting sphere and the  $z$  axis on the line through  $O$  and the real charge  $q$ . The electric potential outside the sphere,  $r \geq a$ , is obtained from (S-2.31) by replacing  $a$  by  $r$ , and  $q'$  and  $d'$  by their values of (S-2.37). We have

$$\varphi(r, \theta) = k_e \left[ \frac{q}{\sqrt{r^2 + d^2 - 2dr \cos \theta}} - \frac{q \frac{a}{d}}{\sqrt{r^2 + \frac{a^4}{d^2} - 2r \frac{a^2}{d} \cos \theta}} \right], \quad (\text{S-2.28})$$

independent of  $\phi$ . The electric field at  $r = a^+$ , on the outer surface of the sphere, is

$$E_\perp(a^+, \theta) = -\partial_r V(r, \theta)|_{r=a}, \quad (\text{S-2.29})$$

and the surface charge density on the sphere is

$$\sigma(\theta) = \frac{1}{4\pi k_e} E_\perp(a^+, \theta). \quad (\text{S-2.30})$$

The actual evaluation does not pose particular difficulties, but is rather involved, and we neglect it here. But we can use the same arguments as in point b). The only real charges of the problem are the real charge  $q$ , and the surface

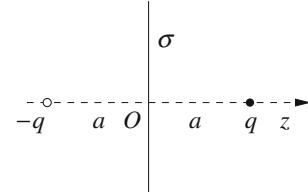


Fig. S-2.6

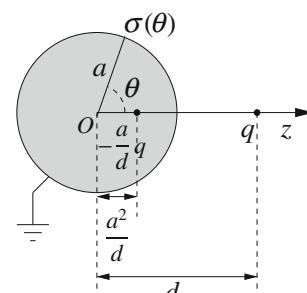


Fig. S-2.7

charge distribution of the sphere. There is no net field inside the sphere, and the field for  $r > 0$  is equivalent to the field of  $q$ , plus the field of an image charge  $-qa/d$  located at  $z = a^2/d$ . Thus, the surface charge distribution alone generates a field equivalent to a charge  $-q$  located at  $z = d$  inside the sphere, and a field equivalent to the field of the image charge  $-qa/d$ , located at  $z = a^2/d$ , outside the sphere.

## S-2.4 A Point Charge in Front of a Conducting Sphere

**a)** We have a conducting grounded sphere of radius  $a$ , and an electric charge  $q$  located at a distance  $d > a$  from its center  $O$ . Again, the whole space is divided into two regions: the inside ( $A$ ) and the outside ( $B$ ) of the sphere. The electrostatic potential is uniformly equal to zero in region  $A$  because the sphere is grounded. We try to solve the potential problem in region  $B$  by locating an image charge  $q'$  inside the sphere, on the line through  $O$  and  $q$ , at a distance  $d'$  from the center  $O$ . The problem is solved if we can find values for  $q'$  and  $d'$  such that the electric potential  $\varphi$  is zero everywhere on the surface of the sphere. This would replicate the boundary conditions for region  $B$ , with  $\varphi = 0$  both on the surface of the sphere and at infinity, and

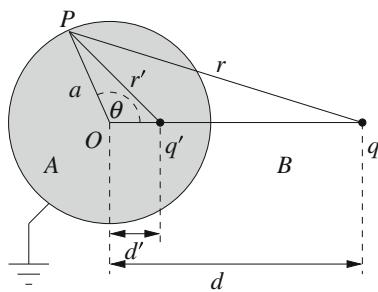


Fig. S-2.8

only the real charge  $q$  in between. Let us evaluate the potential  $\varphi(P)$  at a generic point  $P$  of the sphere surface, such that the line segment  $OP$  forms an angle  $\theta$  with the line segment  $Oq$ , as shown in Fig. S-2.8. We must have

$$\begin{aligned} 0 = \varphi(P) &= k_e \left( \frac{q}{r} + \frac{q'}{r'} \right) \\ &= k_e \left( \frac{q}{\sqrt{a^2 + d^2 - 2ad \cos \theta}} \right. \\ &\quad \left. + \frac{q'}{\sqrt{a^2 + d'^2 - 2ad' \cos \theta}} \right), \end{aligned} \quad (\text{S-2.31})$$

where  $r$  is the distance from  $P$  to  $q$ ,  $r'$  the distance from  $P$  to  $q'$ , and we have used the cosine rule. We see that the sign of  $q'$  must be the opposite of the sign of  $q$ . If we take the square of (S-2.31) we have

$$q^2(a^2 + d'^2 - 2ad' \cos \theta) = q'^2(a^2 + d^2 - 2ad \cos \theta), \quad (\text{S-2.32})$$

which must hold for any  $\theta$ . We must thus have separately

$$q^2(a^2 + d'^2) = q'^2(a^2 + d^2), \quad \text{and} \quad (\text{S-2.33})$$

$$2q^2 ad' \cos \theta = 2q'^2 ad \cos \theta. \quad (\text{S-2.34})$$

Equation (S-2.34) leads to

$$q'^2 = q^2 \frac{d'}{d}, \quad q' = -q \sqrt{\frac{d'}{d}}, \quad (\text{S-2.35})$$

which can be inserted into (S-2.33), leading to

$$dd'2 - (a^2 + d'^2)d' + a^2d = 0, \quad (\text{S-2.36})$$

which has the two solutions  $d' = d$  and  $d' = a^2/d$ . The first solution is not acceptable because it is larger than the radius of the sphere  $a$  (it actually corresponds to the trivial solution of superposing a charge  $-q$  to the charge  $q$ ). Thus we are left with  $d' = a^2/d$ , which can be substituted into (S-2.35), leading to our final solution

$$q' = q \frac{a}{d}, \quad d' = \frac{a^2}{d}. \quad (\text{S-2.37})$$

If the sphere is isolated and has a net charge  $Q$ , the problem in region  $B$  is solved by placing an image charge  $q'$  at  $d'$ , as above, and a further point charge  $q'' = Q - q'$  in  $O$ , so that the potential is uniform over the sphere surface, and the total charge of the sphere is  $Q$ . The case  $Q = 0$  corresponds to an uncharged, isolated sphere.

**b)** The total force  $\mathbf{f}$  on  $q$  equals the sum of the forces exerted on  $q$  by the image charge  $q'$  located in  $d'$ ,  $q'' = -q'$  and  $Q$ , both located in  $O$ . Thus  $\mathbf{f} = \mathbf{f}' + \mathbf{f}'' + \mathbf{f}'''$ , with

$$f' = k_e \frac{qq'}{(d-d')^2} = -k_e \frac{q^2 ad}{(d^2-a^2)^2}, \quad f'' = k_e \frac{q^2}{d^3}, \quad f''' = k_e \frac{qQ}{d^2}. \quad (\text{S-2.38})$$

with  $f'' = f''' = 0$  if the sphere is grounded.

**c)** The electrostatic energy  $U$  of the system equals the work of the electric field if the real charge  $q$  is moved to infinity. When  $q$  is at a distance  $x$  from  $O$  we evaluate the force on it by simply replacing  $d$  by  $x$  in (S-2.38). The work is thus the sum of the three terms

$$\begin{aligned} W_1 &= \int_d^\infty f' dx = k_e \left[ \frac{q^2 a}{2(x^2 - a^2)} \right]_d^\infty = -k_e \frac{q^2 a}{2(d^2 - a^2)}, \\ W_2 &= \int_d^\infty f'' dx = -k_e \left[ \frac{q^2 a}{2x^2} \right]_d^\infty = k_e \frac{q^2 a}{2d^2}, \\ W_3 &= \int_d^\infty f''' dx = k_e \frac{qQ}{d}. \end{aligned} \quad (\text{S-2.39})$$

Thus we have  $U = W_1$  for the grounded sphere,  $U = W_1 + W_2$  for the isolated chargeless sphere, and  $U = W_1 + W_2 + W_3$  for the isolated charged sphere.

It is interesting to compare this result for the energy of the isolated chargeless sphere with the electrostatic energy  $U_{\text{real}}$  of a system comprising three real charges  $q$ ,  $q'$ , and  $-q'$ , located in  $d$ ,  $d'$  and  $O$ , respectively:

$$\begin{aligned} U_{\text{real}} &= k_e \sum_{i<j} \frac{q_i q_j}{r_{ij}} = k_e \left( \frac{qq'}{d-d'} - \frac{qq'}{d} - \frac{q'^2}{d'} \right) \\ &= k_e \left( -\frac{q^2 a}{d^2 - a^2} + \frac{q^2 a}{d^2} - \frac{q^2}{d'} \right). \end{aligned} \quad (\text{S-2.40})$$

We see that  $U$  is obtained from  $U_{\text{real}}$  by halving the interaction energies of the real charge with the two image charges, and neglecting the interaction energy between the two image charges.

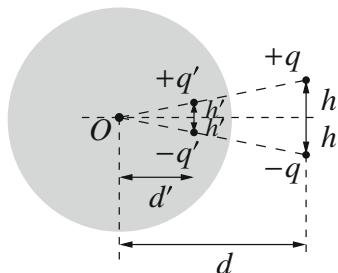
## S-2.5 Dipoles and Spheres

a) We consider the case of the grounded sphere first, so that its potential is zero. We can treat the dipole as a system of two point charges  $\pm q$ , separated by a distance  $2h$  as in Fig. S-2.9. Eventually, we shall let  $q$  approach  $\infty$ , and  $h$  approach zero, with the product  $p = 2hq$  remaining constant. Following Problem 2.4, the two charges induce two images

$$\pm q' = \mp q \frac{a}{\sqrt{d^2 + h^2}}, \quad (\text{S-2.41})$$

respectively, each at a distance

$$d' = \frac{a^2}{\sqrt{d^2 + h^2}} \quad (\text{S-2.42})$$



from the center of the sphere  $O$ , each lying on the straight line passing through  $O$  and the corresponding real charge. Since we are interested in the limit  $h \rightarrow 0$  (thus,  $h \ll d$ ), we can use the approximations

$$\pm q' \approx \mp q \frac{a}{d}, \quad \text{and} \quad d' = \frac{a^2}{d}. \quad (\text{S-2.43})$$

**Fig. S-2.9**

The two image charges are separated by a distance

$$2h' = 2h \frac{d'}{d} = h \left( \frac{a}{d} \right)^2, \quad (\text{S-2.44})$$

so that the moment of the image dipole is

$$\mathbf{p}' = 2q' \mathbf{h}' = -2q \mathbf{h} \left( \frac{a}{d} \right)^3 = -\mathbf{p} \left( \frac{a}{d} \right)^3. \quad (\text{S-2.45})$$

The image dipole is antiparallel to the real dipole, i.e., the two dipoles lie on parallel straight lines, but point in opposite directions. The sum of the image charges, which equals the total induced charge on the sphere surface, is zero. Therefore this solution is valid also for an isolated uncharged sphere.

**b)** Also in this case, we consider the grounded sphere first. Again, the dipole can be treated as a system of two charges  $\pm q$ , separated by a distance  $h = p/q$ . This time the charge  $+q$  is at distance  $d$  from the center of the sphere  $O$ , while  $-q$  is at distance  $d+h$ .

Thus, the images  $q'$  of  $+q$ , and  $q''$  of  $-q$ , have different absolute values, and are located at different distances from  $O$ ,  $d'$  and  $d''$ , respectively. We have

$$q' = -q \frac{a}{d} = -\frac{p}{h} \frac{a}{d}, \quad d' = \frac{a^2}{d}, \quad (\text{S-2.46})$$

$$q'' = +q \frac{a}{d+h} = +\frac{p}{h} \frac{a}{d+h}, \quad d'' = \frac{a^2}{d+h}.$$

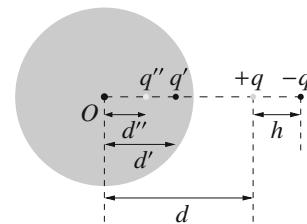


Fig. S-2.10

The absolute values of  $q'$  and  $q''$  remain different from each other also at the limits  $h \rightarrow 0, q \rightarrow \infty$ , so that a net image charge  $q'''$  is superposed to the image dipole

$$q''' = \lim_{h \rightarrow 0} (q' + q'') = \lim_{h \rightarrow 0} -p \frac{a}{h} \frac{h}{d(d+h)} = -p \frac{a}{d^2}. \quad (\text{S-2.47})$$

The moment of the image electric dipole can be calculated as the limit of the absolute value of  $q'$  times  $(d' - d'')$

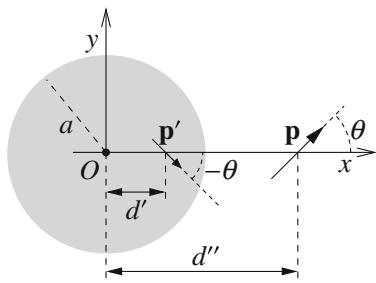
$$p' = \lim_{h \rightarrow 0} |q'| (d' - d'') = \lim_{h \rightarrow 0} \frac{p}{h} \frac{a}{d} \frac{a^2 h}{d(d+h)} = p \left( \frac{a}{d} \right)^3, \quad (\text{S-2.48})$$

the same result is obtained by evaluating the limit of  $q''(d' - d'')$ . Thus the real dipole  $\mathbf{p}$  and the image dipole  $\mathbf{p}'$  lie on the same straight line and point in the same direction. The image dipole is located at a distance  $a^2/d$  from  $O$ .

Since a net charge  $q'''$  is needed to have zero potential on the surface of the sphere, this solution is valid only in the case of a grounded sphere. The solution for an isolated uncharged sphere requires an image charge  $-q''' = +pa/d^2$  at the center of the sphere, so that the total image charge is zero and the surface of the sphere is equipotential.

**c)** We start from the case of the grounded sphere, and use a Cartesian reference frame with the origin located at the center of the sphere,  $O$ , the  $x$  axis passing through the dipole  $\mathbf{p}$ , and the  $y$  axis lying in the plane of the dipole. We denote by  $\theta$  the angle between the electric dipole  $\mathbf{p}$  and the  $x$  axis, as in Fig. S-2.11. We can decompose the dipole into the vector sum of its  $x$  and  $y$  components

$$\mathbf{p}_x = p \cos \theta \hat{\mathbf{x}}, \quad \text{and} \quad \mathbf{p}_y = p \sin \theta \hat{\mathbf{y}}. \quad (\text{S-2.49})$$

**Fig. S-2.11**

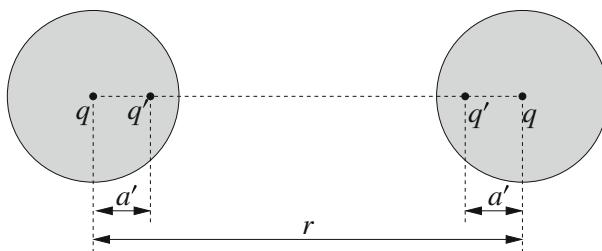
Both components generate images located on the  $x$  axis at a distance  $d' = a^2/d$  from  $O$ . According to a) and b),  $\mathbf{p}_y$  and  $\mathbf{p}_y'$  generate the images

$$\begin{aligned}\mathbf{p}'_y &= -\left(\frac{a}{d}\right)^3 p \sin\theta \hat{\mathbf{y}} \\ \mathbf{p}'_x &= \left(\frac{a}{d}\right)^3 p \cos\theta \hat{\mathbf{x}},\end{aligned}\quad (\text{S-2.50})$$

resulting in an image dipole  $\mathbf{p}'$ , of modulus  $p' = p(a/d)^3$ , forming an angle  $-\theta$  with the  $x$  axis, and superposed to a net charge  $q''' = +p \cos\theta(a/d^2)$ , since now it is the “tail” of  $\mathbf{p}_x$  which points toward  $O$ . In the case of an isolated uncharged sphere, we must add a point charge  $-q'''$  in  $O$ , so that the net charge of the sphere is zero.

## S-2.6 Coulomb’s Experiment

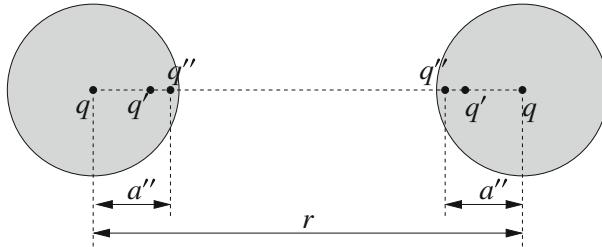
a) The zeroth-order solution is obtained by neglecting the induction effects, considering the charges as uniformly distributed over the surfaces of the two spheres. Thus, at zeroth order, the force between the two spheres equals the force between two point charges, each equal to  $Q$ , located at their centers. In order to evaluate higher-order solutions, it is convenient to introduce the dimensionless parameter  $\alpha = (a/r) < 1$ , where  $a$  is the radius of the two spheres, and  $r$  the distance between their centers. The solution of order  $n$  is obtained by locating inside each sphere a point charge  $q$  of the same order of magnitude as  $Q$  at its center, plus increasingly smaller point charges  $q'$ ,  $q''$ , ...,  $q^{(n)}$  at appropriate positions, with orders of magnitude  $|q'| \sim \alpha Q$ ,  $|q''| \sim \alpha^2 Q$ , ...,  $|q^{(n)}| \sim \alpha^n Q$ . The charges must obey the normalization condition  $q + q' + q'' + \dots + q^{(n)} = Q$ .

**Fig. S-2.12**

At the first order, the point charge  $q$  at the center of each sphere induces an image charge  $q' = -aq$  inside the other sphere, located at a distance  $a' = a^2/r = ra^2$  from its center (see Problem 2.4), as shown in Fig. S-2.12. Thus, by solving the simultaneous

equations  $q' = -\alpha q$ , and  $q + q' = Q$ , we obtain for the values of the two charges

$$q = \frac{1}{1-\alpha} Q, \quad q' = -\frac{\alpha}{1-\alpha} Q. \quad (\text{S-2.51})$$



**Fig. S-2.13**

At the second order, the first-order charge  $q'$  inside each sphere induces an image charge  $q''$  inside the other sphere, located a distance  $a''$  from its center, as shown in Fig. S-2.13. Since the distance of  $q'$  from the center of the other sphere is  $r - a' = r(1 - \alpha^2)$ , we have

$$q'' = -q' \frac{a}{r - a'} = -q' \frac{\alpha}{1 - \alpha^2}, \quad a'' = \frac{a^2}{r - a'} = r \frac{\alpha^2}{1 - \alpha^2}. \quad (\text{S-2.52})$$

Combining the above equation for  $q''$  with equations  $q' = -\alpha q$  and  $q + q' + q'' = Q$ , we finally obtain

$$q = Q \frac{1 - \alpha^2}{1 - \alpha + \alpha^3}, \quad q' = -Q \frac{\alpha(1 - \alpha^2)}{1 - \alpha + \alpha^3}, \quad q'' = Q \frac{\alpha^2}{1 - \alpha + \alpha^3}. \quad (\text{S-2.53})$$

Higher order approximations are obtained by iterating the procedure. Thus we obtain a sequence of image charges  $q, q', q'', q''', \dots$  inside each sphere. At each iteration, the new image charge is of the order of  $\alpha$  times the charge added at the previous iteration. Therefore, the smaller the value of  $\alpha = a/r$ , the sooner one may truncate the sequence obtaining a good approximation.

**b)** We obtain the first order approximation of the force between the two spheres by considering only the charges of (S-2.51) for each sphere. To this approximation, the force between the spheres is the sum of four terms. The first term is the force between the two zeroth-order charges  $q$ , at a distance  $r$  from each other. The second and third terms are the forces between the zeroth order charge  $q$  of one sphere and the first-order charge  $q'$  of the other. The distance between these charges is  $r - a' = r(1 - \alpha^2)$ . The fourth term is the force between the two first-order charges  $q'$ , at a distance  $r - 2a' = r(1 - 2\alpha^2)$  from each other. Summing up all these contributions we obtain

$$F = k_e \frac{Q^2}{r^2} \frac{1}{(1-\alpha)^2} \left[ 1 - \frac{2\alpha}{(1-\alpha^2)^2} + \left( \frac{\alpha}{1-2\alpha^2} \right)^2 \right]. \quad (\text{S-2.54})$$

From the Taylor expansion, valid for  $x < 1$ ,

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + O(x^4), \quad (\text{S-2.55})$$

we obtain, to the fourth order,

$$\frac{1}{(1-\alpha^2)^2} = 1 + 2\alpha^2 + 3\alpha^4 + O(\alpha^6), \quad (\text{S-2.56})$$

and

$$\frac{1}{(1-2\alpha^2)^2} = 1 + 4\alpha^2 + 12\alpha^4 + O(\alpha^6), \quad (\text{S-2.57})$$

so that

$$\begin{aligned} F &= k_e \frac{Q^2}{r^2} (1 + 2\alpha + 3\alpha^2 + 4\alpha^3 + \dots) (1 - 2\alpha + \alpha^2 - 4\alpha^3 + \dots) \\ &= k_e \frac{Q^2}{r^2} [1 - 4\alpha^3 + O(\alpha^4)], \end{aligned} \quad (\text{S-2.58})$$

since all the terms of order  $\alpha$  and  $\alpha^2$  vanish. The first non vanishing correction to the “Coulomb” force is thus at the third order in  $a/r$ ,

$$F = k_e \frac{Q^2}{r^2} \left( 1 - 4 \frac{a^3}{r^3} \right). \quad (\text{S-2.59})$$

This result can be interpreted in terms of multipole expansions of the charge distributions of the spheres. The first two multipole moments of the charge distribution of each sphere are a monopole equal to the total charge  $Q$ , and an electric dipole  $\mathbf{p} = -q'a'\hat{\mathbf{r}} = -(\alpha Q)(\alpha^2 r)\hat{\mathbf{r}} = -\alpha^3 Qr\hat{\mathbf{r}}$ , with  $\hat{\mathbf{r}}$  pointing toward the center of the opposite sphere. The contribution of the monopole moments to the total force is  $F_{\text{mm}} = k_e Q^2/r^2$ . Now we need the force exerted by the monopole terms of each sphere on the dipole term of the other. The monopole of, say, the left sphere generates a field  $\mathbf{E}^{(0)} = k_e Q/r^2$  at the center of the right sphere. We can consider the dipole moment of the right sphere as the limit for  $h \rightarrow 0$  of two charges,  $-q'$  located at  $r-h$  from the center of the left sphere, and  $q'$  located at  $r$ , with  $q'h = |p|$ . The force between the left monopole and the right dipole is thus

$$\begin{aligned} F_{\text{md}} &= \lim_{h \rightarrow 0} k_e Qq' \left[ -\frac{1}{(r-h)^2} + \frac{1}{r^2} \right] \simeq \lim_{h \rightarrow 0} k_e Qq' \left( -\frac{1}{r^2} - \frac{2h}{r^3} + \frac{1}{r^2} \right) \\ &= -\lim_{h \rightarrow 0} k_e Qq' \frac{2h}{r^3} = -2k_e \frac{Qp}{r^3} = -2k_e \alpha^3 \frac{Q^2}{r^3}, \end{aligned} \quad (\text{S-2.60})$$

where we have used the first-order Taylor expansion of  $(r - h)^{-2}$ . Adding the force between the right monopole and the left dipole, the total force is thus

$$F = F_{\text{mm}} + 2F_{\text{md}} = k_e \frac{Q^2}{r^2} \left(1 - 4 \frac{a^3}{r^3}\right), \quad (\text{S-2.61})$$

in agreement with (S-2.59). The same result can be obtained by applying the formula for the force between a point charge and an electric dipole,  $\mathbf{F} = (\mathbf{p} \cdot \nabla) \mathbf{E}$ . See also Problem 1.10 on this subject.

From (S-2.59) we find that a ratio  $a/r \simeq 0.13$  is enough to reduce the systematic deviation from the pure inverse-square law below 1%.

## S-2.7 A Solution Looking for a Problem

- a) The total electric potential in a point of position vector  $\mathbf{r}$  is the sum of the dipole potential and of the potential of the external uniform electric field,

$$\varphi(\mathbf{r}) = k_e \frac{\mathbf{p} \cdot \mathbf{r}}{r^3} - Ez = k_e \frac{p \cos \theta}{r^2} - Er \cos \theta, \quad (\text{S-2.62})$$

where  $\theta$  is the angle between  $\mathbf{r}$  and the  $z$  axis. Note that it is not possible to take the reference point for the electrostatic potential at infinity, since the potential of our uniform electric field diverges for  $z \rightarrow \pm\infty$ . Thus we have chosen  $\varphi = 0$  on the  $xy$  plane, which is an equipotential surface both for the dipole and for the uniform electric field. Now we look for a possible further equipotential surface on which  $\varphi = 0$ . On this surface we must have

$$\varphi = k_e \frac{p \cos \theta}{r^2} - Er \cos \theta = 0, \quad (\text{S-2.63})$$

and, in addition to the solution  $\theta = \pi/2$ , corresponding to the  $xy$  plane, we have the  $\theta$ -independent solution

$$r = k_e^{1/3} \left(\frac{p}{E}\right)^{1/3} \equiv R, \quad (\text{S-2.64})$$

corresponding to a sphere of radius  $R$ . Note that the two equipotential surfaces intersect each other on the circumference  $x^2 + y^2 = R^2$  on the  $z = 0$  plane. This is possible because the electric field of the dipole on the intersection circumference is

$$\mathbf{E}_{\text{dip}} = k_e \frac{3(\mathbf{p} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}}{r^3} = -k_e \frac{\mathbf{p}}{R^3} = -\mathbf{E}, \quad (\text{S-2.65})$$

so that the total field on the circumference is zero, i.e., the only field that can be perpendicular to both equipotential surfaces.

**b)** We must find a solution for the potential  $\varphi$  that satisfies the condition  $\varphi = 0$  at the surface of the conducting sphere, i.e.  $\varphi(|\mathbf{r}| = a) = 0$ , and such that at large distance from the conductor the field is  $\mathbf{E}_0$ .

According to point **a)**, the field outside the sphere must equal  $\mathbf{E}_0$  plus the field of an electric dipole  $\mathbf{p}_i$ , parallel to  $\mathbf{E}_0$  and located at the center of the sphere. The moment of the dipole is obtained by substituting  $R = a$  into (S-2.64),

$$\mathbf{p}_i = k_e^{-1} a^3 \mathbf{E}_0 = \frac{3}{4\pi k_e} V_a \mathbf{E}_0 , \quad (\text{S-2.66})$$

where  $V_a$  is the volume of the sphere. The potential for  $r \geq a$  is thus

$$\varphi = k_e \frac{\mathbf{p}_i \cdot \mathbf{r}}{r^3} - E_0 z , \quad (\text{S-2.67})$$

while  $\varphi = 0$  for  $r \leq a$ . The total charge induced on the sphere is zero, so that the solution is the same for a grounded and for an isolated, uncharged sphere. The solution is identical to the one obtained in Problem 2.1 via a different (heuristic) approach.

**c)** For the dipole at the center of a spherical conducting cavity, the boundary condition is  $\varphi = 0$  at  $r = b$ . The polarization charges on the inner surface must generate a uniform field  $\mathbf{E}_i$  parallel to  $\mathbf{p}_0$  and, according to (S-2.64), of intensity

$$E_i = k_e \frac{p_0}{b^3} = k_e \frac{4\pi p_0}{3V_b} = k_e \frac{p_0}{b^3} . \quad (\text{S-2.68})$$

As in the preceding case, the total induced charge is zero and thus it does not matter whether the shell is grounded, or isolated and uncharged.

**d)** We can think of the dipole as a system of two point charges  $\pm q$ , respectively located at  $z = \pm d$ , with  $p = 2qd$ , as in Fig. S-2.14. According to the method of the image charges, the charge  $+q$  modifies the charge distribution of the inner surface of the shell, so that it generates a field inside the sphere, equivalent to the field of an image charge  $q' = -qb/d$ , located at  $z = d' = b^2/d$ . Also the presence of the charge  $-q$  affects the surface charge distribution, so that the total field inside the shell is the sum of the fields of the two real charges, plus the field of two image charges  $\mp qb/d$  located at  $z = \pm b^2/d$ , respectively. Letting  $d \rightarrow 0$  and  $q \rightarrow \infty$ , keeping the product  $2qd = p$  constant, the field of the real charges approaches the field of a dipole  $\mathbf{p} = p\hat{\mathbf{z}}$  located at the center of the shell, while the field of the image charges approaches a uniform field. Let us evaluate the field of the image charges at the center of the shell:

$$E_c = 2k_e \frac{qb}{d} \left( \frac{d}{b^2} \right)^2 = k_e \frac{2qd}{b^3} = k_e \frac{p}{b^3} , \quad (\text{S-2.69})$$

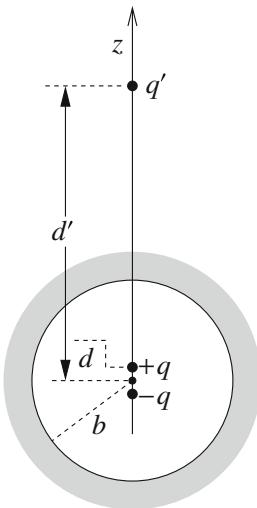


Fig. S-2.14

in agreement with the result of point **c**). The method of the image charges can also be used to obtain the result of point **b**).

## S-2.8 Electrically Connected Spheres

**a)** To the zeroth order in  $a/d$  and  $b/d$ , we assume the surface charges to be uniformly distributed. The electrostatic potential generated by each sphere outside its volume is thus equal to the potential of a point charge located at the center of the sphere. Let us denote by  $Q_a$  and  $Q_b$  the charges on each sphere, with  $Q_a + Q_b = Q$ . The charge on the wire is negligible because we have assumed that its capacitance is negligible. The electrostatic potentials of the spheres with respect to infinity are

$$V_a \simeq k_e \frac{Q_a}{a}, \quad V_b \simeq k_e \frac{Q_b}{b}, \quad (\text{S-2.70})$$

respectively. Since the spheres are electrically connected,  $V_a = V_b \equiv V$ . Solving for the charges we obtain

$$Q_a \simeq Q \frac{a}{a+b}, \quad Q_b \simeq Q \frac{b}{a+b}, \quad (\text{S-2.71})$$

so that  $Q_a > Q_b$ .

**b)** From the results of point **a)** it follows

$$V \simeq k_e \frac{Q}{a+b}, \quad C \simeq \frac{a+b}{k_e}. \quad (\text{S-2.72})$$

**c)** The electric fields at the sphere surfaces are

$$E_a \simeq k_e \frac{Q_a}{a^2} = k_e \frac{Q}{a(a+b)}, \quad E_b \simeq k_e \frac{Q_b}{b^2} = k_e \frac{Q}{b(a+b)}, \quad (\text{S-2.73})$$

with  $E_b > E_a$ . At the limit  $b \rightarrow 0$  we have  $E_a \rightarrow k_e Q/a^2$ , while  $E_b \rightarrow \infty$ .

**d)** We proceed as in Problem 2.6. To zeroth order, we consider the field of each sphere outside its volume as due to a point charge at the sphere center. We denote by  $q_a$  and  $q_b$  the values of these point charges. To the first orders in

$a/d$  and  $b/d$ , we consider that each zeroth-order charge induces an image charge inside the other sphere, with values

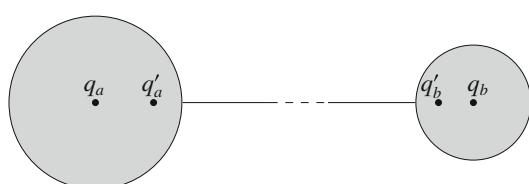


Fig. S-2.15

$$q'_a = -q_b \frac{a}{d}, \quad q'_b = -q_a \frac{b}{d}, \quad (\text{S-2.74})$$

at distances  $a^2/d$  and  $b^2/d$  from the centers, respectively. At each successive order, we add the images of the images added at the previous order. This leads to image charges of higher and higher orders in  $a/d$  and  $b/d$ .

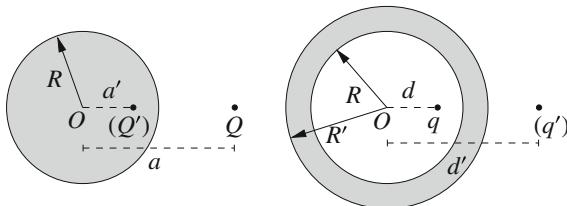
Up to the first order, we thus have four point charges with the condition  $q_a + q_b + q'_a + q'_b = Q$ . A further condition is that the potentials at the sphere surfaces are

$$V_a \approx k_e \frac{q_a}{a}, \quad V_b \approx k_e \frac{q_b}{b}, \quad (\text{S-2.75})$$

since, at the surface of each sphere, the potentials due to the external zeroth-order charge and to the internal first-order charge cancel each other. Finally, we must have  $V_a = V_b$ , because the spheres are connected by the wire, so that

$$q_a \approx \frac{Q}{1 + b/a - 2b/d}, \quad q_b \approx \frac{Q}{1 + a/b - 2a/d}. \quad (\text{S-2.76})$$

## S-2.9 A Charge Inside a Conducting Shell



**Fig. S-2.16**

the origin in  $O$ . We shall need only the radial coordinate  $r$ .

We have seen that the boundary conditions for  $r \geq R$  are replicated by locating an image charge  $Q' = Q(a/R)$  inside the sphere, at a distance  $a' = R^2/a$  from  $O$ , on the line joining  $O$  and  $Q$ . In the present case we are dealing with the reverse problem, and we can obtain the solution in the region  $r \leq R$  by reversing the roles of the real and image charges. The real charge  $q$  is now inside the cavity of a spherical conducting, grounded shell of internal radius  $R$ , at a distance  $d < R$  from the center  $O$ . The boundary conditions inside the cavity are replicated by locating an *external* image charge  $q' = q(R/d)$  at a distance  $d' = R^2/d$  from  $O$ , on the straight line through  $O$  and  $q$ , as in Fig. S-2.16. Thus, the electric potential inside the cavity equals the sum of the potentials of  $q$  and  $q'$ . The potential  $\varphi$  in the region  $R \leq r \leq R'$  is constant because here we are inside a conductor in static conditions, and equal to zero because the shell is grounded. We have  $\varphi \equiv 0$  also for  $r \geq R'$ , because  $\varphi = 0$  both on the spherical surface at  $r = R$ , and at infinity, and there are no charges in between.

**a)** Let us first recall Problem 2.4, now with a point charge  $Q$  at a distance  $a$  from the center  $O$  of a conducting, grounded sphere, of radius  $R < a$ . We introduce a spherical coordinate system, with

- b)** The force between  $q$  and the shell equals the Coulomb force between  $q$  and its image charge  $q'$ , and is attractive

$$F = k_e \frac{qq'}{(d'-d)^2} = -k_e \frac{q^2 R d}{(R^2 - d^2)^2}. \quad (\text{S-2.77})$$

**c)** Let us consider a spherical surface of radius  $R''$ , centered in  $O$ , with  $R < R'' < R'$ . The flux of the electric field through this closed surface is zero, because the field is zero everywhere inside a conductor. The total charge inside the sphere must thus be zero according to Gauss's law. This implies that the charge induced on the inner surface of the shell is  $-q$ , as may be verified directly by calculating the surface charge and integrating over the whole surface.

**d)** The electric potential must still be constant for  $R \leq r \leq R'$ , but it is no longer constrained to be zero. The electric potential in the region  $r \leq R$  is still equivalent to the potential generated by the charges  $q$  and  $q'$  of point **a**), plus a constant quantity  $\varphi_0$  to be determined. The electric field in the region  $R \leq r \leq R'$  is still zero, so that the potential is constant and equal to  $\varphi_0$ . Since the total charge on the shell must be zero, we must distribute a charge  $q$  over its external surface, of radius  $R'$ , to compensate the charge  $-q$  distributed over the internal surface, of radius  $R$ . Since the real charge  $q$ , and the charge  $-q$  distributed over the surface of radius  $R$  generate a constant potential for  $r \geq R$ , the charge  $q$  must be distributed uniformly over the external surface in order to keep the total potential constant in the region  $R \leq r \leq R'$ .

The potential in the region  $r \geq R'$  is equivalent to the potential generated by a point charge  $q$  located in  $O$ . Thus we have  $\varphi(r) = k_e q / r$  for  $r \geq R'$ , if we choose  $\varphi(\infty) = 0$ . Thus  $\varphi_0 = \varphi(R') = k_e q / R'$ , and  $\varphi(r) = \varphi_0$  for  $R \leq r \leq R'$ . For  $r \leq R$  we have

$$\varphi(r) = k_e \left( \frac{q}{r_q} + \frac{q'}{r_{q'}} \right) + \varphi_0, \quad (\text{S-2.78})$$

where  $r_q$  is the distance of the point from the real charge  $q$ , and  $r_{q'}$  is the distance of the point from the image charge  $q'$ . The field inside the cavity is the same for a grounded or for an isolated shell.

## S-2.10 A Charged Wire in Front of a Cylindrical Conductor

- a)** We have  $r = \sqrt{(x+a)^2 + y^2}$  and  $r' = \sqrt{(x-a)^2 + y^2}$ ,  $x$  and  $y$  being the coordinates of  $Q$ . Thus, squaring the equation  $r/r' = K$  we get

$$\begin{aligned}
 \frac{(x+a)^2+y^2}{(x-a)^2+y^2} &= K^2 \\
 x^2+2ax+a^2+y^2 &= K^2x^2-2K^2ax+K^2a^2+K^2y^2 \\
 -(x^2+y^2)(K^2-1) + 2ax(K^2+1) &= a^2(K^2-1) \\
 x^2+y^2-2\frac{K^2+1}{K^2-1}ax &= a^2. \tag{S-2.79}
 \end{aligned}$$

On the other hand, the equation of a circumference centered at  $(x_0, 0)$  and radius  $R$  is

$$\begin{aligned}
 (x-x_0)^2+y^2 &= R^2 \\
 x^2+y^2-2x_0x &= R^2-x_0^2. \tag{S-2.80}
 \end{aligned}$$

Comparing (S-2.80) to (S-2.79) we see that the curves defined by the equation  $r/r' = K$  are circumferences centered at

$$x_0(K) = \frac{K^2+1}{K^2-1}a, \quad y_0 = 0, \tag{S-2.81}$$

of radius

$$R(K) = \frac{2K}{|K^2-1|}a. \tag{S-2.82}$$

Note that

$$x_0\left(\frac{1}{K}\right) = -x_0(K), \quad \text{and} \quad R\left(\frac{1}{K}\right) = R(K). \tag{S-2.83}$$

Thus, we may restrict ourselves to  $K > 1$ , so that  $x_0(K) > a > 0$ , and omit the absolute-value sign in the expression for  $R(K)$ . The circumferences corresponding to  $0 < K < 1$  are obtained by reflection across the  $y$  axis of the circumferences corresponding to  $1/K$ .

**b)** According to Gauss's law, the electrostatic field and potential generated by an infinite straight wire with linear charge density  $\lambda$  are

$$E(r) = \frac{\lambda}{2\pi\epsilon_0 r} \quad \text{and} \quad \varphi(r) = -\frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{r}{r_0}\right), \tag{S-2.84}$$

where  $r$  is the distance from the wire and  $r_0$  an arbitrary constant, corresponding to the distance at which we pose  $\varphi = 0$ . The potential generated by two parallel wires of charge densities  $\lambda$  and  $-\lambda$ , respectively, is

$$\varphi = -\frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{r}{r_0}\right) + \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{r'}{r'_0}\right) = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{r'}{r}\right) + \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{r_0}{r'_0}\right), \tag{S-2.85}$$

where  $r'_0$  is a second arbitrary constant, analogous to  $r_0$ . The term

$$\frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{r_0}{r'_0}\right) \tag{S-2.86}$$

is actually a single arbitrary constant, which we can set equal to zero. With this choice the electrostatic potential is zero on the  $x = 0$  plane of a Cartesian reference frame where the two wires lie on the straight lines ( $x = -a, y = 0$ ) and ( $x = +a, y = 0$ ). The equation for the equipotential surfaces in this reference frame is

$$\frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{r'}{r}\right) = \varphi, \quad (\text{S-2.87})$$

which leads to

$$\frac{r}{r'} = e^{-2\pi\epsilon_0\varphi/\lambda}. \quad (\text{S-2.88})$$

Thus we can substitute  $K = e^{-2\pi\epsilon_0\varphi/\lambda}$  into (S-2.81) and (S-2.82). We see that the equipotential surfaces are infinite cylindrical surfaces whose axes have the equations

$$x_0(\varphi) = \frac{e^{-4\pi\epsilon_0\varphi/\lambda} + 1}{e^{-4\pi\epsilon_0\varphi/\lambda} - 1}, \quad y_0 = 0, \quad (\text{S-2.89})$$

and their radii are

$$R(\varphi) = \frac{2e^{-2\pi\epsilon_0\varphi/\lambda}}{|e^{-4\pi\epsilon_0\varphi/\lambda} - 1|} a. \quad (\text{S-2.90})$$

By multiplying the numerators and denominators of the above expressions by  $e^{2\pi\epsilon_0\varphi/\lambda}$  we finally obtain

$$x_0(\varphi) = \frac{e^{-2\pi\epsilon_0\varphi/\lambda} + e^{2\pi\epsilon_0\varphi/\lambda}}{e^{-2\pi\epsilon_0\varphi/\lambda} - e^{2\pi\epsilon_0\varphi/\lambda}} a = -a \coth\left(\frac{2\pi\epsilon_0\varphi}{\lambda}\right) \quad (\text{S-2.91})$$

and

$$R(\varphi) = \frac{2}{|e^{-2\pi\epsilon_0\varphi/\lambda} - e^{2\pi\epsilon_0\varphi/\lambda}|} a = \left| \frac{a}{\sinh(2\pi\epsilon_0\varphi/\lambda)} \right|. \quad (\text{S-2.92})$$

If the negative wire is located on the ( $x = -a, y = 0$ ) straight line, the  $\varphi > 0$  equipotential cylinders are located in the  $x < 0$  half space ( $r < r'$  in Fig. 2.8), and the  $\varphi < 0$  equipotentials in the  $x > 0$  half space.

- c) We can solve the problem by locating an image wire with charge density  $\lambda' = -\lambda$  inside the cylinder. In Fig. (2.8), let the real wire intersect the  $xy$  plane at  $P \equiv (-a, 0)$ , and the image wire at  $P' \equiv (a, 0)$ . The surface of the conducting cylinder intersects the  $xy$  plane on one of the circumferences  $r/r' = K$ . This is always possible as far as  $d > R$ . With these locations of the real and image wires the potential of the cylinder surface is constant and equal to a certain value  $\varphi_0$ . Given  $R$  and  $d$ , we can find the values of  $a$  and  $d'$  by first defining the dimensionless constant  $\varphi' = 2\pi\epsilon_0\varphi_0/\lambda$ , and then solving the simultaneous equations

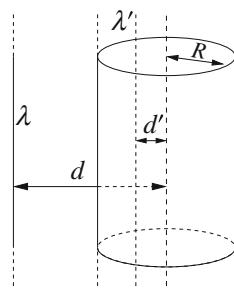


Fig. S-2.17

$$2a + d' = d, \quad a + d' = x_0 = a \coth \varphi', \quad \frac{a}{\sinh \varphi'} = R. \quad (\text{S-2.93})$$

From the first equation we obtain  $a = (d - d')/2$ , which we substitute into the other two equations

$$\frac{d + d'}{2} = \frac{d - d'}{2} \coth \varphi', \quad \frac{d - d'}{2} = R \sinh \varphi', \quad (\text{S-2.94})$$

and the latter equation leads to

$$\sinh \varphi' = \frac{d - d'}{2R}, \quad (\text{S-2.95})$$

independent of  $\lambda$ . From the relations

$$\cosh^2 x - \sinh^2 x = 1, \quad \text{and} \quad \coth x = \frac{\cosh x}{\sinh x},$$

we obtain

$$\coth \varphi' = \frac{\sqrt{4R^2 + (d - d')^2}}{d - d'}, \quad (\text{S-2.96})$$

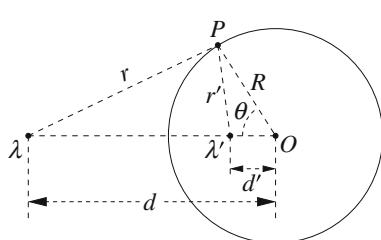
which, substituted into the first of (S-2.93) leads to

$$\frac{d + d'}{2} = \frac{d - d'}{2} \frac{\sqrt{4R^2 + (d - d')^2}}{d - d'}. \quad (\text{S-2.97})$$

Disregarding the trivial solution  $d' = d$  (corresponding to two superposed wires of linear charge density  $\lambda$  and  $-\lambda$ , generating zero field in the whole space), we have

$$d' = \frac{R^2}{d}, \quad a = \frac{d^2 + R^2}{2d}, \quad \varphi' = \operatorname{arccosh} \left( \frac{d^2 + 3R^2}{d^2 + R^2} \right). \quad (\text{S-2.98})$$

Alternatively, may proceed analogously to the well-known problem of the potential of a point charge in front of a grounded, conducting sphere.



**Fig. S-2.18**

Figure S-2.18 shows the intersection with the  $xy$  plane of the conducting cylinder of radius  $R$ , the real charged wire at distance  $d$  from the cylinder axis, and the image wire at distance  $d'$  from the axis. We have translational symmetry perpendicularly to the figure. The potential  $\varphi$  generated by the real wire of linear charge density  $\lambda$ , and by the image wire of linear of linear charge density  $\lambda'$  must be constant over the cylinder surface.

The potential at a generic point  $P$  of the surface is

$$\varphi = -\frac{\lambda}{2\pi\epsilon_0} \ln r - \frac{\lambda'}{2\pi\epsilon_0} \ln r' = \text{const} \quad (\text{S-2.99})$$

where  $r$  is the distance of  $P$  from the real wire and  $r'$  the distance of  $P$  from the image wire. Multiplying by  $-2\pi\epsilon_0$  we obtain

$$\lambda \ln r + \lambda' \ln r' = \text{const}, \quad (\text{S-2.100})$$

which can be rewritten by expressing  $r$  and  $r'$  in terms of  $d, d', R$  and the angle  $\theta$  between  $r$  and the radius joining  $P$  to the intersection of the cylinder axis with the  $xy$  plane,  $O$ , and applying the law of cosines,

$$\lambda \ln \left( \sqrt{d^2 + R^2 - 2Rd \cos \theta} \right) + \lambda' \ln \left( \sqrt{d'^2 + R^2 - 2Rd' \cos \theta} \right) = \text{const}. \quad (\text{S-2.101})$$

Differentiating with respect to  $\theta$  we obtain

$$\frac{\lambda R d \sin \theta}{d^2 + R^2 - 2Rd \cos \theta} = -\frac{\lambda' R d' \sin \theta}{d'^2 + R^2 - 2Rd' \cos \theta}, \quad (\text{S-2.102})$$

implying that  $\lambda$  and  $\lambda'$  must have opposite signs. Dividing both sides by  $R \sin \theta$  we obtain, after some algebra,

$$\begin{aligned} \lambda d(d'^2 + R^2 - 2Rd' \cos \theta) &= -\lambda' d'(d^2 + R^2 - 2Rd \cos \theta) \\ \lambda(d'd'^2 + dR^2 - 2Rdd' \cos \theta) &= -\lambda' (d'd^2 + d'R^2 - 2Rdd' \cos \theta), \end{aligned} \quad (\text{S-2.103})$$

which requires  $\lambda' = -\lambda$  in order to make the equation independent of  $\theta$ , and, disregarding the trivial solution  $d' = d$ , we finally obtain

$$d' = \frac{R^2}{d}. \quad (\text{S-2.104})$$

## S-2.11 Hemispherical Conducting Surfaces

- a) We choose a cylindrical coordinate system  $(r, \phi, z)$  with the symmetry axis of the problem as  $z$  axis, so that the point charge is located in  $(a \sin \theta, \phi, a \cos \theta)$ , with  $\phi$  a given fixed angle, as in Fig. S-2.19. The conductor surface, comprising the hemispherical boss and the plane part, is equipotential with  $\varphi = 0$ . If the conductor surface were simply plane, with no boss, the problem would be solved by locating an image charge  $q_1 = -q$  in  $(a \sin \theta, \phi, -a \cos \theta)$ , as in Fig. S-2.19. On the other hand, if the conductor were a grounded spherical surface of radius  $R$ , the problem would be solved by locating an image charge  $q_2 = -q(R/a)$  in  $(a' \sin \theta, \phi, a' \cos \theta)$ , with  $a' = R^2/a$ . The real charge  $q$ , with the two image charges  $q_1$  and  $q_2$ , gives origin to a potential

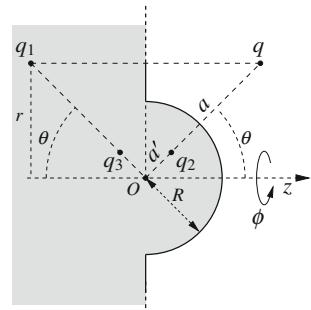


Fig. S-2.19

$\varphi(\mathbf{r}) = \varphi_q(\mathbf{r}) + \varphi_{q_1}(r) + \varphi_{q_2}(r)$  which is different from zero both on the plane surface, where it equals  $\varphi_{q_2}(r)$ , since  $\varphi_q(\mathbf{r}) + \varphi_{q_1}(r) = 0$  on the plane, and on the hemispherical surface, where it equals  $\varphi_{q_1}(r)$ . The problem is solved by adding a third image charge  $q_3 = q(R/a)$  at  $(a' \sin\theta, \phi, -a' \cos\theta)$ , so that the pairs  $\{q, q_1\}$  and  $\{q_2, q_3\}$  generate a potential  $\varphi = 0$  on the plane surface, and the pairs  $\{q, q_2\}$  and  $\{q_1, q_3\}$  generate a potential  $\varphi = 0$  on the spherical (and hemispherical!) surface. According to Gauss's law, the total charge induced on the conductor equals the sum of the image charges

$$q_{\text{ind}} = q_1 + q_2 + q_3 = -q + \left(-\frac{R}{a}q\right) + \left(\frac{R}{a}q\right) = -q. \quad (\text{S-2.105})$$

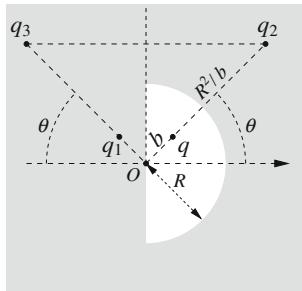


Fig. S-2.20

Note that, since the electric field generated by the real charge plus the three image charges is always perpendicular to the conductor surface, it must be zero on the circumference  $(R, \phi, 0)$ , here with  $\phi$  any, where the hemisphere joins the plane.

**b)** Now the real charge  $q$  is located at  $(b \sin\theta, \phi, b \cos\theta)$  inside the hemispherical cavity of radius  $R > b$  in the conductor, as in Fig. S-2.20. The solution is analogous to the solution of point **a**): we locate three image charges in the conductor, outside of the cavity, namely,  $q_1 = -q$  in  $(b \sin\theta, \phi, -b \cos\theta)$ ,  $q_2 = -(R/b)q$  in  $(b' \sin\theta, \phi, b' \cos\theta)$ , with  $b' = R^2/b > R$ , and  $q_3 = -q_2 = (R/b)q$  in  $(b' \sin\theta, \phi, -b' \cos\theta)$ .

## S-2.12 The Force between the Plates of a Capacitor

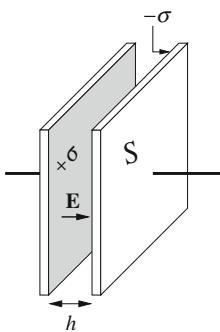


Fig. S-2.21

We present this simple problem in order to point out, and prevent, two typical recurrent errors. The first error regards the electrostatic pressure at the surface of a conductor, the second the derivation of the force from the energy of a system.

**a)** Let us consider the electrostatic pressure first. If  $Q$  is the charge of the capacitor, and  $S$  the surface of its plates, the surface charge density (which is located on the *inner* surfaces only!) is  $\pm\sigma = \pm Q/S$ . Within our approximations, the electric field is uniform between the two charged surfaces,  $E = 4\pi k_e \sigma$ , and zero everywhere else. This leads to an electrostatic pressure

$$P = \frac{1}{2} \sigma E = 2\pi k_e \sigma^2. \quad (\text{S-2.106})$$

Here, the typical mistake is to forget the  $1/2$  factor and to write  $P \stackrel{w}{=} \sigma E$  (the “w” on the “=” sign stands for wrong!). In fact, only one half of the electric field is due to the charge on the other plate. The force  $F$  is attractive because the two plates have opposite charges, and we can write

$$F = -PS = -2\pi k_e \frac{Q^2}{S^2} S = -2\pi k_e \frac{Q^2}{S}. \quad (\text{S-2.107})$$

Thus the force depends on  $Q$  only, and is independent of the distance  $h$  between the plates. (S-2.107) is valid both for an isolated capacitor, and for a capacitor connected to a voltage source maintaining a fixed potential difference  $V$ . But, in the latter case, the charge is no longer constant, and it is convenient to replace  $Q$  by the product  $CV$ , remembering that the capacity of a parallel-plate capacitor is  $C = S/(4\pi k_e h)$ . Thus

$$F = -2\pi k_e \frac{(CV)^2}{S} = -\frac{V^2 S}{8\pi k_e h^2}. \quad (\text{S-2.108})$$

**b)** In the case of an isolated capacitor, the force between the plates can also be evaluated as minus the derivative of the electrostatic energy  $U_{\text{es}}$  of the capacitor with respect to the distance between the plates,  $h$ . It is convenient to write  $U_{\text{es}}$  as a function of the charge  $Q$ , which is constant for an isolated capacitor,

$$U_{\text{es}} = \frac{Q^2}{2C} = 2\pi k_e \frac{Q^2 h}{2S}, \quad (\text{S-2.109})$$

so that the force between the plates is

$$F = -\partial_h U_{\text{es}} = -2\pi k_e \frac{Q^2}{S}, \quad (\text{S-2.110})$$

in agreement with (S-2.107).

If the capacitor is connected to a voltage source, the potential difference  $V$  between the plates is the constant quantity. Thus, it is more convenient to write  $U_{\text{es}}$  as a function of  $V$

$$U_{\text{es}} = \frac{1}{2} CV^2 = \frac{1}{8\pi k_e} \frac{V^2 S}{h}. \quad (\text{S-2.111})$$

At this point, it is tempting, but wrong, to evaluate the force between the plates as minus the derivative of  $U_{\text{es}}$  with respect to  $h$ . We would get

$$F \stackrel{w}{=} -\partial_h U_{\text{es}} = +\frac{1}{8\pi k_e} \frac{V^2 S}{h^2}, \quad (\text{S-2.112})$$

and, if the “+” sign were correct, now the force would be repulsive, although equal in magnitude to (S-2.108)! Of course, this cannot be true, since the plates have opposite charges and attract each other. The error is that the force equals minus the gradient of the potential energy of an *isolated system*, which now includes also the

voltage source. And the voltage source has to do some work to keep the potential difference of the capacitor constant while the capacity is changing. Let us consider an infinitesimal variation of the plate separation,  $dh$  which leads to an infinitesimal variation of the capacity,  $dC$ . The voltage source must move a charge  $dQ = VdC$  across the potential difference  $V$ , in order to keep  $V$  constant. The source thus does a work

$$dW = VdQ = V^2 dC , \quad (\text{S-2.113})$$

and its internal energy (whatever its nature: mechanical, chemical, ...) must change by the amount

$$dU_{\text{source}} = -dW = -V^2 dC . \quad (\text{S-2.114})$$

Since at the same time the electrostatic energy of the capacitor changes by  $\frac{1}{2} V^2 dC$ , the variation of the *total* energy of the isolated system,  $dU_{\text{tot}}$ , is

$$dU_{\text{tot}} = dU_{\text{source}} + dU_{\text{es}} = -V^2 dC + \frac{V^2}{2} dC = -\frac{V^2}{2} dC = -dU_{\text{es}} . \quad (\text{S-2.115})$$

Thus, the force is

$$F = -\partial_h U_{\text{tot}} = +\partial_h U_{\text{es}} = -\frac{V^2 S}{8\pi k_e h^2} , \quad (\text{S-2.116})$$

in agreement with (S-2.108).

### S-2.13 Electrostatic Pressure on a Conducting Sphere

**a)** The surface charge is  $\sigma = Q/S$ , where  $S = 4\pi a^2$  is the surface of the sphere. The electric field at the surface is  $E = 4\pi k_e \sigma$ , so that the pressure is

$$P = \frac{1}{2} \sigma E = 2\pi k_e \sigma^2 = k_e \frac{Q^2}{8\pi a^4} . \quad (\text{S-2.117})$$

**b)** According to Gauss's law, the electric field of the sphere is

$$E(r) = \begin{cases} 0, & r < a, \\ k_e \frac{Q}{r^2}, & r > a, \end{cases} \quad (\text{S-2.118})$$

and thus the electrostatic energy is

$$U_{\text{es}} = \int \frac{1}{8\pi k_e} E^2(\mathbf{r}) d^3 \mathbf{r} = \int_a^\infty \frac{k_e}{8\pi} \left( \frac{Q}{r^2} \right)^2 4\pi r^2 dr = k_e \frac{Q^2}{2a} . \quad (\text{S-2.119})$$

The derivative of  $U_{\text{es}}$  with respect to  $a$ , which has the dimensions of a force, can be interpreted as the integral of the electrostatic pressure over the surface of the sphere. Since the pressure is uniform for symmetry reasons, we can write

$$P = \frac{1}{4\pi a^2} \left( -\frac{dU_{\text{es}}}{da} \right) = \frac{1}{4\pi a^2} \frac{k_e Q^2}{2a^2} = k_e \frac{Q^2}{8\pi a^4}, \quad (\text{S-2.120})$$

in agreement with (S-2.117).

- c) This problem is equivalent to locating a charge  $Q$  on the sphere, such that the potential difference between the sphere and infinity is  $V$ . The problem can also be seen as a spherical capacitor with internal radius  $a$  and external radius  $b$ , potential difference  $V$ , at the limit of  $b$  approaching infinity. The capacity is

$$C = \lim_{b \rightarrow \infty} \frac{1}{k_e} \frac{ab}{b-a} = \frac{a}{k_e}, \quad (\text{S-2.121})$$

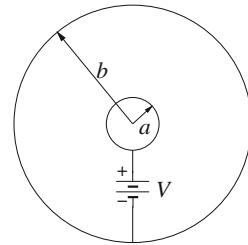


Fig. S-2.22

while the electric potential inside the capacitor is

$$\varphi(r) = \begin{cases} V & (r < a) \\ V \frac{a}{r} & (r > a) \end{cases} \quad (\text{S-2.122})$$

so that the charge on the sphere of radius  $a$  is  $Q = aV/k_e$ . By substituting  $Q$  in (S-2.117) we obtain

$$P = \frac{V^2}{8\pi k_e a^2}. \quad (\text{S-2.123})$$

Alternatively, we can write the electrostatic energy (S-2.119) as a function of  $V$ ,

$$U_{\text{es}} = \frac{1}{2} CV^2 = \frac{aV^2}{2k_e}, \quad (\text{S-2.124})$$

and remember from Problem 2.12 that, if the radius  $a$  is increased by  $da$  at constant voltage, the electrostatic energy of our “capacitor” changes by  $dU_{\text{es}}$ , and, simultaneously, the voltage source does a work  $dW = 2dU_{\text{es}}$ , so that the variation of the “total” energy is

$$dU_{\text{tot}} = dU_{\text{es}} - dW = -dU_{\text{es}}, \quad (\text{S-2.125})$$

and the pressure is

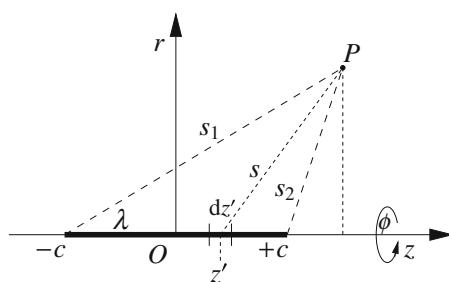
$$P = \frac{1}{4\pi a^2} \left( -\frac{dU_{\text{tot}}}{da} \right) = \frac{1}{4\pi a^2} \left( \frac{dU_{\text{es}}}{da} \right) = \frac{V^2}{8\pi k_e a^2}, \quad (\text{S-2.126})$$

in agreement with (S-2.123).

## S-2.14 Conducting Prolate Ellipsoid

a) Let us consider a line segment of length  $2c$ , of uniform linear electric charge density  $\lambda$ , so that the total charge of the segment is  $Q = 2c\lambda$ . We start using a system of cylindrical coordinates  $(r, \phi, z)$ , such that the end points of the segment have coordinates  $(0, \phi, \pm c)$ , the value of  $\phi$  being irrelevant when  $r = 0$ . The electric potential  $\varphi(P)$  of a generic point  $P$ , of coordinates  $(r, \phi, z)$ , is

$$\varphi(P) = k_e \int_{-c}^{+c} \frac{\lambda dz'}{s} = k_e \lambda \int_{-c}^{+c} \frac{dz'}{\sqrt{(z-z')^2 + r^2}}, \quad (\text{S-2.127})$$



where  $s$  is the distance from  $P$  to the point of the charged segment of coordinate  $z'$ , as shown in Fig. S-2.23. The indefinite integral is

$$\begin{aligned} & \int \frac{dz'}{\sqrt{(z-z')^2 + r^2}} \\ &= -\ln \left[ 2 \sqrt{(z-z')^2 + r^2} + 2z - 2z' \right] \\ &+ C, \end{aligned} \quad (\text{S-2.128})$$

Fig. S-2.23

as can be checked by evaluating the derivative, leading to

$$\varphi(P) = k_e \lambda \ln \left[ \frac{\sqrt{(z+c)^2 + r^2} + z + c}{\sqrt{(z-c)^2 + r^2} + z - c} \right] = k_e \frac{Q}{2c} \ln \left( \frac{s_1 + z + c}{s_2 + z - c} \right), \quad (\text{S-2.129})$$

where  $s_1 = \sqrt{(z+c)^2 + r^2}$  and  $s_2 = \sqrt{(z-c)^2 + r^2}$  are the distances of  $P$  from the end points of the charged line segment, as shown in Fig. S-2.23. We now introduce the elliptic coordinates  $u$  and  $v$

$$u = \frac{s_1 + s_2}{2c}, \quad v = \frac{s_1 - s_2}{2c}, \quad (\text{S-2.130})$$

so that

$$s_1 = c(u+v), \quad s_2 = c(u-v),$$

and

$$uv = \frac{s_1^2 - s_2^2}{4} = \frac{z}{c}. \quad (\text{S-2.131})$$

Because of (S-2.130), we have  $u \geq 1$ , and  $-1 \leq v \leq 1$ . The surfaces  $u = \text{const}$  are confocal ellipsoids of revolution, and the surfaces  $v = \text{const}$  are confocal hyperboloids

of revolution, as shown in Fig. S-2.24. The surface  $u = 1$  is the degenerate case of an ellipsoid with major radius  $a = c$  and minor radius  $b = 0$ , coinciding with segment  $(-c, c)$ . The surface  $v = 0$  is the degenerate case of the plane  $z = 0$ , while  $v = \pm 1$  correspond to the degenerate cases of hyperboloids collapsed to the half-lines  $(c, +\infty)$  and  $(-c, -\infty)$ . In terms of  $u$  and  $v$ , equation (S-2.129) becomes

$$\begin{aligned}\varphi(P) &= k_e \frac{Q}{2c} \ln \left[ \frac{c(u+v) + cuv + c}{c(u-v) + cuv - c} \right] = k_e \frac{Q}{2c} \ln \left[ \frac{(u+1)(v+1)}{(u-1)(v+1)} \right] \\ &= k_e \frac{Q}{2c} \ln \left( \frac{u+1}{u-1} \right),\end{aligned}\quad (\text{S-2.132})$$

Thus, the electric potential depends only on the elliptical coordinate  $u$ , and is constant on the ellipsoidal surfaces  $u = \text{const}$ . The surfaces  $v = \text{const}$  are perpendicular to the equipotential surfaces  $u = \text{const}$ , so that the intersections of the surfaces  $v = \text{const}$  with the planes  $\phi = \text{const}$  (confocal hyperbolae) are the field lines of the electric field. If we let  $u$  approach infinity, i.e., for  $s_1 + s_2 \gg c$ , we have  $s_1 \approx s_2$  and

$$\begin{aligned}\frac{u+1}{u-1} &\approx 1 + \frac{2}{u}, \\ \ln \left( 1 + \frac{2}{u} \right) &\approx \frac{2}{u},\end{aligned}\quad (\text{S-2.133})$$

and

$$\begin{aligned}\lim_{u \rightarrow \infty} \varphi(P) &= k_e \frac{Q}{2c} \frac{2}{u} \\ &= k_e \frac{Q}{cu} \approx k_e \frac{Q}{s_1},\end{aligned}\quad (\text{S-2.134})$$

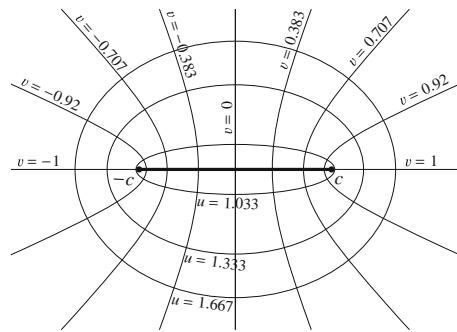


Fig. S-2.24

since  $s_1 \approx s_2$ . This is what expected for a point charge. In other words, the ellipsoidal equipotential surfaces approach spheres as  $u \rightarrow \infty$ .

**b)** For a prolate ellipsoid of revolution of major and minor radii  $a$  and  $b$ , respectively, the distance between the center  $O$  and a focal point,  $c$ , is

$$c = \sqrt{a^2 - b^2}. \quad (\text{S-2.135})$$

At each point of the surface of the ellipsoid we have  $s_1 + s_2 = 2a$ , so that the equation of the surface in elliptic coordinates

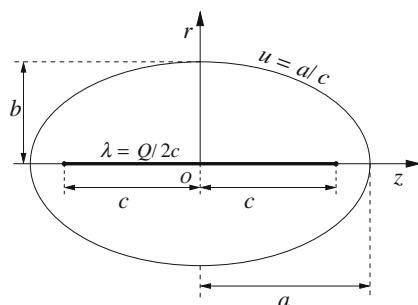


Fig. S-2.25

is  $u = a/c$ . A uniformly charged line segment with end points at  $(0, \phi, -c)$  and  $(0, \phi, c)$ , and linear charge density  $\lambda = Q/(2c)$ , generates a constant electric potential  $\varphi(a, b)$  on the surface of the ellipsoid

$$\varphi(a, b) = k_e \frac{Q}{2c} \ln\left(\frac{u+1}{u-1}\right) = k_e \frac{Q}{2\sqrt{a^2-b^2}} \ln\left(\frac{a+\sqrt{a^2-b^2}}{a-\sqrt{a^2-b^2}}\right). \quad (\text{S-2.136})$$

On the other hand, the potential generated by the charged segment at infinity is zero, and there are no charges between the surface of the ellipsoid and infinity. The flux of the electric field through any closed surface containing the ellipsoid is  $Q$ . Thus, the potential, and the electric field, generated by the charged segment outside the surface of the ellipsoid equal the potential, and the electric field, generated by the conducting ellipsoid carrying a charge  $Q$ , and this solves the problem. The capacity of the ellipsoid is thus

$$C = \frac{Q}{\varphi(a, b)} = \frac{2\sqrt{a^2-b^2}}{k_e} \left[ \ln\left(\frac{a+\sqrt{a^2-b^2}}{a-\sqrt{a^2-b^2}}\right) \right]^{-1}. \quad (\text{S-2.137})$$

The denominator of the argument of the logarithm can be rationalized, leading to

$$\frac{a+\sqrt{a^2-b^2}}{a-\sqrt{a^2-b^2}} = \frac{(a+\sqrt{a^2-b^2})^2}{a-a^2+b^2} = \left( \frac{a+\sqrt{a^2-b^2}}{b} \right)^2 \quad (\text{S-2.138})$$

and the capacity of the prolate ellipsoid can be rewritten

$$C = \frac{\sqrt{a^2-b^2}}{k_e} \left[ \ln\left(\frac{a+\sqrt{a^2-b^2}}{b}\right) \right]^{-1}. \quad (\text{S-2.139})$$

The plates of a confocal ellipsoidal capacitor are the surfaces of two prolate ellipsoids of revolution, sharing the same focal points located at  $\pm c$  on the  $z$  axis, and of major radii  $a_1$  and  $a_2$ , respectively, with  $a_1 < a_2$ . According to (S-2.135) and (S-2.136) the potential on the two plates are

$$\varphi_{1,2} = k_e \frac{Q}{2c} \ln\left(\frac{a_{1,2}+c}{a_{1,2}-c}\right) \quad (\text{S-2.140})$$

so that the capacity is

$$C = \frac{Q}{\varphi_1 - \varphi_2} = \frac{2c}{k_e \ln\left(\frac{a_1+c}{a_1-c} \frac{a_2-c}{a_2+c}\right)} = \frac{2c}{k_e \ln\left(\frac{a_1a_2-c^2+c(a_2-a_1)}{a_1a_2-c^2-c(a_2-a_1)}\right)}. \quad (\text{S-2.141})$$

c) A straight wire of length  $h$  and diameter  $2b$ , with  $h \gg b$ , can be approximated by an ellipsoid prolate in the extreme, with major radius  $a = h/2$  and minor radius  $b$ , with, of course,  $b \ll a$ . From

$$\sqrt{a^2 - b^2} \simeq a - \frac{b^2}{a} \quad \text{valid for } b \ll a, \quad (\text{S-2.142})$$

and (S-2.139) we have

$$C_{\text{wire}} \simeq \frac{a}{2k_e \ln(2a/b)} = \frac{h}{k_e \ln(h/b)}. \quad (\text{S-2.143})$$

# Chapter S-3

## Solutions for Chapter 3

### S-3.1 An Artificial Dielectric

**a)** According to (S-2.6) of Problem 2.1, a metal sphere in a uniform external field  $\mathbf{E}$  acquires a dipole moment

$$\mathbf{p} = \frac{a^3}{k_e} \mathbf{E} = \frac{3}{4\pi k_e} V \mathbf{E}, \quad (\text{S-3.1})$$

where  $V = \frac{4}{3}\pi a^3$  is the volume of the sphere. The polarization of our suspension is

$$\mathbf{P} = n \mathbf{p} = \frac{3n}{4\pi k_e} V \mathbf{E}. \quad (\text{S-3.2})$$

In SI units we have  $\mathbf{P} = \epsilon_0 \chi \mathbf{E}$ , and  $\chi = 3f$ , while in Gaussian units we have  $\mathbf{P} = \chi \mathbf{E}$ , and  $\chi = 3f/(4\pi)$ . In both cases  $f = nV$  is the fraction of the volume occupied by the spheres. Since the minimum distance between the centers of two spheres is  $2a$ , we have

$$f \leq \frac{4\pi a^3}{3} \frac{1}{8a^3} = \frac{\pi}{6}, \quad (\text{S-3.3})$$

leading to  $\chi \leq \pi/2$  in SI units, and  $\chi \leq 1/8$  in Gaussian units.

**b)** The average distance  $\ell$  between two sphere centers is of the order of  $n^{-1/3}$ . The electric field of a dipole at a distance  $\ell$  is of the order of

$$E_{\text{dip}} \simeq k_e \frac{p}{\ell^3} \simeq k_e \frac{a^3}{k_e} E n = a^3 E n. \quad (\text{S-3.4})$$

Thus, the condition  $E_{\text{dip}} \ll E$  requires  $n \ll 1/a^3$ .

### S-3.2 Charge in Front of a Dielectric Half-Space

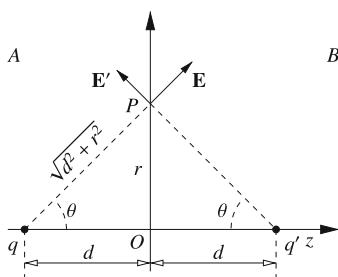


Fig. S-3.1

We denote by  $A$  the  $z < 0$ , vacuum, half-space, containing the real charge  $q$ , and by  $B$  the  $z > 0$ , dielectric, half-space, containing no free charge. We shall treat the two half spaces separately, making educated guesses, in order to apply the uniqueness theorem for the Poisson equation. We use cylindrical coordinates  $(r, \phi, z)$ , with the real charge located at  $(0, \phi, -d)$ . All our formulas will be independent of the azimuthal coordinate  $\phi$ , which is not determined, and not relevant, when  $r = 0$ .

- a)** We treat the field in the half-space  $A$  assuming vacuum in the whole space, including the half-space  $B$ . As ansatz, we locate an image charge  $q'$ , of value to be determined, at  $(0, 0, +d)$ , in the half space that we are not considering, as in Fig. S-3.1. Now we evaluate the electric field  $\mathbf{E}^{(-)}$  in a generic point  $P \equiv (r, \phi, 0^-)$  of the plane  $z = 0^-$ . The distance between  $P$  and  $q$  is  $\sqrt{d^2 + r^2}$  and forms an angle  $\theta = \arccos(d/\sqrt{d^2 + r^2})$  with the  $z$  axis. Also the distance between  $P$  and  $q'$  will be  $\sqrt{d^2 + r^2}$ . The field at  $P$ ,  $\mathbf{E}^{(-)}$ , is the vector sum of the fields  $\mathbf{E}$  due to the real charge  $q$ , and  $\mathbf{E}'$  due to the image charge  $q'$ . The components of  $\mathbf{E}^{(-)}$ , perpendicular and parallel to the  $z = 0$  plane are, respectively

$$\begin{aligned} E_{\perp}^{(-)} &= k_e \frac{q}{d^2 + r^2} \cos \theta - k_e \frac{q'}{d^2 + r^2} \cos \theta = k_e \frac{d}{(d^2 + r^2)^{3/2}} (q - q') \\ E_{\parallel}^{(-)} &= k_e \frac{q}{d^2 + r^2} \sin \theta + k_e \frac{q'}{d^2 + r^2} \sin \theta = k_e \frac{r}{(d^2 + r^2)^{3/2}} (q + q') \end{aligned} \quad . \quad (\text{S-3.5})$$

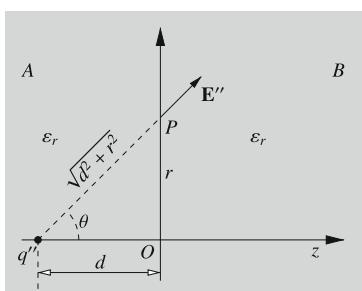


Fig. S-3.2

We treat the half-space  $B$  assuming that the whole space, including the half-space  $A$ , is filled by a dielectric medium of relative permittivity  $\epsilon_r$ . We are not allowed to introduce charges or alter anything in  $B$ , but, as an educated guess, we replace the real charge  $q$ , located in the half-space  $A$  that we are not treating, by a charge  $q''$ , of value to be determined (Fig. S-3.2). We evaluate the field  $\mathbf{E}^{(+)}$  at the same point  $P$  as before, but on the  $z = 0^+$  plane. The components of  $\mathbf{E}^{(+)}$  perpendicular and parallel to the  $z = 0$  plane are

$$\begin{aligned} E_{\perp}^{(+)} &= \frac{k_e}{\epsilon_r} \frac{q''}{d^2 + r^2} \cos \theta = \frac{k_e}{\epsilon_r} \frac{d}{(d^2 + r^2)^{3/2}} q'' \\ E_{\parallel}^{(+)} &= \frac{k_e}{\epsilon_r} \frac{q''}{d^2 + r^2} \sin \theta = \frac{k_e}{\epsilon_r} \frac{r}{(d^2 + r^2)^{3/2}} q''. \end{aligned} \quad (\text{S-3.6})$$

If our educated guesses are correct, the dielectric boundary conditions must hold at  $z = 0$ . This implies  $E_{\perp}^{(-)} = \epsilon_r E_{\perp}^{(+)}$  and  $E_{\parallel}^{(-)} = E_{\parallel}^{(+)}$ , corresponding to the equations

$$q - q' = q'', \quad \text{and} \quad q + q' = \frac{q''}{\epsilon_r}, \quad (\text{S-3.7})$$

with solutions

$$q' = -\frac{\epsilon_r - 1}{\epsilon_r + 1} q, \quad \text{and} \quad q'' = \frac{2\epsilon_r}{\epsilon_r + 1} q. \quad (\text{S-3.8})$$

We can easily check that, at the limit  $\epsilon_r \rightarrow 1$  (vacuum in the whole space), we have  $q' \rightarrow 0$  and  $q'' \rightarrow q$ , i.e., in the whole space we have the field of charge  $q$  only. At the limit  $\epsilon_r \rightarrow \infty$  (dielectric  $\rightarrow$  conductor limit) we have  $q' \rightarrow -q$  and  $q'' \rightarrow 2q$ , i.e., the field of the real charge  $q$  and its image  $-q$  in the half-space  $A$ , and zero field in the half space  $B$ , as at point **a**) of Problem 2.2. The finite value of  $q''$  is irrelevant for the field in the half-space  $B$ , because of the infinite value of  $\epsilon_r$ .

Notice that we can also write equations (S-3.6) without  $\epsilon_r$  in the denominators, thus including the dielectric bound charge into  $q''$ . This leads to the equations

$$q - q' = \epsilon_r q'', \quad \text{and} \quad q + q' = q'' \quad (\text{S-3.9})$$

with solutions

$$q' = -\frac{\epsilon_r - 1}{\epsilon_r + 1} q, \quad \text{and} \quad q'' = \frac{2}{\epsilon_r + 1} q, \quad (\text{S-3.10})$$

which give the same expressions for the electric field as for the choice (S-3.6).

**b)** The polarization charge density on the  $z = 0$  plane,  $\sigma_b(r)$ , is

$$\begin{aligned} \sigma_b(r) &= -\frac{1}{4\pi k_e} (E_{\perp}^{(-)} - E_{\perp}^{(+)}) = -\frac{1}{4\pi} \frac{d}{(d^2 + r^2)^{3/2}} \left( q - q' - \frac{q''}{\epsilon_r} \right) \\ &= -\frac{1}{2\pi} \frac{d}{(d^2 + r^2)^{3/2}} \frac{\epsilon_r - 1}{\epsilon_r + 1} q = \frac{1}{2\pi} \frac{d}{(d^2 + r^2)^{3/2}} q'. \end{aligned} \quad (\text{S-3.11})$$

The total polarization charge on the  $z = 0$  plane is

$$q_p = \int_0^\infty \sigma_b(r) 2\pi r dr = -\frac{\epsilon_r - 1}{\epsilon_r + 1} q \int_0^{\pi/2} \cos \theta d\theta = q', \quad (\text{S-3.12})$$

where we have substituted  $\cos \theta = d / \sqrt{d^2 + r^2}$ ,  $r = d / \cos \theta$  and  $dr = d d\theta / \cos^2 \theta$ .

- c) The polarization charge of the  $z = 0$  plane generates an electric field equal to the field of a charge  $q' = -q(\epsilon_r - 1)/(\epsilon_r + 1)$  located at  $(0, 0, +d)$  in the half space  $z < 0$ , and equal to the field of a charge  $q'$  located at  $(0, 0, -d)$  in the half space  $z > 0$ .

### S-3.3 An Electrically Polarized Sphere

- a) Since the polarization  $\mathbf{P}$  of the sphere is uniform, we have no volume bound-charge density, according to  $\varrho_b = \nabla \cdot \mathbf{P}$ . If we choose a spherical coordinate system  $(r, \theta, \phi)$  with the azimuthal axis parallel to  $\mathbf{P}$ , as shown in Fig. S-3.3, we see that the surface bound-charge density of the sphere is  $\sigma_b = P \cos \theta$ , according to  $\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$ . Thus, in principle, we can evaluate the electric field everywhere in space as the field generated by the bound-charge distribution on the sphere surface.

However, it is easier to consider the polarized sphere as the superposition of two uniformly charged spheres, both of radius  $a$ , one of volume charge density  $\varrho$ , and one of volume charge density  $-\varrho$ . The centers of the two spheres are separated by a small distance  $\delta$ , as in Fig. 1.1 of Problem 1.1. Thus, two initially superposed infinitesimal volume elements  $d^3r$  of the two spheres, of charge  $\pm \varrho d^3r$ , respectively, give origin to an infinitesimal electrical dipole moment  $d\mathbf{p} = \delta \varrho d^3r$  after the displacement.

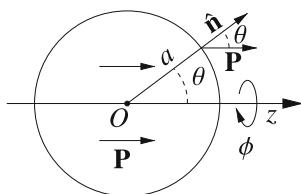


Fig. S-3.3

This corresponds to a polarization  $d\mathbf{p}/d^3r = \varrho \delta$ , we must have  $\mathbf{P} = \varrho \delta$ , and are interested in the limit  $|\delta| \rightarrow 0$ ,  $\rho \rightarrow \infty$ , with  $\varrho \delta = \mathbf{P} = \text{constant}$ . Now we can follow the solution of Problem 1.1. According to (S-1.1), the electric field inside the sphere is uniform and equals

$$\mathbf{E}_{\text{in}} = -\frac{4\pi k_e}{3} \varrho \delta = -\frac{4\pi k_e}{3} \mathbf{P}. \quad (\text{S-3.13})$$

The problem of the field outside the sphere is solved at point b) of Problem 1.1, we have

$$\mathbf{E}_{\text{ext}}(\mathbf{r}) = k_e \frac{3\hat{\mathbf{r}}(\mathbf{p} \cdot \hat{\mathbf{r}}) - \mathbf{p}}{r^3}, \quad (\text{S-3.14})$$

where  $\mathbf{p} = \mathbf{P}(4\pi a^3/3)$  is the total dipole moment of the sphere.

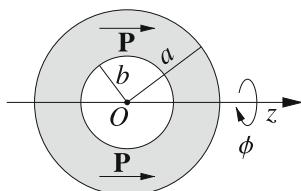


Fig. S-3.4

- b) The problem can be solved by the superposition principle. The hole of radius  $b$  can be regarded as a sphere of uniform electrical polarization  $-\mathbf{P}$  superposed to the sphere of radius  $a$  and polarization  $\mathbf{P}$ . The sphere of radius  $b$  generates a field

$$\mathbf{E}_{\text{in}}^{(b)} = \frac{4\pi k_e}{3} \mathbf{P} \quad (\text{S-3.15})$$

at its interior. Thus, the total field inside the spherical hole is  $\mathbf{E}_{\text{in}}^{(a+b)} = 0$ . The field inside the spherical shell  $b < r < a$  is the sum of the uniform field (S-3.13) and the field generated by an electric dipole of moment  $\mathbf{p}^{(b)}$  located at the center  $O$ , with

$$\mathbf{p}^{(b)} = -\frac{4\pi b^3}{3} \mathbf{P}. \quad (\text{S-3.16})$$

Finally, the external field ( $r > a$ ) equals the field generated by a single dipole  $\mathbf{p}^{(a+b)}$  located in  $O$  with

$$\mathbf{p}^{(a+b)} = \frac{4\pi(a^3 - b^3)}{3} \mathbf{P}. \quad (\text{S-3.17})$$

### S-3.4 Dielectric Sphere in an External Field

a) As an educated guess. Let us assume that the external field induces a uniform electric polarization  $\mathbf{P}$  in the sphere. We have seen in Problem 3.3 that a sphere of uniform

electric polarization  $\mathbf{P}$  generates a uniform electric field  $\mathbf{E}^{\text{pol}} = -(4\pi k_e/3)\mathbf{P}$  at its interior. The difference is that in the present case  $\mathbf{P}$  is not permanent but it is induced by the local electric field, and

$$\mathbf{P} = \frac{\epsilon_r - 1}{4\pi k_e} \mathbf{E}^{\text{diel}}, \quad (\text{S-3.18})$$

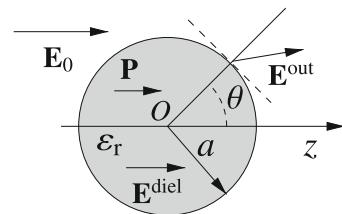


Fig. S-3.5

where  $\mathbf{E}^{\text{diel}}$  is the field inside the dielectric sphere, which is the sum of the external and the induced fields:

$$\mathbf{E}^{\text{diel}} = \mathbf{E}_0 + \mathbf{E}^{\text{pol}}. \quad (\text{S-3.19})$$

We thus have

$$\mathbf{E}^{\text{diel}} = \mathbf{E}_0 - \frac{4\pi k_e}{3} \mathbf{P} = \mathbf{E}_0 - \frac{\epsilon_r - 1}{3} \mathbf{E}^{\text{diel}}, \quad (\text{S-3.20})$$

which can be solved for  $\mathbf{E}^{\text{diel}}$ :

$$\mathbf{E}^{\text{diel}} = \frac{3}{\epsilon_r + 2} \mathbf{E}_0. \quad (\text{S-3.21})$$

Since  $\epsilon_r > 1$ , the field inside the dielectric sphere is smaller than  $\mathbf{E}_0$ .

The electric field outside the sphere  $\mathbf{E}^{\text{out}}$  will be given by the sum of  $\mathbf{E}_0$  and the field of a dipole

$$\mathbf{p} = \frac{4\pi a^3}{3} \mathbf{P} = \frac{a^3}{3k_e} \frac{\epsilon_r - 1}{\epsilon_r + 2} \mathbf{E}_0 \quad (\text{S-3.22})$$

located at the center of the of the sphere. Thus

$$\mathbf{E}^{\text{out}} = \mathbf{E}_0 + k_e \frac{3(\mathbf{p} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}}{r^3} = \mathbf{E}_0 + \frac{a^3}{3k_e} \frac{\epsilon_r - 1}{\epsilon_r + 2} [3(\mathbf{E}_0 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{E}_0]. \quad (\text{S-3.23})$$

It is instructive, and useful for the following, to check that the above solution satisfies the boundary conditions at the surface of the sphere. Let us then restart the problem by assuming that the field  $\mathbf{E}^{\text{diel}}$  inside the sphere ( $r < a$ ) is uniform and parallel to the external field  $\mathbf{E}_0$ , and that the field  $\mathbf{E}^{\text{out}}$  outside the sphere ( $r > a$ ) is the sum of the external field and that of a dipole  $\mathbf{p}$  located at the center of the sphere and also parallel to  $\mathbf{E}_0$ . Thus we can write  $\mathbf{E}^{\text{diel}} = \alpha \mathbf{E}_0$  and  $\mathbf{p} = \eta \mathbf{E}_0$ , with the constants  $\alpha$  and  $\eta$  to be determined by the boundary conditions at  $r = a$ . Choosing a spherical coordinate system with the origin  $O$  at the center of the sphere, and the polar axis  $z$  parallel to  $\mathbf{E}_0$ , we have

$$\begin{aligned} E_r^{\text{diel}} &= \alpha E_0 \cos \theta & E_r^{\text{out}} &= E_0 \cos \theta + k_e \eta E_0 \frac{2 \cos \theta}{r^3} \\ E_\theta^{\text{diel}} &= \alpha E_0 \sin \theta & E_\theta^{\text{out}} &= E_0 \sin \theta - k_e \eta E_0 \frac{\sin \theta}{r^3} \\ E_\phi^{\text{diel}} &= 0 & E_\phi^{\text{out}} &= 0. \end{aligned} \quad (\text{S-3.24})$$

The boundary conditions at the surface of the sphere are  $\epsilon_r E_{\perp}^{\text{diel}} = E_{\perp}^{\text{out}}$  and  $E_{\parallel}^{\text{diel}} = E_{\parallel}^{\text{out}}$  which yields, in spherical coordinates,

$$\epsilon_r E_r^{\text{diel}}(r = a^-) = E_r^{\text{out}}(r = a^+) \quad E_\theta^{\text{diel}}(r = a^-) = E_\theta^{\text{out}}(r = a^+). \quad (\text{S-3.25})$$

Using (S-3.24) and deleting the common factors we obtain

$$\epsilon_r \alpha = 1 + \frac{2k_e}{a^3} \eta, \quad \alpha = 1 - \frac{k_e}{a^3} \eta, \quad (\text{S-3.26})$$

whose solutions for  $\alpha$  and  $\eta$  are

$$\alpha = 3/(\epsilon_r + 2), \quad \eta = \frac{a^3}{k_e} \frac{\epsilon_r - 1}{\epsilon_r + 2}, \quad (\text{S-3.27})$$

and we eventually recover (S-3.21) and (S-3.22).

**b)** We make an educated guess analogous to the one of the previous point, i.e., we assume that the field inside the cavity,  $\mathbf{E}^{\text{cav}}$ , is uniform and parallel to  $\mathbf{E}_d$ , and that the field in the dielectric medium,  $\mathbf{E}^{\text{diel}}$ , is the sum of  $\mathbf{E}_d$  and the field of an electric dipole  $\mathbf{p}_c$ , located at the center of the cavity and parallel to  $\mathbf{E}_d$ . Thus we can write

$$\mathbf{E}^{\text{cav}} = \alpha \mathbf{E}_d, \quad \mathbf{p}_c = \eta \mathbf{E}_d; \quad (\text{S-3.28})$$

where, again,  $\alpha$  and  $\eta$  are constants to be determined by the boundary conditions. Using again spherical coordinates with the origin at the center of the spherical cavity and the  $z$  axis parallel to  $\mathbf{E}_d$ , the expressions analogous to (S-3.24) are

$$\begin{aligned} E_r^{\text{cav}} &= \alpha E_d \cos \theta & E_r^{\text{med}} &= E_d \cos \theta + k_e \eta E_d \frac{2 \cos \theta}{r^3} \\ E_\theta^{\text{cav}} &= \alpha E_d \sin \theta & E_\theta^{\text{med}} &= E_d \sin \theta - k_e \eta E_d \frac{\sin \theta}{r^3} \\ E_\phi^{\text{cav}} &= 0 & E_\phi^{\text{med}} &= 0, \end{aligned} \quad (\text{S-3.29})$$

with the boundary conditions

$$E_r^{\text{cav}}(r = a^-) = \varepsilon_r E_r^{\text{med}}(r = a^+) \quad E_\theta^{\text{cav}}(r = a^-) = E_\theta^{\text{med}}(r = a^+). \quad (\text{S-3.30})$$

The values for  $\alpha$  and  $\eta$  may thus be easily obtained by solving a linear system of two equations as in point **a**). However, we can immediately obtain the solution by noticing that (S-3.29) and (S-3.30) are identical to (S-3.24) and (S-3.25) but for the replacements  $E_d \leftrightarrow E_0$ ,  $E^{\text{cav}} \leftrightarrow E^{\text{diel}}$ ,  $E^{\text{med}} \leftrightarrow E^{\text{out}}$ , and  $\varepsilon_r \leftrightarrow 1/\varepsilon_r$ . Thus, the solutions for  $\mathbf{E}^{\text{cav}}$  and  $\mathbf{p}_c$  are obtained from those for  $\mathbf{E}^{\text{diel}}$  and  $\mathbf{p}$ , (S-3.21) and (S-3.22), by substituting  $E_d$  for  $E_0$  and  $1/\varepsilon_r$  for  $\varepsilon_r$ :

$$\mathbf{E}^{\text{cav}} = \frac{3}{1/\varepsilon_r + 2} \mathbf{E}_d = \frac{3\varepsilon_r}{1 + 2\varepsilon_r} \mathbf{E}_d, \quad (\text{S-3.31})$$

$$\mathbf{p}_c = \frac{a^3}{3k_e} \frac{1/\varepsilon_r - 1}{1/\varepsilon_r + 2} \mathbf{E}_d = \frac{a^3}{3k_e} \frac{1 - \varepsilon_r}{1 + 2\varepsilon_r} \mathbf{E}_d. \quad (\text{S-3.32})$$

Thus  $E^{\text{cav}} > E_d$ , i.e. the field inside the cavity is stronger than that outside it, and  $\mathbf{p}_c$  is antiparallel to  $\mathbf{E}_d$ .

### S-3.5 Refraction of the Electric Field at a Dielectric Boundary

**a)** First, we note that the electric field  $\mathbf{E}_0$  outside the dielectric slab equals the field that we would have in vacuum in the absence of the slab. Neglecting the boundary effects, the bound surface charge densities of slab are analogous to the surface charge densities of a parallel-plate capacitor. These generate a uniform electric field inside the capacitor, but no field outside. Thus, the electric field inside the slab is the sum of  $\mathbf{E}_0$  and the field generated by the surface polarization charge densities. If we denote by  $\mathbf{E}'$  the

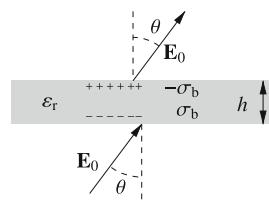
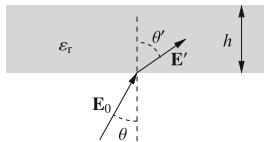


Fig. S-3.6

internal electric field, the boundary conditions at the dielectric surfaces are

$$E_{0\perp} = \epsilon_r E'_{\perp}, \quad E_{0\parallel} = \epsilon_r E'_{\parallel}, \quad (\text{S-3.33})$$

where the subscripts  $\perp$  and  $\parallel$  denote the field components perpendicular and parallel to the interface surface, respectively. In terms of the angles  $\theta$  and  $\theta'$  of Fig. S-3.6



we have

$$\begin{aligned} E_0 \cos \theta &= \epsilon_r E' \cos \theta' \\ E_0 \sin \theta &= E' \sin \theta'. \end{aligned} \quad (\text{S-3.34})$$

If we divide the second of (S-3.34) by the first we obtain

$$\frac{1}{\epsilon_r} \tan \theta' = \tan \theta, \quad (\text{S-3.35})$$

and, since  $\epsilon_r > 1$ , we have  $\theta' > \theta$ .

**b)** From Gauss's law we obtain

$$\sigma_b = \frac{1}{4\pi k_e} (E_{0,\perp} - E'_{\perp}) = \frac{1}{4\pi k_e} E_0 \cos \theta \left( 1 - \frac{1}{\epsilon_r} \right). \quad (\text{S-3.36})$$

**c)** The electrostatic energy density inside the slab is

$$\begin{aligned} u_{es} &= \frac{\epsilon_r}{8\pi k_e} \mathbf{E}'^2 = \frac{\epsilon_r}{8\pi k_e} (E'_{\perp}^2 + E'_{\parallel}^2) = \frac{\epsilon_r}{8\pi k_e} E_0^2 \left( \frac{\cos^2 \theta}{\epsilon_r^2} + \sin^2 \theta \right) \\ &= \frac{1}{8\pi k_e \epsilon_r} E_0^2 [(\epsilon_r^2 - 1) \sin^2 \theta + 1], \end{aligned} \quad (\text{S-3.37})$$

so that  $u_{es}$  increases with increasing  $\theta$ , and we expect a torque  $\tau$  tending to rotate the slab toward the angle of minimum energy, i.e.,  $\theta = 0$ . Neglecting boundary effects, the total electrostatic energy of the slab is  $U_{es} = V u_{es}$ , where  $V$  is the volume of the slab, and the torque exerted by the electric field is

$$\tau = -\frac{\partial U_{es}}{\partial \theta} = -\frac{1}{8\pi k_e \epsilon_r} E_0^2 V (\epsilon_r^2 - 1) \sin 2\theta < 0. \quad (\text{S-3.38})$$

## S-3.6 Contact Force between a Conducting Slab and a Dielectric Half-Space

a) Neglecting boundary effects at the edges of the slab, the electric field is parallel to the  $x$  axis in all the regions of interest because of symmetry reasons. Thus, we can omit the vector notation, and we shall use positive numbers for vectors whose unit vector is  $\hat{x}$ , negative numbers otherwise.

According to Gauss's law, a uniformly charged plane with surface charge density  $\sigma_a$  generates uniform fields at both its sides, of intensities  $E_a = \pm\sigma_a/2\epsilon_0$ , respectively. In our problem we have three charged parallel plane surfaces: we denote by  $\sigma_1$  the surface charge density on the left surface of the slab, by  $\sigma_2$  the density on its right surface, and by  $\sigma_b$  the *bound* surface charge density of the dielectric material on its surface, as shown in Fig. S-3.8. Since the total *free* charge on the slab is  $Q$ , we have

$$\sigma_1 + \sigma_2 = \frac{Q}{S} = \sigma_{\text{tot}} . \quad (\text{S-3.39})$$

At any point in space the total electric field is the sum of the fields generated by the three surface charges. Now, the electric field must be zero inside the conducting slab. Thus the sum of all surface charge densities (including *both* free and bound charges) at the left of the slab must equal the sum of all surface charge densities at the right, so that their respective fields cancel out inside the slab. This conclusion holds both when the slab is in contact with the dielectric, and when there is a vacuum gap between them. Thus, we have

$$\sigma_1 = \sigma_2 + \sigma_b . \quad (\text{S-3.40})$$

The electric field  $E_d$  inside the dielectric medium is  $E_d = 4\pi k_e(\sigma_2 + \sigma_b)$ . This implies for the dielectric polarization of the medium  $P$

$$P = \frac{\epsilon_r - 1}{4\pi k_e} E_d = (\epsilon_r - 1)\sigma_1 . \quad (\text{S-3.41})$$

Since we also have  $\sigma_b = -\mathbf{P} \cdot \hat{x} = -P$ , we obtain the additional relation

$$\sigma_b = -(\epsilon_r - 1)(\sigma_2 + \sigma_b) , \quad (\text{S-3.42})$$

that leads to

$$\sigma_b = -\frac{\epsilon_r - 1}{\epsilon_r} \sigma_2 . \quad (\text{S-3.43})$$

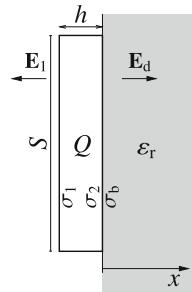


Fig. S-3.8

From (S-3.39), (S-3.40) and (S-3.43) we finally obtain

$$\sigma_1 = \frac{1}{\varepsilon_r + 1} \sigma_{\text{tot}}, \quad \sigma_2 = \frac{\varepsilon_r}{\varepsilon_r + 1} \sigma_{\text{tot}}, \quad \sigma_b = -\frac{\varepsilon_r - 1}{\varepsilon_r + 1} \sigma_{\text{tot}}. \quad (\text{S-3.44})$$

The magnitudes of the electric field at the left of the slab  $E_1$ , and of the electric field inside the dielectric medium  $E_d$ , can be evaluated from Gauss's law, recalling that the field is zero inside the slab. We have

$$E_1 = -4\pi k_e \sigma_1 = -\frac{4\pi k_e}{\varepsilon_r + 1} \sigma_{\text{tot}} \quad \text{and} \quad E_d = 4\pi k_e (\sigma_2 + \sigma_b) = -E_1. \quad (\text{S-3.45})$$

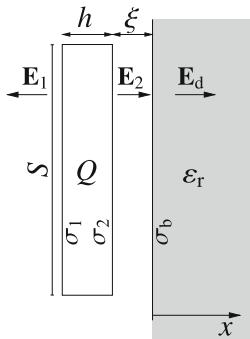
In the case of a vacuum gap between the conducting slab and the dielectric medium, as shown in Fig. S-3.9, the field  $E_2$  in the gap is

$$E_2 = 4\pi k_e \sigma_2 = 4\pi k_e \frac{\varepsilon_r}{\varepsilon_r + 1} \sigma_{\text{tot}} = -\varepsilon_r E_1. \quad (\text{S-3.46})$$

The values of  $E_1$  and  $E_d$  are not affected by the presence of the vacuum gap.

As an alternative approach we can assume, following Problem 3.2, that the free charge layers  $\sigma_1$  and  $\sigma_2$  induce image charge layers  $\sigma'_1$  and  $\sigma'_2$  in the dielectric,

$$\sigma'_1 = -\frac{\varepsilon_r - 1}{\varepsilon_r + 1} \sigma_1, \quad \sigma'_2 = -\frac{\varepsilon_r - 1}{\varepsilon_r + 1} \sigma_2, \quad (\text{S-3.47})$$



with the image planes located in position symmetrical with respect to the dielectric surface. Due to Gauss's law the bound surface charge density is the sum of the image charge densities,

$$\sigma_p = \sigma'_1 + \sigma'_2 = -\frac{\varepsilon_r - 1}{\varepsilon_r + 1} \frac{Q}{S}. \quad (\text{S-3.48})$$

The free charge densities can be now found by requiring the field to vanish inside the slab: omitting a common multiplying factor we have

$$0 = \sigma_1 - \sigma_2 - \sigma'_1 - \sigma'_2 = 2 \frac{\varepsilon_r \sigma_1 - \sigma_2}{\varepsilon_r + 1}, \quad (\text{S-3.49})$$

from which we obtain  $\varepsilon_r \sigma_1 = \sigma_2$ , and we eventually recover the free and bound surface charge densities of (S-3.44).

**b)** In order to evaluate the electrostatic force acting on the conducting slab, we first assume the presence of a small vacuum gap of width  $\xi$  between the slab and the dielectric medium, as shown in Fig. S-3.9.

We can evaluate the total electrostatic force  $F$  acting on the conducting slab in three equivalent ways:

- (i) We can evaluate the variation of the total electrostatic energy  $U_{\text{es}}$  when the slab is displaced by an infinitesimal amount  $dx$  toward the right, thus decreasing the gap. In this case  $U_{\text{es}}$  increases by  $E_1^2 S dx / (8\pi k_e)$  at the left of the slab, because the width of region “filled” by the field  $E_1$  increases by  $dx$ , and correspondingly decreases by  $E_2^2 S dx / (8\pi k_e)$  at its right. Thus,  $dU_{\text{es}} = (E_1^2 - E_2^2) S dx / (8\pi k_e)$ , from which we obtain the force per unit surface

$$\begin{aligned} F &= -\frac{dU_{\text{es}}}{dx} = \frac{S}{8\pi k_e} (E_1^2 - E_2^2) = \frac{S}{8\pi k_e} (\varepsilon_r^2 - 1) \left( \frac{4\pi k_e}{\varepsilon_r + 1} \right)^2 \left( \frac{Q}{S} \right)^2 \\ &= 2\pi k_e \frac{\varepsilon_r - 1}{\varepsilon_r + 1} \frac{Q^2}{S}. \end{aligned} \quad (\text{S-3.50})$$

We have  $F > 0$ , meaning that the slab is attracted by the dielectric medium.

- (ii) We can multiply the charge of the slab  $Q$  by the *local* field, i.e., by the field generated by all charges excluding the charges of the slab. In our case the local field is the field  $E_p$  generated by the bound surface charge density  $\sigma_b$ . We have

$$E_p = -2\pi k_e \sigma_b = 2\pi k_e \frac{\varepsilon_r - 1}{\varepsilon_r + 1} \sigma_{\text{tot}}, \quad \text{and} \quad F = 2\pi k_e \frac{\varepsilon_r - 1}{\varepsilon_r + 1} \frac{Q^2}{S}. \quad (\text{S-3.51})$$

- (iii) We can evaluate the force on the slab by summing the forces  $F_1$  on its left and  $F_2$  on its right surface. These are obtained by multiplying the respective charges  $Q_1 = S\sigma_1$  and  $Q_2 = S\sigma_2$  by the average fields at the surfaces

$$\begin{aligned} F &= F_1 + F_2 = Q_1 \frac{E_1}{2} + Q_2 \frac{E_2}{2} = -\frac{Q}{\varepsilon_r + 1} \frac{2\pi k_e}{\varepsilon_r + 1} \sigma_{\text{tot}} + \frac{\varepsilon_r Q}{\varepsilon_r + 1} \frac{\varepsilon_r 2\pi k_e}{\varepsilon_r + 1} \sigma_{\text{tot}} \\ &= 2\pi k_e \frac{\varepsilon_r^2 - 1}{(\varepsilon_r + 1)^2} \frac{Q^2}{S} = 2\pi k_e \frac{\varepsilon_r - 1}{\varepsilon_r + 1} \frac{Q^2}{S}. \end{aligned} \quad (\text{S-3.52})$$

The force  $F$  is independent of  $\xi$ , thus the above result should be valid also at the limit  $\xi \rightarrow 0$ , i.e., when there is contact between the metal slab and the dielectric. One may argue, however, that in these conditions the field at  $x = 0^+$ , i.e., at the right of the slab, is given by  $E_d = -E_1$ , so that following the approach (iii) one would write

$$F = F_1 + F_2 \stackrel{?}{=} Q_1 \frac{E_1}{2} + Q_2 \frac{E_d}{2} \neq Q_1 \frac{E_1}{2} + Q_2 \frac{E_2}{2}. \quad (\text{S-3.53})$$

This discrepancy comes out because actually the average field on the *free* charges located on the right surface of the slab is *not*  $E_d/2$ , which is the average field across the two merging layers of free and bound charges; however, the force on the slab must be calculated by taking the average field on free charges only.

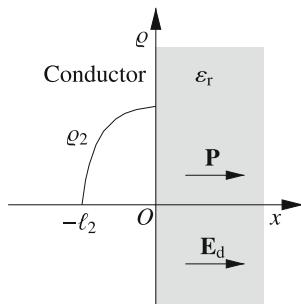


Fig. S-3.10

To illustrate this issue, let us assume for a moment the free charges at the slab surfaces to be distributed in a layer of small but finite width, so that we can localize exactly where free charges are without merging them with bound charges. In particular, let the free surface charge layer have thickness  $\ell_2$  and volume charge density  $\varrho_2(x)$  such that

$$\int_{-\ell_2}^0 \varrho_2 dx = \sigma_2, \quad (\text{S-3.54})$$

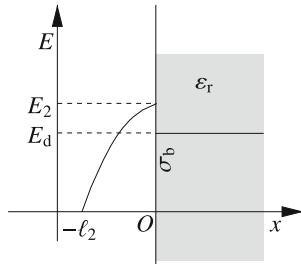


Fig. S-3.11

as shown in Fig. S-3.10. The electric field is still directed along the  $x$  axis for symmetry reasons. Gauss's law in one dimension gives  $\partial_x E = 4\pi k_e \varrho$ . Since  $E(-\ell_2) = 0$  (as deep into the conductor the field should vanish) we have for the electric field in the  $-\ell_2 \leq x \leq 0$  region

$$E(x) = 4\pi k_e \int_{-\ell_2}^x \varrho_2(x') dx'. \quad (\text{S-3.55})$$

The total force *on the free charges only* can thus be evaluated as

$$\begin{aligned} F_2 &= S \int_{-\ell_2}^0 E(x) \varrho_2(x) dx \\ &= \frac{S}{4\pi k_e} \int_{-\ell_2}^0 E(x) \partial_x E(x) dx = \frac{S}{8\pi k_e} \int_{-\ell_2}^0 \partial_x E^2(x) dx \\ &= \frac{S}{8\pi k_e} E_2^2 = \frac{S(4\pi k_e \sigma_2)^2}{8\pi k_e} = 2\pi k_e S \sigma_2^2, \end{aligned} \quad (\text{S-3.56})$$

the electric field at  $x = 0^-$  is  $E_2$ , as shown in Fig. S-3.11, and the resulting electrostatic pressure is  $p_2 = F_2/S = 2\pi k_e \sigma_2^2$ , independent of the particular distribution  $\varrho_2(x)$ , and in agreement with the previous result (S-3.52). However, the electric field at  $x = 0^+$  is  $E_d$  because of the presence of the surface bound charge.<sup>1</sup>

c) If the dielectric medium is actually a slab limited at  $x = w$ , as shown in Fig. S-3.12, a further bound surface charge density  $-\sigma_b$ , opposite to the density  $\sigma_b$  at  $x = 0$ , appears at its  $x = w$  surface. This charge distribution is identical to that of a plane capacitor, so that the bound charges generate *no* field outside the dielectric

<sup>1</sup>We might assume that also the polarization charge fills a layer of small, but finite width  $\ell_d$  at the surface of the dielectric. However, this would only imply that the field becomes  $E_d$  at  $x \geq \ell_d$ , and would not affect our conclusions on the forces on the conductor.

slab. As trivial consequences the surface charge densities on the conducting slab are  $\sigma_1 = -\sigma_2 = Q/2S$ , the field inside the dielectric is  $E_1/\epsilon_r = Q/(2\epsilon_0\epsilon_r S)$ , and there is no force between the slab and the dielectric. Moreover, this result is independent of  $w$ , and therefore should be valid also in the limit  $w \rightarrow \infty$ .

The apparent contradiction with the results of points **a**) and **b**) is that, in two attempts to approximate real conditions by objects of infinite size, we are assuming different boundary conditions at infinity. To discuss this issue let us look again at Fig. 3.5, showing the slab of charge  $Q$  located in front of a dielectric hemisphere of radius  $R$ . At the limit  $R \rightarrow \infty$ , the field in the dielectric half-space approaches the field that we would have if the dielectric medium filled the whole space, and the surface  $S$  had surface charge density  $\sigma'' = 2\epsilon_r\sigma/(\epsilon_r + 1)$ , see Problem 3.2. Thus, the field, the polarization, and the polarization surface charge density all approach zero at the hemispherical surface. Part b) of Fig. 3.5 is an enlargement of the area enclosed in the dashed rectangle of part a) of the same figure, and the vanishing charge density on the hemisphere surface does not contribute to the field in this area, according to the result of Problem 1.11. This motivates the boundary condition assumed in points **a**) and **b**). In contrast, in point c) the bound surface charge density does not vanish at infinity and generates a uniform field, which in vacuum cancels out the field generated by the dielectric surface at  $x = 0$ .

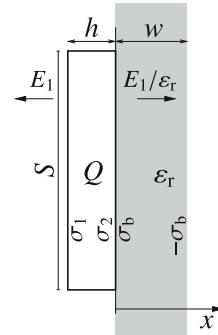


Fig. S-3.12

## S-3.7 A Conducting Sphere between two Dielectrics

**a)** We use a spherical coordinate system  $(r, \theta, \phi)$  with the origin  $O$  at the center of the sphere, and the zenith direction perpendicular to the plane separating the two dielectric media, as shown in Fig. S-3.13. The electric field inside the conducting sphere is zero. The electric field outside the sphere,  $\mathbf{E}(r, \theta, \phi)$ , is independent of  $\phi$  because of the symmetry of our problem. Since the sphere is conducting, the electric field  $\mathbf{E}(R^+, \theta, \phi)$  must be perpendicular to its surface, and its only nonzero component is  $E_r$ . If we write Maxwell's equation  $\nabla \times \mathbf{E} = 0$  in spherical coordinates over the spherical surface  $r = R^+$  (see Table A.1 of the Appendix), we see that the  $r$  and  $\theta$  components of the curl are automatically zero because  $E_\phi = 0$ ,  $E_\theta = 0$ , and all derivatives with respect to  $\phi$  are zero. The condition that also the  $\phi$  component of the curl must be zero is

$$\partial_\theta E_r = \partial_r(rE_\theta) = E_\theta + r\partial_r E_\theta . \quad (\text{S-3.57})$$

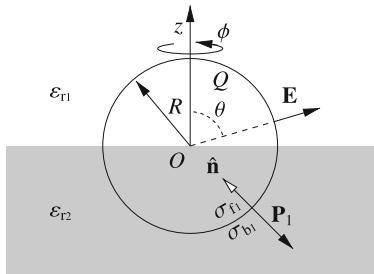


Fig. S-3.13

The right-hand side of (S-3.57) is zero because  $E_\theta(R^-) = E_\theta(R^+) = 0$ , implying that also  $\partial_r E_\theta(R) = 0$ . Thus,  $\partial_\theta E_r(R^+) = 0$ , and  $E_r(R^+)$  does not depend on  $\theta$  (and, consequently, on the dielectric medium). If we denote by  $\sigma_{\text{tot}}$  the sum of the free surface charge density  $\sigma_f$  and the bound charge density  $\sigma_b$ , the relation

$$\sigma_{\text{tot}} = \frac{E(R^+, \theta, \phi)}{4\pi k_e} = \frac{E(R^+)}{4\pi k_e} \quad (\text{S-3.58})$$

shows that  $\sigma_{\text{tot}}$  is constant over the whole surface of the sphere. Thus the electric field in the whole space outside the sphere equals the field of a point charge  $Q_{\text{tot}}$  located in  $O$ , with  $Q_{\text{tot}} = Q + Q_b$ , where  $Q_b$  is the total polarization (bound) charge:

$$\mathbf{E}(r, \theta, \phi) = \mathbf{E}(r) = 4\pi k_e \sigma_{\text{tot}} \frac{R^2}{r^2} \hat{\mathbf{r}}, \quad r > R, \quad (\text{S-3.59})$$

since the field depends on  $r$  only. The polarization charge densities on the surfaces of the two dielectrics in contact with the sphere are, respectively,

$$\begin{aligned} \sigma_{b1} &= \hat{\mathbf{n}} \cdot \mathbf{P}_1 = -\frac{\epsilon_{r1} - 1}{4\pi k_e} E(R) = -(\epsilon_{r1} - 1) \sigma_{\text{tot}} \\ \sigma_{b2} &= \hat{\mathbf{n}} \cdot \mathbf{P}_2 = -\frac{\epsilon_{r2} - 1}{4\pi k_e} E(R) = -(\epsilon_{r2} - 1) \sigma_{\text{tot}} \end{aligned} \quad (\text{S-3.60})$$

where the unit vector  $\hat{\mathbf{n}}$  points toward the center of the sphere. The free surface charge densities in the regions in contact with the two dielectrics,  $\sigma_{f1}$  and  $\sigma_{f2}$ , are, respectively,

$$\begin{aligned} \sigma_{f1} &= \sigma_{\text{tot}} - \sigma_{b1} = \epsilon_{r1} \sigma_{\text{tot}} \\ \sigma_{f2} &= \sigma_{\text{tot}} - \sigma_{b2} = \epsilon_{r2} \sigma_{\text{tot}}. \end{aligned} \quad (\text{S-3.61})$$

Since  $2\pi R^2(\sigma_{f1} + \sigma_{f2}) = Q$ , we finally obtain

$$\begin{aligned} \sigma_{\text{tot}} &= \frac{Q}{2\pi R^2(\epsilon_{r1} + \epsilon_{r2})} & \mathbf{E}(r) &= 2k_e \frac{Q}{(\epsilon_{r1} + \epsilon_{r2}) r^2} \hat{\mathbf{r}} \\ \sigma_{f1} &= \frac{\epsilon_{r1} Q}{2\pi R^2(\epsilon_{r1} + \epsilon_{r2})} & \sigma_{f2} &= \frac{\epsilon_{r2} Q}{2\pi R^2(\epsilon_{r1} + \epsilon_{r2})} \\ \sigma_{b1} &= -\frac{(\epsilon_{r1} - 1) Q}{2\pi R^2(\epsilon_{r1} + \epsilon_{r2})} & \sigma_{b2} &= -\frac{(\epsilon_{r2} - 1) Q}{2\pi R^2(\epsilon_{r1} + \epsilon_{r2})}. \end{aligned} \quad (\text{S-3.62})$$

**b)** The electrostatic pressures on the two hemispherical surfaces equal the electrostatic energy densities in the corresponding dielectric media, and are, respectively,

$$\begin{aligned}\mathcal{P}_{fi} &= \frac{2\pi k_e}{\epsilon_{ri}} \sigma_{fi}^2 = \frac{2\pi k_e}{\epsilon_{ri}} \left[ \frac{\epsilon_{ri} Q}{2\pi R^2 (\epsilon_{r1} + \epsilon_{r2})} \right]^2 \\ &= k_e \frac{\epsilon_{ri} Q^2}{2\pi R^4 (\epsilon_{r1} + \epsilon_{r2})^2} \quad (\text{S-3.63})\end{aligned}$$

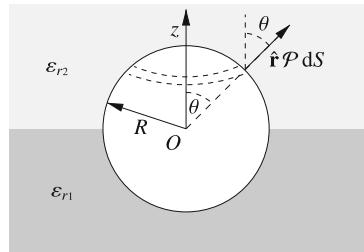


Fig. S-3.14

with  $i = 1, 2$ . Thus  $\mathcal{P}_1 > \mathcal{P}_2$  because  $\epsilon_{r1} > \epsilon_{r2}$ ,

and the pressure pushes the sphere towards the medium of higher permittivity. The force on the sphere surface element  $dS = R^2 \sin \theta d\theta d\phi$  is  $dF = \hat{r} \mathcal{P}_{fi} dS$ , with  $i = 1$  if  $\theta > \pi/2$ , and  $i = 2$  if  $\theta < \pi/2$ . The total force acting on the upper hemisphere ( $\theta < \pi/2$ ) is thus

$$\begin{aligned}\mathbf{F}_2 &= \hat{\mathbf{z}} \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta R^2 \sin \theta \cos \theta k_e \frac{\epsilon_{r2} Q^2}{2\pi(\epsilon_{r1} + \epsilon_{r2})^2 R^4} \\ &= \hat{\mathbf{z}} \pi R^2 k_e \frac{\epsilon_{r2} Q^2}{2\pi(\epsilon_{r1} + \epsilon_{r2})^2 R^4} = \hat{\mathbf{z}} \pi R^2 \mathcal{P}_2 = \hat{\mathbf{z}} k_e \frac{\epsilon_{r2} Q^2}{2(\epsilon_{r1} + \epsilon_{r2})^2 R^2}, \quad (\text{S-3.64})\end{aligned}$$

directed upwards, since the force components perpendicular to the  $z$  axis cancel out. Note that  $\mathbf{F}_2$  simply equals  $\mathcal{P}_{f2}$  times the section of the sphere  $\pi R^2$ . The total force acting on the lower hemisphere ( $\theta > \pi/2$ ) is, analogously,

$$\mathbf{F}_1 = -\hat{\mathbf{z}} k_e \frac{\epsilon_{r1} Q^2}{2(\epsilon_{r1} + \epsilon_{r2})^2 R^2}. \quad (\text{S-3.65})$$

The total electrostatic force acting on the conducting sphere is thus

$$\mathbf{F}_{\text{tot}} = \mathbf{F}_1 + \mathbf{F}_2 = -\hat{\mathbf{z}} k_e \frac{(\epsilon_{r1} - \epsilon_{r2}) Q^2}{2(\epsilon_{r1} + \epsilon_{r2})^2 R^2}. \quad (\text{S-3.66})$$

If the sphere is at equilibrium when half of its volume is submerged,  $\mathbf{F}_{\text{tot}}$  plus the sphere weight must balance Archimedes' buoyant force

$$k_e \frac{(\epsilon_{r1} - \epsilon_{r2}) Q^2}{2(\epsilon_{r1} + \epsilon_{r2})^2 R^2} = g \frac{2\pi R^3}{3} (\rho_1 + \rho_2 - 2\rho), \quad (\text{S-3.67})$$

where  $g$  is the gravitational acceleration. Thus, at equilibrium, the electric charge on the sphere must be

$$Q = \sqrt{\frac{4\pi R^5 (\varepsilon_{r1} + \varepsilon_{r2})^2 (\varrho_1 + \varrho_2 - 2\varrho) g}{3k_e(\varepsilon_{r1} - \varepsilon_{r2})}}. \quad (\text{S-3.68})$$

### S-3.8 Measuring the Dielectric Constant of a Liquid

The partially filled capacitor is equivalent to two capacitors connected in parallel, one with vacuum between the plates, and the other filled by the dielectric liquid. The two capacitors have the same internal and external radii,  $a$  and  $b$ , but different lengths,  $\ell - h$  and  $h$ , respectively. The total capacitance is

$$C = \frac{\ell - h}{2k_e \ln(b/a)} + \frac{\varepsilon_r h}{2k_e \ln(b/a)} = \frac{\ell + (\varepsilon_r - 1)h}{2k_e \ln(b/a)}, \quad (\text{S-3.69})$$

and the electrostatic energy of the capacitor is

$$U_{es} = \frac{1}{2} CV^2 = \frac{\ell + (\varepsilon_r - 1)h}{4k_e \ln(b/a)} V^2. \quad (\text{S-3.70})$$

If the liquid raises by an amount  $dh$  the capacity increases by

$$dC = \frac{(\varepsilon_r - 1)dh}{2k_e \ln(b/a)}, \quad (\text{S-3.71})$$

and, if the potential difference  $V$  across the capacitor plates is kept constant, the electrostatic energy of the capacitor increases by an amount

$$dU_{es} = \frac{1}{2} V^2 dC = \frac{(\varepsilon_r - 1)dh}{4k_e \ln(b/a)} V^2. \quad (\text{S-3.72})$$

Simultaneously the voltage source does a work

$$dW = V dQ = V^2 dC = \frac{(\varepsilon_r - 1)dh}{2k_e \ln(b/a)} V^2, \quad (\text{S-3.73})$$

because the charge of the capacitor must increase by  $dQ = VdC$  in order to keep the potential difference across the plates constant, and this implies moving a charge  $dQ$  from one plate to the other. The energy of the voltage source changes by

$$dU_{source} = -\frac{(\varepsilon_r - 1)dh}{2k_e \ln(b/a)} V^2 = -2dU_{es}. \quad (\text{S-3.74})$$

We must still evaluate the increase in gravitational potential energy of the liquid. When the liquid raises by  $dh$  an infinitesimal annular cylinder of mass  $dm = \rho\pi(b^2 - a^2)dh$  is added at its top, and the gravitational energy increases by

$$dU_g = g\rho\pi(b^2 - a^2)h dh. \quad (\text{S-3.75})$$

The *total* energy variation is thus

$$\begin{aligned} dU_{\text{tot}} &= dU_{\text{es}} + dU_{\text{source}} + dU_g = -dU_{\text{es}} + dU_g \\ &= -\frac{(\varepsilon_r - 1)dh}{4k_e \ln(b/a)} V^2 + g\rho\pi(b^2 - a^2)h dh, \end{aligned} \quad (\text{S-3.76})$$

and the total force is

$$F = -\frac{\partial U_{\text{tot}}}{\partial h} = \frac{\varepsilon_r - 1}{4k_e \ln(b/a)} V^2 - g\rho\pi(b^2 - a^2)h. \quad (\text{S-3.77})$$

At equilibrium we have  $F = 0$ , which corresponds to

$$h = \frac{(\varepsilon_r - 1)V^2}{4\pi k_e \rho(a^2 - b^2)\ln(b/a)} \quad \text{and} \quad \varepsilon_r = 1 + \frac{4\pi k_e \rho(a^2 - b^2)\ln(b/a)}{V^2} h. \quad (\text{S-3.78})$$

For the electric susceptibility  $\chi$ , we have in SI units  $\chi = \varepsilon_r - 1$ , and

$$\chi = \frac{g\rho(a^2 - b^2)\ln(b/a)}{\varepsilon_0 V^2} h, \quad (\text{S-3.79})$$

while in Gaussian units we have  $\chi = (\varepsilon_r - 1)/4\pi$ , and

$$\chi = \frac{g\rho(a^2 - b^2)\ln(b/a)}{V^2} h. \quad (\text{S-3.80})$$

### S-3.9 A Conducting Cylinder in a Dielectric Liquid

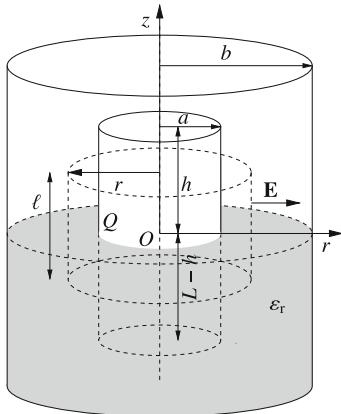
- a) We choose a cylindrical coordinate system  $(r, \phi, z)$  with the longitudinal axis  $z$  superposed to the axis of the conducting cylinder, and the origin  $O$  at the height of the boundary surface between the dielectric liquid and the vacuum above it. The azimuthal angle  $\phi$  is irrelevant for the present problem. The electric field  $\mathbf{E}(r, \phi, z) \equiv [E_r(r, z), 0, 0]$  is perpendicular to the surface of the cylinder, thus we have  $\mathbf{E}(r, \phi, z) \equiv [E_r(r, z), 0, 0]$ . The field is continuous at the dielectric-vacuum boundary surface, since it is parallel to it. We thus have  $E_r(r, z) = E_r(r)$ , independently of  $z$ . Let us denote by  $\sigma_1$  and  $\sigma_2$  the free-charge surface densities on the cylinder lateral surface for  $z > 0$  and  $z < 0$ , respectively. Quantities  $\sigma_1$  and  $\sigma_2$  are related to the electric field at the cylinder

surface,  $E_r(a)$ , to  $\varepsilon_r$  and  $Q$  by

$$\begin{aligned}\sigma_1 &= \frac{E_r(a)}{4\pi k_e}, & \sigma_2 &= \frac{\varepsilon_r E_r(a)}{4\pi k_e}, \\ Q &= 2\pi a[\sigma_1 h + \sigma_2(L - h)].\end{aligned}\quad (\text{S-3.81})$$

We thus have

$$\begin{aligned}E_r(a) &= Q \frac{2k_e}{a[h + \varepsilon_r(L - h)]} \\ &= Q \frac{2k_e}{a[\varepsilon_r L - (\varepsilon_r - 1)h]}.\end{aligned}\quad (\text{S-3.82})$$



**b)** The electric field  $E_r(r)$  in the region  $a < r < b$  can be evaluated by applying Gauss's law to a closed cylindrical surface of radius  $r$  and height  $\ell \ll L$ , coaxial to the conducting cylinder. Neglecting the boundary effects, the flux of the electric field through the bases of the Gaussian surface is zero, and we have

$$\begin{aligned}2\pi r \ell E_r(r) &= 4\pi k_e Q_{\text{int}}, \\ E_r(r) &= \frac{2k_e Q_{\text{int}}}{r \ell},\end{aligned}\quad (\text{S-3.83})$$

**Fig. S-3.15**

where  $Q_{\text{tot}}$  is the *total* charge inside the Gaussian surface, including both free and polarization charges. If we let  $r$  approach  $a$  keeping  $\ell$  constant,  $Q_{\text{int}}$  remains constant and we have

$$\lim_{r \rightarrow a} E_r = \frac{2k_e Q_{\text{int}}}{a \ell} = E_r(a), \quad \text{so that} \quad E_r(r) = E_r(a) \frac{a}{r} \quad (\text{S-3.84})$$

and, inserting (S-3.82),

$$E_r(r) = \frac{2k_e Q}{r[\varepsilon_r L - (\varepsilon_r - 1)h]}. \quad (\text{S-3.85})$$

**c)** The electrostatic energy of the system is

$$\begin{aligned}
U_{\text{es}} &\simeq \frac{1}{8\pi k_e} \left[ \varepsilon_r \int_0^{L-h} dz \int_a^b E_r^2(r) 2\pi r dr + \int_{L-h}^L dz \int_a^b E_r^2(r) 2\pi r dr \right] \\
&= \frac{1}{8\pi k_e} \left[ \frac{2k_e Q}{[\varepsilon_r L - (\varepsilon_r - 1)h]} \right]^2 \left\{ [\varepsilon_r(L-h) + h] \int_a^b \frac{2\pi}{r} dr \right\} \\
&= k_e \left[ \frac{Q}{[\varepsilon_r L - (\varepsilon_r - 1)h]} \right]^2 [\varepsilon_r(L-h) + h] \ln \left( \frac{b}{a} \right) \\
&= k_e \frac{Q^2 \ln(b/a)}{\varepsilon_r L - (\varepsilon_r - 1)h}, \tag{S-3.86}
\end{aligned}$$

i.e., the electrostatic energy of two cylindrical capacitors connected in parallel, with total charge  $Q$ . Both capacitors have internal radius  $a$  and external radius  $b$ , one has length  $L-h$  and is filled with the dielectric material, the other has length  $h$  and vacuum between the plates. The electrostatic force, directed along  $z$ , is

$$F_{\text{es}} = -\frac{dU_{\text{es}}}{dh} = -k_e \frac{(\varepsilon_r - 1) \ln(b/a) Q^2}{[\varepsilon_r L - (\varepsilon_r - 1)h]^2} < 0. \tag{S-3.87}$$

The electrostatic forces tends to decrease  $h$ , i.e., to sink the cylinder into the liquid.

**d)** The sum of the gravitational and buoyant (due to Archimedes' principle) forces on the cylinder is

$$F_g = -Mg + \rho g(L-h)\pi a^2, \tag{S-3.88}$$

and the cylinder is in equilibrium when  $F_{\text{es}} + F_g = 0$ , i.e., when

$$\rho g(L-h)\pi a^2 - Mg = k_e \frac{(\varepsilon_r - 1) \ln(b/a) Q^2}{[\varepsilon_r L - (\varepsilon_r - 1)h]^2}. \tag{S-3.89}$$

Given  $L, h$  and  $\varepsilon_r$ , we have equilibrium for

$$Q = [\varepsilon_r L - (\varepsilon_r - 1)h] \sqrt{\frac{\rho g(L-h)\pi a^2 - Mg}{k_e(\varepsilon_r - 1) \ln(b/a)}}. \tag{S-3.90}$$

### S-3.10 A Dielectric Slab in Contact with a Charged Conductor

**a)** Within our approximations, the electric fields are perpendicular to the conducting surface. We choose a Cartesian reference frame with the origin on the conductor surface and the  $x$  axis perpendicular the surface, as in Fig. S-3.16, so that the only nonzero component of the electric fields is

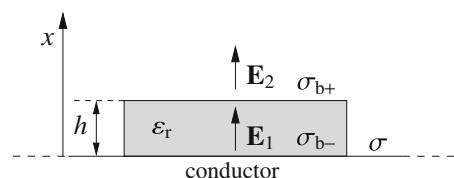
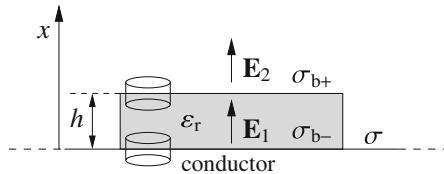


Fig. S-3.16

their  $x$  component. We denote by  $E_1$  the electric field field inside the dielectric slab, and by  $E_2$  the electric field in vacuum, while the field will be zero inside the conductor.

The fields  $E_1$  and  $E_2$  can be evaluated by applying equation  $\nabla \cdot (\epsilon_r \mathbf{E}) = 4\pi k_e \rho_f$  to two Gaussian “pillboxes”, crossing the  $x = 0$  and the  $x = h$  surfaces, respectively,



as in Fig. S-3.17. We see that  $\epsilon_r E$  is discontinuous at  $x = 0$  surface, and continuous at  $x = h$ :

$$\begin{aligned} \epsilon_r E_1 &= 4\pi k_e \sigma, \\ E_2 &= \epsilon_r E_1, \end{aligned} \quad (\text{S-3.91})$$

Fig. S-3.17

which lead to

$$E_1 = \frac{4\pi k_e}{\epsilon_r} \sigma, \quad E_2 = 4\pi k_e \sigma. \quad (\text{S-3.92})$$

**b)** We denote by  $\sigma_{b-}$  and  $\sigma_{b+}$  the surface polarization charge densities at  $x = 0$  and  $x = h$ , respectively. These quantities can be calculated by applying Gauss's law  $\nabla \cdot \mathbf{E} = 4\pi k_e (\rho_f + \rho_b)$  to the two “pillboxes” of Fig. S-3.17, and obtaining

$$E_1 = 4\pi k_e (\sigma + \sigma_{b-}), \quad E_2 - E_1 = 4\pi k_e \sigma_{b+}, \quad (\text{S-3.93})$$

introducing (S-3.91) into (S-3.93) we finally have

$$\sigma_{b+} = -\sigma_{b-} = \left(1 - \frac{1}{\epsilon_r}\right) \sigma = \frac{\epsilon_r - 1}{\epsilon_r} \sigma. \quad (\text{S-3.94})$$

**c)** In the vacuum region between the conductor and the dielectric slab the field is  $E = 4\pi k_e \sigma = E_2$ , independent of  $s$ . The electric field inside the dielectric slab, and above the slab, are  $E_1$  and  $E_2$ , respectively, as in the case of  $s = 0$ , thus independently of  $s$ .

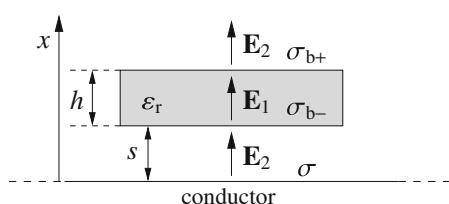


Fig. S-3.18

The net electrostatic force on the slab is zero, independently of  $s$ , since the forces on the upper and lower surfaces of the slab are exactly opposite. Further, if we evaluate the electrostatic energy of the system as the volume integral of  $\epsilon_r E^2 / (8\pi k_e)$ , we see that also this quantity is independent of  $s$ , within our approximations.

### S-3.11 A Transversally Polarized Cylinder

We choose a cylindrical coordinate system  $(r, \phi, z)$ , with the cylinder axis as  $z$  axis, and the reference plane (the plane from which the angle  $\phi$  is measured) parallel to  $\mathbf{P}$ . We have translational symmetry along  $z$ , so that, mathematically, the problem is two-dimensional. The surface charge polarization density of the cylinder is  $\sigma(\phi) = \mathbf{P} \cdot \hat{\mathbf{n}}$ , where  $\hat{\mathbf{n}}$  is the outgoing unit vector perpendicular to the cylinder surface, thus

$$\sigma(\phi) = P \cos \phi. \quad (\text{S-3.95})$$

Similarly to Problem 1.1, our transversally polarized cylinder can be considered as the limit for  $h \rightarrow 0$  and  $\varrho \rightarrow \infty$  of two partially overlapping cylinders, of volume charge density  $\pm \varrho$ , respectively. The two cylinder axes are the straight lines  $x = \pm h/2$ , both out of paper in Fig. S-3.19. The product  $\varrho h$  is constant, and equals the polarization  $P$  of the original cylinder. The electrostatic potential  $\varphi_{\pm}^{\text{ext}}(A)$ , generated by each charged cylinder at an external point  $A \equiv (r, \phi, z)$ , equals the potential of an infinite line charge of linear charge density  $\lambda_{\pm} = \pm \pi a^2 \varrho$ , located on the cylinder axis,

$$\varphi_{\pm}^{\text{ext}}(A) = \mp 2k_e \pi a^2 \varrho \ln \left( \frac{r_{\pm}}{R_{\pm}} \right), \quad (\text{S-3.96})$$

where

$$r_{\pm} \simeq r \mp \frac{h}{2} \cos \phi \quad (\text{S-3.97})$$

are the distances of  $A$  from the axes of the two cylinders, see Fig. S-3.20. Quantities  $R_{\pm}$  are two arbitrary constants, such that  $\varphi_{\pm}^{\text{ext}}(r_{\pm}, \phi, z) = 0$  on the cylindrical surfaces  $r_{\pm} = R_{\pm}$ . It is convenient to choose  $R_+ = R_-$ , so that  $\ln(R_+/R_-) = 0$  will cancel out in the following computations, leaving the potential equal to zero at  $r = \infty$ . The electrostatic potential generated by both cylinders is thus

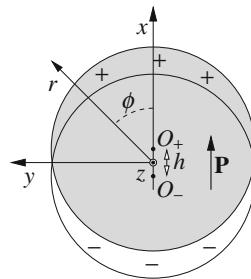


Fig. S-3.19

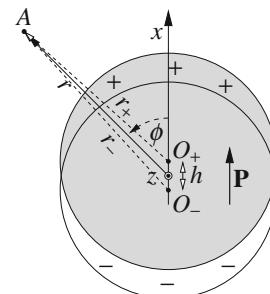


Fig. S-3.20

$$\begin{aligned}
\varphi^{\text{ext}}(A) &= \varphi_+^{\text{ext}}(A) + \varphi_-^{\text{ext}}(A) = 2k_e \pi a^2 \varrho \ln\left(\frac{r_-}{r_+}\right) + 2k_e \pi a^2 \varrho \ln\left(\frac{R_+}{R_-}\right) \\
&\simeq 2k_e \pi a^2 \varrho \ln\left[\frac{r + (h/2)\cos\phi}{r - (h/2)\cos\phi}\right] = 2k_e \pi a^2 \varrho \ln\left[\frac{1 + (h/2r)\cos\phi}{1 - (h/2r)\cos\phi}\right] \\
&= 2k_e \pi a^2 \varrho \left[ \ln\left(1 + \frac{h}{2r} \cos\phi\right) - \ln\left(1 - \frac{h}{2r} \cos\phi\right) \right] \\
&\simeq 2k_e \pi a^2 \frac{h}{r} \cos\phi = 2k_e \pi a^2 \frac{P \cos\phi}{r} = 2k_e \pi a^2 \frac{\mathbf{P} \cdot \hat{\mathbf{r}}}{r}, \tag{S-3.98}
\end{aligned}$$

where  $\hat{\mathbf{r}}$  is the unit vector of the cylindrical coordinate  $r$ . Thus, the potential of our *two-dimensional electric dipole* decreases as  $r^{-1}$ , while the potential of the ordinary electric dipole decreases as  $r^{-2}$ . In Cartesian coordinates we have

$$\varphi^{\text{ext}}(x, y, z) = 2k_e \pi a^2 \frac{Px}{x^2 + y^2}, \tag{S-3.99}$$

where the  $x$  and  $y$  axes are the ones shown in Fig. S-3.19.

The external electric field is obtained by evaluating  $\mathbf{E}^{\text{ext}} = -\nabla\varphi^{\text{ext}}$ . The cylindrical components are, from Table A.1 of the Appendix,

$$\begin{aligned}
E_r^{\text{ext}} &= -\partial_r \varphi^{\text{ext}} = 2k_e \pi a^2 \frac{P \cos\phi}{r^2}, \\
E_\phi^{\text{ext}} &= -\frac{1}{r} \partial_\phi \varphi^{\text{ext}} = 2k_e \pi a^2 \frac{P \sin\phi}{r^2}, \\
E_z^{\text{ext}} &= -\partial_z \varphi^{\text{ext}} = 0, \tag{S-3.100}
\end{aligned}$$

the field decreases proportionally to  $r^{-2}$ , while the field of the usual electric dipole decreases as  $r^{-1}$ . The Cartesian components of the field are

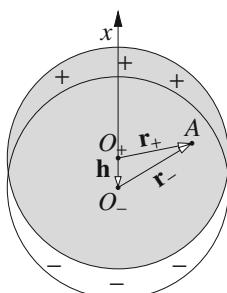


Fig. S-3.21

$$\begin{aligned}
E_x^{\text{ext}} &= -\partial_x \varphi^{\text{ext}} = 2k_e \pi a^2 P \frac{x^2 - y^2}{(x^2 + y^2)^2}, \\
E_y^{\text{ext}} &= -\partial_y \varphi^{\text{ext}} = 2k_e \pi a^2 P \frac{2xy}{(x^2 + y^2)^2}, \\
E_z^{\text{ext}} &= -\partial_z \varphi^{\text{ext}} = 0. \tag{S-3.101}
\end{aligned}$$

The electric field generated by each cylinder at its interior is, according to Gauss's law,  $\mathbf{E}_\pm^{\text{int}} = \pm 2\pi k_e \varrho \mathbf{r}_\pm$ , where  $\mathbf{r}_\pm$  is the distance from the respective axis, see Fig. S-3.21. The two contributions sum up to a uniform internal field

$$\mathbf{E}^{\text{int}}(A) = 2\pi k_e \varrho (\mathbf{r}_+ - \mathbf{r}_-) = -2\pi k_e \varrho h \hat{\mathbf{x}} = -2\pi k_e \mathbf{P}. \tag{S-3.102}$$

The electrostatic potential inside the cylinder, in Cartesian and cylindrical coordinates, is thus

$$\varphi^{\text{int}} = 2\pi k_e x + C = 2\pi k_e r \cos \phi + C, \quad (\text{S-3.103})$$

where  $C$  is an arbitrary constant. Since the potential must be continuous, we must have, in cylindrical coordinates,

$$\varphi^{\text{int}}(a, \phi, z) = \varphi^{\text{ext}}(a, \phi, z), \quad (\text{S-3.104})$$

which is verified if we choose  $C = 0$ .

# Chapter S-4

## Solutions for Chapter 4

### S-4.1 The Tolman-Stewart Experiment

a) The equation of motion for the “free” (conduction) electrons in a metal is, according to the Drude model,

$$m \frac{d\langle v \rangle}{dt} = \mathbf{F} - m\eta \langle v \rangle, \quad (\text{S-4.1})$$

where  $\langle v \rangle$  is the “average” electron velocity,  $\mathbf{F}$  is the external force on the electrons, and  $m\eta \langle v \rangle$  is a phenomenological friction force. In a steady state ( $d\langle v \rangle / dt = 0$ ) in the presence of an external electric field  $\mathbf{E}$ , so that  $\mathbf{F} = -e\mathbf{E}$ , the electrons have a constant average velocity

$$\langle v \rangle = -\frac{e}{m\eta} \mathbf{E}. \quad (\text{S-4.2})$$

The current density is  $\mathbf{J} = -en_e \langle v \rangle$ , where  $n_e$  is the volume density of free electrons. From this we obtain the microscopic form of Ohm’s law

$$\mathbf{J} = \frac{n_e e^2}{m\eta} \mathbf{E} \equiv \sigma \mathbf{E}. \quad (\text{S-4.3})$$

The value of the damping frequency  $\eta$  for copper is

$$\eta = \frac{n_e e^2}{m\sigma} = \frac{8.5 \times 10^{28} (1.6 \times 10^{-19})^2}{9.1 \times 10^{-31} \times 10^7} \simeq 2.4 \times 10^{14} \text{ s}^{-1}, \quad (\text{S-4.4})$$

( $m = m_e = 9.1 \times 10^{-31}$  kg).

At  $t = 0$  the electron tangential velocity is  $v_0 = a\omega$ . For  $t > 0$ , due to the absence of external forces the solution of Eq. (S-4.1) is

$$v = v_0 e^{-\eta t}. \quad (\text{S-4.5})$$

The total current is thus given by

$$I = I_0 e^{-\eta t}, \quad I_0 = -(en_e v_0) S, \quad (\text{S-4.6})$$

and the decay time is  $\tau = 1/\eta \simeq 4 \times 10^{-15}$  s.

**b)** The total charge flown in the ring is

$$Q = \int_0^\infty I(t) dt = \frac{I_0}{\eta} = -\frac{m}{e} \sigma S v_0. \quad (\text{S-4.7})$$

Thus, measuring  $\sigma$ ,  $S$ ,  $v_0$  and  $Q$  the value of  $e/m$  can be obtained. In the original experiment, Tolman and Stewart were able to measure  $Q$  using a ballistic galvanometer in a circuit coupled with a rotating coil.

## S-4.2 Charge Relaxation in a Conducting Sphere

**a)** For symmetry reasons the electric field is radial, and it is convenient to use a spherical coordinate system  $(r, \theta, \phi)$  with the origin located at the center of the sphere. Coordinates  $\theta$  and  $\phi$  are irrelevant for this problem. Let us denote by  $q(r, t)$  the electric charge contained inside the sphere  $r < a$ , at time  $t \geq 0$ . If we apply Gauss's law to the surface of our sphere we obtain

$$E(r, t) = k_e \frac{q(r, t)}{r^2}. \quad (\text{S-4.8})$$

According to the continuity equation, the flux of the current density  $\mathbf{J} = \sigma \mathbf{E}$  through our spherical surface equals the time derivative of  $q(r, t)$ :

$$\oint \mathbf{J} \cdot d\mathbf{S} = 4\pi r^2 J(r, t) = 4\pi r^2 \sigma E(r, t) = -\partial_t q(r, t). \quad (\text{S-4.9})$$

By substituting (S-4.8) into (S-4.9) we obtain

$$\partial_t q(r, t) = -4\pi k_e \sigma q(r, t), \quad (\text{S-4.10})$$

with solution

$$q(r, t) = q(r, 0) e^{-t/\tau}, \quad \text{where} \quad \tau = \frac{1}{4\pi k_e \sigma}. \quad (\text{S-4.11})$$

Since at  $t = 0$  the charge density  $\varrho(r, t)$  is uniform all over the volume of the sphere of radius  $a$ , we have

$$\varrho(r, 0) = \varrho_0 = Q \frac{3}{4\pi a^3}, \quad r < a, \quad \text{so that} \quad q(r, 0) = Q \frac{r^3}{a^3}. \quad (\text{S-4.12})$$

Thus, according to (S-4.11), the density  $\varrho(r, t)$  remains uniform over the sphere volume (independent of  $r$ ) at any time  $t > 0$

$$\varrho(r, t) = \varrho(t) = \varrho_0 e^{-t/\tau}. \quad (\text{S-4.13})$$

The *surface* charge density  $q_s(t)$  (we have already used the Greek letter  $\sigma$  for the conductivity) can also be evaluated from the continuity equation, since

$$\partial_t q_s(t) = +J(a, t) = \sigma E(a, t) = k_e \sigma \frac{Q}{a^2} e^{-t/\tau} = \frac{Q}{4\pi a^2 \tau} e^{-t/\tau}, \quad (\text{S-4.14})$$

so that, asymptotically,

$$q_s(\infty) = \int_0^\infty \partial_t q_s dt = \frac{Q}{4\pi a^2 \tau} \int_0^\infty e^{-t/\tau} dt = \frac{Q}{4\pi a^2}. \quad (\text{S-4.15})$$

The equation for the time evolution of the electric field inside the sphere ( $r < a$ ) is

$$E(r, t) = k_e \frac{q(r, t)}{r^2} = k_e Q \frac{r}{a^3} e^{-t/\tau}, \quad r < a, \quad (\text{S-4.16})$$

while the electric field is independent of time outside the sphere

$$E(r, t) = E(r) = k_e \frac{Q}{r^2}, \quad r > a. \quad (\text{S-4.17})$$

The time constant  $\tau = 1/(4\pi k_e \sigma)$  is extremely short in a good conductor. For copper we have (in SI units)  $\sigma \approx 6 \times 10^7 \Omega^{-1} m^{-1}$  at room temperature, thus

$$\tau = \frac{\epsilon_0}{\sigma} \approx \frac{8.854 \times 10^{-12}}{6 \times 10^7} s \sim 1.4 \times 10^{-19} s = 0.14 \text{ as} \quad (\text{S-4.18})$$

(1 as = 1 *attosecond* =  $10^{-18}$  s: *atten* means *eighteen* in Danish). This extremely short value should be not surprising, since there is no need for the electrons to travel distances even of the order of the atomic spacing within the relaxation time; a very small collective displacement of the electrons is sufficient to reach a condition of mechanical equilibrium (see also Problem 2.1).

**b)** We can easily evaluate the variation of electrostatic energy  $\Delta U_{es}$  during the charge relaxation by noticing that the electric field  $E(r, t)$  is constant outside the conducting sphere ( $r > a$ ). The electric field inside the sphere decays from the ini-

tial profile  $E(r, 0) = (Q/4\pi\varepsilon_0)(r/a^3)$  to  $E(r, \infty) = 0$ . Thus, using the “energy density”  $u_{\text{es}} = E^2/(8\pi k_e)$  we can write

$$\begin{aligned}\Delta U_{\text{es}} &= -\frac{1}{8\pi k_e} \int_{\text{sphere}} E^2(r, 0) d^3x = -\frac{1}{8\pi k_e} \left( \frac{k_e^2 Q^2}{a^3} \right) \int_0^a r^2 4\pi r^2 dr \\ &= -\frac{k_e}{10} \frac{Q^2}{a}.\end{aligned}\quad (\text{S-4.19})$$

c) The time derivative of the electrostatic energy can be written as

$$\begin{aligned}\partial_t U_{\text{es}} &= \frac{1}{8\pi k_e} \partial_t \int_0^\infty E^2(r, t) 4\pi r^2 dr = \frac{1}{8\pi k_e} \int_0^a \partial_t E^2(r, t) 4\pi r^2 dr \\ &= \frac{1}{8\pi k_e} \int_0^a \left( -\frac{2}{\tau} \right) E^2(r, t) 4\pi r^2 dr = -\frac{1}{4\pi k_e \tau} \int_0^a k_e^2 Q^2 \frac{r^2}{a^6} e^{-2t/\tau} 4\pi r^2 dr \\ &= -\frac{k_e}{5\tau} \frac{Q^2}{a} e^{-2t/\tau} = -\frac{4\pi k_e^2 \sigma Q^2}{5a} e^{-2t/\tau},\end{aligned}\quad (\text{S-4.20})$$

where we used (S-4.16) and (S-4.17). The power loss due to Joule heating is

$$\begin{aligned}P_d &= \int_0^\infty \mathbf{J} \cdot \mathbf{E} 4\pi r^2 dr = \int_0^a \sigma E^2(r, t) 4\pi r^2 dr \\ &= \frac{4\pi k_e^2 \sigma Q^2}{5a} e^{-2t/\tau},\end{aligned}\quad (\text{S-4.21})$$

since  $\mathbf{J} = \sigma \mathbf{E}$  for  $r < a$ , and  $\mathbf{J} = 0$  for  $r > a$ . Thus  $P_d = -\partial_t U_{\text{es}}$ , and all the electrostatic energy lost by the sphere during the relaxation process is turned into Joule heat.

### S-4.3 A Coaxial Resistor

a) We use a cylindrical coordinate system  $(r, \phi, z)$ , with the  $z$  axis coinciding with the common axis of the cylindrical plates. The material between the plates can be considered as a series of infinitesimal cylindrical-shell resistors, each of internal and external radii  $r$  and  $r + dr$  and of height  $h$ . The resistance of the cylindrical shell between  $r$  and  $r + dr$  is

$$dR = \rho \frac{dr}{S(r)} = \rho \frac{dr}{2\pi r h}, \quad (\text{S-4.22})$$

since  $dr$  is the “length” of our resistor, and  $S(r) = 2\pi r$  its “cross-sectional area”. The resistance of the material is thus

$$R = \frac{\rho}{2\pi h} \int_a^b \frac{dr}{r} = \rho \frac{\ln(b/a)}{2\pi h}. \quad (\text{S-4.23})$$

**b)** The capacity of a cylindrical capacitor of radii  $a$  and  $b$ , and length  $h$ , is

$$C_0 = \frac{h}{2k_e \ln(b/a)}, \quad (\text{S-4.24})$$

assuming that the space between the plates is filled with vacuum. Thus, in our case we have

$$R = \frac{\rho}{4\pi k_e C_0}. \quad (\text{S-4.25})$$

Equation (S-4.25) is actually of much more general validity, and is a very good approximation for evaluating the resistance between two electrodes of high conductivity and calculable capacity immersed in a medium of known resistivity. As an example, consider two highly-conducting square plates immersed in an ohmic medium, and connected to a voltage source by insulated cables, as in Fig. S-4.1. The current that flows, for instance, from the left plate, can be written

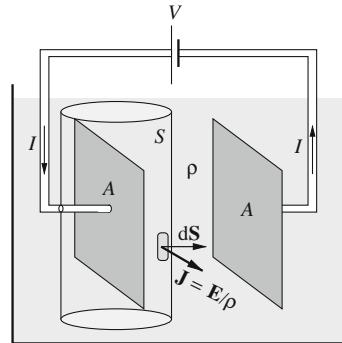


Fig. S-4.1

$$I = \int \mathbf{J} \cdot d\mathbf{S} = \frac{1}{\rho} \int \mathbf{E} \cdot d\mathbf{S}, \quad (\text{S-4.26})$$

where the flux is calculated through a surface enclosing the electrode, except for the area through which the current enters it, like the cylindrical closed surface of Fig. S-4.1. In most cases the contribution of the excluded area to the flux of  $\mathbf{E}$  is negligible in an electrostatic problem, while, according to Gauss's law, we have

$$\oint \mathbf{E} \cdot d\mathbf{S} = 4\pi k_e Q \quad (\text{S-4.27})$$

where  $Q$  is the charge on the electrode that would produce the field  $\mathbf{E}$ . Within the approximation of considering the last integral of (S-4.26) as equal to the integral through the whole closed surface, we have

$$I = \frac{4\pi k_e}{\rho} Q. \quad (\text{S-4.28})$$

On the other hand, if we consider the two electrodes as the plates of a capacitor of capacitance  $C_0$  we have

$$Q = C_0 V, \quad (\text{S-4.29})$$

where  $V$  is the potential difference between them. We thus have

$$I = \frac{4\pi k_e}{\rho} C_0 V = \frac{V}{R}, \quad (\text{S-4.30})$$

from which (S-4.25) follows.

## S-4.4 Electrical Resistance between two Submerged Spheres (1)

**a)** We start by evaluating the capacitance  $C_0$  of the two spheres in vacuum, with the same geometry of the problem. Let the sphere of radius  $a$  carry a charge  $Q$ , and the sphere of radius  $b$  a charge  $-Q$ . With our assumptions  $a \ll x$  and  $b \ll x$  the electric potentials  $\varphi_a$  and  $\varphi_b$  of the two spheres are given approximately by

$$\varphi_a \simeq k_e Q \left( \frac{1}{a} - \frac{1}{x} \right) \quad \text{and} \quad \varphi_b \simeq k_e Q \left( -\frac{1}{b} + \frac{1}{x} \right), \quad (\text{S-4.31})$$

where we have assumed the potential  $\varphi$  to be zero at infinity, and have neglected the induction effects between the two spheres, discussed in Problem 2.6. The capacitance of the two spheres can thus be approximated as

$$C_0 = \frac{Q}{\varphi_a - \varphi_b} \simeq \frac{1}{k_e} \left( \frac{1}{a} + \frac{1}{b} - \frac{2}{x} \right)^{-1}, \quad (\text{S-4.32})$$

and, according to (S-4.25), the resistance between them is

$$R = \frac{\rho}{4\pi k_e C_0} \simeq \frac{\rho}{4\pi} \left( \frac{1}{a} + \frac{1}{b} - \frac{2}{x} \right), \quad (\text{S-4.33})$$

which can be further approximated to

$$R \simeq \frac{\rho}{4\pi} \left( \frac{1}{a} + \frac{1}{b} \right), \quad (\text{S-4.34})$$

independent of the distance between  $x$  between the centers of the spheres.

**b)** In this case the resistance between the spheres is twice the value found at point **a**), since at point **a**) we can introduce a horizontal plane passing through the centers of the spheres, which divides the fluid into two equivalent halves, each of resistance  $2R$ , so that, in parallel, they are equivalent to a resistance  $R$ . In the present case the upper half is replaced by vacuum, so that only the resistance  $2R$  of the lower half remains. This problem is of interest in connection with electrical circuits that use the ground as a return path. In this case  $\rho$  is the resistivity of the earth (of course, the assumption that  $\rho$  is uniform is a very rough approximation). In practical applications, the resistivity of the earth in the neighborhood of the electrodes can be decreased by moistening the ground around them.

## S-4.5 Electrical Resistance between two Submerged Spheres (2)

**a)** According to (S-4.32) of Problem 4.4, and remembering that now the medium has a relative dielectric permittivity  $\epsilon_r$ , the charge of each sphere is

$$\begin{aligned} Q &\simeq \epsilon_r C_0 V = \frac{\epsilon_r}{k_e} \left( \frac{2}{a} - \frac{2}{\ell} \right)^{-1} V = \frac{\epsilon_r}{k_e} \frac{\ell}{\ell-a} \frac{a}{2} V \\ &\simeq \frac{\epsilon_r a}{2k_e} \left( 1 + \frac{a}{\ell} \right) V \simeq \frac{\epsilon_r a}{2k_e} V, \end{aligned} \quad (\text{S-4.35})$$

where the last two terms are the first and the zeroth order approximations in  $a/\ell$ .

**b)** According to (S-4.33) and (S-4.34) we have

$$R \simeq \frac{\rho}{4\pi} \left( \frac{2}{a} - \frac{2}{\ell} \right) \simeq \frac{\rho}{2\pi a}, \quad (\text{S-4.36})$$

again to the zeroth order in  $a/\ell$ . The current  $I$  is thus

$$I = \frac{V}{R} = \frac{2\pi a}{\rho} V. \quad (\text{S-4.37})$$

This result can be checked by introducing a cylindrical coordinate system  $(r, \phi, z)$  with the  $z$  axis through the centers of the two spheres and the origin  $O$  so that the sphere centers are at  $(0, \phi, -\ell/2)$  and  $(0, \phi, +\ell/2)$ , respectively, and evaluating the flux of the current density  $\mathbf{J}$  through the plane  $z = 0$

$$\begin{aligned} I &= \int \mathbf{J} \cdot d\mathbf{S} \\ &= \frac{1}{\rho} \int_0^\infty E_z(r) 2\pi r dr, \end{aligned}$$

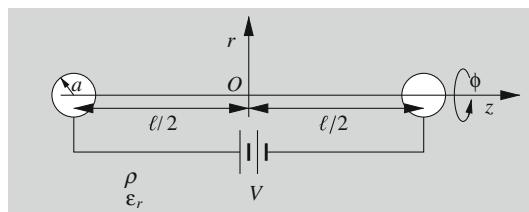


Fig. S-4.2

where

$$E_z(r) = \frac{2k_e Q}{\epsilon_r} \frac{\ell/2}{[(\ell/2)^2 + r^2]^{3/2}}, \quad (\text{S-4.38})$$

so that

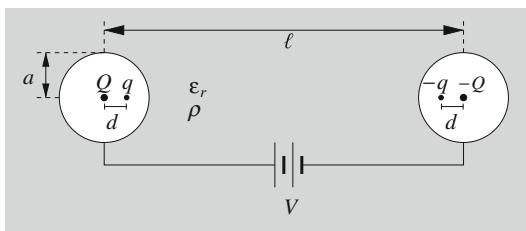
$$I = \frac{k_e Q}{\rho \epsilon_r} \int_0^\infty \frac{\ell r}{[(\ell/2)^2 + r^2]^{3/2}} dr = \frac{4\pi k_e}{\rho \epsilon_r} Q = \frac{2\pi a}{\rho} V. \quad (\text{S-4.39})$$

c) Having a system equivalent to a capacitor in parallel with a resistor, we expect an exponential decay of the charge  $Q$ , with a time constant  $\tau = RC$ . The time constant is independent of the geometry of the problem because the capacitance  $C$  of the system is  $\epsilon_r C_0$ , where  $C_0$  is the capacitance when the medium is replaced by vacuum, while, according to (S-4.25), the resistance is  $R = \rho/(4\pi k_e C_0)$ , so that

$$\tau = RC = \frac{\epsilon_r \rho}{4\pi k_e} . \quad (\text{S-4.40})$$

This relations holds for any “leaky capacitor”, if the discharge occurs only through leakage. In the present case we obtain from the continuity equation

$$\frac{dQ}{dt} = -I = -\frac{4\pi k_e}{\epsilon_r \rho} Q, \quad Q(t) = Q(0) e^{-t/\tau}, \quad \tau = \frac{\epsilon_r \rho}{4\pi k_e} . \quad (\text{S-4.41})$$



**Fig. S-4.3**

centers, on the line connecting the two centers, each toward the other sphere, as in the figure.

Thus, to the first order, the potential of each sphere is  $\approx \pm k_e Q / (\epsilon_r a)$ , since the contribution of the charge  $\mp Q$  on the other sphere is canceled by the image charge  $\pm q$  present in the sphere. We have  $Q \approx \epsilon_r a V / (2k_e)$ , while the absolute value of the total charge on each sphere is  $|Q + q| = Q(1 + a/l)$ . The capacitance of the system is thus

$$C = \frac{Q}{V} = \frac{\epsilon_r a}{2k_e} \left(1 + \frac{a}{l}\right) . \quad (\text{S-4.42})$$

The same result is obtained from (S-4.32) of Problem 4.4

$$C = \epsilon_r C_0 = \frac{\epsilon_r}{k_e} \left(\frac{2}{a} - \frac{2}{l}\right)^{-1} = \frac{\epsilon_r a}{2k_e} \frac{l}{l-a} \approx \frac{\epsilon_r a}{2k_e} \left(1 + \frac{a}{l}\right) , \quad (\text{S-4.43})$$

where the image charges have been disregarded, but the effect of the charge on each sphere on the potential of the other has been taken into account.

d) To the first order in  $a/l$  the electrostatic induction effects can be described by regarding the electric field outside the two spheres as due to two charges  $\pm Q$  located at the centers of the spheres, and two charges  $\pm q = \pm(a/l)Q$  located at distances  $d = a^2/l$  from the centers,

at distances  $d = a^2/l$  from the centers.

According to (S-4.33) the resistance now becomes

$$R = \frac{\rho}{4\pi k_e C_0} = \frac{\rho}{2\pi a} \frac{\ell - a}{a}, \quad (\text{S-4.44})$$

so that the time constant  $\tau = RC = \varepsilon_r \rho / (4\pi k_e)$  is unchanged.

## S-4.6 Effects of non-uniform resistivity

- a) We use a cylindrical coordinate system  $(r, \phi, z)$ , with the  $z$  axis along the common axis of the two cylinders, and the origin  $O$  on the surface separating the two cylinders as in Fig. S-4.4. We denote the volume charge density by  $q_v$ , since the Greek letter  $\rho$  is already used to denote the resistivities. In a steady state we must have  $\partial_t q_v = 0$  everywhere, otherwise the volume charge density would increase, or decrease, indefinitely. Thus, according to the continuity equation, we have also

$$\nabla \cdot \mathbf{J} = -\partial_t q_v = 0. \quad (\text{S-4.45})$$

On the other hand, from  $\nabla \cdot \mathbf{E} = 4\pi k_e q_v$  and  $\mathbf{J} = \mathbf{E}/\rho$  we obtain

$$0 = \nabla \cdot \mathbf{J} = \frac{1}{\rho} \nabla \cdot \mathbf{E} = \frac{4\pi k_e}{\rho} q_v, \quad (\text{S-4.46})$$

showing that also the volume charge density  $q_v$  must be zero everywhere inside a conductor in stationary conditions. This does not exclude the presence of surface charge densities on the surfaces delimiting a conductor.

If we assume that  $h \gg a$ , it follows from  $\nabla \cdot \mathbf{J} = 0$  and  $\nabla \times \mathbf{E} = 0$  that  $\mathbf{J}$  is uniform inside the cylinders, pointing downwards along the  $z$  direction. Since  $\mathbf{E}$  and  $\mathbf{J}$  are proportional to each other inside each cylinder, it follows that also  $\mathbf{E}$  is uniform inside each cylinder. The current density  $\mathbf{J}$  must be continuous through the surface separating the two cylinders, otherwise charge would accumulate indefinitely on the surface. Thus,  $\mathbf{J}$  is uniform throughout the whole conductor, and the current is  $I = J\pi a^2$ .

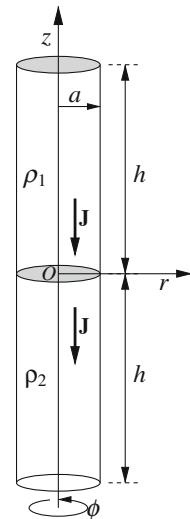


Fig. S-4.4

The resistances  $R_{1,2}$  of the two cylinders are, respectively,

$$R_{1,2} = \rho_{1,2} \frac{h}{\pi a^2}, \quad (\text{S-4.47})$$

leading to a total resistance  $R$  of the system

$$R = R_1 + R_2 = (\rho_1 + \rho_2) \frac{h}{\pi a^2}. \quad (\text{S-4.48})$$

The current, and the current density, flowing in the system are

$$I = \frac{V}{R} = \frac{\pi a^2 V}{h(\rho_1 + \rho_2)}, \quad J = \frac{V}{h(\rho_1 + \rho_2)}. \quad (\text{S-4.49})$$

Since we have the same current density in two conductors of different resistivities, and  $\mathbf{E} = \rho \mathbf{J}$ , the electric fields in the two conductors must be different, namely

$$E_1 = \rho_1 J = \frac{\rho_1 V}{h(\rho_1 + \rho_2)}, \quad E_2 = \rho_2 J = \frac{\rho_2 V}{h(\rho_1 + \rho_2)}. \quad (\text{S-4.50})$$

**b)** The surface charge density on the surface separating the two cylinders can be evaluated from Gauss's law

$$\sigma = \frac{1}{4\pi k_e} (E_2 - E_1) = \frac{1}{4\pi k_e} \frac{(\rho_2 - \rho_1)V}{h(\rho_1 + \rho_2)}. \quad (\text{S-4.51})$$

Assuming that the electric field is zero above the upper base and below the lower base of the conductor, the surface charge densities at the two bases are also obtained from Gauss's law as

$$\sigma_1 = \frac{E_1}{4\pi k_e} = \frac{1}{4\pi k_e} \frac{\rho_1 V}{h(\rho_1 + \rho_2)}, \quad \sigma_2 = -\frac{E_2}{4\pi k_e} = -\frac{1}{4\pi k_e} \frac{\rho_2 V}{h(\rho_1 + \rho_2)}. \quad (\text{S-4.52})$$

## S-4.7 Charge Decay in a Lossy Spherical Capacitor

**a)** We use a spherical coordinate system  $(r, \theta, \phi)$ , with the origin  $O$  at the center of the capacitor. We have  $\mathbf{E} = 0$  for  $r < a$  and  $r > b$ . For symmetry reasons, the electric field  $\mathbf{E}$  is radial and depends on  $r$  and  $t$  only in the spherical shell  $a < r < b$ . The flux of  $\epsilon_r \mathbf{E}$  through a spherical surface centered in  $O$  and of radius  $r$  is independent of  $r$  and equals

$$\epsilon_r \oint \mathbf{E}(r, t) \cdot d\mathbf{S} = 4\pi k_e Q(t), \quad (\text{S-4.53})$$

where  $Q(t)$  is the *free* charge contained in the surface, i.e., the free surface charge of the conducting sphere of radius  $a$ . Thus we have

$$\mathbf{E}(r, t) = \frac{k_e}{\epsilon_r} \frac{Q(t)}{r^2} \mathbf{r}_c. \quad (\text{S-4.54})$$

In addition to the free charge, our system contains surface polarization charges at  $r = a$  and  $r = b$ , of values  $\mp Q(\epsilon_r - 1)/\epsilon_r$ , respectively. No volume polarization charge is present, because

$$\nabla \cdot \mathbf{P} = \frac{\epsilon_r - 1}{4\pi k_e} \nabla \cdot \mathbf{E}(r, t) = 0. \quad (\text{S-4.55})$$

The electric field  $\mathbf{E}(r, t)$ , in the presence of an electrical conductivity  $\sigma$ , gives origin to a current density  $\mathbf{J}$

$$\mathbf{J} = \sigma \mathbf{E} = \sigma \frac{k_e}{\epsilon_r} \frac{Q(t)}{r^2} \hat{\mathbf{r}}, \quad (\text{S-4.56})$$

so that we have a total charge flux rate (electric current) through the surface

$$I = \frac{dQ}{dt} = \oint \mathbf{J} \cdot d\mathbf{S} = \frac{4\pi\sigma k_e}{\epsilon_r} Q(t). \quad (\text{S-4.57})$$

The charge crossing the surface is subtracted from the free charge on the internal conducting sphere, so that

$$\frac{dQ(t)}{dt} = -\frac{4\pi\sigma k_e}{\epsilon_r} Q(t), \quad (\text{S-4.58})$$

leading to

$$Q(t) = Q_0 e^{-t/\tau}, \quad \text{with} \quad \tau = \frac{\epsilon_r}{4\pi\sigma k_e}, \quad (\text{S-4.59})$$

and the decay constant is independent of the sizes of the capacitor, in agreement with (S-4.40).

**b)** The power dissipated over the volume of the capacitor is

$$\begin{aligned} P_d &= \int \mathbf{J} \cdot \mathbf{E} d^3x = \sigma \int \mathbf{E}^2 d^3x = \sigma \int_a^b \left[ \frac{k_e}{\epsilon_r} \frac{Q(t)}{r^2} \right]^2 4\pi r^2 dr \\ &= \frac{4\pi\sigma k_e^2}{\epsilon_r^2} Q_0^2 e^{-2t/\tau} \int_a^b \frac{dr}{r^2} = \frac{4\pi\sigma k_e^2 (b-a)}{\epsilon_r^2 ab} Q_0^2 e^{-2t/\tau}. \end{aligned} \quad (\text{S-4.60})$$

The electrostatic energy of the capacitor is

$$U_{\text{es}} = \frac{1}{2} \frac{Q^2(t)}{C} = \frac{k_e(b-a)}{2\epsilon_r ab} Q_0^2 e^{-2t/\tau}, \quad (\text{S-4.61})$$

so that

$$\frac{dU_{\text{es}}}{dt} = -\frac{k_e(b-a)}{\tau \epsilon_r ab} Q_0^2 e^{-2t/\tau} = -\frac{4\pi\sigma k_e^2(b-a)}{\epsilon_r^2 ab} Q_0^2 e^{-2t/\tau} = -P_d. \quad (\text{S-4.62})$$

Thus, the electrostatic energy of the capacitor is dissipated into Joule heating.

## S-4.8 Dielectric-Barrier Discharge

- a)** We denote by  $E_1$  and  $E_2$  the electric fields in the gas and in the dielectric layers, respectively. Since the voltage drop between the plates is  $V$ , we must have

$$E_1 d_1 + E_2 d_2 = V. \quad (\text{S-4.63})$$

In the absence of free surface charges the normal component of  $\epsilon_r \mathbf{E}$  is continuous through the surface separating the two layers, so that

$$E_1 = \epsilon_r E_2. \quad (\text{S-4.64})$$

Combining (S-4.63) and (S-4.64) we obtain

$$E_1 = \frac{\epsilon_r V}{\epsilon_r d_1 + d_2}, \quad E_2 = \frac{V}{\epsilon_r d_1 + d_2}. \quad (\text{S-4.65})$$

- b)** In steady-state conditions the current density in the gas,  $\mathbf{J}$ , must be zero, otherwise the free charge on the surface separating the gas and the dielectric material would increase steadily. Since the current density is  $J = E_1/\rho$ , we must have  $E_1 = 0$ . On the other hand (S-4.63) still holds, so that  $E_2 = V/d_2$ . The free charge density on the surface separating the layers in steady conditions is

$$\sigma_s = \frac{1}{4\pi k_e} (\epsilon_r E_2 - E_1) = \frac{\epsilon_r}{4\pi k_e} E_2 = \frac{\epsilon_r}{4\pi k_e} \frac{V}{d_2}. \quad (\text{S-4.66})$$

- c)** The continuity equation for  $\sigma$  and  $J$  is

$$\partial_t \sigma = J = \frac{E_1}{\rho}. \quad (\text{S-4.67})$$

From (S-4.66), now with  $E_1 \neq 0$  (discharge conditions), we have

$$E_1 = \epsilon_r E_2 - 4\pi k_e \sigma, \quad (\text{S-4.68})$$

which, combined with (S-4.63), leads to

$$E_1 = \epsilon_r \left( \frac{V}{d_2} - \frac{d_1}{d_2} E_1 \right) - 4\pi k_e \sigma, \quad \text{i.e.,} \quad E_1 = \frac{\epsilon_r V}{\epsilon_r d_1 + d_2} - \frac{4\pi k_e d_2}{\epsilon_r d_1 + d_2} \sigma. \quad (\text{S-4.69})$$

Equation (S-4.69), substituted into (S-4.67), gives

$$\partial_t \sigma = -\frac{4\pi k_e d_2}{\rho(\epsilon_r d_1 + d_2)} \sigma + \frac{\epsilon_r V}{\rho(\epsilon_r d_1 + d_2)}. \quad (\text{S-4.70})$$

with solution

$$\sigma = \frac{\epsilon_r V}{4\pi k_e d_2} (1 - e^{-t/\tau}) \equiv \sigma_s (1 - e^{-t/\tau}), \quad \text{where} \quad \tau = \frac{\rho(\epsilon_r d_1 + d_2)}{4\pi k_e d_2}. \quad (\text{S-4.71})$$

This problem shows the concept of the “dielectric-barrier discharge” (DBD). This scheme, where the dielectric layer acts as a current limiter, is used in various electrical discharge devices, for example in plasma TV displays, where the discharge acts as an ultraviolet micro-source to activate the phosphors in each pixel of the screen.

## S-4.9 Charge Distribution in a Long Cylindrical Conductor

a) As we saw in point a) of Problem 4.6, the volume charge density  $q_v$  is zero everywhere inside our conducting cylinder, while  $\mathbf{E}$  and  $\mathbf{J}$  are uniform. The presence of an electric field requires the presence of a charge distribution generating it, and, since there cannot be volume charge densities inside a conductor in steady conditions, the charges generating the fields must be distributed on the conductor surfaces. Consider the thin cylindrical conductor shown in Fig. S-4.5, of radius  $a$  and length  $2h$ , with  $h \ll a$ , connected to a voltage source  $V_0$ . In this case, neglecting boundary effects, the surface charge densities  $\sigma_B$  and  $-\sigma_B$  on the two bases are sufficient to generate the uniform electric field  $\mathbf{E}$  inside the conductor. This leads also to a uniform current density  $\mathbf{J} = \mathbf{E}/\rho$ . Neglecting the boundary effects we have

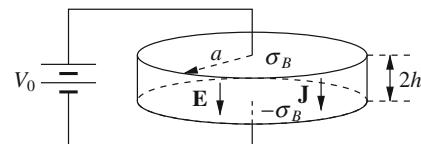


Fig. S-4.5

$$E = \frac{V_0}{2h}, \quad \sigma_B = \frac{E}{4\pi k_e} = \frac{V_0}{8\pi k_e h}. \quad (\text{S-4.72})$$

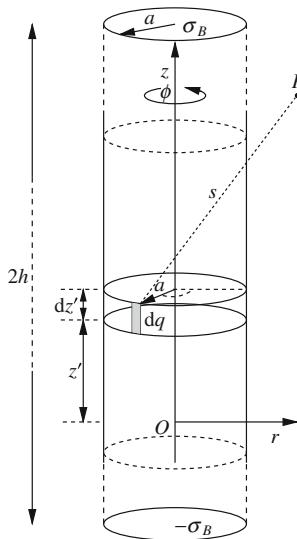


Fig. S-4.6

But here we are dealing with the opposite case, when the potential difference \$V\_0\$ is applied to the bases of a very elongated cylinder, with \$h \gg a\$. Without loss of generality, we assume the potential to be \$+V\_0/2\$ at the upper base, and \$-V\_0/2\$ at the lower base. With this geometry, the surface charge densities \$\pm\sigma\_B\$ on the two bases alone cannot generate a uniform electric field inside the whole conductor. We need another charge density \$\sigma\_L\$, not necessarily uniform, distributed on the lateral surface of the cylinder. In order to treat the problem, we introduce a cylindrical coordinate system \$(r, \phi, z)\$, with the \$z\$ axis coinciding with the axis of the cylinder, the origin \$O\$ being located so that the upper and lower bases are at \$z = \pm h\$, respectively (this is not apparent from Fig. S-4.6 for practical reasons).

Because of symmetry reasons, \$\sigma\_L\$ cannot depend on \$\phi\$. And it cannot be constant along the lateral surface, otherwise, neglecting the boundary effects, it would generate no field inside the conductor. Thus,

$\sigma_L$  must be a function of \$z\$, and \$z\$ only. As an educated guess, we assume that \$\sigma\_L\$ is proportional to \$z\$, so that we have

$$\sigma_L(z) = \gamma z, \quad (\text{S-4.73})$$

with \$\gamma\$ a constant. This choice leads to \$\sigma\_L(0) = 0\$ at \$z = 0\$, and \$|\sigma\_L|\$ increasing, with opposite signs toward the upper and lower bases. Let us evaluate the electric potential in a point \$P \equiv (r, 0, z)\$, with \$r \ll h\$, not necessarily inside the conductor. The choice of \$\phi = 0\$ does not affect the generality of the approach because of the rotational symmetry around the \$z\$ axis. The contribution of the charge element \$dq = \gamma z' a d\phi dz'\$, located on the lateral surface of the conductor at \$(a, \phi, z')\$, to the potential \$\Phi(r, 0, z)\$ is

$$d\Phi = k_e \frac{dq}{s},$$

where \$s\$ is the distance between the points \$(a, \phi, z')\$ and \$(r, 0, z)\$. The distance \$s\$ can be evaluated by the cosine formula,

$$\begin{aligned} s &= \sqrt{(z' - z)^2 + a^2 + r^2 - 2ar \cos \phi} \\ &= \sqrt{(z' - z)^2 + a^2 \left(1 + \frac{r^2}{a^2} - 2\frac{r}{a} \cos \phi\right)} \\ &= \sqrt{(z' - z)^2 + a^2 f(r, \phi)}, \end{aligned} \quad (\text{S-4.74})$$

where we have defined

$$f(r, \phi) = \left( 1 + \frac{r^2}{a^2} - 2 \frac{r}{a} \cos \phi \right). \quad (\text{S-4.75})$$

We thus have

$$d\Phi = k_e a \gamma z' \frac{d\phi dz'}{\sqrt{(z' - z)^2 + a^2 f(r, \phi)}}, \quad (\text{S-4.76})$$

and the electric potential in  $P$  is

$$\Phi(P) = k_e a \gamma \int_0^{2\pi} d\phi \int_{-h}^h \frac{z' dz'}{\sqrt{(z' - z)^2 + a^2 f(r, \phi)}}. \quad (\text{S-4.77})$$

In order to evaluate the integral we introduce a new variable  $\zeta = z' - z$ , so that

$$\Phi(P) = k_e a \gamma \int_0^{2\pi} d\phi \int_{-h-z}^{h-z} \frac{(z + \zeta) d\zeta}{\sqrt{\zeta^2 + a^2 f(r, \phi)}}. \quad (\text{S-4.78})$$

The indefinite integrals needed in the formula are

$$\int \frac{d\zeta}{\sqrt{\zeta^2 + b}} = \ln \left( 2\zeta + 2\sqrt{\zeta^2 + b} \right), \quad \text{and} \quad \int \frac{\zeta d\zeta}{\sqrt{\zeta^2 + b}} = \sqrt{\zeta^2 + b}. \quad (\text{S-4.79})$$

We can split  $\Phi(P)$  into the sum of two terms  $\Phi(P) = \Phi_1(P) + \Phi_2(P)$ , where

$$\begin{aligned} \Phi_1(P) &= k_e a \gamma z \int_0^{2\pi} d\phi \int_{-h-z}^{h-z} \frac{d\zeta}{\sqrt{\zeta^2 + a^2 f(r, \phi)}} \\ &= k_e a \gamma z \int_0^{2\pi} d\phi \ln \left[ \frac{h-z+\sqrt{(h-z)^2+a^2 f(r, \phi)}}{-h-z+\sqrt{(h+z)^2+a^2 f(r, \phi)}} \right] \end{aligned} \quad (\text{S-4.80})$$

and

$$\begin{aligned} \Phi_2(P) &= k_e a \gamma \int_0^{2\pi} d\phi \int_{-h-z}^{h-z} \frac{\zeta d\zeta}{\sqrt{\zeta^2 + a^2 f(r, \phi)}} \\ &= k_e a \gamma \int_0^{2\pi} d\phi \left[ \sqrt{(h-z)^2 + a^2 f(r, \phi)} - \sqrt{(h+z)^2 + a^2 f(r, \phi)} \right]. \end{aligned} \quad (\text{S-4.81})$$

The square roots appearing in the integrals can be approximated as

$$\sqrt{(h \pm z)^2 + a^2 f(r, \phi)} \approx h \pm z + \frac{a^2}{2(h \pm z)} f(r, \phi) \quad (\text{S-4.82})$$

up to the second order in  $a/h$  and  $r/h$ . The second order is needed only in the denominator of the argument of the logarithm appearing in (S-4.80), where the first order cancels out with  $-h - z$ . Thus,  $\Phi_1(P)$  can be approximated as

$$\begin{aligned} \Phi_1(P) &\approx k_e a \gamma z \int_0^{2\pi} d\phi \ln \left\{ \frac{2(h-z)}{a^2 f(r, \phi) / [2(h+z)]} \right\} \\ &= 2\pi k_e a \gamma z \int_0^{2\pi} d\phi \ln \left[ \frac{4(h^2 - z^2)}{a^2 f(r, \phi)} \right] \approx 2\pi k_e a c z \int_0^{2\pi} d\phi \ln \left[ \frac{4h^2}{a^2 f(r, \phi)} \right], \end{aligned} \quad (\text{S-4.83})$$

while the approximation for  $\Phi_2(P)$  is

$$\Phi_2(P) \approx -k_e a \gamma \int_0^{2\pi} 2z d\phi = -4\pi k_e a c z. \quad (\text{S-4.84})$$

The two contributions sum up to

$$\begin{aligned} \Phi(P) &= \Phi_1(P) + \Phi_2(P) \approx 2\pi k_e a \gamma z \left\{ \int_0^{2\pi} d\phi \ln \left[ \frac{4h^2}{a^2 f(r, \phi)} \right] - 2 \right\} \\ &= 2\pi k_e a \gamma z \left\{ 2\pi \ln \left( \frac{h^2}{a^2} \right) + \int_0^{2\pi} d\phi \ln \left[ \frac{4}{f(r, \phi)} \right] - 2 \right\} \\ &= 2\pi k_e a \gamma z \left\{ 4\pi \ln \left( \frac{h}{a} \right) + \int_0^{2\pi} d\phi \ln \left[ \frac{4}{f(r, \phi)} \right] - 2 \right\}. \end{aligned} \quad (\text{S-4.85})$$

If  $h$  is sufficiently large, the first terms in braces is dominant, and we have

$$\Phi(r, z) \approx 8\pi^2 k_e a \gamma \ln \left( \frac{h}{a} \right) z, \quad (\text{S-4.86})$$

thus independent of  $r$ , within our approximations, as expected. Since we have assumed  $\Phi(r, h) = V_0/2$ , we must have

$$\frac{V_0}{2} = 8\pi^2 k_e a \gamma \ln \left( \frac{h}{a} \right) h, \quad (\text{S-4.87})$$

which leads to

$$\gamma = \frac{V_0}{16\pi^2 k_e a h \ln(h/a)} \quad \text{and} \quad \sigma_L(z) = \frac{V_0}{16\pi^2 k_e a h \ln(h/a)} z. \quad (\text{S-4.88})$$

## S-4.10 An Infinite Resistor Ladder

Let us denote by  $R_L$  the resistance measured between the terminals  $A$  and  $B$ . If a further unit of three resistors is added to the left of the ladder, as in Fig. S-4.7, the "new" resistance measured between terminals  $A'$  and  $B'$  must equal the "old" resistance  $R_L$ . The "old" resistor ladder at the right of terminals  $A$  and  $B$  can be replaced by the equivalent resistance  $R_L$ , leading to the configuration of Fig. S-4.8. We see that the resistance between terminals  $A'$  and  $B'$  is the solution of

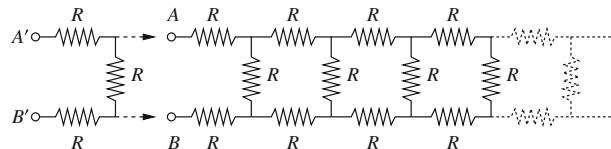


Fig. S-4.7

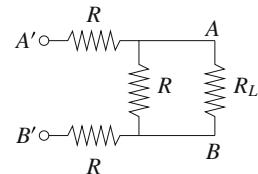


Fig. S-4.8

$$\begin{aligned} R_L &= 2R + \frac{RR_L}{R+R_L} \\ RR_L + R_L^2 &= 2R^2 + 2RR_L + RR_L \\ R_L^2 - 2RR_L - 2R^2 &= 0, \end{aligned} \quad (\text{S-4.89})$$

and, disregarding the negative solution, we have

$$R_L = R(1 + \sqrt{3}). \quad (\text{S-4.90})$$

# Chapter S-5

## Solutions for Chapter 5

### S-5.1 The Rowland Experiment

**a)** Neglecting the boundary effects, the electric field  $\mathbf{E}_0$  in the regions between the disk and the plates is uniform, perpendicular to the disk surfaces, and its magnitude is  $E_0 = V_0/h$  in both regions. In both regions, the field is directed outwards from the disk, according to the polarity of the source shown in Fig. 5.1. The charge densities of the lower and upper surfaces of the disk,  $\sigma$ , are equal in modulus and sign, because the field must be zero inside the disk. Thus we have  $\sigma = E_0/(4\pi k_e) = V_0/(4\pi k_e h)$ . In SI units we have  $\sigma = \epsilon_0 V_0/h$ , with  $\epsilon_0 = 8.85 \times 10^{-12}$ ,  $V_0 = 10^4$  V,  $h = 5 \times 10^{-3}$  m, resulting in  $\sigma = 1.77 \times 10^{-5}$  C/m<sup>2</sup>. In Gaussian units we have  $\sigma = V_0/(4\pi h)$ , with  $V_0 = 33.3$  statV and  $h = 0.5$  cm, resulting in  $\sigma = 5.3$  statC/cm<sup>2</sup>.

**b)** We evaluate the magnetic field  $\mathbf{B}_c$  at the center of the disk by dividing its upper and lower surfaces into annuli of radius  $r$  (with  $0 < r < a$ ) and width  $dr$ . On each surface, each annulus carries a charge  $dq = \sigma dS = 2\pi\sigma r dr$ . Due to the rotation of the disk, each annulus is equivalent to a coil with a current intensity  $dI = \omega dq/(2\pi)$ , that generates at its center a magnetic field  $d\mathbf{B}_c = 2\pi k_m dI/r \hat{\omega}$ , perpendicular to the disk plane. The total field at the center of the disk is thus given by the integral

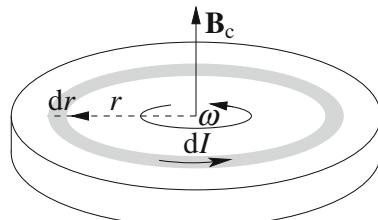


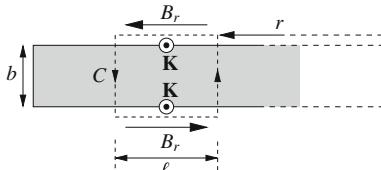
Fig. S-5.1

$$B_c = 2 \int_0^a 2\pi k_m \frac{dI}{r} = 4\pi k_m \sigma \omega \int_0^a \frac{r dr}{r}$$

$$= 4\pi k_m \omega \sigma a = \begin{cases} \mu_0 \omega \sigma a \simeq 1.4 \times 10^{-9} \text{ T} & \text{SI} \\ \frac{4\pi}{c} \omega \sigma a \simeq 1.4 \times 10^{-5} \text{ G} & \text{Gaussian,} \end{cases} \quad (\text{S-5.1})$$

where the factor of 2 in front of the first integral is due to contribution of both the upper and the lower surfaces of the disk to the magnetic field.

The magnetic field component  $B_r$ , parallel to the disk surface and close to it, can be evaluated by applying Ampère's law to the closed rectangular path  $C$  shown Fig. S-5.2. The path is placed at a distance  $r$  from the rotation axis, with the sides parallel to the disk surfaces having length  $\ell \ll r$ , so that  $\mathbf{B}_r$  is approximately constant along the sides. The contribution of the vertical paths to the line integral cancel each other, thus



$$4\pi k_m I_c = \oint_C \mathbf{B} \cdot d\ell \approx 2B_r \ell, \quad (\text{S-5.2})$$

where  $I_c$  is the current flowing through the rectangular loop  $C$ , and the antisymmetry of  $B_r$  with respect to the midplane has been used. The rotation of the disk leads to a surface current density  $\mathbf{K} = \sigma \mathbf{v} = \sigma \omega r \hat{\phi}$ , resulting in a total current flowing through the rectangular loop  $I_c = 2K\ell = 2\sigma\omega r\ell$ . Thus, according to (S-5.2),

$$B_r(r) = \frac{2\pi k_m I_c}{\ell} = \frac{2\pi k_m}{\ell} 2\sigma\omega r\ell = 4\pi k_m \sigma \omega r. \quad (\text{S-5.3})$$

The maximum value of  $B_r(r)$  occurs at  $r = a$ , where  $B_r(a) = B_c$ .

c) The deviation angle of the needle is given by  $\tan \theta = B/B_{\oplus}$ , hence

$$\theta \approx \frac{B}{B_{\oplus}} = 2.8 \times 10^{-5} \text{ rad} = 1.6 \times 10^{-3} \text{ deg}. \quad (\text{S-5.4})$$

The expected angle is very small, and its measurement requires exceptional care.

## S-5.2 Pinch Effect in a Cylindrical Wire

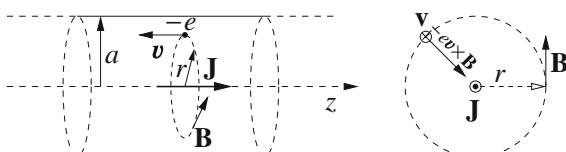


Fig. S-5.3

The magnetic field  $\mathbf{B}$  is azimuthal for symmetry reasons. Its only component  $B_\phi(r)$  can easily be evaluated by applying Ampère's circuital law to a circular closed path coaxial with the cylinder axis, as shown in figure. We have

a) We use a cylindrical coordinate system  $(r, \phi, z)$  with the  $z$  axis along the axis of the cylinder. The vectors  $\mathbf{J}$  and  $\mathbf{v}$  are along  $z$ . If we assume  $J > 0$  we have  $v < 0$  since  $J = -n_e e v$ .

$$2\pi r B_\phi = 4\pi^2 r^2 k_m J = \begin{cases} \mu_0 \pi r^2 J, & \text{SI} \\ \frac{4\pi^2 r^2}{c} J, & \text{Gaussian,} \end{cases} \quad (\text{S-5.5})$$

so that

$$B_\phi = 2\pi k_m J r = -2\pi k_m n_e e v, \quad (\text{S-5.6})$$

and  $B_\phi > 0$ , since  $v < 0$ . Thus the field lines of  $\mathbf{B}$  are oriented counterclockwise with respect to the  $z$  axis.

The magnetic force  $\mathbf{F}_m = -eb_m(\mathbf{v} \times \mathbf{B})$  is radial and directed towards the  $z$  axis

$$\mathbf{F}_m = -2\pi k_m b_m n_e e^2 v^2 = \begin{cases} -\frac{\mu_0 n_e e^2 v^2}{2} \mathbf{r}, & \text{SI} \\ -\frac{2\pi n_e e^2 v^2}{c^2} \mathbf{r}, & \text{Gaussian.} \end{cases} \quad (\text{S-5.7})$$

Thus the magnetic force pulls the charge carriers toward the axis of the wire, independently of their sign. A beam of charged particles always gives origin to a magnetic field that tends to “pinch” the beam, i.e., to shrink it toward its axis. However, if the beam is propagating in vacuum, the Coulomb repulsion between the charged particles is dominant. In our case, or in the case of a plasma, the medium is globally neutral, and, initially, the positive and negative charge densities are uniform over the medium, so that the pinch effect can be observed, at least in principle.

**b)** The Lorentz force is  $\mathbf{F}_L = -e(\mathbf{E} + b_m \mathbf{v} \times \mathbf{B})$ . At equilibrium the  $r$  component of  $\mathbf{F}_L$  must be zero in the presence of conduction electrons (see Problem 1.9), so that the electrons flow only along the  $z$  axis. Thus the  $r$  component of the electric field,  $E_r$ , must be

$$E_r = -b_m v B = 2\pi k_m b_m n_e e v^2 r, \quad (\text{S-5.8})$$

while  $E_z = J/\sigma$ , where  $\sigma$  is the conductivity of the material. According to Gauss's law, a charge density  $\varrho$ , uniform over the cylinder volume, generates a field  $\mathbf{E} = 2\pi k_e \varrho \mathbf{r}$ , and the required field  $E_r$  is generated by the charge density

$$\varrho = \frac{k_m b_m}{k_e} n_e e v^2 = n_e e \frac{v^2}{c^2}, \quad (\text{S-5.9})$$

independent of the system of units. On the other hand, the global charge density is  $\varrho = e(Zn_i - n_e)$ , so that

$$n_e = \frac{Zn_i}{1 - v^2/c^2}. \quad (\text{S-5.10})$$

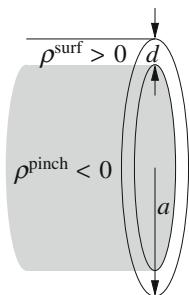
Thus, the electron density is uniform over the wire volume, but it exceeds the value  $n_{e0} = Zn_i$ , corresponding to  $\varrho = 0$ . This means that the number density of the electrons is increased by a factor  $(1 - v^2/c^2)^{-1}$ , and  $\varrho$  is negative, *inside* the wire. The “missing” positive charge is uniformly distributed over the surface of the conductor.

**c)** For electrons in a usual Ohmic conductor we have  $v \approx 1 \text{ cm/sec} = 10^{-2} \text{ m/sec}$ , corresponding to  $(v/c)^2 \approx 10^{-21}$ , and the resulting “pinch” effect is so small that it cannot be observed. On the other hand, the effect may be strong in high density particle beams or plasma columns, where  $v$  is not negligible with respect to  $c$ .

In order to get further insight into the size of the effect, let us consider an Ohmic cylindrical conductor (wire) of radius  $a$ . We assume that the electron density is increased in a central cylindrical region of radius  $a - d$ , where  $n_e^{\text{pinch}} = Zn_i/(1 - v^2/c^2)$ , and the volume charge density is

$$\varrho^{\text{pinch}} = e(Zn_i - n_e^{\text{pinch}}) = -eZn_i \frac{v^2}{c^2} \frac{1}{1 - v^2/c^2} < 0, \quad (\text{S-5.11})$$

while the cylindrical shell between  $r = a - d$  and  $r = a$  is depleted of conduction electrons, so that its charge density  $\varrho^{\text{surf}}$  is  $\varrho^{\text{surf}} = eZn_i$ . The thickness  $d$  of the depleted cylindrical shell can be estimated by the constraint of charge conservation.



A slice of wire of length  $\ell$  must be globally neutral, thus, assuming  $d \ll a$ , we must have

$$\begin{aligned} \pi(a-d)^2\ell\varrho^{\text{pinch}} &= -2\pi ad\ell\varrho^{\text{surf}} \\ \pi(a-d^2)eZn_i \frac{v^2}{c^2} \frac{1}{1-v^2/c^2} &= 2\pi adeZn_i \\ (a^2 - 2ad + d^2) \frac{v^2}{c^2} \frac{1}{1-v^2/c^2} &= 2ad, \end{aligned} \quad (\text{S-5.12})$$

Fig. S-5.4

and, since  $v \ll c$  and  $d \ll a$ , we can approximate

$$a^2 \frac{v^2}{c^2} \approx 2ad, \quad \text{so that} \quad d \approx \frac{a}{2} \frac{v^2}{c^2}. \quad (\text{S-5.13})$$

Remembering that  $v^2/c^2$  is of the order of  $10^{-21}$ , we see that a value of  $d$  of the order of the crystal lattice spacing ( $\approx 10^{-10} \text{ m}$ ) would require a wire of radius  $a \approx 10^{11} \text{ m}$ , a remarkably large radius!

### S-5.3 A Magnetic Dipole in Front of a Magnetic Half-Space

**a)** The analogy between the magnetostatic equations in the absence of free currents ( $\nabla \times \mathbf{H} = 0$ ,  $\nabla \cdot \mathbf{B} = 0$ ) with those for the electrostatics of dielectric in the absence of free charges ( $\nabla \times \mathbf{E} = 0$ ,  $\nabla \cdot \mathbf{D} = 0$ ) indicates that the solution of this problem will be

similar to that of Problem (3.2). Thus, analogously to Problem (3.2), we treat the vacuum half-space and the medium-filled half-space separately, with separate educated guesses for the magnetic field in each half-space.

This in order to exploit the uniqueness theorem for the Poisson equation (5.5). Our guess for half-space 1 ( $x < 0$ ) is that the field is same as if the magnetic medium were removed from half-space 2 (thus, vacuum in the whole space), and replaced by an image magnetic dipole  $\mathbf{m}'$  located symmetrically to  $\mathbf{m}$  with respect to the  $x = 0$  plane, at  $x = d$ . Our guess for half-space 2 ( $x > 0$ ) is that the field is the same as if the magnetic medium filled the whole space, and the magnetic dipole  $\mathbf{m}$  were replaced by a different magnetic dipole  $\mathbf{m}''$ , placed at the same location. Thus we look for values of  $\mathbf{m}'$  and  $\mathbf{m}''$  originating a magnetic field  $\mathbf{B}_1$  in half-space 1, and a magnetic field  $\mathbf{B}_2$  in half-space 2, satisfying the interface conditions at  $x = 0$

$$B_{1\perp}(x = 0^-) = B_{2\perp}(x = 0^+), \quad B_{1\parallel}(x = 0^-) = \frac{1}{\mu_r} B_{2\parallel}(x = 0^+), \quad (\text{S-5.14})$$

The subscripts  $\parallel$  and  $\perp$  stand for parallel and perpendicular to the  $x = 0$  plane, respectively. Thus, at a generic point  $P \equiv (0, y, z)$  of the  $x = 0$  plane, we must have

$$\begin{aligned} B_x(0^-, y, z) &= B_x(0^+, y, z) \\ B_y(0^-, y, z) &= \frac{1}{\mu_r} B_y(0^+, y, z) \\ B_z(0^-, y, z) &= \frac{1}{\mu_r} B_z(0^+, y, z). \end{aligned} \quad (\text{S-5.15})$$

The field generated by a magnetic dipole  $\mathbf{m}$  in a medium of relative magnetic permittivity  $\mu_r$  is

$$\mathbf{B}(\mathbf{r}) = \frac{k_m}{b_m} \mu_r \frac{3(\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}}{r^3} \quad (\text{S-5.16})$$

where  $\mathbf{r}$  is the distance vector directed from  $\mathbf{m}$  to the point where we evaluate the field, and  $\hat{\mathbf{r}} = \mathbf{r}/r$  is the unit vector along  $\mathbf{r}$ . Note that, differently from Problem (3.2), here we do not have cylindrical symmetry around the  $x$  axis, because the real magnetic dipole  $\mathbf{m}$  is not lying on  $x$ . It is convenient to introduce the angles  $\theta = \arcsin(d/r)$  and  $\phi = \arctan(z/y)$ , and write the Cartesian components of  $\mathbf{B}$  separately

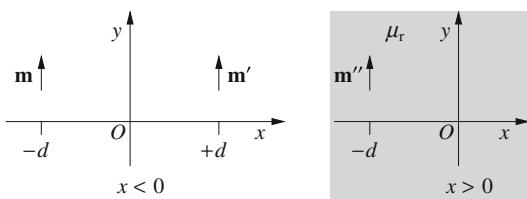


Fig. S-5.5

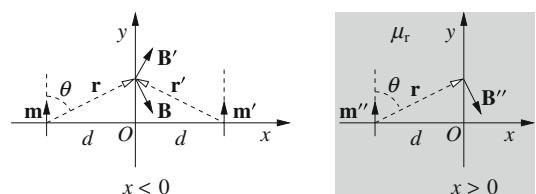


Fig. S-5.6

$$\begin{aligned} B_x &= \frac{k_m}{b_m} \mu_r \frac{3m \cos \theta \sin \theta}{r^3} \\ B_y &= \frac{k_m}{b_m} \mu_r \frac{3m \cos^2 \theta \cos \phi - m}{r^3} \\ B_z &= \frac{k_m}{b_m} \mu_r \frac{3m \cos^2 \theta \sin \phi}{r^3}, \end{aligned} \quad (\text{S-5.17})$$

where  $r = \sqrt{d^2 + y^2 + z^2}$ . If we replace (S-5.17) into (S-5.15) and divide by  $(k_m/b_m)$  the boundary conditions become

$$\begin{aligned} \frac{3m \cos \theta \sin \theta}{r^3} - \frac{3m' \cos \theta \sin \theta}{r^3} &= \mu_r \frac{3m'' \cos \theta \sin \theta}{r^3} \\ \frac{3m \cos^2 \theta \cos \phi - m}{r^3} + \frac{3m' \cos^2 \theta \cos \phi - m'}{r^3} &= \frac{3m'' \cos^2 \theta \cos \phi - m''}{r^3} \\ \frac{3m \cos^2 \theta \sin \phi}{r^3} + \frac{3m' \cos^2 \theta \sin \phi}{r^3} &= \frac{3m'' \cos^2 \theta \sin \phi}{r^3}, \end{aligned} \quad (\text{S-5.18})$$

which can be further simplified into

$$\begin{aligned} \mathbf{m} - \mathbf{m}' &= \mu_r \mathbf{m}'' \\ \mathbf{m} + \mathbf{m}' &= \mathbf{m}'' \end{aligned} \quad (\text{S-5.19})$$

leading to the solution

$$\mathbf{m}' = -\frac{\mu_r - 1}{\mu_r + 1} \mathbf{m}, \quad \mathbf{m}'' = \frac{2}{\mu_r + 1} \mathbf{m}. \quad (\text{S-5.20})$$

As expected, the expressions for  $m'$  and  $m''$  as functions of  $m$  are identical to (S-3.8) for the image charges  $q'$  and  $q''$  as functions of the real charge  $q$  (although Problem 3.2 involves point charges, the generalization to electric dipoles is immediate).

**b)** The force exerted by the magnetic half-space on  $\mathbf{m}$  equals the force that would be exerted on  $\mathbf{m}$  by a real magnetic dipole  $\mathbf{m}'$  located at  $x = +d$ . The force between two magnetic dipoles at a distance  $r$  from each other is

$$\mathbf{f} = -\frac{k_m}{b_m} \nabla \left[ \frac{\mathbf{m} \cdot \mathbf{m}' - 3(\mathbf{m} \cdot \hat{\mathbf{r}})(\mathbf{m}' \cdot \hat{\mathbf{r}})}{r^3} \right], \quad (\text{S-5.21})$$

with, in our case,  $\hat{\mathbf{r}} = \hat{\mathbf{x}}$ ,  $r = 2d$ , and  $\mathbf{m} \cdot \hat{\mathbf{r}} = \mathbf{m}' \cdot \hat{\mathbf{r}} = 0$ , so that the force on  $\mathbf{m}$  is

$$\mathbf{f} = -\frac{k_m}{b_m} \frac{3m^2}{r^4} \left( \frac{\mu_r - 1}{\mu_r + 1} \right) \hat{\mathbf{x}} = \frac{k_m}{b_m} \frac{3m^2}{16d^4} \left( \frac{\mu_r - 1}{\mu_r + 1} \right) \hat{\mathbf{x}}. \quad (\text{S-5.22})$$

The force is repulsive (antiparallel to  $\hat{\mathbf{x}}$ ) for  $\mu_r < 1$  (diamagnetic material), and attractive (parallel to  $\hat{\mathbf{x}}$ ) for  $\mu_r > 1$  (paramagnetic material). At the limit  $\mu_r \rightarrow 0$  we have a perfect diamagnetic material (superconductor), and  $\mathbf{m}' \rightarrow \mathbf{m}$ , the two dipoles are parallel and the force is repulsive, as expected. In this case  $\mathbf{m}'' \neq 0$ , so that  $\mathbf{H} \neq 0$

in the half-space 2, where, however,  $\mu_r = 0$  so that  $\mathbf{B} = \mu_0 \mu_r \mathbf{H} = 0$ . The situation is *opposite* to that of a perfect conductor in electrostatics, where an electric dipole would induce an *opposite* image dipole, and the force would be *attractive*.

At the limit of  $\mu_r \rightarrow \infty$  (perfect ferromagnetic material), we have  $\mathbf{m}' \rightarrow -\mathbf{m}$ , corresponding to an *attractive* force, while  $\mathbf{m}'' \rightarrow 0$  and  $\mathbf{H} \rightarrow 0$  inside the material. This situation is analogous to the case of a conductor in electrostatics. Notice that  $\mathbf{B}$  is finite inside the material (since  $\mu_r \mathbf{m}'' \rightarrow 2\mathbf{m}$ ) and given by

$$\mathbf{B} = \frac{k_m}{b_m} \mu_r \frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{m}'') - \mathbf{m}''}{4\pi r^3} \rightarrow 2 \frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{m}) - \mathbf{m}}{4\pi r^3}, \quad (x > 0), \quad (\text{S-5.23})$$

so that the paramagnetic material doubles the value of the magnetic field in vacuum in the limit  $\mu_r \rightarrow \infty$ .

## S-5.4 Magnetic Levitation

a) The radial component  $B_r$  of the magnetic field close to the  $z$  axis can be evaluated by applying Gauss's law  $\nabla \cdot \mathbf{B} = 0$  to a small closed cylinder of radius  $r$ , coaxial with the  $z$  axis, and with the bases at  $z$  and  $z + \Delta z$ , as shown in Fig. S-5.7. The flux of  $\mathbf{B}$  through the total surface of the cylinder must be zero, thus we have

$$0 = \oint_{\text{cylinder}} \mathbf{B} \cdot d\mathbf{S} \quad (\text{S-5.24})$$

$$= 2\pi r \Delta z B_r(r) + \pi r^2 [B_z(z + \Delta z) - B_z(z)],$$

leading to

$$B_r = -\frac{r[B_z(z + \Delta z) - B_z(z)]}{2\Delta z} \quad (\text{S-5.25})$$

$$\simeq -\frac{B_0}{2L} r$$

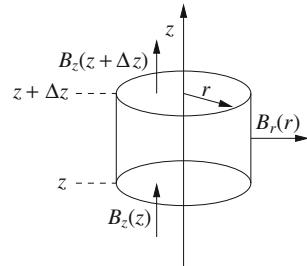


Fig. S-5.7

b) According to Table 5.1, the force exerted by an external magnetic field  $\mathbf{B}$  on a magnetic dipole  $\mathbf{m}$  is  $\mathbf{f} = (\mathbf{m} \cdot \nabla) \mathbf{B}$ . If we assume that the dipole is moving in a region free of electric current densities, so that  $\nabla \times \mathbf{B} = 0$ , the work done on the dipole when it performs an infinitesimal displacement  $d\mathbf{r} \equiv (dx, dy, dz)$  is<sup>1</sup>

<sup>1</sup>We have

$$dW = [(\mathbf{m} \cdot \nabla) \mathbf{B}] \cdot d\mathbf{r} = \sum_{i,j} m_i \partial_i B_j dx_j = \sum_{i,j} m_i \partial_j B_i dx_j = \sum_i m_i dB_i = \mathbf{m} \cdot d\mathbf{B}, \quad (\text{S-5.26})$$

where, as usual,  $x_{1,2,3} = x, y, z$ , and  $\partial_{1,2,3} = \partial_x, \partial_y, \partial_z$ . We have used the property  $\partial_i B_j = \partial_j B_i$ , trivial for  $i = j$ , while the condition  $\nabla \times \mathbf{B} = 0$  implies  $\partial_i B_j - \partial_j B_i = 0$  also for  $i \neq j$ .

$$dW = \mathbf{f} \cdot d\mathbf{r} = [(\mathbf{m} \cdot \nabla) \mathbf{B}] \cdot d\mathbf{r} = \mathbf{m} \cdot d\mathbf{B}. \quad (\text{S-5.27})$$

For a permanent magnetic dipole this leads to the well known expression for the potential energy of a dipole located at  $\mathbf{r}$

$$U(\mathbf{r}) = -\mathbf{m} \cdot \mathbf{B}(\mathbf{r}). \quad (\text{S-5.28})$$

Here, however, the magnetic dipole is not permanent. Rather, we have an induced dipole  $\mathbf{m} = \alpha \mathbf{B}$ . Thus we have

$$\begin{aligned} U(\mathbf{r}_2) - U(\mathbf{r}_1) &= - \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{m} \cdot d\mathbf{B} = -\alpha \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{B} \cdot d\mathbf{B} = -\frac{\alpha}{2} \int_{\mathbf{r}_1}^{\mathbf{r}_2} dB^2 \\ &= \frac{\alpha}{2} [B^2(\mathbf{r}_1) - B^2(\mathbf{r}_2)] = \frac{1}{2} [\mathbf{m}(\mathbf{r}_1) \cdot \mathbf{B}(\mathbf{r}_1) - \mathbf{m}(\mathbf{r}_2) \cdot \mathbf{B}(\mathbf{r}_2)]. \end{aligned} \quad (\text{S-5.29})$$

and the potential energy for the induced dipole at  $\mathbf{r}$  is written

$$U(\mathbf{r}) = -\frac{1}{2} \mathbf{m}(\mathbf{r}) \cdot \mathbf{B}(\mathbf{r}). \quad (\text{S-5.30})$$

For the present problem, this leads to

$$U(\mathbf{r}) = -\frac{1}{2} \mathbf{m}(\mathbf{r}) \cdot \mathbf{B}(\mathbf{r}) = -\frac{1}{2} \alpha B^2(\mathbf{r}) = -\frac{1}{2} \frac{\alpha B_0^2}{L^2} \left( z^2 + \frac{r^2}{4} \right). \quad (\text{S-5.31})$$

- c) The potential energy  $U$  has a minimum in the origin ( $r = 0, z = 0$ ) if  $\alpha < 0$  (diamagnetic particle). The force is

$$\mathbf{f} = -\nabla U = -\frac{1}{2} |\alpha| \frac{2B_0^2}{L^2} \left( z \hat{\mathbf{z}} + \frac{r}{4} \mathbf{r} \right). \quad (\text{S-5.32})$$

Thus, we have a harmonic force both for radial and axial displacements, with corresponding oscillation frequencies

$$\omega_z = \sqrt{\frac{|\alpha| B_0^2}{ML^2}}, \quad \omega_r = \frac{\omega_z}{2}. \quad (\text{S-5.33})$$

## S-5.5 Uniformly Magnetized Cylinder

a) The volume magnetization current (bound current) density  $\mathbf{J}_m$  is zero all over the cylinder volume because the cylinder magnetization  $\mathbf{M}$  is uniform, and  $\mathbf{J}_m = \nabla \times \mathbf{M}/b_m$ . For the surface magnetization current density  $\mathbf{K}_m$  we have  $\mathbf{K}_m = \mathbf{M} \times \hat{\mathbf{n}}/b_m$ , where  $\hat{\mathbf{n}}$  is the unit vector perpendicular to the cylinder lateral surface, and  $b_m$  is the system dependent constant defined in (5.1). Since  $\mathbf{M}$  and  $\hat{\mathbf{n}}$  are perpendicular to each other, we have  $K_m = |\mathbf{K}_m| = |\mathbf{M}|/b_m$ .

b) The magnetized cylinder is equivalent to a solenoid with  $nI = K_m$ , where  $n$  is the number of coils per unit length, and  $I$  is the electric current circulating in each coil. Thus, at the  $h \gg R$  limit, the magnetic field is uniformly zero outside the cylinder, and it is uniform and equal to

$$B = \frac{4\pi k_m}{b_m} nI = \frac{4\pi k_m}{b_m} K_m = \frac{4\pi k_m}{b_m} M = \begin{cases} \mu_0 M, & \text{SI} \\ 4\pi M, & \text{Gaussian,} \end{cases} \quad (\text{S-5.34})$$

inside. The auxiliary field  $\mathbf{H}$  is zero both inside and outside the cylinder because

$$\mathbf{H}_{in} = \begin{cases} \frac{\mathbf{B}_{in}}{\mu_0} - \mathbf{M} = 0, & \text{SI} \\ \mathbf{B}_{in} - 4\pi \mathbf{M} = 0, & \text{Gaussian.} \end{cases} \quad \mathbf{H}_{out} = \begin{cases} \frac{\mathbf{B}_{out}}{\mu_0} = 0, & \text{SI} \\ \mathbf{B}_{out} = 0, & \text{Gaussian.} \end{cases} \quad (\text{S-5.35})$$

c) At the “flat cylinder” limit,  $R \gg h$ , the cylinder is equivalent to a single coil of radius  $R$  carrying a current  $I = hK_m = hM/b_m$ . Thus we have for the field at its center

$$B_0 = 2\pi k_m \frac{I}{R} = \begin{cases} \frac{\mu_0 I}{2R} = \frac{\mu_0 Mh}{2R}, & \text{SI} \\ \frac{2\pi I}{cR} = \frac{2\pi Mh}{cR}, & \text{Gaussian,} \end{cases} \quad (\text{S-5.36})$$

and  $\mathbf{B}_0$  approaches zero as  $h/r \rightarrow 0$ .

d) The equivalent magnetic charge density is defined as  $\varrho_m = -\nabla \cdot \mathbf{M}$ , thus  $\varrho_m \equiv 0$  inside the cylinder volume, while the two bases of the cylinder carry surface magnetic charge densities  $\sigma_m = \mathbf{M} \cdot \hat{\mathbf{n}} = \pm M$ . Therefore our flat magnetized cylinder is the “magnetostatic” equivalent of an electrostatic parallel-plate capacitor. The equivalent magnetic charge “generates” the auxiliary magnetic field  $\mathbf{H}$ , which is uniform, and equal to  $H = -\sigma_m = -M$ , inside the volume of the flat cylinder, and zero outside. Thus  $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$  is zero everywhere (more realistically, it is zero far from the boundaries).

The field of a magnetized cylinder and its electrostatic analog are further discussed in Problem 13.1.

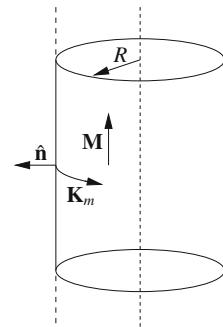


Fig. S-5.8

## S-5.6 Charged Particle in Crossed Electric and Magnetic Fields

a) We choose a Cartesian laboratory frame of reference  $xyz$  with the  $y$  axis parallel to the electric field  $\mathbf{E}$ , the  $z$  axis parallel to the magnetic field  $\mathbf{B}$ , and the origin  $O$  located so that the particle is initially at rest in  $O$ . The Lorentz force on the particle

$$\mathbf{f} = q[\mathbf{E} + b_m \mathbf{v} \times \mathbf{B}]$$

has no  $z$  component, and the motion of the particle occurs in the  $xy$  plane. The equations of motion are thus

$$\begin{aligned} m\ddot{x} &= b_m q B \dot{y}, \\ m\ddot{y} &= -b_m q B \dot{x} + qE. \end{aligned} \quad (\text{S-5.37})$$

It is convenient to introduce two new variables  $x'$ , and  $y'$ , such that

$$x = x' + v_0 t, \quad y = y', \quad (\text{S-5.38})$$

where  $v_0$  is a constant velocity, which we shall determine in order to simplify the equations of motion. The initial conditions for the primed variables are

$$\begin{aligned} \dot{x}'(0) &= 0, & \dot{x}'(0) &= -v_0, \\ \dot{y}'(0) &= 0, & \dot{y}'(0) &= -0. \end{aligned} \quad (\text{S-5.39})$$

Differentiating (S-5.38) with respect to time we obtain

$$\begin{aligned} \dot{x} &= v_0 + \dot{x}', & \ddot{x} &= \ddot{x}', \\ \dot{y} &= \dot{y}', & \ddot{y} &= \ddot{y}', \end{aligned} \quad (\text{S-5.40})$$

which we substitute into (S-5.37), thus obtaining the following equations for the time evolution of the primed variables

$$\begin{aligned} m\ddot{x}' &= b_m q B \dot{y}', \\ m\ddot{y}' &= -b_m q B v_0 - b_m q B \dot{x}' + qE. \end{aligned} \quad (\text{S-5.41})$$

Now we choose the constant velocity  $v_0$  to be

$$v_0 = \frac{E}{b_m B} = \begin{cases} \frac{E}{B}, & \text{SI,} \\ \frac{E}{B} c, & \text{Gaussian,} \end{cases} \quad (\text{S-5.42})$$

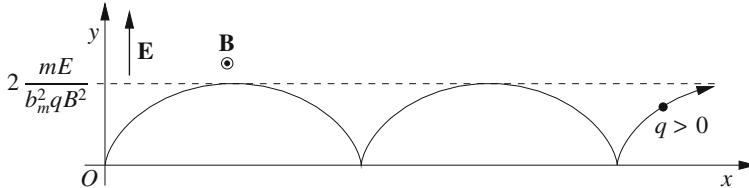
independently of the charge and mass of the particle, so that the terms  $qE$  and  $-b_m q B v_0$  cancel each other in the second of (S-5.41). The equations reduce to

$$\begin{aligned}\ddot{x}' &= b_m \frac{qB}{m} \dot{y}', \\ \ddot{y}' &= -b_m \frac{qB}{m} \dot{x}',\end{aligned}\quad (\text{S-5.43})$$

which are the equations of a uniform circular motion with angular velocity  $\omega = -b_m q B / m$ . The rotation is clockwise if  $q > 0$ , counterclockwise if  $q < 0$ . Since, according to (S-5.39),  $\dot{x}'(0) = -v_0$  and  $\dot{y}'(0) = 0$ , the radius of the circular path is

$$r = \frac{mv_0}{b_m q B} = \frac{mE}{b_m^2 q B^2}. \quad (\text{S-5.44})$$

The time evolution of the primed variables is thus



**Fig. S-5.9**

$$x' = x'_0 + r \cos(\omega t + \phi) = r \sin(\omega t) = -\frac{mE}{b_m^2 q B^2} \sin\left(\frac{b_m q B}{m} t\right), \quad (\text{S-5.45})$$

$$y' = y'_0 + r \sin(\omega t + \phi) = r - r \cos(\omega t) = \frac{mE}{b_m^2 q B^2} \left[ 1 - \cos\left(\frac{b_m q B}{m} t\right) \right],$$

where we have chosen the constants  $\phi = -\pi/2$ ,  $x'_0 = 0$ , and  $y'_0 = -r$ , in order to reproduce the initial conditions. The time evolution of the unprimed variables is

$$\begin{aligned}x &= \frac{E}{b_m B} t - \frac{mE}{b_m^2 q B^2} \sin\left(\frac{b_m q B}{m} t\right), \\ y &= \frac{mE}{b_m^2 q B^2} \left[ 1 - \cos\left(\frac{b_m q B}{m} t\right) \right],\end{aligned}\quad (\text{S-5.46})$$

and the observed motion is a cycloid, as shown in Fig. S-5.9 for a positive charge.

- b)** From the results of point **a)**, we know that the motion of the electron will be a cycloid starting from the negative plate, and reaching a maximum distance  $2r = 2mE/(b_m^2 q B^2)$  from it, where  $E = V/h$ . The condition for the electron not reaching the positive plate is thus

$$\frac{2mE}{b_m^2 q B^2} = \frac{2mV}{b_m^2 h q B^2} < h, \quad \text{corresponding to} \quad B > \frac{1}{b_m h} \sqrt{\frac{2m_e V}{e}}, \quad (\text{S-5.47})$$

where  $m_e$  is the electron mass, and  $e$  the absolute value of the electron charge.

### S-5.7 Cylindrical Conductor with an Off-Center Cavity

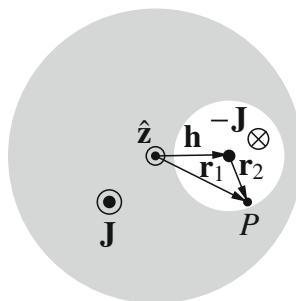


Fig. S-5.10

According to the superposition principle, a current density  $\mathbf{J}$  flowing uniformly through the cross section of the conductor in the positive  $z$  direction is equivalent to a uniform current density  $\mathbf{J}$ , flowing through the whole circular section of radius  $a$ , superposed to a current density  $-\mathbf{J}$ , flowing in the negative  $z$  direction through the cavity.

The magnetic field generated by an infinite, straight wire of radius  $a$  and uniformly distributed current density  $\mathbf{J} = J\hat{z}$  has azimuthal symmetry. Using a cylindrical coordinate system  $(r, \phi, z)$  with the  $z$  axis coinciding with the axis of the wire, the magnetic field  $\mathbf{B} = B_\phi(r)\hat{\phi}$  can be evaluated using (5.3): the line integral calculated over the circle  $C$  of radius  $r$  is

$$2\pi r B_\phi(r) = 4\pi k_m \oint_C \mathbf{J} \cdot d\mathbf{S} = 4\pi k_m J \times \begin{cases} \pi r^2, & r < a, \\ \pi a^2, & r > a. \end{cases} \quad (\text{S-5.48})$$

We thus obtain

$$B_\phi(r) = \begin{cases} \frac{2\pi k_m J r}{r}, & r < a, \\ \frac{2\pi k_m J a^2}{r}, & r > a. \end{cases} \quad (\text{S-5.49})$$

It is possible, and useful for the following, to write the above expressions in a compact vectorial form. Since  $\hat{r} \times \hat{z} = \hat{\phi}$  we have for the field  $\mathbf{B}_w = \mathbf{B}_w(\mathbf{r}; a)$  of the infinite wire or radius  $a$  at a distance  $\mathbf{r}$  from the axis

$$\mathbf{B}_w(\mathbf{r}; a) = \begin{cases} \frac{2k_m \mathbf{r} \times \mathbf{J}}{r}, & r < a, \\ \frac{2k_m a^2 \mathbf{r} \times \mathbf{J}}{r^2}, & r > a. \end{cases} \quad (\text{S-5.50})$$

Coming back to the cylindrical conductor with a cavity, the magnetic field in a point  $P$  is the sum of the field generated by a wire of radius  $a$  with current  $\mathbf{J}$  and a wire of radius  $b$  with current  $-\mathbf{J}$ , with the distance between the axes of the two wires equal to  $\mathbf{h}$ . Let  $\mathbf{r}_1$  and  $\mathbf{r}_2$  be the distance of  $P$  from the axes of the first and the second

wire, respectively, we have  $\mathbf{r}_1 - \mathbf{r}_2 = \mathbf{h}$ . We thus have  $\mathbf{B}(P) = \mathbf{B}_w(\mathbf{r}_1; a) + \mathbf{B}_w(\mathbf{r}_2; b)$ . In particular, inside the cavity we have  $r_1 < a$  and  $r_2 < b$ , thus

$$\mathbf{B}(P) = 2k_m \mathbf{r}_1 \times \mathbf{J} + 2k_m \mathbf{r}_2 \times (-\mathbf{J}) = 2k_m (\mathbf{r}_1 - \mathbf{r}_2) \times \mathbf{J} = 2k_m \mathbf{h} \times \mathbf{J}, \quad (\text{S-5.51})$$

which is a constant vector. Thus, inside the cavity the magnetic field is uniform and perpendicular to both  $\mathbf{J}$  and  $\mathbf{h}$ .

## S-5.8 Conducting Cylinder in a Magnetic Field

a) We use a cylindrical coordinate system  $(r, \phi, z)$ , with the  $z$  axis along the cylinder axis. The centrifugal force,  $\mathbf{F}_c$ , and the magnetic force,  $\mathbf{F}_m$ , are both directed along  $\hat{\mathbf{r}}$  and depend on  $r$  only:

$$\mathbf{F}_c = m_e \omega^2 \mathbf{r}, \quad \mathbf{F}_m = -e \mathbf{v} \times \mathbf{B}_0 = -e \omega B_0 \mathbf{r}, \quad \frac{|\mathbf{F}_c|}{|\mathbf{F}_m|} = \frac{m_e \omega}{e B_0} \simeq 7.2 \times 10^{-5}. \quad (\text{S-5.52})$$

The magnetic force is dominant, and we shall neglect the centrifugal force in the following.

b) In static conditions the magnetic force must be compensated by an electric field  $\mathbf{E}$

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B}_0 = -\omega B_0 \mathbf{r}. \quad (\text{S-5.53})$$

The existence of this electric field implies a uniform charge density

$$\varrho = \frac{1}{4\pi k_e} \nabla \cdot \mathbf{E} = \frac{E(r)}{2\pi k_e r} = -\frac{\omega B_0}{2\pi k_e}. \quad (\text{S-5.54})$$

Since the cylinder carries no net charge, its lateral surface must have a charge density

$$\sigma = -\frac{\pi a^2 h \varrho}{2\pi a h} = -\frac{a \varrho}{2} = \frac{\omega a B_0}{4\pi k_e}. \quad (\text{S-5.55})$$

c) The volume charge density  $\varrho$  is associated to a volume rotational current density  $\mathbf{J}(r)$  due to the cylinder rotation

$$\mathbf{J}(r) = \varrho \omega r \hat{\phi} = -\frac{\omega^2 r B_0}{2\pi k_e} \hat{\phi}. \quad (\text{S-5.56})$$

The contribution of  $\mathbf{J}(r)$  to the magnetic field on the cylinder axis,  $B_J$ , can be evaluated by dividing the

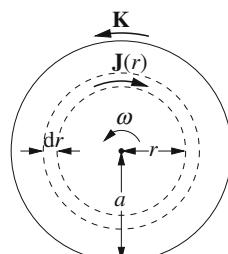


Fig. S-5.11

cylinder into infinitesimal coaxial cylindrical shells between  $r$  and  $r + dr$ . Each shell is equivalent to a solenoid of radius  $r$  and product  $nI = J(r)dr$ , contributing  $dB_J = 4\pi k_m J(r)dr$  to the field at its inside. The total contribution of  $J(r)$  at a distance  $r$  from the axis is thus

$$\begin{aligned} B_J(r) &= 4\pi k_m \int_r^a J(r') dr' = -4\pi k_m \frac{\omega^2 B_0}{2\pi k_e} \int_r^a r' dr' \\ &= -4\pi k_m \frac{\omega^2 B_0}{2\pi k_e} \left[ \frac{r'^2}{2} \right]_r^a = -\frac{k_m}{k_e} \omega^2 B_0 (a^2 - r^2). \end{aligned} \quad (\text{S-5.57})$$

Now we must add the contribution  $B_K$  of the surface current density  $K = \sigma \omega a$

$$B_K = 4\pi k_m \sigma \omega a = 4\pi k_m \omega a \frac{\omega a B_0}{4\pi k_e} = \frac{k_m}{k_e} \omega^2 a^2 B_0 \quad (\text{S-5.58})$$

and the total magnetic field  $B_1(r)$  due to the rotational currents is

$$B_1(r) = B_J(r) + B_K = \frac{k_m}{k_e} \omega^2 B_0 r^2, \quad (\text{S-5.59})$$

which is zero on the axis and reaches its maximum value at  $r = a^-$ . We thus have

$$\frac{B_1(a^-)}{B_0} = \frac{k_m}{k_e} \omega^2 a^2 = \frac{\omega^2 a^2}{c^2} \simeq (2.1 \times 10^{-7})^2 \ll 1. \quad (\text{S-5.60})$$

### S-5.9 Rotating Cylindrical Capacitor

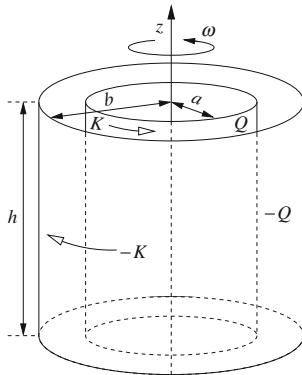


Fig. S-5.12

a) We use cylindrical coordinates  $(r, \phi, z)$  with the  $z$  axis coinciding with capacitor axis. We assume  $\omega = \omega \hat{z}$ , with  $\omega = 2\pi/T > 0$ . The surface currents due to the capacitor rotation are thus

$$K = \sigma v = \frac{Q}{2\pi ah} \omega a = \frac{Q}{hT}, \quad (\text{S-5.61})$$

where  $\sigma = Q/(2\pi ah)$  is the surface charge density on the inner shell, and  $-K$  on the outer shell, independently of  $a$  and  $b$ . Thus the two cylindrical shells are equivalent to two solenoids with  $nI$  products  $nI = \pm K$ , respectively. The outer shell gives origin to a magnetic field  $\mathbf{B}_b = -4\pi k_m K \hat{z}$  in the region  $r < b$ , and to no field in the region  $r > b$ . The inner shell

gives origin to a field  $\mathbf{B}_a = -B_b$  in the region  $r < a$ , and to no field in the region  $r > b$ . The total field  $\mathbf{B} = \mathbf{B}_a + \mathbf{B}_b$  is thus

$$\mathbf{B} = \begin{cases} -4\pi k_m K \hat{\mathbf{z}} = -4\pi k_m \frac{Q}{hT} \hat{\mathbf{z}}, & a < r < b, \\ 0, & r < a, r > b. \end{cases} \quad (\text{S-5.62})$$

**b)** The electric field is zero for  $r < a$  and  $r > b$ , while it is  $\mathbf{E}(r) = \hat{\mathbf{r}} 2k_e Q/(hr)$  for  $a < r < b$ , and the force between the two shells is attractive. The electrostatic force per unit area on, for instance, the external shell is thus

$$\mathbf{f}_s^{(e)} = \sigma_b \frac{\mathbf{E}(b)}{2} = -\hat{\mathbf{r}} \frac{Q}{2\pi b h} \frac{2k_e Q}{hb} = -\hat{\mathbf{r}} \frac{k_e Q^2}{\pi b^2 h^2}, \quad (\text{S-5.63})$$

where  $\sigma_b$  is the surface charge density on the shell. The magnetic force per unit area on the same shell is

$$\mathbf{f}_s^{(m)} = \sigma_b \mathbf{v} \times \mathbf{B} = \hat{\mathbf{r}} \frac{Q}{2\pi b h} \frac{2\pi b}{T} 4\pi k_m \frac{Q}{hT} = 4\pi k_m \frac{Q^2}{h^2 T^2}, \quad (\text{S-5.64})$$

directed opposite to the electrostatic force. The ratio  $f_s^{(m)}/f_s^{(e)}$  on the outer shell is

$$\frac{f_s^{(m)}}{f_s^{(e)}} = 4\pi k_m \frac{Q^2}{h^2 T^2} \frac{\pi b^2 h^2}{k_e Q^2} = \frac{k_m}{k_e} \left( \frac{2\pi b}{T} \right)^2 = \frac{v_b^2}{c^2} \quad (\text{S-5.65})$$

where  $v_b = 2\pi b/T$  is the tangential velocity of the outer shell. The ratio (S-5.65) is thus negligibly small in all practical cases.

## S-5.10 Magnetized Spheres

**a)** The quickest way to obtain the solution is to exploit the analogy of the magnetostatic equations  $\nabla \times \mathbf{H} = 0$ ,  $\nabla \cdot \mathbf{B} = 0$  with the electrostatic ones  $\nabla \times \mathbf{E} = 0$ ,  $\nabla \cdot \mathbf{D} = 0$  (see also Problem 5.3), along with the definitions (3.4) and (5.19). The spatial distribution of  $\mathbf{M}$  is the same as that of  $\mathbf{P}$  in Problem (3.3), and the boundary conditions for  $\mathbf{H}$  are the same as for  $\mathbf{E}$ . Thus from (S-3.13) we immediately obtain that inside the sphere ( $r < R$ ) the field  $\mathbf{H}$  is uniform with constant value  $\mathbf{H}^{(\text{int})}$ , given by

$$\mathbf{H}^{(\text{int})} = \begin{cases} -\frac{\mathbf{M}}{3}, & \text{SI,} \\ -\frac{4\pi \mathbf{M}}{3}, & \text{Gaussian.} \end{cases} \quad (\text{S-5.66})$$

Using (5.19) we obtain for the magnetic field inside the sphere

$$\mathbf{B}^{(\text{int})} = \begin{cases} \frac{2\mu_0 \mathbf{M}}{3}, & \text{SI,} \\ \frac{8\pi \mathbf{M}}{3}, & \text{Gaussian.} \end{cases} \quad (\text{S-5.67})$$

Outside the sphere the field is that of a magnetic dipole  $\mathbf{m} = \mathbf{M}(4\pi R^3/3)$  located at the center of the sphere ( $r = 0$ ).

- b)** The rotation of the sphere with uniform surface charge  $\sigma = Q/(4\pi R^2)$  generates an azimuthal surface current

$$\mathbf{K}_{\text{rot}} = \sigma \mathbf{v} = \sigma R \omega \sin \theta \hat{\phi}, \quad (\text{S-5.68})$$

where  $\theta = 0$  corresponds to the direction of  $\mathbf{M}$ . This surface current distribution is analogous to that of the magnetization current distribution (5.17) for the magnetized sphere of point  $\mathbf{a}$ ),

$$\mathbf{K}_m = \mathbf{M} \times \hat{\mathbf{r}} = M \sin \theta \hat{\phi}. \quad (\text{S-5.69})$$

Thus, the magnetic field generated by  $\mathbf{K}_{\text{rot}}$  is the same as that generated by  $\mathbf{K}_m$ , with the replacement  $\mathbf{M} = \sigma R \omega = Q \omega / (4\pi R)$ .

- c)** Analogously to Problem 3.4 for a dielectric sphere in an external electric field, we assume that the induced magnetization  $\mathbf{M} = \chi_m \mathbf{H}$  is uniform and parallel to  $\mathbf{B}_0$ . The total field will be the sum of the external field  $\mathbf{B}_0 \equiv [\mu_0] \mathbf{H}_0$  (with  $[\mu_0]$  replaced by unity for Gaussian units) and of that generated by the magnetization. Thus, inside the sphere  $\mathbf{H}$  is uniform and has the value  $\mathbf{H}^{(\text{int})}$  given by

$$\mathbf{H}^{(\text{int})} = \begin{cases} \mathbf{H}_0 - \frac{\mathbf{M}}{3} = \mathbf{H}_0 - \frac{\chi_m}{3} \mathbf{H}^{(\text{int})}, & \text{SI,} \\ \mathbf{H}_0 - \frac{4\pi \mathbf{M}}{3} = \mathbf{H}_0 - \frac{4\pi \chi_m}{3} \mathbf{H}^{(\text{int})}, & \text{Gaussian.} \end{cases} \quad (\text{S-5.70})$$

Solving for  $\mathbf{H}^{(\text{int})}$  and finally using  $\mathbf{B}^{(\text{int})} = [\mu_0] \mu_r \mathbf{H}^{(\text{int})}$  we obtain

$$\mathbf{H}^{(\text{int})} = \frac{3}{\mu_r + 2} \mathbf{H}_0, \quad \mathbf{B}^{(\text{int})} = \frac{3\mu_r}{\mu_r + 2} \mathbf{B}_0, \quad (\text{S-5.71})$$

independently of the system of units; it may be interesting to compare the result with (S-3.21) for the dielectric sphere. The magnetization is given by  $\mathbf{M} = \chi_m \mathbf{H}^{(\text{int})}$ .

In the case of a perfectly diamagnetic sphere (a superconducting sphere) we have  $\mu_r = 0$  and  $\mathbf{B}^{(\text{int})} = 0$ , and the magnetization is

$$\mathbf{M} = \frac{3\chi_m}{2} \mathbf{H}_0 = -\frac{3}{8\pi} \frac{b_m}{k_m} \mathbf{B}_0. \quad (\text{S-5.72})$$

Actually, inside the sphere the external field is completely screened by the surface currents (S-5.69) due to the magnetization (S-5.72).

It is instructive to double check the above solution by verifying the boundary conditions at the surface of the sphere, analogously to the dielectric case of Solution S-3.4. We choose a spherical coordinate system  $(r, \theta, \phi)$  with the zenith direction  $z$  parallel to the external magnetic field  $\mathbf{B}_0$ , and the origin at the center of the sphere  $O$ , as shown in Fig. S-5.13. As an educated guess, we look for a solution where i) the magnetic field inside the sphere,  $\mathbf{B}^{(\text{int})}$ , is uniform and proportional to  $\mathbf{B}_0$ , and, accordingly, ii) the magnetization  $\mathbf{M}$  of the sphere, proportional to  $\mathbf{B}^{(\text{int})}$ , is uniform, and iii) the total external field,  $\mathbf{B}^{(\text{ext})}$ , is the superposition of the applied external field  $\mathbf{B}_0$  and of the field  $\mathbf{B}^{(\text{mag})}$ , generated by the sphere magnetization. Thus,  $\mathbf{B}^{(\text{mag})}$  will be the field generated by a magnetic dipole  $\mathbf{m} = \alpha \mathbf{B}_0$  located at the center of the sphere, with  $\alpha$  a constant to be determined. Summing up, we are looking for a solution

$$\begin{aligned}\mathbf{B}^{(\text{int})} &= \psi \mathbf{B}_0, \\ \mathbf{B}^{(\text{ext})} &= \mathbf{B}_0 + \mathbf{B}^{(\text{mag})},\end{aligned}\quad (\text{S-5.73})$$

with  $\psi$  a further constant to be determined.  $\mathbf{B}^{(\text{mag})}$  and its spherical components are

$$\mathbf{B}^{(\text{mag})} = \alpha B_0 \frac{k_m}{b_m} \left[ \left( 3 \frac{\hat{\mathbf{z}} \cdot \mathbf{r}}{r^5} \right) \mathbf{r} - \frac{\hat{\mathbf{z}}}{r^3} \right], \quad (\text{S-5.74})$$

$$B_r^{(\text{mag})} = \alpha B_0 \frac{k_m}{b_m} \frac{2 \cos \theta}{r^3}, \quad (\text{S-5.75})$$

$$B_\theta^{(\text{mag})} = \alpha B_0 \frac{k_m}{b_m} \frac{\sin \theta}{r^3}, \quad (\text{S-5.76})$$

$$B_\phi^{(\text{mag})} = 0, \quad (\text{S-5.77})$$

where  $k_m/b_m = \mu_0/(4\pi)$  in SI units, and  $k_m/b_m = 1$  in Gaussian units. The constants  $\alpha$  and  $\psi$  are determined from the boundary conditions on  $\mathbf{B}$  and  $\mathbf{B}/\mu_r$  at the surface of the sphere

$$B_\perp^{(\text{int})}(R, \theta) = B_\perp^{(\text{ext})}(R, \theta), \quad \frac{B_\parallel^{(\text{int})}(R, \theta)}{\mu_r} = B_\parallel^{(\text{ext})}(R, \theta), \quad (\text{S-5.78})$$

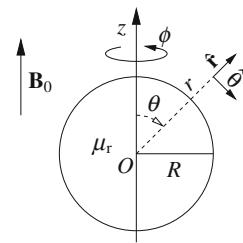


Fig. S-5.13

which lead to

$$\psi B_0 \cos \theta = B_0 \cos \theta + B_r^{(\text{mag})}(R, \theta) = B_0 \cos \theta \left( 1 + \alpha \frac{k_m}{b_m} \frac{2}{R^3} \right), \quad (\text{S-5.79})$$

$$\psi \frac{B_0}{\mu_r} \sin \theta = B_0 \sin \theta - B_\theta^{(\text{mag})}(R, \theta) = B_0 \sin \theta \left( 1 - \alpha \frac{k_m}{b_m} \frac{1}{R^3} \right). \quad (\text{S-5.80})$$

Dividing (S-5.79) by  $B_0 \cos \theta$ , and (S-5.80) by  $B_0 \sin \theta$ , we obtain

$$\psi = 1 + \alpha \frac{k_m}{b_m} \frac{2}{R^3}, \quad \frac{\psi}{\mu_r} = 1 - \alpha \frac{k_m}{b_m} \frac{1}{R^3}, \quad (\text{S-5.81})$$

with solutions

$$\psi = \frac{3\mu_r}{\mu_r + 2}, \quad \alpha = R^3 \frac{b_m}{k_m} \left( \frac{\mu_r - 1}{\mu_r + 2} \right) = \begin{cases} \frac{4\pi R^3}{\mu_0} \left( \frac{\mu_r - 1}{\mu_r + 2} \right), & \text{SI,} \\ R^3 \left( \frac{\mu_r - 1}{\mu_r + 2} \right), & \text{Gaussian,} \end{cases} \quad (\text{S-5.82})$$

which eventually lead to

$$\mathbf{B}^{(\text{int})} = \frac{3\mu_r}{\mu_r + 2} \mathbf{B}_0, \quad \mathbf{m} = \frac{4\pi R^3}{3} \mathbf{M} = R^3 \frac{b_m}{k_m} \left( \frac{\mu_r - 1}{\mu_r + 2} \right) \mathbf{B}_0. \quad (\text{S-5.83})$$

# Chapter S-6

## Solutions for Chapter 6

### S-6.1 A Square Wave Generator

**a)** The motion is periodic, and we choose the origin of time,  $t = 0$ , at an instant when the coil surface is completely in the  $x \geq 0$  half of the  $xy$  plane. With this choice, the flux of the magnetic field through the coil,  $\Phi(t)$ , increases with time when  $2n\pi < \omega t < (2n+1)\pi$ , with  $n$  any integer, and equals  $\Phi(t) = B(\omega t \bmod 2\pi)a^2/2$ . Here,  $(x \bmod y)$  stands for the remainder of the division of  $x$  by  $y$  with an integer quotient. When  $(2n+1)\pi < \omega t < (2n+2)\pi$ , the flux decreases with time and equals  $\Phi(t) = B[2\pi - (\omega t \bmod 2\pi)]a^2/2$ . The electromotive force in the coil,  $\mathcal{E}(t)$ , is thus

$$\begin{aligned}\mathcal{E}(t) &= -b_m \frac{d\Phi(t)}{dt} \\ &= -b_m \frac{Ba^2\omega}{2} \operatorname{sign}[\pi - (\omega t \bmod 2\pi)],\end{aligned}\quad (\text{S-6.1})$$

where  $\operatorname{sign}(x) = x/|x|$  is the sign function. Thus,  $\mathcal{E}$  reverses its sign whenever  $\omega t = n\pi$ , with  $n$  any integer. The current circulating in the coil is  $I = \mathcal{E}/R$ . As shown in Fig. S-6.1,  $I$  (as well as  $\mathcal{E}$ ) is a square wave of period  $T = 2\pi/\omega$ , and amplitude

$$I_0 = \frac{\mathcal{E}_0}{R} = b_m \frac{B\omega a^2}{2R}. \quad (\text{S-6.2})$$

**b)** The external torque applied to the coil in order to keep its angular velocity constant must balance the torque exerted by the magnetic forces. The magnetic force on a current-carrying circuit element  $d\ell$  is  $d\mathbf{f} = b_m I d\ell \times \mathbf{B}$ , and is different from zero

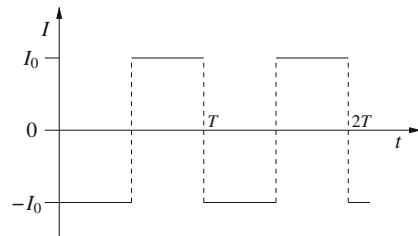


Fig. S-6.1

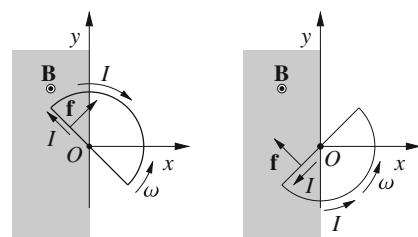


Fig. S-6.2

only in the  $x < 0$  half plane. The corresponding torque  $d\tau = \mathbf{r} \times d\mathbf{f} = b_m^2 I \mathbf{r} \times (d\ell \times \mathbf{B})$ , where  $\mathbf{r}$  is the distance of the coil element  $d\ell$  from the  $z$  axis, is always equal to zero on the circumference arc of the coil because the three vectors of the triple product are mutually perpendicular here. Thus,  $d\tau$  is different from zero only on the half of the straight part of the coil inside the magnetic field, where  $d\ell = dr$ . Here we have  $d\tau = -b_m^2 I_0 r B dr \hat{\mathbf{z}}$ , as shown in Fig. S-6.2, and the total torque on the coil,  $\tau$ , is

$$\tau = \int d\tau = -b_m^2 \omega \frac{B^2 a^2}{2R} \hat{\mathbf{z}} \int_0^a r dr = -b_m^2 \omega \frac{B^2 a^4}{4R} \hat{\mathbf{z}}, \quad (\text{S-6.3})$$

corresponding to a power dissipation

$$P_{\text{diss}} = -\tau \cdot \omega = b_m^2 \omega^2 \frac{B^2 a^4}{4R} = RI_0^2, \quad (\text{S-6.4})$$

that equals the power dissipated by Joule heating. The power dissipation is constant in time, neglecting the “abrupt” transient phases at  $t = n\pi/\omega$ , where  $I$  instantly changes sign. Thus, the external torque must provide the power dissipated by Joule heating.

c) If we take the coil self-inductance  $L$  into account, the equation for the current in the coil becomes

$$\mathcal{E}(t) - L \frac{dI}{dt} = RI, \quad (\text{S-6.5})$$

where  $\mathcal{E}(t)$  is the electromotive force (S-6.1), due to the flux change of the external field only. However “small”  $L$  may be, its contribution is not negligible because, if  $I$  were an ideal square wave, its derivative  $dI/dt$  would diverge whenever  $t = n\pi/\omega$  (instantaneous transition between  $-I_0$  and  $+I_0$ ). The general solution of (S-6.5) is, taking into account that  $\mathcal{E}(t)$  is constant over each half-period,

$$I = \frac{\mathcal{E}}{R} + K e^{-t/t_0}, \quad (\text{S-6.6})$$

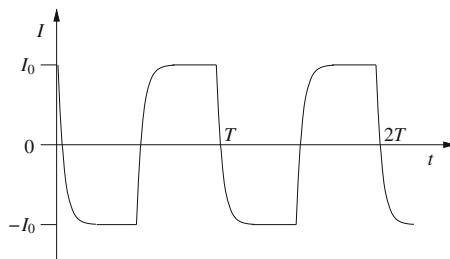


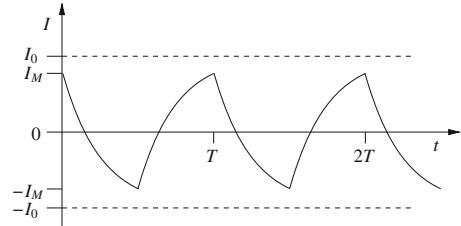
Fig. S-6.3

where  $t_0 = L/R$  is the characteristic time of the loop, and  $K$  is a constant to be determined from the initial conditions. If  $L$  is small enough, we can assume that at time  $t = 0^-$  we have  $\mathcal{E}(t) = +\mathcal{E}_0$  and  $I(t) = +I_0$ . At time  $t = 0$ ,  $\mathcal{E}(t)$  switches instantaneously from  $+\mathcal{E}_0$  to  $-\mathcal{E}_0$ , and the constant  $K$  is determined by the initial condition  $I(0) = I(0^-) = I_0 = \mathcal{E}_0/R$ , leading to  $K = 2\mathcal{E}_0/R$ . Thus, for  $0 < t < \pi/\omega$ ,

$$I(t) = I_0(2e^{-t/t_0} - 1). \quad (\text{S-6.7})$$

At  $t = (\pi/\omega)^-$  we have  $\mathcal{E}(t) = -\mathcal{E}_0$  and we can assume that  $I(t) = -I_0$ . At  $t = \pi/\omega$   $\mathcal{E}(t)$  switches instantaneously from  $-\mathcal{E}_0$  to  $+\mathcal{E}_0$ , and, for  $\pi/\omega < t < 2\pi/\omega$ , we have

$$I(t) = -I_0(2e^{-(t-\pi/\omega)/t_0} - 1), \quad (\text{S-6.8})$$



**Fig. S-6.4**

and so on for the successive periods.

The self-inductance of the coil prevents the current from switching instantaneously between  $+I_0$  and  $-I_0$ : the change occurs following an exponential with characteristic time  $t_0 = L/R$ .

The behavior described by (S-6.7) and (S-6.8) is valid only if  $t_0 \ll T = 2\pi/\omega$ , as in Fig. S-6.3, representing the case of  $t_0 = 0.04T$ . If  $t_0$  is not negligible with respect to  $T$ , the current oscillates between two values  $+I_M$  and  $-I_M$ , with  $I_M < I_0$ . Let us consider the time interval  $0 \leq t \leq \pi/\omega$ . We must have  $I(0) = I_M$  and  $I(\pi/\omega) = -I_M$ . Replacing  $I$  by (S-6.6), we obtain

$$I_M = I_0 \frac{1 - e^{-T/2t_0}}{1 + e^{-T/2t_0}}. \quad (\text{S-6.9})$$

The plot of  $I(t)$  can no longer be approximated by a square wave, as shown in Fig. S-6.4 for the case  $t_0 = 0.25T$ .

## S-6.2 A Coil Moving in an Inhomogeneous Magnetic Field

a) With our assumptions, the flux of the magnetic field through the coil can be approximated as

$$\Phi_B(t) = \Phi_B[z(t)] \simeq \pi a^2 B_0 \frac{z(t)}{L} = \pi a^2 B_0 \frac{z_0 + vt}{L}, \quad (\text{S-6.10})$$

where  $z_0$  is the position of the center of the coil at  $t = 0$ . The rate of change of this magnetic flux is associated to an electromotive force  $\mathcal{E}$ , and to a current  $I = \mathcal{E}/R$  circulating in the coil

$$\mathcal{E} = RI = -b_m \frac{d\Phi}{dt} = -b_m \pi a^2 B_0 \frac{v}{L}. \quad (\text{S-6.11})$$

b) The power dissipated by Joule heating is

$$P = RI^2 = \frac{\mathcal{E}^2}{R} = b_m^2 \frac{(\pi a^2 B_0 v)^2}{L^2 R}. \quad (\text{S-6.12})$$

Thus, in order to keep the coil in motion at constant speed, one must exert an external force  $\mathbf{f}_{\text{ext}}$  on the coil, whose work compensates the dissipated power. We have

$$\mathbf{f}_{\text{ext}} \cdot \mathbf{v} = P = b_m^2 \frac{(\pi a^2 B_0 v)^2}{L^2 R}, \quad (\text{S-6.13})$$

and the coil is submitted to a frictional force proportional to its velocity

$$\mathbf{f}_{\text{frict}} = -\mathbf{f}_{\text{ext}} = -b_m^2 \frac{(\pi a^2 B_0)^2}{L^2 R} \mathbf{v}. \quad (\text{S-6.14})$$

c) The force  $\mathbf{f}_{\text{frict}}$  is actually the net force obtained by integrating the force  $d\mathbf{f}_{\text{frict}} = b_m I d\ell \times \mathbf{B}$  acting on each coil element  $d\ell$ :

$$\mathbf{f}_{\text{frict}} = b_m I \oint_{\text{coil}} d\ell \times \mathbf{B}. \quad (\text{S-6.15})$$

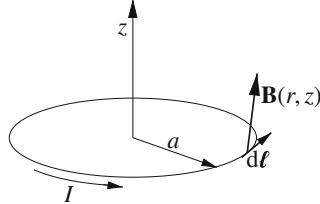


Fig. S-6.5

The contribution of the  $z$  component of  $\mathbf{B}$  is a radial force tending to shrink the coil if  $\partial_t \Phi > 0$ , or to widen it if  $\partial_t \Phi < 0$ , according to Lenz's law; the case represented in Fig. S-6.5 corresponds to the latter case. Thus  $\mathbf{f}_{\text{ext}}$ , directed along  $z$ , is due only to the radial component  $B_r$  of  $\mathbf{B}$ . The component  $B_r$  is not given by the problem, but, as we saw at answer a) of Problem 5.4, it can be evaluated by applying Gauss's law to a closed cylindrical surface of radius  $r$  and height  $\Delta z$ . According to (S-5.25)

$$B_r \simeq -\frac{B_0}{2L} r, \quad \text{thus} \quad d\mathbf{f}_{\text{frict}} = b_m I d\ell \frac{B_0 a}{2L}, \quad (\text{S-6.16})$$

and by substituting (S-6.11) and integrating over the coil we obtain

$$\begin{aligned} \mathbf{f}_{\text{frict}} &= \hat{\mathbf{z}} b_m I \oint_{\text{coil}} d\ell \frac{B_0 a}{2L} = -\hat{\mathbf{z}} b_m \left( b_m \pi a^2 B_0 \frac{v}{L} \right) \left( 2\pi a \frac{B_0 a}{2L} \right) \\ &= -\hat{\mathbf{z}} b_m^2 \frac{(\pi a^2 B_0)^2}{L^2 R} v, \end{aligned} \quad (\text{S-6.17})$$

in agreement with (S-6.14).

### S-6.3 A Circuit with ‘Free-Falling’ Parts

a) We choose the  $x$  axis oriented downwards, with the origin at the location of the upper horizontal bar, as in Fig. S-6.6. The current  $I$  in the rectangular circuit is

$$I = \frac{\mathcal{E}}{R} = -b_m \frac{1}{R} \frac{d\Phi(\mathbf{B})}{dt} = -b_m \frac{Ba}{R} \frac{dx}{dt} = -b_m \frac{Bav}{R}, \quad (\text{S-6.18})$$

where  $x$  is the position of the falling bar, and  $v = \dot{x}$  its velocity. The velocity is positive, and the current  $I$  is negative, i.e., it circulates clockwise, in agreement with Lenz’s law. The magnetic force on the falling bar is  $\mathbf{f}_B = b_m Ba I \hat{\mathbf{x}}$ , antiparallel to the gravitational force  $m\mathbf{g}$ , and the equation of motion is

$$m \frac{dv}{dt} = mg + b_m Ba I = mg - b_m^2 \frac{(Ba)^2}{R} v. \quad (\text{S-6.19})$$

The solution of (S-6.19), with the initial condition  $v(0) = 0$ , is

$$v(t) = v_t (1 - e^{-t/\tau}) \quad (\text{S-6.20})$$

where

$$\tau = \frac{mR}{(b_m Ba)^2} \quad \text{and} \quad v_t = g\tau = \frac{mRg}{(b_m Ba)^2}. \quad (\text{S-6.21})$$

As  $t \rightarrow \infty$ , the falling bar approaches the terminal velocity  $v_t$ .

**b)** When  $v = v_t$ , the power dissipated in the circuit by Joule heating is

$$P_J = RI_t^2 = \frac{(b_m Ba v_t)^2}{R} = \left( \frac{mg}{b_m Ba} \right)^2 R, \quad (\text{S-6.22})$$

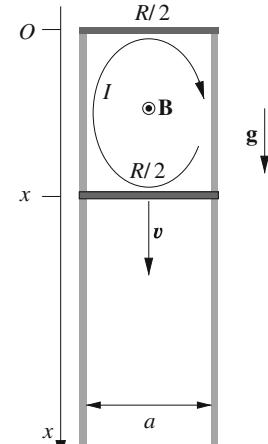


Fig. S-6.6

where  $I_t = -b_m Ba v_t / R$  is the “terminal current”. On the other hand, the work done by the force of gravity per unit time is

$$P_G = m\mathbf{g} \cdot \mathbf{v}_t = mg \frac{mgR}{(b_m Ba)^2} = P_J, \quad (\text{S-6.23})$$

in agreement with energy conservation for the bar moving at constant velocity.

**c)** When both horizontal bars are falling, we denote by  $x_1$  the position of the upper bar, and by  $x_2$  the position of the lower bar, as in Fig. S-6.7, with  $v_1 = \dot{x}_1$  and  $v_2 = \dot{x}_2$ . The current  $I$  circulating in the circuit is

$$\begin{aligned} I &= \frac{\mathcal{E}}{R} = -b_m \frac{1}{R} \frac{d\Phi(\mathbf{B})}{dt} = -b_m \frac{Ba}{R} \frac{d}{dt}(x_2 - x_1) \\ &= -b_m \frac{Ba}{R} (v_2 - v_1), \end{aligned} \quad (\text{S-6.24})$$

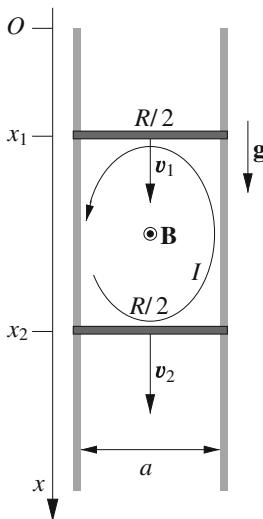


Fig. S-6.7

circulating counterclockwise ( $I > 0$ ) if  $v_1 > v_2$ , and clockwise ( $I < 0$ ) if  $v_1 < v_2$ . The magnetic forces acting on the two falling bars are  $\mathbf{f}_{B1} = -b_m BaI \hat{\mathbf{x}}$  and  $\mathbf{f}_{B2} = b_m BaI \hat{\mathbf{x}}$ , respectively. Independently of the sign of  $I$ , we have  $\mathbf{f}_{B1} = -\mathbf{f}_{B2}$ , so that the net magnetic force on the system comprising the two falling bars is zero. The equations of motion are thus

$$m \frac{dv_1}{dt} = mg + b_m^2 \frac{(Ba)^2}{R} (v_2 - v_1) \quad (\text{S-6.25})$$

$$m \frac{dv_2}{dt} = mg - b_m^2 \frac{(Ba)^2}{R} (v_2 - v_1), \quad (\text{S-6.26})$$

with the initial conditions  $v_1(0) = v_0$  and  $v_2(0) = 0$ . The sum of equations (S-6.25) and (S-6.26) is

$$\frac{d}{dt} (v_1 + v_2) = 2g, \quad \text{with solution} \quad \frac{v_1 + v_2}{2} = \frac{v_0}{2} + gt, \quad (\text{S-6.27})$$

meaning that the center of mass of the two horizontal bars follows a free fall, independent of the magnetic field  $\mathbf{B}$ . On the other hand, the difference of equations (S-6.25) and (S-6.26) is

$$\frac{d}{dt} (v_1 - v_2) = -\frac{2}{\tau} (v_1 - v_2), \quad \text{with solution} \quad v_1 - v_2 = v_0 e^{-2t/\tau}, \quad (\text{S-6.28})$$

where  $\tau = mR/(b_m Ba)^2$ . For the velocities of the two horizontal bars we obtain

$$v_1 = \frac{v_0}{2} \left( 1 + e^{-2t/\tau} \right) + gt, \quad v_2 = \frac{v_0}{2} \left( 1 - e^{-2t/\tau} \right) + gt. \quad (\text{S-6.29})$$

At the steady state limit ( $t \gg \tau$ ) we have

$$\lim_{t \rightarrow \infty} v_1 = \lim_{t \rightarrow \infty} v_2 = \frac{v_0}{2} + gt \quad \text{and} \quad \lim_{t \rightarrow \infty} I = 0, \quad (\text{S-6.30})$$

since, for  $v_1 = v_2$ , the flux of  $\mathbf{B}$  through the loop is constant.

## S-6.4 The Tethered Satellite

- a) To within our approximations, we can assume that the magnetic field is constant over the satellite orbit, and equal to the field at the Earth's equator,  $B_{\text{eq}} \approx 3.2 \times 10^{-5}$  T. The field is parallel to the axis of the satellite orbit, and constant over the tether length. The electromotive force  $\mathcal{E}$  on the tether equals the line integral of the magnetic force along the wire,

$$\mathcal{E} = b_m \int_{\text{tether}} d\ell \cdot \mathbf{v}(r) \times \mathbf{B}_{\text{eq}} = b_m \int_{R_{\oplus}+h}^{R_{\oplus}+h-\ell} dr \omega r B_{\text{eq}}, \quad (\text{S-6.31})$$

where  $\omega = v/r$  is the angular velocity of the satellite. To within our approximations we can also assume that also  $v(r) \simeq v(R_{\oplus}) \simeq 8000 \text{ m/s}$  is constant over the wire length, and obtain

$$\mathcal{E} \simeq b_m v \ell B_{\text{eq}} = \begin{cases} 8000 \times 1000 \times 3.2 \times 10^{-5} \simeq 250 \text{ V}, & \text{SI,} \\ \frac{1}{c} \times 8 \times 10^5 \times 10^5 \times 0.32 \simeq 0.85 \text{ statV,} & \text{Gaussian.} \end{cases} \quad (\text{S-6.32})$$

**b)** Neglecting the resistance of the ionosphere, the current  $I$  circulating in the wire, and the corresponding power dissipated by Joule heating  $P_{\text{diss}}$  are, respectively,

$$I = \frac{\mathcal{E}}{R} = b_m \frac{v \ell B_{\text{eq}}}{R}, \quad \text{and} \quad P_{\text{diss}} = RI^2 = b_m^2 \frac{v^2 \ell^2 B_{\text{eq}}^2}{R}. \quad (\text{S-6.33})$$

The power dissipated in the tether by Joule heating must equal minus the work done by the magnetic force on the wire. This can be easily verified, since the magnetic force acting on the wire is

$$\mathbf{F} = b_m I \ell \hat{\mathbf{r}} \times \mathbf{B}_{\text{eq}} = -b_m^2 \frac{B_{\text{eq}}^2 \ell^2}{R} \mathbf{v}, \quad (\text{S-6.34})$$

and the corresponding work rate is

$$P = \mathbf{F} \cdot \mathbf{v} = -b_m^2 \frac{B_{\text{eq}}^2 \ell^2}{R} v^2 = -P_{\text{diss}}. \quad (\text{S-6.35})$$

If we assume that the tether is a copper wire (conductivity  $\sigma \simeq 10^7 \Omega^{-1} \text{ m}^{-1}$  SI,  $\sigma \simeq 9 \times 10^{16} \text{ s}^{-1}$  Gaussian) of cross section  $A=1 \text{ cm}^2$ , the magnitude of the magnetic drag force on the system is

$$F_{\text{drag}} = b_m^2 \frac{B_{\text{eq}}^2 \ell^2}{\ell / (\sigma A)} v = \begin{cases} \frac{(3.2 \times 10^{-5})^2 \times 1000}{1 / (10^7 \times 10^{-4})} \times 8000 \simeq 8.2 \text{ N,} & \text{SI} \\ \frac{1}{c^2} \frac{(0.32)^2 \times 10^5}{1 / (9 \times 10^{16})} \times 8 \times 10^5 \simeq 8.2 \times 10^5 \text{ dyn,} & \text{Gaussian.} \end{cases} \quad (\text{S-6.36})$$

This problem gives an elementary description of the principle of the “Tethered Satellite System”, investigated in some Space Shuttle missions as a possible generator of electric power for orbiting systems.

## S-6.5 Eddy Currents in a Solenoid

**a)** The time-dependent current in the solenoid generates a time-dependent magnetic field which, in turn, induces a time-dependent contribution to the electric field. The induced electric field is associated to a displacement current density, and, in a conductor, also to a conduction current density  $\mathbf{J} = \sigma\mathbf{E}$ . Both current densities, in turn, affect the magnetic field. According to our symmetry assumptions, in a cylindrical coordinate system  $(r, \phi, z)$ , with the solenoid axis as  $z$  axis, the only non-zero component of the magnetic field is  $B_z$ , and the only non-zero component of the electric field is  $E_\phi$ , therefore the only non-zero component of the conduction current density is  $J_\phi$ . Both  $B_z$  and  $E_\phi$  depend only on  $r$ . In principle we must solve (6.1) and (6.5) which in cylindrical coordinates yield (see Table A.1 of the Appendix),

$$\frac{1}{r}\partial_r(rE_\phi) = -b_m\partial_tB_z, \quad (\text{S-6.37})$$

$$-\partial_rB_z = 4\pi k_m\sigma E_\phi + \frac{k_m}{k_e}\partial_tE_\phi. \quad (\text{S-6.38})$$

Finding the complete solution to (S-6.37) and (S-6.38) is possible but somewhat involved. However, if the angular frequency  $\omega$  of the driving current is low enough, the *slowly varying current approximation* (SVCA) provides a sufficiently accurate solution of the problem.

In the SVCA, we start by calculating  $\mathbf{B}$  as in the static case. Neglecting boundary effects, a DC current  $I$  would generate a uniform magnetic field  $\mathbf{B} = \hat{\mathbf{z}}4\pi k_m\mu_r n I$  inside our solenoid, and  $\mathbf{B} \equiv 0$  outside. Thus, inside the solenoid, we would have  $\mathbf{B} = \hat{\mathbf{z}}\mu_0\mu_r n I$  in SI units, and  $\mathbf{B} = \hat{\mathbf{z}}4\pi\mu_r n I/c$  in Gaussian units. If we replace  $I$  by  $I_0 \cos \omega t$  we obtain

$$\mathbf{B}^{(0)} = \hat{\mathbf{z}}4\pi k_m\mu_r n I_0 \cos \omega t, \quad (\text{S-6.39})$$

which we assume as our zeroth-order approximation for the field inside the solenoid. In the next step of SVCA, we evaluate the first order correction by calculating the electric field  $\mathbf{E}^{(1)}$  induced by (S-6.39), and its associated current densities. These current densities, in turn, contribute to the first order correction to the magnetic field. *A posteriori*, our procedure will be justified if the first order correction to the magnetic field,  $\mathbf{B}^{(1)}$ , is much smaller than  $\mathbf{B}^{(0)}$ . And so on for the successive correction orders. For additional simplicity, we neglect the displacement current, i.e., the last term on the right-hand side of (S-6.38), although its inclusion would not be difficult.

Using (6.1) and the symmetry assumptions, the first-order electric field  $\mathbf{E}^{(1)}(r) = \hat{\phi}E^{(1)}(r)$  can be found from its path integral over the circumference of radius  $r$ ,

$$\oint \mathbf{E}^{(1)} \cdot d\ell = 2\pi r E^{(1)}(r) = -b_m\pi r^2 \partial_t B^{(0)}, \quad (\text{S-6.40})$$

which yields

$$\mathbf{E}^{(1)}(r) = k_m b_m 2\pi \mu_r r n I_0 \omega \sin \omega t \hat{\phi}$$

$$= \begin{cases} \frac{\mu_0 \mu_r}{2} r n I_0 \omega \sin \omega t \hat{\phi}, & \text{SI,} \\ \frac{1}{c^2} 2\pi \mu_r r n I_0 \omega \sin \omega t \hat{\phi}, & \text{Gaussian.} \end{cases} \quad (\text{S-6.41})$$

Notice that the induced electric field also generates an electromotive force  $\mathcal{E}^{(1)}$  in the solenoid coils. We assume the generator producing the current  $I(t) = I_0 \cos \omega t$  to be an ideal one, which maintains the same current against any effect occurring in the circuit (the appearance of  $\mathcal{E}^{(1)}$  will require extra work to maintain the current).

**b)** Due to the conductivity  $\sigma$  of the solenoid core, the electric field  $\mathbf{E}^{(1)}(r)$  originates an azimuthal current density  $\mathbf{J}^{(1)}(r) = \sigma \mathbf{E}^{(1)}(r)$  (eddy currents) in the material. The corresponding Joule dissipation heats up the material. The energy turned into heat per unit volume at each instant  $t$  is

$$\mathbf{J}^{(1)}(r) \cdot \mathbf{E}^{(1)}(r) = \sigma [E^{(1)}(r)]^2 = \sigma (k_m b_m 2\pi \mu_r r n I_0 \omega \sin \omega t)^2, \quad (\text{S-6.42})$$

with a time average

$$\langle \mathbf{J}^{(1)}(r) \cdot \mathbf{E}^{(1)}(r) \rangle = 2\sigma (k_m b_m \pi \mu_r r n I_0 \omega)^2. \quad (\text{S-6.43})$$

The total dissipated power is found by integrating (S-6.43) over the volume of the cylindrical core

$$P_d = \int_{\text{cylinder}} \langle \mathbf{J}^{(1)}(r) \cdot \mathbf{E}^{(1)}(r) \rangle d^3x = 2\sigma (k_m b_m \pi \mu_r r n I_0 \omega)^2 \int_0^R r^2 \ell 2\pi r dr$$

$$= \sigma \pi \ell (k_m b_m \pi \mu_r r n I_0 \omega R^2)^2 = \begin{cases} \frac{\sigma \mu_0^2 \mu_r^2}{16} \pi n^2 I_0^2 \omega^2 \ell R^4, & \text{SI,} \\ \frac{1}{c^4} \pi^3 \sigma \mu_r^2 n^2 I_0^2 \omega^2 \ell R^4, & \text{Gaussian.} \end{cases} \quad (\text{S-6.44})$$

**c)** The induced current density  $\mathbf{J}^{(1)}(r) = \sigma \mathbf{E}^{(1)}(r)$  generates a magnetic field  $\mathbf{B}^{(1)}(r)$  in the cylindrical volume enclosed by the surface of radius  $r$ . Each infinitesimal cylindrical shell between  $r$  and  $r + dr$  of Fig. S-6.9 behaves like a solenoid of radius  $r$ , generating a magnetic field whose value is obtained by replacing the product  $nI$  by the product  $J^{(1)}(r)dr$ . Thus, the contribution to the magnetic field in  $r$  of the infinitesimal shell is

$$d\mathbf{B}_{\text{int}}^{(1)}(r) = \hat{\mathbf{z}} 4\pi k_m \mu_r \sigma E^{(1)}(r). \quad (\text{S-6.45})$$

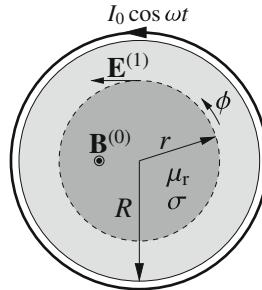
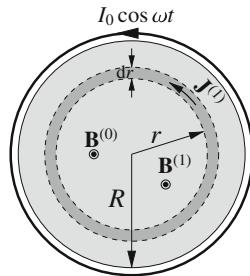


Fig. S-6.8

**Fig. S-6.9**

Also all infinitesimal cylindrical shells between  $r'$  and  $r' + dr'$ , with  $r < r' < R$ , contribute to the field in  $r$ , and the resulting first-order correction to the field in  $r$  is

$$\mathbf{B}^{(1)}(r) = \int_r^R d\mathbf{B}_{\text{int}}^{(1)}(r) = 4\pi k_m \mu_r \sigma \int_r^R E^{(1)}(r') dr'. \quad (\text{S-6.46})$$

If we replace (S-6.41) into (S-6.46) we obtain

$$\begin{aligned} \mathbf{B}^{(1)}(r) &= 8\pi^2 k_m^2 b_m \mu_r^2 \sigma n I_0 \omega \sin \omega t \int_r^R r' dr' = \hat{\mathbf{z}} 4\pi^2 k_m^2 b_m \mu_r^2 \sigma n I_0 (R^2 - r^2) \omega \sin \omega t \\ &= \begin{cases} \hat{\mathbf{z}} \frac{1}{4} \mu_0^2 \mu_r^2 \sigma n I_0 (R^2 - r^2) \omega \sin \omega t, & \text{SI}, \\ \hat{\mathbf{z}} \frac{4\pi^2}{c^3} \mu_r^2 \sigma n I_0 (R^2 - r^2) \omega \sin \omega t, & \text{Gaussian}. \end{cases} \end{aligned} \quad (\text{S-6.47})$$

Thus,  $\mathbf{B}^{(1)}(r)$  is maximum for  $r = 0$ , where all infinitesimal cylindrical shells contribute, and zero for  $r = R$ . Our treatment is justified if  $B^{(1)}(0) \ll B^{(0)}$  for all  $r < R$  and for all  $t$ , i.e., if

$$\frac{\langle B^{(1)}(0) \rangle}{\langle B^{(0)} \rangle} = \pi k_m b_m \mu_r \sigma \omega R^2 \ll 1, \quad (\text{S-6.48})$$

where the angle brackets denote the average over time. This gives the condition on  $\omega$

$$\omega \ll \frac{1}{\pi k_m b_m \mu_r \sigma R^2} = \begin{cases} \frac{4}{\mu_0 \mu_r \sigma R^2}, & \text{SI}, \\ \frac{c^2}{\pi \mu_r \sigma R^2}, & \text{Gaussian}. \end{cases} \quad (\text{S-6.49})$$

Thus, for materials with a high value of the product  $\mu_r \sigma$ , the frequency must be very low. For instance, iron has a relative magnetic permeability  $\mu_r \approx 5000$ , and a conductivity  $\sigma \approx 10^7 \Omega^{-1} m^{-1}$  in SI units. Assuming a solenoid with  $R = 1$  cm, we obtain the following condition on the frequency  $\nu$  of the driving current

$$\nu = \frac{\omega}{2\pi} \ll \frac{4}{8\pi^2 \times 10^{-7} \times 5 \times 10^3 \times 10^7 \times 10^{-4}} \approx 0.10 \text{ Hz}, \quad (\text{S-6.50})$$

which is a very low value. Iron is a good material as the core of an electromagnet, due to its high magnetic permeability, but a poor material as the core of a transformer or of an inductor, due to its high conductivity, which gives origin to high eddy-current losses. On the other hand, manganese-zinc ferrite (a ceramic compound containing iron oxides combined with zinc and manganese compounds) also has a relative magnetic permeability  $\mu_r \approx 5000$ , but a much lower conductivity,

$\sigma \approx 5 \Omega^{-1} m^{-1}$ . The condition on the frequency of the driving current is thus

$$\nu \ll \frac{4}{8\pi^2 \times 10^{-7} \times 5 \times 10^3 \times 5 \times 10^{-4}} \approx 2 \times 10^5 \text{ Hz}, \quad (\text{S-6.51})$$

and ferrite is used in electronics industry to make cores for inductors and transformers, and in various microwave components.

It is also instructive to compare the energy dissipated per cycle,  $U_{\text{diss}} = (2\pi/\omega)P_{\text{diss}}$ , to the total magnetic energy stored in the solenoid,

$$U_M = \left\langle \frac{b_m (B^{(0)})^2}{2k_m \mu_r} \right\rangle \pi R^2 \ell = \begin{cases} \left\langle \frac{(B^{(0)})^2}{2\mu_0 \mu_r} \right\rangle \pi R^2 \ell, & \text{SI,} \\ \left\langle \frac{(B^{(0)})^2}{8\pi \mu_r} \right\rangle \pi R^2 \ell, & \text{Gaussian.} \end{cases} \quad (\text{S-6.52})$$

The ratio is

$$\frac{U_{\text{diss}}}{U_M} \approx \frac{\pi}{4} k_m b_m \mu_r \sigma \omega R^2. \quad (\text{S-6.53})$$

Thus, the condition (S-6.49) is also equivalent to the requirement that the energy loss per cycle due to Joule heating is small compared to the total stored magnetic energy.

## S-6.6 Feynman's "Paradox"

- a) The mutual inductance  $M$  between the charged ring and the superconducting ring is, assuming  $a \ll R$  (see Problem 6.12),

$$M = 4\pi k_m b_m \frac{\pi a^2}{2R}. \quad (\text{S-6.54})$$

Thus, when a current  $I(t)$  is circulating in the smaller ring of radius  $a$ , the magnetic flux through the charged ring is

$$\Phi_I = MI(t) = 4\pi k_m b_m \frac{\pi a^2}{2R} I(t). \quad (\text{S-6.55})$$

If  $\Phi_I$  is time-dependent, it gives origin to an induced electric field  $\mathbf{E}_I$ , whose line-integral around the charged ring is

$$\oint \mathbf{E}_I \cdot d\ell = -b_m \frac{d\Phi_I}{dt} = -4\pi k_m b_m^2 \frac{\pi a^2}{2R} \partial_t I(t). \quad (\text{S-6.56})$$

Due to the symmetry of our problem, field  $\mathbf{E}_I$  is azimuthal on the  $xy$  plane, and independent of  $\phi$ . Its magnitude on the charged ring is thus

$$E_I = \frac{1}{2\pi R} \oint \mathbf{E}_I \cdot d\ell = -k_m b_m^2 \frac{\pi a^2}{R^2} \partial_t I(t), \quad (\text{S-6.57})$$

and the force exerted on an infinitesimal element  $d\ell$  of the charged ring is

$$d\mathbf{f} = \mathbf{E}_I \lambda d\ell = -\hat{\phi} k_m b_m^2 \frac{\pi a^2}{R^2} \lambda d\ell \partial_t I(t), \quad (\text{S-6.58})$$

corresponding to a torque  $d\tau$  about the center of the ring

$$d\tau = \mathbf{r} \times d\mathbf{f} = -\hat{\mathbf{z}} k_m b_m^2 \frac{\pi a^2}{R} \lambda d\ell \partial_t I(t). \quad (\text{S-6.59})$$

The total torque on the charged ring is thus

$$\tau = \int d\tau = -\hat{\mathbf{z}} k_m b_m^2 \frac{\pi a^2}{R} \lambda 2\pi R \partial_t I(t) = -\hat{\mathbf{z}} k_m b_m^2 \frac{\pi a^2}{R} Q \partial_t I(t), \quad (\text{S-6.60})$$

where  $Q = 2\pi R \lambda$  is the total charge of the ring. The equation of motion for the charged ring is thus

$$mR^2 \frac{d\omega}{dt} = \tau = -k_m b_m^2 \frac{\pi a^2}{R} Q \partial_t I(t), \quad (\text{S-6.61})$$

where  $mR^2$  is the moment of inertia of the ring. The solution for  $\omega(t)$  is

$$\omega(t) = -k_m b_m^2 \frac{\pi a^2}{mR^3} Q \int_0^t \partial_t I(t') dt' = k_m b_m^2 \frac{\pi a^2}{mR^3} Q [I_0 - I(t)], \quad (\text{S-6.62})$$

and the final angular velocity is

$$\omega_f = k_m b_m^2 \frac{\pi a^2}{mR^3} Q I_0 = \begin{cases} \frac{\mu_0 a^2 Q}{4mR^3} I_0, & \text{SI,} \\ \frac{\pi a^2 Q}{c^3 m R^3} I_0, & \text{Gaussian,} \end{cases} \quad (\text{S-6.63})$$

corresponding to a final angular momentum

$$L_f = mR^2 \omega_f = k_m b_m^2 \frac{\pi a^2 Q}{R} I_0 = \begin{cases} \frac{\mu_0 a^2 Q}{R} I_0, & \text{SI,} \\ \frac{\pi a^2 Q}{c^3 R} I_0, & \text{Gaussian,} \end{cases} \quad (\text{S-6.64})$$

independent of the mass  $m$  of the ring.

**b)** The rotating charged ring is equivalent to a circular loop carrying a current  $I_{\text{rot}} = Q\omega/2\pi$ . Thus, after the current in the small ring is switched off, there is still a magnetic field due to the rotation of the charged ring. The final magnetic field at the center of the rings is

$$\begin{aligned}\mathbf{B}_c &= \hat{\mathbf{z}} \frac{k_m}{2} \frac{I_{\text{rot}}}{R} = \hat{\mathbf{z}} \frac{k_m}{4\pi} \frac{Q\omega_f}{R} \\ &= \hat{\mathbf{z}} \frac{k_m^2 b_m^2 a^2 Q^2}{4mR^4} I_0 = \begin{cases} \hat{\mathbf{z}} \frac{\mu_0^2 a^2 Q^2}{64\pi^2 m R^4} I_0, & \text{SI,} \\ \hat{\mathbf{z}} \frac{a^2 Q^2}{4c^4 m R^4} I_0, & \text{Gaussian,} \end{cases} \quad (\text{S-6.65})\end{aligned}$$

parallel to the initial field  $\mathbf{B}_0 = \hat{\mathbf{z}} k_m I_0 / (2a)$ , in agreement with Lenz's law. We further have

$$\pi a^2 B_c = M I_{\text{rot}}, \quad (\text{S-6.66})$$

where  $M$  is the mutual inductance of the rings (S-6.54).

**c)** As seen above at point **b**), the rotating charged ring generates a magnetic field all over the space. This field modifies the magnetic flux through the rotating ring itself, giving origin to self-induction. Let  $\mathcal{L}$  be the "self-inductance" of the rotating ring. The magnetic flux generated by the rotating ring through itself is

$$\Phi_{\text{rot}} = \frac{1}{b_m} \mathcal{L} I_{\text{rot}} = \frac{1}{b_m} \mathcal{L} \frac{Q\omega}{2\pi}. \quad (\text{S-6.67})$$

Correspondingly, (S-6.56) for the line integral of the electric field around the charged ring is modified as follows:

$$\oint \mathbf{E}_I \cdot d\ell = -b_m \left( \frac{d\Phi_I}{dt} + \frac{d\Phi_{\text{rot}}}{dt} \right) = -\frac{4\pi^2 k_m b_m^2 a^2}{2R} \partial_t I - \mathcal{L} \frac{Q}{2\pi} \frac{d\omega}{dt}. \quad (\text{S-6.68})$$

The torque on the ring becomes

$$\tau = -\hat{\mathbf{z}} \left( \frac{k_m b_m^2 \pi a^2 Q}{R} \partial_t I + \mathcal{L} \frac{Q^2 a^2}{2\pi} \frac{d\omega}{dt} \right), \quad (\text{S-6.69})$$

and the equation of motion (S-6.61) becomes

$$mR^2 \frac{d\omega}{dt} = -\frac{k_m b_m^2 \pi a^2 Q}{R} \partial_t I - \mathcal{L} \frac{Q^2 a^2}{2\pi} \frac{d\omega}{dt},$$

or

$$\left( mR^2 + \mathcal{L} \frac{Q^2 a^2}{2\pi} \right) \frac{d\omega}{dt} = -\frac{k_m b_m^2 \pi a^2 Q}{R} \partial_t I, \quad (\text{S-6.70})$$

which is equivalent to (S-6.61) if we replace the mass of the charged ring by an effective value

$$m_{\text{eff}} = m + \mathcal{L} \frac{Q^2 a^2}{2\pi R^2}. \quad (\text{S-6.71})$$

Thus we obtain for the dependence of  $\omega$  on  $I(t)$

$$\omega(t) = k_m b_m^2 \frac{\pi a^2 Q}{m_{\text{eff}} R^3} [I_0 - I(t)], \quad (\text{S-6.72})$$

and for its final value

$$\omega_f = k_m b_m^2 \frac{\pi a^2 Q}{m_{\text{eff}} R^3} I_0, \quad (\text{S-6.73})$$

corresponding to a final angular momentum

$$\begin{aligned} L_f &= mR^2 \omega_f = k_m b_m^2 \frac{\pi a^2 Q}{R + \mathcal{L} a^2 Q^2 / (2\pi m R)} I_0 \\ &= \begin{cases} \frac{\mu_0}{4\pi} \frac{\pi a^2 Q}{R + \mathcal{L} a^2 Q^2 / (2\pi m R)} I_0, & \text{SI,} \\ \frac{1}{c^3} \frac{\pi a^2 Q}{R + \mathcal{L} a^2 Q^2 / (2\pi m R)} I_0, & \text{Gaussian.} \end{cases} \end{aligned} \quad (\text{S-6.74})$$

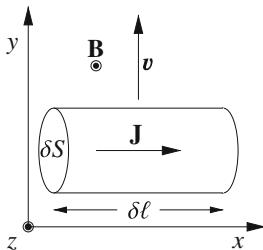
The final magnetic flux through the charged ring is

$$\Phi_f = \frac{1}{b_m} \mathcal{L} \frac{Q \omega_f}{2\pi} = k_m b_m \frac{\mathcal{L} a^2 Q^2}{2mR^3 + \mathcal{L} Q^2 a^2 R / \pi} I_0, \quad (\text{S-6.75})$$

and the approximations of point a) are valid only if

$$\Phi_f \ll \Phi_0 = 4\pi k_m b_m \frac{\pi a^2}{2R} I_0, \quad \text{or} \quad \frac{\mathcal{L} Q^2}{4\pi^2 m R^2 + 2\pi \mathcal{L} Q^2 a^2} \ll 1. \quad (\text{S-6.76})$$

## S-6.7 Induced Electric Currents in the Ocean



**a)** We choose a Cartesian coordinate system with the  $y$  axis parallel to the velocity  $v$  of the fluid and the  $z$  axis parallel to the magnetic field, as shown in Fig. S-6.10. Due to the motion of the fluid, the charge carriers (mainly the  $\text{Na}^+$  and  $\text{Cl}^-$  ions of the dissolved salt) are subject to a force per unit charge equal to  $b_m v \times \mathbf{B}$ , parallel to the  $x$  axis. This is equivalent to an electric field  $\mathbf{E}_{\text{eq}} \equiv b_m v \times \mathbf{B}$ . The induced

Fig. S-6.10

current density is thus

$$\mathbf{J} = \sigma \mathbf{E}_{\text{eq}} = b_m \sigma \mathbf{v} \times \mathbf{B}. \quad (\text{S-6.77})$$

**b)** Inserting the typical values given in the text into (S-6.77) we obtain

$$J \approx \begin{cases} 4 \times 1 \times 5 \times 10^{-5} = 2 \times 10^{-4} \text{ A/m}^2, & \text{SI,} \\ 3.6 \times 10^{10} \times \frac{100}{c} \times 0.5 = 60 \text{ statA/cm}^2, & \text{Gaussian.} \end{cases} \quad (\text{S-6.78})$$

**c)** We evaluate the force on a fluid element of cylindrical shape, with area of the bases  $\delta S$  and height  $|\delta \ell|$ , where  $\ell$  is parallel to  $\mathbf{J}$  and to the  $x$  axis. The current intensity in the cylinder is  $I = J\delta S$ , and the force acting on it is thus  $\delta \mathbf{F} = b_m I \delta \ell \times \mathbf{B} = -b_m B J \delta S \delta \ell \hat{\mathbf{y}} = -b_m B J \delta V \hat{\mathbf{y}}$ , where  $\delta V$  is the volume of the cylinder. The mass of the cylinder is  $\delta m = \rho \delta V$ , with  $\rho = 10^3 \text{ kg/m}^3$  (1 g/cm<sup>3</sup> in Gaussian units), for water. Both  $\mathbf{v}$  and  $\delta \mathbf{F}$  are parallel to the  $y$  direction, and the equation of motion can be written in scalar form

$$\delta m \frac{dv}{dt} = \delta F. \quad (\text{S-6.79})$$

Replacing the values of  $\delta m$  and  $\delta F$  we obtain

$$\begin{aligned} \rho \delta V \frac{dv}{dt} &= -b_m B J \delta V, \\ \rho \frac{dv}{dt} &= -b_m^2 B^2 \sigma v, \end{aligned} \quad (\text{S-6.80})$$

where we have divided both sides by  $\delta V$  and replaced  $J$  by its expression (S-6.77). The solution is a decreasing exponential  $v = v_0 e^{-t/\tau}$  with a time constant

$$\tau = \frac{\rho}{\sigma b_m^2 B^2} \simeq 10^{11} \text{ s} \simeq 3 \times 10^3 \text{ yr.} \quad (\text{S-6.81})$$

## S-6.8 A Magnetized Sphere as Unipolar Motor

**a)** We recall from Problem 5.10 that the magnetic field inside a uniformly magnetized sphere is uniform and equals

$$\mathbf{B} = \frac{8\pi k_m}{3 b_m} \mathbf{M} = \begin{cases} \frac{2\mu_0}{3} \mathbf{M}, & \text{SI,} \\ \frac{8\pi}{3} \mathbf{M}, & \text{Gaussian.} \end{cases} \quad (\text{S-6.82})$$

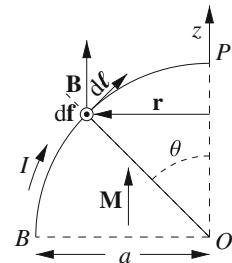


Fig. S-6.11

Outside of the sphere we have the same magnetic field that would be generated by a magnetic dipole of moment  $\mathbf{m} = \mathbf{M}4\pi a^3/3$ , located at the center of the sphere. When an electric current  $I$  flows in the circuit, the magnetic force on an element  $d\ell$  of the “meridian” wire  $BP$  is  $d\mathbf{f} = Id\ell \times \mathbf{B}$ , directed out of paper in Fig. S-6.11. Since the component of  $\mathbf{B}$  perpendicular to  $d\ell$  is continuous across the surface of the sphere, there is no ambiguity. The torque  $d\tau$  on the wire element  $d\ell$  is

$$\begin{aligned} d\tau &= \mathbf{r} \times d\mathbf{f} = I\mathbf{r} \times (d\ell \times \mathbf{B}) = \hat{\mathbf{z}} Ia \sin \theta a d\theta B \cos \theta \\ &= \hat{\mathbf{z}} Ia^2 B \cos \theta \sin \theta d\theta, \end{aligned} \quad (\text{S-6.83})$$

where  $\mathbf{r}$  is the distance of  $d\ell$  from the rotation axis of the sphere ( $r = a \sin \theta$ ), and we have used  $a d\theta = d\ell$ . The total torque on the meridian wire  $BP$  is thus

$$\tau = \int d\tau = \hat{\mathbf{z}} Ia^2 B \int_0^{\pi/2} \sin \theta \cos \theta d\theta = \hat{\mathbf{z}} \frac{1}{2} Ia^2 B, \quad (\text{S-6.84})$$

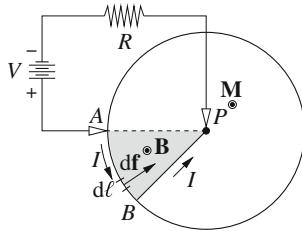


Fig. S-6.12

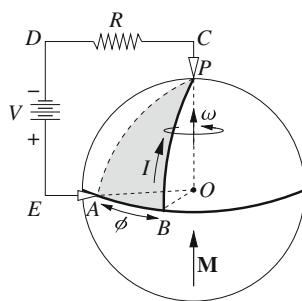


Fig. S-6.13

while the torque on the current-carrying portion  $AB$  of the “equatorial” wire is zero, because the magnetic force is radial, as shown in Fig. S-6.12. Thus, (S-6.84) is the total torque on the sphere.

**b)** When the sphere rotates, the total electromotive force  $\mathcal{E}_{\text{tot}}$  in the circuit is the sum of the electromotive force of the voltage source and the electromotive force  $\mathcal{E}_{\text{rot}}$  due to the rotation of the wires

$$\mathcal{E}_{\text{tot}} = V + \mathcal{E}_{\text{rot}} = V - b_m \frac{d\Phi}{dt}, \quad (\text{S-6.85})$$

where  $\Phi$  is the flux of the magnetic field through any surface bounded by the closed path  $ABPCDEA$  in Fig. S-6.13. Lines  $PC$ ,  $CD$ ,  $DE$  and  $EA$  are coplanar lines, lying on a plane containing also the rotation axis  $OP$  of the sphere and the meridian arc  $PA$ , while  $AB$  is an equatorial arc, and  $BP$  a meridian arc, both lying on the surface of the sphere. The flux of  $\mathbf{B}$  through any surface bounded by the closed path  $ABPCDEA$  is the same, because  $\nabla \cdot \mathbf{B} = 0$ . For simplicity, we choose a surface comprising two parts:

1. the planar surface  $PCDEA$ , its perimeter being closed by the arc  $AP$ , through which the flux is zero, and
2. the spherical polar triangle  $PAB$  shaded in Fig. S-6.13.

The flux through  $PAB$  can be easily calculated remembering that the flux of  $\mathbf{B}$  through any closed surface is zero. Consider the closed surface formed by  $PAB$  and the three circular sectors  $OAP$ ,  $OBP$  and  $OAB$ . The flux through  $OAP$  and  $OBP$  is zero, thus the flux  $\Phi_{PAB}$  through  $PAB$  and the flux  $\Phi_{OAB}$  through  $OAB$  must be equal ( $\Phi_{OAB}$  must be taken with the minus sign when evaluating its contribution to the flux through the total closed surface, since the magnetic field *enters* through  $OAB$  and exits through  $PAB$ ), and we have

$$\Phi_{PAB} = \Phi_{OAB} = \frac{1}{2} Ba^2 \phi, \quad (\text{S-6.86})$$

where  $\phi$  is the angle  $\widehat{AOB}$ . We thus have

$$\mathcal{E}_{\text{tot}} = V - b_m \frac{d\Phi}{dt} = V - b_m \frac{Ba^2}{2} \frac{d\phi}{dt} = V - b_m \frac{Ba^2}{2} \omega, \quad (\text{S-6.87})$$

and the current flowing in the circuit is

$$I = \frac{\mathcal{E}_{\text{tot}}}{R} = \frac{1}{R} \left( V - b_m \frac{Ba^2}{2} \omega \right). \quad (\text{S-6.88})$$

The torque on the sphere is zero when  $I = 0$ , thus the terminal angular velocity of the sphere is

$$\omega_t = \frac{2V}{b_m Ba^2} = \begin{cases} \frac{2V}{Ba^2}, & \text{SI,} \\ \frac{2V}{Ba^2} c, & \text{Gaussian,} \end{cases} \quad (\text{S-6.89})$$

independent of the moment of inertia of the sphere  $\mathcal{I}$  and of the resistance  $R$  of the circuit. The equation of motion for the sphere is

$$\mathcal{I} \frac{d\omega}{dt} = \tau = \frac{1}{2} Ba^2 I = \frac{Ba^2}{2R} \left( V - b_m \frac{Ba^2}{2} \omega \right), \quad (\text{S-6.90})$$

which, using (S-6.89), can be rewritten as

$$\mathcal{I} \frac{d\omega}{dt} = -b_m \frac{(Ba^2)^2}{4\mathcal{I}R} (\omega - \frac{2V}{b_m Ba^2}) \equiv -\frac{1}{\tau} (\omega - \omega_t), \quad (\text{S-6.91})$$

where

$$\tau = \frac{4\mathcal{I}R}{b_m (Ba^2)^2}. \quad (\text{S-6.92})$$

Assuming that the sphere is at rest at  $t = 0$ , the solution is

$$\omega(t) = \omega_t \left( 1 - e^{-t/\tau} \right). \quad (\text{S-6.93})$$

## S-6.9 Induction Heating

a) When the displacement current is neglected, (6.5) can be written as

$$\nabla \times \mathbf{B} = 4\pi k_m (\mathbf{J}_f + \mathbf{J}_m) = 4\pi k_m \mu_r \mathbf{J}_f , \quad (\text{S-6.94})$$

where  $\mathbf{J}_f$  is the free current density and  $\mathbf{J}_m$  is the magnetization current density.

Now, using (A.12),

$$\nabla \times (\nabla \times \mathbf{B}) = -\nabla^2 \mathbf{B} + \nabla(\nabla \cdot \mathbf{B}) = 4\pi k_m \mu_r \nabla \times \mathbf{J}_f , \quad (\text{S-6.95})$$

and recalling that  $\nabla \cdot \mathbf{B} = 0$  and  $\mathbf{J}_f = \sigma \mathbf{E}$  we obtain

$$-\nabla^2 \mathbf{B} = 4\pi k_m \mu_r \sigma \nabla \times \mathbf{E} . \quad (\text{S-6.96})$$

Finally, using  $\nabla \times \mathbf{E} = -b_m \partial_t \mathbf{B}$  we have

$$\partial_t \mathbf{B} = (4\pi k_m b_m \mu_r \sigma)^{-1} \nabla^2 \mathbf{B} \equiv \alpha \nabla^2 \mathbf{B} , \quad (\text{S-6.97})$$

where

$$\alpha = \frac{1}{4\pi k_m b_m \mu_r \sigma} = \begin{cases} \frac{1}{\mu_0 \mu_r \sigma}, & \text{SI,} \\ \frac{c^2}{4\pi \mu_r \sigma}, & \text{Gaussian.} \end{cases} \quad (\text{S-6.98})$$

b) The tangential component of the auxiliary vector  $\mathbf{H}$  must be continuous through the  $x = 0$  plane, thus, the tangential component of  $\mathbf{B}/\mu_r$  must be continuous. In the vacuum half-space ( $x < 0$ ) we have  $\mathbf{B} = \hat{\mathbf{y}} B_0 \cos(\omega t)$ , correspondingly, the field at  $x = 0^+$  (just inside our medium) is

$$\mathbf{B}(0^+, t) = \hat{\mathbf{y}} \mu_r B_0 \cos(\omega t) . \quad (\text{S-6.99})$$

In one dimension, (6.6) is rewritten

$$\partial_t B = \alpha \partial_x^2 B , \quad (\text{S-6.100})$$

and, as an educated guess, we look for a solution of the form  $B(x, t) = \text{Re} [\tilde{B}(x) e^{-i\omega t}]$ . The differential equation for the time-independent function  $\tilde{B}(x)$  is

$$-i\omega \tilde{B} = \alpha \partial_x^2 \tilde{B} , \quad (\text{S-6.101})$$

and we look for an exponential solution of the form  $\tilde{B}(x) = \tilde{B}(0) e^{\gamma x}$ , with  $\tilde{B}(0)$  and  $\gamma$  two constants to be determined. The boundary condition gives  $\tilde{B}(0) = \mu_r B_0$ , and, by substituting into (S-6.101), we have

$$-i\omega \mu_r B_0 e^{\gamma x} = \alpha \gamma^2 \mu_r B_0 e^{\gamma x} , \quad (\text{S-6.102})$$

which leads to  $\alpha\gamma^2 = -i\omega$ , so that

$$\gamma = \pm \sqrt{-i\frac{\omega}{\alpha}} = \pm \frac{1-i}{\sqrt{2}} \sqrt{\frac{\omega}{\alpha}} = \pm(1-i) \frac{1}{\ell_s} \quad (\text{S-6.103})$$

where  $(1-i)/\sqrt{2} = \sqrt{-i}$ , and the quantity

$$\ell_s = \sqrt{\frac{2\alpha}{\omega}} = \sqrt{\frac{2}{4\pi k_m b_m \mu_r \sigma \omega}} = \begin{cases} \sqrt{\frac{2}{\mu_0 \mu_r \sigma \omega}}, & \text{SI}, \\ \sqrt{\frac{c^2}{2\pi \mu_r \sigma \omega}}, & \text{Gaussian}, \end{cases} \quad (\text{S-6.104})$$

which has the dimension of a length, is called the (*resistive*) *skin depth*. We disregard the positive value of  $\gamma$ , which would lead to a magnetic field exponentially increasing with distance into the material, and obtain

$$\mathbf{B} = \hat{\mathbf{y}} \operatorname{Re} [\mu_r B_0 e^{-(1-i)x/\ell_s - i\omega t}] = \hat{\mathbf{y}} \mu_r B_0 e^{-x/\ell_s} \cos\left(\frac{x}{\ell_s} - \omega t\right). \quad (\text{S-6.105})$$

Thus the magnetic field decreases exponentially with distance into the material, with a decay length  $\ell_s$ . A slab of our material can be considered as semi-infinite if its depth is much larger than  $\ell_s$ .

c) The electric field  $\mathbf{E}(x)$  inside the material can be evaluated from  $\nabla \times \mathbf{E} = -b_m \partial_t \mathbf{B}$ . Assuming  $\mathbf{E}(x, t) = \operatorname{Re} [\tilde{\mathbf{E}}(x) e^{-i\omega t}]$  we have

$$\begin{aligned} (\nabla \times \mathbf{E})_y &= -\partial_x \operatorname{Re} [\tilde{E}_z(x) e^{-i\omega t}], \\ \partial_t \mathbf{B} &= \operatorname{Re} [-i\omega \mu_r B_0 e^{-(1-i)x/\ell_s - i\omega t}], \end{aligned} \quad (\text{S-6.106})$$

thus  $\mathbf{E}(x, t) = \hat{\mathbf{z}} \operatorname{Re} [\tilde{E}_z(x) e^{-i\omega t}]$ , with  $\partial_x \tilde{E}_z = -i b_m \omega \mu_r B_0 e^{-(1-i)x/\ell_s}$ . Integrating with respect to  $x$  we obtain

$$\tilde{E}_z = \frac{i}{1-i} b_m \omega \mu_r \ell_s B_0 e^{-(1-i)x/\ell_s} = -\frac{1-i}{2} b_m \omega \mu_r \ell_s B_0 e^{-(1-i)x/\ell_s}. \quad (\text{S-6.107})$$

The dissipated power per unit volume, *due to the free currents only*, is thus

$$\begin{aligned} \langle \mathbf{J}_f \cdot \mathbf{E} \rangle &= \frac{\sigma}{2} |\tilde{E}_z|^2 = \frac{\sigma}{4} b_m^2 \mu_r^2 \omega^2 \ell_s^2 B_0^2 e^{-2x/\ell_s} = \frac{\sigma}{4} b_m^2 \frac{2\mu_r^2 \omega^2 B_0^2}{4\pi k_m b_m \mu_r \sigma \omega} e^{-2x/\ell_s} \\ &= b_m \frac{\mu_r \omega B_0^2}{8\pi k_m} e^{-2x/\ell_s} = \begin{cases} \frac{\mu_r \omega B_0^2}{2\mu_0} e^{-2x/\ell_s}, & \text{SI}, \\ \frac{\mu_r \omega B_0^2}{32\pi^2} e^{-2x/\ell_s}, & \text{Gaussian}, \end{cases} \end{aligned} \quad (\text{S-6.108})$$

where we have substituted (S-6.104) for  $\ell_s$  in the fraction. The total dissipated power per unit surface of the slab is

$$\int_0^\infty \langle \mathbf{J}_f \cdot \mathbf{E} \rangle dx = b_m \frac{\mu_r \omega B_0^2}{16\pi k_m} \ell_s = \frac{B_0^2}{16\pi} \sqrt{\frac{b_m \mu_r \omega}{2k_m \sigma}}. \quad (\text{S-6.109})$$

One might wonder if there is also a contribution of the magnetization volume and surface current densities,  $\mathbf{J}_m$  and  $\mathbf{K}_m$ , to the dissipated power. In the presence of the magnetic field (S-6.105), our medium of relative magnetic permeability  $\mu_r$  acquires a magnetization  $\mathbf{M}$

$$\mathbf{M} = \frac{b_m}{4\pi k_m} \frac{\mu_r - 1}{\mu_r} \mathbf{B} = \hat{\mathbf{y}} \frac{b_m}{4\pi k_m} (\mu_r - 1) \operatorname{Re} [B_0 e^{-(1-i)x/\ell_s - i\omega t}], \quad (\text{S-6.110})$$

which corresponds to

$$\mathbf{J}_m = \frac{1}{b_m} \nabla \times \mathbf{M}. \quad (\text{S-6.111})$$

Taking the symmetry of the problem into account, and introducing the complex amplitudes  $\tilde{\mathbf{J}}_m$  and  $\tilde{J}_m z$  such that  $\mathbf{J}_m = \operatorname{Re}(\tilde{\mathbf{J}}_m e^{-i\omega t}) = \hat{\mathbf{z}} \operatorname{Re}(\tilde{J}_m z e^{-i\omega t})$ , we have

$$\tilde{J}_m z = \frac{\mu_r - 1}{4\pi k_m \mu_r} \partial_x \tilde{B} = -\frac{\mu_r - 1}{4\pi k_m} \frac{1-i}{\ell_s} B_0 e^{-(1-i)x/\ell_s}. \quad (\text{S-6.112})$$

The corresponding power per unit volume is

$$\langle \mathbf{J}_m \cdot \mathbf{E} \rangle = \frac{1}{2} \operatorname{Re}(\tilde{J}_m z \tilde{E}_z^*) = b_m (\mu_r - 1) \frac{\mu_r \omega B_0^2}{8\pi k_m} e^{-2x/\ell_s} = (\mu_r - 1) \langle \mathbf{J}_f \cdot \mathbf{E} \rangle, \quad (\text{S-6.113})$$

and the total power per unit surface is

$$\int_0^\infty \langle \mathbf{J}_m \cdot \mathbf{E} \rangle dx = (\mu_r - 1) \int_0^\infty \langle \mathbf{J}_f \cdot \mathbf{E} \rangle dx = b_m (\mu_r - 1) \frac{\mu_r \omega B_0^2}{16\pi k_m} \ell_s. \quad (\text{S-6.114})$$

However, we also have a surface magnetization current density  $\mathbf{K}_m$  flowing on the  $x = 0$  plane, given by

$$\mathbf{K}_m = \frac{1}{b_m} \mathbf{M}(0^+) \times \hat{\mathbf{n}} = \hat{\mathbf{z}} \frac{\mu_r - 1}{4\pi k_m} \operatorname{Re}(B_0 e^{-i\omega t}) = \hat{\mathbf{z}} K_m z \cos(\omega t), \quad (\text{S-6.115})$$

where  $\hat{\mathbf{n}} = -\hat{\mathbf{x}}$  is the outward-pointing unit vector on the  $x = 0$  boundary plane. This surface current density corresponds to a power per unit surface

$$\langle \mathbf{K}_m(t) \cdot \mathbf{E}(0, t) \rangle = \frac{1}{2} \operatorname{Re} [K_m z \tilde{E}_z(0)] = -b_m (\mu_r - 1) \frac{\mu_r \omega B_0^2}{16\pi k_m} \ell_s, \quad (\text{S-6.116})$$

which cancels out the contribution (S-6.114). Thus, the total dissipated power in the medium is due to the free current only, and given by (S-6.109). Note that the parallel component of the electric field must be continuous at the boundary between two media, so that  $E_z(0, t)$  appearing in (S-6.116) is a well defined quantity.

## S-6.10 A Magnetized Cylinder as DC Generator

a) We can consider the magnetic field as due to the azimuthal magnetization surface current density  $\mathbf{K}_m$ , flowing on the lateral surface of the cylinder. We have  $\mathbf{K}_m = \mathbf{M} \times \hat{\mathbf{n}}/b_m$ , where  $\hat{\mathbf{n}}$  is the outward unit vector normal to the surface. Thus, the magnetized cylinder is equivalent to a solenoid of the same sizes, with  $n$  turns per unit length, current  $I$  per turn, and the product  $nI = K_m$ . Far from the two bases we have an approximately uniform field  $\mathbf{B}_0$ , independent of the radius and height of the cylinder,

$$\begin{aligned}\mathbf{B}_0 &\simeq 4\pi k_m K_m \hat{\mathbf{z}} \\ &= 4\pi \frac{k_m}{b_m} \mathbf{M} = \begin{cases} \mu_0 \mathbf{M}, & \text{SI,} \\ 4\pi \mathbf{M}, & \text{Gaussian.} \end{cases} \quad (\text{S-6.117})\end{aligned}$$

The field at, for instance, the upper base, can be evaluated by considering an “extended” cylinder, obtained by joining an identical, coaxial cylinder, at the base we are considering, as shown in Fig. S-6.14. The total field at the base is now due to both cylinders, and, being far from both bases of the extended cylinder, its value is  $\mathbf{B}_0 \simeq 4\pi(k_m/b_m)\mathbf{M}$ . Both cylinders contribute to this field, and, for symmetry reasons, the  $z$  components  $B_z$  of both contributions are equal, while the radial components cancel each other. The dashed lines of Fig. S-6.14 represent three  $\mathbf{B}$  field lines for each cylinder, one along the axis and two off-axis. Thus, the  $z$  component of the field generated by the single cylinder at its base is

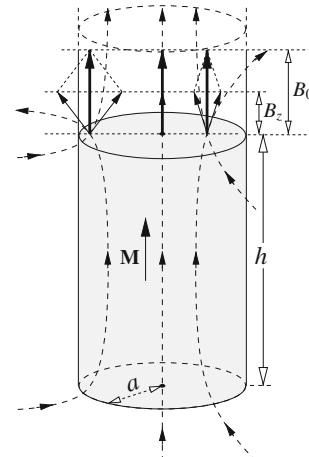


Fig. S-6.14

$$B_z = 2\pi \frac{k_m}{b_m} M = \frac{B_0}{2}. \quad (\text{S-6.118})$$

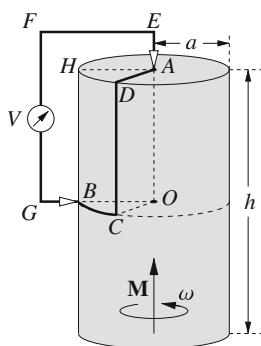


Fig. S-6.15

**b)** We apply Faraday's law of induction to the flux of the magnetic field through the closed path  $AEFGBCDA$ , represented by the thick line in Fig. S-6.15. Points  $A$ ,  $E$ ,  $F$ ,  $G$ , and  $B$  are fixed in the laboratory frame, while points  $C$  and  $D$  rotate with the magnetized cylinder. We have

$$\mathcal{E} = -b_m \frac{d\Phi}{dt}, \quad (\text{S-6.119})$$

where  $\mathcal{E}$  is the electromotive force around the closed path, measured by the voltmeter  $V$ , and  $\Phi$  is the flux of the magnetic field through any surface bounded by the closed path. We choose a surface consisting of three parts:

1. the plane surface bounded by the path  $AEFGBHA$ , fixed in the laboratory frame, through which the flux of  $\mathbf{B}$  is zero;
2. the surface bounded by the path  $BCDHB$ , lying on the lateral surface of the cylinder; and
3. the circular sector  $AHD$  on the upper base, where points  $A$  and  $H$  are fixed, while point  $D$  is rotating.

The flux of  $\mathbf{B}$  through the two surfaces  $BCDH$  and  $AHD$  can be calculated analogously to the flux through the polar spherical triangle  $PAB$  of Fig. S-6.13, Problem 6.8. We consider the closed surface comprising, in addition to  $BCDH$  and  $AHD$ , the circular sector  $OBC$  and the two rectangles  $COAD$  and  $BOAH$ . The flux must be zero through the total closed surface, and is zero through the two rectangles because  $\mathbf{B}$  is parallel to their surfaces. Thus we have

$$\Phi_{AHD} + \Phi_{BCDH} + \Phi_{OBC} = 0, \quad (\text{S-6.120})$$

and

$$\Phi_{AHD} + \Phi_{BCDH} = -\Phi_{OBC} = \frac{1}{2} B_0 a^2 \phi, \quad (\text{S-6.121})$$

where  $\phi$  is the angle  $\widehat{BOC} = \widehat{HAD}$ , and the sign accounts for the fact that the magnetic field is entering the closed surface through  $OBC$ . The electromotive force is

$$\mathcal{E} = -b_m \frac{d\Phi}{dt} = -b_m \frac{1}{2} B_0 a^2 \frac{d\phi}{dt} = 2\pi k_m M a^2 \omega = \begin{cases} \frac{\mu_0}{2} M a^2 \omega, & \text{SI,} \\ \frac{2\pi}{c} M a^2 \omega, & \text{Gaussian.} \end{cases} \quad (\text{S-6.122})$$

The same result can be obtained by evaluating the electromotive force  $\mathcal{E}$  as the integral of  $b_m(\mathbf{v} \times \mathbf{B}) \cdot d\ell$  along the path  $AOB$

$$\begin{aligned}\mathcal{E} &= b_m \int_A^O (\mathbf{v} \times \mathbf{B}) \cdot d\ell + b_m \int_O^B (\mathbf{v} \times \mathbf{B}) \cdot d\ell = b_m \int_O^a \omega r B_0 dr \\ &= b_m \frac{1}{2} B_0 a^2 \omega = 2\pi k_m M a^2 \omega,\end{aligned}\quad (\text{S-6.123})$$

since  $\mathbf{v} = 0$  along the path  $AO$ , which lies on the rotation axis of the cylinder.

## S-6.11 The Faraday Disk and a Self-sustained Dynamo

- a) The magnetic force on the each charge carrier of the rotating disk is  $q b_m \mathbf{v} \times \mathbf{B}$ , where  $q$  is the charge of the carrier ( $-e$  for the electrons), and  $\mathbf{v} = \omega \times \mathbf{r}$  is the velocity of a charge-carrier at a distance  $\mathbf{r}$  from the rotation axis, at rest relative to the disk. At equilibrium, carriers must be at rest relative to the disk, and the magnetic force must be compensated by a static electric field  $\mathbf{E}$  such that  $\mathbf{E} + b_m \mathbf{v} \times \mathbf{B} = 0$ . This corresponds to an electric potential drop  $V$  between the center and the circumference of the disk

$$V = \varphi(a) - \varphi(0) = - \int_0^a \mathbf{E} \cdot d\mathbf{r} = b_m \int_0^a \omega r B dr = b_m \omega B \frac{a^2}{2}. \quad (\text{S-6.124})$$

The rotating disk is thus a voltage source, known as a *Faraday disk*.

- b) In the presence of the brush contacts at points  $O$  and  $A$  of Fig. 6.9, the electromotive force  $\mathcal{E}$  of the circuit equals the voltage drop  $V$  of (S-6.124). The total current  $I$  circulating in the circuit is thus

$$I = \frac{\mathcal{E}}{R} = b_m \frac{\omega B a^2}{2R}. \quad (\text{S-6.125})$$

The power dissipated in the circuit by Joule heating is  $P_d = I^2 R = \mathcal{E}^2 / R$ , and there must be an external torque  $\tau_{\text{ext}}$  providing a mechanical power  $P_m = \tau_{\text{ext}} \cdot \omega = P_d$  in order to maintain a rotation at constant angular velocity. Thus,

$$\tau_{\text{ext}} = \hat{\mathbf{z}} b_m^2 \frac{\omega B^2 a^4}{4R}. \quad (\text{S-6.126})$$

Alternatively, the external torque must compensate the torque of the magnetic forces on the disk. Since the current exits the disk through the brush contact  $A$ , it is difficult to make assumptions on the symmetry of the current density distribution. However, the problem can be tackled as follows. The torque on an infinitesimal volume element,

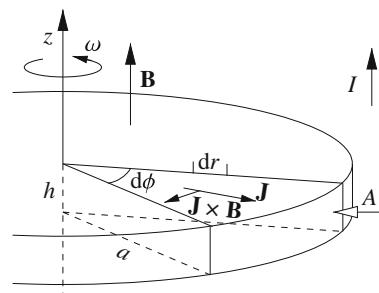


Fig. S-6.16

$r d\phi dr dz$  in cylindrical coordinates (Fig. S-6.16), is  $d\tau = b_m \mathbf{r} \times (\mathbf{J} \times \mathbf{B}) r d\phi dr dz$ , and the total magnetic torque on the disk is obtained by integrating  $d\tau$  over the disk volume

$$\tau_B = b_m \int_0^a dr \int_0^h dz \int_0^{2\pi} r d\phi \mathbf{r} \times (\mathbf{J} \times \mathbf{B}). \quad (\text{S-6.127})$$

The vector triple product in (S-6.127) can be rewritten

$$\mathbf{r} \times (\mathbf{J} \times \mathbf{B}) = \mathbf{J}(\mathbf{r} \cdot \mathbf{B}) - \mathbf{B}(\mathbf{r} \cdot \mathbf{J}) = -\hat{\mathbf{z}} Br J_r, \quad (\text{S-6.128})$$

since  $(\mathbf{r} \cdot \mathbf{B}) = 0$  because  $\mathbf{r}$  and  $\mathbf{B}$  are orthogonal to each other, and  $J_r$  is the  $r$

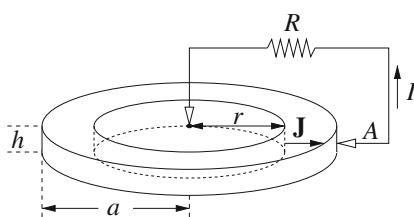


Fig. S-6.17

component of  $\mathbf{J}$ . We further have

$$\int_0^h dz \int_0^{2\pi} r d\phi J_r = I, \quad (\text{S-6.129})$$

independently of  $r$ , since the double integral is the flux of  $\mathbf{J}$  through a lateral cylindrical surface of radius  $r$  and height  $h$ , as shown in Fig. S-6.17. Thus we have for the torque exerted by the magnetic forces on the disk

$$\tau_B = -\hat{\mathbf{z}} b_m BI \int_0^a r dr, \quad (\text{S-6.130})$$

and finally, substituting (S-6.125) for  $I$ ,

$$\tau_b = -\hat{\mathbf{z}} b_m BI \frac{a^2}{2} = -\hat{\mathbf{z}} b_m^2 \frac{\omega B^2 a^4}{4R} = -\tau_{\text{ext}}. \quad (\text{S-6.131})$$

c) If the disk acts as the current source for the solenoid we must have

$$B = 4\pi k_m nI = 4\pi k_m b_m n \frac{\omega Ba^2}{2R}, \quad (\text{S-6.132})$$

from which we find that the frequency must be a function of the circuit parameters

$$\omega = \frac{2R}{4\pi k_m b_m na^2} = \begin{cases} \frac{2R}{\mu_0 na^2}, & \text{SI,} \\ \frac{Rc^2}{2\pi na^2}, & \text{Gaussian,} \end{cases} \quad (\text{S-6.133})$$

independently of the intensity of the magnetic field  $B$ .

## S-6.12 Mutual Induction Between Circular Loops

- a)** We can assume the magnetic field generated by the current  $I$  circulating in loop  $B$  to be uniform and equal to  $B_0 \hat{\mathbf{z}} = \hat{\mathbf{z}} 2\pi k_m I / b$  all over the surface of loop  $A$ , since  $a \ll b$ . The angle between the axis of loop  $A$  and the  $z$  axis is  $\theta = \omega t$ , and the flux of the magnetic field through the surface of loop  $A$  is

$$\Phi = B_0 \pi a^2 \cos \omega t = \frac{2\pi k_m I}{b} \pi a^2 \cos \omega t = \frac{2\pi^2 a^2 k_m I}{b} \cos \omega t. \quad (\text{S-6.134})$$

Thus, according to Faraday's law of induction, there is an induced electromotive force  $\mathcal{E}$  on loop  $A$

$$\mathcal{E} = -\frac{d\Phi}{dt} = \frac{2\pi^2 a^2 k_m I}{b} \omega \sin \omega t, \quad (\text{S-6.135})$$

and the current circulating in loop  $A$  is

$$I_A = \frac{2\pi^2 a^2 k_m I}{Rb} \omega \sin \omega t. \quad (\text{S-6.136})$$

- b)** The power dissipated into Joule heating is

$$P_{\text{diss}} = RI_A^2 = \frac{4\pi^4 a^4 \omega^2 k_m^2 I^2}{Rb^2} \sin^2 \omega t. \quad (\text{S-6.137})$$

- c)** The torque acting on loop  $A$  is  $\tau = \mathbf{m} \times \mathbf{B}_0$ , where  $\mathbf{m} = \hat{\mathbf{n}} I_A \pi a^2$  is the magnetic moment of loop  $A$ , and  $\hat{\mathbf{n}}$  is the unit vector perpendicular to its surface, directed so that its tip sees  $I_A$  circulating counterclockwise. Thus

$$\tau = \frac{2\pi^2 a^2 k_m I}{Rb} \omega \sin \omega t \pi a^2 \frac{2\pi k_m I}{b} \sin \omega t = \frac{4\pi^4 a^4 \omega k_m^2 I^2}{Rb^2} \sin^2 \omega t, \quad (\text{S-6.138})$$

and the corresponding mechanical power is

$$P_{\text{mech}} = \tau \cdot \omega = \frac{4\pi^4 a^4 \omega^2 k_m^2 I^2}{Rb^2} \sin^2 \omega t = P_{\text{diss}}, \quad (\text{S-6.139})$$

and all the mechanical power needed to keep loop  $A$  rotating at constant angular velocity is turned into Joule heating.

- d)** The flux through the surface of loop  $B$  of the magnetic field generated by the current  $I$  circulating in loop  $A$  is

$$\Phi_B = M_{AB} I, \quad (\text{S-6.140})$$

where  $M_{AB}$  is the coefficient of mutual induction between loop  $A$  and loop  $B$ . We know that  $M_{AB} = M_{BA}$ , and from (S-6.134) we have

$$M_{AB} = M_{BA} = \frac{2\pi^2 a^2 k_m}{b} \cos \omega t, \quad (\text{S-6.141})$$

thus

$$\Phi_B = \frac{2\pi^2 a^2 k_m I}{b} \cos \omega t, \quad (\text{S-6.142})$$

and

$$\mathcal{E} = -\frac{d\Phi}{dt} = \frac{2\pi^2 a^2 k_m I}{b} \omega \sin \omega t, \quad (\text{S-6.143})$$

as (S-6.134) and (S-6.135).

### S-6.13 Mutual Induction between a Solenoid and a Loop

**a)** Neglecting boundary effects, the magnetic field inside the solenoid is uniform, parallel to the solenoid axis  $z$ , and equal to

$$\mathbf{B} = 4\pi k_m n I \hat{\mathbf{z}}. \quad (\text{S-6.144})$$

Thus, its flux through the surface  $\mathbf{S}$  of the rotating coil is

$$\Phi_a(t) = \mathbf{B} \cdot \mathbf{S}(t) = 4\pi k_m n I \pi a^2 \cos \omega t = 4\pi^2 a^2 k_m n I \cos \omega t = M_{sl}(t) I, \quad (\text{S-6.145})$$

where

$$M_{sl}(t) = 4\pi^2 a^2 k_m n \cos \omega t \quad (\text{S-6.146})$$

is the coefficient of mutual inductance between solenoid and loop, time dependent because the loop is rotating. The coefficient of mutual inductance is symmetric,  $M_{sl} = M_{ls}$ , i.e., the inductance by the solenoid on the loop equals the inductance by the loop on the solenoid, we shall use this property for the answer to point **c**).

**b)** The electromotive force acquired by the loop equals the rate of change of the magnetic flux through it,

$$\mathcal{E} = -\frac{d\Phi}{dt} = 4\pi^2 a^2 k_m n I \omega \sin \omega t, \quad (\text{S-6.147})$$

and the current circulating in the loop is

$$I_a = \frac{4\pi^2 a^2 k_m n I \omega}{R} \sin \omega t. \quad (\text{S-6.148})$$

The loop dissipates a power  $P_{\text{diss}}$  due to Joule heating

$$P_{\text{diss}} = RI_a^2 = \frac{(4\pi^2 a^2 k_m n I \omega)^2}{R} \sin^2 \omega t. \quad (\text{S-6.149})$$

This power must be provided by the work of the torque  $\tau$  applied to the loop in order to keep it in rotation at constant angular velocity. The time-averaged power is

$$\langle P_{\text{diss}} \rangle = \frac{(4\pi^2 a^2 k_m n I \omega)^2}{2R}, \quad (\text{S-6.150})$$

since  $\langle \sin^2 \omega t \rangle = 1/2$ .

- c) The magnetic field generated by a magnetic dipole  $\mathbf{m}$  is identical to the field generated by a current-carrying loop of radius  $a$  and current  $I_l$  such that  $b_m \pi a^2 I_l = m$ , at distances  $r \gg a$  from the center of the loop. The result of point a) is valid, in particular, in the case  $a \ll b$ . In this case we can replace the magnetic dipole by a loop, and use the symmetry property of the mutual-inductance coefficient. The flux  $\Phi_s$  generated by the dipole through the solenoid is thus

$$\Phi_s = M_{ls}(t) I_l = 4\pi^2 a^2 k_m n I_l \cos \omega t = 4\pi \frac{k_m}{b_m} n m \cos \omega t. \quad (\text{S-6.151})$$

## S-6.14 Skin Effect and Eddy Inductance in an Ohmic Wire

Assuming a very long, straight cylindrical wire, the problem has cylindrical symmetry. We choose a cylindrical coordinate system  $(r, \phi, z)$  with the  $z$  axis along the axis of the wire, and expect that the electric field inside the wire can be written as

$$\mathbf{E} = \hat{\mathbf{z}} E(r, t) = \hat{\mathbf{z}} \operatorname{Re} [\tilde{E}(r) e^{i\omega t}], \quad (\text{S-6.152})$$

where  $\tilde{E}(r)$  is the static complex amplitude associated to the electric field. We start from the two Maxwell equations

$$\nabla \times \mathbf{E} = -b_m \partial_t \mathbf{B}, \quad \nabla \times \mathbf{B} = 4\pi k_m \mathbf{J} + \frac{1}{b_m c^2} \partial_t \mathbf{E}, \quad (\text{S-6.153})$$

where we have assumed  $\epsilon_r = 1$  and  $\mu_r = 1$  inside copper. If we substitute  $\mathbf{J} = \sigma \mathbf{E}$  into the second of (S-6.153) we obtain

$$\begin{aligned}\nabla \times \mathbf{B} &= 4\pi k_m \sigma \mathbf{E} + \frac{1}{b_m c^2} \partial_t \mathbf{E} = 4\pi k_m \sigma \mathbf{E} + \hat{\mathbf{z}} \frac{\omega}{b_m c^2} \operatorname{Re} [\mathrm{i} \tilde{E}(r) e^{i\omega t}] \\ &= \begin{cases} \mu_0 \sigma \mathbf{E} + \hat{\mathbf{z}} \frac{\omega}{c^2} \operatorname{Re} [\mathrm{i} \tilde{E}(r) e^{i\omega t}], & \text{SI}, \\ \frac{4\pi\sigma}{c} \mathbf{E} + \hat{\mathbf{z}} \frac{\omega}{c} \operatorname{Re} [\mathrm{i} \tilde{E}(r) e^{i\omega t}], & \text{Gaussian}. \end{cases} \quad (\text{S-6.154})\end{aligned}$$

In SI units, the conductivity of copper is  $\sigma = 5.96 \times 10^7 \Omega^{-1} \text{m}^{-1}$ , and the product  $\mu_0 \sigma c^2$  is

$$\mu_0 \sigma c^2 = 6.77 \times 10^{18} \text{s}^{-1}. \quad (\text{S-6.155})$$

Alternatively, in Gaussian units, the conductivity of copper is  $\sigma = 5.39 \times 10^{17} \text{s}^{-1}$  and the product  $4\pi\sigma$  is  $6.77 \times 10^{18} \text{s}^{-1}$ . Thus the displacement current is negligible compared to the conduction current  $\mathbf{J}$  for frequencies  $v = \omega/(2\pi) \ll 10^{18} \text{Hz}$ , i.e., up to the ultraviolet. In other words, the displacement current can be neglected compared to the conduction current for all practical purposes in good conductors, and we can rewrite the second of (S-6.153) simply as  $\nabla \times \mathbf{B} = 4\pi k_m \sigma \mathbf{E}$ . Evaluating the curl of both sides of the first of (S-6.153) we have

$$\nabla \times (\nabla \times \mathbf{E}) = -b_m \partial_t (\nabla \times \mathbf{B}) = -4\pi k_m b_m \sigma \partial_t \mathbf{E}, \quad (\text{S-6.156})$$

which, remembering that

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}, \quad (\text{S-6.157})$$

and assuming  $\nabla \cdot \mathbf{E} = 0$ , turns into a diffusion equation for the electric field  $\mathbf{E}$

$$\nabla^2 \mathbf{E} = 4\pi k_m b_m \sigma \partial_t \mathbf{E}. \quad (\text{S-6.158})$$

Introducing our assumption (S-6.152), we have the following equation in cylindrical coordinates for the complex amplitude  $\tilde{E}(r)$ ,

$$\nabla^2 \tilde{E}(r) = \frac{1}{r} \partial_r [r \partial_r \tilde{E}(r)] = \mathrm{i} \omega 4\pi k_m b_m \sigma \tilde{E}(r) \quad (\text{S-6.159})$$

or

$$\frac{1}{r} \partial_r [r \partial_r \tilde{E}(r)] = \mathrm{i} \frac{2}{\delta^2} \tilde{E}(r), \quad (\text{S-6.160})$$

where we have introduced the skin depth

$$\delta = \sqrt{\frac{1}{2\pi k_m b_m \sigma \omega}} = \begin{cases} \sqrt{\frac{2}{\mu_0 \sigma \omega}}, & \text{SI}, \\ \frac{c}{\sqrt{2\pi \sigma \omega}}, & \text{Gaussian}. \end{cases} \quad (\text{S-6.161})$$

Equation (S-6.160), multiplied by  $r^2$ , is Bessel's differential equation with  $n = 0$ . However, in this context, we prefer to find the approximate solutions for the two limiting cases  $\delta \gg r_0$  and  $\delta \ll r_0$ , where  $r_0$  is the radius of the wire. For the weak skin effect, i.e., for  $\delta \gg r_0$ , we write the solution of (S-6.160) as a Taylor series

$$\tilde{E}(r) = E_0 \sum_{n=0}^{\infty} a_n \left( \frac{r}{\delta} \right)^n \quad (\text{S-6.162})$$

which, substituted into the left-hand side of (S-6.160) gives

$$\begin{aligned} \frac{1}{r} \partial_r [r \partial_r \tilde{E}(r)] &= \frac{1}{r} \partial_r \left[ r E_0 \sum_{n=0}^{\infty} \frac{a_n n r^{n-1}}{\delta^n} \right] = \frac{1}{r} E_0 \partial_r \sum_{n=0}^{\infty} \frac{a_n n r^n}{\delta^n} \\ &= \frac{1}{r} E_0 \sum_{n=0}^{\infty} \frac{a_n n^2 r^{n-1}}{\delta^n} = E_0 \sum_{n=0}^{\infty} \frac{a_n n^2 r^{n-2}}{\delta^n}, \end{aligned} \quad (\text{S-6.163})$$

while the right-hand side is

$$i \frac{2}{\delta^2} \tilde{E}(r) = 2i E_0 \sum_{n=0}^{\infty} \frac{a_n r^n}{\delta^{n+2}}. \quad (\text{S-6.164})$$

Comparing the coefficients of the same powers of  $r$  in (S-6.163) and (S-6.164) we obtain the recurrence relation

$$a_{n+2} = \frac{2i}{(n+2)^2} a_n, \quad (\text{S-6.165})$$

which leads to

$$a_{2n} = \frac{i^n}{2^n (n!)^2} \quad \text{and} \quad a_{2n+1} = 0, \quad (\text{S-6.166})$$

for all  $n \geq 0$  and  $n \in \mathbb{N}$ . We thus have

$$\begin{aligned} \tilde{E}(r) &= E_0 \sum_{n=0}^{\infty} \frac{i^n}{2^n (n!)^2} \left( \frac{r}{\delta} \right)^{2n} \\ &= E_0 \left[ 1 + \frac{i}{2} \frac{r^2}{\delta^2} - \frac{1}{16} \frac{r^4}{\delta^4} - \frac{i}{48} \frac{r^6}{\delta^6} + \dots + \frac{i^n}{2^n (n!)^2} \frac{r^{2n}}{\delta^{2n}} + \dots \right]. \end{aligned} \quad (\text{S-6.167})$$

The complex amplitude  $I$  associated to the total current through the wire is

$$\begin{aligned}
I &= \int_0^{r_0} J 2\pi r dr = 2\pi\sigma \int_0^{r_0} \tilde{E}(r) r dr \\
&= 2\pi\sigma E_0 \int_0^{r_0} \left[ 1 + \frac{i}{2} \frac{r^2}{\delta^2} - \frac{1}{16} \frac{r^4}{\delta^4} - \frac{i}{48} \frac{r^6}{\delta^6} + \dots + \frac{i^n}{2^n (n!)^2} \frac{r^{2n}}{\delta^{2n}} + \dots \right] r dr \\
&= 2\pi\sigma E_0 \left[ \frac{r_0^2}{2} + \frac{i}{8} \frac{r_0^4}{\delta^2} - \frac{1}{96} \frac{r_0^6}{\delta^4} - \frac{i}{2304} \frac{r_0^8}{\delta^6} + \dots + \frac{i^n}{2^n (n!)^2 (2n+2)} \frac{r_0^{2n+2}}{\delta^{2n}} \right] \\
&= \pi r_0^2 \sigma E_0 \left[ 1 + \frac{i}{4} \frac{r_0^2}{\delta^2} - \frac{1}{48} \frac{r_0^4}{\delta^4} - \frac{i}{1152} \frac{r_0^6}{\delta^6} + \dots + \frac{i^n}{2^n (n+1)! n!} \frac{r_0^{2n}}{\delta^{2n}} + \dots \right]. \quad (\text{S-6.168})
\end{aligned}$$

We can define the impedance per unit length of the wire,  $Z_\ell = R_\ell + i\omega L_\ell$  (where  $R_\ell$  is the resistance per unit length, and  $L_\ell$  the self-inductance per unit length), as the ratio of the electric field at the wire surface to the total current through the wire, i.e., as

$$\begin{aligned}
Z_\ell &= \underbrace{\frac{1}{\pi r_0^2 \sigma} \left[ 1 + \frac{i}{2} \left( \frac{r_0}{\delta} \right)^2 - \frac{1}{16} \left( \frac{r_0}{\delta} \right)^4 - \frac{i}{48} \left( \frac{r_0}{\delta} \right)^6 + \dots \right]}_A \\
&\times \underbrace{\left[ 1 + \frac{i}{4} \left( \frac{r_0}{\delta} \right)^2 - \frac{1}{48} \left( \frac{r_0}{\delta} \right)^4 - \frac{i}{1152} \left( \frac{r_0}{\delta} \right)^6 + \dots \right]^{-1}}_B, \quad (\text{S-6.169})
\end{aligned}$$

where  $A$  and  $B$  are Taylor expansions in even powers of  $r_0/\delta \ll 1$ , which we have truncated at the 6th order. The first four expansion coefficients of  $B^{-1}$ , i.e.,  $1, b_1, b_2$ , and  $b_3$ ,

$$B^{-1} = \left[ 1 + b_1 \left( \frac{r_0}{\delta} \right)^2 + b_2 \left( \frac{r_0}{\delta} \right)^4 + b_3 \left( \frac{r_0}{\delta} \right)^6 + \dots \right], \quad (\text{S-6.170})$$

can be evaluated by requiring that the product  $BB^{-1}$  equals 1 with a remainder of the order of  $(r_0/\delta)^8$ , i.e.,

$$\begin{aligned}
1 &= BB^{-1} \simeq \left[ 1 + \frac{i}{4} \left( \frac{r_0}{\delta} \right)^2 - \frac{1}{48} \left( \frac{r_0}{\delta} \right)^4 - \frac{i}{1152} \left( \frac{r_0}{\delta} \right)^6 + \dots \right] \\
&\times \left[ 1 + b_1 \left( \frac{r_0}{\delta} \right)^2 + b_2 \left( \frac{r_0}{\delta} \right)^4 + b_3 \left( \frac{r_0}{\delta} \right)^6 + \dots \right] \quad (\text{S-6.171})
\end{aligned}$$

leading to

$$b_1 = -\frac{i}{4}, \quad b_2 = -\frac{1}{24}, \quad \text{and} \quad b_3 = \frac{7i}{1152} \quad (\text{S-6.172})$$

Thus we have for  $Z_\ell$

$$\begin{aligned} Z_\ell &\simeq \frac{1}{\pi r_0^2 \sigma} \left[ 1 + \frac{i}{2} \left( \frac{r_0}{\delta} \right)^2 - \frac{1}{16} \left( \frac{r_0}{\delta} \right)^4 - \frac{i}{48} \left( \frac{r_0}{\delta} \right)^6 + \dots \right] \\ &\quad \times \left[ 1 - \frac{i}{4} \left( \frac{r_0}{\delta} \right)^2 - \frac{1}{24} \left( \frac{r_0}{\delta} \right)^4 + \frac{7i}{1152} \left( \frac{r_0}{\delta} \right)^6 + \dots \right] \\ &\simeq \frac{1}{\pi r_0^2 \sigma} \left[ 1 + \frac{i}{4} \left( \frac{r_0}{\delta} \right)^2 + \frac{1}{48} \left( \frac{r_0}{\delta} \right)^4 - i \frac{23}{1152} \left( \frac{r_0}{\delta} \right)^6 + \dots \right]. \end{aligned} \quad (\text{S-6.173})$$

The zeroth-order term of the expansion,

$$R_\ell^{(0)} = \frac{1}{\pi r_0^2 \sigma}, \quad (\text{S-6.174})$$

is simply the direct-current resistance per unit length of the wire. The third term

$$R_\ell^{(1)} = \frac{r_0^2}{48\pi\sigma\delta^4} = \frac{k_m^2 b_m^2 \pi r_0^2 \sigma \omega^2}{12} = \begin{cases} \frac{\mu_0^2 \pi r_0^2 \sigma \omega^2}{192}, & \text{SI,} \\ \frac{\pi r_0^2 \sigma \omega^2}{12c^4}, & \text{Gaussian} \end{cases} \quad (\text{S-6.175})$$

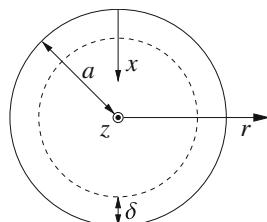
is the lowest order contribution of the weak skin effect to the resistance increase. The second-order term of the expansion can be interpreted as

$$\frac{i}{4\pi r_0^2 \sigma} \left( \frac{r_0}{\delta} \right)^2 = i \omega L_\ell^{(0)}, \quad (\text{S-6.176})$$

leading to

$$L_\ell^{(0)} = \frac{1}{4\pi\sigma\omega r_0^2} \left( \frac{r_0}{\delta} \right)^2 = \frac{1}{2} k_m b_m = \begin{cases} \frac{\mu_0}{8\pi}, & \text{SI,} \\ \frac{1}{2c^2}, & \text{Gaussian,} \end{cases} \quad (\text{S-6.177})$$

which is the DC self-inductance per unit length of a straight cylindrical wire, while the sixth-order term is the lowest order contribution of the weak skin-effect to the self-inductance of the cylindrical wire. Thus, at the low-frequency limit, the current depends on the radial coordinate, but no true skin effect is observed. According to (S-6.168), the current is actually stronger on the axis of the wire than at its surface.



**Fig. S-6.18**

Things are different at the high-frequency limit. As the frequency increases, the skin depth becomes progressively smaller. When  $\delta \ll a$ , the fields will be varying in space on a distance much smaller than the wire radius, so that we expect the effect of the curvature to be negligible. For a strong skin effect, i.e., for  $\delta \ll a$ , the electric field is significantly different from zero only close to the wire surface. Thus, we introduce the variable  $x = a - r$ , shown in Fig. S-6.18, and assume  $r \approx a$  in (S-6.160). Using  $\partial_r = -\partial_x$  we get

$$\partial_x^2 \tilde{E} = i \frac{2}{\delta^2} \tilde{E}. \quad (\text{S-6.178})$$

Substituting  $\tilde{E} = E_0 e^{\alpha x}$ , we have

$$\alpha = \pm \sqrt{i \frac{2}{\delta^2}} = \pm \frac{1+i}{\delta}, \quad (\text{S-6.179})$$

and the solution corresponding to a field decreasing for increasing  $x$  (increasing depth into the wire) is

$$\tilde{E} \approx E_0 e^{-x/\delta} e^{-ix/\delta} = E_0 e^{-(a-r)/\delta} e^{-i(a-r)/\delta}, \quad (\text{S-6.180})$$

where  $E_0 e^{i\omega t}$  is the electric field at the wire surface. The complex amplitude corresponding to the total current current through the wire is thus

$$\begin{aligned} I &= \int_0^a J 2\pi r dr = 2\pi\sigma \int_0^a \tilde{E}(r) r dr = 2\pi\sigma E_0 \int_0^a e^{-(a-r)/\delta} e^{-i(a-r)/\delta} e^{i\omega t} r dr \\ &= 2\pi\sigma E_0 e^{-a(1+i)/\delta + i\omega t} \int_0^a e^{r(1+i)/\delta} r dr. \end{aligned} \quad (\text{S-6.181})$$

Remembering that

$$\int x e^{ax} dx = e^{ax} \left( \frac{x}{a} - \frac{1}{a^2} \right), \quad (\text{S-6.182})$$

and neglecting terms in  $\delta^2$ , we obtain finally

$$I = \pi a \delta \sigma (1-i) E_0. \quad (\text{S-6.183})$$

The impedance per unit length of the wire,  $Z_\ell$ , can again be defined as

$$Z_\ell = R_\ell + iX_\ell = \frac{E_0}{I} = \frac{1}{\pi a \delta \sigma (1-i)} = \frac{1}{2\pi a \delta \sigma} + \frac{i}{2\pi a \delta \sigma}, \quad (\text{S-6.184})$$

so that the magnitudes of the resistance per unit length  $R_\ell$ , and of the reactance per unit length  $X_\ell$ , are equal at the high frequency limit:

$$R_\ell = X_\ell = \frac{1}{2\pi a \delta \sigma}. \quad (\text{S-6.185})$$

The value of  $R_\ell$  shows that the current actually flows through a thin annulus close to the surface (the “skin” of the wire), of width  $\delta$  and approximate area  $2\pi a\delta$ . The reactance per unit length can be considered as due to a self-inductance per unit length  $L_\ell$ , according to  $X_\ell = \omega L_\ell$ , with

$$L_\ell = \frac{1}{2\pi a\delta\sigma\omega} = \sqrt{\frac{k_m b_m}{2\pi a^2\sigma\omega}} = \begin{cases} \sqrt{\frac{\mu_0}{8\pi^2 a^2 \sigma \omega}}, & \text{SI,} \\ \sqrt{\frac{1}{2\pi c^2 a^2 \sigma \omega}}, & \text{Gaussian.} \end{cases} \quad (\text{S-6.186})$$

## S-6.15 Magnetic Pressure and Pinch Effect for a Surface Current

**a)** We use a cylindrical coordinate system  $(r, \phi, z)$ , with the cylinder axis as  $z$  axis. The field lines of  $\mathbf{B}$  are circles around the  $z$  axis because of symmetry. Thus,  $B_\phi(r)$  is the only nonzero component of  $\mathbf{B}$ . According to Ampère's law we have

$$B_\phi(r) = \begin{cases} 0, & r < a, \\ 2k_m \frac{I}{r} = 4\pi k_m K \frac{a}{r}, & r > a. \end{cases} \quad (\text{S-6.187})$$

**b)** First approach (heuristic). The current  $dI$  flowing in an infinitesimal surface strip parallel to  $z$ , of width  $ad\phi$ , is  $dI = Kad\phi$ . The force  $d\mathbf{f}$  exerted by an azimuthal magnetic field  $\mathbf{B} \equiv (0, B_\phi, 0)$  on an infinitesimal strip portion of length  $dz$  is

$$d\mathbf{f} = b_m dz dI \hat{\mathbf{z}} \times \mathbf{B} = -b_m K a B_\phi d\phi dz \hat{\mathbf{r}}, \quad (\text{S-6.188})$$

directed towards the axis, i.e., so to shrink the conducting surface (pinch effect). However, here we must remember that  $B_\phi(r)$  is discontinuous at the cylinder surface, being zero inside. Therefore, we replace the value of  $B_\phi$  in (S-6.188) by its “average” value  $B_\phi^{\text{aver}} = [B_\phi(a^+) - B_\phi(a^-)]/2 = 2\pi k_m K$  (the point is the same as for the calculation of electrostatic pressure on a surface charge layer). Thus, the absolute value of the force acting on an infinitesimal area  $dS = ad\phi dz$  is

$$|d\mathbf{f}| = 2\pi k_m b_m K^2 dS = \begin{cases} \frac{\mu_0}{2} K^2 dS, & \text{SI} \\ \frac{2\pi}{c^2} K^2 dS, & \text{Gaussian,} \end{cases} \quad (\text{S-6.189})$$

and the magnetic pressure on the surface is

$$P = \frac{|\mathbf{d}f|}{\mathbf{d}S} = 2\pi k_m b_m K^2 = k_m b_m \frac{I^2}{2\pi a^2} = \begin{cases} \frac{\mu_0 I^2}{2(2\pi a)^2}, & \text{SI} \\ \frac{1}{c^2} \frac{I^2}{2\pi a^2}, & \text{Gaussian.} \end{cases} \quad (\text{S-6.190})$$

Second method (rigorous). The magnetic force per infinitesimal volume  $d^3r$ , where a current density  $\mathbf{J}$  is flowing in the presence of a magnetic field  $\mathbf{B}$ , is

$$d^3\mathbf{f} = b_m \mathbf{J} \times \mathbf{B} d^3r. \quad (\text{S-6.191})$$

Due to the symmetry of our problem, the term  $(\mathbf{B} \cdot \nabla) \mathbf{B}$  appearing in (6.7) is

$$(\mathbf{B} \cdot \nabla) \mathbf{B} = \left( B_\phi \frac{1}{r} \partial_\phi \right) \mathbf{B} = 0, \quad (\text{S-6.192})$$

where we have used the gradient components in cylindrical coordinates of Table A.1, and the fact that the only nonzero component of  $\mathbf{B}$ , i.e.,  $B_\phi$ , is independent of  $\phi$ . The infinitesimal volume element in cylindrical coordinates is  $d^3r = r dr d\phi dz$ , thus

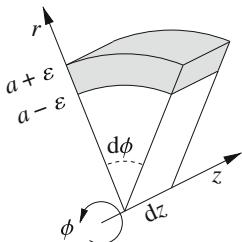


Fig. S-6.19

$$d^3\mathbf{f} = -\hat{\mathbf{r}} \frac{b_m}{8\pi k_m} [\partial_r B_\phi^2(r)] r dr d\phi dz. \quad (\text{S-6.193})$$

Now we integrate (S-6.193) with respect to  $dr$  between  $r = a - \varepsilon$  and  $a + \varepsilon$ , obtaining the force  $d^2\mathbf{f}$  acting on the small shaded volume of Fig. S-6.19, delimited by the two cylindrical surfaces  $r = a - \varepsilon$  and  $r = a + \varepsilon$ , with infinitesimal azimuthal aperture  $d\phi$ , and longitudinal length  $dz$ . Integrating by parts we have

$$\int_{a-\varepsilon}^{a+\varepsilon} [\partial_r B_\phi^2(r)] r dr = [r B_\phi^2(r)]_{a-\varepsilon}^{a+\varepsilon} - \int_{a-\varepsilon}^{a+\varepsilon} B_\phi^2(r) dr, \quad (\text{S-6.194})$$

At the limit  $\varepsilon \rightarrow 0$ , the first term on the right-hand side equals  $B_\phi^2(a^+)$ , because  $B_\phi^2(r) = 0$  for  $r < a$ . At the same limit  $\varepsilon \rightarrow 0$ , the integral on the right-hand side approaches zero because, according to the mean-value theorem, it equals  $2\varepsilon B_\phi^2(\bar{r})$ , with  $\bar{r}$  some value in the range  $(a - \varepsilon, a + \varepsilon)$ . We thus have

$$d^2\mathbf{f} = -\hat{\mathbf{r}} \frac{b_m}{8\pi k_m} B_\phi^2(a^+) a d\phi dz. \quad (\text{S-6.195})$$

where  $a d\phi dz$  is the infinitesimal surface element on which  $d^2\mathbf{f}$  is acting. The pressure is thus

$$P = \frac{b_m}{8\pi k_m} B_\phi^2(a^+) = \frac{b_m}{8\pi k_m} (4\pi k_m K)^2 = 2\pi b_m k_m K^2, \quad (\text{S-6.196})$$

in agreement with (S-6.190). Now we prove (6.7):

$$\begin{aligned} 4\pi k_m (\mathbf{J} \times \mathbf{B})_i &= [(\nabla \times \mathbf{B}) \times \mathbf{B}]_i = \varepsilon_{ijk} \left( \varepsilon_{jlm} \frac{\partial B_m}{\partial x_l} \right) B_k = \varepsilon_{ijk} \varepsilon_{jlm} \frac{\partial B_m}{\partial x_l} B_k \\ &= (\delta_{kl} \delta_{im} - \delta_{km} \delta_{il}) B_k \frac{\partial B_m}{\partial x_l} = B_k \frac{\partial B_i}{\partial x_k} - B_k \frac{\partial B_k}{\partial x_i} \\ &= (\mathbf{B} \times \nabla) B_i - \frac{1}{2} \frac{\partial (B_k B_k)}{\partial x_i} = (\mathbf{B} \times \nabla) B_i - \frac{1}{2} \nabla_i B^2, \end{aligned} \quad (\text{S-6.197})$$

where the subscripts  $i, j, k, l, m$  range from 1 to 3, and  $x_{1,2,3} = x, y, z$ , respectively. The symbol  $\varepsilon_{ijk}$  is the Levi-Civita symbol, defined by  $\varepsilon_{ijk} = 1$  if  $(i, j, k)$  is a cyclic permutation of  $(1, 2, 3)$ ,  $\varepsilon_{i,j,k} = -1$  if  $(i, j, k)$  is an anticyclic permutation of  $(1, 2, 3)$ , and  $\varepsilon_{i,j,k} = 0$  if at least two of the subscripts  $(i, j, k)$  are equal.

c) The magnetic energy  $\Delta U_M$  stored in the infinite layer between  $z$  and  $z + \Delta z$  equals the volume integral

$$\begin{aligned} \Delta U_M &= \int_{\text{layer}} u_M d^3r = \int_{\text{layer}} \frac{b_m}{8\pi k_m} B^2(r) d^3r \\ &= 2\pi \Delta z \int_a^\infty \frac{b_m}{8\pi k_m} B_\phi^2(r) r dr, \end{aligned} \quad (\text{S-6.198})$$

which, involving the integral  $\int_a^\infty r^{-1} dr$ , is infinite. However, if the radius of the cylinder increases by  $da$ , the integrand does not change for  $r > a + da$ , while the integration (Fig. S-6.20) volume decreases. Correspondingly, the (infinite) value of the integral decreases by the finite value

$$d(\Delta U_M) = -\Delta z \frac{b_m}{8\pi k_m} B_\phi^2(a^+) 2\pi a da. \quad (\text{S-6.199})$$

Thus, an expansion of the current carrying surface leads to a *decrease* of the magnetic energy. If the system were isolated, the force  $d\mathbf{f}$  acting on the surface element  $dS = a d\phi dz$  would be directed radially outwards,

leading to an expansion of the cylinder. However, the system is *not* isolated, because a current source is required to keep the current surface density  $\mathbf{K}$  constant. An increase of the radius  $da$  leads to a decrease of the magnetic flux in the layer equal

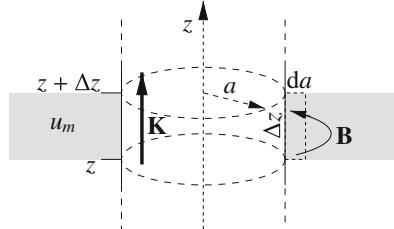


Fig. S-6.20

to  $d(\Delta\Phi) = B_\phi(a^+) \Delta z da$  (see figure), which, in turn, implies the appearance of an electromotive force  $\Delta\mathcal{E}$ . In fact, in order to keep  $\mathbf{K}$  constant during the time interval  $dt$  in which the cylinder radius increases by  $da$ , the source must provide to the layer the energy  $d(\Delta U_{\text{source}})$ , that compensates the work  $d(\Delta W) = \Delta\mathcal{E}I dt$  done by the electromotive force  $\Delta\mathcal{E} = -b_m d(\Delta\Phi)/dt$ , so that

$$\begin{aligned} d(\Delta U_{\text{source}}) &= -b_m I d(\Delta\Phi) = 2\pi b_m a K B_\phi(a^+) \Delta z da \\ &= b_m 2\pi a \frac{1}{4\pi k_m} B_\phi^2(a^+) \Delta z da = -2d(\Delta U_m). \end{aligned} \quad (\text{S-6.200})$$

Thus, the total energy balance for the layer is given by

$$d(\Delta U_{\text{tot}}) = d(\Delta U_{\text{source}}) + d(\Delta U_m) = -d(\Delta U_m), \quad (\text{S-6.201})$$

and the force per unit surface is

$$P = -\frac{1}{2\pi a \Delta z} \frac{d(\Delta U_{\text{tot}})}{da} = +\frac{1}{2\pi a \Delta z} \frac{d(\Delta U_m)}{da}, \quad (\text{S-6.202})$$

in agreement with (S-6.190).

## S-6.16 Magnetic Pressure on a Solenoid

a) The magnetic force  $d\mathbf{f}$  on an infinitesimal coil arc of length  $d\ell$ , carrying a current  $I$ , is

$$d\mathbf{f} = b_m I d\ell \times \mathbf{B}. \quad (\text{S-6.203})$$

Thus, the force  $d\mathbf{F}$  on the surface element  $d\mathbf{S} = d\ell \times dz$  of the solenoid, of width  $d\ell$ , is

$$d\mathbf{F} = b_m I B n d\ell \times dz = b_m I B n d\mathbf{S}, \quad (\text{S-6.204})$$

since the surface element comprises  $ndz$  coil arcs, each of length  $d\ell$ . The force  $d\mathbf{F}$  is directed towards the exterior of the solenoid, and the solenoid tends to expand radially.

The magnetic field  $\mathbf{B}$  is discontinuous at the surface of the solenoid, due to the presence of the electric current in the coils. At the limit of an infinitely long solenoid we have

$$\mathbf{B} = \mathbf{B}_0 = 4\pi k_m n I \hat{\mathbf{z}} = \begin{cases} \mu_0 n I \hat{\mathbf{z}}, & \text{SI,} \\ \frac{4\pi}{c} n I \hat{\mathbf{z}}, & \text{Gaussian,} \end{cases} \quad (\text{S-6.205})$$

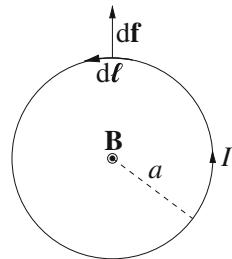
inside, where  $\hat{\mathbf{z}}$  is the unit vector along the solenoid axis, and  $\mathbf{B} = 0$  outside. Thus we substitute the average value

$$\frac{B(a^+) + B(a^-)}{2} = \frac{B_0}{2} = 2\pi k_m n I$$

for  $B$  in (S-6.204), obtaining

$$dF = 2\pi b_m k_m n^2 I^2 dS.$$

Fig. S-6.21



The pressure  $P$  on the solenoid surface is obtained by dividing  $dF$  by  $dS$ , thus

$$P = \frac{dF}{dS} = 2\pi b_m k_m n^2 I^2 = \frac{b_m B_0^2}{8\pi k_m} = \begin{cases} \frac{\mu_0}{2} n^2 I^2 = \frac{B_0^2}{2\mu_0}, & \text{SI,} \\ 2\pi n^2 I^2 = \frac{B_0^2}{8\pi}, & \text{Gaussian.} \end{cases} \quad (\text{S-6.207})$$

**b)** The magnetic energy of the solenoid can be written in terms of the magnetic energy density  $u_M$  associated to the magnetic field  $\mathbf{B}_0$

$$u_M = \frac{b_m}{8\pi k_m} B_0^2 = \begin{cases} \frac{B_0^2}{2\mu_0}, & \text{SI,} \\ \frac{B_0^2}{8\pi}, & \text{Gaussian.} \end{cases} \quad (\text{S-6.208})$$

Neglecting the boundary effects, we obtain the total magnetic energy of the solenoid  $U_M$  by multiplying  $u_M$  by the solenoid volume

$$U_M = \pi a^2 h u_M = \frac{a^2 h b_m B_0^2}{8k_m} = 2\pi^2 a^2 h b_m k_m n^2 I^2, \quad (\text{S-6.209})$$

thus, if the solenoid radius  $a$  increases by  $da$  the energy  $U_M$  increases by

$$dU_M = 4\pi^2 a h b_m k_m n^2 I^2 da, \quad (\text{S-6.210})$$

while  $B_0$ , given by (S-6.205), and thus  $u_M$ , remain constant. This implies an increase in the flux  $\Phi$  of  $\mathbf{B}_0$  through each coil of the solenoid

$$d\Phi = 2\pi a B_0 da = 8\pi^2 k_m a n I da, \quad (\text{S-6.211})$$

corresponding to a total electromotive force (the solenoid comprises  $hn$  coils)

$$\mathcal{E} = -b_m \frac{d\Phi}{dt} = -b_m k_m 8\pi^2 a h n^2 I^2 \frac{da}{dt} \quad (\text{S-6.212})$$

that must be compensated by the current source in order to keep  $I$  constant. The work  $dW_{\text{source}}$  done by the current source is thus

$$dW_{\text{source}} = -\mathcal{E}I dt = b_m k_m 8\pi^2 a n^2 h I^2 da. \quad (\text{S-6.213})$$

Thus the total energy of the system solenoid+current source changes by

$$dU_{\text{tot}} = dU_M - dW_{\text{source}} = -4\pi^2 a h b_m k_m n^2 I^2 da. \quad (\text{S-6.214})$$

The pressure on the solenoid surface is  $P = -dU_{\text{tot}}/dV$ , where  $V = \pi a^2 h$  is the volume of the solenoid. Thus

$$P = -\frac{dU_{\text{tot}}}{dV} = -\frac{1}{2\pi a h} \frac{dU_{\text{tot}}}{da} = 2\pi b_m k_m n^2 I^2, \quad (\text{S-6.215})$$

in agreement with (S-6.207).

## S-6.17 A Homopolar Motor

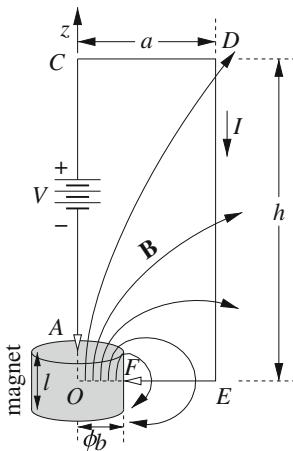


Fig. S-6.22

The motor is schematized in the diagram of Fig. S-6.22, that displays only one “half” of the circuit because of the symmetry of the problem. We use cylindrical coordinates  $(r, \phi, z)$  with the origin  $O$  at the center of the cylindrical magnet, of radius  $b$  and length  $l$ . The  $z$  axis coincides with the axes of the magnet and of the cell, which here is represented by the voltage source  $V$ . The circuit  $ACDEF$  is closed by brush contacts (white arrows in the figure) to the magnet at points  $A \equiv (0, \phi, l/2)$  and  $F \equiv (b, \phi, 0)$ , so that the current  $I$  can flow through the conducting magnet. The circuit is free to rotate around the  $z$  axis. Let  $a > b$  and  $h$  be the horizontal and vertical sizes of the circuit, respectively. We denote by  $\mathbf{B} = \mathbf{B}(r, \phi, z)$  the magnetic field generated by the magnet, independent of  $\phi$ , and with  $B_\phi \equiv 0$ . Some field lines of  $\mathbf{B}$  are sketched in Fig. S-6.22. The magnetic field on the  $z = 0$  plane is parallel to the  $z$  axis, directed upwards for  $r < b$ , and downwards for  $r > b$ . For simplicity, we approximate  $\mathbf{B}(r, \phi, 0) = B_0 \hat{\mathbf{z}}$  for  $r < b$ , with  $B_0$  independent of  $r$ , even if this approximation is valid only for  $l \gg b$ .

$z = 0$  plane is parallel to the  $z$  axis, directed upwards for  $r < b$ , and downwards for  $r > b$ . For simplicity, we approximate  $\mathbf{B}(r, \phi, 0) = B_0 \hat{\mathbf{z}}$  for  $r < b$ , with  $B_0$  independent of  $r$ , even if this approximation is valid only for  $l \gg b$ .

The voltage source drives a current  $I$  through the circuit. When the circuit is at rest we simply have  $I = V/R$ , but, when the circuit rotates, we must take into account the motion of the circuit in the presence of the magnetic field. Since the magnetic field  $\mathbf{B}$  lies on the plane of the circuit, the force  $d\mathbf{f} = I d\ell \times \mathbf{B}$  on an infinitesimal segment of the circuit  $d\ell$  is perpendicular to the plane of the circuit (out of paper in the case represented in Fig. S-6.23). The corresponding infinitesimal torque relative to the  $z$  axis is thus

$$d\tau = \mathbf{r} \times d\mathbf{f} = b_m I \mathbf{r} \times (d\ell \times \mathbf{B}), \quad (\text{S-6.216})$$

where  $\mathbf{r}$  is the distance of  $d\ell$  from the  $z$  axis. The torque  $d\tau$  is always parallel (or antiparallel) to  $\hat{\mathbf{z}}$ , independently of the circuit element  $d\ell$  we are considering. For the vector product  $d\ell \times \mathbf{B}$  we have

$$\begin{aligned} d\ell \times \mathbf{B} &= -\hat{\phi} B d\ell \sin \theta = -\hat{\phi} B d\ell \cos \psi \\ &= -\hat{\phi} \mathbf{B} \cdot \hat{\mathbf{n}} d\ell, \end{aligned} \quad (\text{S-6.217})$$

where  $\theta$  is the angle between  $d\ell$  and  $\mathbf{B}$ ,  $\hat{\mathbf{n}}$  is the unit vector perpendicular to  $d\ell$ , and  $\psi = \theta - \pi/2$  is the angle between  $\mathbf{B}$  and  $\hat{\mathbf{n}}$ , as shown in Fig. S-6.24. Since  $\hat{\mathbf{r}}$  is perpendicular to  $\hat{\phi}$  (unit vectors of the corresponding cylindrical coordinates), we have for the total torque acting on the circuit

$$\tau = b_m I \int_A^F \mathbf{r} \times (d\ell \times \mathbf{B}) = -\hat{\mathbf{z}} b_m I \int_A^F \mathbf{B} \cdot \hat{\mathbf{n}} r d\ell. \quad (\text{S-6.218})$$

The last integral of (S-6.218) can be calculated, within our approximations, if we first demonstrate that the line integral of  $\mathbf{B} \cdot \hat{\mathbf{n}} r$  around the closed path  $OCDEO$  of Fig. S-6.22 is zero, i.e., that

$$\oint \mathbf{B} \cdot \hat{\mathbf{n}} r d\ell = \int_A^F \mathbf{B} \cdot \hat{\mathbf{n}} r d\ell + \int_F^O \mathbf{B} \cdot \hat{\mathbf{n}} r d\ell + \int_O^A \mathbf{B} \cdot \hat{\mathbf{n}} r d\ell = 0. \quad (\text{S-6.219})$$

First, we note that the integral along the whole  $\overline{OC}$  path is zero, both because  $r$  is zero, and because  $\mathbf{B}$  is parallel to  $d\ell$ , thus perpendicular to  $\hat{\mathbf{n}}$ . Thus, the integral of (S-6.219) becomes

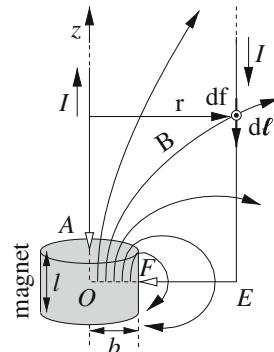


Fig. S-6.23

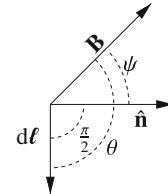
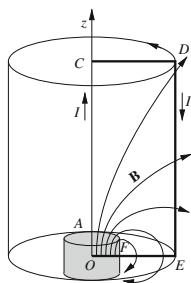


Fig. S-6.24

$$\oint \mathbf{B} \cdot \hat{\mathbf{n}} r d\ell = \int_C^D \mathbf{B} \cdot \hat{\mathbf{n}} r dr - \int_D^E \mathbf{B} \cdot \hat{\mathbf{n}} r dz - \int_E^O \mathbf{B} \cdot \hat{\mathbf{n}} r dr \\ = \int_C^D \mathbf{B} \cdot \hat{\mathbf{n}} r dr + \int_D^E \mathbf{B} \cdot \hat{\mathbf{n}} r dz + \int_O^E \mathbf{B} \cdot \hat{\mathbf{n}} r dr, \quad (\text{S-6.220})$$

since  $d\ell = dr$  along  $\overline{CD}$ ,  $d\ell = -dz$  along  $\overline{DE}$ , and  $d\ell = -dr$  along  $\overline{EO}$ .



As a next step, we generate a cylinder by rotating the  $CDEO$  path around the  $z$  axis, as in Fig. S-6.25. The outgoing flux of the magnetic field  $\mathbf{B}$  through the total surface of the cylinder is

$$\int_{\text{upper base}} \mathbf{B} \cdot \hat{\mathbf{n}} dS + \int_{\text{lateral surface}} \mathbf{B} \cdot \hat{\mathbf{n}} dS + \int_{\text{lower base}} \mathbf{B} \cdot \hat{\mathbf{n}} dS = 0, \quad (\text{S-6.221})$$

since  $\nabla \cdot \mathbf{B} = 0$ . Equation (S-6.221) can be rewritten

$$0 = \int_0^a \mathbf{B}(r, \phi, h) \cdot \hat{\mathbf{n}} 2\pi r dr + \int_0^h \mathbf{B}(a, \phi, z) \cdot \hat{\mathbf{n}} 2\pi r dz \\ + \int_0^a \mathbf{B}(r, \phi, 0) \cdot \hat{\mathbf{n}} 2\pi r dr = 2\pi \oint \mathbf{B} \cdot \hat{\mathbf{n}} r d\ell, \quad (\text{S-6.222})$$

Fig. S-6.25

which demonstrates (S-6.219). For the last integral appearing in (S-6.218) we thus have

$$\int_A^F \mathbf{B} \cdot \hat{\mathbf{n}} r d\ell = - \int_F^O \mathbf{B} \cdot \hat{\mathbf{n}} r d\ell = \int_0^b B_0 r dr = \frac{B_0 b^2}{2}, \quad (\text{S-6.223})$$

where we have remembered that the line integrals are zero on the  $z$  axis, that  $d\ell = -dr$  on the  $\overline{FO}$  line, and that, within our approximations,  $\mathbf{B} \cdot \hat{\mathbf{n}} = -B_0$ , independently of  $r$ , on the  $\overline{FO}$  line. The torque on the rotating circuit is

$$\tau = -\hat{\mathbf{z}} b_m I \int_A^F \mathbf{B} \cdot \hat{\mathbf{n}} r d\ell = -\hat{\mathbf{z}} b_m I \frac{1}{R} \left( V + b_m \omega \frac{B_0 b^2}{2} \right). \quad (\text{S-6.224})$$

This is why sliding contacts are needed in a homopolar motor. If the line segment  $\overline{FO}$  were rotating with the rest of the circuit, the total torque on the complete circuit around the  $z$  axis would be zero, because the torque acting on  $\overline{FO}$  would compensate the torque on the rest of the circuit.

If we denote by  $\mathcal{I}$  the moment of inertia of the rotating circuit and for the moment, neglect frictional effects, the equation of motion is

$$\mathcal{I} \frac{d\omega}{dt} = \tau = -b_m I \frac{B_0 b^2}{2} - \eta\omega, \quad (\text{S-6.225})$$

where we have assumed the presence of a frictional torque  $\tau_{fr} = -\eta\omega$  proportional to the angular velocity. The current  $I$  is determined by the voltage source and by the electromotive force  $\mathcal{E}$ , due to the rotation of the circuit in the presence of the magnetic field  $\mathbf{B}$ ,

$$\begin{aligned} \mathcal{E} &= b_m \int_C^F (\boldsymbol{\omega} \times \mathbf{r}) \cdot \mathbf{B} \cdot d\ell = b_m \int_C^F \omega r \hat{\phi} \times \mathbf{B} \cdot d\ell = -b_m \omega \int_C^F r \hat{\phi} \cdot d\ell \times \mathbf{B} \\ &= b_m \omega \int_C^F r \mathbf{B} \cdot \hat{\mathbf{n}} d\ell = b_m \omega \frac{B_0 b^2}{2}, \end{aligned} \quad (\text{S-6.226})$$

where we have used (S-6.217) and (S-6.223) in the last two steps. The current is thus

$$I = \frac{1}{R} \left( V + b_m \omega \frac{B_0 b^2}{2} \right), \quad (\text{S-6.227})$$

and the equation of motion is

$$\begin{aligned} \mathcal{I} \frac{d\omega}{dt} &= -b_m \frac{1}{R} \left( V + b_m \omega \frac{B_0 b^2}{2} \right) \frac{B_0 b^2}{2} - \eta\omega \\ &= -b_m \frac{VB_0 b^2}{2R} - \omega \left( b_m^2 \frac{B_0^2 b^4}{4R} + \eta \right), \end{aligned} \quad (\text{S-6.228})$$

with solution

$$\omega = -\frac{2b_m VB_0 b^2}{b_m^2 B_0^2 b^4 + 4R\eta} \left( 1 - e^{-t/T} \right), \quad \text{where } T = \frac{4R\mathcal{I}}{b_m^2 B_0^2 b^4 + 4R\eta}. \quad (\text{S-6.229})$$

If we assume negligible frictional torque, i.e.,  $\eta \ll b_m^2 B_0^2 b^4 / (4R)$ , (S-6.229) reduces to

$$\omega = -\frac{2V}{b_m B_0 b^2} \left( 1 - e^{-t/T} \right), \quad \text{where } T = \frac{4R\mathcal{I}}{b_m^2 B_0^2 b^4}, \quad (\text{S-6.230})$$

however, inserting “reasonable values” into (S-6.230), such as  $V = 1.5 \text{ V}$ ,  $B_0 = 100 \text{ Gauss} = 10^{-2} \text{ T}$  and  $b = 0.5 \text{ cm}$  we obtain for the steady state solution

$$\omega_0 = -\frac{2V}{b_m B_0 b^2} \approx -1200 \text{ rad/s}, \quad \text{i.e.,} \quad \nu_0 \approx 190 \text{ s}^{-1}, \quad (\text{S-6.231})$$

which is indeed a very fast rotation! In the absence of friction, the steady state is reached when  $V + \mathcal{E} = 0$ , so that  $I = 0$  and there is no torque acting on the circuit. The final steady-state kinetic energy of the rotating circuit in these conditions is

$$K_{ss} = \frac{1}{2} I \omega_0^2 = \frac{1}{2} I \frac{4V^2}{b_m^2 B_0^2 b^4} = \frac{2V^2 I}{b_m^2 B_0^2 b^4}. \quad (\text{S-6.232})$$

The current flowing in the circuit is

$$I(t) = \frac{1}{R} \left( V + \frac{b_m B_0 b^2 \omega}{2} \right) = \frac{V}{R} e^{-t/T}, \quad (\text{S-6.233})$$

and the total energy provided by the voltage source is

$$U = \int_0^\infty VI dt = \frac{V^2}{R} \int_0^\infty e^{-t/T} dt = \frac{V^2}{R} T = \frac{4V^2 I}{b_m^2 B_0^2 b^4} = 2K_{ss}, \quad (\text{S-6.234})$$

or twice the final kinetic energy. An amount equal to  $K_{ss}$  is dissipated into Joule heat. More realistically, we must take the frictional torque into account. For instance, the steady-state angular velocity is reduced by a factor 10 if we assume  $4R\eta = 9b_m B_0 b^2$ . This, assuming  $R = 1\Omega$ , means

$$\eta \approx 6 \times 10^{-5} \text{ Nms}. \quad (\text{S-6.235})$$

In the presence of friction the steady-state angular velocity is

$$\omega_f = -\frac{2b_m V B_0 b^2}{b_m^2 B_0^2 b^4 + 4R\eta}, \quad (\text{S-6.236})$$

and the power dissipated by friction is

$$P_{fr} = \tau_{fr} \omega_f = \eta \omega_f^2 = \eta \left( \frac{2b_m V B_0 b^2}{b_m^2 B_0^2 b^4 + 4R\eta} \right)^2. \quad (\text{S-6.237})$$

The voltage source drives a current

$$I_f = \frac{V}{R} \left( 1 - \frac{b_m^2 B_0^2 b^4}{b_m^2 B_0^2 b^4 + 4R\eta} \right) = \frac{4V\eta}{b_m^2 B_0^2 b^4 + 4R\eta}, \quad (\text{S-6.238})$$

and provides a power

$$P_{\text{source}} = VI_f = \frac{4V^2\eta}{b_m^2 B_0^2 b^4 + 4R\eta}. \quad (\text{S-6.239})$$

The power dissipated into Joule heat is

$$P_J = RI_f^2 = R \left( \frac{4V\eta}{b_m^2 B_0^2 b^4 + 4R\eta} \right)^2, \quad (\text{S-6.240})$$

and we can easily check that

$$P_J + P_{\text{fr}} = P_{\text{source}}. \quad (\text{S-6.241})$$

# Chapter S-7

## Solutions for Chapter 7

### S-7.1 Coupled RLC Oscillators (1)

a) Assuming the two currents  $I_1$  and  $I_2$  to flow clockwise, and applying Kirchhoff's mesh rule to the two loops of the circuit, we have

$$L \frac{dI_1}{dt} + \frac{Q_1}{C_1} + \frac{Q_0}{C_0} = 0, \quad L \frac{dI_2}{dt} + \frac{Q_2}{C_1} - \frac{Q_0}{C_0} = 0, \quad (\text{S-7.1})$$

where  $Q_1$  is the charge of the left capacitor,  $Q_2$  the charge of the right capacitor, and  $Q_0$  the charge of capacitor  $C_0$ . Charge conservation in the two loops implies

$$\frac{dQ_1}{dt} = I_1, \quad \frac{dQ_2}{dt} = I_2, \quad (\text{S-7.2})$$

while Kirchhoff's junction rule, applied either to junction A or to junction B, leads to

$$\frac{dQ_0}{dt} = I_1 - I_2. \quad (\text{S-7.3})$$

Differentiating (S-7.1), substituting (S-7.2) and (S-7.3), and dividing by  $L$ , we obtain

$$\begin{aligned} \frac{d^2I_1}{dt^2} &= -\frac{1}{LC_1}I_1 - \frac{1}{LC_0}(I_1 - I_2), \quad \text{or} \quad \frac{d^2I_1}{dt^2} = -\omega_1^2 I_1 - \omega_0^2(I_1 - I_2) \\ \frac{d^2I_2}{dt^2} &= -\frac{1}{LC_1}I_2 - \frac{1}{LC_0}(I_2 - I_1), \quad \text{or} \quad \frac{d^2I_2}{dt^2} = -\omega_1^2 I_2 - \omega_0^2(I_2 - I_1), \end{aligned} \quad (\text{S-7.4})$$

where we have introduced the quantities  $\omega_0 = 1/\sqrt{LC_0}$  and  $\omega_1 = 1/\sqrt{LC_1}$ . By substituting  $I_1 = A_1 e^{-i\omega t}$  and  $I_2 = A_2 e^{-i\omega t}$  from (7.2) into (S-7.4), we obtain

$$\begin{aligned} (\omega_1^2 + \omega_0^2 - \omega^2)A_1 - \omega_0^2 A_2 &= 0 \\ -\omega_0^2 A_1 + (\omega_1^2 + \omega_0^2 - \omega^2)A_2 &= 0. \end{aligned} \quad (\text{S-7.5})$$

Non-trivial solutions for this system exist only if the determinant

$$D = D(\omega) = (\omega_1^2 + \omega_0^2 - \omega^2)^2 - \omega_0^4 = (\omega_1^2 - \omega^2)(\omega_1^2 + 2\omega_0^2 - \omega^2) \quad (\text{S-7.6})$$

equals zero. Thus, the frequencies of the normal modes of the circuit are the roots of the equation  $D(\omega) = 0$ , i.e.,

$$\omega = \omega_1 \equiv \Omega_+, \quad \omega = \sqrt{\omega_1^2 + 2\omega_0^2} \equiv \Omega_- . \quad (\text{S-7.7})$$

Substituting these values for  $\omega$  into (S-7.5) we obtain that  $A_1 = A_2$ , i.e.,  $I_1(t) = I_2(t)$ , for the mode of frequency  $\Omega_+$ , and that  $A_1 = -A_2$ , i.e.,  $I_1(t) = -I_2(t)$ , for the mode of frequency  $\Omega_-$ .

The normal modes of this simple case, with only two degrees of freedom, can also be evaluated, more simply, by taking the sum and the difference of (S-7.4), obtaining the harmonic oscillator equations

$$\frac{d^2 I_{\pm}}{dt^2} = -\Omega_{\pm}^2 I_{\pm} . \quad (\text{S-7.8})$$

for the variables  $I_{\pm} \equiv I_1 \pm I_2$ . The currents in the two meshes are  $I_1 = (I_+ + I_-)/2$  and  $I_2 = (I_+ - I_-)/2$ , respectively.

When the circuit is in the mode of frequency  $\Omega_+$ , no current flows through the  $AB$  branch (capacitor  $C_0$ ), where the two currents cancel out because  $I_1 = I_2$ . Frequency  $\Omega_+$  is simply the resonant frequency of a single-loop  $LC$  circuit of inductance  $L$  and capacitance  $C_1$ , i.e., the frequency at which the impedance of the loop is zero

$$Z_{LC}(\omega) = Z_L(\omega) + Z_{C_1}(\omega) = -i\omega L - \frac{1}{i\omega C_1} = 0 . \quad (\text{S-7.9})$$

Since  $Z_{LC}(\Omega_+) = 0$ , the current flows “freely” through each loop.

For the mode of frequency  $\Omega_-$ , we have  $I_1 = -I_2$ , and a current  $2I_1$  flows through the  $AB$  branch. The effective impedance of the circuit is the series of  $Z_{C_0} = (i\omega C_0)^{-1}$  with the parallel of the two impedances  $Z_{LC}$ ,

$$Z = Z_{C_0} + \frac{Z_{LC}Z_{LC}}{Z_{LC} + Z_{LC}} = Z_0 + \frac{Z_{LC}}{2} = -\frac{1}{i\omega C_0} - \frac{1}{2} \left( i\omega L + \frac{1}{i\omega C_1} \right) , \quad (\text{S-7.10})$$

which vanishes if

$$\omega^2 = \frac{1}{L} \left( \frac{2}{C_1} + \frac{1}{C_0} \right) = \Omega_-^2 . \quad (\text{S-7.11})$$

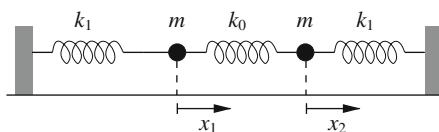


Fig. S-7.1

The circuit is equivalent to the two coupled identical harmonic oscillators of Fig. S-7.1. Each oscillator comprises a mass  $m$ , connected to a fixed wall by a spring of Hooke's constant  $k_1 = m\omega_1^2$ .

The two masses are connected to each other by a third spring of Hooke's constant  $k_0 = m\omega_0^2$ . We assume that all springs have their respective rest lengths when the two masses are at their equilibrium positions. The equations of motions for the two masses are

$$\begin{aligned} m \frac{d^2x_1}{dt^2} &= -k_1 x_1 - k_0(x_1 - x_2), \quad \text{or} \quad \frac{d^2x_1}{dt^2} = -\omega_1^2 x_1 - \omega_0^2(x_1 - x_2) \\ m \frac{d^2x_2}{dt^2} &= -k_1 x_2 - k_0(x_2 - x_1), \quad \text{or} \quad \frac{d^2x_2}{dt^2} = -\omega_1^2 x_2 - \omega_0^2(x_2 - x_1), \end{aligned} \quad (\text{S-7.12})$$

where  $x_1$  and  $x_2$  are the displacements of the two masses from their equilibrium positions. Equations (S-7.12) for  $x_1$  and  $x_2$  are formally equivalent to equations (S-7.4) for  $I_1$  and  $I_2$ , and thus have the same solutions. For the mode at frequency  $\Omega_+$ , the two masses oscillate in phase ( $x_1 = x_2$ ), central spring ( $k_0$ ) has always its rest length, and does not exert forces on the two masses. Thus, frequency  $\Omega_+$  is the characteristic frequency each single harmonic oscillator. For the mode at frequency  $\Omega_-$ , we have  $x_1 = -x_2$  and the two masses oscillate with opposite phases.

**b)** The presence of a nonzero resistance  $R$  in series with each inductor changes Equation (S-7.1) into

$$L \frac{dI_1}{dt} + RI_1 + \frac{Q_1}{C_1} + \frac{Q_0}{C_0} = 0, \quad L \frac{dI_2}{dt} + RI_2 + \frac{Q_2}{C_1} - \frac{Q_0}{C_0} = 0, \quad (\text{S-7.13})$$

By differentiating the equations and proceeding as for (S-7.8) we obtain

$$\frac{d^2I_{\pm}}{dt^2} = -\Omega_{\pm}^2 I_{\pm} - \gamma \frac{dI_{\pm}}{dt}, \quad (\text{S-7.14})$$

with  $\gamma = R/L$ . These are the equations of two damped oscillators. The amplitudes of the normal modes vary in time as  $\exp(-i\Omega_{\pm}t - \gamma t)$ , decaying with a time constant  $\tau = \gamma^{-1}$ . The damping rate of the normal modes can also be found by looking for solutions in the form  $I_{1,2} = A_{1,2} e^{-i\omega t}$ , but allowing  $A_{1,2}$  and  $\omega$  to have imaginary parts. For the equivalent mechanical system, the same equations are obtained by inserting frictional forces  $f_i = -m\gamma dx_i/dt$  in the equations of motion (S-7.12).

**c)** Inserting the voltage source, Equations (S-7.13) are modified as follows:

$$L \frac{dI_1}{dt} + RI_1 + \frac{Q_1}{C_1} + \frac{Q_0}{C_0} = V_0 e^{-i\omega t}, \quad L \frac{dI_2}{dt} + RI_2 + \frac{Q_2}{C_1} - \frac{Q_0}{C_0} = 0, \quad (\text{S-7.15})$$

and, by proceeding as for (S-7.8) and (S-7.14), we have

$$\frac{d^2I_{\pm}}{dt^2} = -\Omega_{\pm}^2 I_{\pm} - \gamma \frac{dI_{\pm}}{dt} - \frac{i\omega V_0}{L} e^{-i\omega t}, \quad (\text{S-7.16})$$

which are the equations of two forced oscillators with a driving term  $-(i\omega V_0/L)e^{-i\omega t}$ . Resonances are observed when  $\omega = \Omega_+$  and for  $\omega = \Omega_-$ , i.e., when the driving frequency equals one of the frequencies of the normal modes.

## S-7.2 Coupled RLC Oscillators (2)

**a)** Proceeding as in Solution S-7.1, we assume  $I_1$  and  $I_2$  to flow clockwise. Applying Kirchhoff's mesh rule to both meshes of the circuit we obtain

$$\begin{aligned} L \frac{dI_1}{dt} + \frac{Q_1}{C} + L_0 \left( \frac{dI_1}{dt} - \frac{dI_2}{dt} \right) &= 0, \\ L \frac{dI_2}{dt} + \frac{Q_2}{C} - L_0 \left( \frac{dI_1}{dt} - \frac{dI_2}{dt} \right) &= 0, \end{aligned} \quad (\text{S-7.17})$$

again with  $I_1 = dQ_1/dt$  and  $I_2 = dQ_2/dt$ . Differentiating (S-7.17) with respect to  $t$  we obtain

$$\begin{aligned} (L + L_0) \frac{d^2 I_1}{dt^2} + \frac{I_1}{C} - L_0 \frac{d^2 I_2}{dt^2} &= 0 \\ (L + L_0) \frac{d^2 I_2}{dt^2} + \frac{I_2}{C} - L_0 \frac{d^2 I_1}{dt^2} &= 0. \end{aligned} \quad (\text{S-7.18})$$

The sum and difference of the two equations of (S-7.18) give the following equations for the new variables  $I_{\pm} \equiv I_1 \pm I_2$

$$\frac{d^2 I_+}{dt^2} = -\frac{I_+}{LC} \equiv -\Omega_+^2 I_+, \quad \frac{d^2 I_-}{dt^2} = -\frac{I_-}{(L + 2L_0)C} \equiv -\Omega_-^2 I_-, \quad (\text{S-7.19})$$

which show that  $I_{\pm}$  are the normal oscillation modes of the circuit, and  $\Omega_{\pm}$  the corresponding frequencies.

**b)** Inserting  $R \neq 0$ , (S-7.17) turn into

$$\begin{aligned} L \frac{dI_1}{dt} + RI_1 + \frac{Q_1}{C} + L_0 \left( \frac{dI_1}{dt} - \frac{dI_2}{dt} \right) + R(I_1 - I_2) &= 0 \\ L \frac{dI_2}{dt} + RI_2 + \frac{Q_2}{C} - L_0 \left( \frac{dI_1}{dt} - \frac{dI_2}{dt} \right) - R(I_1 - I_2) &= 0. \end{aligned} \quad (\text{S-7.20})$$

Performing again the sum and difference of the two equations we obtain

$$\frac{d^2 I_+}{dt^2} = -\gamma_+ \frac{dI_+}{dt} - \Omega_+^2 I_+, \quad \frac{d^2 I_-}{dt^2} = -\gamma_- \frac{dI_-}{dt} - \Omega_-^2 I_-, \quad (\text{S-7.21})$$

with  $\gamma_+ = R/L$ , and  $\gamma_- = 3R/(L + 2L_0)$ . These are the equations for two damped oscillators, with different damping rates  $\gamma_{\pm}$ .

## S-7.3 Coupled RLC Oscillators (3)

**a)** Let us denote by  $Q_1$  and  $Q_2$  the charges of the capacitors on the  $AB$  and on the  $DE$  branches, respectively. According to Kirchhoff's mesh rule we have, for the three meshes of the circuit,

$$L \frac{dI_1}{dt} = -\frac{Q_1}{C}, \quad L \frac{dI_2}{dt} = \frac{Q_1}{C} - \frac{Q_2}{C}, \quad L \frac{dI_3}{dt} = \frac{Q_2}{C}, \quad (\text{S-7.22})$$

and, according to Kirchhoff's junction rule applied to the *A* and *D* junctions,

$$\frac{dQ_1}{dt} = I_1 - I_2, \quad \frac{dQ_2}{dt} = I_2 - I_3. \quad (\text{S-7.23})$$

Differentiating Equations (S-7.22) with respect to *t*, and substituting  $dQ_1/dt$  and  $dQ_2/dt$  from (S-7.23), we obtain

$$\begin{aligned} \frac{d^2I_1}{dt^2} &= \frac{1}{LC}(-I_1 + I_2), \\ \frac{d^2I_2}{dt^2} &= \frac{1}{LC}(I_1 - 2I_2 + I_3), \\ \frac{d^2I_3}{dt^2} &= \frac{1}{LC}(I_2 - I_3). \end{aligned} \quad (\text{S-7.24})$$

Mathematically, the circuit is equivalent to a mechanical system comprising three identical masses *m*, coupled by two identical springs of Hooke's constant *k*, as shown in Fig. S-7.2. If we denote by  $x_1$ ,  $x_2$ , and  $x_3$  the displacement of each mass from its rest position, the equations of motion for the three masses are

$$\begin{aligned} \frac{d^2x_1}{dt^2} &= \frac{k}{m}(x_2 - x_1), \\ \frac{d^2x_2}{dt^2} &= -\frac{k}{m}(x_2 - x_1) + \frac{k}{m}(x_2 - x_3), \\ \frac{d^2x_3}{dt^2} &= -\frac{k}{m}(x_2 - x_3), \end{aligned} \quad (\text{S-7.25})$$

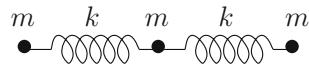


Fig. S-7.2

which are identical to (S-7.25), after substituting  $I_j \rightarrow x_j$ , with  $j = 1, 2, 3$ , and  $1/(LC) \rightarrow k/m$ .

**b)** The frequencies of the normal modes can be found by looking for solutions of (S-7.25) in the form

$$I_j(t) = A_j e^{-i\omega t}. \quad (\text{S-7.26})$$

After substituting (S-7.26) and  $\omega_0^2 = 1/(LC)$  into (S-7.25), and dividing by the common exponential factor, we obtain the system of linear equations in matrix form

$$\begin{pmatrix} (\omega_0^2 - \omega^2) & -\omega_0^2 & 0 \\ -\omega_0^2 & (2\omega_0^2 - \omega^2) & -\omega_0^2 \\ 0 & -\omega_0^2 & (\omega_0^2 - \omega^2) \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = 0, \quad (\text{S-7.27})$$

which has non-trivial solutions only if the determinant of the matrix is zero, i.e., if

$$(\omega_0^2 - \omega^2) [ (2\omega_0^2 - \omega^2)(\omega_0^2 - \omega^2) - \omega_0^4 ] - \omega_0^4 (\omega_0^2 - \omega^2) = 0. \quad (\text{S-7.28})$$

Equation (S-7.28) is a cubic equation in  $\omega^2$ , in the following we shall consider only the corresponding nonnegative values of  $\omega$ . A first solution is  $\omega = \omega_0 = \Omega_1$ . If we substitute  $\omega = \Omega_1$  into (S-7.27) we obtain  $A_1 = -A_3$ , and  $A_2 = 0$ , corresponding to zero current in the central mesh, and  $I_1$  and  $I_2$  oscillating with opposite phases. For the mechanical system of Fig. S-7.2, this solution corresponds to the central mass at rest, while the left and right masses oscillate with opposite phases.

Dividing (S-7.28) by  $(\omega_0^2 - \omega^2)$  we obtain the equation

$$-3\omega_0^2\omega^2 + \omega^4 = 0, \quad (\text{S-7.29})$$

which has the two solutions  $\omega = \sqrt{3}\omega_0 = \Omega_2$  and  $\omega = 0 = \Omega_3$ . The mode of zero frequency ( $\Omega_3$ ) corresponds to a DC current  $I = I_1 = I_2 = I_3$  flowing freely through the inductors, while  $I_1$  and  $I_2$  cancel out in branch  $AB$ , and  $I_2$  and  $I_3$  cancel out in branch  $DE$ . For the mechanical system, this solution correspond to a pure translational motions of the three masses.

Substituting  $\Omega_2$  into (S-7.27) we obtain

$$A_2 = -2A_1, \quad A_3 = A_1, \quad (\text{S-7.30})$$

i.e.,  $I_1$  and  $I_3$  have the same amplitude and oscillate in phase, while  $I_2$  oscillates with double amplitude and opposite phase. The two external masses of Fig. S-7.2 oscillate in phase, at constant distance from each other, while the central mass oscillates with opposite phase and double amplitude, so that the center of mass is at rest.

The three quantities

$$\mathcal{J}_0 = I_1 + I_2 + I_3, \quad \mathcal{J}_1 = I_1 - I_3, \quad \mathcal{J}_2 = I_1 - 2I_2 + I_3, \quad (\text{S-7.31})$$

corresponding to the three normal modes of the circuits, oscillate at the frequencies  $\Omega_0 = 0$ ,  $\Omega_1$ , and  $\Omega_2$ , respectively.

c) Taking the finite resistances into account, (S-7.22) become

$$\frac{dI_1}{dt} + RI_1 = -\frac{\Omega_1}{C}, \quad \frac{dI_2}{dt} + RI_2 = \frac{\Omega_1}{C} - \frac{\Omega_2}{C}, \quad \frac{dI_3}{dt} + RI_3 = \frac{\Omega_2}{C}, \quad (\text{S-7.32})$$

which give for the normal modes

$$\begin{aligned} \frac{d^2\mathcal{J}_0}{dt^2} + \frac{R}{L} \frac{d\mathcal{J}_0}{dt} &= 0, \\ \frac{d^2\mathcal{J}_1}{dt^2} + \frac{R}{L} \frac{d\mathcal{J}_1}{dt} + \Omega_1^2 \mathcal{J}_1 &= 0, \\ \frac{d^2\mathcal{J}_2}{dt^2} + \frac{R}{L} \frac{d\mathcal{J}_2}{dt} + \Omega_2^2 \mathcal{J}_2 &= 0. \end{aligned} \quad (\text{S-7.33})$$

The solution for  $\mathcal{J}_0$  describes a non-oscillating, exponentially decreasing current  $\mathcal{J}_0 = C_0 e^{-\gamma t}$ , with decay rate  $\gamma = R/L$ . The last two equations describe damped oscillating currents  $\mathcal{J}_{1,2} = C_{1,2} \exp(-i\tilde{\Omega}_{1,2}t - \gamma t)$ , with

$$\tilde{\Omega}_{1,2} = \sqrt{\omega_{1,2}^2 - \frac{\gamma^2}{4}}, \quad (\text{S-7.34})$$

where we have assumed  $\tilde{\Omega}_{1,2} > \gamma/2$ .

## S-7.4 The LC Ladder Network

a) Let  $Q_n$  be the charge on the  $n$ th capacitor. Kirchhoff's junction rule at junction  $D$  of Fig. 7.4 implies

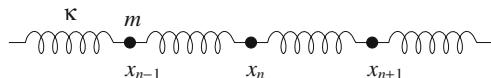
$$\frac{dQ_n}{dt} = I_{n-1} - I_n, \quad (\text{S-7.35})$$

while Kirchhoff's mesh rule applied to mesh  $DEFG$  implies

$$\frac{Q_n}{C} - \frac{Q_{n+1}}{C} = L \frac{dI_n}{dt}. \quad (\text{S-7.36})$$

Now we differentiate (S-7.36) with respect to time, and insert (S-7.35) for the derivatives of  $Q_n$ , obtaining

$$\frac{d^2I_n}{dt^2} = \omega_0^2(I_{n-1} - 2I_n + I_{n+1}), \quad \text{where} \quad \omega_0^2 = \frac{1}{LC}. \quad (\text{S-7.37})$$



**Fig. S-7.3**

The equivalent mechanical system is a linear sequence of  $N$  identical masses  $m$ , each pair of consecutive masses being bound to each other by a spring of Hooke's constant  $\kappa$  (we use the Greek letter  $\kappa$  here because we shall need the letter  $k$  for the wavevector later on), as shown in Fig. S-7.3. We denote by  $x_n$  the displacement of each mass from its equilibrium position, i.e., its position when all springs have their rest length. Thus, the equation of motion of the  $n$ th mass is

$$m \frac{d^2x_n}{dt^2} = -\kappa(x_n - x_{n-1}) + \kappa(x_{n+1} - x_n), \quad (\text{S-7.38})$$

which, divided by  $m$ , and after introducing  $\omega_0^2 = \kappa/m$  becomes

$$\frac{d^2x_n}{dt^2} = \omega_0^2 (x_{n-1} - 2x_n + x_{n+1}), \quad (\text{S-7.39})$$

mathematically equivalent to (S-7.37). This equation can be generalized to the case of a mechanical system where transverse displacements are allowed, in addition to the longitudinal displacements. If the masses can move in three dimensions, and we denote by  $\mathbf{r}_n$  the displacement of the  $n$ th mass from its equilibrium position, the equation of motion is written

$$\frac{d^2\mathbf{r}_n}{dt^2} = \omega_0^2 (\mathbf{r}_{n-1} - 2\mathbf{r}_n + \mathbf{r}_{n+1}),$$

which is separable into three one-dimensional equations, each identical to (S-7.37).

**b)** First, we note that, without loss of generality, we can assume the wavevector  $k$  appearing in Equation (7.3) to be positive ( $k > 0$ ), so that (7.3) represents a wave traveling from left to right. Changing the sign of  $k$  simply gives a wave of the same frequency propagating in the opposite direction, whose dispersion relation is the same as for the forward-propagating wave, because of the inversion symmetry of the problem.

Inserting (7.3) in (S-7.37), and dividing both sides by  $Ce^{-i\omega t}$  we obtain

$$-\omega^2 e^{ikna} = \omega_0^2 [e^{ik(n+1)a} - 2e^{ikna} + e^{ik(n-1)a}], \quad (\text{S-7.40})$$

where, again, we have substituted  $\omega_0^2 = 1/LC$ . Dividing both sides by  $e^{ikna}$  we obtain

$$\omega^2 = \omega_0^2 (2 - e^{ika} - e^{-ika}) = 2\omega_0^2 (1 - \cos ka) = 4\omega_0^2 \sin^2(ka/2), \quad (\text{S-7.41})$$

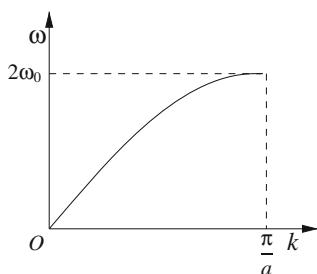


Fig. S-7.4

or, performing the square root,

$$\omega = 2\omega_0 \left| \sin\left(\frac{ka}{2}\right) \right|. \quad (\text{S-7.42})$$

The dispersion relation (S-7.42) is shown in Fig. S-7.4 for  $0 < k \leq \pi/a$ , this range being sufficient to describe all waves propagating in the system. In fact, although (S-7.42) seems to imply that  $\omega(k)$  is a periodic function of  $k$ , with period  $2\pi/a$ , the wavevectors  $k$  and  $k' = k + 2\pi s/a$ , with  $s$  any integer, actually represent the same wave, since

$$e^{ik'na} = e^{i(k+2\pi s/a)na} = e^{ikna} e^{2\pi i s n} = e^{ikna}, \quad (\text{S-7.43})$$

$sn$  being an integer. This is why it is sufficient to consider the range  $0 < k \leq \pi/a$ .

The existence of a maximum wave vector and of a cut-off frequency is related to the discrete periodic nature of the network, which imposes a minimum sampling rate  $a$ . The value  $k_{\max} = \pi/a$  corresponds to  $\lambda_{\min} = 2\pi/k_{\max} = 2a$ , and waves with a smaller wavelength cannot exist. In these waves, the current intensity value is repeated every two meshes of the network, as shown in Fig. S-7.5. A wave with a smaller period cannot exist because of the geometry of the network. One can also note that the direction of wave propagation cannot be determined by observing the wave profile a two instants  $t_1$  and  $t_1 + \pi/\omega$  (half a period later, upper and lower parts of Fig. S-7.5). This is consistent with the group velocity  $v_g(k_{\max}) = (\partial_k \omega)(k_{\max}) = 0$ . The maximum wavevector corresponds to a high cut-off frequency  $\omega_{\max} = 2\omega_0$ . Since higher frequencies cannot be transmitted, the  $LC$  network is a low-pass filter.

- c) The general monochromatic solution of frequency  $\omega$  is a standing wave, i.e., the sum of two waves, one propagating from left to right and the other form right to left

$$I_n(t) = A e^{ikna-i\omega t} + B e^{-ikna-i\omega t}, \quad (\text{S-7.44})$$

where  $\omega$  and  $k$  are related by the dispersion relation (S-7.42). Because of our boundary conditions we must have

$$x_0(t) = 0 \Rightarrow A + B = 0; \quad x_N(t) = 0 \Rightarrow A e^{ikNa} + B e^{-ikNa} = 0. \quad (\text{S-7.45})$$

This gives the condition  $e^{ikNa} - e^{-ikNa} = 2i \sin(kNa) = 0$ , i.e.,  $k = \pi l/Na$  with  $l = 1, 2, 3, \dots, N-1, N$ . We have  $N$  allowed wavevectors  $k_l$  and frequencies  $\omega_l = \omega(k_l)$ . Note that  $k_{\min} = \pi/Na$  corresponds to  $\lambda_{\max} = 2\pi/k_{\min} = 2Na$ , this is a standing wave of wavelength twice the length of the system.

- d) We obtain the limit to a continuous by letting  $a \rightarrow 0$  and  $n \rightarrow \infty$  with  $na = x$  = constant so that

$$\begin{aligned} \lim_{a \rightarrow 0} \frac{I_{n+1}(t) - 2I_n(t) + I_{n-1}(t)}{a^2} &= \lim_{a \rightarrow 0} \frac{I(x+a, t) - 2I(x, t) + I(x-a, t)}{a^2} \\ &= \partial_x^2 I(x, t). \end{aligned} \quad (\text{S-7.46})$$

At this limit we can define a *capacity per unit length*  $C_\ell$ , and an *inductance per unit length*  $L_\ell$ , of the circuit, such that the capacitance and inductance of a circuit segment of length  $\Delta x$  are, respectively,

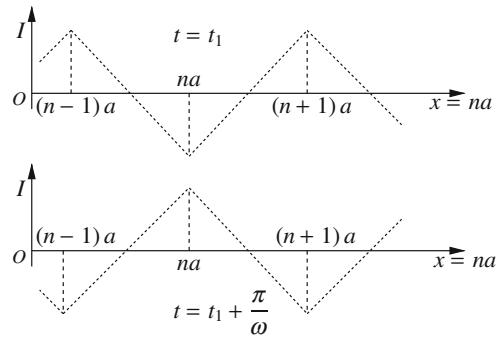


Fig. S-7.5

$$C = C_\ell \Delta x \quad \text{and} \quad L = L_\ell \Delta x. \quad (\text{S-7.47})$$

If we further introduce the quantity

$$v = \sqrt{\frac{1}{L_\ell C_\ell}}, \quad (\text{S-7.48})$$

which has the dimensions of a velocity, (S-7.37) is written for the continuous system

$$\begin{aligned} \partial_t^2 I(x, t) &= \lim_{a \rightarrow 0} \frac{v^2}{a^2} [I(x + a, t) - 2I(x, t) + I(x - a, t)] \\ &= v^2 \partial_x^2 I(x, t). \end{aligned} \quad (\text{S-7.49})$$

This is the equation for a wave propagating with velocity  $v$ , independent of the wave frequency  $\omega$ . At the limit of a continuous system there is no dispersion. This is the case of ideal transmission lines, like parallel wires and coaxial cables with no resistance. See Prob. 7.6 for the case of a realistic transmission line with resistive losses where, however, dispersion can be eliminated.

## S-7.5 The CL Ladder Network

a) We have the same electric potential on the lower horizontal branch of each mesh, and we assume it to be zero. The voltage drop across the  $n$ th capacitor is

$$V_{n-1} - V_n = \frac{Q_n}{C}. \quad (\text{S-7.50})$$

The current in the  $n$ th inductor is  $I_n - I_{n+1}$ , corresponding to a voltage drop across the inductor  $L(dI_n/dt - dI_{n+1}/dt)$ . Thus we have

$$V_{n-1} = L \left( \frac{dI_{n-1}}{dt} - \frac{dI_n}{dt} \right), \quad V_n = L \left( \frac{dI_n}{dt} - \frac{dI_{n+1}}{dt} \right), \quad (\text{S-7.51})$$

which, inserted into (S-7.50), give

$$L \left( \frac{dI_{n-1}}{dt} - 2 \frac{dI_n}{dt} + \frac{dI_{n+1}}{dt} \right) = \frac{Q_n}{C}. \quad (\text{S-7.52})$$

Differentiating (S-7.52) with respect to time, and using  $dQ_n/dt = I_n$ , we obtain

$$L \left( \frac{d^2 I_{n-1}}{dt^2} - 2 \frac{d^2 I_n}{dt^2} + \frac{d^2 I_{n+1}}{dt^2} \right) = \frac{I_n}{C}, \quad (\text{S-7.53})$$

which is (7.4).

**b)** By substituting  $I_n = Ae^{ikna-i\omega t}$  and  $I_{n\pm 1} = Ae^{ik(n\pm 1)a-i\omega t}$  into (S-7.53), defining  $\omega_0^2 = (LC)^{-1}$ , and dividing both sides by  $LA e^{ikna-i\omega t}$ , we obtain

$$-\omega^2 \left( e^{ika} - 2 + e^{-ika} \right) = \omega_0^2 . \quad (\text{S-7.54})$$

The left-hand side can be rewritten

$$\begin{aligned} -\omega^2 \left( e^{ika} - 2 + e^{-ika} \right) &= -\omega^2 [2 \cos(ka) - 2] = -2\omega^2 [\cos(ka) - 1] \\ &= -2\omega^2 \left[ \cos(ka) - \cos^2\left(\frac{ka}{2}\right) - \sin^2\left(\frac{ka}{2}\right) \right] \\ &= -2\omega^2 \left[ \cos^2\left(\frac{ka}{2}\right) - \sin^2\left(\frac{ka}{2}\right) - \cos^2\left(\frac{ka}{2}\right) - \sin^2\left(\frac{ka}{2}\right) \right] \\ &= 4\omega^2 \sin^2\left(\frac{ka}{2}\right) . \end{aligned} \quad (\text{S-7.55})$$

Substituting into (S-7.54) we have

$$\omega^2 = \frac{\omega_0^2}{4 \sin^2(ka/2)} , \quad (\text{S-7.56})$$

or

$$\omega = \frac{\omega_0}{2|\sin(ka/2)|} . \quad (\text{S-7.57})$$

Fig. S-7.6 shows the plot of the dispersion relation. Compare this behavior with the dispersion relation shown in Fig. S-7.4 for an *LC* network, where capacitors and inductors are swapped

with respect to the present case (Problem 7.4). In the *LC* network  $2\omega_0$  is an upper cut-off frequency. Here, in the *CL* network, we have a lower cut-off frequency  $\omega_0/2$ , and the *CL* ladder network acts as a low-pass filter.

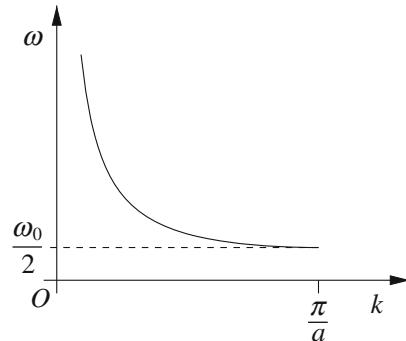


Fig. S-7.6

## S-7.6 A non-dispersive transmission line

**a)** The voltage drop from  $x$  to  $x + dx$  is

$$V(x, t) - V(x + dx, t) = \partial_t I(x, t)L + I(x, t)R , \quad (\text{S-7.58})$$

which yields, after replacing  $R$  by  $R_\ell dx$  and  $L$  by  $L_\ell dx$ ,

$$\partial_x V = -L_\ell \partial_t I - R_\ell I . \quad (\text{S-7.59})$$

The charge associated to the capacitance per unit length is  $Q = Q(x, t) = CV(x, t)$ , and charge conservation yields

$$\partial_t Q(x, t) = I(x - dx, t) - I(x, t) - I_L(x, t), \quad (\text{S-7.60})$$

with the leakage current given by

$$I_L = I_L(x, t) = V(x, t)/R_p = V(x, t) G_\ell dx. \quad (\text{S-7.61})$$

We thus obtain, by eliminating  $Q$  and replacing  $C$  by  $C_\ell dx$ ,

$$C_\ell \partial_t V = -\partial_x I - G_\ell V. \quad (\text{S-7.62})$$

Now we eliminate  $V$  by calculating

$$\begin{aligned} \partial_x^2 I &= -C_\ell \partial_t \partial_x V - G_\ell \partial_x V \\ &= +L_\ell C_\ell \partial_t^2 I + C_\ell R_\ell \partial_t I + G_\ell L_\ell \partial_t I + G_\ell R_\ell I, \end{aligned} \quad (\text{S-7.63})$$

which yields Eq. (7.6).

**b)** By substituting (7.7) in (7.6) we obtain

$$-k^2 + \frac{\omega^2}{v_0^2} = -i\omega(R_\ell C_\ell + L_\ell G_\ell) + R_\ell G_\ell, \quad (\text{S-7.64})$$

where  $v_0^2 = (L_\ell C_\ell)^{-1}$ . Thus, the wavevector  $k$  is a complex number. Writing  $k = k_r + ik_i$  we obtain

$$k_r^2 - k_i^2 = \frac{\omega^2}{v_0^2} - R_\ell G_\ell, \quad (\text{S-7.65})$$

$$2k_r k_i = \omega(R_\ell C_\ell + L_\ell G_\ell). \quad (\text{S-7.66})$$

The wave is thus evanescent,

$$I(x, t) = I_0 e^{-k_i x} e^{ik_r x - i\omega t}, \quad (\text{S-7.67})$$

where the acceptable values for  $k_i$  are positive. Since in general  $k_r = k_r(\omega)$  if  $R_\ell \neq 0$  or  $G_\ell \neq 0$ , resistive effects make the line to be dispersive, so that a wavepacket is distorted along its propagation.

**c)** If we assume that  $k_i^2 = R_\ell G_\ell$  in (S-7.65), then  $k_r = \omega/v_0$ , which means that the propagation is non-dispersive: the phase velocity  $v_p = \omega/k_r = v_0$  is independent of frequency. In addition, since  $k_i$  does not depend on  $\omega$ , the evanescence length  $k_i^{-1}$  is also frequency-independent. By substituting  $k_i = \sqrt{R_\ell G_\ell}$  and  $k_r = \omega(L_\ell C_\ell)^{1/2}$  in (S-7.66) we obtain the condition

$$2\sqrt{R_\ell G_\ell} \sqrt{L_\ell C_\ell} = R_\ell C_\ell + L_\ell G_\ell. \quad (\text{S-7.68})$$

Squaring both sides and rearranging the terms yields  $(R_\ell C_\ell - L_\ell G_\ell)^2 = 0$ , which leads to the simple, equivalent condition

$$R_\ell C_\ell = L_\ell G_\ell . \quad (\text{S-7.69})$$

This is the condition for a non-dispersive or *distortionless* transmission line due to O. Heaviside.

If the input current at one side of the line, say at  $x = 0$ , is

$$I(0, t) = I_0(t) = \int \tilde{I}_0(\omega) e^{-i\omega t} d\omega , \quad (\text{S-7.70})$$

where  $\tilde{I}_0(\omega)$  is the Fourier transform, then the current along the line will be given by

$$\begin{aligned} I(x, t) &= \int \tilde{I}_0(\omega) e^{ik_r x - i\omega t} e^{-k_i x} d\omega = e^{-k_i x} \int \tilde{I}_0(\omega) e^{-i\omega(t-x/v_0)} d\omega \\ &= e^{-k_i x} I_0(t - x/v_0) , \end{aligned} \quad (\text{S-7.71})$$

since  $k_i$  is independent on  $\omega$ . This is equivalent to state that the general solution of (7.6) with the condition (S-7.69) has the form (7.8) with  $v = v_0$  and  $\kappa = k_i$ .

The same conclusion may be obtained by direct substitution of (7.8) into Eq. (7.6). The partial derivatives are given by

$$\begin{aligned} \partial_t I &= -v e^{-\kappa x} f'(x - vt) , \\ \partial_t^2 I &= v^2 e^{-\kappa x} f''(x - vt) , \\ \partial_x I &= -\kappa e^{-\kappa x} f(x - vt) + e^{-\kappa x} f'(x - vt) , \\ \partial_x^2 I &= \kappa^2 e^{-\kappa x} f(x - vt) - 2\kappa e^{-\kappa x} f'(x - vt) + e^{-\kappa x} f''(x - vt) , \end{aligned} \quad (\text{S-7.72})$$

where  $f'(x) = df(x)/dx$  and  $f''(x) = d^2 f(x)/dx^2$ . Thus Eq. (7.6) becomes

$$(\kappa^2 - R_\ell G_\ell) f + (v(R_\ell C_\ell + L_\ell G_\ell) - 2\kappa) f' + (1 - L_\ell C_\ell v^2) f'' = 0 . \quad (\text{S-7.73})$$

For this equation to be true for arbitrary  $f$ , the coefficients of  $f$ ,  $f'$  and  $f''$  must be all zero. Thus

$$\kappa^2 = R_\ell G_\ell , \quad 2\kappa = v(R_\ell C_\ell + L_\ell G_\ell) , \quad v^2 = (L_\ell C_\ell)^{-1} , \quad (\text{S-7.74})$$

which bring again the conditions on the line parameters found above.

## S-7.7 An “Alternate” LC Ladder Network

a) Let  $Q_n$  be the charge of the capacitor at the right of mesh  $n$ . Applying Kirchhoff's mesh rule to the even and odd meshes of the ladder network we have, respectively,

$$-\frac{Q_{2n-1}}{C} + L_2 \frac{dI_{2n}}{dt} + \frac{Q_{2n}}{C} = 0 , \quad -\frac{Q_{2n}}{C} + L_1 \frac{dI_{2n+1}}{dt} + \frac{Q_{2n+1}}{C} = 0 , \quad (\text{S-7.75})$$

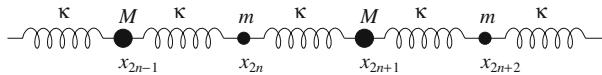
while Kirchhoff's junction rule gives

$$\frac{dQ_{2n-1}}{dt} = I_{2n-1} - I_{2n}, \quad \frac{dQ_{2n}}{dt} = I_{2n} - I_{2n+1}. \quad (\text{S-7.76})$$

Differentiating (S-7.75) with respect to time, and inserting (S-7.76), we obtain

$$\begin{aligned} L_2 \frac{d^2 I_{2n}}{dt^2} &= \frac{1}{C} \left( \frac{dQ_{2n-1}}{dt} - \frac{d^2 Q_{2n}}{dt^2} \right) = \frac{1}{C} (I_{2n-1} - 2I_{2n} + I_{2n+1}) \\ L_1 \frac{d^2 I_{2n+1}}{dt^2} &= \frac{1}{C} \left( \frac{dQ_{2n}}{dt} - \frac{d^2 Q_{2n+1}}{dt^2} \right) = \frac{1}{C} (I_{2n} - 2I_{2n+1} + I_{2n+2}), \end{aligned} \quad (\text{S-7.77})$$

identical to (7.9).



**Fig. S-7.7**

A mechanical equivalent to our network is the one-dimensional sequence of masses and springs shown in Fig. S-7.7, where the masses have, alternately, the values  $M$  and  $m$ , while all springs are identical, with Hooke's constant  $\kappa$ . If we denote by  $x_{2n+1}$  the positions of the odd masses  $M$ , and by  $x_{2n}$  the positions of even masses  $m$ , the equations of motion for the system are

$$\begin{aligned} m \frac{d^2 x_{2n}}{dt^2} &= -\kappa(x_{2n} - x_{2n+1}) + \kappa(x_{2n-1} - x_{2n}) = \kappa(x_{2n-1} - 2x_{2n} + x_{2n+1}) \\ M \frac{d^2 x_{2n+1}}{dt^2} &= -\kappa(x_{2n+1} - x_{2n+2}) + \kappa(x_{2n} - x_{2n+1}) = \kappa(x_{2n} - 2x_{2n+1} + x_{2n+2}), \end{aligned} \quad (\text{S-7.78})$$

which, after the substitutions  $m \rightarrow L_2$ ,  $M \rightarrow L_1$ ,  $x \rightarrow I$ , and  $\kappa \rightarrow 1/C$ , are identical to (S-7.77).

**b)** Substituting (7.10) into (S-7.77), and dividing both sides by  $e^{-i\omega t}$ , we obtain

$$\begin{aligned} -\omega^2 L_2 I_e e^{i(2n)ka} &= \frac{1}{C} (I_o e^{i(2n-1)ka} - 2I_e e^{i(2n)ka} + I_o e^{i(2n+1)ka}) \\ -\omega^2 L_2 I_o e^{i(2n+1)ka} &= \frac{1}{C} (I_e e^{i(2n)ka} - 2I_o e^{i(2n+1)ka} + I_e e^{i(2n+2)ka}). \end{aligned} \quad (\text{S-7.79})$$

Now we define the two angular frequencies  $\omega_o = 1/\sqrt{L_1 C}$  and  $\omega_e = 1/\sqrt{L_2 C}$ , and divide (S-7.79) by  $e^{i(2n)ka}$ , obtaining

$$\begin{aligned} (2\omega_e^2 - \omega^2) I_e - 2\omega_e^2 \cos(ka) I_o &= 0 \\ 2\omega_o^2 \cos(ka) I_e - (2\omega_o^2 - \omega^2) I_o &= 0. \end{aligned} \quad (\text{S-7.80})$$

This system of linear equations has non-trivial solutions if and only if its determinant is zero, i.e., if

$$(2\omega_e^2 - \omega^2)(2\omega_0^2 - \omega^2) - 4\omega_0^2\omega_e^2 \cos^2(ka) = 0, \quad (\text{S-7.81})$$

the solution of this quadratic equation in  $\omega^2$  is

$$\omega^2 = \omega_e^2 + \omega_0^2 \pm \sqrt{(\omega_e^2 + \omega_0^2)^2 - 4\omega_0^2\omega_e^2 \sin^2(ka)}. \quad (\text{S-7.82})$$

Both solutions are physically acceptable: the system allows for *two* types of propagating waves, described by two different dispersion relations.

At the limit  $L_2 \ll L_1$  (or  $m \ll M$ , for the equivalent mechanical system) we have  $\omega_0^2 \ll \omega_e^2$ , and (S-7.82) can be approximated as

$$\omega^2 \simeq \omega_e^2 + \omega_0^2 \pm \omega_e^2 \sqrt{1 + 2\frac{\omega_0^2}{\omega_e^2} - 4\frac{\omega_0^2}{\omega_e^2} \sin^2(ka)} \quad (\text{S-7.83})$$

where we have disregarded the fourth-order term  $\omega_0^4/\omega_e^4$  inside the square root. If we further use the approximation  $\sqrt{1+x} \simeq 1 + x/2$ , valid for  $x \ll 1$ , (S-7.83) becomes

$$\omega^2 \simeq \omega_e^2 + \omega_0^2 \pm \omega_e^2 \left\{ 1 + \frac{\omega_0^2}{\omega_e^2} [1 - 2 \sin^2(ka)] \right\}, \quad (\text{S-7.84})$$

corresponding to the two dispersion relations

$$\omega \simeq \begin{cases} \sqrt{2(\omega_e^2 + \omega_0^2) - 2\omega_0^2 \sin^2(ka)} \\ \sqrt{2}\omega_0 \sin(ka) \end{cases} \quad (\text{S-7.85})$$

The lower branch can propagate for frequencies between 0 and  $\omega_1 = \sqrt{2}\omega_0$ , while the upper branch lies between  $\omega_2 = \omega_e \sqrt{2(1 - \omega_0^2/\omega_e^2)}$  and  $\omega_3 = \sqrt{2}\omega_e$ . Thus, there is a gap of “forbidden” frequencies between  $\omega_1$  and  $\omega_2$ . Figure S-7.8 shows the exact solution (continuous lines), and the approximate solution (dashed lines), still in good agreement, for  $\omega_0^2/\omega_e^2 = 0.25$ .

Of course, the two branches are present also in the case of the alternating mechanical oscillators, and provide a model for an effect known in solid state physics. The vibrations

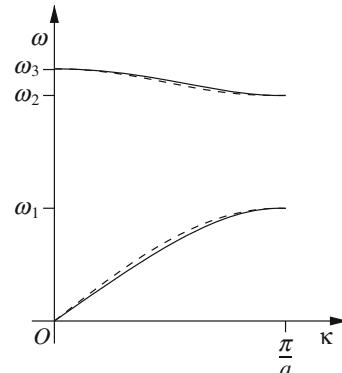


Fig. S-7.8

of a lattice formed by identical ions have a single branch (Problem 7.4), with a dispersion relation similar to the lower branch, which is named “acoustic branch”. In an ionic crystal, formed by two ion species alternating on the sites of the lattice, we observe also the upper branch, named “optical branch”.

## S-7.8 Resonances in an LC Ladder Network

a) According to Problem 7.4, the current flowing in the  $n$ th mesh is

$$\frac{d^2 I_n}{dt^2} = \omega_0^2 (I_{n+1} - 2I_n + I_{n-1}). \quad (\text{S-7.86})$$

We are looking for a propagating wave solution, and define the phase

$$\phi \equiv ka, \quad (\text{S-7.87})$$

where  $a$  is the length of a single mesh, to be substituted into (S-7.44), writing  $I_n(t)$  as

$$I_n(t) = A e^{in\phi-i\omega t}. \quad (\text{S-7.88})$$

Substituting (S-7.88) into (S-7.86), and dividing by  $e^{-i\omega t}$ , we get

$$-\omega^2 e^{in\phi} = \omega_0^2 [e^{i(n+1)\phi} - 2e^{in\phi} + e^{i(n-1)\phi}], \quad (\text{S-7.89})$$

from which we obtain the dispersion relation

$$\omega^2 = \omega_0^2 (2 - e^{i\phi} - e^{-i\phi}) = 2\omega_0^2 (1 - \cos \phi) = 4\omega_0^2 \sin^2(\phi/2), \quad (\text{S-7.90})$$

whose inverse is

$$\sin\left(\frac{\phi}{2}\right) = \frac{\omega}{2\omega_0}, \quad \text{or} \quad \phi = 2 \arcsin\left(\frac{\omega}{2\omega_0}\right), \quad (\text{S-7.91})$$

that shows that  $\phi$  is a real number if  $\omega < 2\omega_0$ .

Due to the presence of the current source, (S-7.88) holds if the current in the 0th mesh is

$$I_0(t) = I_s e^{-i\omega t}, \quad (\text{S-7.92})$$

thus we must have  $A = I_s$ , and the final expression for  $I_n(t)$  is

$$I_n(t) = I_s e^{in\phi-i\omega t}, \quad (\text{S-7.93})$$

where  $\phi$  is given by (S-7.91)

**b)** If  $\omega > 2\omega_0$  the current wave cannot propagate in the ladder. We look for a solution of the form suggested by the hint. Substituting (7.12) into (S-7.86) we obtain

$$-\omega^2 \alpha^{-n} = \omega_0^2 [\alpha^{-(n+1)} - 2\alpha^{-n} + \alpha^{-(n-1)}], \quad (\text{S-7.94})$$

which, multiplied by  $\alpha^n/\omega_0^2$ , turns into

$$\alpha^2 + \left( \frac{\omega^2}{\omega_0^2} - 2 \right) \alpha + 1 = 0. \quad (\text{S-7.95})$$

The solutions are

$$\alpha = 1 - \frac{\omega^2}{2\omega_0^2} \pm \sqrt{\left( 1 - \frac{\omega^2}{2\omega_0^2} \right)^2 - 1}. \quad (\text{S-7.96})$$

We must have  $|\alpha| < 1$  for an infinite ladder, otherwise the current would grow indefinitely in successive meshes. Thus, we keep the solution with the plus sign, because  $\omega > 2\omega_0$  implies that all solutions of (S-7.96) are negative, obtaining

$$I_n(t) = I_s (-1)^n |\alpha|^n e^{-i\omega t}, \quad |\alpha| = \frac{\omega^2}{2\omega_0^2} - 1 - \sqrt{\left( \frac{\omega^2}{2\omega_0^2} - 1 \right)^2 - 1}, \quad (\text{S-7.97})$$

that we can rewrite as

$$I_n(t) = I_s e^{-\gamma n - i\omega t}, \quad \text{where } \gamma = i\pi + \ln |\alpha|. \quad (\text{S-7.98})$$

**c)** We consider the case of the propagating wave ( $\omega < 2\omega_0$ ) first. If the ladder comprises  $N$  meshes numbered as in Fig. 7.8, the boundary condition at the right end is  $I_N(t) \equiv 0$  (mesh number  $N$  does not exist!). The most general solution is the sum of two counterpropagating waves

$$I_n(t) = A e^{in\phi - i\omega t} + B e^{-in\phi - i\omega t}. \quad (\text{S-7.99})$$

Imposing the conditions  $I_0 = I_s$  and  $I_N = 0$ , we obtain

$$A + B = I_s, \quad A e^{iN\phi} + B e^{-iN\phi} = 0, \quad (\text{S-7.100})$$

with solutions

$$A = +\frac{i}{2} I_s \frac{e^{-iN\phi}}{\sin(N\phi)}, \quad B = -\frac{i}{2} I_s \frac{e^{+iN\phi}}{\sin(N\phi)}, \quad (\text{S-7.101})$$

where  $\phi = \phi(\omega)$  depends on  $\omega$  according to (S-7.91). We observe resonances when  $\sin(N\phi) = 0$ , i.e., for  $\phi = m\pi/N$  with  $m$  an integer. Remembering (S-7.87)

$$N = m \frac{\pi}{\phi} = m \frac{\pi}{ka} = m \frac{\pi}{a} \frac{\lambda}{2\pi} = m \frac{1}{a} \frac{\lambda}{2}, \quad (\text{S-7.102})$$

and, multiplying both sides by  $a$

$$L = Na = m \frac{\lambda}{2}, \quad (\text{S-7.103})$$

where  $L$  is the total length of the ladder network. This corresponds to the case when the frequency of the current source equals the frequency of one of the standing waves allowed in the network, i.e., when the length of the ladder network is an integer multiple of a half wavelength.

If  $\omega > 2\omega_0$ , the general solution is

$$I_n(t) = A\alpha_+^n e^{-i\omega t} + B\alpha_-^n e^{-i\omega t}, \quad (\text{S-7.104})$$

where  $\alpha_{\pm} = \alpha_{\pm}(\omega)$  are the two solutions of (S-7.96). Here also the case  $|\alpha| > 1$  is allowed, because  $|\alpha|^n$  cannot diverge if  $n$  is limited. The boundary conditions are

$$A + B = I_s, \quad A\alpha_+^N + B\alpha_-^N = 0, \quad (\text{S-7.105})$$

with solutions

$$A = +I_s \frac{\alpha_-^N}{\alpha_-^N - \alpha_+^N}, \quad B = -I_s \frac{\alpha_+^N}{\alpha_-^N - \alpha_+^N}. \quad (\text{S-7.106})$$

The  $A$  and  $B$  coefficients diverge if  $\alpha_- = \alpha_+ = 1$ , i.e., if  $\omega = 2\omega_0$ . Thus, for  $\omega > 2\omega_0$  there are no resonances, but the response of the system diverges as the frequency approaches the cut-off value, i.e. as  $\omega \rightarrow 2\omega_0$ .

## S-7.9 Cyclotron Resonances (1)

a) The rotating electric field can be written as

$$\mathbf{E} = \mathbf{E}(t) = E_0 (\hat{x} \cos \omega t \pm \hat{y} \sin \omega t), \quad (\text{S-7.107})$$

where the positive (negative) sign indicates counterclockwise (clockwise) rotation. From the equation of motion

$$m \frac{d\mathbf{v}}{dt} = q \left( \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right), \quad (\text{S-7.108})$$

we see that  $dv_z/dt = 0$ , thus, if we assume that  $v_z(0) = 0$ , the motion occurs in the  $(x, y)$  plane. The equations of motion along the  $x$  and  $y$  axes are

$$\begin{aligned}\frac{dv_x}{dt} &= +qv_y \frac{B_0}{mc} + \frac{qE_0}{m} \cos \omega t \\ \frac{dv_y}{dt} &= -qv_x \frac{B_0}{mc} \pm \frac{qE_0}{m} \sin \omega t.\end{aligned}\quad (\text{S-7.109})$$

In principle, we can differentiate both equations with respect to time, and then substitute the expressions for  $dv_{x,y}/dt$ , thus obtaining two uncoupled second-order differential equations for a driven harmonic oscillator.

But we prefer a more “elegant” approach, introducing the complex variable  $\zeta = v_x + iv_y$ . The velocity is thus represented by a complex vector in the  $(\text{Re}\zeta, \text{Im}\zeta)$  plane. Adding the second of (S-7.109), multiplied by  $i$ , to the first, we obtain

$$\frac{d\zeta}{dt} = -i\omega_c \zeta + \frac{qE_0}{m} e^{\pm i\omega t}, \quad (\text{S-7.110})$$

where  $\omega_c = qB_0/mc$  is the cyclotron (or Larmor) frequency. The solution of the associated homogeneous equation is

$$\zeta(t) = A e^{-i\omega_c t}, \quad (\text{S-7.111})$$

where  $A$  is an arbitrary complex constant. Equation (S-7.111) describes the motion in the absence of the electric field, when the velocity rotates clockwise with frequency  $\omega_c$  in the  $\zeta$  plane. We then search for a particular integral of the inhomogeneous equation in the form

$$\zeta = \zeta_0 e^{\pm i\omega t},$$

and find, by direct substitution,

$$\zeta_0 = -i \frac{qE_0}{m(\omega_c \pm \omega)}. \quad (\text{S-7.112})$$

Thus the general solution of (S-7.110) is

$$\zeta(t) = A e^{-i\omega_c t} - i \frac{qE_0}{m(\omega_c \pm \omega)} e^{\pm i\omega t}. \quad (\text{S-7.113})$$

Assuming  $\omega_c > 0$ , we observe a resonance at  $\omega = \omega_c$  only if the field rotates clockwise. In this case the electric field accelerates the particle along the direction of its “natural” motion.

**b)** At resonance ( $\omega = \omega_c$ ), we search for a non-periodic solution of the form

$$\zeta(t) = \zeta_R(t) e^{-i\omega t}, \quad (\text{S-7.114})$$

which, substituted into (S-7.110), gives

$$\left( \frac{d\zeta_R}{dt} \right) e^{-i\omega t} - i\omega \zeta_R e^{-i\omega t} = -i\omega_c \zeta_R e^{-i\omega t} + \frac{qE_0}{m} e^{-i\omega t}, \quad (\text{S-7.115})$$

and, since  $\omega_c = \omega$ ,

$$\left( \frac{d\zeta_R}{dt} \right) = \frac{qE_0}{m}. \quad (\text{S-7.116})$$

The solution of (S-7.116) is

$$\zeta_R(t) = \zeta(0) + \frac{qE_0}{m} t, \quad (\text{S-7.117})$$

which gives

$$\zeta(t) = \left[ \zeta(0) + \frac{qE_0}{m} t \right] e^{-i\omega t}. \quad (\text{S-7.118})$$

The trajectory is a spiral, with the radial velocity increasing linearly with time.

c) Introducing a viscous force  $\mathbf{f}_v = -m\gamma\mathbf{v}$  we obtain the following equation for  $\zeta$

$$\frac{d\zeta}{dt} = -i\omega_c \zeta - \gamma\zeta + \frac{qE_0}{m} e^{-i\omega t}. \quad (\text{S-7.119})$$

The solution has the form of (S-7.113) with  $\omega_c$  replaced by  $(\omega_c - i\gamma)$ ,

$$\zeta = -i \frac{qE_0}{m(\omega_c - \omega - i\gamma)} e^{-i\omega t} + A e^{-i\omega_c t - \gamma t}, \quad (\text{S-7.120})$$

where the second term undergoes an exponential decay, and any memory of the initial conditions is lost after a transient phase, while the periodic part of the solution does not diverge at resonance, due to the presence of  $i\gamma$  in the denominator. Thus, the steady-state solution at resonance is

$$\zeta_R = \frac{qE_0}{m\gamma} e^{-i\omega t}. \quad (\text{S-7.121})$$

The average dissipated power is the time average of the instantaneous dissipated power over a period

$$P = \langle \mathbf{f} \cdot \mathbf{v} \rangle = \langle q\mathbf{E} \cdot \mathbf{v} \rangle. \quad (\text{S-7.122})$$

The components of the particle velocity in the steady state are

$$\begin{aligned} v_x &= \operatorname{Re}(\zeta) = \frac{qE_0\gamma}{m[(\omega_c - \omega)^2 + \gamma^2]} \cos \omega t - \frac{qE_0(\omega_c - \omega)}{m[(\omega_c - \omega)^2 + \gamma^2]} \sin \omega t, \\ v_y &= \operatorname{Im}(\zeta) = -\frac{qE_0\gamma}{m[(\omega_c - \omega)^2 + \gamma^2]} \sin \omega t - \frac{qE_0(\omega_c - \omega)}{m[(\omega_c - \omega)^2 + \gamma^2]} \cos \omega t. \end{aligned} \quad (\text{S-7.123})$$

Thus, inserting (S-7.123) and the relations

$$\begin{aligned} E_x &= E_0 \cos \omega t, & E_y &= -E_0 \sin \omega t, & \langle \cos^2 \omega t \rangle &= \langle \sin^2 \omega t \rangle = \frac{1}{2}, \\ \langle \cos \omega t \sin \omega t \rangle &= 0, \end{aligned} \quad (\text{S-7.124})$$

into (S-7.122), we obtain for the average dissipated power

$$P = \frac{q^2 E_0^2 \gamma}{m [(\omega_c - \omega)^2 + \gamma^2]}. \quad (\text{S-7.125})$$

At resonance we have

$$P = \frac{q^2 E_0^2}{m \gamma}. \quad (\text{S-7.126})$$

## S-7.10 Cyclotron Resonances (2)

a) The equations of motion are

$$\frac{dv_x}{dt} = +\omega_c v_y + \frac{qE_0}{m} \cos \omega t, \quad \frac{dv_y}{dt} = -\omega_c v_x, \quad (\text{S-7.127})$$

where  $\omega_c = qB_0/m$ . By differentiating (S-7.127) with respect to time, and substituting the values for  $v_x$  and  $v_y$  from (S-7.127) itself, we obtain the two equations

$$\begin{aligned} \frac{d^2v_x}{dt^2} &= +\omega_c \frac{dv_y}{dt} - \frac{qE_0 \omega}{m} \sin \omega t = -\omega_c^2 v_x - \frac{qE_0 \omega}{m} \sin \omega t, \\ \frac{d^2v_y}{dt^2} &= -\omega_c \frac{dv_x}{dt} = -\omega_c^2 v_y - \frac{qE_0 \omega_c}{m} \cos \omega t, \end{aligned} \quad (\text{S-7.128})$$

each of which describes the velocity of a driven harmonic oscillator. The steady state solutions are

$$v_x = \frac{qE_0 \omega}{m(\omega^2 - \omega_c^2)} \sin \omega t, \quad v_y = \frac{qE_0 \omega_c}{m(\omega^2 - \omega_c^2)} \cos \omega t. \quad (\text{S-7.129})$$

We observe a resonance if  $\omega = |\omega_c|$ , independently on the signs of  $q$  and  $B_0$ . With respect to Problem 7.9, where a rotating electric field was assumed, here a resonance is always found because the linearly oscillating electric field can be decomposed into two counter-rotating fields of the same amplitude, of which one will excite the resonance.

b) In the presence of a frictional force  $\mathbf{f} = -m\gamma\mathbf{v}$  the equations of motion become

$$\frac{dv_x}{dt} = +\omega_c v_y - \gamma v_x + \frac{qE_0}{m} \cos \omega t, \quad \frac{dv_y}{dt} = -\omega_c v_x - \gamma v_y, \quad (\text{S-7.130})$$

and cannot be uncoupled by the procedure of point a). Analogously to Problem 7.9, we introduce the complex quantity  $\zeta = v_x + iv_y$ , obtaining the single equation

$$\frac{d\zeta}{dt} = -i\omega_c \zeta - \gamma \zeta + \frac{qE_0}{2m} (e^{i\omega t} + e^{-i\omega t}), \quad (\text{S-7.131})$$

where we have used Euler's formula for the cosine. Differently from Problem 7.9, now we search for a steady-state solution of the form

$$\zeta = Ae^{-i\omega t} + Be^{i\omega t}, \quad (\text{S-7.132})$$

where  $A$  and  $B$  are two complex constants to be determined. By direct substitution into (S-7.131) we have

$$-i\omega A e^{-i\omega t} + i\omega B e^{i\omega t} = -(i\omega_c + \gamma)A e^{-i\omega t} - (i\omega_c + \gamma)B e^{i\omega t} + \frac{qE_0}{2m} (e^{i\omega t} + e^{-i\omega t}),$$

which is separable into two equations relative, respectively, to the terms rotating clockwise and counterclockwise in the complex plane

$$-i\omega A = -(i\omega_c + \gamma)A + \frac{qE_0}{2m}, \quad i\omega B = -(i\omega_c + \gamma)B + \frac{qE_0}{2m}. \quad (\text{S-7.133})$$

The solutions for  $A$  and  $B$  are

$$\begin{aligned} A &= \frac{qE_0}{2m[i(\omega_c - \omega) + \gamma]} = \frac{qE_0\gamma}{2m[(\omega_c - \omega)^2 + \gamma^2]} - i \frac{qE_0(\omega_c - \omega)}{2m[(\omega_c - \omega)^2 + \gamma^2]} \\ B &= \frac{qE_0}{2m[i(\omega_c + \omega) + \gamma]} = \frac{qE_0\gamma}{2m[(\omega_c + \omega)^2 + \gamma^2]} - i \frac{qE_0(\omega_c + \omega)}{2m[(\omega_c + \omega)^2 + \gamma^2]}, \end{aligned} \quad (\text{S-7.134})$$

from which we obtain the stationary-state velocity components of the particle

$$\begin{aligned} v_x &= [\operatorname{Re}(A) + \operatorname{Re}(B)] \cos \omega t + [\operatorname{Im}(A) - \operatorname{Im}(B)] \sin \omega t \\ v_y &= [\operatorname{Im}(A) + \operatorname{Im}(B)] \cos \omega t - [\operatorname{Re}(A) - \operatorname{Re}(B)] \sin \omega t. \end{aligned} \quad (\text{S-7.135})$$

The average absorbed power is

$$\begin{aligned} P &= \langle q\mathbf{v} \cdot \mathbf{E} \rangle = \langle qv_x E_x \rangle = q[\operatorname{Re}(A) + \operatorname{Re}(B)]E_0 \frac{1}{2} \\ &= \frac{q^2 E_0^2 \gamma}{4m[(\omega_c - \omega)^2 + \gamma^2]} + \frac{q^2 E_0^2 \gamma}{4m[(\omega_c + \omega)^2 + \gamma^2]}, \end{aligned} \quad (\text{S-7.136})$$

since  $E_x = E_0 \cos \omega t$ ,  $\langle \cos^2 \omega t \rangle = 1/2$ , and  $\langle \cos \omega t \sin \omega t \rangle = 0$ . Thus, again, we observe a resonance at  $\omega = |\omega_c|$ , independently of the signs of  $q$  and  $B_0$ . Assuming  $\gamma \ll \omega_c$  the power absorbed at resonance is

$$P_{\max} \simeq \frac{q^2 E_0^2}{4m\gamma}. \quad (\text{S-7.137})$$

## S-7.11 A Quasi-Gaussian Wave Packet

We need to evaluate the inverse transform

$$\begin{aligned} f(x) &= A \int_{-\infty}^{+\infty} e^{-L^2(k-k_0)^2} e^{i\phi(k)} e^{ikx} dk \\ &\simeq \int_{-\infty}^{+\infty} \exp \left[ -L^2(k-k_0)^2 + i\phi_0 + i\phi'_0(k-k_0) + \right. \\ &\quad \left. + \frac{i}{2}\phi''_0(k-k_0)^2 + i(k-k_0)x + ik_0x \right] dk, \end{aligned} \quad (\text{S-7.138})$$

where, for brevity, we wrote  $x$  instead of  $(x - vt)$ , and  $\phi_0, \phi'_0, \dots$  instead of  $\phi(k_0), \phi'(k_0), \dots$  By using (7.1) we obtain

$$\begin{aligned} f(x) &\simeq A e^{ik_0x+i\phi_0} \int_{-\infty}^{+\infty} \exp \left[ -L^2(k-k_0)^2 \left( 1 - i\frac{\phi''_0}{2L^2} \right) + i(k-k_0)(x+\phi'_0) \right] dk \\ &= C \exp \left[ -\frac{(x+\phi'_0)^2}{4L^2(1-i\phi''_0/2L^2)} \right], \end{aligned} \quad (\text{S-7.139})$$

where  $C$  is a constant, whose value is not relevant for our purposes. By substituting

$$\frac{1}{1 - i\phi''_0/(2L^2)} = \frac{1 + i\phi''_0/(2L^2)}{1 + \phi''_0^2/(4L^4)} \quad (\text{S-7.140})$$

we obtain the wave packet profile as

$$f(x-vt) = C \exp \left\{ -\frac{(x-vt+\phi'_0)^2 [1 + i\phi''_0/(2L^2)]}{L^2 [1 + \phi''_0^2/(4L^4)]} \right\}. \quad (\text{S-7.141})$$

We thus see that the packet is wider than the purely Gaussian case, since  $L^2[1 + \phi''_0^2/(4L^4)] > L^2$ . In addition, the center of the packet is shifted from  $(x - vt)$  to  $(x - vt + \phi'_0)$ , and there is an aperiodic (anharmonic) modulation due to the factor  $(i\phi''_0/2L^2)$  in the numerator of the exponent.

### S-7.12 A Wave Packet Traveling along a Weakly Dispersive Line

**a)** There is no dispersion if  $b = 0$ . In these conditions the signal propagates at velocity  $v$  keeping its shape:

$$f(x - vt) = Ae^{-i\omega_0(t-x/v)}e^{-(t-x/v)^2/\tau^2}. \quad (\text{S-7.142})$$

**b)** The phase velocity and the group velocity are, by definition,

$$v_\phi = \frac{\omega}{k} = v(1 + bk), \quad v_g = \frac{\partial \omega}{\partial k} = v(1 + 2bk). \quad (\text{S-7.143})$$

We can write  $v_\phi$  and  $v_g$  as functions of  $\omega$  by first inverting (7.17), obtaining for  $k = k(\omega)$

$$k = \sqrt{\frac{1}{(2b)^2} + \frac{\omega}{bv}} - \frac{1}{2b}. \quad (\text{S-7.144})$$

Then we expand the square root to the second order in  $\omega/v$ , obtaining

$$k \approx \frac{\omega}{v} - \frac{\omega^2 b}{v^2}. \quad (\text{S-7.145})$$

The same result can also be obtained by an iterative procedure, by inserting the first order value for  $k$ , i.e.,  $k = \omega/v$ , into the bracket at the right hand side of (7.17). Thus, the phase and group velocities to the first order are, using (S-7.143),

$$v_{\phi 0} \approx v + b\omega_0, \quad v_{g0} \approx v + 2b\omega_0. \quad (\text{S-7.146})$$

**c)** The peak of the signal propagates at the group velocity, thus  $t_x = x/v_{g0}$ . The spectral width of the wave packet may be estimated as  $\Delta\omega \approx 1/\tau$ , which corresponds to a spread in the propagation velocity of its Fourier components

$$\Delta v \approx v \left( \frac{2b}{v} \Delta\omega \right) \approx \frac{2b}{\tau}. \quad (\text{S-7.147})$$

Thus the spread of the wave packet in time and space can be estimated as

$$\Delta t \approx \frac{\partial t_x}{\partial v_g} \Delta v = t_x \frac{\Delta v}{v_g}, \quad \Delta x \approx v_g \Delta t = \frac{2bx}{v_g \tau}. \quad (\text{S-7.148})$$

**d)** We approximate

$$k(\omega) \approx k_0 + k'_0(\omega - \omega_0) + \frac{1}{2}k''_0(\omega - \omega_0)^2, \quad (\text{S-7.149})$$

where

$$k'_0 = \frac{\partial k}{\partial \omega} \Big|_{\omega_0} \simeq \frac{1}{v} - 2 \frac{\omega_0 b}{v^2} \simeq \frac{1}{v_g}, \quad k''_0 = \frac{\partial^2 k}{\partial \omega^2} \Big|_{\omega_0} \simeq -2 \frac{b}{v^2}. \quad (\text{S-7.150})$$

The spectrum of the wave packet (i.e., its Fourier transform) is

$$\tilde{f}(\omega) = \sqrt{\pi} \tau A e^{-[\omega - \omega_0]^2 \tau^2 / 4}. \quad (\text{S-7.151})$$

Since we are only interested in the behavior of the function, we evaluate the following integral forgetting proportionality constants,

$$\begin{aligned} f(x, t) &\sim \int \exp \left[ ik(\omega)x - i\omega t - \frac{(\omega - \omega_0)^2 \tau^2}{4} \right] d\omega \\ &\sim \int \exp \left[ ik_0 x + ik'_0 x(\omega - \omega_0) + i \frac{k''_0 x}{2} (\omega - \omega_0)^2 - \frac{(\omega - \omega_0)^2 \tau^2}{4} \right] d\omega \\ &\sim \exp(ik_0 x - i\omega_0 t) \int \exp \left[ -i(t - k'_0 x) \omega' + \left( -\frac{\tau^2}{4} + i \frac{k''_0 x}{2} \right) \omega'^2 \right] d\omega' \\ &\sim \exp \left[ ik_0 x - i\omega_0 t - \frac{(t - k'_0 x)^2}{\tau^2 - 2ik''_0 x} \right]. \end{aligned} \quad (\text{S-7.152})$$

The factor which describes the envelope of the wave packet (recalling that  $k'_0 = 1/v_g$ ) is

$$\begin{aligned} \exp \left[ -\frac{(t - x/v_g)^2}{\tau^2 - 2ik''_0 x} \right] &= \exp \left[ -(t - x/v_g)^2 \frac{\tau^2 + 2ik''_0 x}{\tau^4 + (2k''_0 x)^2} \right] \\ &= \exp \left\{ -(t - x/v_g)^2 \frac{1 + 2ik''_0 x/\tau^2}{\tau^2 [1 + (2k''_0 x/\tau)^2]} \right\}. \end{aligned} \quad (\text{S-7.153})$$

The temporal width of the wave packet increases during the propagation as

$$\Delta t(x) = \tau \sqrt{1 + \left( \frac{2k''_0 x}{\tau} \right)^2}. \quad (\text{S-7.154})$$

# Chapter S-8

## Solutions for Chapter 8

### S-8.1 Poynting Vector(s) in an Ohmic Wire

For symmetry reasons, the magnetic field is azimuthal and depends only on the radial coordinate  $r$ . Applying Ampère's law to a circular path of radius  $r < a$  around the wire axis yields

$$2\pi rB = \frac{4\pi}{c}(\pi r^2 J), \quad (\text{S-8.1})$$

which leads to

$$\mathbf{B} = \frac{2\pi}{c}r\sigma E \hat{\phi}, \quad (\text{S-8.2})$$

where  $\hat{\phi}$  is the azimuthal unit vector. Thus, the Poynting vector at a distance  $r$  from the axis is

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} = -\mathbf{r} \frac{\sigma}{2} E^2. \quad (\text{S-8.3})$$

The energy flux  $\Phi_S \equiv \Phi(\mathbf{S})$  through the surface of a cylinder of radius  $r < a$  and length  $h$  and coaxial to the wire is thus

$$\begin{aligned} \Phi_S &= \oint \mathbf{S} \cdot d\mathbf{A} = -2\pi rhS(r) \\ &= -\pi r^2 h \sigma E^2, \end{aligned} \quad (\text{S-8.4})$$

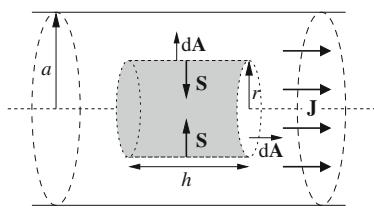


Fig. S-8.1

where  $d\mathbf{A}$  is the vector surface element of the cylinder. The energy flows inwards, and is entirely dissipated into Joule heating inside the cylinder volume, as we can check by calculating

$$W = \int \mathbf{J} \cdot \mathbf{E} dV = \pi r^2 h JE = \pi r^2 h \sigma E^2, \quad (\text{S-8.5})$$

where the integral is extended to the volume the cylinder. The equality  $W = -\Phi_S$  satisfies Poynting's theorem since there is no variation in time of the EM energy.

Note that, in the approximation of an infinitely long wire, the electric field is uniform also for  $r > a$  (in the case of a finite wire of length  $2h \gg a$ , this is a good approximation in the central region for  $r \ll h$ , see Problem 4.9), while the magnetic field  $B = 2\pi Ja^2/rc$ . Within this approximation,  $\mathbf{S} = -(a^2\sigma E^2/2r)\hat{\mathbf{r}}$  for  $r > a$ , so that the energy flux is independent of  $r$  and it is still equal to minus the total dissipated power:

$$\Phi_S = -2\pi rhS(r) = -\pi a^2 h \sigma E^2 \quad (r > a). \quad (\text{S-8.6})$$

**b)** We must show that  $\nabla \cdot (\mathbf{S} - \mathbf{S}') = 0$ , i.e., that  $\mathbf{S} - \mathbf{S}' = \nabla \times \mathbf{f}$ , where  $\mathbf{f}$  is a vector function of the coordinates. Let us substitute  $\mathbf{E} = -\nabla\varphi$  into (8.7)

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} = -\frac{c}{4\pi} \nabla\varphi \times \mathbf{B}. \quad (\text{S-8.7})$$

Now from the vector identity

$$\nabla \times (\varphi \mathbf{B}) = \nabla\varphi \times \mathbf{B} + \varphi \nabla \times \mathbf{B} = \nabla\varphi \times \mathbf{B} + \varphi \left( \frac{4\pi}{c} \mathbf{J} \right) \quad (\text{S-8.8})$$

we obtain

$$\nabla\varphi \times \mathbf{B} = \nabla \times (\varphi \mathbf{B}) - \varphi \left( \frac{4\pi}{c} \mathbf{J} \right), \quad (\text{S-8.9})$$

which can be substituted into (S-8.7), leading to

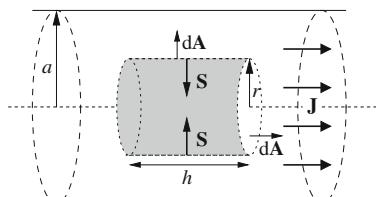
$$\mathbf{S} = \varphi \mathbf{J} - \frac{c}{4\pi} \nabla \times (\varphi \mathbf{B}). \quad (\text{S-8.10})$$

Thus, we are free to redefine the Poynting vector as

$$\mathbf{S}' = \varphi \mathbf{J}, \quad (\text{S-8.11})$$

since

$$\mathbf{S} - \mathbf{S}' = \nabla \times \left( -\frac{c}{4\pi} \varphi \mathbf{B} \right). \quad (\text{S-8.12})$$



We can show that  $\mathbf{S}'$  is equivalent to  $\mathbf{S}$  by computing its flux through the same cylindrical surface as above. Since  $\mathbf{S}'$  is parallel to the wire axis, only the two base surfaces contribute to the flux  $\Phi_{S'} \equiv \Phi(\mathbf{S}')$

Fig. S-8.2

$$\begin{aligned}\Phi_{S'} &= \pi a^2 [-S(z) + S(z+h)] \\ &= \pi a^2 J [\varphi(z+h) - \varphi(z)].\end{aligned}\quad (\text{S-8.13})$$

Since  $\varphi = -Ez$ , we finally obtain

$$\Phi_{S'} = \pi a^2 [-S(z) + S(z+h)] = -\pi a^2 h JE, \quad (\text{S-8.14})$$

which gives again minus the total dissipated power.

## S-8.2 Poynting Vector(s) in a Capacitor

- a)** The magnetic field has azimuthal symmetry, i.e.,  $\mathbf{B} = B(r)\hat{\phi}$ , and can be evaluated from the equation  $c\nabla \times \mathbf{B} = \partial_t \mathbf{E}$ , which, with our assumption  $E = E_0 t/\tau$ , leads to

$$B(r) = \frac{r}{2c} \partial_t E = \frac{r}{2c\tau} E_0. \quad (\text{S-8.15})$$

- b)** The corresponding Poynting vector  $\mathbf{S}$  is

$$\mathbf{S} = \frac{c}{4\pi} E \hat{\mathbf{z}} \times (B \hat{\phi}) = -\frac{r}{8\pi} (E \partial_t E) (\hat{\mathbf{z}} \times \hat{\phi}) = -\frac{1}{2} \partial_t \left( \frac{E^2}{8\pi} \right) \mathbf{r}. \quad (\text{S-8.16})$$

We evaluate the flux of  $\mathbf{S}$  through the smallest closed cylindrical surface enclosing our capacitor, shown in Fig. S-8.3. Since  $\mathbf{S}$  is radial, only the lateral surface of the cylinder contributes to the flux, and we have

$$\Phi(\mathbf{S}) = -2\pi ah S(a) = -\pi a^2 h \partial_t \left( \frac{E^2}{8\pi} \right). \quad (\text{S-8.17})$$

Quantity  $(E^2 + B^2)/8\pi$  is the energy density associated to the EM field, and, since in our case  $B$  does not depend on time, is also the total EM energy density inside the capacitor. Thus,  $\Phi(\mathbf{S})$  equals minus the time derivative of the energy stored in the capacitor. For a general dependence of  $E_z(t)$  on time,  $\mathbf{B}$  is also time-dependent, and the flux of  $\mathbf{S}$  equals the time derivative of the electrostatic energy to the first order, within the slowly varying current approximation.

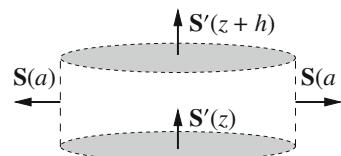


Fig. S-8.3

- c)** The electric potential is  $\varphi = -Ez$ . By substituting  $\mathbf{E} = -\nabla\varphi$  into (8.7) we obtain

$$\mathbf{S} = \frac{c}{4\pi} (-\nabla\varphi) \times \mathbf{B} = -\frac{c}{4\pi} [\nabla \times (\varphi \mathbf{B}) - \varphi \nabla \times \mathbf{B}]. \quad (\text{S-8.18})$$

Thus, the vector

$$\mathbf{S}' \equiv \frac{c}{4\pi} \varphi \nabla \times \mathbf{B} = \frac{1}{4\pi} \varphi \partial_t \mathbf{E} = \mathbf{S} + \frac{c}{4\pi} \nabla \times (\varphi \mathbf{B}) \quad (\text{S-8.19})$$

equals  $\mathbf{S}$  plus the curl of a vector function, and is thus another suitable Poynting vector. Since  $\mathbf{S}'$  is perpendicular to the capacitor plates, its flux through our closed cylindrical surface is (see Fig. S-8.3)

$$\Phi(\mathbf{S}') = \pi a^2 [S'(z+h) - S'(z)] = -\pi a^2 h \left( \frac{E \partial_t E}{4\pi} \right) = -\pi a^2 h \partial_t \left( \frac{E^2}{8\pi} \right), \quad (\text{S-8.20})$$

in agreement with (S-8.17).

### S-8.3 Poynting's Theorem in a Solenoid

**a)** We take a cylindrical coordinate system with the  $z$  axis along the solenoid axis. Inside an infinite solenoid the magnetic field is uniform and equals  $\mathbf{B} = B \hat{\mathbf{z}} = (4\pi/c)nI \hat{\mathbf{z}}$ . According to Faraday's law of induction, the rate of change of  $\mathbf{B} = \mathbf{B}(t)$ , due to the time dependence of  $I = I(t)$ , generates an electric field  $\mathbf{E}$  associated to the induced electromotive force. For symmetry reasons, the field lines of  $\mathbf{E}$  are circles coaxial to the solenoid, i.e., we have  $\mathbf{E} = E(r)\hat{\phi}$ . Applying Faraday's law to a circle of radius  $r < a$ , coaxial to the solenoid, we have

$$2\pi r E(r) = -\pi r^2 \frac{1}{c} \partial_t B = -\pi r^2 \frac{4\pi n I_0}{c^2 \tau}, \quad (\text{S-8.21})$$

from which  $E(r) = -2\pi n I_0 r / (c^2 \tau)$ .

**b)** The Poynting vector inside the solenoid ( $r < a$ ) is

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} = -\frac{2\pi(nI_0)^2 rt}{(c\tau)^2} (\hat{\phi} \times \hat{\mathbf{z}}) = -\frac{2\pi(nI_0)^2 t}{(c\tau)^2} \mathbf{r}. \quad (\text{S-8.22})$$

Thus, the flux of  $\mathbf{S} = \mathbf{S}(r)$  through the surface of a closed cylinder of radius  $r$  and height  $h$  is nonzero only through the lateral surface, and we have

$$\Phi(\mathbf{S}) = 2\pi r h \mathbf{S} \cdot \hat{\mathbf{r}} = -\left( \frac{2\pi n I_0 r}{c\tau} \right)^2 ht. \quad (\text{S-8.23})$$

The magnetic energy enclosed by the cylinder surface is

$$U_M = u_M V = \frac{B^2}{8\pi} \pi r^2 h = 2\pi^2 r^2 h \left( \frac{nI_0 t}{c\tau} \right)^2, \quad (\text{S-8.24})$$

where  $V$  is the volume of the cylinder, thus

$$\frac{dU_M}{dt} = 4\pi^2 r^2 ht \left( \frac{nI_0}{c\tau} \right)^2 = -\Phi(\mathbf{S}), \quad (\text{S-8.25})$$

according to Poynting's theorem, since the electric field is constant in time, and  $\mathbf{J} \cdot \mathbf{E} = 0$  for  $r < a$ , i.e., inside the solenoid.

c) Outside the solenoid ( $r > a$ ) we have  $\mathbf{B} = 0$ . Correspondingly, also  $\mathbf{S} = 0$  and  $\Phi(\mathbf{S}) = 0$ . The rate of change of the magnetic energy is given by (S-8.25) with  $r = a$ , and must equal the volume integral of  $\mathbf{J} \cdot \mathbf{E}$ , which is the work done by the induced field on the current flowing in the coils (notice that this is different from the electric field driving the current and causing Joule heating in the coils, see Problem 13.18). In our representation, the current is distributed on the surface  $r = a$ , thus  $\mathbf{J} d^3r$  is replaced by  $nI dS = nIa d\phi dz$  in the integral, and  $E$  is evaluated at  $r = a$ . We thus obtain

$$\begin{aligned} \int_V \mathbf{J} \cdot \mathbf{E} d^3r &= \int_S nI E(a) dS = -2\pi a h \left( nI_0 \frac{t}{\tau} \right) \left( \frac{2\pi n I_0 a}{c^2 \tau} \right) \\ &= -4\pi a^2 ht \left( \frac{nI_0}{c\tau} \right)^2 = -\frac{dU_M}{dt} \Big|_{r=R}. \end{aligned} \quad (\text{S-8.26})$$

## S-8.4 Poynting Vector in a Capacitor with Moving Plates

a) We use a cylindrical coordinate system  $(r, \phi, z)$ , with the  $z$  axis along the symmetry axis of the capacitor, and the origin on the fixed plate. Thus, within the limits of our approximations, the electric field is uniform and parallel to  $\hat{\mathbf{z}}$  inside the capacitor, whose capacitance is

$$C = \frac{\pi a^2}{4\pi h(t)} = \frac{a^2}{4(h_0 + vt)}. \quad (\text{S-8.27})$$

In the case of the isolated plates the charge is constant and equal to  $Q_0$ , while the voltage between the plates  $V$  and the electric field  $E$  between the plates are, respectively,

$$V = \frac{Q_0}{C} = Q_0 \frac{4(h_0 + vt)}{a^2}, \quad E = \frac{V}{h} = \frac{4Q_0}{a^2}. \quad (\text{S-8.28})$$

In the case of constant voltage between the plates,  $V = V_0$ , the charge  $Q$  of the capacitor and the electric field  $E$  are, respectively.

$$Q = CV_0 = V_0 \frac{a^2}{4(h_0 + vt)}, \quad E = \frac{V_0}{h_0 + vt}. \quad (\text{S-8.29})$$

In the case of constant charge, the electrostatic force between the plates  $F_{\text{es}}$  is also constant and equals

$$F_{\text{es}} = -Q \frac{E}{2} = -\frac{2Q_0^2}{a^2}, \quad (\text{S-8.30})$$

while in the case of constant voltage we have

$$F_{\text{es}} = -Q \frac{E}{2} = -V_0^2 \frac{a^2}{8(h_0 + vt)^2}, \quad (\text{S-8.31})$$

in both cases the minus signs means that the force is attractive. In both cases the applied external force  $\mathbf{F}_{\text{mech}}$  must cancel the electrostatic force, i.e., we must have  $F_{\text{mech}} = -F_{\text{es}}$ , for the plates to move at constant velocity.

**b)** The electrostatic energy can be written as

$$U = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} CV^2, \quad (\text{S-8.32})$$

so that at constant charge we have

$$U = Q_0^2 \frac{2(h_0 + vt)}{a^2}, \quad \frac{dU}{dt} = \frac{2vQ_0^2}{a^2} > 0, \quad (\text{S-8.33})$$

while at constant voltage we have

$$U = V_0^2 \frac{a^2}{8(h_0 + vt)}, \quad \frac{dU}{dt} = -\frac{a^2 v V_0^2}{8(h_0 + vt)^2} < 0. \quad (\text{S-8.34})$$

**c)** At constant charge, the electric field  $\mathbf{E} = E_0 \hat{\mathbf{z}}$  is also constant, with  $E_0 = Q_0/(\pi a^2)$ , therefore the displacement current density  $\mathbf{J}_D = \partial_t \mathbf{E}/c$  is zero. Also the conduction current density  $\mathbf{J}_C$  is zero between the plates (actually, there is a conduction current localized on the moving plate, we shall come back to this point below), so that also the magnetic field  $\mathbf{B}$  is zero between the plates.

At constant voltage, the electric field is  $\mathbf{E} = \hat{\mathbf{z}} V_0/h(t) = \hat{\mathbf{z}} V_0/(h_0 + vt)$ , implying the presence of a displacement current along  $\hat{\mathbf{z}}$ . The magnetic field can be calculated by taking the path integral of  $\mathbf{B}$  over a circumference of radius  $r < a$  coaxial with, and located between, the plates, which equals the flux of the displacement current through the enclosed circle. Due to the cylindrical symmetry of the system, the only nonzero component of  $\mathbf{B}$  is azimuthal,  $\mathbf{B} = B(r, t)\hat{\phi}$ , and calculating its path integral over the circle of radius  $r$  corresponding to a field line we have for  $B = B(r, t)$

$$2\pi r B = -\frac{\pi r^2}{c} \partial_t E = -\frac{\pi r^2}{c} \frac{V_0 v}{(h_0 + vt)^2}, \quad (\text{S-8.35})$$

so that

$$B = -\frac{r}{2c} \frac{V_0 v}{(h_0 + vt)^2}. \quad (\text{S-8.36})$$

**d)** At constant charge we have  $\mathbf{B} = 0$ , and the Poynting vector  $\mathbf{S} = (c/4\pi)\mathbf{E} \times \mathbf{B}$  is also zero. In this case, (S-8.30) and (S-8.33) tell us that the rate of work done against the electric force  $W_{\text{mech}}$

$$W_{\text{mech}} = \mathbf{F}_{\text{mech}} \cdot \mathbf{v} = -\mathbf{F}_{\text{es}} \cdot \mathbf{v} = \frac{2Q_0^2 v}{a^2} \quad (\text{S-8.37})$$

equals the rate of change the electrostatic energy  $dU/dt$ . This rate of work must also equal minus the integral of  $\mathbf{J} \cdot \mathbf{E}$  over the whole space, according to Poynting's theorem. We verify this at the end of this answer.

At constant voltage, the Poynting vector is radial,  $\mathbf{S} = S \hat{\mathbf{r}}$ , and, according to (S-8.29) and (S-8.36), we have

$$S = -\frac{c}{4\pi} E_z B_\phi = \frac{V_0^2 v r}{8\pi (h_0 + vt)^3}. \quad (\text{S-8.38})$$

Evaluating the flux of  $\mathbf{S}$  through the minimum closed surface enclosing the capacitor, of lateral surface  $2\pi a(h_0 + vt)$ , we obtain

$$\Phi_S = 2\pi a(h_0 + vt) \frac{V_0^2 v a}{8\pi (h_0 + vt)^3} = \frac{a^2 V_0^2 v}{4(h_0 + vt)^2}. \quad (\text{S-8.39})$$

Through (S-8.31) and (S-8.34) we can verify that

$$-\Phi_S = \frac{dU}{dt} + Fv. \quad (\text{S-8.40})$$

Note also that, in this case,  $\Phi_S$  equals the power *absorbed* by the voltage source. In fact, the current flowing through the circuit is

$$I = \frac{dQ}{dt} = -\frac{a^2 v}{4(h_0 + vt)^2} V_0 , \quad (\text{S-8.41})$$

where we have inserted the first of (S-8.29), corresponding to a power absorption by the source

$$W = -V_0 I = \frac{a^2 v}{4(h_0 + vt)^2} V_0^2 = \Phi_s . \quad (\text{S-8.42})$$

We avoided so far to discuss the role of the conduction current circulating in the plates [the following discussion will require some familiarity with the distributions  $\delta(x)$  and  $\Theta(x)$ , where  $\Theta(x)$  is the Heaviside step function, defined by  $\Theta(x) = 1$  for  $x > 0$  and  $\Theta(x) = 0$  for  $x < 0$ ; notice that  $d\Theta(x)/dx = \delta(x)$ ]. Let us consider the constant charge case. Since the upper plate has a charge  $Q_0$  distributed on the surface  $z = -h_0 + vt$  and moves with velocity  $v$ , there is actually a current density

$$\mathbf{J}_C = \frac{Q_0}{\pi a^2} v \delta(z - h_0 - vt) . \quad (\text{S-8.43})$$

On the other hand, the electric field between the plates may be written as

$$\mathbf{E} = -\frac{4Q_0}{a^2} [\Theta(z) - \Theta(z - h_0 - vt)] \hat{\mathbf{z}} , \quad (\text{S-8.44})$$

where  $\Theta(z)$  is the Heaviside step function, defined by  $\Theta(z) = 1$  for  $z > 0$  and  $\Theta(z) = 0$  for  $z < 0$ . This expression takes into account the fact that at each time  $t$  the field exists only in the  $0 < z < vt$  region, so it is actually a time-dependent field. Since  $d\Theta(z)/dz = \delta(z)$ , the displacement current is

$$\mathbf{J}_D = \frac{1}{c} \partial_t \mathbf{E} = -\frac{4Q_0 v}{a^2 c} \delta(z - h_0 - vt) \hat{\mathbf{z}} = -\frac{4\pi}{c} \mathbf{J}_C , \quad (\text{S-8.45})$$

so that the source term for the magnetic field  $(4\pi/c) \mathbf{J}_C + \mathbf{J}_D$  is zero. It also follows that

$$\mathbf{J}_C \cdot \mathbf{E} = -\frac{1}{4\pi} (\partial_t \mathbf{E}) \cdot \mathbf{E} = -\frac{1}{8\pi} \partial_t E^2 , \quad (\text{S-8.46})$$

which ensures energy conservation, since the work done on the current equals the rate of change of the electrostatic energy. In detail, we have

$$\partial_t E^2 = (4\pi\sigma_{\text{up}})^2 \partial_t \Theta(z - h_0 - vt) = -v \left( \frac{4Q_0}{a^2} \right)^2 \delta(z - h_0 - vt), \quad (\text{S-8.47})$$

thus

$$\begin{aligned} \int \mathbf{J}_C \cdot \mathbf{E} d^3r &= \frac{v(4\pi\sigma)^2}{8\pi} \int \delta(z - h_0 - vt) d^3r \\ &= 2\pi^2 a^2 \sigma^2 v = \frac{2vQ_0^2}{a^2} = \frac{dU}{dt}. \end{aligned} \quad (\text{S-8.48})$$

## S-8.5 Radiation Pressure on a Perfect Mirror

- a) We consider the case of perpendicular incidence first, and choose a Cartesian reference frame with the  $x$  axis perpendicular to the mirror surface. The incident plane wave packet has duration  $\tau$  (with  $\tau \gg 2\pi/\omega$ , the laser period), corresponding to a length  $c\tau$ , and propagates along  $\hat{\mathbf{x}}$ . We want to calculate how much momentum is transferred to an area  $A$  of the mirror surface during the reflection of the whole wave packet. The momentum transferred per unit time and area is the pressure exerted by the radiation.

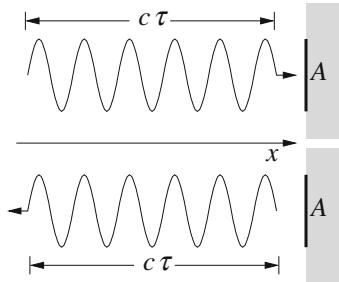


Fig. S-8.4

The momentum density of an EM field is  $\mathbf{S}/c^2$ , where  $\mathbf{S} = c\mathbf{E} \times \mathbf{B}/4\pi$  is the Poynting vector. Thus the total momentum delivered by the incident wave packet on the area  $A$  is

$$\mathbf{p}_i = \left\langle \frac{\mathbf{S}_i}{c^2} \right\rangle c\tau A = \hat{\mathbf{x}} \frac{I}{c} \tau A \quad (\text{S-8.49})$$

where the angle brackets denote the average over one cycle,  $\mathbf{S}_i$  is the Poynting vector of the incident packet, and  $I = |\langle \mathbf{S}_i \rangle|$  is the intensity of the incident pulse (the average flux of energy per unit time and area), according to Poynting's theorem of energy conservation.

The reflected wave packet carries a total momentum, over the area  $A$ ,

$$\mathbf{p}_r = \left\langle \frac{\mathbf{S}_r}{c^2} \right\rangle c\tau A = -\hat{\mathbf{x}} \frac{I}{c} \tau A \quad (\text{S-8.50})$$

where  $\mathbf{S}_r = -\mathbf{S}_i$  is the Poynting vector of the reflected packet. The momentum transferred to the mirror over the surface area  $A$  during the time interval  $\tau$  is thus

$$\Delta \mathbf{p} = \mathbf{p}_i - \mathbf{p}_r = |\Delta \mathbf{p}| \hat{\mathbf{x}} \quad (\text{S-8.51})$$

and the corresponding pressure is

$$P_{\text{rad}} = \frac{|\Delta \mathbf{p}|}{\tau A} = 2 \frac{I}{c}. \quad (\text{S-8.52})$$

Using a similar heuristic argument, it is quite straightforward to find the radiation pressure for oblique incidence at an angle  $\theta$  from the normal to the mirror surface. In fact, in this case the momentum transferred to the mirror along the normal is

$$\Delta \mathbf{p} = \mathbf{p}_i - \mathbf{p}_r = 2 \hat{\mathbf{x}} \frac{I}{c^2} c \tau A \cos \theta = 2 \hat{\mathbf{x}} \frac{I}{c} \tau A \cos \theta, \quad (\text{S-8.53})$$

and the area of incidence is now  $A / \cos \theta$ . Thus

$$P_{\text{rad}} = |\mathbf{p}_i - \mathbf{p}_r| \frac{\cos \theta}{\tau A} = 2 \frac{I}{c} \tau A \cos \theta \frac{\cos \theta}{\tau A} = 2 \frac{I}{c} \cos^2 \theta. \quad (\text{S-8.54})$$

**b)** The mechanical force on a closed system of charges, currents and fields is given by the following integral over the volume of the system

$$\mathbf{F}_{\text{mech}} = \frac{d\mathbf{p}_{\text{mech}}}{dt} = \int_V \left( \varrho \mathbf{E} + \frac{1}{c} \mathbf{J} \times \mathbf{B} \right) d^3 r. \quad (\text{S-8.55})$$

From now on, we shall consider the case of perpendicular incidence only, and leave the case of oblique incidence as a further exercise for the reader. In the present case,  $\varrho = 0$  everywhere and only the magnetic term contributes. Thus, in plane geometry the time-averaged force on a planar surface of area  $A$  is

$$\langle F_{\text{mech}} \rangle = \int_0^{+\infty} \left\langle \frac{1}{c} \mathbf{J} \times \mathbf{B} \right\rangle A dx \quad (\text{S-8.56})$$

and is directed along  $\hat{\mathbf{x}}$  for symmetry reasons.

The current in a perfect mirror is localized on the surface, where the magnetic field is discontinuous. Here we assume that the wave fields  $\mathbf{B}$  and  $\mathbf{E}$  are parallel to  $\hat{\mathbf{z}}$  and  $\hat{\mathbf{y}}$ , respectively. Let  $\mathbf{E}_i(x, t) = \hat{\mathbf{y}} E_i \cos(kx - \omega t)$  be the incident electric field. The total field  $\mathbf{E}(x, t)$  is the sum of  $\mathbf{E}_i$  and the field  $\mathbf{E}_r(x, t) = -\hat{\mathbf{y}} E_i \cos(-kx - \omega t)$  of the reflected wave, so that  $\mathbf{E}(0, t) = 0$ . Thus the total fields for  $x < 0$  have the form of standing waves

$$E_y(x, t) = 2E_i \sin(kx) \sin \omega t , \quad (\text{S-8.57})$$

$$B_z(x, t) = 2E_i \cos(kx) \cos \omega t . \quad (\text{S-8.58})$$

The discontinuity of  $B_z$  leads to a surface current  $J_y = K_y \delta(x)$  where

$$K_y = -\frac{c}{4\pi} [B_z(0^+, t) - B_z(0^-, t)] = \frac{c}{4\pi} B_z(0^-, t) = \frac{c}{2\pi} E_i \cos \omega t , \quad (\text{S-8.59})$$

where we have used Stokes' theorem and  $B_z(0^+, t) = 0$ . The force per unit surface, i.e., the pressure, is given by the surface current multiplied by the mean value of the field across the current layer (the argument is identical to the one used for calculating the electrostatic pressure on a surface charge layer in electrostatics):

$$\begin{aligned} P_{\text{rad}} &= \left\langle K_y \frac{1}{2c} [B_z(0^+, t) + B_z(0^-, t)] \right\rangle = \frac{c}{8\pi} \left\langle B_z^2(0^-, t) \right\rangle \\ &= \frac{c}{8\pi} \left( \frac{1}{2} 4|E_i|^2 \right) = 2 \frac{I}{c} , \end{aligned} \quad (\text{S-8.60})$$

since  $B_z(0^-, t) = 2E_i \cos \omega t$ , and  $I = (c/4\pi)(|E_i|^2/2)$ . This is equivalent to evaluate the integral in (S-8.56) as

$$\begin{aligned} \int_{0^-}^{+\infty} J_y B_z dx &= - \int_{0^-}^{+\infty} \left( \frac{c}{4\pi} \partial_x B_z - \frac{1}{4\pi} \partial_t E_y \right) B_z dx = - \frac{c}{4\pi} \int_{0^-}^{+\infty} \frac{1}{2} \partial_x B_z^2 dx \\ &= \frac{c}{8\pi} B_z^2(0^-, t) , \end{aligned} \quad (\text{S-8.61})$$

where we used the fact that  $E_y = 0$  and  $\partial_t E_y = 0$  for  $x \geq 0^-$ .

c) The momentum conservation theorem (8.8) states that, for a closed system of charges, currents and EM fields bounded by a closed surface  $S$ , the following balance equation holds:

$$\frac{d}{dt} (\mathbf{p}_{\text{mech}} + \mathbf{p}_{\text{EM}})_i = \oint_S \sum_j \mathbf{T}_{ij} \hat{n}_j d^2 r , \quad (\text{S-8.62})$$

where  $i, j = x, y, z$ ,  $\mathbf{T}_{ij}$  is the Maxwell stress tensor, and  $\hat{n}_j$  is the  $j$  component of the outward-pointing unit vector locally normal to  $S$ . Thus, the integral on the right-hand side is the outward flux of the vector  $\mathbf{T} \cdot \hat{\mathbf{n}}$  through  $S$ . In (S-8.62),  $\mathbf{p}_{\text{mech}}$  is the mechanical momentum of the system, while the momentum associated to the EM field is

$$\mathbf{p}_{\text{EM}} = \int_V \mathbf{g} d^3 r , \quad (\text{S-8.63})$$

where  $\mathbf{g} = \mathbf{S}/c^2$  is the momentum density (8.9), and the integral is evaluated over the volume bounded by  $\mathcal{S}$ .

In our case, we take the front surface  $A$  of the mirror and close it by adding a surface extending deep into the mirror, where the fields are zero. Thus, the amount of EM momentum which flows into the mirror (and “transformed” into mechanical momentum) is given by the integral

$$\int_A \sum_j T_{ij} \hat{n}_j d^2r = A \sum_j T_{ij}(0^-, t) \hat{n}_j . \quad (\text{S-8.64})$$

The radiation pressure on the mirror is the time-averaged momentum flow per unit area,

$$P_{\text{rad}} = \left\langle \sum_j T_{1j}(0^-, t) \hat{n}_j \right\rangle = -\langle T_{11}(0^-, t) \rangle, \quad (\text{S-8.65})$$

since, in our case,  $\hat{\mathbf{n}} = (-1, 0, 0)$ . Thus we actually need to evaluate  $T_{11}(0, t)$  only:

$$T_{11}(0, t) = -\frac{1}{8\pi} B_z^2(0^-, t) . \quad (\text{S-8.66})$$

The radiation pressure is thus

$$P_{\text{rad}} = -\langle T_{11}(0, t) \rangle = \frac{1}{8\pi} \left\langle B_z^2(0^-, t) \right\rangle = \frac{1}{4\pi} |E_i|^2 = 2\frac{I}{c} . \quad (\text{S-8.67})$$

## S-8.6 Poynting Vector for a Gaussian Light Beam

a) The divergence of the electric field in vacuum is zero. With our geometry, this means that, since we have assumed  $E_y = 0$ , we have

$$0 = \nabla \cdot \mathbf{E} = \partial_x E_x + \partial_z E_z . \quad (\text{S-8.68})$$

From (S-8.68) and (8.14) we obtain

$$\partial_z E_z = -\partial_x E_x = -2E_0 x e^{-r^2/r_0^2} \cos(kz - \omega t) , \quad (\text{S-8.69})$$

where the divergence is calculated in the generic point  $(x, y, z)$ , and we have used  $r^2 = x^2 + y^2$ . Integrating with respect to  $z$ , we have

$$E_z = -E_0 \frac{2x}{kr_0^2} e^{-r^2/r_0^2} \sin(kz - \omega t) . \quad (\text{S-8.70})$$

Analogously, we obtain for the longitudinal component of  $\mathbf{B}$

$$B_z = -B_0 \frac{2y}{kr_0^2} e^{-r^2/r_0^2} \sin(kz - \omega t). \quad (\text{S-8.71})$$

Let us verify if these fields are consistent with Maxwell's equations. First, we check if  $\partial_t E_z = c(\nabla \times \mathbf{B})_z = c \partial_x B_y$  holds. We have

$$\partial_x B_y = -B_0 \frac{2x}{r_0^2} e^{-r^2/r_0^2} \cos(kz - \omega t), \quad (\text{S-8.72})$$

$$\partial_t E_z = -E_0 \omega \frac{2x}{kr_0^2} e^{-r^2/r_0^2} \cos(kz - \omega t), \quad (\text{S-8.73})$$

which implies  $B_0 = (\omega/kc) E_0 = E_0$ . Analogously we can check that  $\partial_t B_z = -c(\nabla \times \mathbf{E})_z$ .

**b)** The Poynting vector is

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} = \frac{c}{4\pi} (\hat{\mathbf{x}} E_z B_y - \hat{\mathbf{y}} E_x B_z + \hat{\mathbf{z}} E_x B_y), \quad (\text{S-8.74})$$

and its components averaged over one cycle are

$$\langle S_x \rangle = \frac{c}{4\pi} \frac{2x}{kr_0^2} E_0^2 e^{-2r^2/r_0^2} \langle \sin(kz - \omega t) \cos(kz - \omega t) \rangle = 0, \quad (\text{S-8.75})$$

$$\langle S_y \rangle = -\frac{c}{4\pi} \frac{2y}{kr_0^2} E_0^2 e^{-2r^2/r_0^2} \langle \cos(kz - \omega t) \sin(kz - \omega t) \rangle = 0, \quad (\text{S-8.76})$$

$$\langle S_z \rangle = \frac{c}{4\pi} E_0^2 e^{-2r^2/r_0^2} \langle \cos^2(kz - \omega t) \rangle = \frac{c}{8\pi} E_0^2 e^{-2r^2/r_0^2}. \quad (\text{S-8.77})$$

Thus, we can define the local intensity and the total power of the beam as

$$I(r) = \langle S_z \rangle, \quad P = \int_0^\infty I(r) 2\pi r dr. \quad (\text{S-8.78})$$

**c)** We have

$$\nabla^2 E_x = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial E_x}{\partial r} \right) - \frac{\partial^2 E_x}{\partial z^2} = \left[ \frac{4}{r_0^2} \left( \frac{r^2}{r_0^2} - 1 \right) - k^2 \right] E_x \quad (\text{S-8.79})$$

(see Table A.1 for the Laplacian operator in cylindrical coordinates; notice that here the fields are independent of  $\phi$ ). We can easily check that  $(\nabla^2 + \omega^2/c^2) E_x \neq 0$ ; the “extra” terms being of the order of  $\sim 1/(kr_0)^2$ . Thus we expect our approximate expressions for the fields to be accurate as long as  $r_0 \gg 1/k = \lambda/2\pi$ , i.e., if the beam is much wider than one wavelength.

It is known that a beam with finite width actually undergoes diffraction. The width of a Gaussian beam doubles after a typical distance, called Rayleigh length,

$r_R = kr_0^2$ . This corresponds to an aperture angle

$$\theta_d \approx \frac{r_0}{z_R} = \frac{1}{kr_0} \approx \frac{\lambda}{r_0}. \quad (\text{S-8.80})$$

It might be interesting to notice that this result may be inferred from the values for the longitudinal field components obtained at point **a**). In fact, the beam may be obtained as a linear superposition of plane waves of the same frequency but different wavevectors. For the plane wave, the electric and magnetic field are perpendicular to the wavevector  $\mathbf{k}$ . Thus, the typical ratio  $E_z/E_y \sim 2/(kr_0)$  (at  $r = r_0$ ) also corresponds to a typical value  $k_x/k_z \sim 2/(kr_0)$ , which should determine the typical angular spread of the wavevector spectrum, hence the spreading angle of the beam.

## S-8.7 Intensity and Angular Momentum of a Light Beam

**a)** First, we define the shorthand symbols  $C = \cos(kz - \omega t)$ ,  $S = \sin(kz - \omega t)$ , and  $E'_0 = \partial_r E_0(r)$ , that we shall use throughout the solution of the problem. We have for the intensity of the beam

$$\begin{aligned} I(r) \equiv S_z &= \frac{c}{4\pi} (E_x B_y - E_y B_x) = \frac{c}{4\pi} E_0^2(r) [CC - (-SS)] \\ &= \frac{c}{4\pi} E_0^2(r) [C^2 + S^2] = \frac{c}{4\pi} E_0^2(r). \end{aligned} \quad (\text{S-8.81})$$

**b)** The divergence of the fields in vacuum must be zero. For the electric field we have

$$0 = \nabla \cdot \mathbf{E} = \partial_x E_x + \partial_y E_y + \partial_z E_z, \quad (\text{S-8.82})$$

thus

$$\partial_z E_z = -\partial_x E_x - \partial_y E_y = -\frac{x}{r} E'_0(r) C + \frac{y}{r} E'_0(r) S,$$

and, integrating with respect to  $z$ ,

$$E_z = -\frac{1}{kr} E'_0(r) [xS + yC]. \quad (\text{S-8.83})$$

Analogously, we can evaluate  $B_z$ :

$$\begin{aligned} \partial_z B_z &= -\partial_x B_x - \partial_y B_y = -\frac{x}{rc} E'_0(r) S - \frac{y}{rc} E'_0(r) C, \\ B_z &= +\frac{1}{krc} E'_0(r) [xC - yS]. \end{aligned} \quad (\text{S-8.84})$$

**c)** The  $x$  and  $y$  components of the Poynting vector are

$$\begin{aligned}
S_x &= \frac{c}{4\pi} (E_y B_z - E_z B_y) \\
&= \frac{c}{4\pi} \left\{ (-E_0 S) \left[ \frac{E'_0}{krc} (xC - yS) \right] - \left[ -\frac{E'_0}{kr} (xS + yC) \right] \frac{1}{c} E_0 C \right\} \\
&= \frac{c}{4\pi} \frac{E_0 E'_0}{kr} (-xSC + yS^2 + xSC + yC^2) = \frac{c}{4\pi} E_0 E'_0 \frac{y}{kr}. \quad (\text{S-8.85})
\end{aligned}$$

$$\begin{aligned}
S_y &= \frac{c}{4\pi} (E_z B_x - E_x B_z) \\
&= \frac{c}{4\pi} \left\{ \left[ -\frac{E'_0}{kr} (xS + yC) \right] \frac{1}{c} E_0 S - (E_0 C) \left[ \frac{E'_0}{krc} (xC - yS) \right] \right\} \\
&= \frac{c}{4\pi} \frac{E_0 E'_0}{kr} (-xS^2 - yCS - xC^2 + yCS) = -\frac{c}{4\pi} E_0 E'_0 \frac{x}{kr} \quad (\text{S-8.86})
\end{aligned}$$

Since we have

$$\frac{c}{4\pi} E_0(r) E'_0(r) = \frac{c}{8\pi} \partial_r E_0^2(r) = \frac{1}{2} \partial_r I(r), \quad (\text{S-8.87})$$

the Poynting vector can be written

$$\mathbf{S} = \left( \frac{y}{2kr} \partial_r I(r), -\frac{x}{2kr} \partial_r I(r), I(r) \right). \quad (\text{S-8.88})$$

Assuming a Gaussian beam, we have  $E_0(r) \propto e^{-r^2/r_0^2}$ , and  $S_{x,y} \propto S_z/(kr_0) \propto \theta_d S_z$ , with  $\theta_d$  the diffraction angle of (S-8.80).

**d)** We have

$$\begin{aligned}
\ell_z &= \frac{1}{c^2} (xS_y - yS_x) = \frac{-x^2 - y^2}{2krc^2} \partial_r I(r) = -\frac{r}{2kc^2} \partial_r I(r) \\
&= -\frac{r}{2c\omega} \partial_r I(r). \quad (\text{S-8.89})
\end{aligned}$$

We eventually obtain the total angular momentum by integrating the above expression by parts,

$$\begin{aligned}
L_z &= \int_0^\infty \ell_z(r) 2\pi r dr = - \int_0^\infty \frac{r}{2c\omega} \partial_r I(r) 2\pi r dr \\
&= \frac{1}{c\omega} \int_0^\infty I(r) 2\pi r dr = \frac{W}{c\omega}. \quad (\text{S-8.90})
\end{aligned}$$

## S-8.8 Feynman's Paradox solved

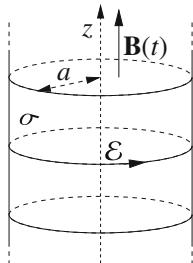


Fig. S-8.5

**a)** We use a cylindrical coordinate system  $(r, \phi, z)$  with the cylinder axis as  $z$  axis. The induced electric field  $\mathbf{E}_{\text{ind}}$  has azimuthal symmetry, i.e.,  $\mathbf{E}_{\text{ind}} = E_\phi(r, t)\hat{\phi}$ , and can be obtained from Faraday's law by equating its line integral over the circumference of radius  $r$  to the temporal derivative of the magnetic field flux through the circle:

$$E_\phi = -\frac{r}{2c} \partial_t B_{\text{ext}}(t). \quad (\text{S-8.91})$$

(We assumed the slowly varying current approximation, whose validity is ensured by the  $t_f \gg a/c$  condition.)

On an infinitesimal surface element of the cylindrical surface  $dS = ad\phi dz$  the induced electric field exerts a force

$$d\mathbf{f} = \phi d\mathbf{f} = \phi \sigma E_\phi(r = a) dS = -\phi \sigma \frac{a}{2c} \partial_t B_{\text{ext}}(t) dS, \quad (\text{S-8.92})$$

where  $\sigma = Q/(2\pi a)$  is the surface charge density. The corresponding mechanical torque is  $d\tau = \hat{\mathbf{z}} a d\mathbf{f}$ . By integrating over the whole surface of the cylinder we obtain for the total torque

$$\tau = -\frac{\pi a^3 h \sigma}{c} \partial_t \mathbf{B}_{\text{ext}}(t). \quad (\text{S-8.93})$$

The equation of motion for the rotation of the cylinder is

$$I \frac{d\omega}{dt} = \tau = -\frac{\pi a^3 h \sigma}{c} \partial_t \mathbf{B}_{\text{ext}}(t), \quad (\text{S-8.94})$$

with solution (the total time derivative being trivially equivalent to the partial derivative when applied to  $\mathbf{B}_{\text{ext}}(t)$ )

$$\omega(t) = -\frac{\pi a^3 h}{I c} \sigma [\mathbf{B}_{\text{ext}}(t) - \mathbf{B}_{\text{ext}}(0)] = -\frac{a^2 Q}{2 I c} [\mathbf{B}_{\text{ext}}(t) - \mathbf{B}_{\text{ext}}(0)]. \quad (\text{S-8.95})$$

The angular momentum is  $L_c(t) = I\omega(t)$ . The final values depend only on the initial value of  $\mathbf{B}_{\text{ext}}$  and not on its temporal profile,

$$\omega(t_f) = -\frac{a^2}{2 I c} Q \mathbf{B}_0. \quad (\text{S-8.96})$$

**b)** The rotation of the charged cylinder leads to a surface current  $\mathbf{K}$  at  $r = a$ ,

$$\mathbf{K} = \sigma v = \sigma a \omega \hat{\mathbf{z}}. \quad (\text{S-8.97})$$

This current generates a uniform magnetic field  $\mathbf{B}_{\text{ind}}$  inside the long cylinder (equivalent to a solenoid where  $nI = K$ ),

$$\mathbf{B}_{\text{ind}} = \frac{4\pi}{c} K \hat{\mathbf{z}} = \frac{4\pi}{c} \sigma a \omega. \quad (\text{S-8.98})$$

We now proceed as in point **a)** but adding the induced field  $\mathbf{B}_{\text{ind}}$  to the external field  $\mathbf{B}_{\text{ext}}$ :

$$\begin{aligned} I \frac{d\omega}{dt} = \tau &= -\frac{\pi a^3 h \sigma}{c} \partial_t [\mathbf{B}_{\text{ext}}(t) + \mathbf{B}_{\text{ind}}(t)] \\ &= -\frac{a^2 Q}{2c} \partial_t \left[ \mathbf{B}_{\text{ext}}(t) + \frac{4\pi}{c} \sigma a \omega \right] \\ &= -\frac{a^2 Q}{2c} \partial_t \mathbf{B}_{\text{ext}}(t) - \frac{a^2 Q^2}{hc^2} \frac{d\omega}{dt}, \end{aligned} \quad (\text{S-8.99})$$

which can be rewritten as

$$I' \frac{d\omega}{dt} = \tau = -\frac{\pi a^3 h \sigma}{c} \partial_t \mathbf{B}_{\text{ext}}(t), \quad (\text{S-8.100})$$

$$I' = I + \frac{a^2 Q^2}{hc^2}. \quad (\text{S-8.101})$$

Equation (S-8.100) is identical to (S-8.94) but for the replacement  $I \rightarrow I'$ , which means that the effects of the rotation-induced magnetic field  $\mathbf{B}_{\text{ind}}$  are equivalent to an additional inertia of the cylinder. The final velocity becomes

$$\omega'(t_f) = -\frac{a^2}{2I'c} Q \mathbf{B}_0. \quad (\text{S-8.102})$$

Notice that the total magnetic field does not vanish inside the cylinder at  $t = t_f$ , being equal to the induced field

$$\mathbf{B}_{\text{tot}}(t_f) = \mathbf{B}_{\text{ind}}(t_f) = \frac{4\pi}{c} \sigma a \omega'(t_f). \quad (\text{S-8.103})$$

**c)** For a magnetic field  $\mathbf{B} = B_z \hat{\mathbf{z}}$  and a configuration with cylindrical symmetry the density of EM angular momentum (8.18) becomes

$$\boldsymbol{\ell} \equiv \mathbf{r} \times \mathbf{g} = -\frac{1}{4\pi} r E_r B_z \hat{\mathbf{z}}. \quad (\text{S-8.104})$$

The contribution of the induced electric field  $E_\phi$  vanishes in the vector product. However, the angular momentum is not zero because of the radial electrostatic field inside the cylinder, which is easily found from Gauss's theorem:

$$E_r(r) = \begin{cases} \frac{2\lambda}{r} = -\frac{2Q}{hr} & (r < a) \\ 0 & (r > a) \end{cases}. \quad (\text{S-8.105})$$

Thus,  $\ell \neq 0$  inside the cylinder ( $r < a$ ). The total EM angular momentum is thus

$$\mathbf{L}_{\text{EM}} = \frac{1}{4\pi} B_z \int_0^a r \frac{2\lambda}{r} 2\pi r h dr = \frac{1}{2} B_z \lambda h a^2 \hat{\mathbf{z}} = \frac{Qa^2}{2} \mathbf{B}. \quad (\text{S-8.106})$$

Notice that  $\mathbf{B}$  represents the total field inside the cylinder and that the equation for  $\mathbf{L}_{\text{EM}}$  is valid at any time. Now, (S-8.95) can be rewritten (using the total field) as

$$I\omega(t) + \frac{a^2 Q}{2c} \mathbf{B}_{\text{ext}}(t) = \frac{a^2 Q}{2c} \mathbf{B}_{\text{ext}}(0), \quad (\text{S-8.107})$$

which is equivalent to

$$\mathbf{L}_c(t) + \mathbf{L}_{\text{EM}}(t) = \mathbf{L}_{\text{EM}}(0), \quad (\text{S-8.108})$$

thus showing that the total angular momentum of the system is conserved, since  $\mathbf{L}_c(0) = 0$ . The “paradox” thus consists in ignoring that a static EM field can contain a finite angular momentum. Similar considerations hold for Problem 6.6 where, however, the EM angular momentum is more difficult to calculate.<sup>1</sup>

## S-8.9 Magnetic Monopoles

- a) We build a magnetic dipole  $\mathbf{m}$  by locating two magnetic charges (magnetic monopoles)  $+q_m$  and  $-q_m$  at a distance  $\mathbf{h}$  from each other, so that  $\mathbf{m} = q_m \mathbf{h}$ . The magnetic field at distances  $r \gg h$  can be evaluated from (8.19), using the same approximations as for the field of an electric dipole, obtaining

$$\mathbf{B}_{\text{dip}} = \alpha \frac{(\mathbf{m} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - \mathbf{m}}{r^3}. \quad (\text{S-8.109})$$

---

<sup>1</sup>The present explanation of Feynman's “paradox” is taken from J. Belcher and K. McDonald ([http://cosmology.princeton.edu/~mcdonald/examples/feynman\\_cylinder.pdf](http://cosmology.princeton.edu/~mcdonald/examples/feynman_cylinder.pdf)) who further discuss subtle aspects of this problem.

On the other hand, the field of a usual magnetic dipole  $\mathbf{m} = I\mathbf{S}$ , consisting of a small circular loop of surface  $\mathbf{S}$  carrying a current  $I$ , with the head of  $\mathbf{S}$  pointing so that it “sees”  $I$  circulating counterclockwise, is

$$\mathbf{B}_{\text{dip}} = k_m \frac{(\mathbf{m} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - \mathbf{m}}{r^3}. \quad (\text{S-8.110})$$

Comparing the formulas, we obtain  $\alpha = k_m$ , i.e.,  $\alpha = \mu_0/4\pi = 1/4\pi\epsilon_0 c^2$  in SI units, and  $\alpha = 1/c$  in Gaussian units.

The magnetic force on an electric charge  $q_e$ , moving with velocity  $\mathbf{v}$  in the presence of a magnetic field  $\mathbf{B}$ , is  $\mathbf{f}_L = q_e b_m \mathbf{v} \times \mathbf{B}$ . The force exerted by a magnetic field  $\mathbf{B}$  on a magnetic monopole of charge  $q_m$  is  $\mathbf{f}_m = q_m \mathbf{B}$ . Thus the physical dimensions of the magnetic charge  $q_m$  are

$$[q_m] = [q_e b_m v] = \begin{cases} [q_e v], & \text{SI} \\ [q_e], & \text{Gaussian} \end{cases} \quad (\text{S-8.111})$$

i.e., the same physical dimensions as an electric charge in Gaussian units, and the dimensions of an electric charge times a velocity in SI units.

**b)** In analogy with the equation  $\nabla \cdot \mathbf{E} = 4\pi k_e \varrho_e$ , where  $\varrho_e$  is the volume density of electric charge, Maxwell's equation  $\nabla \cdot \mathbf{B} = 0$  is modified as

$$\nabla \cdot \mathbf{B} = 4\pi k_m \varrho_m = \begin{cases} \mu_0 \varrho_m & \text{SI} \\ \frac{4\pi}{c} \varrho_m & \text{Gaussian}, \end{cases} \quad (\text{S-8.112})$$

where  $\varrho_m$  is the volume density of magnetic charge. Equation (S-8.112) can be proved by first observing that, in the presence of magnetic charges, Gauss's law for the magnetic field is

$$\oint \mathbf{B} \cdot d\mathbf{S} = 4\pi k_m Q_m = 4\pi k_m \int \varrho_m d^3x, \quad (\text{S-8.113})$$

where the flux of  $B$  is evaluated through any closed surface, and  $Q_m$  is the net magnetic charge inside the surface, then applying the divergence theorem.

The conservation of magnetic charge is expressed by the continuity equation

$$\nabla \cdot \mathbf{J}_m = -\partial_t \varrho_m. \quad (\text{S-8.114})$$

Maxwell's equation for  $\nabla \times \mathbf{E}$  (describing Faraday's law of induction) must be completed in order to take the magnetic current density into account, by writing

$$\nabla \times \mathbf{E} = \eta \mathbf{J}_m - b_m \partial_t \mathbf{B}. \quad (\text{S-8.115})$$

The constant  $\eta$  can be determined, for instance, by applying the divergence operator to both sides of the equation, remembering the divergence of the curl of any vector field is always zero,

$$0 = \nabla \cdot (\nabla \times \mathbf{E}) = \eta \nabla \cdot \mathbf{J}_m - b_m \partial_t \nabla \cdot \mathbf{B} = \eta \nabla \cdot \mathbf{J}_m - 4\pi k_m \partial_t \varrho_m, \quad (\text{S-8.116})$$

from which  $\eta = -4\pi k_m$  follows. We thus obtain

$$\nabla \times \mathbf{E} = -4\pi k_m \mathbf{J}_m - b_m \partial_t \mathbf{B} = \begin{cases} -\mu_0 \mathbf{J}_m - \partial_t \mathbf{B}, & \text{SI} \\ -\frac{4\pi}{c} \mathbf{J}_m - \frac{1}{c} \mathbf{B}, & \text{Gaussian.} \end{cases} \quad (\text{S-8.117})$$

c) We choose a cylindrical reference frame  $(r, \phi, z)$  with the  $z$  axis coinciding with the axis of the beam. Because of the cylindrical symmetry of our magnetic charge distribution, the only non-zero component of the magnetic field is  $B_r$ . Applying Gauss's law to a cylindrical surface coaxial with the beam we obtain

$$B_r = \begin{cases} 2\pi k_m n q_m r, & r \leq a \\ \frac{2\pi k_m n q_m a^2}{r}, & r \geq a. \end{cases} \quad (\text{S-8.118})$$

The electric field  $\mathbf{E}$  is solenoidal and can be obtained by applying Kelvin-Stokes theorem to a circular path of radius  $r$  coaxial with the beam

$$\oint \mathbf{E} \cdot d\ell = 2\pi r E_\phi = \int \nabla \times \mathbf{E} \cdot d\mathbf{S} = \begin{cases} -\pi r^2 4\pi k_m n q_m v, & r \leq a \\ -\pi r^2 4\pi k_m n q_m \frac{a^2}{r} v, & r \geq a, \end{cases} \quad (\text{S-8.119})$$

leading finally to

$$E_\phi = \begin{cases} \frac{2\pi k_m n q_m v r}{r} & \text{if } r \leq a \\ \frac{2\pi k_m n q_m v a^2}{r} & \text{if } r \geq a. \end{cases} \quad (\text{S-8.120})$$

Thus, for instance for  $r \leq a$ , we have

$$E_\phi = \begin{cases} \frac{\mu_0 n q_m v r}{2}, & \text{SI} \\ \frac{2\pi n q_m v r}{c}, & \text{Gaussian.} \end{cases} \quad (\text{S-8.121})$$

# Chapter S-9

## Solutions for Chapter 9

### S-9.1 The Fields of a Current-Carrying Wire

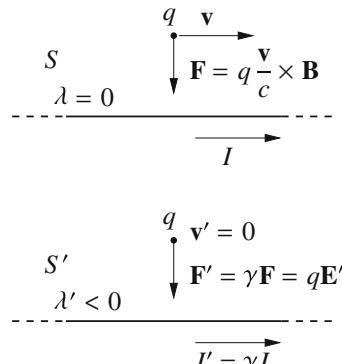
a) In the  $S$  reference frame the wire generates an azimuthal magnetic field  $\mathbf{B} = B_\phi(r)\hat{\phi}$ . In cylindrical coordinates we have  $B_\phi = B_\phi(r) = (2I/rc)$ . The Lorentz force on the charge  $q$  is

$$\mathbf{F} = q \frac{\mathbf{v}}{c} \times \mathbf{B} = \hat{\mathbf{r}} F_r = -\hat{\mathbf{r}} q B_\phi(r) \frac{v}{c} = -\hat{\mathbf{r}} \frac{2qIv}{rc^2}. \quad (\text{S-9.1})$$

The  $S'$  frame moves with velocity  $\mathbf{v}$  with respect to  $S$ . Applying the Lorentz transformations, in  $S'$  the force on  $q$  is  $\mathbf{F}' = \hat{\mathbf{r}} F'_r = \hat{\mathbf{r}} \gamma F_r$  (where  $\gamma = 1/\sqrt{1-\beta^2}$ , and  $\beta = v/c$  with  $v = |\mathbf{v}|$ , see Fig. S-9.1). Since  $q$  is at rest in  $S'$ , the force  $\mathbf{F}'$  is due to the electric field  $\mathbf{E}'$  only, with  $\mathbf{E}' = \hat{\mathbf{r}} E'_r = \hat{\mathbf{r}} F'_r/q$ . This corresponds to the transformation  $\mathbf{E}'_\perp = \hat{\mathbf{r}} E'_r = -\hat{\mathbf{r}} \gamma \beta B_\phi$  or, in vector form,

$$\mathbf{E}'_\perp = \gamma(\beta \times \mathbf{B}), \quad (\text{S-9.2})$$

where the subscript “ $\perp$ ” refers to the direction perpendicular to  $\mathbf{v}$ . At the limit  $|\mathbf{v}| \ll c$  (for which  $\mathbf{F} = \mathbf{F}'$ ) we get  $\mathbf{E}'_\perp \simeq \beta \times \mathbf{B}$ , which is



**Fig. S-9.1**

correct up to first order in  $\beta = v/c$ , and may be called the “Galilei” transformation of the field. The electric field<sup>1</sup>

$$E'_r = -\gamma\beta B_\phi(r') = -2\gamma\beta I/(r'c) \quad (\text{S-9.3})$$

is generated by a uniform linear charge density  $\lambda' = -\beta\gamma I/c$  on the wire, as can be easily verified by applying Gauss's law. Thus *the wire is negatively charged in S'*.<sup>2</sup>

Since the force is purely magnetic in  $S$  and purely electric in  $S'$ , at this point we cannot say much about the magnetic field in  $S'$ .

**b)** We know that  $J = (\rho c, \mathbf{J})$  is a four-vector. The cross-section  $W$  of the wire is invariant for a Lorentz boost along the wire axis, thus the linear charge density  $\lambda = W\rho$  and the electric current  $\mathbf{I} = W\mathbf{J}$  transform like  $\rho$  and  $\mathbf{J}$ . Therefore the linear charge density of the wire in  $S'$  is

$$\lambda' = \gamma\left(\lambda - \beta\frac{I}{c}\right) = -\gamma\beta\frac{I}{c}, \quad (\text{S-9.4})$$

which, according to Gauss's law, generates the radial electric field  $E'_r = 2\lambda'/r$ , in agreement with our result of point (a). We also obtain the current intensity in  $S'$ ,

$$I' = \gamma(I - \beta c \lambda) = \gamma I, \quad (\text{S-9.5})$$

which generates the magnetic field  $B'_\phi = 2I'/(r'c) = \gamma B_\phi$ .

The same results can be obtained through the transformation of the four-potential  $(\phi, \mathbf{A})$ . In  $S$ , we have obviously  $\phi = 0$ , since there is no net charge, while the vector potential  $\mathbf{A}$  satisfies the equation

$$\nabla^2 \mathbf{A} = -\frac{4\pi}{c} \mathbf{J} \quad (\text{S-9.6})$$

<sup>1</sup>In general, the complete transformation is  $\mathbf{E}'_\perp(\mathbf{r}', t) = \gamma\beta \times \mathbf{B}[\mathbf{r}(\mathbf{r}', t'), t(\mathbf{r}', t')]$ , where  $\mathbf{r} = \mathbf{r}(\mathbf{r}', t')$  and  $t = t(\mathbf{r}', t')$ , according to the Lorentz transformations of the coordinates. Since in cylindrical coordinates  $B_\phi$  depends on  $r$  only, and for the coordinates in the plane transverse to the boost velocity  $\mathbf{r}'_\perp = \mathbf{r}_\perp$ , in the present case we have the trivial transformation  $r' = r$ .

<sup>2</sup>It might seem that the law of charge conservation is violated in the transformation from  $S$  to  $S'$ . Actually, this a consequence of the somewhat “pathological” nature of currents which are not closed in a loop, as in the case of an infinite wire. In fact, strictly speaking, the infinite current-carrying wire is not a steady system, since charges of opposite sign are accumulating at the two “ends” of the wire, i.e., at  $z = \pm\infty$ . If we introduce “return” currents to close the loop in  $S$ , e.g., if we assume the wire to be the inner conductor of a coaxial cable, or if we add a second wire carrying the current  $-I$  at some distance, we find that the return currents would appear as opposite charge densities in  $S'$ , as required by charge conservation.

Thus,  $\mathbf{A}$  is parallel to the wire and its only non-zero component is  $A_z$ , which can be evaluated from the equation

$$\nabla^2 A_z = -\frac{4\pi}{c} I \delta(r). \quad (\text{S-9.7})$$

This is mathematically identical to the Poisson equation for the electrostatic potential of a uniformly charged wire, thus the solution is

$$A_z = -\frac{2I}{c} \ln\left(\frac{r}{a}\right), \quad (\text{S-9.8})$$

where  $a$  is an arbitrary constant. It is straightforward to verify that  $B_\phi = -\partial_r A_z$ .

The scalar potential in  $S'$  is

$$\phi' = \gamma(\phi - \beta A_z) = -\gamma\beta A_z = -\frac{2\gamma\beta I}{c} \ln\left(\frac{r}{a}\right), = -2\lambda' \ln\left(\frac{r}{a}\right), \quad (\text{S-9.9})$$

where  $\lambda' = -\beta\gamma I/c$ . The electric field is evaluated from  $\mathbf{E}' = -\nabla\phi'$ , obtaining the same result of point **a**). For the vector potential in  $S'$ , trivially  $A'_z = \gamma(A_z - \beta\phi) = \gamma A_z$  from which we get  $B'_\phi = \gamma B_\phi$  again.

These results are in agreement with the explicit formulas for the transformation of the EM field (9.3), which, in our case, lead to  $\mathbf{E}' = \gamma\beta \times \mathbf{B}$  and  $\mathbf{B}' = \gamma\mathbf{B}$ .

**c)** Let us first consider the linear charge densities of both ions ( $\lambda_i = Ze n_i W$ ) and electrons ( $\lambda_e = -en_e W$ ) in  $S$ , where  $\exists$  and  $n_e$  are the ion and electron volume densities, respectively. Since there is no net charge on the wire in  $S$ , we have  $\lambda_i = -\lambda_e$ .

Let us evaluate the charge densities  $\lambda'_i$  and  $\lambda'_e$  in  $S'$  from relativistic kinematics. In  $S$ , a wire segment of length  $\Delta L$  carries an ion charge  $\Delta Q = \lambda_i \Delta L$ . In  $S'$ , the segment has the same charge as in  $S$  (the charge is a Lorentz invariant), but the length undergoes a Lorentz contraction,  $\Delta L' = \Delta L/\gamma$ . Thus we have a higher charge density  $\lambda'_i = \Delta Q/\Delta L' = \gamma\lambda_i$ . This is a quite general result: in a frame where a fluid moves at velocity  $v$ , the fluid has a higher density (by a factor  $\gamma$ ) than in its rest frame.

On the other hand, the electrons are *not* at rest in  $S$ : they move along the wire with a velocity  $v_e < 0$  such that  $I = -en_e v_e W = \lambda_e v_e = -\lambda_i v_e$ . Thus, their density is already *higher* by a factor  $\gamma_e = 1/\sqrt{1-v_e^2/c^2}$  than the density  $\lambda_{e0}$  in the rest frame of the electrons: we have  $\lambda_{e0} = \lambda_e/\gamma_e$ . In  $S'$ , the electrons drift with a velocity  $v'_e$

$$v'_e = \frac{v_e - v}{1 - v_e v/c^2}, \quad (\text{S-9.10})$$

according to Lorentz transformations. Thus, the electron density in  $S'$  is

$$\lambda'_e = \gamma'_e \lambda_{e0} = \frac{\gamma'_e}{\gamma_e} \lambda_e , \quad (\text{S-9.11})$$

where  $\gamma'_e = 1/\sqrt{1 - v_e'^2/c^2}$ . The expression for  $\gamma'$  can be put in a more convenient form by some algebra:

$$\begin{aligned} \gamma'_e &= \frac{1}{\sqrt{1 - \frac{(v_e - v)^2}{c^2 \left(1 - \frac{v_e v}{c^2}\right)^2}}} = \sqrt{\frac{\left(1 - \frac{v_e v}{c^2}\right)^2}{\left(1 - \frac{v_e v}{c^2}\right)^2 - \frac{(v_e - v)^2}{c^2}}} \\ &= \left(1 - \frac{v_e v}{c^2}\right) \frac{1}{\sqrt{1 - 2 \frac{v_e v}{c^2} + \frac{v_e^2 v^2}{c^4} - \frac{v_e^2}{c^2} + 2 \frac{v_e v}{c^2} - \frac{v^2}{c^2}}} \\ &= \left(1 - \frac{v_e v}{c^2}\right) \frac{1}{\sqrt{\left(1 - \frac{v_e^2}{c^2}\right)\left(1 - \frac{v^2}{c^2}\right)}} \\ &= \left(1 - \frac{v_e v}{c^2}\right) \gamma_e \gamma . \end{aligned} \quad (\text{S-9.12})$$

We thus obtain for the *total* charge density in  $S'$

$$\begin{aligned} \lambda' &= \lambda'_i + \lambda'_e = \lambda_i \left(\gamma - \frac{\gamma'_e}{\gamma_e}\right) = \lambda_i \gamma \left(1 - 1 + \frac{v_e v}{c^2}\right) = \lambda_i \gamma \frac{v_e v}{c^2} \\ &= -\gamma v \frac{I}{c^2} , \end{aligned} \quad (\text{S-9.13})$$

as previously found on the basis of Lorentz transformations for the forces, charge and current densities, and EM fields.

It might be interesting to remark that there is an issue of charge conservation already in the  $S$  frame. The wire is electrically neutral, thus its ion and electron charge densities are exactly equal and opposite when it is disconnected from any voltage or current source, and in the absence of external fields. Now assume that we drive a steady current  $I$  through the wire, keeping the conduction electrons in motion with a velocity  $v_e$  along the wire axis. If the wire is still electrically neutral, as we assumed, the absolute values of the charge densities of ions and electrons must still be equal and opposite. However, while the charge density of the ions, at rest, has not changed, the charge density of the moving electrons undergoes a “relativistic increase” by a factor  $\gamma_e$ . If the total charge density does not change (the wire must

still be neutral), some electrons must have left the wire.<sup>3</sup> We can explain where the missing electrons have gone only by recalling that the wire is not “open”, but must be part of a closed current loop, with specific boundary conditions and how the circuit is closed.

## S-9.2 The Fields of a Plane Capacitor

**a)** We choose a Cartesian coordinate system with the  $y$  axis perpendicular to the plates, so that the lower plate is at  $y = 0$  and the upper plate at  $y = h$ , and the  $x$  axis parallel to  $v$ , so that  $v = \beta c \hat{x}$ . The only non-vanishing component of the EM field in  $S$  is  $E_y = 4\pi\sigma$ . By applying a Lorentz transformation we find for the fields in  $S'$

$$E'_y = \gamma E_y = 4\pi\gamma\sigma, \quad B'_z = -\beta\gamma E_y = -4\pi\beta\gamma\sigma. \quad (\text{S-9.14})$$

**b)** In  $S'$  the electric field  $E'_y$  is generated by the surface charge densities  $\pm\sigma' = \pm E'_y/4\pi = \pm\gamma\sigma$  on the capacitor plates. Similarly, the magnetic field  $B'_z$  is generated by the two surface current densities  $\pm\mathbf{K}' = \pm K'_x \hat{x}$  with  $K'_x = cB'_z/(4\pi) = -\beta\gamma\sigma c$ , flowing on the two capacitor plates.

These results are in agreement with the Lorentz transformation of the four-vector

$$K_\mu = (c\sigma, \mathbf{K}). \quad (\text{S-9.15})$$

We can check that  $K_\mu$  is actually a four-vector, by imagining two volume four-current densities  $J_\mu = (c\rho, \pm\mathbf{J})$  distributed over the two thin layers,  $|y| < \delta/2$  and  $|h - y| < \delta/2$ , around the capacitor plates, such that  $\sigma = \rho\delta$  and  $\mathbf{K} = \mathbf{J}\delta$ . Since  $\delta$  is invariant for transformations with velocity parallel to  $\mathbf{J}$ , it follows that also  $K_\mu \equiv J_\mu \delta$  transforms as a four-vector:

$$\sigma' = \gamma(\sigma - \beta K_x/c) = \gamma\sigma, \quad K'_x = \gamma(K_x - \beta c\sigma) = -\beta\gamma\sigma c. \quad (\text{S-9.16})$$

**c)** In  $S$  there is a perpendicular force per unit surface  $p = \sigma E_y/2 = 2\pi\sigma^2$  on the internal surfaces of the plates, such that the plates attract each other. In  $S'$ , the force per unit surface is the sum of two terms of electrostatic and magnetic nature, respectively,

$$p' = \frac{1}{2}\sigma'E'_y + \frac{1}{2}K'_x B'_z = 2\pi\sigma^2\gamma^2 - 2\pi\sigma^2\beta^2\gamma^2 = 2\pi\sigma^2\gamma^2(1 - \beta^2) = 2\pi\sigma^2 \\ = p. \quad (\text{S-9.17})$$

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<sup>3</sup>Of course, the effect is negligibly small for ordinary conduction in metals, for which the typical electron velocities  $v_e$  are of the order of  $10^{-10} c$ . On the other hand, this issue is very important for relativistic hydrodynamics, i.e., for contexts where fluids move at velocities close to  $c$ .

The invariance of  $p$  is also proven from the equivalent expression

$$p' = \frac{1}{8\pi} E_y'^2 - \frac{1}{8\pi} B_z'^2 = \frac{1}{8\pi} (\mathbf{E}'^2 - \mathbf{B}'^2), \quad (\text{S-9.18})$$

which is a Lorentz invariant.

In  $S$ , the total force is  $F = pA$ . In  $S'$ , due to the Lorentz contraction of lengths,  $A' = (L/\gamma)L = A/\gamma$ , so that  $F' = p'A' = pA/\gamma = F/\gamma$ .

### S-9.3 The Fields of a Solenoid

**a)** We choose a Cartesian reference frame with the solenoid axis as  $z$  axis, and the  $x$  axis such that  $\mathbf{v} = v_x \hat{\mathbf{x}}$ . In addition, we shall also use a cylindrical reference frame sharing the  $z$  axis with the Cartesian frame, and with the azimuthal coordinate  $\phi$  such that the  $\phi = 0$  plane coincides with the  $xz$  plane. In  $S$ , the magnetic field inside the solenoid is longitudinal and uniform,  $\mathbf{B} = B \hat{\mathbf{z}}$ , with  $B = 4\pi nI/c$ , and the force on  $q$  is  $\mathbf{F} = q\mathbf{v} \times \mathbf{B}/c = -q\beta B \hat{\mathbf{y}}$ .

In the  $S'$  frame the charge  $q$  is at rest, thus the force on it must be due to an electric field only. According to the Lorentz transformations of the fields we have

$$\begin{aligned} E'_x &= E_x = 0, & B'_x &= B_x = 0, \\ E'_y &= \gamma(E_y - \beta B_z) = -\gamma\beta B, & B'_y &= \gamma(B_y + \beta E_z) = 0, \\ E'_z &= \gamma(E_z + \beta B_y) = 0, & B'_z &= \gamma(B_z - \beta E_y) = \gamma B, \end{aligned} \quad (\text{S-9.19})$$

and the force on  $q$  is thus  $\mathbf{F}' = qE'_y \hat{\mathbf{y}} = -q\gamma\beta B_z \hat{\mathbf{y}} = \gamma\mathbf{F}$ .

**b)** Since we are assuming  $\beta \ll 1$ , we have  $\gamma = 1/\sqrt{1-\beta^2} = 1 + \beta^2/2 + \dots \approx 1$  up to the first order in  $\beta$ , and we can neglect the relativistic contraction of lengths. Thus the cross-section of the solenoid remains circular in  $S'$  to within our approximations. The electric field outside the solenoid is zero (we discuss this point further below), thus the electric field component perpendicular to the solenoid winding surface is discontinuous, implying the presence of surface charge density  $\sigma'$ . We have from Gauss's theorem

$$\sigma' = \frac{E'_\perp}{4\pi} = \frac{E'_y}{4\pi} \sin\phi = -\beta \frac{B}{4\pi} \sin\phi = -\beta n \frac{I}{c} \sin\phi, \quad (\text{S-9.20})$$

where the subscript  $\perp$  means perpendicular to the solenoid winding surface.

This result is in agreement with the transformation laws for the four-vector  $K_\mu = (c\sigma, \mathbf{K})$ , where  $\mathbf{K}$  is the surface current density on the walls of the solenoid (see Problem 9.2). In  $S$  we have  $\mathbf{K} = nI\hat{\phi} = nI(-\hat{\mathbf{x}} \sin\phi + \hat{\mathbf{y}} \cos\phi)$ , and in  $S'$

$$\sigma' = \gamma \left( \sigma - \beta \frac{K_x}{c} \right) \approx -\beta \frac{K_x}{c} = \beta n \frac{I}{c} \sin\phi. \quad (\text{S-9.21})$$

A surface charge density varying as  $\sin\phi$  on the lateral surface of an infinite cylinder generates a uniform electrostatic field inside the cylinder, as seen in the solution of Problem 3.11. But there we also saw that surface charge density generates a “two-dimensional dipole” field *outside* the cylinder. This might seem in contradiction with the fact that, since the external EM field is zero in the  $S$  frame, it must be zero in  $S'$  as well. But there are *not only static fields* in  $S'$ , because the transverse motion of the solenoid generates a time-dependent magnetic field, which, in turn, is related to a non-conservative electric field and to boundary conditions which are different from the static case.

We start by noting that an electric field that is uniform (and nonzero) inside the solenoid, and zero outside, is not conservative. Let us choose a rectangular path  $C$  of sides  $a$  and  $b$  crossing the solenoid winding as in Fig. S-9.2. The path  $C$  is at rest in  $S'$ , while the solenoid moves toward the left with velocity  $-v$ . At  $t = 0$  the upper side of length  $a$  is tangent to the winding at its central point, and  $a$  is sufficiently small for the enclosed winding arc to be well approximated by a straight line segment. We also have  $b \ll a$ . The field  $\mathbf{E}'$  is not conservative because the line integral of  $\mathbf{E}'$  along  $C$  does not vanish:

$$\oint_C \mathbf{E}' \cdot d\ell = E'_\parallel a = E'_y a \cos\phi. \quad (\text{S-9.22})$$

This is consistent with the fact that the flux of  $\mathbf{B}'$  through the rectangle enclosed by the path  $C$  is time-dependent. The winding arc enclosed by the rectangle moves towards the lower side of length  $a$  with velocity  $v \cos\phi$ , and the flux of  $\mathbf{B}'$  through the rectangle is

$$\Phi_C(\mathbf{B}') = \int_C \mathbf{B}' \cdot d\mathbf{S} = B' a [b - (v \cos\phi)t], \quad (\text{S-9.23})$$

corresponding to a line integral

$$-\frac{1}{c} \frac{d\Phi_C(\mathbf{B}')}{dt} = \frac{v}{c} B' a \cos\phi = E' y a \cos\phi, \quad (\text{S-9.24})$$

in agreement with (S-9.22).

## S-9.4 The Four-Potential of a Plane Wave

- a) The fields of the plane wave may be written in complex notation as

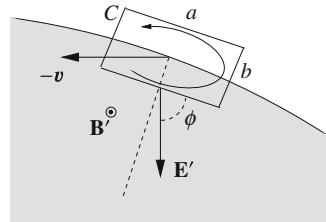


Fig. S-9.2

$$\mathbf{E} = \hat{\mathbf{y}} E_0 e^{ikx-i\omega t}, \quad \mathbf{B} = \hat{\mathbf{z}} B_0 e^{ikx-i\omega t}, \quad (\text{S-9.25})$$

with  $E_0 = B_0$ . A vector potential of the form  $\mathbf{A} = \hat{\mathbf{y}} A_0 e^{ikx-i\omega t}$  generates an electric field along  $\hat{\mathbf{y}}$  and a magnetic field along  $\hat{\mathbf{z}}$  given by

$$\mathbf{E} = -\frac{1}{c} \partial_t \mathbf{A} - \nabla \varphi, \quad \mathbf{B} = \nabla \times \mathbf{A}. \quad (\text{S-9.26})$$

In the absence of electric charges we have  $\varphi \equiv 0$ , and we obtain from (S-9.26)

$$A_0 = -\frac{ic}{\omega} E_0, \quad A_0 = -\frac{i}{k} B_0, \quad (\text{S-9.27})$$

which are equivalent since  $\omega = kc$ . The vector potential  $\mathbf{A} = \hat{\mathbf{y}} A_0 e^{ikx-i\omega t}$  obviously satisfies the wave equation in vacuum, and also respects the Lorenz gauge condition.

**b)** The Lorentz transformations from  $S$  to  $S'$  give  $\omega' = \gamma\omega$  (transverse Doppler effect), and  $k'_x = k_x = k$ ,  $k'_y = -\omega'v/c^2 = -\gamma\beta k_x$ . The nonzero components of the fields in  $S'$  are  $E'_y = E_y$ ,  $E'_x = \gamma\beta B_z$ , and  $B'_z = \gamma B_z$ . We may thus write

$$\mathbf{E}' = (\hat{\mathbf{x}}\gamma\beta + \hat{\mathbf{y}})E_0 e^{i(k'_x x' + k'_y y' - \omega' t')}, \quad \mathbf{B}' = \hat{\mathbf{z}}\gamma B_0 e^{i(k'_x x' + k'_y y' - \omega' t')}, \quad (\text{S-9.28})$$

The polarization is linear and directed along the unit vector  $\boldsymbol{\epsilon} = \beta\hat{\mathbf{x}} + \hat{\mathbf{y}}/\gamma$ .

**c)** Assuming  $\varphi' = 0$ , we have in  $S'$

$$\mathbf{E}' = -\frac{1}{c} \partial'_t \mathbf{A}', \quad \mathbf{B}' = \nabla' \times \mathbf{A}', \quad (\text{S-9.29})$$

which are both satisfied if we choose

$$\mathbf{A}' = -\frac{c}{\omega'} \mathbf{E}' = \left( \hat{\mathbf{x}}\beta + \hat{\mathbf{y}}\frac{1}{\gamma} \right) A_0 e^{i(k'_x x' + k'_y y' - \omega' t')}, \quad (\text{S-9.30})$$

being  $\omega' = \gamma\omega$ .

**d)** The Lorentz transformation from  $S$  to  $S'$  for the four-potential  $A_\mu = (\varphi, \mathbf{A}) = (0, 0, A_y, 0)$  gives

$$\bar{A}'_\mu = (-\gamma\beta A_y, 0, \gamma A_y, 0) \equiv (\bar{\varphi}', 0, \bar{A}'_y, 0). \quad (\text{S-9.31})$$

The fields derived from this four-potential are

$$\bar{E}'_x = -\partial'_x \bar{\varphi}' = -ik'_x \bar{\varphi}' = ikc\gamma\beta A_y = \gamma\beta(i\omega A_y) = \gamma\beta E_y = E'_x, \quad (\text{S-9.32})$$

$$\begin{aligned} \bar{E}'_y &= -\frac{1}{c}\partial'_t \bar{A}'_y - \partial'_y \bar{\varphi}' = i\frac{\omega'}{c}\bar{A}'_y - ik'_y \bar{\varphi}' = i\left(\gamma\frac{\omega}{c}\right)\gamma A_y - i(-\gamma\beta k)(-\gamma\beta c)A_y \\ &= i\frac{\omega}{c}\gamma^2(1-\beta^2)A_y = E_y = E'_y, \end{aligned} \quad (\text{S-9.33})$$

$$\bar{B}'_z = \partial'_x \bar{A}'_y = ik'_x \bar{A}'_y = ik\gamma A_y = \gamma B_z = B'_z, \quad (\text{S-9.34})$$

in agreement with the results of point **c**).

- e)** The expressions  $A'_\mu = (0, \mathbf{A}')$  and  $\bar{A}'_\mu = (\bar{\varphi}', \bar{\mathbf{A}}')$  are two possible choices for the four-potential. Thus they must differ at most by a gauge transformation, i.e., there must be a scalar function  $f = f(x', t')$  such that

$$\mathbf{A}' = \bar{\mathbf{A}}' + \nabla' f, \quad \varphi' = \bar{\varphi}' - \frac{1}{c}\partial'_t f. \quad (\text{S-9.35})$$

Since  $\varphi' = 0$  we find  $\partial'_t f/c = \bar{\varphi}'$ , i.e.,

$$f = \frac{c}{\omega'}\bar{\varphi}' = -\frac{ic}{\omega'}\gamma\beta A_y, \quad (\text{S-9.36})$$

Now, since

$$\nabla' f = (\hat{\mathbf{x}}ik'_x + \hat{\mathbf{y}}ik'_y)f = (\hat{\mathbf{x}}\beta - \hat{\mathbf{y}}\gamma\beta^2)A_y, \quad (\text{S-9.37})$$

we also have that

$$\bar{\mathbf{A}}' + \nabla' f = [\hat{\mathbf{x}}\beta + \hat{\mathbf{y}}(\gamma - \gamma\beta^2)]A_y = \left(\hat{\mathbf{x}}\beta + \hat{\mathbf{y}}\frac{1}{\gamma}\right)A_y = \mathbf{A}'. \quad (\text{S-9.38})$$

## S-9.5 The Force on a Magnetic Monopole

- a)** In the reference frame  $S'$ , where the magnetic monopole is at rest ( $\mathbf{v}' = 0$ ), the magnetic field is

$$\mathbf{B}' = -\gamma \frac{\mathbf{v}}{c} \times \mathbf{E}, \quad (\text{S-9.39})$$

thus the force on the monopole is  $\mathbf{F}' = q_m \mathbf{B}'$ . On the other hand we must have  $\mathbf{F}' = \gamma \mathbf{F}$ , since  $\mathbf{F}$  is perpendicular to  $\mathbf{v}$ , so that in the laboratory frame  $S$  we have

$$\mathbf{F} = \frac{q_m}{\gamma} \mathbf{B}' = -q_m \frac{\mathbf{v}}{c} \times \mathbf{E}, \quad (\text{S-9.40})$$

which proves (9.8).

- b)** The equation of motion for a magnetic monopole in the presence of a uniform electric field  $\mathbf{E} = \hat{\mathbf{z}} E$  alone is identical to the equation of motion for a an electric charge in the presence of a uniform magnetic field  $\mathbf{B} = \hat{\mathbf{z}} B$ , after replacing  $-q_m \mathbf{E}$  by  $q \mathbf{B}$ . The solution is a helicoidal motion, with a constant drift velocity parallel to  $\mathbf{E}$ , and a constant angular velocity  $\omega_m = \hat{\mathbf{z}} q_m E / mc$ . (Notice that, for a magnetic monopole, the angular velocity vector is parallel to  $\mathbf{E}$ , while it is antiparallel to  $\mathbf{B}$  in the case of an electric charge.)

In the case of crossed electric and magnetic fields, the condition  $E > B$  ensures that there is a reference frame  $S'$  where the magnetic field vanishes. In fact, taking a Lorentz boost with  $\beta = (\mathbf{E} \times \mathbf{B})/E^2$  we have

$$\mathbf{B}' = \gamma(\mathbf{B} - \beta \times \mathbf{E}) = \gamma \left( \mathbf{B} + \frac{E^2 \mathbf{B} - (\mathbf{E} \cdot \mathbf{B}) \mathbf{E}}{E^2} \right) = 0, \quad (\text{S-9.41})$$

since  $\mathbf{E} \cdot \mathbf{B} = 0$ . Thus, in the boosted frame there is only the electric field

$$\mathbf{E}' = \gamma(\mathbf{E} + \beta \times \mathbf{B}) = \gamma \left( \mathbf{E} - \frac{B^2}{E^2} \mathbf{E} \right) = \frac{\mathbf{E}}{\gamma}, \quad (\text{S-9.42})$$

since  $\gamma = 1/\sqrt{1-\beta^2} = 1/\sqrt{1-B^2/E^2}$ . Thus the motion in  $S'$  is a circular orbit with angular frequency  $\omega' = (q_m E / \gamma c)$ . By transforming back to the laboratory frame  $S$  we add a drift velocity  $-c\beta$ , and the trajectory in  $S$  is a cycloid.

## S-9.6 Reflection from a Moving Mirror

- a)** As an ansatz, we write the total electromagnetic field as the sum of the fields of the incident wave and the fields of a reflected wave of the same frequency and polarization, but opposite direction

$$\mathbf{E}(x, t) = \hat{\mathbf{y}} E_y(x, t), \quad \mathbf{B} = \hat{\mathbf{z}} B_z(x, t), \quad (\text{S-9.43})$$

$$E_y(x, t) = \operatorname{Re} \left( E_i e^{ikx-i\omega t} + E_r e^{-ikx-i\omega t} \right), \quad (\text{S-9.43})$$

$$B_z(x, t) = \operatorname{Re} \left( E_i e^{ikx-i\omega t} - E_r e^{-ikx-i\omega t} \right). \quad (\text{S-9.44})$$

The amplitude of the reflected wave  $E_r$  must be determined by the boundary condition at the mirror surface  $x = 0$ . We may already know that the electric field component parallel to the bounding surface between two media is continuous across the surface, i.e., that  $E_{||}(0^-) = E_{||}(0^+)$ . However, here we prefer to derive this result in detail, because this will help the discussion of the reflection at the surface of a *moving* mirror, which we shall consider in the following. Evaluating the line integral of  $\mathbf{E}$  over a closed rectangular loop across the boundary, as in Fig. S-9.3, yields

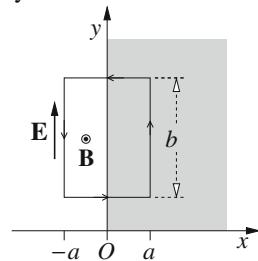


Fig. S-9.3

$$\oint \mathbf{E} \cdot d\ell = [E_y(a, t) - E_y(-a, t)]b \\ = -\frac{1}{c} \frac{d\Phi(\mathbf{B})}{dt} = -\frac{b}{c} \int_{-a}^0 \partial_t B_z dx \\ = \frac{i\omega b}{c} \int_{-a}^0 B_z dx = \frac{i\omega}{c} \bar{B}_z ab, \quad (\text{S-9.45})$$

where  $\bar{B}_z$  is the mean value of  $B_z$  in the  $(-a, a)$  interval. If  $B_z$  is finite, the “rightmost RHS” of (S-9.45) vanishes at the limit  $a \rightarrow 0$ , and  $E_y(0^+, t) = E_y(0^-, t)$ .

For a perfect mirror we must have  $E_y(0^+, t) = 0$ , and the boundary condition implies that also  $E_y(0^-, t) = 0$ . Thus we obtain

$$E_y(0, t) = (E_i + E_r) e^{-i\omega t} = 0, \quad E_r = -E_i. \quad (\text{S-9.46})$$

The total electric field for  $x \leq 0$  is thus a standing wave

$$E_y = E_i (e^{ikx-i\omega t} - e^{-ikx-i\omega t}) = 2iE_y \sin(kx)e^{-i\omega t}, \quad (\text{S-9.47})$$

with nodes where  $\sin kx = 0$  and maximum amplitude  $2E_i$ . Recalling that  $\omega/k = c$ , the magnetic field of the wave is

$$B_z = 2E_i \cos(kx)e^{-i\omega t}. \quad (\text{S-9.48})$$

Thus,  $B_z$  is *discontinuous* at the  $x = 0$  surface. This implies the presence of a surface current density  $\mathbf{K} = \hat{\mathbf{y}} K_y(t)$  at  $x = 0$ , corresponding to a volume current density  $\mathbf{J} = \mathbf{K}\delta(x) = \hat{\mathbf{y}} K_y(t)\delta(x)$ . By evaluating the line integral of  $\mathbf{B}$  over a closed path crossing the mirror surface we find the boundary condition

$$B_z(0^+, t) - B_z(0^-, t) = \frac{4\pi}{c} K_y(t), \quad (\text{S-9.49})$$

and the surface current density on the surface of a perfect mirror is

$$K_y(t) = -\frac{c}{4\pi} B_z(0^-, t) = -\frac{cE_i}{2\pi} e^{-i\omega t}. \quad (\text{S-9.50})$$

**b)** Let  $\beta = v/c$  (in what follows  $v$ , and, consequently,  $\beta$ , may have both positive or negative values, depending on whether the wave and the mirror velocity are parallel or antiparallel, respectively). We know that  $(\omega/c, \mathbf{k})$  is a four-vector, and that  $\mathbf{k}$  is parallel to  $\mathbf{v}$ . Thus the frequency of the incident wave in  $S'$  is

$$\omega'_i = \gamma(\omega - \mathbf{k} \cdot \mathbf{v}) = \gamma\omega(1 - \beta), \quad (\text{S-9.51})$$

where  $k = \omega/c$  has been used. The magnitude of the incident wave vector in  $S'$  is  $k'_i = \omega'_i/c$ . If  $v > 0$  ( $v < 0$ ) we have  $\omega'_i < \omega$  ( $\omega'_i > \omega$ ).

The Lorentz transformations give the following amplitudes for the fields in  $S'$

$$E'_{iy} = \gamma(E_{iy} - \beta B_{iz}) = \gamma(1 - \beta) E_i, \quad (\text{S-9.52})$$

$$B'_{iz} = \gamma(B_{iz} - \beta E_{iy}) = \gamma(1 - \beta) E_i, \quad (\text{S-9.53})$$

since  $B_{iz} = E_{iy}$ . In the  $S'$  frame the reflected wave has frequency  $\omega'_r = \omega'_i$ , and field amplitudes  $E'_{ry} = -E'_{iy}$ ,  $B'_{rz} = B'_{iz}$ .

**c)** The frequency  $\omega_r$  of the reflected wave in the laboratory frame  $S$  can be evaluated by applying the inverse transformation from  $S'$  to  $S$

$$\begin{aligned} \omega_r &= \gamma(\omega'_r + \mathbf{k}_r \cdot \mathbf{v}) = \gamma(\omega'_r - k_r v) = \gamma\omega'_r(1 - \beta) = \omega\gamma^2(1 - \beta)^2 \\ &= \omega \frac{1 - \beta}{1 + \beta}. \end{aligned} \quad (\text{S-9.54})$$

The electric and magnetic field amplitudes of the reflected wave in  $S'$  are  $E'_r = -E'_i = -\gamma(1 - \beta)E_i$  and  $B'_r = B'_i = \gamma(1 - \beta)E_i$ . We thus have in  $S$

$$E_{ry} = \gamma(E'_{ry} + \beta B'_{rz}) = -\gamma(1 - \beta)E'_i = -\gamma^2(1 - \beta)^2 E_i = -\frac{1 - \beta}{1 + \beta} E_i, \quad (\text{S-9.55})$$

$$B_{rz} = \gamma(B'_{rz} + \beta E'_{ry}) = \gamma(1 - \beta)E'_i = \gamma^2(1 - \beta)^2 E_i = \frac{1 - \beta}{1 + \beta} E_i. \quad (\text{S-9.56})$$

If  $\beta < 0$  we have  $|E_r| > |E_i|$ : in  $S$  the reflected wave has a higher amplitude than the incident wave.

d) The complete expressions for the fields in  $S$  are

$$E_y(x, t) = E_i e^{ikx - i\omega t} - \frac{1 - \beta}{1 + \beta} E_i e^{-ik_r x - i\omega_r t}, \quad (\text{S-9.57})$$

$$B_z(x, t) = E_i e^{ikx - i\omega t} + \frac{1 - \beta}{1 + \beta} E_i e^{-ik_r x - i\omega_r t}, \quad (\text{S-9.58})$$

thus, also  $E_y$  has a finite value at the mirror surface  $x(t) = vt$ , and is therefore *discontinuous*:

$$E_y[x(t), t] = \frac{2\beta}{1 + \beta} E_i e^{-i(1-\beta)\omega t}, \quad (\text{S-9.59})$$

$$B_z[x(t), t] = \frac{2}{1 + \beta} E_i e^{-i(1-\beta)\omega t}. \quad (\text{S-9.60})$$

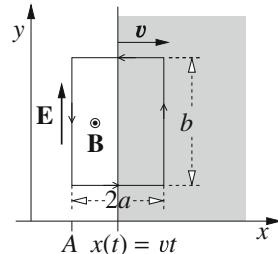


Fig. S-9.4

This can be seen by considering again the line integral of the electric field  $\mathbf{E}$  along a closed rectangular path of sides  $2a$  and  $b$ , at rest in  $S$ . We assume that the left vertical side of the path is on the  $x = A$  line, that at time  $t$  the mirror surface cuts the two horizontal sides, as in Fig. S-9.4, and that  $a \ll \lambda$ , where  $\lambda$  is the wavelength in  $S$ . The flux of the magnetic field through the rectangular path at time  $t$  is thus

$$\Phi(t) \approx B_z[x(t), t][x(t) - A]b = B_z[x(t), t](vt - A)b, \quad (\text{S-9.61})$$

so that

$$-\frac{1}{c} \frac{d\Phi(t)}{dt} \approx -\frac{1}{c} [\partial_t B_z[x(t), t](vt - A)b + B_z[x(t), t]vb]. \quad (\text{S-9.62})$$

At the limit  $a \rightarrow 0, A \rightarrow vt$ , the first term of the right-hand side vanishes, and we are left with

$$-\frac{1}{c} \frac{d\Phi(t)}{dt} \approx -\frac{1}{c} B_z[x(t), t]vb = -B_z[x(t), t]\beta b. \quad (\text{S-9.63})$$

On the other hand, the line integral of  $\mathbf{E}$  along the closed rectangular path of Fig. S-9.4 is

$$\oint \mathbf{E} \cdot d\ell = -E_y[x(t), t]b = -\frac{2\beta}{1 + \beta} b E_i e^{-i(1-\beta)\omega t} = -B_z[x(t), t]\beta b. \quad (\text{S-9.64})$$

## S-9.7 Oblique Incidence on a Moving Mirror

**a)** We choose a Cartesian reference frame  $S$  where  $v$  is parallel to the  $x$  axis, the mirror surface lies on the  $yz$  plane and the wave vector  $\mathbf{k}_i$  of the incident wave lies in the  $xy$  plane. The Lorentz transformations to the frame  $S'$  give

$$k'_{ix} = \gamma \left( k_{ix} - \omega \frac{v}{c^2} \right) = \gamma \frac{\omega}{c} (\cos \theta_i - \beta), \quad (\text{S-9.65})$$

$$k'_{iy} = k_{iy}, \quad (\text{S-9.66})$$

$$\omega'_i = \gamma(\omega_i - k_x v) = \gamma\omega_i(1 - \beta \cos \theta_i), \quad (\text{S-9.67})$$

$$\tan \theta'_i = \frac{k'_{iy}}{k'_{ix}} = \frac{k_{ix} \tan \theta_i}{\gamma(\omega_i/c)(\cos \theta_i - \beta)} = \frac{k_{ix} \sin \theta_i}{\gamma k_{ix}(\cos \theta_i - \beta)} = \frac{\sin \theta_i}{\gamma(\cos \theta_i - \beta)}, \quad (\text{S-9.68})$$

where, as usual,  $\beta = v/c$  and  $\gamma = 1/\sqrt{1-\beta^2}$ . In  $S'$  the reflection angle  $\theta'_r$  equals the incidence angle  $\theta'_i$ , thus

$$k'_{rx} = -k'_{ix}, \quad k'_{ry} = k'_{iy}, \quad \omega'_r = \omega'_i. \quad (\text{S-9.69})$$

**b)** By performing the Lorentz transformations back to the laboratory frame  $S$  we obtain

$$k_{ry} = k_{iy}, \quad (\text{S-9.70})$$

$$\begin{aligned} k_{rx} &= \gamma \left( k'_{rx} + \omega' \frac{v}{c^2} \right) = -\gamma^2 \left[ k_{ix} (1 + \beta^2) - 2\omega_i \frac{\beta}{c} \right] \\ &= -2\gamma^2 \frac{\omega}{c} [(1 + \beta^2) \cos \theta_i - 2\beta], \end{aligned} \quad (\text{S-9.71})$$

$$\omega_r = \gamma(\omega'_r + k'_{rx} v) = \gamma^2 \left[ \omega (1 + \beta^2) - 2k_{ix} v \right] = \gamma^2 \frac{\omega}{c} (1 + \beta^2 - 2\beta \cos \theta_i), \quad (\text{S-9.72})$$

from which

$$\tan \theta_r \equiv -\frac{k_{ry}}{k_{rx}} = \frac{\sin \theta_i}{\gamma^2 [2\beta - (1 + \beta^2) \cos \theta_i]}, \quad (\text{S-9.73})$$

For  $\cos \theta_i = v/c = \beta$  the denominator of the “rightmost right-hand side” of (S-9.68) is zero, and the incidence angle  $\theta'_i$  in  $S'$  is a right angle. This means that, in  $S'$ , the incident wave propagates parallel to the mirror surface, without hitting the mirror, and no reflection occurs. For incidence angles such that  $\cos \theta_i > \beta$ , all the above formulas are meaningless, since they would imply  $k'_{ix} < 0$ , i.e., that the wave is incident on the other side of the mirror.

## S-9.8 Pulse Modification by a Moving Mirror

**a)** The number of oscillations in the wave packet is a relativistic invariant, and the Lorentz transformations are linear in the EM fields. Thus, in the reference frame  $S'$ , where the mirror is at rest, the incident wave packet is still square and comprises the same number of oscillations. On the other hand, as already seen in Problem 9.6, the frequency  $\omega'_i$  and the amplitude  $E'_i$  are

$$\omega'_i = \gamma(1-\beta)\omega_i, \quad E'_i = \gamma(1-\beta)E_i, \quad (\text{S-9.74})$$

where  $\beta = v/c$ . In  $S'$ , the reflected packet has the same shape, duration, and frequency of the incident packet, but opposite amplitude and direction.

$$E'_r = -E'_i, \quad \omega'_r = \omega'_i, \quad \tau'_r = \tau'_i = N \frac{2\pi}{\omega'_i} = N \frac{2\pi}{\gamma(1-\beta)\omega_i} = \frac{\tau_i}{\gamma(1-\beta)}. \quad (\text{S-9.75})$$

Back-transforming to  $S$  (see also Problem 9.6) we have

$$E_r = -\frac{1-\beta}{1+\beta} E_i, \quad \omega_r = \gamma(1-\beta)\omega'_r = \gamma^2(1-\beta^2)\omega_i = \frac{1-\beta}{1+\beta}\omega_i \quad (\text{S-9.76})$$

The duration of the reflected wave packet is thus

$$\tau_r = N \frac{2\pi}{\omega_r} = N \frac{2\pi}{\omega} \frac{1+\beta}{1-\beta} = \frac{1+\beta}{1-\beta} \tau_i. \quad (\text{S-9.77})$$

If  $\beta > 0$ , i.e., if the mirror velocity is parallel to the packet propagation direction, the reflected packet has a longer duration than the incident packet, while the reflected packet is shorter if the mirror velocity is antiparallel.

**b)** The energy per unit surface of each packet is given by its intensity  $I$  times its duration  $\tau$ . The intensity is proportional to the square of the electric field amplitude, thus the relation between the reflected and incident intensities is

$$I_r = \left(\frac{1-\beta}{1+\beta}\right)^2 I_i, \quad (\text{S-9.78})$$

and the relation between the energies per unit surface of the whole reflected and incident packets is

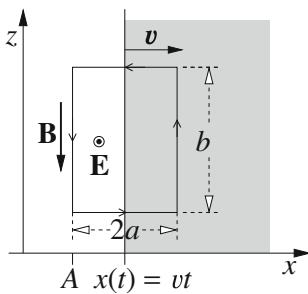
$$U_r = I_r \tau_r = \left(\frac{1-\beta}{1+\beta}\right)^2 I_i \frac{1+\beta}{1-\beta} \tau_i = \frac{1-\beta}{1+\beta} I_i \tau_i = \frac{1-\beta}{1+\beta} U_i. \quad (\text{S-9.79})$$

We see that  $U_r \neq U_i$ , hence some work per unit surface is needed in order to keep the mirror moving at constant velocity, namely

$$W = U_r - U_i = -\frac{2\beta}{1+\beta} U_i . \quad (\text{S-9.80})$$

Thus a mirror with  $\beta < 0$ , i.e., moving in the direction opposite to the incident wave packet, transfers some energy to the packet.

c) As a first step, we determine the distribution of the current density  $\mathbf{J}$ . Since all the fields are null inside the mirror, i.e., for  $x > x(t) = vt$ , the current must be localized on the mirror surface,  $\mathbf{J}(x, t) = \mathbf{K}(t) \delta(x - vt)$ . We can evaluate the surface current



evaluate the surface current density  $\mathbf{K}(t)$  on the mirror surface by considering the fields close to the surface. By calculating the line integral of  $\mathbf{B}$  over a closed rectangular path, fixed in  $S$ , of sides  $b$ , parallel to  $\mathbf{B}$  and to the mirror surface, and  $2a$ , perpendicular to, and crossing the mirror surface, as in Fig. S-9.5, we obtain

$$\oint_{\text{path}} \mathbf{B} \cdot d\ell = \frac{4\pi}{c} \Phi(\mathbf{J}) + \frac{1}{c} \frac{d\Phi(\mathbf{E})}{dt} , \quad (\text{S-9.81})$$

Fig. S-9.5

where  $\Phi(\mathbf{J})$  and  $\Phi(\mathbf{E})$  are the fluxes through the surface delimited by the path  $Jv$  and  $\mathbf{E}$ , respectively. At the limit  $a \rightarrow 0$  and  $A \rightarrow vt$ , we have

$$\oint_{\text{path}} \mathbf{B} \cdot d\ell \simeq B(vt)b , \quad \Phi(\mathbf{J}) = K(t)b , \quad (\text{S-9.82})$$

$$\frac{d\Phi(\mathbf{E})}{dt} \simeq \partial_t E(vt)(vt - A)b + E(vt)bv \simeq E(vt)bv . \quad (\text{S-9.83})$$

From the knowledge of  $E$  and  $B$  at the mirror surface (Prob. 9.6) we obtain

$$\begin{aligned} K(t) &= \frac{c}{4\pi} \left[ B(vt) - \frac{v}{c} E(vt) \right] = \frac{c}{4\pi} (1 - \beta^2) \frac{2E_i}{1 + \beta} e^{-i(1-\beta)\omega t} \\ &= \frac{cE_i}{2\pi} (1 - \beta) e^{-i(1-\beta)\omega t} . \end{aligned} \quad (\text{S-9.84})$$

Thus,  $\mathbf{K}$  and  $\mathbf{E}$  are in phase. In order to evaluate the total mechanical work per unit surface on the mirror, we first switch back to the real quantities

$$K(t) = \frac{cE_i}{2\pi} (1 - \beta) \cos[(1 - \beta)\omega t] , \quad (\text{S-9.85})$$

$$E(vt) = \frac{2\beta}{1 + \beta} E_i \cos[(1 - \beta)\omega t] , \quad (\text{S-9.86})$$

and evaluate the integral over the mirror depth

$$\int_{vt}^{\infty} \mathbf{J} \cdot \mathbf{E} dx = \frac{1}{2} K(t) E(vt) = \frac{cE_i^2}{2\pi} \frac{\beta(1-\beta)}{1+\beta} \cos^2[(1-\beta)\omega t]. \quad (\text{S-9.87})$$

We have inserted the factor 1/2 to account for the discontinuity of  $E$  at  $x = vt$  (see also Prob. 2.12). Equation (S-9.87) gives the mechanical power per unit surface exerted on the mirror. To find the mechanical work, (S-9.87) must be integrated over the time interval for which  $K(t) \neq 0$ , i.e., for the time needed by the wave packet to undergo a complete reflection. If the front of the wave packet reaches the mirror at  $t = 0$ , the end of the packet will leave the mirror at  $t = \tau/(1-\beta)$ , which is different from the pulse duration  $\tau$  because the mirror moves while the wave train is reflected. We thus need the integral

$$\int_0^{\tau/(1-\beta)} \cos^2[(1-\beta)\omega t] dt = \frac{1}{\omega(1-\beta)} \int_0^{\omega\tau} \cos^2 x dx = \frac{\pi N}{(1-\beta)\omega}, \quad (\text{S-9.88})$$

since  $\omega\tau = 2\pi N$ , and the integral of  $\cos^2 x$  over one period equals  $\pi$ . We thus obtain

$$W = \int \frac{1}{2} K(t) E(vt) dt = \frac{cE_i^2}{2\pi} \frac{\beta(1-\beta)}{1+\beta} \frac{\pi N}{(1-\beta)\omega} = \frac{cE_i^2}{4\pi} \frac{\beta}{1+\beta} \tau \quad (\text{S-9.89})$$

$$= 2I_i \tau \frac{\beta}{1+\beta} = \frac{2\beta}{1+\beta} I_i, \quad (\text{S-9.90})$$

in agreement with (S-9.80).

The work  $W$  divided by the reflection time gives, the mechanical power per unit surface

$$\mathcal{P} = W \frac{1-\beta}{\tau} = \frac{2\beta(1-\beta)}{1+\beta} I_i = \frac{2(1-\beta)}{1+\beta} I_i \frac{v}{c}, \quad (\text{S-9.91})$$

which must be equal to the pressure exerted on the moving mirror times its velocity  $v$ . We thus obtain that the radiation pressure on a moving mirror is

$$P_{\text{rad}} = \frac{\mathcal{P}}{v} = \frac{2I_i}{c} \frac{1-\beta}{1+\beta}, \quad (\text{S-9.92})$$

a result which can also be obtained in different ways (see Problems 13.7 & 13.8).

## S-9.9 Boundary Conditions on a Moving Mirror

- a) We can assume the wave to be linearly polarized along  $y$ , without loss of generality. We choose the origin of the frame  $S'$ , where the mirror is at rest, so that the mirror surface is on the  $x' = 0$  plane. In  $S'$  the total fields at the mirror surface are

$$\begin{aligned}\mathbf{E}'_s(t') &= \hat{\mathbf{y}}' E'_s(t') \equiv 0, \\ \mathbf{B}'_s(t') &= \hat{\mathbf{z}}' B'_s(t') e^{-i\omega'_i t'} = -\hat{\mathbf{z}}' 2E'_i e^{-i\omega'_i t'},\end{aligned}\quad (\text{S-9.93})$$

respectively, where

$$E'_i = \gamma(1-\beta)E_i, \quad \omega'_i = \gamma(1-\beta)\omega_i, \quad (\text{S-9.94})$$

are the amplitude and frequency of the incident wave in  $S'$ , as seen in Problem 9.6. Notice that  $\mathbf{E}'$  is continuous at  $x' = 0$ , while  $\mathbf{B}'$  is not. By transforming the field amplitudes at the mirror surface back to  $S$  we obtain

$$\begin{aligned}E_s &= \gamma(E'_s + \beta B'_s) = \gamma\beta B'_s = -2\gamma^2\beta(1-\beta)E_i, \\ B_s &= \gamma(B'_s - \beta E'_s) = \gamma B'_s = -2\gamma^2(1-\beta)E_i,\end{aligned}\quad (\text{S-9.95})$$

where  $\beta = v/c$  and  $\gamma = 1/\sqrt{1-\beta^2}$ . Thus, in general, in  $S$  we have both  $E_s \neq 0$  and  $B_s \neq 0$ , while the fields are zero inside the mirror.

**b)** The EM fields are related to the vector potential by

$$\mathbf{E} = -\frac{1}{c} \partial_t \mathbf{A}, \quad \mathbf{B} = \nabla \times \mathbf{A}. \quad (\text{S-9.96})$$

Thus, the only nonzero component of the vector potential is  $A_y$ , and we have

$$E_s = -\frac{1}{c} \partial_t A_y, \quad B_s = \partial_x A_y. \quad (\text{S-9.97})$$

The total derivative of  $\mathbf{A}$  appearing in (9.9) can be rewritten

$$\left. \frac{d\mathbf{A}}{dt} \right|_{x=x(t)} = \left[ \partial_t A_y + v \partial_x A_y \right]_{x=x(t)} = c E_s - v B_s = c(E_s - \beta B_s) = 0, \quad (\text{S-9.98})$$

according to (S-9.95). Thus the equations (S-9.93) and (S-9.95) imply  $d\mathbf{A}/dt = 0$  on the mirror surface in  $S$ .

**c)** The total vector potential in  $S$  is the sum of the vector potentials of the incident and the reflected waves,

$$\mathbf{A}(x, t) = \hat{\mathbf{y}} \left[ A_i e^{ik_i x - i\omega_i t} + A_r e^{-ik_r x - i\omega_r t} \right] = \hat{\mathbf{y}} \left[ A_i e^{ik_i(x-ct)} + A_r e^{-ik_r(x+ct)} \right], \quad (\text{S-9.99})$$

where  $A_i = i c E_i / \omega_i$ ,  $k_i = \omega_i / c$ , and  $k_r = \omega_r / c$ . The boundary condition gives

$$0 = A_y(vt, t) = A_i e^{-ik_i(c-v)t} + A_r e^{-ik_r(c+v)t}. \quad (\text{S-9.100})$$

This equation is satisfied if

$$A_r = -A_i, \quad \frac{k_r}{k_i} = \frac{\omega_r}{\omega_i} = \frac{c-v}{c+v} = \frac{1-\beta}{1+\beta}. \quad (\text{S-9.101})$$

For the total electric field we find

$$\begin{aligned} E_y &= -\frac{1}{c} \partial_t A_y = i \frac{\omega_i}{c} A_i e^{ik_i x - i\omega_i t} - i \frac{\omega_r}{c} A_r e^{-ik_r x - i\omega_r t} \\ &= E_i e^{ik_i x - i\omega_i t} - \frac{1-\beta}{1+\beta} E_i e^{-ik_r x - i\omega_r t}. \end{aligned} \quad (\text{S-9.102})$$

# Chapter S-10

## Solutions for Chapter 10

### S-10.1 Cyclotron Radiation

**a)** The electric dipole moment  $\mathbf{p} = -er$  rotates in the  $xy$  plane with frequency  $\omega_L$ , which is also the frequency of the emitted radiation. The dipole approximation is valid if the dimensions of the radiating source are much smaller than the emitted wavelength  $\lambda$ . Here this corresponds to the condition  $2r_L = 2v/\omega_L \ll \lambda = 2\pi c/\omega_L$ , always true for non-relativistic velocities.

The rotating dipole can be written as  $\mathbf{p} = p_0(\hat{\mathbf{x}}\cos\omega_L t + \hat{\mathbf{y}}\sin\omega_L t)$ . For the electric field of the dipole radiation observed in a direction of unit vector  $\hat{\mathbf{n}}$ , we have  $\mathbf{E} \propto -(\mathbf{p} \times \hat{\mathbf{n}}) \times \hat{\mathbf{n}}$ . If  $\hat{\mathbf{n}} = \hat{\mathbf{z}}$ , then  $\mathbf{E} \propto \hat{\mathbf{x}}\cos\omega_L t + \hat{\mathbf{y}}\sin\omega_L t$  (circular polarization); if  $\hat{\mathbf{n}} = \hat{\mathbf{x}}$  or  $\hat{\mathbf{n}} = \hat{\mathbf{y}}$ , we have  $\mathbf{E} \propto -\hat{\mathbf{y}}\sin\omega_L t$  and  $\mathbf{E} \propto -\hat{\mathbf{x}}\cos\omega_L t$ , respectively (linear polarization).

Since  $\ddot{\mathbf{r}} = \mathbf{v} \times \omega_L$  (where  $\omega_L = \hat{\mathbf{z}}\omega_L$ ), the radiated power can be written as

$$P_{\text{rad}} = \frac{2}{3} \frac{|e\ddot{\mathbf{r}}|^2}{c^3} = \frac{2}{3} \frac{e^2 v^2 \omega_L^2}{c^3}. \quad (\text{S-10.1})$$

**b)** We assume that the energy loss due to radiation is small enough to cause a variation of the orbit radius  $\Delta r_c \ll r_c$  during a single period, so that, during a single period, the motion is still approximately circular. Thus the magnitude of the electron velocity  $v = v(t)$  can be written as  $v \simeq \omega_L r$ , where  $r = r(t)$  is the radius of the orbit at time  $t$ . The electron energy is

$$U = \frac{m_e v^2}{2} = \frac{m_e \omega_L^2 r^2}{2}, \quad (\text{S-10.2})$$

and the equation for the energy loss,  $dU/dt = -P_{\text{rad}}$ , becomes

$$\frac{d}{dt} \left( \frac{m_e \omega_L^2 r^2}{2} \right) = -\frac{2}{3c^3} (e^2 \omega_L^4 r^2) = -\frac{2r_e m_e \omega_L^4}{3c} r^2, \quad (\text{S-10.3})$$

where  $r_e = e^2/(m_e c^2)$  is the classical electron radius. Substituting the relation  $d(r^2)/dt = 2r dr/dt$  into (S-10.3) we obtain

$$\frac{dr}{dt} = -\frac{2r_e \omega_L^2}{3c} r \equiv -\frac{r}{\tau}, \quad \text{with} \quad \tau = \frac{3c}{2r_e \omega_L^2} = \frac{3m_e c^3}{2e^2 \omega_L^2}, \quad (\text{S-10.4})$$

whose solution is

$$r(t) = r(0) e^{-t/\tau}, \quad (\text{S-10.5})$$

and the trajectory of the electron is a spiral with a decay time  $\tau$ . Inserting the expressions for  $r_e$  and  $\omega_L$  we have

$$\tau = \frac{3 m_e^3 c^5}{2 e^4 B_0^2} = \frac{5.2 \times 10^5}{B_0^2} \text{ s} \quad (\text{S-10.6})$$

where the magnetic field  $B_0$  is in G. The condition  $\tau \gg \omega_L^{-1}$  implies

$$\frac{3 m_e^3 c^5}{2 e^4 B_0^2} \gg \frac{m_e c}{e B_0}, \quad \text{or} \quad B_0 \ll \frac{3 m_e^2 c^4}{2 e^3} = 9.2 \times 10^{13} \text{ G}, \quad (\text{S-10.7})$$

a condition well verified in all experimental conditions: such high fields can be found only on neutron stars! (see Problem 10.5)

c) We insert a frictional force  $\mathbf{f}_{\text{fr}} = -m_e \eta \mathbf{v}$  into the equation of motion, obtaining

$$m_e \frac{d\mathbf{v}}{dt} = -\frac{e}{c} \mathbf{v} \times \mathbf{B}_0 - m_e \eta \mathbf{v}. \quad (\text{S-10.8})$$

This corresponds to the following two coupled equations for the the  $x$  and  $y$  components the electron velocity

$$\dot{v}_x = -\omega_L v_y - \eta v_x, \quad \dot{v}_y = \omega_L v_x - \eta v_y. \quad (\text{S-10.9})$$

An elegant method to solve these equations is to combine the  $x$  and  $y$  coordinates of the electron into a single complex variable  $R = x + iy$ , and the velocity components into the complex variable  $V = v_x + iv_y$ . The two equations (S-10.9) are thus combined into the single complex equation

$$\dot{V} = (i\omega_L - \eta)V, \quad \text{with solution} \quad V = V(0) e^{i\omega_L t - \eta t} = v_0 e^{i\omega_L t - \eta t}. \quad (\text{S-10.10})$$

For the electron position we have

$$R = \int V dt + C = \frac{v_0}{i\omega_L - \eta} e^{i\omega_L t - \eta t} + C = -\frac{(\eta + i\omega_L) v_0}{\omega_L^2 + \eta^2} e^{i\omega_L t - \eta t} + C, \quad (\text{S-10.11})$$

where  $C$  is a complex constant depending on our choice of the origin of the coordinates. We choose  $C = 0$ , and rewrite (S-10.11) as

$$R = -\frac{v_0}{\sqrt{\omega_L^2 + \eta^2}} (\cos \phi + i \sin \phi) e^{i\omega_L t - \eta t} = -\frac{v_0}{\sqrt{\omega_L^2 + \eta^2}} e^{i(\omega_L t + \phi) - \eta t}, \quad (\text{S-10.12})$$

where

$$\cos \phi = \frac{\eta}{\sqrt{\omega_L^2 + \eta^2}}, \quad \sin \phi = \frac{\omega_L}{\sqrt{\omega_L^2 + \eta^2}}, \quad \phi = \arctan\left(\frac{\omega_L}{\eta}\right). \quad (\text{S-10.13})$$

Going back to the real quantities we have

$$v_x = \operatorname{Re}(V) = v_0 e^{-\eta t} \cos \omega_L t, \quad (\text{S-10.14})$$

$$v_y = \operatorname{Im}(V) = v_0 e^{-\eta t} \sin \omega_L t, \quad (\text{S-10.15})$$

$$x = \operatorname{Re}(R) = -\frac{v_0}{\sqrt{\omega_L^2 + \eta^2}} e^{-\eta t} \cos(\omega_L t + \phi), \quad (\text{S-10.16})$$

$$y = \operatorname{Im}(R) = -\frac{v_0}{\sqrt{\omega_L^2 + \eta^2}} e^{-\eta t} \sin(\omega_L t + \phi). \quad (\text{S-10.17})$$

Thus, the velocity rotates with frequency  $\omega_L$ , while its magnitude decays exponentially,  $|v(t)| = v_0 e^{-\eta t}$ . For the radius of the trajectory we have

$$r(t) = |R(t)| = \frac{v_0}{\sqrt{\omega_L^2 + \eta^2}} e^{-\eta t}. \quad (\text{S-10.18})$$

Thus, choosing  $\eta = 1/\tau$ , the motion with frictional force is identical to the motion with radiative power loss, and

$$\mathbf{f}_{\text{fr}} \cdot \mathbf{v} = -m_e \eta v^2 = -\frac{m_e v^2}{\tau} = -m_e v^2 \frac{2e^2 \omega_L^2}{3m_e c^3} = -\frac{2e^2 v^2 \omega_L^2}{3c^3} = -P_{\text{rad}}. \quad (\text{S-10.19})$$

A drawback of this approach is that the frictional coefficient inserted here is not universal but is dependent on the force on the electron (in this case, via the dependence on  $\omega_L$ ). See Problem 10.12 for a more general approach to radiation friction.

## S-10.2 Atomic Collapse

**a)** An electron describing a circular orbit of radius  $a_0$  (Bohr radius) around a proton corresponds to a counterrotating electric dipole  $\mathbf{p}(t)$  of magnitude  $p_0 = ea_0$ . The angular velocity of the orbit  $\omega$  can be evaluated by considering that the centripetal acceleration is due to the Coulomb force,

$$\omega^2 a_0 = \frac{1}{m_e} \frac{e^2}{a_0^2}, \quad (\text{S-10.20})$$

from which we obtain

$$\omega = \sqrt{\frac{e^2}{m_e a_0^3}} = 4.1 \times 10^{16} \text{ rad/s}. \quad (\text{S-10.21})$$

Actually, the strongest emission from the hydrogen atom occurs at a frequency smaller by about one order of magnitude.

Since  $\mathbf{p}$  is perpendicular to  $\omega$ , we have  $\dot{\mathbf{p}} = (\mathbf{p} \times \omega) \times \omega$  and  $|\dot{\mathbf{p}}|^2 = (\omega^2 p_0)^2$  (the same result can be obtained by considering the rotating dipole as the superposition of two perpendicularly oscillating dipoles). Thus the radiated power is

$$P_{\text{rad}} = \frac{2}{3c^3} |\dot{\mathbf{p}}|^2 = \frac{2}{3} \frac{\omega^4 e^2 a_0^2}{c^3} = \frac{2}{3} \frac{e^2 r_e^2 c}{a_0^4}, \quad (\text{S-10.22})$$

where  $r_e$  is the classical electron radius.

**b)** We assume that, due to the emission of radiation, the electron loses its energy according to  $dU/dt = -P_{\text{rad}}$ , where  $U = K + V$  is the total electron energy,  $K$  and  $V$  being the kinetic and potential energy, respectively. If the energy lost per period is small with respect to the total energy, we may assume that the electron the orbit is almost circular during a period, with the radius *slowly* decreasing with time,  $r = r(t)$  with  $\dot{r}/r \ll \omega$ .

Since the velocity is  $v = r\omega$ , the total energy can be written as a function of  $a$ :

$$U = K + V = \frac{m_e v^2}{2} - \frac{e^2}{r} = -\frac{e^2}{2r}. \quad (\text{S-10.23})$$

Therefore

$$\frac{dU}{dt} \simeq -\frac{e^2}{2} \frac{d}{dr} \left( \frac{1}{r} \right) = \frac{e^2}{2r^2} \frac{dr}{dt}. \quad (\text{S-10.24})$$

Since

$$P_{\text{rad}} = \frac{2}{3} \frac{e^2 r_e^2 c}{r^4} \quad (\text{S-10.25})$$

the equation  $dU/dt = -P_{\text{rad}}$  can be written as

$$\frac{e^2}{2r^2} \frac{dr}{dt} = -\frac{2}{3} \frac{e^2 r_e^2 c}{r^4} \Rightarrow r^2 \frac{dr}{dt} = -\frac{4}{3} r_e^2 c \Rightarrow \frac{1}{3} \frac{dr^3}{dt} = \frac{4}{3} r_e^2 c \quad (\text{S-10.26})$$

The solution, assuming  $r(0) = a_0$ , is

$$r^3 = a_0^3 - 4r_e^2 c t, \quad (\text{S-10.27})$$

giving for the time need by the electron to fall on the nucleus

$$\tau = \frac{a_0^3}{4r_e^2 c} \simeq 1.6 \times 10^{-11} \text{ s}. \quad (\text{S-10.28})$$

This is a well-known result, showing that a classical “Keplerian” atom is not stable. It is however interesting to notice that the value of  $\tau$  is of the same order of magnitude of the lifetime of the first excited state, i.e., of the time by which the excited state decays to the ground state emitting radiation.

### S-10.3 Radiative Damping of the Elastically Bound Electron

a) The solution of (10.5) with the given initial conditions and  $\eta = 0$  is

$$\mathbf{r} = \mathbf{s}_0 \cos \omega_0 t. \quad (\text{S-10.29})$$

The corresponding average radiated power in the dipole approximation is

$$P_{\text{rad}} = \frac{2}{3c^3} \langle -e |\dot{\mathbf{r}}|^2 \rangle = \frac{2e^2}{3c^3} \omega_0^4 s_0^2 \langle \cos^2 \omega_0 t \rangle = \frac{e^2}{3c^3} \omega_0^4 s_0^2. \quad (\text{S-10.30})$$

The radiated power is emitted at the expense of the energy of the oscillating electron. Thus, the total mechanical energy of electron must decrease in time, and the harmonic-oscillator solution of (S-10.29) cannot be exact. Assuming that the energy of the oscillator decays very slowly, i.e., with a decay constant  $\tau \gg \omega_0^{-1}$ , we can approximate (S-10.29) as

$$\mathbf{r} \simeq \mathbf{s}(t) \cos \omega_0 t. \quad (\text{S-10.31})$$

where  $s(t)$  is a decreasing function of time to be determined. Consequently, we must replace  $s_0$  by  $s(t)$  also in equation (S-10.30) for the actual average radiated power.

**b)** At time  $t$ , the total energy of the oscillating electron is  $U(t) = m_e \omega_0^2 s^2(t)$ . The time decay constant  $\tau$  is defined as

$$\tau = \frac{U(t)}{P_{\text{rad}}(t)} = \frac{3m_e c^3}{2e^2 \omega_0^2} = \frac{3c}{2r_e \omega_0^2}, \quad (\text{S-10.32})$$

and is thus independent of  $t$ . Since the classical electron radius is  $r_e \approx 2.82 \times 10^{-15}$  m, the condition  $\tau > 2\pi/\omega_0$  leads to

$$\omega_0 < \frac{3}{4\pi} \frac{c}{r_e} \approx 3 \times 10^{22} \text{ rad/s}. \quad (\text{S-10.33})$$

For a comparison, estimating  $\omega_0$  as the frequency of the 1S  $\leftarrow$  2P Lyman-alpha emission line of the hydrogen atom, we have  $\omega_0 \approx 3 \times 10^{16}$  rad/s.

**c)** We look for a solution of the form  $\mathbf{r} = \text{Re}(\mathbf{s}_0 e^{-i\omega t})$ , with *complex*  $\omega$ . Substituting this into (10.5), the characteristic equation becomes

$$\omega^2 + i\eta\omega + \omega_0^2 = 0, \quad (\text{S-10.34})$$

whose solution is

$$\omega = -i\frac{\eta}{2} \pm \sqrt{\omega_0^2 - \frac{\eta^2}{4}} \approx -i\frac{\eta}{2} \pm \omega_0, \quad (\text{S-10.35})$$

where we have neglected the terms of the order  $(\eta/\omega_0)^2$  and higher. Thus, the approximated solution for the electron position is

$$\mathbf{r} \approx \mathbf{s}_0 e^{-\eta t/2} \cos \omega_0 t. \quad (\text{S-10.36})$$

Actually, this approximation gives an initial velocity  $\dot{\mathbf{r}}(0) = -\eta \mathbf{s}_0/2$  instead of zero. However, this discrepancy can be neglected if  $\eta \ll \omega_0$ . The maximum speed reached by the electron is  $v_{\text{max}} \approx \omega_0 s_0$ , and  $\eta s_0/2 \ll \omega_0 s_0$ .

The time-dependent total energy of the electron and average radiated power are

$$U(t) \approx \frac{m_e}{2} \omega_0^2 s_0^2 e^{-\eta t}, \quad \text{and} \quad P_{\text{rad}}(t) \approx \frac{e^2}{3c^3} \omega_0^4 s_0^2 e^{-\eta t}. \quad (\text{S-10.37})$$

The condition  $dU/dt = -P_{\text{rad}}$  leads to

$$\eta = \frac{2r_e \omega_0^2}{3c} = \frac{1}{\tau}. \quad (\text{S-10.38})$$

## S-10.4 Radiation Emitted by Orbiting Charges

**a)** Let us denote by  $\mathbf{r}_1$  and  $\mathbf{r}_2$  the location vectors of the two charges with respect to the center of their common circular orbit. In polar coordinates we have

$$\mathbf{r}_1 \equiv [R, \phi_1(t)] , \quad \text{and} \quad \mathbf{r}_2 \equiv [R, \phi_2(t)] . \quad (\text{S-10.39})$$

Defining  $\Delta\phi = \phi_2 - \phi_1$  and choosing an appropriate origin of time, the equations of motion in polar coordinates are

$$\mathbf{r}_1 \equiv \left( R, \omega t - \frac{\Delta\phi}{2} \right) , \quad \text{and} \quad \mathbf{r}_2 \equiv \left( R, \omega t + \frac{\Delta\phi}{2} \right) . \quad (\text{S-10.40})$$

In Cartesian coordinates we have

$$\mathbf{r}_1 \equiv [x_1(t), y_1(t)] , \quad \text{and} \quad \mathbf{r}_2 \equiv [x_2(t), y_2(t)] , \quad (\text{S-10.41})$$

with, respectively,

$$x_1(t) = R \cos\left(\omega t - \frac{\Delta\phi}{2}\right) , \quad y_1(t) = R \sin\left(\omega t - \frac{\Delta\phi}{2}\right) , \quad (\text{S-10.42})$$

$$x_2(t) = R \cos\left(\omega t + \frac{\Delta\phi}{2}\right) , \quad y_2(t) = R \sin\left(\omega t + \frac{\Delta\phi}{2}\right) . \quad (\text{S-10.43})$$

The dipole moment of the system is  $\mathbf{p} = q(\mathbf{r}_1 + \mathbf{r}_2)$ , with Cartesian components

$$p_x = qR \left[ \cos\left(\omega t - \frac{\Delta\phi}{2}\right) + \cos\left(\omega t + \frac{\Delta\phi}{2}\right) \right] = 2qR \cos\left(\frac{\Delta\phi}{2}\right) \cos \omega t , \quad (\text{S-10.44})$$

$$p_y = qR \left[ \sin\left(\omega t - \frac{\Delta\phi}{2}\right) + \sin\left(\omega t + \frac{\Delta\phi}{2}\right) \right] = 2qR \cos\left(\frac{\Delta\phi}{2}\right) \sin \omega t , \quad (\text{S-10.45})$$

i.e.,  $\mathbf{p}$  has constant magnitude  $p = 2qR \cos(\Delta\phi/2)$ , and rotates in the  $z = 0$  plane with angular frequency  $\omega$ .

**b)** In the dipole approximation, the electric field of the radiation emitted along a direction of unit vector the  $\hat{\mathbf{n}}$  is parallel to the vector

$$(\mathbf{p} \times \hat{\mathbf{n}}) \times \hat{\mathbf{n}} = \mathbf{p}_{\perp} . \quad (\text{S-10.46})$$

Since for a dipole rotating in the  $z = 0$  plane

$$(\mathbf{p} \times \hat{\mathbf{x}}) \times \hat{\mathbf{x}} \text{ is parallel to } \hat{\mathbf{y}} , \quad \text{and} \quad (\mathbf{p} \times \hat{\mathbf{y}}) \times \hat{\mathbf{y}} \text{ is parallel to } \hat{\mathbf{x}} , \quad (\text{S-10.47})$$

the polarization of the radiation observed in the  $\hat{\mathbf{x}}$  ( $\hat{\mathbf{y}}$ ) direction is linear and along  $\hat{\mathbf{y}}$  ( $\hat{\mathbf{x}}$ ). For radiation observed the  $\hat{\mathbf{z}}$  direction

$$(\mathbf{p} \times \hat{\mathbf{z}}) \times \hat{\mathbf{z}} \text{ is parallel to } \mathbf{p}, \quad (\text{S-10.48})$$

and the observed polarization is circular.

The total radiated power is

$$P_{\text{rad}} = \frac{2}{3c^3} |\ddot{\mathbf{p}}|^2 = \frac{4q^2 R^2 \omega^4}{3c^3} \cos^2\left(\frac{\Delta\phi}{2}\right), \quad (\text{S-10.49})$$

which obviously vanishes when  $\mathbf{p} = 0$ , i.e., for  $\Delta\phi = \pi$  (charges on opposite ends of a rotating diameter), and has a maximum for  $\Delta\phi = 0$  (superposed charges).

- c)** In this case charges are superposed to each other every half turn. We choose the coordinates and the time origin so that the charges are superposed at  $t = 0$  we have  $\mathbf{r}_1 = \mathbf{r}_2 = (R, 0)$ . Thus the trajectories can be written as

$$r_1 = r_2 = R, \quad \phi_1(t) = \omega t, \quad \phi_2(t) = -\omega t, \quad (\text{S-10.50})$$

in polar coordinates, and as

$$\begin{aligned} x_1(t) &= R \cos \omega t, & y_1(t) &= R \sin \omega t, \\ x_2(t) &= R \cos \omega t, & y_2(t) &= -R \sin \omega t, \end{aligned} \quad (\text{S-10.51})$$

in Cartesian coordinates. The total dipole moment is thus  $\mathbf{p} = (2qR \cos \omega t)\hat{\mathbf{x}}$ . No radiation is emitted along  $x$ , while the radiation emitted along all other directions is linearly polarized. The total average radiated power is

$$P_{\text{rad}} = \frac{2}{3c^3} |\ddot{\mathbf{p}}|^2 = \frac{4q^2 R^2 \omega^4}{3c^3}. \quad (\text{S-10.52})$$

- d)** With an appropriate choice of the time origin the equations of motion of the three charges can be written, in polar coordinates, as

$$\begin{aligned} r_1 = r_2 = r_3 &= R, & \phi_1(t) &= \omega t, \\ \phi_2(t) &= \omega t + \Delta\phi_2, & \phi_3(t) &= \omega t + \Delta\phi_3, \end{aligned} \quad (\text{S-10.53})$$

and, in Cartesian coordinates,

$$x_i = R \cos \phi_i(t), \quad y_i = R \sin \phi_i(t), \quad (i = 1, 2, 3). \quad (\text{S-10.54})$$

The electric dipole moment vanishes if the three charges are on the vertices of a rotating equilateral triangle ( $\Delta\phi_2 = -\Delta\phi_3 = 2\pi/3$ ), and has its maximum value when the three charges are overlapped ( $\Delta\phi_2 = \Delta\phi_3 = 0$ ).

- e)** The magnetic dipole moment for a point charge  $q$ , traveling at angular velocity  $\omega$  on a circular orbit of radius  $R$ , is defined by

$$\mathbf{m} = \frac{1}{2c} \int \mathbf{r} \times \mathbf{J} d^3x = \frac{qR^2\omega}{2c}, \quad (\text{S-10.55})$$

and is constant (notice that  $\mathbf{m}$  is proportional to the angular momentum of the orbiting charge). Thus the magnetic dipole does not contribute to radiation, because the radiation fields are proportional to  $\dot{\mathbf{m}}$ .

This problem explains why a circular coil carrying a constant current does not radiate, although we may consider the current as produced by charges moving on circular orbits, and thus subject to acceleration.

## S-10.5 Spin-Down Rate and Magnetic Field of a Pulsar

**a)** Due to the nonzero angle  $\alpha$  between the magnetic moment and the rotation axis of the pulsar, the component of  $\mathbf{m}$  perpendicular to  $\omega$  rotates with frequency  $\omega$ . Thus the Pulsar emits magnetic dipole radiation of frequency  $\omega$ . The total power is

$$P = \frac{2}{3c^3} |\dot{\mathbf{m}}_{\perp}|^2 = \frac{2}{3} \frac{m_{\perp}^2 \omega^4}{c^3}, \quad (\text{S-10.56})$$

where  $m_{\perp} = m \sin \alpha$ .

**b)** The mechanical energy is  $U = I\omega^2/2$ , where  $I = 2MR^2/5 \approx 1.1 \times 10^{43} \text{ g cm}^2$  is the moment of inertia of the pulsar, assuming a uniform mass distribution over the volume of a sphere of radius  $R$ . Assuming that the energy loss is due to radiation emission only, we can write

$$\frac{dU}{dt} = \frac{d}{dt} \left( \frac{I\omega^2}{2} \right) = I\omega \dot{\omega} = -P, \quad (\text{S-10.57})$$

and, substituting (S-10.56), we have

$$I\omega \dot{\omega} = -\frac{2}{3} \frac{m_{\perp}^2 \omega^4}{c^3} \Rightarrow \frac{\dot{\omega}}{\omega^3} = -\frac{2m_{\perp}^2}{3Ic^3}. \quad (\text{S-10.58})$$

By integrating over time from 0 to  $t$  we obtain

$$\frac{1}{2\omega^2(t)} - \frac{1}{2\omega^2(0)} = \frac{2m_{\perp}^2}{3Ic^3} t, \quad (\text{S-10.59})$$

and thus

$$\omega(t) = \frac{\omega(0)}{\sqrt{1 + \frac{t}{\tau}}}, \quad \text{where} \quad \tau = \frac{3Ic^3}{4m_{\perp}^2 \omega^2(0)}. \quad (\text{S-10.60})$$

**c)** We can rewrite  $\dot{\omega}/\omega^3$  as  $\dot{\omega}/\omega^3 = -T\dot{T}/4\pi^2$ , where  $T = 2\pi/\omega$  is the rotation period of the pulsar, and  $\dot{T} = -2\pi\dot{\omega}/\omega^2$ . Thus we can obtain the magnetic dipole moment  $m = m_\perp$  of the pulsar as a function of the experimentally measured parameters from (S-10.58):

$$m = \sqrt{\frac{3Ic^3}{8\pi^2} T \dot{T}} \simeq 3.3 \times 10^{36} \sqrt{T \dot{T}} \text{ erg/G}, \quad (\text{S-10.61})$$

where  $T$  is in seconds. The magnetic field immediately outside of the pulsar surface is the field of a magnetic dipole located at the pulsar center:

$$\mathbf{B} = \frac{3(\hat{\mathbf{r}} \cdot \mathbf{m}) \hat{\mathbf{r}} - \mathbf{m}}{r^3}, \quad (\text{S-10.62})$$

and thus  $B_{\max} = 2m/R^3$ . Thus we obtain the practical formula

$$B_{\max} \simeq 6.6 \times 10^{21} \sqrt{T \dot{T}} \text{ G}, \quad (\text{S-10.63})$$

Inserting the experimental values for  $T$  and  $\dot{T}$  we obtain

$$B_{\max} \approx (9.6 \pm 0.25) \times 10^{16} \text{ G}. \quad (\text{S-10.64})$$

## S-10.6 A Bent Dipole Antenna

**a)** If we divide the antenna into a series of infinitesimal resistors, each of length  $dz$  and resistance  $dR = (R/a)dz$ , we can write the dissipated power as

$$P_{\text{diss}} = \int \langle I^2 \rangle dR = \int_{-a}^{+a} \frac{I_0^2}{2} \left(1 - \frac{|z|}{a}\right)^2 \frac{R}{a} dz = \frac{I_0^2 R}{3}. \quad (\text{S-10.65})$$

**b)** The linear charge density on the antenna  $q_\ell$  can be obtained from the continuity equation  $\partial_t q_\ell = -\partial_z I$ , obtaining

$$q_\ell = \pm \frac{iI_0}{a\omega} e^{-i\omega t}, \quad (\text{S-10.66})$$

where the signs + and - apply to  $z > 0$  and  $z < 0$ , respectively. The linear charge density is uniform (independent of  $z$ ) on each half of the antenna. For symmetry reasons, the only non-vanishing component of the electric dipole  $\mathbf{p}$  is along  $z$  and it is given by

$$p_z = \int_{-a}^{+a} z q_\ell dz = 2 \int_0^{+a} \frac{iI_0}{a\omega} e^{-i\omega t} z dz = \frac{iI_0 a}{\omega} e^{-i\omega t}. \quad (\text{S-10.67})$$

c) The average radiated power, in the dipole approximation, is

$$P_{\text{rad}} = \frac{1}{3c^3} \langle |\vec{p}_z|^2 \rangle = \frac{I_0^2 a^2 \omega^2}{6c^3} = \frac{2\pi^2 a^2 I_0^2}{3c\lambda^2}, \quad (\text{S-10.68})$$

where  $\lambda = 2\pi c/\omega$  is the radiation wavelength. Thus

$$\frac{P_{\text{rad}}}{P_{\text{diss}}} = \frac{2\pi^2 a^2}{c\lambda^2 R}, \quad (\text{S-10.69})$$

where we recall that  $R$  has the dimensions of the inverse of a velocity in Gaussian units.

- d) The angular distribution of the radiated power is proportional to  $\sin^2 \theta$ , where  $\theta$  is the angle between the observation direction and  $\mathbf{p}$ . Thus the emitted radiation intensity is zero along the  $z$  axis and maximum for observation in the  $xy$  plane.
- e) The bent antenna has a linear charge density  $\pm(iI_0/a\omega)e^{-i\omega t}$  on its horizontal and vertical arms, respectively. Thus the electric dipole moment has two components

$$p_x = \int_0^a \frac{iI_0}{a\omega} e^{-i\omega t} x dx = \frac{iI_0 a}{2\omega} e^{-i\omega t}, \quad (\text{S-10.70})$$

$$p_z = - \int_{-a}^0 \frac{iI_0}{a\omega} e^{-i\omega t} z dz = \frac{iI_0 a}{2\omega} e^{-i\omega t}. \quad (\text{S-10.71})$$

Since the components are perpendicular to each other, the cycle-averaged radiated power can be calculated as the sum of the powers from each dipole:

$$P_{\text{rad}} = \frac{1}{3c^3} \langle |\vec{p}_x|^2 + |\vec{p}_z|^2 \rangle = \frac{1}{12c^3} (I_0 \omega a)^2, \quad (\text{S-10.72})$$

which is one half of the value for the linear antenna, while the dissipated power  $P_{\text{diss}}$  does not change.

The electric dipole of the bent antenna lies along the diagonal direction, which thus corresponds to the direction of zero emitted intensity. The intensity is maximum in the plane perpendicular to the dipole.

## S-10.7 A Receiving Circular Antenna

- a) We choose a Cartesian reference frame such that the wave is propagating in the  $z$  direction, its electric field  $\mathbf{E}$  is along the  $x$  axis, and its magnetic field  $\mathbf{B}$  is along the  $y$  axis. The current  $I$  flowing in the antenna is  $I = \mathcal{E}_{\text{circ}}/R$ , where  $\mathcal{E}_{\text{circ}} = -(1/c)d\Phi(\mathbf{B})/dt$  is the electromotive force, and  $\Phi(\mathbf{B})$  is the flux of  $\mathbf{B}$  through the circle delimited by the antenna. Since we have assumed  $\lambda \gg a$ ,  $\mathbf{B}$  is practically uniform over the whole surface of the circle, and  $\Phi(\mathbf{B}) \approx \pi a^2 \mathbf{B} \cdot \hat{\mathbf{n}}$ , where  $\hat{\mathbf{n}}$  is unit

vector perpendicular to the circle surface. Thus the circular antenna must lie on the  $xz$  plane in order to maximize  $\Phi(\mathbf{B})$ . With a proper choice of the time origin the magnetic field on the circle surface can be written as  $\mathbf{B} \approx \hat{\mathbf{y}} B_0 \cos \omega t$ , and

$$\mathcal{E}_{\text{circ}} = \pi a^2 \frac{\omega}{c} E_0 \sin(\omega t), \quad (\text{S-10.73})$$

since  $B_0 = E_0$  in Gaussian units.

**b)** The electromotive force on a linear antenna parallel to the  $x$  axis is practically  $\mathcal{E}_{\text{lin}} = \ell E_0 \cos(\omega t + \phi)$ , where  $\ell$  is the length of the antenna and  $\phi$  is a phase angle. The ratio of the average electromotive force of the circular antenna to the average electromotive force of the linear antenna is thus

$$\frac{\langle \mathcal{E}_{\text{circ}} \rangle}{\langle \mathcal{E}_{\text{lin}} \rangle} \approx \frac{\langle \mathcal{E}_{\text{circ}} \rangle}{E_0 \ell} = \frac{\pi a^2 \omega}{\ell c} = 2\pi^2 \frac{a^2}{\ell \lambda}. \quad (\text{S-10.74})$$

In the range  $10^2 \text{ cm} < \lambda < 10^3 \text{ cm}$ , and with our assumptions  $\ell \approx 50 \text{ cm}$  and  $a \approx 25 \text{ cm}$ , this ratio varies between 2.5 and 0.25. The circular antenna is more convenient for shorter wavelengths.

**c)** The radiation emission from the circular antenna is dominated by the magnetic dipole term. The dipole moment of the antenna is

$$\mathbf{m} = \frac{1}{c} \mathcal{I} \pi a^2 \hat{\mathbf{n}}, \quad (\text{S-10.75})$$

where  $\mathcal{I}$  is the current circulating in the antenna due to the electromotive force induced by the incident wave. The corresponding time-averaged radiated power is

$$\begin{aligned} P_{\text{rad}} &= \frac{2}{3c^3} \langle |\ddot{\mathbf{m}}|^2 \rangle = \frac{2}{3c^5} (\pi a^2)^2 \omega^4 \langle \mathcal{I}^2 \rangle \\ &= \frac{(\pi a^2)^4 \omega^6}{3c^7 R^2} E_0^2. \end{aligned} \quad (\text{S-10.76})$$

In Gaussian units, the intensity of the incoming wave is  $I = cE_0^2/4\pi$ , and (S-10.76) can be rewritten

$$P_{\text{rad}} = \frac{4\pi(\pi a^2)^4 \omega^6}{3c^8 R^2} I = \frac{2(2\pi)^7 (\pi a^2)^4}{3c^2 R^2 \lambda^6} I. \quad (\text{S-10.77})$$

The factor multiplying  $I$ ,

$$\sigma_{\text{scatt}} = \frac{2(2\pi)^7 (\pi a^2)^4}{3c^2 R^2 \lambda^6}, \quad (\text{S-10.78})$$

has the dimensions of a surface ( $R$  has the dimensions of an inverse velocity in Gaussian units), and is the radiative scattering cross section for our circular antenna, in the magnetic dipole approximation.

The time-averaged power dissipated by Joule heating is  $P_{\text{diss}} = R\langle \mathcal{I}^2 \rangle$ , so that

$$\frac{P_{\text{rad}}}{P_{\text{diss}}} = \frac{2}{3c^5R}(\pi a^2)^2\omega^4 = \frac{2(2\pi)^4(\pi a^2)^2}{3Rc\lambda^4}. \quad (\text{S-10.79})$$

## S-10.8 Polarization of Scattered Radiation

- a) We choose a Cartesian reference frame with the origin located on the scattering particle, and the  $z$  axis parallel to the wave vector of  $\mathbf{k}$  the incident wave. In order to have complete rotational symmetry around the  $z$  axis it is convenient to assume that the incident wave is circularly polarized. The electric field of the incoming wave can thus be written

$$\mathbf{E}_i = E_0(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})e^{ikz-i\omega t}. \quad (\text{S-10.80})$$

Thus, the dipole moment of the scatterer is  $\mathbf{p} = \alpha\mathbf{E}_i = \alpha E_0(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})e^{ikz-i\omega t}$ .

Because of the rotational symmetry of the problem around the  $z$  axis, it is sufficient to consider the scattered radiation with the wave vector  $\mathbf{k}_d$  lying in the  $yz$  plane and forming an angle  $\theta$  with the  $z$  axis, as shown in Fig. S-10.1. Disregarding a proportionality factor depending on  $\alpha$  and  $\theta$ , the electric field  $\mathbf{E}_d$  of the scattered radiation can be written

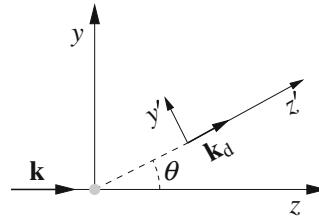


Fig. S-10.1

$$\begin{aligned} \mathbf{E}_d &\propto -(\mathbf{p} \times \hat{\mathbf{n}}) \times \hat{\mathbf{n}} \\ &\propto -[(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}) \times \hat{\mathbf{n}}] \times \hat{\mathbf{n}}, \end{aligned} \quad (\text{S-10.81})$$

where  $\hat{\mathbf{n}} = (0, \sin\theta, \cos\theta)$  is the unit vector parallel to  $\mathbf{k}_d$ . Now, recalling that

$$(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}) \times \hat{\mathbf{n}} = (\pm i\cos\theta, -\cos\theta, \sin\theta), \quad (\text{S-10.82})$$

$$[(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}) \times \hat{\mathbf{n}}] \times \hat{\mathbf{n}} = (-1, \mp i\cos^2\theta, \pm i\sin\theta\cos\theta), \quad (\text{S-10.83})$$

we find that

$$\mathbf{E}_d \propto (1, \pm i\cos^2\theta, \mp i\sin\theta\cos\theta). \quad (\text{S-10.84})$$

Since an observer would measure the polarization of the scattered radiation with respect to the direction  $\hat{\mathbf{n}}$ , we calculate the components of the field in the rotated

coordinate system  $(x', y', z')$ , rotated by an angle  $\theta$  around the  $x$  axis, so that  $x' = x$  and  $z'$  is along  $\hat{\mathbf{n}}$ :

$$E'_{dx} = E_{dx} \propto 1, \quad (\text{S-10.85})$$

$$\begin{aligned} E'_{dy} &= E_{dy} \cos \theta - E_{dz} \sin \theta \propto \pm I_i \cos^3 \theta \pm i \sin^2 \theta \cos \theta \\ &= \pm i \cos \theta (\cos^2 \theta + \sin^2 \theta) = \pm i \cos \theta, \end{aligned} \quad (\text{S-10.86})$$

$$E'_{dz} = E_{dy} \sin \theta + E_{dz} \cos \theta \propto \pm i \sin \theta \cos^2 \theta \mp i \sin \theta \cos^2 \theta = 0. \quad (\text{S-10.87})$$

The last equality is a check that the radiation field is transverse. We thus obtain

$$\mathbf{E}_d \propto \hat{\mathbf{x}}' \pm i \cos \theta \hat{\mathbf{y}}', \quad (\text{S-10.88})$$

which gives the dependence of the polarization on the scattering angle  $\theta$ . In addition, the angular distribution or the radiated power is given by

$$\frac{dP_{\text{scatt}}}{d\Omega} \propto |\mathbf{E}_d|^2 \propto 1 + \cos^2 \theta. \quad (\text{S-10.89})$$

- b)** The radiation from most sources (sunlight is a typical example) is usually incoherent. This means that its phase and electric field direction change randomly at time intervals not much longer than the oscillation period. Thus, the radiation is effectively unpolarized at direct observation, in the sense that it is not possible to measure a definite polarization because of its fast variations. However, (S-10.88) shows that, independently of the source polarization, the radiation scattered at  $90^\circ$  ( $\cos \theta = 0$ ) is always linearly polarized (in the direction perpendicular to both the wave vector of the incoming light and the observation direction). Hence, incoherent radiation that has undergone scattering (as the blue light from the sky) tends to be polarized, even if the radiation from the primary source (in this case the sun) is unpolarized. A measurement of the polarization might help, then, to localize the position of the Sun on a cloudy day.

## S-10.9 Polarization Effects on Thomson Scattering

- a)** Equation (10.7) leads to the following two equations for the velocity components of the electron,  $v_x$  and  $v_y$ ,

$$m_e v_x = -e E_0 \cos \theta \cos(kz - \omega t), \quad m_e v_x = -e E_0 \sin \theta \sin(kz - \omega t), \quad (\text{S-10.90})$$

where  $m_e$  is the electron mass. We search for a steady-state solution of the form

$$v_x = V_{0x} \sin(kz - \omega t), \quad v_y = V_{0y} \cos(kz - \omega t), \quad (\text{S-10.91})$$

with  $V_{0x}$  and  $V_{0y}$  two real constants to be determined. Substituting into (S-10.90) we obtain

$$V_{0x} = \frac{eE_0 \cos \theta}{m_e \omega}, \quad V_{0y} = -\frac{eE_0 \sin \theta}{m_e \omega}. \quad (\text{S-10.92})$$

The second derivative of the electric dipole moment of the electron with respect to time is

$$\ddot{\mathbf{p}} = -e\dot{\mathbf{v}} = -\frac{e^2 E_0}{m_e} [(\hat{\mathbf{x}} \cos \theta \cos(kz - \omega t) + \hat{\mathbf{y}} \sin \theta \sin(kz - \omega t)], \quad (\text{S-10.93})$$

and the electron radiates at frequency  $\omega$ . The polarization for scattered radiation propagating in a generic direction of unit vector  $\hat{\mathbf{n}}$  direction is parallel to the projection of the dipole moment onto the plane perpendicular to  $\hat{\mathbf{n}}$ , i.e., to  $\mathbf{p}_\perp = (\ddot{\mathbf{p}} \times \hat{\mathbf{n}}) \times \hat{\mathbf{n}}$ . Thus, we observe linear polarization parallel to  $\hat{\mathbf{y}}$  for the radiation emitted along  $\hat{\mathbf{x}}$ , and linear polarization parallel to  $\hat{\mathbf{x}}$  for the radiation emitted along  $\hat{\mathbf{y}}$ , and elliptical polarization for the radiation emitted along  $\hat{\mathbf{z}}$ .

If  $0 < \theta < \pi/4$ , so that  $\sin \theta < \cos \theta$ , we choose the observation-direction unit vector  $\hat{\mathbf{n}} = (\sin \psi, 0, \cos \psi)$ , lying in the  $xz$  plane, and forming an angle  $\psi$  with the  $z$  axis, as shown in Fig. S-10.2, where  $\mathbf{k}_i$  is the wave vector of the incident wave. Now we choose a Cartesian reference frame  $x', y', z'$ , with  $y' \equiv y$  and  $z'$  along  $\hat{\mathbf{n}}$ , so that the scattered radiation of interest is propagating along  $z'$ .

If we perform an orthogonal projection onto the  $x'y'$  plane of an ellipse lying on the  $xy$  plane, of half axes  $\cos \theta$  parallel to  $x$ , and  $\sin \theta$  parallel to  $y$ , we obtain an ellipse of half-axes  $\cos \theta \cos \psi$  along  $x'$ , and  $\sin \theta$  along  $y'$ . Thus we observe a circular polarization if  $\cos \theta \cos \psi = \sin \theta$ , i.e., if  $\cos \psi = \tan \theta$ . Analogously, if  $\pi/4 < \theta < \pi/2$ , so that  $\sin \theta > \cos \theta$ , we choose the observation-direction unit vector  $\hat{\mathbf{n}} = (0, \sin \psi, \cos \psi)$ , lying in the  $yz$  plane, and we observe circular polarization if  $\sin \theta \cos \psi = \cos \theta$ , i.e., if  $\cos \psi = \cot \theta$ . b) The average total scattered power is

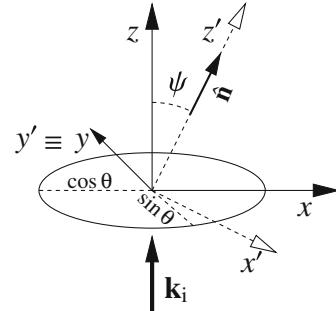


Fig. S-10.2

$$P = \frac{2}{3c^3} \langle |\ddot{\mathbf{p}}|^2 \rangle = \frac{2e^4}{3m_e^2 c^3} \langle |\mathbf{E}|^2 \rangle, \quad (\text{S-10.94})$$

where

$$\langle |\mathbf{E}|^2 \rangle = \langle E_x^2 + E_y^2 \rangle = \frac{1}{2} E_0^2 (\cos^2 \theta + \sin^2 \theta) = \frac{1}{2} E_0^2. \quad (\text{S-10.95})$$

Thus, the total scattered power is independent of  $\theta$  and can be written as

$$P = \frac{e^4 E_0^2}{3m_e^2 c^3} = \frac{c E_0^2}{3} r_e^2 = \frac{4\pi}{3} r_e^2 I, \quad (\text{S-10.96})$$

where

$$r_e = \frac{e^2}{m_e c^2}, \quad \text{and} \quad I = \frac{c E_0^2}{4\pi},$$

are the classical electron radius and the intensity of the incident wave, respectively.

c) The magnetic field of the wave is

$$\mathbf{B} = E_0 [-\hat{x} \sin \theta \sin(kz - \omega t) + \hat{y} \cos \theta \cos(kz - \omega t)]. \quad (\text{S-10.97})$$

The only non-vanishing component of  $\mathbf{v} \times \mathbf{B}$  is in the  $\hat{z}$  direction, and the magnetic force on the electron can be written as

$$\begin{aligned} F_z &= -\frac{e}{c} (\mathbf{v} \times \mathbf{B})_z = -\frac{e}{c} (v_x B_y - v_y B_x) \\ &= -\frac{e^2 E_0^2}{2cm_e \omega} (\cos^2 \theta - \sin^2 \theta) \sin(2kz - 2\omega t), \end{aligned} \quad (\text{S-10.98})$$

this quantity vanishes for  $\theta = \pi/4$ , when  $\cos \theta = \sin \theta$ , i.e., for circular polarization.

d) The magnetic force  $F_z$  drives dipole oscillations along the  $z$  axis at frequency  $2\omega$ . Thus, in addition to the scattered radiation of frequency  $\omega$  discussed at points a) and b), we observe also scattered radiation of frequency  $2\omega$ , angularly distributed as  $\sin^2 \psi$  around the  $z$  axis. Since the dipole oscillating at  $2\omega$  is perpendicular to the dipole oscillating at  $\omega$ , we can simply add the corresponding scattered powers. Now we want to evaluate the power emitted at frequency  $2\omega$ .

The equation of motion for the electron along the  $z$  axis is (we put  $\cos^2 \theta - \sin^2 \theta = \cos 2\theta$ )

$$m_e \ddot{v}_z = F_z = -\frac{e^2 E_0^2}{2cm_e \omega} \cos 2\theta \sin(2kz - 2\omega t). \quad (\text{S-10.99})$$

Once more, we search for a steady-state solution of the form

$$v_z = V_{0z} \cos(2kz - 2\omega t), \quad (\text{S-10.100})$$

with  $V_{0z}$  a constant. Substituting into (S-10.99) we obtain

$$V_{0z} = -\frac{e^2 E_0^2}{4cm_e^2 \omega^2} \cos(2\theta) \quad (\text{S-10.101})$$

and

$$\dot{v}_z = -\frac{e^2 E_0^2}{2cm_e^2 \omega} \cos(2\theta) \sin(2kz - 2\omega t) . \quad (\text{S-10.102})$$

The total average power emitted by the dipole oscillating at  $2\omega$  is

$$\begin{aligned} P_{2\omega} &= \frac{2}{3c^3} \left\langle |\dot{\mathbf{p}}_{2\omega}|^2 \right\rangle = \frac{2}{3c^3} \left\langle |ev_z|^2 \right\rangle = \frac{e^6 E_0^4}{12c^5 m_e^4 \omega^2} \cos^2(2\theta) \\ &= \frac{4\pi}{3} \frac{e^2 E_0^2}{4c^2 m_e^2 \omega^2} \cos^2(2\theta) r_e^2 I \\ &= \frac{4\pi}{3} \frac{V_{0z}}{c} \cos^2(2\theta) r_e^2 I . \end{aligned} \quad (\text{S-10.103})$$

## S-10.10 Scattering and Interference

- a)** With a proper choice of the time origin, the electric field of the incident plane wave at  $x = \pm d/2$  can be written as

$$\mathbf{E}_i \left( \pm \frac{d}{2}, t \right) = E_0 e^{\pm ikd/2 - i\omega t} \hat{\mathbf{z}} , \quad (\text{S-10.104})$$

and the phase difference between the two scatterers is

$$\phi_+ - \phi_- = kd . \quad (\text{S-10.105})$$

We denote by  $r_{\pm}$  the optical paths between the observation point  $P$  and the scatterers located at  $(\pm d/2, 0, 0)$ , as shown in Fig. 10.5. The difference between the two optical paths is

$$\Delta r = (r_+ - r_-) \simeq -d \sin \theta , \quad (\text{S-10.106})$$

where  $\theta$  is the angle between the  $y$  axis and the line joining the origin to  $P$ , as shown in Fig. 10.5. The approximation is valid for  $L \gg d$ . The phase difference between the two scattered waves in  $P$  is obtained by combining (S-10.105) and (S-10.106),

$$\Delta\phi = kd(1 - \sin \theta) . \quad (\text{S-10.107})$$

- b)** If we neglect the difference between the magnitudes of the scattered fields  $\mathbf{E}_+$  and  $\mathbf{E}_-$  in  $P$ ,  $\mathbf{E}_{\pm}$  being the field of the wave scattered at  $(\pm d/2, 0, 0)$ , the total scattered intensity  $I_s$  in  $P$  is proportional to

$$\begin{aligned} I_s &\propto |\mathbf{E}_+ + \mathbf{E}_-|^2 \propto |\mathbf{E}_+|^2 \left| e^{ikd(1-\sin\theta)/2} + e^{-ikd(1-\sin\theta)/2} \right|^2 \\ &\propto \frac{1}{r^2} \cos^2 \left[ \frac{kd}{2} (1 - \sin\theta) \right]. \end{aligned} \quad (\text{S-10.108})$$

Since  $r \cos\theta = L$ , we can also write

$$I_s \propto \frac{\cos^2\theta}{L^2} \cos^2 \left[ \frac{kd}{2} (1 - \sin\theta) \right]. \quad (\text{S-10.109})$$

We denote by  $u = (kd/2)(1 - \sin\theta)$  the argument of the second  $\cos^2$  appearing in (S-10.109). For  $-\pi/2 < \theta < \pi/2$  the variable  $u$  varies continuously and monotonically from  $kd$  to 0. If  $kd \ll 1$  (i.e., if  $d \ll \lambda/2\pi$ ), then  $\cos^2 u \approx 1$  and  $I_s(\theta) \sim \cos^2\theta$ , as if a single scatterer was present. If  $kd < \pi/2$  the function  $\cos^2 u$  has no zeros, meaning that interference fringes are not observed if the distance between the scatterers is less than  $\lambda/4$ . If

$$\frac{\pi}{2} < \frac{kd}{2} < (n+1) \frac{\pi}{2},$$

with  $n$  an integer number and  $n \geq 1$ , the function  $\cos^2 u$  has  $n$  zeros, and one observes  $n$  scattered-intensity minima and  $n+1$  maxima as  $\theta$  varies from  $-\pi/2$  to  $+\pi/2$ . The intensity of the maxima is modulated by the function  $\cos^2\theta$ .

## S-10.11 Optical Beats Generating a “Lighthouse Effect”

**a)** On the  $z = 0$  plane the electric fields  $\mathbf{E}_{\pm}$  emitted by the two dipoles are parallel to  $\hat{\mathbf{z}}$  (perpendicular to the plane), and their amplitudes are independent of  $\phi$ . Since for each dipole  $\mathbf{E}_{\pm} \propto -\omega_{\pm}^2 \mathbf{p}_0$ , the field amplitudes are  $E_+ \approx E_-$ , equal to each other up to the first order in  $\delta\omega/\omega_0$ . The difference between the optical paths from the two dipoles to  $P$  is  $\delta r \approx d \sin\phi = (\pi c/\omega_0) \sin\phi$ , which yields a phase difference of  $\pi \sin\phi$ . The total field may be thus written as

$$\begin{aligned} E &= E_0 \cos[(\omega_0 + \delta\omega/2)t + \pi \sin\phi/2] + E_0 \cos[(\omega_0 - \delta\omega/2)t - \pi \sin\phi/2] \\ &= 2E_0 \cos(\omega_0 t) \cos(\delta\omega t + \pi \sin\phi). \end{aligned} \quad (\text{S-10.110})$$

**b)** The EM energy flux in the radiation zone is given by Poynting's vector  $\mathbf{S}$ , which is proportional to the square modulus of the electric field. Thus

$$S \propto 4 \cos^2(\omega_0 t) \cos^2(\delta\omega t + \pi \sin\phi). \quad (\text{S-10.111})$$

Using the “fast”, or “instantaneous”, detector, only the factor  $\cos^2(\omega_0 t)$  is averaged, and the measured signal is proportional to

$$\langle S \rangle \propto 2 \cos^2(\delta\omega t + \pi \sin\phi), \quad \text{since} \quad \langle \cos^2(\omega_0 t) \rangle = \frac{1}{2}. \quad (\text{S-10.112})$$

At time  $t$ , the direction of maximum flux intensity is determined by the condition

$$\delta\omega t + \pi \sin \phi = \begin{cases} 0 \\ \pi \end{cases}, \quad (\text{S-10.113})$$

which means that the direction of maximum flux  $\phi_{\max}$  rotates in the  $z = 0$  plane, similarly to a lighthouse beam, according to

$$\phi_{\max}(t) = \arcsin\left(-\frac{\delta\omega}{\pi} t\right). \quad (\text{S-10.114})$$

If the EM flux is measured with the “slow” detector, i.e., averaging over times longer than  $2\pi/\delta\omega$ , both  $\cos^2$  terms of (S-10.111) are averaged to 1/2, and the total flux is the sum of the two independent fluxes from the two dipoles.

- c) Now the observation point  $P$  is on the  $x = 0$  plane, at a distance  $\mathbf{r}$  from the origin, as in Fig. S-10.3. The angle between the  $z$  axis and  $\mathbf{r}$  is  $\theta$ . Within our approximations, the intensities of the two electric fields  $\mathbf{E}_+$  and  $\mathbf{E}_-$  in  $P$  are equal and proportional to  $\sin\theta$ . Thus the two separate intensities are dependent on  $\theta$ , while they are independent of  $\phi$  on the  $z = 0$  plane. The amplitude of the Poynting vector is proportional to

$$S \propto 2 \sin^2 \theta \cos^2(\omega_0 t) \cos^2(\delta\omega t + \pi \sin \theta). \quad (\text{S-10.115})$$

Thus the “fast” detector still measures a rotation of the direction of maximum emission, but the intensity is modulated by a  $\sin^2 \theta$  factor.

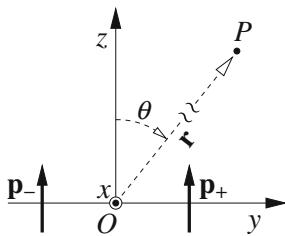


Fig. S-10.3

## S-10.12 Radiation Friction Force

- a) We insert (10.11) for  $\mathbf{F}_{\text{rad}}$  into (10.9), obtaining

$$\begin{aligned} \int_t^{t+T} \mathbf{F}_{\text{rad}}(t) \cdot \mathbf{v}(t) dt &= m_e \tau \int_t^{t+T} \frac{d^2 \mathbf{v}(t)}{dt^2} \cdot \mathbf{v}(t) dt \\ &= m_e \tau \left[ \frac{d\mathbf{v}(t)}{dt} \cdot \mathbf{v}(t) \right]_t^{t+T} - m_e \tau \int_t^{t+T} \left| \frac{d\mathbf{v}(t)}{dt} \right|^2 dt, \quad (\text{S-10.116}) \end{aligned}$$

where we have used integration by parts in the second line. The first term vanishes since the motion is periodic<sup>1</sup>:

$$\left[ \frac{d\mathbf{v}(t)}{dt} \cdot \mathbf{v}(t) \right]_t^{t+T} = \frac{1}{2} \left[ \frac{d}{dt} \mathbf{v}^2(t+T) - \frac{d}{dt} \mathbf{v}^2(t) \right] = 0. \quad (\text{S-10.117})$$

We thus obtain

$$\int_t^{t+T} \mathbf{F}_{\text{rad}}(t) \cdot \mathbf{v}(t) dt = -m_e \tau \int_t^{t+T} \left| \frac{d\mathbf{v}}{dt} \right|^2 dt. \quad (\text{S-10.118})$$

Substituting Larmor's formula (10.10) into the right-hand side of (10.9) we obtain

$$-\int_0^t P_{\text{rad}}(t') dt' = -\int_0^t \frac{2e^2}{3c^2} \left| \frac{d\mathbf{v}}{dt'} \right|^2 dt', \quad (\text{S-10.119})$$

and (10.9) is verified if we choose

$$\tau = \frac{2e^2}{3m_e c^2}. \quad (\text{S-10.120})$$

Apart from the 2/3 factor,  $\tau$  is the time needed by light to travel a distance equal to the classical electron radius  $r_e = 2.82 \times 10^{-13}$  cm, and we have  $\tau \sim 10^{-23}$  s.

**b)** After substituting (10.12) into (10.8), we search for a steady-state solution of the form  $\mathbf{v}(t) = \mathbf{v}_0 e^{-i\omega t}$ , and find

$$\mathbf{v}_0 = -\frac{ie\mathbf{E}_0}{m_e \omega (1 + i\omega \tau)}. \quad (\text{S-10.121})$$

Analogously, the steady-state solution of (10.13) is

$$\mathbf{v}_0 = -\frac{ie\mathbf{E}_0}{m_e \omega \left(1 + i\frac{\eta}{\omega}\right)}. \quad (\text{S-10.122})$$

The two solutions are identical if we choose  $\eta = \omega^2 \tau$ . The same result can be obtained by a direct comparison of  $F_{\text{rad}}$  to the frictional force  $-m_e \eta \mathbf{v}$ .

Equation (10.8) represents the first attempt to derive an expression for the “radiation friction” or “radiation reaction” force which is deeply related to the back-action of the electron on itself, since the electron interacts with the electric field it generates (self-force<sup>9</sup>). However, (10.8) is considered unsatisfactory for two reasons: (i) it increases the order of the equation of motion, and, consequently, one needs a further initial condition for the acceleration; and ii) it has unphysical “runaway”

<sup>1</sup>Actually it is not strictly necessary for the motion to be periodic, it is sufficient that  $d\mathbf{v}^2(t)/dt$  vanishes at the initial and final instants of the time interval considered.

solutions in the absence of an external field, such as  $\mathbf{a}(t) = \mathbf{a}_0 e^{t/\tau}$  with  $\mathbf{a} = d\mathbf{v}/dt$ . This problem has a long and still open history. Additional discussion may be found in textbooks and in the literature, also in very recent works related to highly relativistic electrons in ultraintense laser fields (for which the radiation friction effect becomes important).

# Chapter S-11

## Solutions for Chapter 11

### S-11.1 Wave Propagation in a Conductor at High and Low Frequencies

a) We determine the conductivity of the metal by searching for a steady-state solution in complex form,  $v = \tilde{v} e^{-i\omega t}$ , of (11.5) in the presence of an oscillating electric field  $\mathbf{E}(\mathbf{r}, t) = \tilde{\mathbf{E}} e^{-i\omega t}$ . We find

$$\tilde{v} = -\frac{ie}{m_e(\omega + i\eta)} \tilde{\mathbf{E}}, \quad (\text{S-11.1})$$

corresponding to a current density

$$\tilde{\mathbf{J}} = -en_e \tilde{v} = \frac{ie^2 n_e}{m_e \omega (\omega + i\eta)} \tilde{\mathbf{E}} = \frac{i\omega_p^2}{4\pi(\omega + i\eta)} \tilde{\mathbf{E}} \equiv \sigma(\omega) \tilde{\mathbf{E}}, \quad (\text{S-11.2})$$

where  $\omega_p$  is the plasma frequency of the metal. At the limits of high frequencies  $\omega \gg \eta$ , and of low frequencies  $\omega \ll \eta$ , we have

$$\sigma(\omega) \simeq \begin{cases} \frac{i\omega_p^2}{4\pi\omega}, & \text{for } \omega \gg \eta, \\ \frac{\omega_p^2}{4\pi\eta}, & \text{for } \omega \ll \eta. \end{cases} \quad (\text{S-11.3})$$

The DC conductivity is thus  $\sigma_{\text{DC}} = \sigma(0) = \omega_p^2 / 4\pi\eta$ . In a metal, typically we have  $\omega_p \sim 10^{16} \text{ s}^{-1}$ , since  $n_e \sim 10^{23} \text{ cm}^{-3}$  and  $\eta \sim 10^{13} \text{ s}^{-1}$ . It is thus a very good approximation to assume  $\sigma$  to be purely imaginary for optical frequencies, i.e., for  $\omega \sim 10^{15} \text{ s}^{-1}$ , and to be purely real and equal to  $\sigma_{\text{DC}}$  (i.e. independent of frequency) for microwaves and longer wavelengths.

**b)** Assuming plane geometry and monochromatic waves, in the absence of sources at  $x = +\infty$ , the electric field of the wave for  $x > 0$  can be written as (in complex notation)

$$\mathbf{E}(x, t) = \mathbf{E}_t e^{ikx - i\omega t}, \quad (\text{S-11.4})$$

where the wave vector  $k$  is determined by the general dispersion relation (11.4) in a medium where the refractive index  $n = n(\omega)$  or, equivalently, the permittivity  $\epsilon = \epsilon(\omega) = n^2$  are known. For an incident wave of amplitude  $\mathbf{E}_i$ , the electric field at the surface is given by the Fresnel formula

$$\mathbf{E}_t = \frac{2}{1+n} \mathbf{E}_i. \quad (\text{S-11.5})$$

The permittivity  $\epsilon(\omega)$  is related to the complex conductivity of the medium by (11.3). Inserting (S-11.3) for  $\sigma(\omega)$ , if  $\omega \gg \eta$  we have  $\epsilon \approx 1 - \omega_p^2/\omega^2$ , and  $k^2$  is real, so that the wave is propagating. For  $k^2 < 0$ , i.e., for  $\omega_p > \omega$ , we have  $ikx = -|k|x = -x/\ell_p$ , with  $\ell_p = c/\sqrt{\omega_p^2 - \omega^2}$ , and the wave is evanescent:

$$\mathbf{E}(x, t) = \mathbf{E}_t e^{-x/\ell_p - i\omega t} \quad (\text{S-11.6})$$

(the solution  $\propto e^{x/\ell_p}$  has been disregarded as unphysical because it is divergent for  $x \rightarrow \infty$ ). For a metal, the condition  $\omega < \omega_p$  implies that the metal is reflecting for frequencies in the optical range, while it becomes transparent for ultraviolet frequencies.

If  $\omega \ll \eta$ , we have that also  $\sigma_{DC} \ll \eta$ , so that  $\epsilon \approx 4\pi i\sigma_{DC}/\omega$  is an imaginary number. In this case, since  $k = \pm(1+i)/\ell_c$  with  $\ell_c = \sqrt{\omega\sigma_{DC}/2c}$ , the evanescent solution is

$$\mathbf{E}(x, t) = \mathbf{E}_t e^{-x/\ell_c - ix/\ell_c - i\omega t}. \quad (\text{S-11.7})$$

**c)** The net flux of energy through the surface is given by the time average of the  $x$ -component of the Poynting vector  $\mathbf{S} = (c/4\pi)\mathbf{E} \times \mathbf{B}$  at  $x = 0$ . We obtain the magnetic field of the wave from the relation  $\partial_t \mathbf{B} = -c\nabla \times \mathbf{E}$ . Thus the complex field amplitudes for  $x \geq 0$  can be written as

$$\tilde{E}_y = E_t e^{ik_0 n x}, \quad \tilde{B}_z = n E_t e^{ik_0 n x}, \quad (\text{S-11.8})$$

where  $k_0 = \omega/c$ . Thus we need to evaluate

$$\langle S_x(0) \rangle = \frac{1}{2} \frac{c}{4\pi} \operatorname{Re} [\tilde{E}_y(0) \tilde{B}_z^*(0)] = \frac{c}{8\pi} |E_t|^2 \operatorname{Re}(n). \quad (\text{S-11.9})$$

At the limit  $\omega \gg \eta$ ,  $n$  is purely imaginary, as found above, and  $\langle S_x(0) \rangle = 0$ , and there is no energy dissipated into the metal (it can be easily shown that the reflection coefficient obtained from the Fresnel formulas has unity modulus, i.e., all the incident energy is reflected). At the limit  $\omega \gg \eta$  we obtain

$$\langle S_x(0) \rangle = \frac{c}{8\pi} |E_t|^2 \sqrt{\frac{2\pi\sigma_{DC}}{\omega}} \simeq \frac{c}{16\pi} |E_i|^2 \sqrt{\frac{\omega}{2\pi\sigma_{DC}}}, \quad (\text{S-11.10})$$

where in the latter expression  $|1 + n|^2 \simeq |n|^2 = 2(2\pi\sigma_{DC}/\omega)$  has been assumed.

The energy dissipated per unit volume is

$$\begin{aligned} \langle \mathbf{J} \cdot \mathbf{E} \rangle &= \frac{1}{2} \operatorname{Re}(\sigma \tilde{E}_y \tilde{E}_y^*) = \frac{|E_t|^2}{2} \operatorname{Re}(\sigma e^{ik_0 n x} e^{-ik_0 n^* x}) \\ &= \frac{|E_t|^2}{2} \operatorname{Re}(\sigma) \exp[-2k_0 \operatorname{Im}(n)x]. \end{aligned} \quad (\text{S-11.11})$$

If  $\sigma$  is imaginary then there is no dissipation, consistently with what found above. In the  $\omega \ll \eta$  regime, the total energy dissipated per unit surface is given by the integral

$$\begin{aligned} \int_0^\infty \langle \mathbf{J} \cdot \mathbf{E} \rangle dx &= \frac{|E_t|^2}{2} \frac{\sigma_{DC}}{2k_0 \operatorname{Im}(n)} = \frac{|E_t|^2}{2} \frac{\sigma_{DC}}{2\omega_c \sqrt{2\pi\sigma_{DC}/\omega}} \\ &= \frac{c}{8\pi} |E_t|^2 \sqrt{\frac{2\pi\sigma_{DC}}{\omega}}, \end{aligned} \quad (\text{S-11.12})$$

which is equal to the EM energy flux of (S-11.10).

## S-11.2 Energy Densities in a Free Electron Gas

a) We use the complex representation for all fields,  $A(x, t) = \operatorname{Re}(\tilde{A}e^{ikx-i\omega t})$ , where  $A$  is the considered field. For the electric field of the wave we have  $\tilde{\mathbf{E}} = \mathbf{E}_0$ , where  $E_0$  can be considered as a real quantity. The equation of motion for an electron, neglecting the nonlinear magnetic term, is

$$m_e \frac{d^2 \mathbf{r}}{dt^2} = m_e \frac{d\mathbf{v}}{dt} = -e\mathbf{E}, \quad (\text{S-11.13})$$

which has the steady-state solution for the electron velocity and position

$$\tilde{\mathbf{v}} = -\frac{ie}{m_e \omega} \mathbf{E}_0, \quad \tilde{\mathbf{r}} = \frac{e}{m_e \omega^2} \mathbf{E}_0. \quad (\text{S-11.14})$$

The polarization density is

$$\tilde{\mathbf{P}} = -en_e \tilde{\mathbf{r}} = -\frac{n_e e^2}{m_e \omega^2} \mathbf{E}_0 = -\frac{1}{4\pi} \frac{\omega_p^2}{\omega^2} \mathbf{E}_0 \equiv \chi(\omega) \mathbf{E}_0 , \quad (\text{S-11.15})$$

corresponding to a dielectric permittivity of the medium

$$\varepsilon(\omega) = 1 + 4\pi\chi(\omega) = 1 - \frac{\omega_p^2}{\omega^2} . \quad (\text{S-11.16})$$

Using (11.4), the dispersion relation is obtained as

$$\omega^2 = \frac{k^2 c^2}{\varepsilon(\omega)} = \omega_p^2 + k^2 c^2 . \quad (\text{S-11.17})$$

The phase and group velocities are

$$v_\varphi = \frac{\omega}{k} = \frac{c}{\sqrt{1 - \frac{\omega_p^2}{\omega^2}}} , \quad v_g = \frac{\partial \omega}{\partial k} = c \sqrt{1 - \frac{\omega_p^2}{\omega^2}} , \quad (\text{S-11.18})$$

so that both  $v_\varphi$  and  $v_g$  are real if  $\omega > \omega_p$ , and  $v_\varphi v_g = c^2$ . Finally, using the equation  $c\mathbf{v} \times \mathbf{E} = -\partial_t \mathbf{B}$ . i.e.,  $ikc\tilde{\mathbf{E}} = i\omega \tilde{\mathbf{B}}$ , we obtain  $E_0 = (v_\varphi/c)B_0$ .

**b)** From the definition of the EM energy density

$$\begin{aligned} u_{\text{EM}} &= \left\langle \frac{1}{8\pi} (\mathbf{E}^2 + \mathbf{B}^2) \right\rangle = \frac{1}{16\pi} (E_0^2 + B_0^2) = \frac{1}{16\pi} E_0^2 \left( 1 + \frac{c^2}{v_\varphi^2} \right) \\ &= \frac{1}{16\pi} E_0^2 \left( 2 - \frac{\omega_p^2}{\omega^2} \right) . \end{aligned} \quad (\text{S-11.19})$$

**c)** From the definition of the kinetic energy density

$$\begin{aligned} u_K &= \left\langle n_e \frac{m_e}{2} \mathbf{v}^2 \right\rangle = n_e \frac{m_e}{2} \frac{1}{2} \left| \frac{eE_0}{m_e \omega} \right|^2 = \frac{1}{4} \frac{n_e e^2}{m_e \omega^2} E_0^2 \\ &= \frac{1}{16\pi} E_0^2 \frac{\omega_p^2}{\omega^2} . \end{aligned} \quad (\text{S-11.20})$$

Thus

$$u = u_{\text{EM}} + u_K = \frac{1}{8\pi} E_0^2 , \quad (\text{S-11.21})$$

independently of  $n_e$ .

**d)** In our case (11.6) can be rewritten

$$v_g E_t^2 = c(E_0^2 - E_r^2). \quad (\text{S-11.22})$$

Using Fresnel formulas as functions of the phase velocity  $v_\varphi = c/n$ , with  $n = \sqrt{\epsilon_r}$ , we obtain

$$E_r = \frac{v_\varphi - c}{v_\varphi + c} E_0, \quad E_t = \frac{2v_\varphi}{v_\varphi + c} E_0, \quad (\text{S-11.23})$$

leading to

$$4v_g v_\varphi^2 = 4c^2 v_\varphi, \quad (\text{S-11.24})$$

which is equivalent to  $v_g v_\varphi = c^2$ .

### S-11.3 Longitudinal Waves

**a)** We obtain from Maxwell's equations, assuming  $\mathbf{B} = 0$ ,

$$0 = \nabla \times \mathbf{B} = \frac{1}{c} (4\pi \mathbf{J} + \partial_t \mathbf{E}) = \frac{1}{c} (4\pi \partial_t \mathbf{P} + \partial_t \mathbf{E}) = \frac{1}{c} \partial_t (4\pi \mathbf{P} + \mathbf{E}). \quad (\text{S-11.25})$$

where  $\mathbf{P}$  is the polarization density of the medium and  $\mathbf{J} = \partial_t \mathbf{P}$  the associated polarization current. Assuming all fields to have an harmonic dependence  $\sim e^{-i\omega t}$ , we have  $\mathbf{P} = \chi(\omega) \mathbf{E}$  with  $\chi = [\epsilon_r(\omega) - 1]/(4\pi)$ . Now, using (11.7), we can write

$$0 = -i\omega (4\pi \mathbf{P} + \mathbf{E}) = -i\omega \{[\epsilon_r(\omega) - 1] \mathbf{E} + \mathbf{E}\} = -i\omega \epsilon_r(\omega) \mathbf{E}, \quad (\text{S-11.26})$$

implying  $\epsilon_r(\omega) = 0$ .

**b)** The total charge and current densities in the medium can be obtained from  $\mathbf{E}$  using the equations

$$\varrho = \frac{1}{4\pi} \nabla \cdot \mathbf{E}, \quad \mathbf{J} = -\frac{1}{4\pi} \partial_t \mathbf{E}, \quad (\text{S-11.27})$$

which also imply the continuity equation  $4\pi \partial_t \varrho = -\nabla \cdot \mathbf{J}$ . For  $\mathbf{E}$  given by (11.7) we obtain

$$\varrho = \frac{ik}{4\pi} E_0 e^{ikx-i\omega t}, \quad \mathbf{J} = \hat{\mathbf{x}} \frac{i\omega}{4\pi} E_0 e^{ikx-i\omega t}. \quad (\text{S-11.28})$$

c) Assuming electrons moving with negligible friction, the equation of motion for the single electron is

$$m_e \frac{d^2 \mathbf{r}}{dt^2} = -m_e \omega_0^2 \mathbf{r} - e \mathbf{E}, \quad (\text{S-11.29})$$

where  $m_e$  is the electron mass, and  $\mathbf{r}$  is the distance of the electron from its equilibrium position. For a monochromatic field  $\mathbf{E} = \mathbf{E}_0 e^{-i\omega t}$  the stationary solution is

$$\mathbf{r} = \frac{e \mathbf{E}}{m_e(\omega^2 - \omega_0^2)}. \quad (\text{S-11.30})$$

The polarization density of the medium is

$$\mathbf{P} = -en_e \mathbf{r} = -\frac{n_e e^2}{m_e(\omega^2 - \omega_0^2)} \mathbf{E} \equiv \chi(\omega) \mathbf{E}, \quad (\text{S-11.31})$$

where  $n_e$  is the number of electrons per unit volume, and

$$\chi(\omega) = -\frac{n_e e^2}{m_e(\omega^2 - \omega_0^2)} = -\frac{\omega_p^2}{\omega^2 - \omega_0^2}, \quad (\text{S-11.32})$$

is the dielectric susceptibility of the medium, and  $\omega_p = \sqrt{4\pi n_e e^2 / m_e}$  is its plasma frequency. The dielectric permittivity is thus

$$\epsilon_r(\omega) = 1 + 4\pi\chi(\omega) = 1 - \frac{\omega_p^2}{\omega^2 - \omega_0^2}, \quad (\text{S-11.33})$$

and the longitudinal-wave condition  $\epsilon_r(\omega) = 0$  leads to

$$\omega = \sqrt{\omega_p^2 + \omega_0^2}. \quad (\text{S-11.34})$$

It is important to notice that the wavevector  $\mathbf{k}$  is *not* determined by this equation; it may have any value, and the phase velocity may thus be arbitrary (lower or greater than  $c$ ). Longitudinal waves in condensed matter physics are also called *polaritons*. In a free electron medium where  $\omega_0 = 0$  (a simple metal, a ionized gas or a plasma), we have  $\omega = \omega_p$ ; in this case the waves are called plasma waves or *plasmons*.

## S-11.4 Transmission and Reflection by a Thin Conducting Foil

a) Since the problem of determining the transmission and reflection coefficient is linear, and the medium is isotropic, the choice of polarization is arbitrary. For definiteness, we assume linear polarization, with the electric field  $\mathbf{E}$  of the incoming wave parallel to the  $y$  axis, and the magnetic field  $\mathbf{B}$  parallel to the  $z$  axis.

We apply Stokes's theorem to a closed rectangular path  $C$ , delimiting a surface area  $A$ , twice: once for  $\mathbf{E}$  and once for  $\mathbf{B}$ . In both cases the base of the path extends from  $x = -h/2$  to  $x = +h/2$ , while the height, of length  $\ell$ , is parallel to the  $y$  axis for the electric field  $\mathbf{E}$ , as shown in Fig. S-11.1, and to the  $z$  axis for the magnetic field  $\mathbf{B}$ . For the electric field we have

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = \left[ E\left(\frac{h}{2}\right) - E\left(-\frac{h}{2}\right) \right] \ell = +i \frac{\omega}{c} \int_A \mathbf{B} \cdot d\mathbf{A} = i \frac{\omega}{c} \bar{B} \ell h, \quad (\text{S-11.35})$$

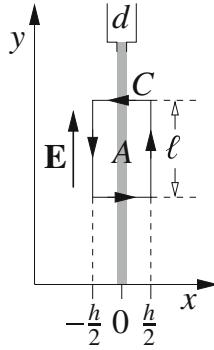


Fig. S-11.1

where  $\bar{B}$  is the amplitude of  $\mathbf{B}$  at some point of the surface  $A$ , according to the mean value theorem. Since  $\mathbf{B}$  is limited, at the limit  $h \rightarrow 0$  we have  $\bar{B}h \rightarrow 0$ , and the first of (11.8) is proved. For the magnetic field we have

$$\begin{aligned} \oint_C \mathbf{B} \cdot d\mathbf{l} &= \left[ B\left(\frac{h}{2}\right) - B\left(-\frac{h}{2}\right) \right] \ell = \int_A \left( \frac{4\pi}{c} \mathbf{J} - i \frac{\omega}{c} \mathbf{E} \right) \cdot d\mathbf{A} \\ &= \int_A \frac{4\pi}{c} K \delta(x) dx dz - i \int_A \frac{\omega}{c} \mathbf{E} \cdot d\mathbf{A} \\ &= \frac{4\pi}{c} \ell \bar{K} + -i \frac{\omega}{c} \bar{E} \ell h, \end{aligned} \quad (\text{S-11.36})$$

where, in the second line, we have replaced  $\mathbf{J}$  by  $\mathbf{K}\delta(x)$ , and, in the third line,  $\bar{K}$  is a value assumed by  $K$  somewhere on the segment of length  $\ell$ . Since, again,  $\bar{E}$  is limited, the product  $\bar{E} \ell h \rightarrow 0$  as  $h \rightarrow 0$ , and the second of (11.8) is proved.

b) The most general expression for the field is the sum of the incident and the reflected wave for  $x < 0$ , and the transmitted wave only for  $x > 0$ :

$$E(x, t) = \begin{cases} E_i e^{ikx-i\omega t} + E_r e^{-ikx-i\omega t}, & x < 0, \\ E_t e^{ikx-i\omega t}, & x > 0. \end{cases} \quad (\text{S-11.37})$$

The amplitudes  $E_r$  and  $E_t$  must be determined as functions of  $E_i$  and other parameters, by imposing (11.8) as boundary conditions. Noticing that  $K = \sigma dE(0) = \sigma dE_t$  and that  $c\partial_x E(x, t) = -\partial_t B(x, t)$ , we have

$$E_t - E_i - E_r = 0, \quad E_t - E_i + E_r = -4\pi \frac{\sigma d}{c} E_t, \quad (\text{S-11.38})$$

so that, writing  $2\pi\sigma d/c = \eta$  as a shorthand, we have

$$E_r = -\frac{\eta}{1+\eta} E_i, \quad E_t = \frac{1}{1+\eta} E_i. \quad (\text{S-11.39})$$

**c)** At the limit  $\eta \gg \omega$  the conductivity is given by  $\sigma = n_e e^2 / m_e \eta$  and is a real number (Ohmic conductor). The mechanical power  $P$  is the cycle average of  $\mathbf{J} \cdot \mathbf{E}$  integrated over the volume of the foil, thus we obtain (per unit surface)

$$P = \frac{1}{2} |E(0)|^2 d = \frac{1}{2} \frac{\sigma d}{(1+\eta)^2} E_i^2 = \frac{c}{4\pi} \frac{\eta}{(1+\eta)^2} E_i^2, \quad (\text{S-11.40})$$

(notice that  $E_i$  can be taken as a real quantity).

At the limit  $\eta \ll \omega$  the conductivity is  $\sigma = i n_e e^2 / m_e \omega = i \omega_p^2 / 4\pi \omega$  and is thus imaginary, corresponding to a real permittivity  $\epsilon = 1 - \omega_p^2 / \omega^2$ . Accordingly,  $\mathbf{J}$  and  $\mathbf{E}$  have opposite phase, and  $\langle \mathbf{J} \cdot \mathbf{E} \rangle = 0$ , as can be directly verified.

**d)** The energy flux through the foil is given by the difference between the values of the Poynting flux at the two surfaces (here we switch back to real fields for simplicity),

$$S(0^+) - S(0^-) = \frac{c}{4\pi} [E(0^+)B(0^+) - E(0^-)B(0^-)]. \quad (\text{S-11.41})$$

Inserting the boundary conditions we may write

$$E(0^+)B(0^+) - E(0^-)B(0^-) = E(0)[B(0^+) - B(0^-)] = -E(0) \frac{4\pi}{c} J d, \quad (\text{S-11.42})$$

so that

$$S(0^+) - S(0^-) = -JE(0)d = -KE(0), \quad (\text{S-11.43})$$

i.e., the energy flux through the foil equals the mechanical power dissipated in the foil (all quantities have been defined per unit surface).

Alternatively, we may compute the energy flux directly and compare it to the mechanical power. For the cycle-averaged Poynting vector at the two surfaces we have

$$\langle S(0^+) \rangle = \frac{c}{4\pi} \langle E^2(0^+) \rangle = \frac{2\pi}{c} |E_t|^2 = \frac{2\pi}{c} \frac{1}{|1+\eta|^2} E_i^2, \quad (\text{S-11.44})$$

$$\begin{aligned} \langle S(0^-) \rangle &= \epsilon_0 c^2 \langle E(0^-) B(0^-) \rangle = \frac{2\pi}{c} \operatorname{Re} [(E_i + E_r)(E_i^* - E_r^*)] \\ &= \frac{2\pi}{c} \frac{1}{|1+\eta|^2} \operatorname{Re} (2\eta^* + 1) E_i^2. \end{aligned} \quad (\text{S-11.45})$$

If  $\eta \ll \omega$ , then  $\eta$  is purely imaginary and  $S(0^-) = S(0^+)$ : there is no net energy flux inside the foil, consistently with the vanishing of the mechanical power.

If  $\eta \gg \omega$ , then  $\eta$  is real and the net flux of energy is

$$\langle S(0^+) \rangle - \langle S(0^-) \rangle = \frac{2\pi}{c} \frac{1-(2\eta+1)}{(1+\eta)^2} E_i^2 = -\frac{c}{4\pi} \frac{\eta}{(1+\eta)^2} E_i^2, \quad (\text{S-11.46})$$

which is equal to minus the absorbed power (S-11.40).

## S-11.5 Anti-Reflection Coating

**a)** In the absence of sources at  $x = +\infty$ , the general solution can be written as (omitting the common time dependence  $e^{-i\omega t}$ )

$$E = \begin{cases} E_i e^{ikx} + E_r e^{-ikx} & (x < 0), \\ E_+ e^{in_1 kx} + E_- e^{-in_1 kx} & (0 < x < d), \\ E_t e^{in_2 kx} & (x > d), \end{cases} \quad (\text{S-11.47})$$

where  $k = \omega/c$ ,  $E_i$  is the amplitude of the incident wave,  $E_r$  the amplitude of the wave reflected at  $x = 0$ ,  $E_+$  and  $E_-$  the amplitudes of the waves propagating along  $+\hat{x}$  and  $-\hat{x}$ , respectively, in the  $0 < x < d$  layer, and  $E_t$  the amplitude of the wave propagating along  $+\hat{x}$  in the  $x > d$  half-space. The subscripts of the electric fields  $E$  in (S-11.47) are in agreement with the subscripts of the wave vectors  $\mathbf{k}$  in Fig. 11.2.

**b)** The matching conditions require the electric field and its derivative with respect to  $x$  (which is proportional to the magnetic field) to be continuous at the planes  $x = 0$  and  $x = d$ . We thus obtain

$$E_i + E_r = E_+ + E_-, \quad (\text{S-11.48})$$

$$E_i - E_r = n_1(E_+ - E_-), \quad (\text{S-11.49})$$

$$E_+ e^{+in_1 kd} + E_- e^{-in_1 kd} = E_t e^{in_2 kd}, \quad (\text{S-11.50})$$

$$n_1(E_+ e^{+in_1 kd} - E_- e^{-in_1 kd}) = n_2 E_t e^{in_2 kd}. \quad (\text{S-11.51})$$

c) Since we require that there is no reflected wave in vacuum,  $E_r$  must be zero. Posing  $E_r = 0$  in (S-11.48)-(S-11.51), the latter can be regarded as an homogeneous linear system in  $E_i$ ,  $E_+$ ,  $E_-$  and  $E_t$ . Such system has non-trivial solutions only if its determinant is zero, i.e. if

$$e^{2in_1kd} = \frac{n_1 + n_2}{n_1 - n_2} \frac{n_1 - 1}{n_1 + 1}. \quad (\text{S-11.52})$$

In the case of a layer of thickness  $d$  with vacuum at both sides,  $n_2 = 1$  and the right-hand side of (S-11.52) equals unity, thus  $e^{2in_1kd} = 1$ . This implies  $2n_1kd = 2m\pi$ , with  $m$  any integer. Thus, there is no reflected wave when the layer thickness is  $d = m\lambda/2n_1$  (since  $k = 2\pi/\lambda$ ), i.e. when the “optical depth”  $nd$  equals an half-integer number of wavelengths.

d) In the general case, the left-hand side of (S-11.52) is a complex number of modulus 1, while the right-hand side is always real number if  $n_1$  and  $n_2$  are real as we assumed. Thus, we have solutions only if  $e^{2in_1kd} = \pm 1$ . The case  $e^{2in_1kd} = +1$  is the case of  $n_2 = 1$ , considered above at the end of point c). In the second case  $e^{2in_1kd} = -1$  we have the condition

$$2n_1kd = (2m+1)\pi, \quad \frac{n_1 + n_2}{n_1 - n_2} \frac{n_1 - 1}{n_1 + 1} = -1, \quad (\text{S-11.53})$$

the second equation implying  $n_2 = \sqrt{n_1}$ . The thickness of the layer must be

$$d = (2m+1) \frac{\lambda}{4n_1}, \quad (\text{S-11.54})$$

with  $m$ , again, any integer. The smallest possible thickness is  $d = \lambda/(4n_1)$ , corresponding to  $m = 0$ . This shows that, with a suitable choice of materials and of layer thickness, we can produce an “anti-reflection” coating on an optical element (such as a window or lens) from which we do not want any reflection to occur.

## S-11.6 Birefringence and Waveplates

a) The incident wave can be considered as the superposition of two waves having, respectively,  $P$  and  $S$  polarization, i.e., one having the electric field lying in the  $xy$  plane, and the other parallel to  $z$ . The difference between the refractive indices for  $P$  and  $S$  polarization,  $n_p$  and  $n_s$ , gives origin to two different refraction angles,  $\theta_{t,p}$  and  $\theta_{t,s}$ , according to Snell's law. With our assumptions, the refraction angles are

$$\sin \theta_{t,p} = \frac{\sin \theta_i}{n_p} = \sin(\theta_t - \alpha), \quad \sin \theta_{t,s} = \frac{\sin \theta_i}{n_s} = \sin(\theta_t + \alpha), \quad (\text{S-11.55})$$

at the limit  $\alpha \ll 1$  we can approximate  $\sin \alpha \simeq \alpha$  and  $\cos \alpha \simeq 1$ , obtaining

$$\sin(\theta_t \pm \alpha) = \sin \theta_t \cos \alpha \pm \cos \theta_t \sin \alpha \simeq \sin \theta_t \pm \alpha \cos \theta_t . \quad (\text{S-11.56})$$

The refractive indices are  $n_p = \bar{n} + \delta n$ , and  $n_s = \bar{n} - \delta n$ , respectively, with  $\delta n \ll \bar{n}$ . For  $P$  polarization we have, up to the first order in  $\delta n/\bar{n}$ ,

$$\sin \theta_t - \alpha \cos \theta_t = \frac{\sin \theta_i}{\bar{n} + \delta n} \simeq \frac{\sin \theta_i}{\bar{n}} \left( 1 - \frac{\delta n}{\bar{n}} \right) , \quad (\text{S-11.57})$$

and, analogously, for  $S$  polarization we have

$$\sin \theta_{t,s} \simeq \frac{\sin \theta_i}{\bar{n}} \left( 1 + \frac{\delta n}{\bar{n}} \right) . \quad (\text{S-11.58})$$

The above results lead to

$$\bar{n} = \frac{\sin \theta_i}{\sin \theta_t} , \quad \text{and} \quad \alpha = 2 \frac{\sin \theta_i}{\cos \theta_t} \frac{\delta n}{\bar{n}^2} = 2 \delta n \frac{\sin^2 \theta_t}{\cos \theta_t \sin \theta_i} . \quad (\text{S-11.59})$$

**b)** In order to have exiting circularly polarized light, the exiting  $P$ - and  $S$ -polarized components must be phase-shifted by  $\delta\phi = \pi/2$ . This can be obtained making use of the difference between the two optical path lengths,  $n_p d$  and  $n_s d$ . The condition for circularly polarized light is thus

$$\delta\phi = k 2 \delta n d = \frac{4\pi \delta n d}{\lambda} \doteq \frac{\pi}{2} , \quad (\text{S-11.60})$$

i.e.  $d = \lambda/(8\delta n)$ . This is called a *quarter-wave plate*. If  $\delta\phi = \pi$  instead, i.e., if  $d = \lambda/(4\delta n)$ , there is a relative change of sign between the two components, which leads to a polarization rotation of  $\pi/2$ ; this is an *half-wave plate*.

## S-11.7 Magnetic Birefringence and Faraday Effect

**a)** Neglecting the effect of the magnetic field of the wave, much smaller than the external field  $\mathbf{B}_0$ , the equation of motion for the electrons is

$$m_e \frac{d^2 \mathbf{r}}{dt^2} = -e \mathbf{E} - e \frac{\mathbf{v}}{c} \times \mathbf{B}_0 - m_e \omega_0^2 \mathbf{r} . \quad (\text{S-11.61})$$

The electric field of the circularly polarized EM wave can be written, in complex notation, as

$$\mathbf{E}_{\pm} = E(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}) e^{ikz-i\omega t} , \quad (\text{S-11.62})$$

where the plus and minus signs correspond to left-handed (clockwise) and right-handed (counter-clockwise) circular polarizations, respectively. We look for solutions of (S-11.61) of the form

$$\mathbf{r}_\pm = r_\pm (\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}) e^{ikz-i\omega t}, \quad \mathbf{v}_\pm = v_\pm (\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}) e^{ikz-i\omega t}, \quad (\text{S-11.63})$$

with  $v_\pm = -i\omega r_\pm$ . The vector product  $\mathbf{v}_\pm \times \mathbf{B}_0$  is

$$\mathbf{v}_\pm \times \mathbf{B}_0 = v_\pm B_0 (\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}) \times \hat{\mathbf{z}} = v_\pm B_0 (-\hat{\mathbf{y}} \pm i\hat{\mathbf{x}}) = \pm i v_\pm B_0 (\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}), \quad (\text{S-11.64})$$

thus (S-11.61) leads to the equation for  $\mathbf{r}_\pm$

$$(\omega_0^2 - \omega^2) r_\pm = -\frac{e}{m_e} E \mp ie \frac{v_\pm}{m_e c} B_0 = -\frac{e}{m_e} E \mp \omega \omega_c r_\pm, \quad (\text{S-11.65})$$

where  $\omega_c = eB_0/m_ec$  is the cyclotron frequency. The solution for  $r_\pm$  is

$$r_\pm = \frac{eE}{m_e (\omega^2 - \omega_0^2 \mp \omega \omega_c)}. \quad (\text{S-11.66})$$

Thus, we have a different polarization of the medium  $\mathbf{P}_\pm$ , and a corresponding different dielectric susceptibility  $\chi_\pm$ , for each each circular-polarization state of the EM wave,

$$\mathbf{P}_\pm = -en_e \mathbf{r}_\pm \equiv \chi_\pm \mathbf{E}_\pm. \quad (\text{S-11.67})$$

In turn, this gives two different dielectric constants  $\epsilon_\pm = 1 + 4\pi\chi_\pm$

$$\epsilon_\pm = 1 - \frac{\omega_p^2}{\omega^2 - \omega_0^2 \mp \omega \omega_c}, \quad (\text{S-11.68})$$

where  $\omega_p = \sqrt{4\pi e^2 n_e / m_e}$  is the plasma frequency of the medium. The propagation of the wave requires  $\epsilon_\pm > 0$ , i.e.,  $\omega > \omega_{co\pm}$ , where the two cutoff frequencies  $\omega_{co\pm}$  depend on the polarization of the wave

$$\omega_{co\pm} = \sqrt{\omega_0^2 + \omega_p^2 + \frac{\omega_c^2}{4}} \pm \frac{\omega_c}{2}. \quad (\text{S-11.69})$$

The magnetized medium is thus birefringent. For waves of frequency in the range  $\omega_{co-} < \omega < \omega_{co+}$ , only one state of circular polarization can propagate in the medium, while we have an evanescent wave for the opposite polarization. The two resonant frequencies  $\omega_{res\pm}$ , defined by  $\chi(\omega_{res\pm}) \rightarrow \infty$ , also depend on polarization:

$$\omega_{\text{res}\pm} = \sqrt{\omega_0^2 + \frac{\omega_c^2}{4}} \pm \frac{\omega_c}{2}. \quad (\text{S-11.70})$$

Notice that in the case  $\omega_0 = 0$ , i.e., for a magnetized free-electron medium, there is a single resonance at  $\omega = \omega_c$ , for *only one* circular polarization (see Problem 7.9).

The knowledge of the permittivity (or, equivalently, of the refraction index) for the two independent states of circular polarization is sufficient to study the propagation of a transverse wave of arbitrary polarization, since the latter can be always expressed as a linear superposition of circularly polarized states. Notice that if we had searched for linearly polarized solutions, we would have found a *mixing* of polarization vectors directed along  $\hat{x}$  and  $\hat{y}$ , i.e. the permittivity would have been a matrix instead of a number. It can be shown that such matrix can be diagonalized, with circularly polarized states as eigenvectors and (S-11.68) as eigenvalues.

**b)** The linearly polarized wave can be considered as a superposition of the two states of circular polarization, so that at  $z = 0$  the electric field of the wave can be written

$$\mathbf{E}(z = 0, t) = \hat{x} E e^{-i\omega t} = \frac{E}{2} [(\hat{x} + i\hat{y}) + (\hat{x} - i\hat{y})] e^{-i\omega t}. \quad (\text{S-11.71})$$

The two circularly polarized components travel at different phase velocities  $v_{\pm} = c/n_{\pm}$ , where  $n_{\pm} = \sqrt{\epsilon_{\pm}}$  is the refractive index associated to each polarization state. At  $z = \ell$ , the electric field of the wave is

$$\mathbf{E}(z = \ell, t) = \frac{E}{2} [(\hat{x} + i\hat{y}) e^{ik_+ \ell} + (\hat{x} - i\hat{y}) e^{-ik_- \ell}] e^{-i\omega t}, \quad (\text{S-11.72})$$

where  $k_{\pm} = \omega/v_{\pm} = (\omega/c)n_{\pm}$ . To first order in  $\omega_c/\omega$ , we can write  $n_{\pm} \approx n_0 \pm \delta n$ , where  $n_0 = n(\omega_c = 0)$  and

$$\delta n = \frac{\omega \omega_c \omega_p^2}{2n_0(\omega^2 - \omega_0^2)^2}. \quad (\text{S-11.73})$$

Thus, the wave vectors for the two polarizations can be written  $k_{\pm} \approx k_0 \pm \delta k$ , where  $k_0 = (\omega/c)n_0$  and  $\delta k = (\omega/c)\delta n$ . The electric field at  $z = \ell$  can be rewritten as

$$\begin{aligned} \mathbf{E}(z = \ell, t) &= \frac{E}{2} [(\hat{x} + i\hat{y}) e^{i\delta k \ell} + (\hat{x} - i\hat{y}) e^{-i\delta k \ell}] e^{ik_0 \ell - i\omega t} \\ &\propto \hat{x} \cos(\delta k \ell) - \hat{y} \sin(\delta k \ell). \end{aligned} \quad (\text{S-11.74})$$

The polarization has thus rotated by an angle  $\phi = \delta k \ell$ , proportional to the intensity of the magnetic field.

## S-11.8 Whistler Waves

The dielectric permittivity of a magnetized free electron gas for circularly polarized transverse waves, propagating along the magnetic field, is (see Problem 11.7)

$$\varepsilon = \varepsilon_{\pm}(\omega) = 1 - \frac{\omega_p^2}{\omega(\omega \mp \omega_c)}, \quad (\text{S-11.75})$$

where  $\omega_p = \sqrt{4\pi e^2 n_e / m_e}$  is the plasma frequency of the medium,  $\omega_c = eB_0 / m_e c$  is the cyclotron (Larmor) frequency, and the plus and minus signs refer to left-handed (counterclockwise) and right-handed (clockwise) circular polarizations, respectively. Since, in general, the dispersion relation is  $\omega^2 = k^2 c^2 / \varepsilon(\omega)$ , (11.11) implies that  $\varepsilon = c^2 / \alpha \omega$ . For  $\omega \ll \omega_c$  and  $\omega \ll \omega_p^2 / \omega_c$ , (S-11.75) reduces to

$$\varepsilon_{\pm} \simeq \pm \frac{\omega_p^2}{\omega \omega_c}. \quad (\text{S-11.76})$$

Wave propagation requires  $\varepsilon > 0$ . Thus, only left-handed polarized waves can propagate in the presence of a dispersion relation given by (11.11), with  $\alpha = c^2 \omega_c / \omega_p^2$ .

Assuming the values of  $n_e$  and  $B_0$  given in the text, we estimate  $\omega_p \simeq 5.6 \times 10^6 \text{ s}^{-1}$  and  $\omega_c \simeq 8.8 \times 10^6 \text{ s}^{-1}$ . A typical frequency for which (S-11.76) holds is  $\omega \sim 10^5 \text{ s}^{-1}$ .

**b)** First, we notice that, in general, (11.11) implies  $v_g = \partial_k \omega = 2ak = 2\omega/k = 2v_\varphi$ . Thus, the phase velocity depends on frequency as

$$v_\varphi = \frac{\omega}{k} = \sqrt{\alpha \omega} = \sqrt{\frac{\omega_c \omega}{\omega_p^2}} c \ll c. \quad (\text{S-11.77})$$

For  $\omega = 10^5$ , and the above values of  $\omega_p$  and  $\omega_c$ , we obtain  $v_\varphi \simeq 0.03c$ .

**c)** With a spectral range from  $\omega_1$  to  $2\omega_1$ , the frequency components travel with velocities differing by a factor up to 2, so that the wave packet generated by the lightning will spread out and increase its length during its propagation. The higher frequencies travel faster, and are thus received earlier by the observer, than the slower frequencies. This is the origin of name “whistlers”.

In order to estimate the spread of the packet after a distance  $L = 10^9 \text{ cm}$ , we assume that the center of the wave packet travels with a group velocity  $v_g \simeq 0.06c$ , reaching a distance  $L$  after a time  $\tau = L/v_g = 0.56 \text{ s}$ . The “extreme” frequencies  $\omega_1$  and  $\omega_2$  will have group velocities  $v_1 \simeq 0.04c$  and  $v_2 \simeq 0.08c$ , respectively, and the pulse duration may be roughly estimated as the difference  $\Delta\tau = \tau_1 - \tau_2 = L/v_1 - L/v_2 \simeq (0.83 - 0.42) \text{ s} = 0.41 \text{ s}$ , provided that the duration at the emission is much shorter than  $\Delta\tau$ . This rough estimate neglects the deformation of the wave packet due to the strong dispersion.

## S-11.9 Wave Propagation in a “Pair” Plasma

Actually, it is convenient to calculate the dispersion relation in the presence of an external magnetic field  $\mathbf{B}_0$  first, then, the answer to point **a)** is simply obtained as a special case with  $\mathbf{B}_0 = 0$ . We assume  $\mathbf{B}_0 = B_0 \hat{\mathbf{z}}$  and a wave linearly polarized along  $\hat{\mathbf{x}}$  in a Cartesian reference frame  $xyz$ . The differential equations for the velocities of positrons,  $v_+$ , and electrons,  $v_-$ , are respectively

$$\frac{dv_{x\pm}}{dt} = \pm \frac{e}{m_e c} (E_x + v_{y\pm} B_0), \quad \frac{dv_{y\pm}}{dt} = \mp \frac{e}{m_e c} (v_{x\pm} B_0), \quad (\text{S-11.78})$$

where we have assumed  $v_{z\pm} = 0$ . Differentiating the first of (S-11.78) once more with respect to  $t$ , and substituting the second of (S-11.78) for  $dv_{y\pm}/dt$ , we obtain

$$\frac{d^2 v_{x\pm}}{dt^2} = \mp i\omega \frac{e}{m_e} E_x \pm \frac{eB_0}{m_e c} \frac{dv_{y\pm}}{dt} = \mp i\omega \frac{e}{m_e} E_x + \omega_c^2 v_{x\pm}, \quad (\text{S-11.79})$$

where  $\omega_c = \sqrt{eB_0/m_e c}$  is the cyclotron frequency. Substituting  $E_x = E_0 e^{-i\omega t}$  we obtain

$$v_{x\pm} = \mp i\omega \frac{e}{m_e(\omega_c^2 - \omega^2)} E_0. \quad (\text{S-11.80})$$

Analogously, for  $v_{y\pm}$  we have

$$v_{y\pm} = \pm \frac{eB_0}{m_e} v_{x\mp} = -i\omega \frac{e}{m_e} E_0, \quad (\text{S-11.81})$$

which has the same value for both electrons and positrons. The components of the current density are thus

$$\begin{aligned} J_x &= n_0 e (v_{x+} - v_{x-}) = - \frac{2i\omega n_0 e^2}{m_e(\omega_c^2 - \omega^2)} E_0, \\ J_y &= n_0 e (v_{y+} - v_{y-}) = 0. \end{aligned} \quad (\text{S-11.82})$$

The dielectric permittivity of the pair plasma,  $\epsilon(\omega)$ , is obtained from the usual definitions  $\mathbf{J} = \sigma \mathbf{E} = -i\omega \chi \mathbf{E}$  and is

$$\epsilon(\omega) = 1 - \frac{2\omega_p^2}{\omega^2 - \omega_c^2}. \quad (\text{S-11.83})$$

The same result can be obtained for circular polarization, both for left-handed and right-handed waves, confirming that there is no magnetically induced birefringence

in a pair plasma. This is different from the case of a medium containing free electrons only, considered in Problem 11.7.

For case **a**), where  $\mathbf{B}_0 = 0$ , we set  $\omega_c = 0$ , and obtain a cut-off frequency at  $\omega = 2\omega_p$ .

For case **b**), there is a resonance at  $\omega = \omega_c$ , while wave propagation is forbidden for frequencies in the range  $\omega_c < \omega < \sqrt{\omega_c^2 + 2\omega_p^2}$ .

## S-11.10 Surface Waves

**a)** In a dielectric medium described by  $\epsilon = \epsilon(\omega)$ , a monochromatic EM field of frequency  $\omega$  satisfies the Helmholtz equation. Thus we have for the magnetic field

$$\left( \nabla^2 + \epsilon \frac{\omega^2}{c^2} \right) B_z = 0. \quad (\text{S-11.84})$$

Substituting (11.12) for  $B_z$  into the Helmholtz equation, we obtain

$$q^2 - k^2 + \frac{\omega^2}{c^2} \epsilon = 0. \quad (\text{S-11.85})$$

**b)** From the equation  $c\nabla \times \mathbf{B} = 4\pi\mathbf{J} + \partial_t \mathbf{E}$  and the definition of  $\epsilon$  we obtain (for monochromatic waves in complex notation)  $c\nabla \times \mathbf{B} = -i\omega\epsilon\mathbf{E}$ . By substituting (11.12) for  $\mathbf{B}$  we obtain

$$-i\omega\epsilon\mathbf{E} = (\hat{\mathbf{x}}\partial_y - \hat{\mathbf{y}}\partial_x)B_z c = (ik\hat{\mathbf{x}} - q\hat{\mathbf{y}})B_z c, \quad (\text{S-11.86})$$

which gives for the electric field

$$\mathbf{E} = -(k\hat{\mathbf{x}} + iq\hat{\mathbf{y}}) \frac{c}{\epsilon\omega} B_z. \quad (\text{S-11.87})$$

**c)** From the definition of  $\mathbf{S} = c\mathbf{E} \times \mathbf{B}/(4\pi)$  we find that  $\mathbf{S}$  has components both along  $x$  and along  $y$ , given by

$$S_x = \frac{c}{4\pi} E_y B_z = \frac{qcB_0^2}{4\pi\epsilon\omega} e^{2qx} \cos(ky - \omega t) \sin(ky - \omega t), \quad (\text{S-11.88})$$

$$S_y = -\frac{c}{4\pi} E_x B_z = \frac{kcB_0^2}{4\pi\epsilon\omega} e^{2qx} \cos^2(ky - \omega t). \quad (\text{S-11.89})$$

However, averaging over one oscillation period we obtain  $\langle S_x \rangle = 0$ , thus the net energy flux is in the  $y$ -direction only, since  $\langle S_y \rangle \neq 0$ .

**d)** The tangential component of the magnetic field at the interface between two media must be continuous. Thus, from  $B_z(0^-) = B_z(0^+)$  we get  $B_1 = B_2$ .

**e)** Also the tangential component of the electric must be continuous at the interface, thus  $E_y(0^-) = E_y(0^+)$ . Using the results of points **b)** and **d)** we obtain

$$\frac{q_1}{\varepsilon_1} = -\frac{q_2}{\varepsilon_2}. \quad (\text{S-11.90})$$

Since both  $q_1 > 0$  and  $q_2 > 0$ ,  $\varepsilon_1$  and  $\varepsilon_2$  must have opposite signs.

**f)** Using the relationship  $(q_1/\varepsilon_1)^2 = (q_2/\varepsilon_2)^2$  and the result of point **a)** we obtain

$$\varepsilon_2^2 \left( k^2 - \frac{\omega^2}{c^2} \varepsilon_1 \right) = \varepsilon_1^2 \left( k^2 - \frac{\omega^2}{c^2} \varepsilon_2 \right), \quad (\text{S-11.91})$$

from which it follows that

$$\omega^2 = k^2 c^2 \frac{\varepsilon_2^2 - \varepsilon_1^2}{\varepsilon_2^2 \varepsilon_1 - \varepsilon_1^2 \varepsilon_2} = k^2 c^2 \frac{\varepsilon_2 + \varepsilon_1}{\varepsilon_2 \varepsilon_1}. \quad (\text{S-11.92})$$

Since wave can propagate only if  $k^2 > 0$ , and  $\varepsilon_1 \varepsilon_2 < 0$ , we get the additional condition  $\varepsilon_1 + \varepsilon_2 < 0$ .

**g)** Since  $\varepsilon_2 < -\varepsilon_1 = -1$  must hold, we may choose a metal, or a free electron gas, or an ideal plasma . . . , for which  $\varepsilon_2 = 1 - \omega_p^2/\omega^2$ , and a frequency such that  $\omega < \omega_p / \sqrt{2}$ .

The above described EM modes are *surface waves* (also named *surface plasmons*). These waves propagate along the surface of a conductor and are evanescent along the perpendicular direction, so that the EM energy is confined in a narrow layer, thinner than the wavelength in vacuum. Surface waves are a building block of *plasmonics*, a discipline oriented to develop optical and electronic devices on a nanometric scale.<sup>1</sup>

## S-11.11 Mie Resonance and a “Plasmonic Metamaterial”

**a)** The incident field can be written, in complex notation, as

$$\mathbf{E}_i = \mathbf{E}_i(x, t) = \mathbf{E}_0 e^{ikx-i\omega t}. \quad (\text{S-11.93})$$

Since  $a \ll \lambda$ , the electric field can be considered as uniform over the volume of the sphere, thus  $\mathbf{E}_i \simeq \mathbf{E}_0 e^{-i\omega t}$ , assuming the center of the sphere to be located at  $x = 0$ .

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<sup>1</sup> See e.g. W. L. Barnes et al., “Surface plasmon subwavelength optics”, Nature **424**, 824 (2003); E. Ozbay, “Plasmonics: merging photonics and electronics at nanoscale dimensions”, Science **311**, 189 (2006).

Now we introduce a spherical coordinate system  $(r, \theta, \phi)$  with the origin at the center of the sphere, and the zenith direction parallel to  $\mathbf{E}_i$ . At the surface of the sphere,  $r = a$ , we have the usual boundary conditions at the interface between two media

$$E_{\perp}(a^+, \theta) - E_{\perp}(a^-, \theta) = 4\pi\sigma(\theta), \quad E_{\parallel}(a^+, \theta) - E_{\parallel}(a^-, \theta) = 0, \quad (\text{S-11.94})$$

where  $\sigma(\theta)$  is the surface charge density on the sphere, independent of  $\phi$  within our approximations. The problem is thus analogous to the case of a dielectric sphere in a static uniform external field, treated in Problem 3.4. We can extend the results for the internal field and polarization to the case of an oscillating field as follows

$$\mathbf{E}_{\text{int}} = \frac{3\mathbf{E}_0}{\varepsilon_r(\omega) + 2}, \quad \mathbf{P} = \chi\mathbf{E}_{\text{int}} = \frac{3(\varepsilon_r(\omega) - 1)}{4\pi(\varepsilon_r(\omega) + 2)}\mathbf{E}_0. \quad (\text{S-11.95})$$

The difference with the electrostatic case is that now  $\varepsilon_r$  depends on frequency, and is not necessarily positive and greater than one, so that the internal field  $\mathbf{E}_{\text{int}}$  can be greater than the external applied field  $\mathbf{E}_0$ . A resonance appears when the real part of the denominator vanishes. Setting  $\eta = 0$  for simplicity, the resonance condition is

$$\varepsilon_r(\omega) + 2 = 3 - \frac{\omega_p^2}{\omega^2 - \omega_0^2} = 0, \quad (\text{S-11.96})$$

which yields

$$\omega^2 = \omega_0^2 + \frac{\omega_p^2}{3}. \quad (\text{S-11.97})$$

The physical meaning of the resonance is particularly clear for  $\omega_0 = 0$ , e.g., for a metallic (nano)sphere in a high-frequency (optical) field. In this case the resonance frequency is

$$\omega = \frac{\omega_p}{\sqrt{3}}, \quad (\text{S-11.98})$$

that is the natural frequency of the collective “Mie oscillations” of the electron sphere treated in Problem 1.5, also known as the lowest-order surface plasmon of the sphere. The resonance thus corresponds to the excitation of this oscillation mode.

**b)** The *macroscopic* polarization is given by the dipole moment of each nanosphere,  $\mathbf{p}_{\text{sphere}} = \mathbf{P}V$ , with  $V = (4\pi/3)a^3$  the volume of the sphere, times the number of nanospheres per unit volume,  $n_s$ :

$$\mathbf{P}_{\text{macro}} = n_s \mathbf{p}_{\text{sphere}} = -\frac{3n_s V \omega_p^2}{3\omega^2 - \omega_p^2} \mathbf{E}_0. \quad (\text{S-11.99})$$

This is equivalent to a *macroscopic* dielectric function

$$\varepsilon_r(\omega) = 1 - \frac{3n_s V \omega_p^2}{3\omega^2 - \omega_p^2}. \quad (\text{S-11.100})$$

Wave propagation requires  $\varepsilon_r$  to be positive, i.e.,

$$\omega < \frac{\omega_p}{\sqrt{3}}, \quad \omega > \frac{\omega_p}{\sqrt{3}} \sqrt{1 + 3n_s V}. \quad (\text{S-11.101})$$

This is a simple example of an artificial “metamaterial”, where the plasmonic properties of the nanostructures composing the material determine the optical response.

# Chapter S-12

## Solutions for Chapter 12

### S-12.1 The Coaxial Cable

a) Since the capacitance has been defined assuming *static* conditions and boundary effects are negligible for an “infinite” wire, we evaluate the capacitance per unit length of the cable,  $C$ , as for a cylindrical capacitor assuming the charge density to be constant in time and uniformly distributed. For symmetry reasons the electrostatic field between the two conductors is radial and independent of  $z$  and  $\phi$ , and it is obtained easily from Gauss’s law as

$$\mathbf{E} = \frac{2\lambda}{r} \hat{\mathbf{r}}, \quad a < r < b. \quad (\text{S-12.1})$$

Thus, the potential drop between the two conductors is

$$V = - \int_a^b E_r(r) dr = -2\lambda \ln\left(\frac{b}{a}\right), \quad (\text{S-12.2})$$

so that we obtain

$$C = \frac{\lambda}{|V|} = \frac{1}{2 \ln(b/a)}. \quad (\text{S-12.3})$$

Similarly, a static current  $I$  uniformly distributed on the inner conductor generates a magnetic field

$$\mathbf{B} = B_\phi(r) \hat{\phi} = \frac{2I}{cr} \hat{\phi}. \quad (\text{S-12.4})$$

The inductance per unit length of the cable can be obtained by evaluating the flux of  $\mathbf{B}$

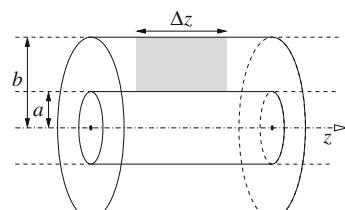


Fig. S-12.1

through a rectangle of width  $\Delta z$ , lying on a plane containing the  $z$  axis, and extending from  $r = a$  to  $r = b$ , as highlighted in Fig. S-12.1. The flux is

$$\Phi(\mathbf{B}) = \int_a^b B_\phi(r) \Delta z dr = \frac{2I}{c} \ln\left(\frac{b}{a}\right) \Delta z, \quad (\text{S-12.5})$$

corresponding to an inductance per unit length  $\mathcal{L}$

$$\mathcal{L} = \frac{\Phi(\mathbf{B})}{I c \Delta z} = \frac{2}{c^2} \ln\left(\frac{b}{a}\right). \quad (\text{S-12.6})$$

The same result can be obtained by calculating the magnetic energy in a cable section of length  $\Delta z$ , and inductance  $\Delta z \mathcal{L}$ ,

$$\frac{1}{2} \Delta z \mathcal{L} I^2 \equiv \Delta z \int_a^b \frac{B^2}{8\pi} 2\pi r dr = \frac{I^2}{c^2} \ln\left(\frac{b}{a}\right) \Delta z. \quad (\text{S-12.7})$$

**b)** The coaxial cable is a continuous system with finite capacitance and inductance per unit length, thus we know from Problem 7.4 that a current signal propagates along the wire according to the wave equation (S-7.49), with velocity

$$v = \frac{1}{\sqrt{\mathcal{L}C}} = c. \quad (\text{S-12.8})$$

The general solution for the propagating current signal is thus  $I(z, t) = I(z - vt)$ , and propagation occurs with no dispersion. The associated charge signal  $\lambda(z, t)$  is related to  $I(z, t)$  by the continuity equation,

$$\partial_t \lambda(z, t) = -\partial_z I(z, t) = -I'(z - ct), \quad (\text{S-12.9})$$

where  $I'$  denotes the derivative of  $I$  with respect to its argument. Since  $\partial_t \lambda(z - ct) = -c \lambda'(z - ct)$ , we obtain

$$\lambda(z, t) = \lambda(z - ct) = \frac{1}{c} I(z - ct). \quad (\text{S-12.10})$$

**c)** A transverse electric field  $\mathbf{E}$  must be radial for symmetry reasons,  $\mathbf{E} = E_r(r, z, t) \hat{\mathbf{r}}$ . Applying Gauss's law to a cylindrical surface of radius  $a < r < b$ , infinitesimal height  $\Delta z$ , and coaxial to the cable, we find  $E_r = 2\lambda(z, t)/r$ . Again for symmetry reasons, a transverse magnetic field must be azimuthal,  $\mathbf{B} = B_\phi(r, z, t) \hat{\phi}$ . Applying Stokes' law to a circle of radius  $a < r < b$ , coaxial to the cable, we obtain  $B_\phi = 2I(z, t)/rc$ . The displacement current does not contribute to the flux through the circle, since  $\mathbf{E}$  is radial. Thus, the fields have the same dependence on  $\lambda$  and  $I$  as the static fields, the only difference being that here both  $\lambda = \lambda(z, t)$  and  $I = I(z, t)$  depend on  $z$  and  $t$ . Notice that it is such peculiar character of the TEM configuration which

allows to use the capacitance and inductance calculated for static fields to obtain the propagation velocity of electromagnetic signals along the cable, a result also true for any transmission line in TEM mode.

We can check that the above fields constitute a solution to Maxwell's equations by verifying that

$$\begin{aligned}\nabla \times \mathbf{E} &= \partial_z E_r \hat{\phi} = \frac{2}{r} \partial_z \lambda(z-ct) \hat{\phi} = \frac{2}{r} \lambda'(z-ct) \hat{\phi} \\ &= \frac{2}{rc} I'(z-ct) \hat{\phi} = -\frac{2}{rc^2} \partial_t I(z-ct) \hat{\phi} \\ &= -\frac{1}{c} \partial_t \mathbf{B}.\end{aligned}\quad (\text{S-12.11})$$

**d)** The source at  $z = 0$  must do a work  $W(t)$  in order to drive the current between the inner and outer conductors,

$$W(t) = V(0,t) I(0,t) = -2c\lambda^2(0,t) \ln\left(\frac{b}{a}\right). \quad (\text{S-12.12})$$

The local flux of energy at any point  $(r, \phi, z)$ , with  $a < r < b$  and  $z > 0$ , is

$$\begin{aligned}\mathbf{S}(r,z,t) &= \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} = \hat{\mathbf{z}} \frac{c}{4\pi} \frac{2\lambda(z-ct)}{r} \frac{2I(z-ct)}{rc} \\ &= \hat{\mathbf{z}} \frac{c}{\pi r^2} \lambda^2(z-ct),\end{aligned}\quad (\text{S-12.13})$$

corresponding to a total flow of energy at  $z$

$$\Phi(z,t) = \int_a^b S_z 2\pi r dr = 2c\lambda^2(z-ct) \ln\left(\frac{b}{a}\right) = -W(z-ct). \quad (\text{S-12.14})$$

This shows that the energy flow is sustained by the source.

**e)** The expressions for the fields, and for the capacitance and inductance per unit length, are, in the presence of generic values of  $\epsilon$  and  $\mu$ ,

$$E_r = 2 \frac{\lambda}{\epsilon r}, \quad B_\phi = \frac{2\mu I}{rc}, \quad (\text{S-12.15})$$

$$C = \frac{\epsilon}{2 \ln(b/a)}, \quad \mathcal{L} = \frac{2}{c^2} \ln\left(\frac{b}{a}\right), \quad (\text{S-12.16})$$

corresponding to a wave velocity  $v = c/\sqrt{\epsilon\mu} < c$ . In general, however, both  $\epsilon$  and  $\mu$  can depend on frequency, and the cable becomes a dispersive transmission line with phase velocity  $v_\phi(\omega) = c/\sqrt{\epsilon(\omega)\mu(\omega)}$ .

## S-12.2 Electric Power Transmission Line

**a)** The continuity equation is  $\partial_t \lambda = -\partial_z I$ . Writing  $\lambda$  in the form  $\lambda = \lambda_0 e^{ikz-i\omega t}$ , we obtain

$$-i\omega \lambda_0 = -ikI_0, \quad \text{or} \quad \lambda_0 = \frac{k}{\omega} I_0 = \frac{I_0}{v_\varphi}, \quad (\text{S-12.17})$$

where  $v_\varphi$  is the phase velocity of the signal.

**b)** The electric field  $\mathbf{E}$  can be calculated by applying Gauss's law to a cylindrical surface coaxial to the wire, obtaining

$$E_r(r, z, t) = \frac{2\lambda(z, t)}{r}. \quad (\text{S-12.18})$$

The magnetic field  $\mathbf{B}$  can be obtained from the equation  $c\nabla \times \mathbf{B} = 4\pi\mathbf{J} + \partial_t \mathbf{E}$ . If we choose a circle of radius  $r$  coaxial to the wire and apply Stokes' theorem we have

$$\oint \mathbf{B} \cdot d\ell = \frac{1}{c} \int (4\pi\mathbf{J} + \partial_t \mathbf{E}) \cdot d\mathbf{S}. \quad (\text{S-12.19})$$

The  $\partial_t \mathbf{E}$  term is radial and thus does not contribute to the flux at the right-hand side, so that

$$2\pi r B_\phi = 4\pi \frac{I}{c}, \quad \text{and} \quad B_\phi(r, z, t) = \frac{2I(z, t)}{rc}. \quad (\text{S-12.20})$$

The equations for  $E_r(r, \phi, z)$  and  $B_\phi(r, \phi, z)$  have the same form as in the static case of a wire with constant and uniform charge density and current, respectively. We also have  $|E_r|/|B_\phi| = c/v_\varphi$ . These are typical properties of the TEM (transverse electromagnetic) mode for the transmission lines. Maxwell's equation  $c\nabla \times \mathbf{E} = -\partial_t \mathbf{B}$  gives  $c\partial_z E_r = -\partial_t B_\phi$  leads to

$$ik\lambda_0 = i\omega \frac{I_0}{c^2} \quad \Rightarrow \quad k \frac{I_0}{v_\varphi} = \omega \frac{I_0}{c^2} \quad \Rightarrow \quad \frac{\omega}{v_\varphi^2} = \frac{\omega}{c^2} \quad \Rightarrow \quad v_\varphi = c, \quad (\text{S-12.21})$$

where we have used (S-12.17) and  $k = \omega/v_\varphi$ .

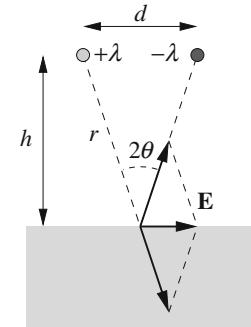
In SI units we have

$$E_r = \frac{\lambda}{2\pi\epsilon_0}, \quad B_\phi = \frac{\mu_0 I}{2\pi r}, \quad \frac{|E_r|}{|B_\phi|} = \frac{c^2}{v_\varphi}. \quad (\text{S-12.22})$$

c) Consider a line on the midplane, at a distance  $h$  from the plane containing the two wires, as in Figs. S-12.2 and S-12.3. The distance of the line from each wire is  $r = \sqrt{h^2 + d^2}/4$ . The electric and magnetic fields generated by the two wires sum up to

$$\mathbf{E} = 2\hat{\mathbf{x}} \frac{2I_0}{rc} \sin\theta e^{-i\omega t}, \quad (\text{S-12.23})$$

$$\mathbf{B} = 2\hat{\mathbf{y}} \frac{2I_0}{rc} \sin\theta e^{-i\omega t}, \quad (\text{S-12.24})$$



with  $\hat{\mathbf{x}}$  and  $\hat{\mathbf{y}}$  the unit vectors parallel and perpendicular to the plane containing the wires, respectively.

Since  $\sin\theta = d/(2r)$ , we obtain

$$|E_x| = |B_y| = \frac{2I_0 d}{r^2 c}. \quad (\text{S-12.25})$$

The corresponding expressions in SI units are

$$|E_x| = |B_y|c = \frac{I_0 d}{2\pi c \epsilon_0 r^2} = \frac{\mu_0 I_0 d c}{2\pi r^2}. \quad (\text{S-12.26})$$

Thus

$$|B_y| = \frac{4\pi \times 10^{-7} \times 10^3 \times 5}{2\pi \times (30^2 + 5^2/2^2)} \text{ T} \simeq 10^{-6} \text{ T}, \quad (\text{S-12.27})$$

Fig. S-12.2

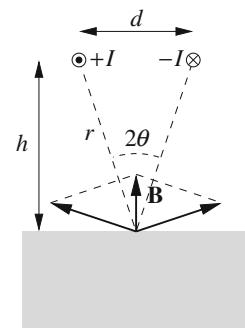


Fig. S-12.3

and

$$|E_x| \simeq 3 \times 10^2 \text{ V/m}. \quad (\text{S-12.28})$$

For a comparison, the average magnetic field at the Earth's surface is  $\sim 5 \times 10^{-5}$  T, while the electric field is  $\sim 1.5 \times 10^2$  V/m. Possible screening effects by the Earth's surface have been neglected.

### S-12.3 TEM and TM Modes in an “Open” Waveguide

a) Inserting (12.23) into the wave equation for  $\mathbf{B}$

$$\left( \nabla^2 - \frac{1}{c^2} \partial_t^2 \right) B_z = 0, \quad (\text{S-12.29})$$

and recalling that  $\partial_z B_z = 0$ , we obtain the following relation between  $k_x$ ,  $k_y$  and  $\omega$

$$k_x^2 + k_y^2 - \frac{\omega^2}{c^2} = 0. \quad (\text{S-12.30})$$

**b)** The electric field of the wave can be obtained from

$$\begin{aligned} \partial_t \mathbf{E} &= c \nabla \times \mathbf{B} = c (\hat{\mathbf{x}} \partial_y - \hat{\mathbf{y}} \partial_x) B_z, \\ -i\omega \mathbf{E} &= c B_0 [ -\hat{\mathbf{x}} k_y \sin(k_y y) - \hat{\mathbf{y}} i k_x \cos(k_y y) ] e^{ik_x x - i\omega t}, \end{aligned} \quad (\text{S-12.31})$$

which leads to

$$E_x = -i \frac{k_y c}{\omega} B_0 \sin(k_y y) e^{ik_x x - i\omega t}, \quad (\text{S-12.32})$$

$$E_y = \frac{k_x c}{\omega} B_0 \cos(k_y y) e^{ik_x x - i\omega t}. \quad (\text{S-12.33})$$

**c)** The parallel component  $E_{||}$  of the electric field  $\mathbf{E}$  must vanish at the boundary with a perfectly conducting surface, thus we must have  $E_x(y = \pm a/2) = 0$ . This implies that  $\sin(k_y a/2) = 0$ , and  $k_y a = 2m\pi$ , with  $m \in \mathbb{N}$ . By substitution into (S-12.30) we obtain

$$\omega^2 = k_x^2 c^2 + \left( \frac{\pi c}{a} \right)^2 (2m)^2. \quad (\text{S-12.34})$$

The  $m = 0$  mode corresponds to  $E_x = 0$  and to  $E_y$  and  $B_z$  independent of  $y$ . The fields are thus uniform over any cross-section of the waveguide parallel to the  $yz$  plane, and we have  $\omega = k_x c$ . This is the TEM mode typical of transmission lines. The  $m = 1$  mode has frequency

$$\omega = \sqrt{k_x^2 c^2 + \left( \frac{2\pi c}{a} \right)^2} > \frac{2\pi c}{a} \equiv \omega_{co}, \quad (\text{S-12.35})$$

where  $\omega_{co} \equiv 2\pi c/a$  is the cut-off frequency.

**d)** The energy flux is given by Poynting's vector, parallel to the  $z = 0$  plane,

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} = \frac{c}{4\pi} (E_y B_z \hat{\mathbf{x}} - E_x B_z \hat{\mathbf{y}}). \quad (\text{S-12.36})$$

By averaging over one full cycle we find  $\langle S_y \rangle = 0$ , i.e., there is no net energy flux along  $y$ . Averaging  $S_x$  over one cycle we obtain

$$\langle S_x \rangle = \frac{c^2}{8\pi} \frac{k_x}{\omega} B_0^2 \cos^2(k_y y). \quad (\text{S-12.37})$$

The group velocity of the wave is

$$v_g = \partial_k \omega = \frac{k_x c^2}{\sqrt{k_x^2 c^2 + \omega_{co}^2}} = \frac{k_x c^2}{\omega}, \quad (\text{S-12.38})$$

thus we can also write

$$\langle S_x \rangle = v_g \frac{B_0^2}{8\pi} \cos^2(k_y y). \quad (\text{S-12.39})$$

## S-12.4 Square and Triangular Waveguides

a) The electric field must satisfy the wave equation in vacuum

$$\left( \nabla^2 - \frac{1}{c^2} \partial_t^2 \right) \mathbf{E} = 0, \quad (\text{S-12.40})$$

and, substituting (12.4) for  $\mathbf{E}$ , we obtain the time-independent Helmholtz's equation for the only nonzero component of the electric field,  $\tilde{E}_x$ ,

$$\left( \partial_x^2 + \partial_y^2 - k_z^2 + \frac{\omega^2}{c^2} \right) \tilde{E}_x = 0. \quad (\text{S-12.41})$$

In vacuum we must also have  $\nabla \cdot \mathbf{E} = 0$ , this condition is automatically satisfied if we assume that  $\tilde{E}_x$  is independent of  $x$ ,  $\tilde{E}_x = \tilde{E}_x(y)$ , and (S-12.41) reduces to

$$\left( \partial_y^2 - k_z^2 + \frac{\omega^2}{c^2} \right) \tilde{E}_x(y) = 0. \quad (\text{S-12.42})$$

According to the boundary conditions, the parallel component of  $\mathbf{E}$  must be zero at the perfectly reflecting walls of the waveguide  $y = 0$  and  $y = a$ . This condition is satisfied if we assume

$$\tilde{E}_x(y) = E_{0x} \sin(k_y y), \quad \text{with} \quad k_y = n \frac{\pi}{a}, \quad n = 1, 2, 3, \dots, \quad (\text{S-12.43})$$

where  $E_{0x}$  is an arbitrary, constant amplitude. The electric field of our  $\hat{\mathbf{x}}$  polarized wave can thus be written

$$\mathbf{E} = \hat{\mathbf{x}} \tilde{E}_x(y) e^{ik_z z - i\omega t} = \hat{\mathbf{x}} E_{0x} \sin\left(\frac{n\pi}{a} y\right) e^{ik_z z - i\omega t}. \quad (\text{S-12.44})$$

Substituting (S-12.43) for  $\tilde{E}_x$  into (S-12.42) leads to

$$\left(-k_y^2 - k_z^2 + \frac{\omega^2}{c^2}\right)E_{0x} = 0, \quad (\text{S-12.45})$$

which, disregarding the trivial case  $E_{0x} = 0$ , is true only if

$$k_y^2 + k_z^2 - \frac{\omega^2}{c^2} = 0, \quad \text{or} \quad k_z = \sqrt{\frac{\omega^2}{c^2} - k_y^2} = \sqrt{\frac{\omega^2}{c^2} - n^2 \frac{\pi^2}{a^2}}. \quad (\text{S-12.46})$$

The wave can propagate only if  $k_z$  is real, thus we must have

$$\omega > n \frac{\pi c}{a}. \quad (\text{S-12.47})$$

The cutoff frequency  $\omega_a$  is the lowest value of  $\omega$  at which wave propagation occurs. Since we must have  $n \geq 1$ , we have  $\omega_a = \pi c/a$ . If we choose a frequency such that  $\pi c/a < \omega < 2\pi c/a$ , only the  $n = 1$  mode can propagate in the guide.

The cross-section of the waveguide being square, the conditions for a  $\hat{\mathbf{y}}$  polarized TE wave are obtained by interchanging the roles of  $x$  and  $y$  in all the above formulae, and the electric field is

$$\mathbf{E} = \hat{\mathbf{y}} E_{0y} \sin(k_x x) e^{ik_z z - i\omega t} = \hat{\mathbf{y}} E_{0y} \sin\left(\frac{m\pi}{a} x\right) e^{ik_z z - i\omega t}, \quad (\text{S-12.48})$$

with, again,  $E_{0y}$  an arbitrary amplitude,  $m = 1, 2, 3, \dots$ , and the same dispersion relation as between  $\omega$  and  $k_z$  as above. Modes with  $m = n$  are degenerate, sharing the same wavevector  $k_z$ .

In general, a monochromatic TE wave propagating in the guide will be a superposition of the two polarizations. The electric field will be

$$\mathbf{E} = \left[ \hat{\mathbf{x}} E_{0x} \sin\left(\frac{n\pi}{a} y\right) + \hat{\mathbf{y}} E_{0y} \sin\left(\frac{n\pi}{a} x\right) \right] e^{ik_z z - i\omega t}. \quad (\text{S-12.49})$$

**b)** In the case of the triangular waveguide, the parallel component of the electric field  $\mathbf{E}$  must be zero on the three  $x = 0$ ,  $y = 0$ , and  $y = x$  planes. A field of the form (S-12.49) already satisfies the boundary conditions at the  $x = 0$  and  $y = 0$  planes. The additional condition at the  $y = x$  plane is  $\mathbf{E}(x, x) \cdot \hat{\mathbf{n}} = 0$ , where  $\hat{\mathbf{n}} = (-1, 1, 0)/\sqrt{2}$  is the unit vector perpendicular to the  $y = x$  plane. Thus

$$\mathbf{E} \cdot \mathbf{h} = E_{0x} \sin\left(\frac{n\pi}{a} x\right) - E_{0y} \sin\left(\frac{n\pi}{a} x\right) = 0, \quad (\text{S-12.50})$$

which is satisfied if  $E_{0y} = -E_{0x} \equiv E_0$ , so that we eventually obtain

$$\mathbf{E} = E_0 \left[ \hat{\mathbf{x}} \sin\left(\frac{n\pi}{a} y\right) - \hat{\mathbf{y}} \sin\left(\frac{n\pi}{a} x\right) \right] e^{ik_z z - i\omega t}. \quad (\text{S-12.51})$$

## S-12.5 Waveguide Modes as an Interference Effect

**a)** The electrostatic potential  $\phi$  must be zero on the two conducting planes at  $y = \pm a$ , and the electric fields at  $y = a^-$  and  $y = -a^+$  must be perpendicular to their surfaces (parallel to  $\hat{y}$ ). The real dipole  $\mathbf{p}$  is located at the origin of our coordinate system, thus, we need an image dipole equal to  $\mathbf{p}$ , located at  $(0, 2a, 0)$  and represented by  $\mathbf{p}_1$  in Fig. S-12.4, in order to fulfill these conditions at the generic point  $A$  of the  $y = +a$  plane. Analogously, the real dipole  $\mathbf{p}$  requires a further image dipole  $\mathbf{p}$  located at  $(0, -2a, 0)$ , represented by  $\mathbf{p}_{-1}$  in Fig. S-12.4, in order to fulfill the conditions at the  $y = -a$  conducting plane. But now the three dipoles  $\mathbf{p}$ ,  $\mathbf{p}_1$ , and  $\mathbf{p}_{-1}$  together do not generate a potential equal to zero on either plane. We can readjust the potential at  $y = +a$  by adding a new image dipole equal to  $\mathbf{p}$ , symmetrical to  $\mathbf{p}_{-1}$ , at  $(0, 4a, 0)$ , represented by  $\mathbf{p}_2$ . But this requires adding a further image dipole  $\mathbf{p}_{-2}$ , and so on. Thus, the exact solution requires two infinite sets of equal image dipoles,  $\mathbf{p}_n$  and  $\mathbf{p}_{-n}$ , with  $n = 1, 2, 3, \dots$ , located respectively at  $(0, 2na, 0)$  and  $(0, -2na, 0)$ . The resulting electrostatic potential between the plates is finite because, for high  $n$  values, the contribution of  $\pm n$ th dipole is proportional to  $(2na)^{-2}$ .

**b)** In order to fulfill the boundary conditions, all the image dipoles must oscillate in phase with the real dipole. Consider the radiation emitted by each dipole in the  $\hat{\mathbf{n}} \equiv (\sin \theta, \cos \theta, 0)$  direction in the  $z = 0$  plane, with wavevector  $\mathbf{k} = (\omega/c)\hat{\mathbf{n}}$ . In the following we consider wavevectors lying in the  $z = 0$  plane, but our considerations apply to wavevectors lying in any plane containing the  $y$  axis, due to the rotational symmetry of the problem. The optical path difference between the waves emitted by two neighboring dipoles (real or images) is  $\Delta\ell = 2a \cos \theta$ , as shown in Fig. S-12.5 for the case of the real dipole  $\mathbf{p}$  and the image  $\mathbf{p}_1$ . This corresponds to a phase difference  $\Delta\varphi = k\Delta\ell$ , and the condition for constructive interference is

$$k\Delta\ell = \frac{2\omega a}{c} \cos \theta = 2\pi m, \quad \theta = \arccos\left(m \frac{\pi c}{\omega a}\right), \quad (\text{S-12.52})$$

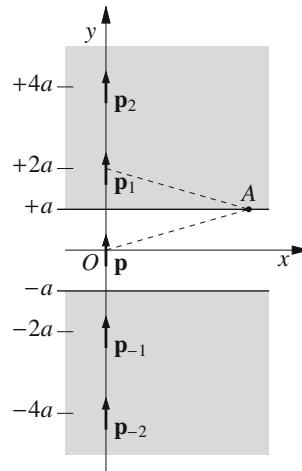


Fig. S-12.4

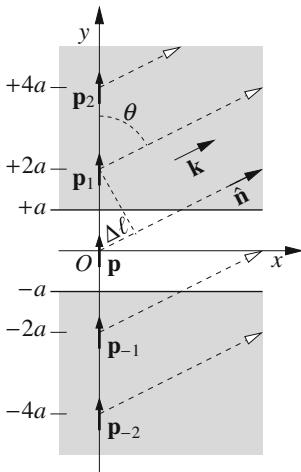


Fig. S-12.5

with  $m = 0, 1, 2, \dots$ . Due to the mirror symmetry of the system for reflections through the  $y = 0$  plane (actually, *antisymmetry*, since all dipoles are inverted by the reflection), if an angle  $\theta$  satisfies (S-12.52) for constructive interference, so does  $\pi - \theta$ . In other words, at large distance from the oscillating dipole, each interference order  $m > 0$  corresponds to the superposition of two waves with wavevectors  $\mathbf{k}_{\pm} \equiv (\sin \theta, \pm \cos \theta, 0) \omega/c$ , respectively.

The  $m = 0$  condition corresponds to  $\theta = \pi/2$ , and the waves travel along the  $x$  axis. For  $m > 0$ , we can write

$$\begin{aligned} k_x &= \frac{\omega}{c} \sin \theta = \frac{\omega}{c} \sqrt{1 - \cos^2 \theta} = \frac{\omega}{c} \sqrt{1 - \left(m \frac{\pi c}{\omega a}\right)^2} \\ &= \sqrt{\frac{\omega^2}{c^2} - \left(m \frac{\pi}{a}\right)^2}, \end{aligned} \quad (\text{S-12.53})$$

and  $k_x$  is real only if  $\omega > m\pi c/a$ . Thus, given a frequency  $\omega$ , we observe only the modes with  $m < \omega a/(\pi c)$ . If  $\omega < \pi c/a$ , corresponding to a wavelength  $\lambda > 2a$ , only the mode  $m = 0$  can propagate.

c) Both magnetic fields must satisfy the wave equation

$$(c^2 \nabla^2 - \partial_t^2) \mathbf{B}_i = (c^2 \partial_x^2 + c^2 \partial_y^2 - \partial_t^2) \mathbf{B}_i = 0, \quad i = 0, 1, \quad (\text{S-12.54})$$

from which we obtain, denoting by  $\mathbf{k}_0$  and  $\mathbf{k}_1$  the respective wavevectors,

$$k_{0x}^2 c^2 = \omega^2, \quad k_{1x}^2 c^2 + k_{1y}^2 c^2 = \omega^2. \quad (\text{S-12.55})$$

d) Assuming electric fields of the form  $\mathbf{E} = \tilde{\mathbf{E}} e^{-i\omega t}$ , where  $\tilde{\mathbf{E}}$  depends on the space coordinates only, Maxwell's equation in vacuum,  $\partial_t \mathbf{E} = c \nabla \times \mathbf{B}$ , gives

$$-i\omega \mathbf{E} = c(\hat{\mathbf{x}} \partial_y B_z - \hat{\mathbf{y}} \partial_x B_z). \quad (\text{S-12.56})$$

For the wave of type “0” we obtain

$$\mathbf{E}_0 = \hat{\mathbf{y}} \frac{k_{0x} c}{\omega} B_0 e^{ik_{0x} x - i\omega t} = \hat{\mathbf{y}} B_0 e^{ik_{0x} x - i\omega t}. \quad (\text{S-12.57})$$

For the wave of type “1” we obtain

$$\mathbf{E}_1 = \frac{ic}{\omega} B_1 [\hat{\mathbf{y}} k_{1x} \cos(k_{1y} y) - \hat{\mathbf{x}} k_{1y} \sin(k_{1y} y)] e^{ik_{1x} x - i\omega t}. \quad (\text{S-12.58})$$

**e)** The “type-0” wave has the three vectors  $\mathbf{E}_0$ ,  $\mathbf{B}$  and  $\mathbf{k}$  perpendicular to one another, analogously to a plane wave in the free space (TEM mode). Further,  $\mathbf{E}_0$  is perpendicular to the two conducting surfaces, automatically satisfying the boundary conditions. Thus, the frequency  $\omega$  and the wavevector  $\mathbf{k} = \hat{\mathbf{x}}k_{0x}$ , with  $k_{0x} = \omega/c$ , are subject to no constraint.

On the other hand, the electric field of the “type-1” wave has a component  $E_x$  parallel to the two conducting surfaces, in addition to the transverse  $E_y$  component (the mode is TM rather than TEM). The boundary conditions at  $y = \pm a$  require that  $E_x(y = \pm a) = 0$ . Thus we must have  $\sin(\pm k_y a) = 0$ , or  $k_y = m\pi/a$ , with  $m = 1, 2, 3, \dots$ , leading to

$$k_x = \sqrt{\frac{\omega^2}{c^2} - \left(m \frac{\pi}{a}\right)^2}. \quad (\text{S-12.59})$$

The  $m$ -th mode can propagate only if the corresponding  $k_x$  is real, and has a lower cut-off frequency  $\omega_{\text{co}}(m) = 2\pi mc/a$ .

A comparison to point **(b)** shows that the type-0 wave (TEM mode) corresponds to the  $m = 0$  interference order, while the type-1 waves (TM modes) correspond to the  $> 0$  interference orders. Actually, more precisely, we need not single dipoles, but “dipole layers”, spread parallel to the  $z$  axis, in order to generate waves with fields independent of  $z$ . If the real dipole of points **(a)** and **(b)** is parallel, rather than perpendicular, to the conducting planes, the different boundary conditions would lead to TE, rather than TM modes [1].

## S-12.6 Propagation in an Optical Fiber

**a)** The electric field (12.6) corresponds to the sum of two plane waves of the same frequency and different wavevectors,  $\mathbf{k}_1$  and  $\mathbf{k}_2$ , propagating in the medium. For both waves the dispersion relation is  $\omega = kc/n$ , where  $n = n(\omega)$  is the refractive index of the medium. Both waves impinge on the medium-vacuum interface at the angle  $\theta$ , and the condition for total reflection is, according to Snell’s law,

$$\sin\theta > \frac{1}{n}. \quad (\text{S-12.60})$$

**b)** The internal reflections at the  $y = \pm a/2$  planes turn the wave of type “1” into a wave of type “2”, and vice versa. Thus the field amplitudes of the two waves at the interface are related by the amplitude reflection coefficient  $r$

$$\begin{aligned} E_2(x, y = +a/2, t) &= rE_1(x, y = +a/2, t), \\ E_1(x, y = -a/2, t) &= rE_2(x, y = -a/2, t). \end{aligned} \quad (\text{S-12.61})$$

For total reflection there is no transmission of energy through the  $y = \pm a/2$  planes, thus the amplitudes of the incident and the reflected fields must be equal but for a change of phase. For  $S$ -polarization ( $\mathbf{E}$  parallel to the interface, as in our case)  $r$  is written, according to the Fresnel equations,

$$r = \frac{n \cos \theta - i \sqrt{n^2 \sin^2 \theta - 1}}{n \cos \theta + i \sqrt{n^2 \sin^2 \theta - 1}}, \quad (\text{S-12.62})$$

and, if  $n \sin \theta > 1$ , the square roots are real and  $|r| = 1$ . Thus we can write

$$r = e^{i\delta} = \cos \delta + i \sin \delta, \quad \tan \frac{\delta}{2} = -\frac{\sqrt{n^2 \sin^2 \theta - 1}}{n \cos \theta}. \quad (\text{S-12.63})$$

Substituting  $r = e^{i\delta}$  into (S-12.61) we obtain the following conditions at the  $y = \pm a/2$  planes

$$E_2 e^{-ik_y a/2} = E_1 e^{+ik_y a/2} e^{i\delta}, \quad E_1 e^{-ik_y a/2} = E_2 e^{+ik_y a/2} e^{i\delta}. \quad (\text{S-12.64})$$

By calculating the determinant of the homogeneous system for  $E_1$  and  $E_2$  we obtain the condition

$$1 = e^{2i(k_y a + \delta)}, \quad (\text{S-12.65})$$

true if

$$2k_y a + 2\delta = 2m\pi, \quad m = 0, 1, 2, \dots. \quad (\text{S-12.66})$$

The implicit relation determining the allowed frequencies is

$$k_x^2 = \frac{\omega^2}{c^2} n^2 - k_y^2 > 0. \quad (\text{S-12.67})$$

If  $n \sin \theta \gg 1$  then  $\delta \approx -2\theta$ , and if  $\theta \rightarrow \pi/2$  then

$$k_y \rightarrow (m+1) \frac{\pi}{a}. \quad (\text{S-12.68})$$

c) All the above results are valid also for  $P$ -polarization, where the electric field of the wave lies in the  $xy$  plane. Only (S-12.62) must be replaced by

$$r_{||} = e^{i\delta_{||}} = \frac{-n^2 \cos \theta + i \sqrt{\sin^2 \theta - n^2}}{n^2 \cos \theta + i \sqrt{\sin^2 \theta - n^2}}, \quad (\text{S-12.69})$$

corresponding to a different dependence of  $r$  and  $\delta$  on  $\theta$ .

## S-12.7 Wave Propagation in a Filled Waveguide

**a)** The electric field  $\mathbf{E}$  of a monochromatic EM wave of frequency  $\omega$  propagating in a medium of refractive index  $n = n(\omega)$  satisfies Helmholtz's equation

$$\left( \nabla^2 + n^2(\omega) \frac{\omega^2}{c^2} \right) \mathbf{E} = 0. \quad (\text{S-12.70})$$

We are considering a TE mode with  $\mathbf{E} = \hat{\mathbf{z}} E_z(y) e^{ikx-i\omega t}$ , thus we have

$$\left( \partial_y^2 - k^2 + n^2(\omega) \frac{\omega^2}{c^2} \right) E_z(y) = 0, \quad (\text{S-12.71})$$

whose general solution has the form  $E_z(y) = A \cos(qy) + B \sin(qy)$ , with  $A$  and  $B$  two arbitrary constants. The electric field being parallel to the conducting walls at  $y = \pm a/2$ , the boundary conditions are  $E_z(y = \pm a/2) = 0$ , from which we obtain

$$E_z(y) = E_0 \begin{cases} \cos(q_n y), & n = 1, 3, 5 \dots \\ \sin(q_n y), & n = 2, 4, 6 \dots \end{cases}, \quad q_n = n \frac{\pi}{a}, \quad (\text{S-12.72})$$

and (S-12.71) turns into

$$q_n^2 + k^2 - n^2(\omega) \frac{\omega^2}{c^2} = 0. \quad (\text{S-12.73})$$

The wave can propagate only if  $k$  is real, i.e., if  $\omega > q_n c \equiv \omega_n$ .

In the case of a plasma

$$q_n^2 + k^2 - \frac{\omega^2 - \omega_p^2}{c^2} = 0, \quad (\text{S-12.74})$$

and the cut-off frequencies are

$$\omega'_n = \sqrt{q_n^2 c^2 + \omega_p^2}. \quad (\text{S-12.75})$$

**b)** The incident wave must be in the  $n = 1$  mode, and its electric field is

$$\mathbf{E}_i = \hat{\mathbf{z}} E_0 \cos(q_1 y) e^{ik_1 x - i\omega t}, \quad (\text{S-12.76})$$

where  $k_1 = \sqrt{\omega^2/c^2 - q_1^2}$ . The total electric field is the sum of the incident field  $\mathbf{E}_i$  and the reflected field  $\mathbf{E}_r$  for  $x < 0$ , while only the transmitted field  $\mathbf{E}_t$  is present in the  $x > 0$  region. The boundary conditions at  $x = 0$  is  $(E_{iz} + E_{rz})|_{x=0} = E_{tz}|_{x=0}$ , thus all the waves must have the same dependence on  $t$  and  $y$ . The total field must thus be

$$E_z = \begin{cases} (E_0 e^{ik_1 x} + E_r e^{-ik_1 x}) \cos(q_1 y) e^{-i\omega t}, & x < 0, \\ E_t \cos(q_1 y) e^{ik_t x - i\omega t}, & x > 0, \end{cases} \quad (\text{S-12.77})$$

where  $k_t = \sqrt{n(\omega)\omega^2/c^2 - q_1^2}$ . The boundary condition on the electric field yields

$$E_0 + E_r = E_t. \quad (\text{S-12.78})$$

In addition, the magnetic field must be also continuous at  $x = 0$ . From  $\partial_t \mathbf{B} = -c \nabla \times \mathbf{E}$  we obtain

$$B_x = \frac{i\pi c}{\omega a} \times \begin{cases} (E_0 e^{ik_1 x} + E_r e^{-ik_1 x}) \sin(q_1 y) e^{-i\omega t}, & x < 0, \\ E_t \sin(q_1 y) e^{ik_t x - i\omega t}, & x > 0, \end{cases} \quad (\text{S-12.79})$$

$$B_y = -\frac{c}{\omega} \times \begin{cases} k(E_0 e^{ik_1 x} - E_r e^{-ik_1 x}) \cos(q_1 y) e^{-i\omega t}, & x < 0, \\ k_t E_t \cos(q_1 y) e^{ik_t x - i\omega t}, & x > 0, \end{cases} \quad (\text{S-12.80})$$

We notice that the continuity of  $B_x$  is ensured by the condition  $E_0 + E_r = E_t$ , while the continuity of  $B_y$  yields

$$k_1(E_0 - E_r) = k_t E_t. \quad (\text{S-12.81})$$

Eventually, we obtain

$$E_r = \frac{k_1 - k_t}{k_1 + k_t} E_0, \quad E_t = \frac{2k_1}{k_1 + k_t} E_0, \quad (\text{S-12.82})$$

which are identical to Fresnel's formulas for  $S$ -polarization. In fact, the field of the incoming wave (S-12.76) can be written as

$$\begin{aligned} \mathbf{E}_i &= \hat{\mathbf{z}} E_0 \cos(q_1 y) e^{ik_1 x - i\omega t} = \hat{\mathbf{z}} \frac{E_0}{2} (e^{iq_1 y} + e^{-iq_1 y}) e^{ik_1 x - i\omega t} \\ &= \hat{\mathbf{z}} \frac{E_0}{2} e^{ik_1 x + iq_1 y - i\omega t} + \hat{\mathbf{z}} \frac{E_0}{2} e^{ik_1 x - iq_1 y - i\omega t}, \end{aligned} \quad (\text{S-12.83})$$

which is the superposition of two  $z$ -polarized plane waves of equal amplitude, and wavevectors of equal magnitude, but opposite  $y$  component,  $\mathbf{k} = \hat{\mathbf{x}} k_1 \pm \hat{\mathbf{y}} q_1$ . Thus both plane waves impinge on the vacuum-medium interface at the same incidence angle  $|\theta| = \arctan(q_1/k_1)$ .

## S-12.8 Schumann Resonances

- a) Substituting the electric field (12.9) into the periodic boundary conditions (12.8) we obtain

$$e^{ik_x L} = 1, \quad e^{ik_y L} = 1, \quad (\text{S-12.84})$$

solved by

$$k_x = m \frac{2\pi}{L}, \quad k_y = n \frac{2\pi}{L}, \quad m, n = 0, 1, 2, \dots \quad (\text{S-12.85})$$

where  $m$  and  $n$  are not allowed to be zero simultaneously, and  $L = 2\pi R_{\oplus}$ . Since the wave equations gives us  $\omega^2 = k^2 c^2$ , we have

$$\omega^2 = \left( \frac{c}{R_{\oplus}} \right)^2 (m^2 + n^2). \quad (\text{S-12.86})$$

The lowest frequency corresponds to  $m = 1, n = 0$  or  $m = 0, n = 1$ , and its value is

$$\nu_{\min} = \frac{\omega_{\min}}{2\pi} = \frac{\omega_{10}}{2\pi} = \frac{c}{2\pi R_{\oplus}} \simeq 7.5 \text{ s}^{-1}, \quad (\text{S-12.87})$$

corresponding to a wavelength  $\lambda_{\max} = 2\pi R_{\oplus} \simeq 40000 \text{ km}$ , the length of a great circle of the Earth. The experimentally observed value is  $\nu_{\min} \simeq 8 \text{ s}^{-1}$ .

**b)** An ohmic conductor can be considered as perfectly reflecting at a frequency  $\omega$  if its conductivity  $\sigma(\omega)$ , assumed to be real, fulfills the condition  $\sigma(\omega) \gg \omega/4\pi k_e$ , where  $k_e = 1$  in Gaussian units, and  $k_e = 1/(4\pi\epsilon_0)$  in SI units. Heuristically, the condition corresponds to the conduction current  $\mathbf{J}$  being much larger than Maxwell's displacement current. Since  $\epsilon_0 = 8.854 \times 10^{-12}$  SI units, and  $\sigma/\omega \approx 0.6 \text{ s } \Omega^{-1} \text{ m}^{-1}$ , sea water can be considered as a perfect conductor in the frequency range of the Schumann resonances. In Gaussian units, the low-frequency conductivity of sea water is  $\sigma \simeq 4 \times 10^{10} \text{ s}^{-1}$ .

A discussion of Schumann resonances based on a “realistic” spherical geometry can be found in Reference [2], Section 8.9 and Problem 8.7. Nevertheless, our simplified approach reveals the essential point that the characteristic length  $L$  of the system, which determines the maximum wavelength for a standing wave ( $\lambda \approx L$ ), is the Earth’s circumference, rather than the height of the ionosphere above the the Earth’s surface.

## References

1. R.P. Feynman, R.B. Leighton, M. Sands, *The Feynman Lectures on Physics*, Addison-Wesley Publishing Company, Reading, MA 2006. Volume II, Section 24–8
2. J.D. Jackson, *Classical Electrodynamics*, § 9.2 and 9.4, 3rd Ed., Wiley, New York, London, Sidney (1998)

# Chapter S-13

## Solutions for Chapter 13

### S-13.1 Electrically and Magnetically Polarized Cylinders

(a) Long cylinders. In the “magnetic case”, the parallel component of the auxiliary field,  $\mathbf{H} = \mathbf{B}/[(\mu_0)\mu_r]$  (here, and the following, the parentheses mean that  $\mu_0$  appears in SI units only, not in Gaussian units) is continuous at the lateral surface of the cylinder. Thus the magnetic field inside the cylinder,  $\mathbf{B}_i$ , is

$$\mathbf{B}_i = \mu_r \mathbf{B}_0. \quad (\text{S-13.1})$$

The interface condition for the electric field is that the parallel component of  $\mathbf{E}$  must be continuous at the lateral surface, thus we have for the internal field

$$\mathbf{E}_i = \mathbf{E}_0. \quad (\text{S-13.2})$$

These results are consistent with the analogy between the equations for  $\mathbf{E}$  in electrostatics and  $\mathbf{H}$  in magnetostatics and in the absence of free currents, i.e.,  $\nabla \times \mathbf{E} = 0$  and  $\nabla \times \mathbf{H} = 0$ .

(b) Flat cylinders. In the “magnetic case”, the perpendicular component of  $\mathbf{B}$  is continuous at the bases, thus we have

$$\mathbf{B}_i = \mathbf{B}_0. \quad (\text{S-13.3})$$

In the “electric case”, the perpendicular component of the auxiliary vector  $\mathbf{D}$  must be continuous at the interface, thus internal field is

$$\mathbf{E}_i = \frac{1}{\epsilon_r} \mathbf{E}_0. \quad (\text{S-13.4})$$

These results are consistent with the analogy between the equations for  $\mathbf{B}$  and for  $\mathbf{D}$  in electrostatics and in the absence of free charges, i.e.,  $\nabla \cdot \mathbf{B} = 0$  and  $\nabla \cdot \mathbf{D} = 0$ .

(c) Let us assume (S-13.2) as zero-order solution for the case of the “long” dielectric cylinder. According to (3.1) the cylinder acquires a uniform electric polarization

$$\mathbf{P} = \frac{\epsilon_r - 1}{4\pi k_e} \mathbf{E}_i = \frac{\epsilon_r - 1}{4\pi k_e} \mathbf{E}_0, \quad (\text{S-13.5})$$

corresponding to two bound surface charge densities  $\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = \pm P$  at the cylinder bases. When evaluating the field at the cylinder center, due to the condition  $a \ll h$  the total bound charges on the two bases can be approximated by two point charges  $\pm Q$ , with

$$Q = \pi a^2 P = \frac{a^2(\epsilon_r - 1)}{4k_e} E_0 = \begin{cases} \frac{\pi a^2 \epsilon_0 (\epsilon_r - 1)}{4} E_0, & \text{SI,} \\ \frac{a^2(\epsilon_r - 1)}{4} \epsilon_r E_0, & \text{Gaussian,} \end{cases} \quad (\text{S-13.6})$$

located at distances  $\pm h/2$ . Thus, at the cylinder center we have an additional field

$$E_b \approx -2k_e \frac{Q}{(h/2)^2} = -2(\epsilon_r - 1) E_0 \left( \frac{a}{h} \right)^2, \quad (\text{S-13.7})$$

corresponding to a second-order correction. The electric field up to the second order in  $(a/h)$  is thus

$$\mathbf{E}_i^{(2)} = \mathbf{E}_i + \mathbf{E}_b = \mathbf{E}_0 \left[ 1 - 2(\epsilon_r - 1) \left( \frac{a}{h} \right)^2 \right]. \quad (\text{S-13.8})$$

In the corresponding “magnetic case”, the formal analogy between  $\mathbf{H}$  and  $\mathbf{E}$  leads to a second-order correction to the auxiliary field  $\mathbf{H}_i$  at the cylinder center

$$\mathbf{H}_b = -2(\mu_r - 1) \mathbf{H}_0 \left( \frac{a}{h} \right)^2, \quad (\text{S-13.9})$$

where  $H_0 = B_0/(\mu_0)$ . Because of the formal analogy between  $\mathbf{H}$  and  $\mathbf{E}$ , the correction to  $\mathbf{H}$  at the center of the cylinder can be interpreted as due to the presence of *fictitious equivalent magnetic charges*  $Q_m = \pm \pi a^2 M$  on the two cylinder bases. The fictitious magnetic charge densities  $\sigma_m = \pm M$  at the two bases are associated to the magnetization  $\mathbf{M} = \chi_m \mathbf{H}_0$ , where  $\chi_m$  is given by (5.22) in terms of  $\mu_r$ . Each magnetic charge gives origin to an auxiliary field

$$\mathbf{H} = \begin{cases} \frac{1}{4\pi} \frac{Q_m}{r^2} \hat{\mathbf{r}}, & \text{SI,} \\ \frac{1}{c} \frac{Q_m}{r^2} \hat{\mathbf{r}}, & \text{Gaussian.} \end{cases} \quad (\text{S-13.10})$$

Recalling that, in SI units,  $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$ , we obtain for the magnetic field at the cylinder center

$$\mathbf{B} = \mu_0(\mathbf{H}_0 + \mathbf{M}) + \mu_0\mathbf{H}_b = \mu_r H_0 + \mu_0 H_b \equiv \mathbf{B}_i + \mathbf{B}_b, \quad (\text{S-13.11})$$

and the second-order correction is

$$\mathbf{B}_b = -2(\mu_r - 1)\mathbf{B}_0 \left( \frac{a}{h} \right)^2. \quad (\text{S-13.12})$$

In Gaussian units we have  $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}$ , the second order correction remaining the same as in (S-13.12).

Notice that it would have been *wrong* to write

$$\mathbf{B}_b = (\mu_0)\mu_r \mathbf{H}_b \quad (\text{wrong!}), \quad (\text{S-13.13})$$

as it would have been wrong to write

$$E_b \simeq -2 \frac{k_e}{\varepsilon_r} \frac{Q}{(h/2)^2} = -2 \frac{(\varepsilon_r - 1)}{\varepsilon_r} E_0 \left( \frac{a}{h} \right)^2 \quad (\text{wrong!}), \quad (\text{S-13.14})$$

instead of (S-13.7), because we are considering the fields generated by polarization charges, and inserting  $\mu_r$  or  $\varepsilon_r$  would mean taking the effects of the medium polarization into account twice.

Alternatively, we can recall that the zero-order approximation of the cylinder magnetization is

$$\mathbf{M} = \chi_m \mathbf{H}_i = \chi_m \frac{\mathbf{B}_i}{(\mu_0)\mu_r} = \chi_m \frac{\mathbf{B}_0}{(\mu_0)}, \quad (\text{S-13.15})$$

again,  $\mu_0$  appearing in SI units only. The magnetization is associated to a surface magnetization current density  $\mathbf{K}_m = \mathbf{M} \times \hat{\mathbf{n}}/b_m$  on the lateral surface of the cylinder

$$\mathbf{K}_m = \frac{\chi_m}{b_m} \frac{B_0}{(\mu_0)} \hat{\phi}, \quad (\text{S-13.16})$$

where  $\hat{\phi}$  is the azimuthal unit vector of the cylindrical coordinates with the cylinder axis as longitudinal axis. Thus, the cylinder is equivalent to a finite solenoid of height  $h$  and radius  $a$ , with the product  $nI$  equal to a  $K_m$ . The magnetic field of a finite solenoid on its axis is

$$\begin{aligned} B_M &= 2\pi k_m n I (\cos \alpha_1 - \cos \alpha_2) = 2\pi k_m K_m (\cos \alpha_1 - \cos \alpha_2) \\ &= 2\pi k_m \frac{\chi_m}{b_m} \frac{B_0}{(\mu_0)} (\cos \alpha_1 - \cos \alpha_2) = (\mu_r - 1) \frac{B_0}{2} (\cos \alpha_1 - \cos \alpha_2), \end{aligned} \quad (\text{S-13.17})$$

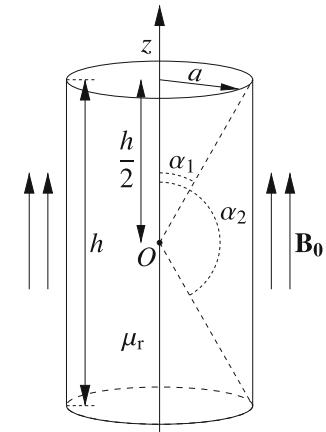


Fig. S-13.1

where the angles  $\alpha_1$  and  $\alpha_2$  are shown in Fig. S-13.1. At the solenoid center we have

$$\cos \alpha_1 = -\cos \alpha_2 = \frac{h/2}{\sqrt{a^2 + (h/2)^2}} \simeq 1 - \frac{1}{2} \left( \frac{2a}{h} \right)^2 = 1 - 2 \left( \frac{a}{h} \right)^2, \quad (\text{S-13.18})$$

thus

$$B_M \simeq (\mu_r - 1) B_0 \left[ 1 - 2 \left( \frac{a}{h} \right)^2 \right]. \quad (\text{S-13.19})$$

The total field at the cylinder center equals the external field  $B_0$  plus the field due to the cylinder magnetization

$$\begin{aligned} B(0) &= B_0 + B_M = \mu_r B_0 - 2\mu_r B_0 \left( \frac{a}{h} \right)^2 + 2B_0 \left( \frac{a}{h} \right)^2 \\ &= \mu_r B_0 - 2(\mu_r - 1) B_0 \left( \frac{a}{h} \right)^2, \end{aligned} \quad (\text{S-13.20})$$

in agreement with (S-13.19).

The correction to the field at the center of the magnetic “flat” cylinder can be evaluated as due to a circular loop of radius  $a$  carrying an electric current  $I_s = K_m h$ :

$$B_b = \frac{2\pi k_m I_s}{a} = 2\pi k_m K_m \frac{h}{a} = 2\pi \frac{k_m}{b_m} M \frac{h}{a}. \quad (\text{S-13.21})$$

At zeroth order we have

$$\mathbf{H}_i \simeq \frac{\mathbf{B}_0}{(\mu_0)\mu_r}, \quad \text{thus} \quad \mathbf{M} \simeq \chi_m \frac{\mathbf{B}_0}{(\mu_0)\mu_r}, \quad (\text{S-13.22})$$

and we get

$$B_b = \frac{\mu_r - 1}{2\mu_r} \frac{a}{h} B_0. \quad (\text{S-13.23})$$

The auxiliary field  $\mathbf{H}$  is given by (5.19), thus we have, up to the second order

$$H_i + H_b = \frac{B_0 + B_b}{(\mu_0)\mu_r} = \begin{cases} \frac{B_0 + B_b}{\mu_0} - M = H_i + \frac{B_b}{\mu_0} & \text{SI,} \\ B_0 + B_b - 4\pi M = H_i + B_b & \text{Gaussian.} \end{cases} \quad (\text{S-13.24})$$

Thus we have

$$H_b = \frac{B_b}{(\mu_0)} = \frac{\mu_r - 1}{2\mu_r} \frac{a}{h} H_0. \quad (\text{S-13.25})$$

Due to the formal analogy between  $\mathbf{H}$  and  $\mathbf{E}$  we have for the flat dielectric cylinder

$$E_b = \frac{\epsilon_r - 1}{2\epsilon_r} \frac{a}{h} E_0. \quad (\text{S-13.26})$$

## S-13.2 Oscillations of a Triatomic Molecule

(a) The equations of motion for the two lateral masses are

$$m\ddot{x}_1 = -k(x_1 - x_c + \ell), \quad m\ddot{x}_2 = -k(x_2 - x_c - \ell); \quad (\text{S-13.27})$$

from (13.1) we obtain for the position of the central mass

$$x_c = -\frac{m}{M}(x_1 + x_2), \quad (\text{S-13.28})$$

which, substituted into (S-13.27) after dividing by  $m$ , leads to a system of two equations of motion involving  $x_1$  and  $x_2$  only

$$\ddot{x}_1 = -k\left(\frac{1}{m} + \frac{1}{M}\right)x_1 - \frac{k}{M}x_2 - \frac{k}{m}\ell, \quad (\text{S-13.29})$$

$$\ddot{x}_2 = -k\left(\frac{1}{m} + \frac{1}{M}\right)x_2 - \frac{k}{M}x_1 + \frac{k}{m}\ell. \quad (\text{S-13.30})$$

Adding and subtracting these equations we obtain

$$\ddot{x}_1 + \ddot{x}_2 = -k\left(\frac{1}{m} + \frac{2}{M}\right)(x_1 + x_2) = -k\frac{M_{\text{tot}}}{mM}(x_1 + x_2) \quad (\text{S-13.31})$$

$$\ddot{x}_1 - \ddot{x}_2 = -\frac{k}{m}(x_1 - x_2 + 2\ell), \quad (\text{S-13.32})$$

where  $M_{\text{tot}} = M + 2m$  is the total mass of the molecule. Thus, introducing the new variables

$$x_+ = x_1 + x_2 \quad \text{and} \quad x_- = x_1 - x_2 + 2\ell, \quad (\text{S-13.33})$$

we obtain the following equations for the normal longitudinal modes of the molecule

$$\ddot{x}_{\pm} = -\omega_{\pm}^2 x_{\pm}, \quad \text{where} \quad \omega_+ = \sqrt{\frac{kM_{\text{tot}}}{mM}} \quad \text{and} \quad \omega_- = \sqrt{\frac{k}{m}}. \quad (\text{S-13.34})$$

Frequency  $\omega_+$  corresponds to an antisymmetric (!) motion of the masses: while the lateral masses move, for instance, to the right by the same amount, the central mass moves to the left, and vice versa, so that  $x_{\text{cm}} = 0$ . Frequency  $\omega_-$  corresponds to a

symmetric motion: the lateral masses perform opposite oscillations, while the central mass does not move.

(b) The electric dipole moment of the molecule is parallel to the molecular axis and its magnitude is

$$p = -qx_1 + 2qx_c - qx_2 = -q\left(1 + \frac{2m}{M}\right)(x_1 + x_2) = -q \frac{M_{\text{tot}}}{M} x_+. \quad (\text{S-13.35})$$

Thus, the dipole oscillates in the antisymmetric mode at frequency  $\omega_+$ . The dipole moment is zero when the molecule oscillates in the symmetric mode, and radiation at frequency  $\omega_-$  is due only to quadrupole emission, which is weaker than dipole emission.

(c) The initial conditions for  $x_+$  are

$$x_+(0) = x_1(0) + x_2(0) = d_1 + d_2, \quad \dot{x}_+(0) = 0, \quad (\text{S-13.36})$$

thus for  $t > 0$

$$x_+(t) = (d_1 + d_2) \cos \omega_+ t. \quad (\text{S-13.37})$$

The symmetric mode is also excited, but does not contribute to the dipole radiation. The instantaneous radiated power is

$$P = \frac{2}{3c^3} |\vec{p}|^2 = \frac{2q^2}{3c^3} \left(\frac{M_{\text{tot}}}{M}\right)^2 \omega_+^2 (d_1 + d_2)^2 \cos^2 \omega_+ t. \quad (\text{S-13.38})$$

### S-13.3 Impedance of an Infinite Ladder Network

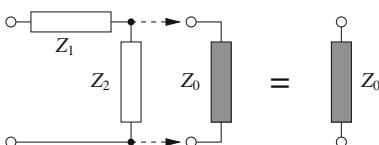


Fig. S-13.2

(a) Our infinite network is a sequence of identical sections. As we did for Problem 4.10, we note that adding a further L-section to the left of Fig. 13.3 does not change the impedance of the ladder network. Thus we must have (see Fig. S-13.2).

$$Z_0 = Z_1 + \frac{Z_2 Z_0}{Z_2 + Z_0}, \quad (\text{S-13.39})$$

from which  $Z_0^2 - Z_1Z_0 - Z_1Z_2 = 0$  follows. The solution is

$$Z_0 = \frac{Z_1}{2} + \sqrt{\frac{Z_1^2}{4} + Z_1Z_2}, \quad (\text{S-13.40})$$

The other solution of the quadratic equation has been discarded because in the case of real, positive impedances (the purely resistive case of Problem 4.10) it would give an unphysical negative value. Thus, a finite ladder of  $N$  sections, terminated by an impedance  $Z_0$  as shown in Fig. S-13.3, is equivalent to the infinite ladder.

(b) In Fig. 13.3, current  $I_n$  flows through the  $Z_1$  impedance of the  $(n+1)$ -th section, thus, the voltage drop across the impedance,  $V_n - V_{n+1}$ , must equal  $I_nZ_1$ . On the other hand,  $I_n$  is input into the semi-infinite ladder network starting at node  $n$ , thus we must have  $I_n = V_n/Z_0$ . The two conditions give

$$V_n - V_{n+1} = \frac{V_n}{Z_0} Z_1, \quad (\text{S-13.41})$$

so that we obtain for the ratio of the voltages at adjacent nodes

$$\alpha \equiv \frac{V_{n+1}}{V_n} = 1 - \frac{Z_1}{Z_0}. \quad (\text{S-13.42})$$

If  $V_0(t) = V_0 e^{-i\omega t}$  is the input voltage, we have  $V_n = \alpha^n V_0 e^{-i\omega t}$  at the  $n$ -th node. For a purely resistive network we have

$$Z_0 \equiv R_0 = \frac{R_1}{2} + \sqrt{\frac{R_1^2}{4} + R_1R_2}, \quad (\text{S-13.43})$$

which is a real number, and  $\alpha = 1 - R_1/R_0 < 1$ . At each successive node the signal is damped by a factor  $\alpha$ .

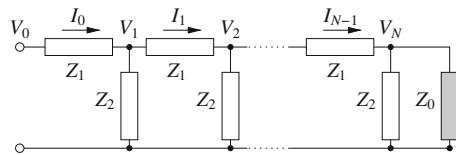


Fig. S-13.3

(c) For the  $LC$  network we have

$$\begin{aligned} Z_0 &= -\frac{i\omega L}{2} + \sqrt{-\frac{\omega^2 L^2}{4} + \frac{i\omega L}{i\omega C}} = -\frac{i\omega L}{2} + \sqrt{\frac{L}{C} - \frac{\omega^2 L^2}{4}} \\ &= \sqrt{\frac{4L^2}{4LC} - \frac{\omega^2 L^2}{4}} - \frac{i\omega L}{2} = \frac{L}{2} \sqrt{\frac{4}{LC} - \omega^2} - \frac{i\omega L}{2} \\ &= \frac{L}{2} \left( \sqrt{\omega_{co}^2 - \omega^2} - i\omega \right), \end{aligned} \quad (\text{S-13.44})$$

where  $\omega_{co} \equiv 2/\sqrt{LC}$ . Thus

$$\alpha = 1 - \frac{Z_1}{Z_0} = 1 + \frac{2i\omega L}{L \left( \sqrt{\omega_{co}^2 - \omega^2} - i\omega \right)} = \frac{\sqrt{\omega_{co}^2 - \omega^2} + i\omega}{\sqrt{\omega_{co}^2 - \omega^2} - i\omega}. \quad (\text{S-13.45})$$

If  $\omega < \omega_{co}$ , the square roots are real and  $\alpha$  is the ratio of a complex number to its own complex conjugate, therefore  $|\alpha| = 1$ , and we can write  $\alpha = e^{i\phi}$  with

$$\tan\left(\frac{\phi}{2}\right) = \frac{\omega}{\sqrt{\omega_{co}^2 - \omega^2}}. \quad (\text{S-13.46})$$

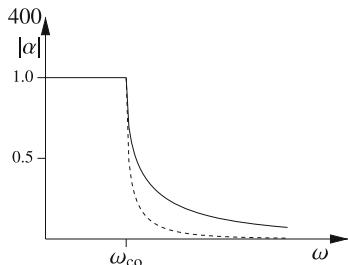
Thus the voltage at node  $n$  is  $V_n = V_0 e^{in\phi - i\omega t}$ , and the signal propagates along the network without damping. The above equation also gives the dispersion relation

$$\omega = \omega_{co} \left| \sin\left(\frac{\phi}{2}\right) \right|. \quad (\text{S-13.47})$$

This is analogous to the dispersion relation (S-7.42) found in Problem 7.4, when we substitute  $\phi$  for  $ka$ .

If  $\omega > \omega_{co}$ ,  $Z_0$  is a purely imaginary number,

$$Z_0 = \pm i \sqrt{\omega^2 - \omega_{co}^2}, \quad (\text{S-13.48})$$



and  $\alpha$  is real

$$\alpha = \frac{\pm \sqrt{\omega^2 - \omega_{co}^2} + \omega}{\pm \sqrt{\omega^2 - \omega_{co}^2} - \omega}. \quad (\text{S-13.49})$$

Inserting the negative root into (S-13.49) leads to  $|\alpha| < 1$ , and the signal is damped. The

Fig. S-13.4

positive root would lead to an unphysical  $|\alpha| > 1$ , implying an amplification of the signal along the network, without an external energy source.

Thus the  $LC$  network behaves as a low-pass filter, since signals at frequencies  $\omega > \omega_{\text{co}}$  are attenuated by a factor  $|\alpha|^N$  after  $N$  nodes. The dependence of the network transmission on frequency approaches an ideal low-pass filter, for which transmission is zero for  $\omega > \omega_{\text{co}}$ , at high numbers of circuit sections  $N$ . Figure S-13.4 shows  $|\alpha|$  (solid line) and  $|\alpha|^2$  (dashed line) as a functions of the signal frequency  $\omega$ .

(d) For the  $CL$  network (Problem 7.5) we proceed analogously to point (c) for the  $LC$  network, and obtain

$$Z_0 = \frac{i}{2\omega C} + \sqrt{-\frac{1}{4\omega^2 C^2} + \frac{\omega L}{\omega C}} = \frac{1}{2C} \left[ \sqrt{\frac{1}{\omega_{\text{co}}^2} - \frac{1}{\omega^2}} + \frac{i}{\omega} \right], \quad (\text{S-13.50})$$

and

$$\alpha = \frac{\sqrt{\omega_{\text{co}}^{-2} - \omega^{-2}} - i/\omega}{\sqrt{\omega_{\text{co}}^{-2} - \omega^{-2}} + i/\omega}. \quad (\text{S-13.51})$$

We have undamped propagation for  $|\alpha| = 1$ , i.e., when  $\omega > \omega_{\text{co}}$ . For  $\omega < \omega_{\text{co}}$  the signals are damped, and the network acts as a high-pass filter.

## S-13.4 Discharge of a Cylindrical Capacitor

(a) We use cylindrical coordinates  $(r, \phi, z)$ . For symmetry reasons, assuming  $h \gg b$ , the electric field between the capacitor plates is radial, and easily evaluated from Gauss's law as

$$E_r = E_r(r) = \frac{2Q_0}{hr}, \quad (\text{Gaussian units}). \quad (\text{S-13.52})$$

The potential difference  $V$  across the plates is

$$V = \left| \int_a^b \mathbf{E} \cdot d\mathbf{s} \right| = \frac{2Q_0}{h} \int_a^b \frac{dr}{r} = \frac{2Q_0}{h} \ln(b/a), \quad (\text{S-13.53})$$

and the capacity of our cylindrical capacitor is

$$C = \frac{Q_0}{V} = \frac{h}{2 \ln(b/a)} \quad (\text{S-13.54})$$

The initial electrostatic energy is  $U_{\text{es}}(0) = Q_0^2/2C$ .

After the plates are connected through the resistor at  $t = 0$ , the system is an  $RC$  circuit, and the capacitor charge at time  $t$  is

$$Q(t) = Q_0 e^{-t/\tau}, \quad \text{where} \quad \tau = RC = \frac{Rh}{2\ln(b/a)} \quad (\text{S-13.55})$$

Assuming that the charge densities remain uniform over the plates during the discharge, the absolute value of the charge of each plate between its bottom,  $z = 0$ , and any height  $z < h$  (see Fig. S-13.5 for the case of the inner plate of Fig. 13.4) is

$$\Delta Q(z, t) = Q(t) \frac{z}{h}. \quad (\text{S-13.56})$$

The decay of the charge implies a current flowing over each plate, along the  $\hat{\mathbf{z}}$  direction. Let  $I_a(z, t)$  and  $I_b(z, t)$  be the currents in the inner and outer plate, respectively, which can be obtained from the continuity equation: for the inner plate

$$I_a(z, t) = -\frac{d[\Delta Q(z, t)]}{dt} = \frac{Q(t)}{\tau} \frac{z}{h} = \frac{Q_0}{\tau} \frac{z}{h} e^{-t/\tau}. \quad (\text{S-13.57})$$

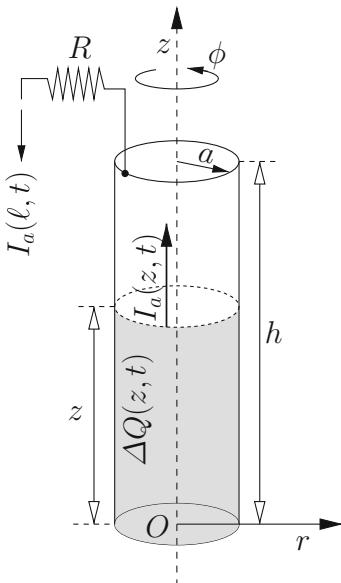


Fig. S-13.5

Since, in the assumption of uniform charge densities, the charge on the outer plate is  $-\Delta Q(z, t)$ , then  $I_b(z, t) = -I_a(z, t)$ .

We can evaluate  $\mathbf{B}$  in the  $a < r < b$  region from Maxwell's equation

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \partial_t \mathbf{E}. \quad (\text{S-13.58})$$

The only nonzero component of  $\mathbf{J}$  is along  $z$  and the only nonzero component of  $\mathbf{E}$  is along  $r$ , given by

$$\mathbf{E} = \hat{\mathbf{r}} \frac{2Q(t)}{hr}, \quad (\text{S-13.59})$$

while  $\mathbf{B}$  must be independent of  $\phi$  because of the symmetry of our problem. Thus, according to the curl components in cylindrical coordinates of Table A.1 of the Appendix we have

$$\begin{aligned}(\nabla \times \mathbf{B})_r &= -\partial_z B_\phi = \frac{1}{c} \partial_t E_r, \\ (\nabla \times \mathbf{B})_z &= \frac{1}{r} \partial_r (r B_\phi) = \frac{4\pi}{c} J_z,\end{aligned}\quad (\text{S-13.60})$$

and we see that the only nonzero component of  $\mathbf{B}$  is  $B_\phi$ , which can be evaluated from either of (S-13.60). We choose the second of (S-13.60), and apply Stokes' theorem to a circle  $C$  of radius  $a < r < b$ , coaxial to the capacitor and located at height  $0 < z < h$ ,

$$\oint_C \mathbf{B}(r, z, t) \cdot d\ell = 2\pi r B_\phi(r, z, t) = \frac{4\pi}{c} I_a(z, t), \quad (\text{S-13.61})$$

$$B_\phi(r, z, t) = \frac{2}{c} \frac{I_a(z, t)}{r} = \frac{2}{ch\tau} \frac{z}{r} Q_0 e^{-t/\tau}. \quad (\text{S-13.62})$$

**(b)** The Poynting vector is

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} = \hat{\mathbf{z}} \frac{Q_0^2}{\pi h^2 \tau} \frac{z}{r^2} e^{-2t/\tau}, \quad a < r < b, \quad (\text{S-13.63})$$

and  $\mathbf{S} = 0$  if  $r < a$  or  $r > b$ . The flux of  $\mathbf{S}$  through a plane perpendicular to  $z$  at height  $0 < z < h$  is thus

$$\Phi_S(z, t) = \frac{Q_0^2 z}{\pi h^2 \tau} e^{-2t/\tau} \int_a^b \frac{1}{r^2} 2\pi r dr = \frac{2Q_0^2 z \ln(b/a)}{h^2 \tau} e^{-2t/\tau}. \quad (\text{S-13.64})$$

The electrostatic energy associated to the volume between the bottom of the capacitor ( $z = 0$ ) and height  $z$  at time  $t$  is

$$\Delta U_{es}(z, t) = \frac{z}{h} \frac{Q^2(t)}{2C} = \frac{z}{h} \frac{Q_0^2 e^{-2t/\tau}}{2C} = \frac{z}{h} \frac{Q_0^2 \ln(b/a)}{h} e^{-2t/\tau}, \quad (\text{S-13.65})$$

because the electric field does not depend on  $z$ . Thus we have

$$\frac{d[\Delta U_{es}(z, t)]}{dt} = -\frac{2\Delta U_{es}(z, t)}{\tau} = -\Phi_S(z, t). \quad (\text{S-13.66})$$

**(c)** The assumptions of slowly varying currents and of uniform charge density are closely related. In fact, the capacitor can be viewed as a portion of a coaxial cable along which charge and current signals are propagating in TEM mode, at velocity  $c$ . In these conditions, the charge density can be assumed as uniform if the propagation of the signals is “instantaneous” with respect to the duration of the discharge, i.e., if the propagation time  $h/c \ll \tau$ . This is equivalent to assuming that the wavelengths corresponding to the frequency spectrum of the signal are much larger than  $h$ , so that the field can be considered as uniform along  $z$ .

We can reach the same conclusion by checking that the electric field  $\mathbf{E}_1$ , generated by the magnetic induction, is much smaller than the electrostatic field  $\mathbf{E}_0$ . From Maxwell's equation

$$\nabla \times \mathbf{E}_1 \simeq -\frac{1}{c} \partial_t \mathbf{B}, \quad (\text{S-13.67})$$

where the only nonzero component of  $\mathbf{B}$  is  $B_\phi$ , we obtain

$$\begin{aligned} \partial_z E_{1r} &= \frac{1}{c} \frac{2}{ch\tau^2} \frac{z}{r} Q_0 e^{-t/\tau} \\ E_{1r} &= \frac{Q_0}{c^2 h \tau^2} \frac{z^2}{r} e^{-t/\tau} = \frac{1}{2} \left( \frac{z}{c\tau} \right)^2 E_{0r}. \end{aligned} \quad (\text{S-13.68})$$

where  $E_{0r}$  is from the second of (S-13.59). Thus  $E_{1r} \ll E_{0r}$  if  $h \ll c\tau$ .

### S-13.5 Fields Generated by Spatially Periodic Surface Sources

(a) In this case fields and potential are electrostatic. The potential  $\varphi = \varphi(x, y)$  is a solution of the 2D Laplace's equation

$$(\partial_x^2 + \partial_y^2)\varphi = 0 \quad \text{for } y \neq 0, \quad (\text{S-13.69})$$

and, due to the symmetry of the source, must be an even function of  $y$ . We attempt to find a solution by the method of separation of variables, i.e., we look for a solution of the form  $\varphi = X(x)Y(y)$ , where  $X$  depends only on  $x$  and  $Y$  only on  $y$ . Equation (S-13.69) becomes

$$X''(x)Y(y) + X(x)Y''(y) = 0, \quad (\text{S-13.70})$$

where the double primes denote the second derivatives. Dividing by  $X(x)Y(y)$  we obtain

$$\frac{Y''(y)}{Y(y)} = -\frac{X''(x)}{X(x)}, \quad (\text{S-13.71})$$

which must hold for every  $x, y$ , implying that both sides of the equation must equal some constant value, which, for convenience, we denote by  $\alpha^2$ ,

$$\frac{Y''(y)}{Y(y)} = \alpha^2, \quad \frac{X''(x)}{X(x)} = -\alpha^2, \quad (\text{S-13.72})$$

whose solutions are

$$Y(y) = A_y e^{+\alpha y} + B_y e^{-\alpha y}, \quad \text{and} \quad X(x) = A_x e^{+i\alpha x} + B_x e^{-i\alpha x}, \quad (\text{S-13.73})$$

where  $A_x$ ,  $A_y$ ,  $B_x$ , and  $B_y$  are constants to be determined. Discarding the solutions that diverge for  $|y| \rightarrow \infty$ , and fitting the  $x$  dependence to the dependence of  $\sigma$ , which implies  $\alpha = k$ , we obtain

$$\varphi = \varphi_0 e^{-k|y|} \cos(kx), \quad (\text{S-13.74})$$

where  $\varphi_0$  is a constant to be determined. The nonzero components of the electric field are

$$\begin{aligned} E_x &= -\partial_x \varphi = k\varphi_0 e^{-k|y|} \sin(kx), \\ E_y &= -\partial_y \varphi = \operatorname{sgn}(y) k\varphi_0 e^{-k|y|} \cos(kx). \end{aligned} \quad (\text{S-13.75})$$

The component  $E_x$  is continuous at the  $y = 0$  plane, as expected, since  $\nabla \times \mathbf{E} = 0$ . We can obtain the relation between  $E_y$  at  $y = 0$  and the surface charge density by using Gauss's law,

$$E_y(x, y = 0^+) - E_y(x, y = 0^-) = 4\pi\sigma(x), \quad (\text{S-13.76})$$

from which we obtain the value of  $\varphi_0$ , namely  $\varphi_0 = 2\pi\sigma_0/k$ , and, finally

$$\varphi = \frac{2\pi\sigma_0}{k} e^{-k|y|} \cos(kx). \quad (\text{S-13.77})$$

**(b)** Here we have magnetostatic fields. Due to the analogy between the Poisson equations for the vector potential  $\nabla^2 \mathbf{A} = -4\pi\mathbf{J}/c$ , and for the scalar potential  $\nabla^2 \varphi = -4\pi\rho$ , we can use (S-13.77) for obtaining the vector potential  $\mathbf{A}$  as

$$\mathbf{A} = \hat{\mathbf{z}} A_0 e^{-k|y|} \cos(kx), \quad \text{where } A_0 = \frac{2\pi K_0}{kc}. \quad (\text{S-13.78})$$

The nonzero components of the magnetic field are

$$\begin{aligned} B_x &= \partial_y A_z = -\operatorname{sgn}(y) k A_0 e^{-k|y|} \cos(kx) = -\operatorname{sgn}(y) \frac{2\pi K_0}{c} e^{-k|y|} \cos(kx), \\ B_y &= -\partial_x A_z = k A_0 e^{-k|y|} \sin(kx) = \frac{2\pi K_0}{c} e^{-k|y|} \sin(kx). \end{aligned} \quad (\text{S-13.79})$$

Thus,  $B_y$  is continuous at  $y = 0$ , as expected from  $\nabla \cdot \mathbf{B} = 0$ . Further we have

$$B_x(x, y = 0^+) - B_x(x, y = 0^-) = -\frac{4\pi}{c} K_0 \cos(kx), \quad (\text{S-13.80})$$

in agreement with Ampère's law.

**(c)** Since  $\sigma = 0$ , also the scalar potential is zero,  $\varphi = 0$ . The inhomogeneous wave equation for the vector potential  $\mathbf{A}$  is, in the Lorentz gauge condition,

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi}{c} \mathbf{J} = -\hat{\mathbf{z}} \frac{4\pi}{c} \delta(y) K_0 e^{-i\omega t} \cos(kx). \quad (\text{S-13.81})$$

As an educated guess, we search for a solution of the form

$$\mathbf{A} = \hat{\mathbf{z}} A_0 e^{-q|y|-i\omega t} \cos(kx), \quad (\text{S-13.82})$$

which, for  $y \neq 0$ , leads to

$$\left( -k^2 + q^2 + \frac{\omega^2}{c^2} \right) \mathbf{A} = 0, \quad \text{or} \quad q^2 = k^2 - \left( \frac{\omega}{c} \right)^2. \quad (\text{S-13.83})$$

Thus, if  $\omega < kc$ ,  $q$  is real and  $\mathbf{A}$  decays exponentially with  $|y|$ . If  $\omega > kc$ ,  $q$  is imaginary and the waves propagates,  $\mathbf{A}$  being proportional to  $e^{i|q||y|-i\omega t}$ . If we integrate (S-13.81) in  $dy$  from  $-h$  to  $+h$  we obtain

$$\lim_{h \rightarrow 0} \int_{-h}^{+h} \left( \frac{\partial^2 \mathbf{A}}{\partial x^2} + \frac{\partial^2 \mathbf{A}}{\partial y^2} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} \right) dy = -\hat{\mathbf{z}} \frac{4\pi}{c} K_0 e^{-i\omega t} \cos(kx). \quad (\text{S-13.84})$$

Now, both  $\partial^2 \mathbf{A} / \partial x^2$  and  $\partial^2 \mathbf{A} / \partial t^2$  are continuous at  $y = 0$  and don't contribute to the integral at the limit  $h \rightarrow 0$ . Thus, the left-hand side of (S-13.84) is

$$\begin{aligned} \lim_{h \rightarrow 0} \int_{-h}^{+h} \frac{\partial^2 \mathbf{A}}{\partial y^2} dy &= \lim_{h \rightarrow 0} [\partial_y \mathbf{A}]_{-h}^{+h} = -\hat{\mathbf{z}} A_0 q \cos(kx) \lim_{h \rightarrow 0} [\operatorname{sgn}(y) e^{-q|y|-i\omega t}]_{-h}^{+h} \\ &= -\hat{\mathbf{z}} 2A_0 e^{-i\omega t} q \cos(kx), \end{aligned} \quad (\text{S-13.85})$$

which must equal the right-hand side of (S-13.84), leading to

$$A_0 = \frac{2\pi}{qc} K_0 \quad (\text{S-13.86})$$

which, at the static limit  $\omega \rightarrow 0$ ,  $q \rightarrow k$ , equals (S-13.78).

The nonzero components of the magnetic field are

$$\begin{aligned} B_x &= \partial_y A_z = -\operatorname{sgn}(y) \frac{2\pi}{c} K_0 e^{-q|y|-i\omega t} \cos(kx), \\ B_y &= -\partial_x A_z = \frac{2\pi k}{qc} K_0 e^{-q|y|-i\omega t} \sin \omega t, \end{aligned} \quad (\text{S-13.87})$$

which, at the static limit  $\omega \rightarrow 0$ ,  $q \rightarrow k$ , equal (S-13.79). The electric field is obtained from  $\mathbf{E} = -\partial_t \mathbf{A} / c = i\omega \mathbf{A} / c$ , and its only nonzero component is

$$E_z = -\frac{2\pi i\omega K_0}{qc^2} e^{-q|y|-i\omega t} \cos(kx). \quad (\text{S-13.88})$$

(d) In this context, given a function  $f = f(x, t)$ , we denote its time average by angle brackets, and its space average by a bar, as follows

$$\langle f \rangle = \frac{\omega}{2\pi} \int_{-\pi/\omega}^{+\pi/\omega} f dt, \quad \bar{f} = \frac{k}{2\pi} \int_{-\pi/k}^{+\pi/k} f dx. \quad (\text{S-13.89})$$

Thus we write the average power dissipated per unit time and unite surface on the  $y = 0$  plane as

$$\overline{\langle K_z E_z \rangle} = \frac{1}{2} \operatorname{Re} \left[ K_0 \left( \frac{2\pi i \omega}{qc} K_0 \right)^* \right] \overline{\cos^2(kx)} = \frac{\pi \omega}{4c} |K_0|^2 \operatorname{Re} \left( \frac{-i}{q} \right). \quad (\text{S-13.90})$$

If  $q$  is real we have  $\overline{\langle K_z E_z \rangle} = 0$ , consistently with the fields being evanescent for  $|y| \rightarrow \infty$ . There is no energy flow out of the  $y = 0$  plane, and the work done by the currents is zero on average. On the other hand, if  $q$  is imaginary, we have

$$\overline{\langle K_z E_z \rangle} = -\frac{\pi \omega |K_0|^2}{2|q|c}, \quad (\text{S-13.91})$$

which equals minus the flux of electromagnetic energy out of the  $y = 0$  plane. In fact, the averaged Poynting vector is

$$\begin{aligned} \overline{\langle S_y \rangle} &= \frac{c}{4\pi} \overline{\langle E_z B_x \rangle} = \frac{1}{2} \frac{c}{4\pi} \operatorname{Re} \left[ \frac{2\pi i \omega K_0}{2qc} \left( \operatorname{sgn}(y) q^* \frac{2\pi K_0^*}{q^* c} \right) \right] \overline{\cos^2(kx)} \\ &= \operatorname{sgn}(y) \frac{\pi \omega |K_0|^2}{4|q|c}, \end{aligned} \quad (\text{S-13.92})$$

where we have used  $\operatorname{Re}(i/q) = 1/|q|$  (for imaginary  $q$ ). The flux of energy out of the  $y = 0$  plane is thus  $2|\overline{\langle S_y \rangle}| = -\overline{\langle K_z E_z \rangle}$ .

## S-13.6 Energy and Momentum Flow Close to a Perfect Mirror

(a) The total electric field in front of the mirror is the sum of the fields of the incident ( $\mathbf{E}_i$ ) and of the reflected ( $\mathbf{E}_r$ ) waves, which have equal amplitude and frequency, but opposite polarizations and wavevectors,

$$\begin{aligned}
\mathbf{E} &= \mathbf{E}_i + \mathbf{E}_r = \hat{\mathbf{y}} E_\epsilon [\cos(kx - \omega t) - \cos(-kx - \omega t)] \\
&\quad - \hat{\mathbf{z}} \epsilon E_\epsilon [\sin(kx - \omega t) - \sin(-kx - \omega t)] \\
&= \hat{\mathbf{y}} E_\epsilon [\cos(kx) \cos(\omega t) + \sin(kx) \sin(\omega t) - \cos(kx) \cos(\omega t) + \sin(kx) \sin(\omega t)] \\
&\quad - \hat{\mathbf{z}} \epsilon E_\epsilon [\sin(kx) \cos(\omega t) - \cos(kx) \sin(\omega t) + \sin(kx) \cos(\omega t) + \cos(kx) \sin(\omega t)] \\
&= \hat{\mathbf{y}} 2E_\epsilon \sin(kx) \sin(\omega t) - \hat{\mathbf{z}} 2\epsilon E_\epsilon \sin(kx) \cos(\omega t), \tag{S-13.93}
\end{aligned}$$

where  $E_\epsilon \equiv E_0 / \sqrt{1 + \epsilon^2}$ . We can obtain the magnetic field from Maxwell's equation

$$\begin{aligned}
\partial_t \mathbf{B} &= -c \nabla \times \mathbf{E} = \hat{\mathbf{y}} c \partial_x E_z - \hat{\mathbf{z}} c \partial_x E_y \\
&= -\hat{\mathbf{y}} 2\epsilon E_\epsilon c k \cos(kx) \cos(\omega t) - \hat{\mathbf{z}} 2E_\epsilon c k \cos(kx) \sin(\omega t), \tag{S-13.94}
\end{aligned}$$

which yields, after integration in  $dt$ ,

$$\begin{aligned}
\mathbf{B} &= -\hat{\mathbf{y}} 2\epsilon E_\epsilon \frac{ck}{\omega} \cos(kx) \sin(\omega t) + \hat{\mathbf{z}} 2E_\epsilon \frac{ck}{\omega} \cos(kx) \cos(\omega t) \\
&= -\hat{\mathbf{y}} 2\epsilon E_\epsilon \cos(kx) \sin(\omega t) + \hat{\mathbf{z}} 2E_\epsilon \cos(kx) \cos(\omega t), \tag{S-13.95}
\end{aligned}$$

where we have used  $k = \omega/c$ . The Poynting vector is

$$\begin{aligned}
\mathbf{S} &= \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} = \hat{\mathbf{x}} \frac{c}{4\pi} (E_y B_z - E_z B_y) \\
&= \hat{\mathbf{x}} \frac{c E_\epsilon^2}{\pi} [\sin(kx) \cos(kx) \sin(\omega t) \cos(\omega t) - \epsilon^2 \sin(kx) \cos(kx) \cos(\omega t) \sin(\omega t)] \\
&= \hat{\mathbf{x}} \frac{c E_\epsilon^2}{\pi} \sin(kx) \cos(kx) \sin(\omega t) \cos(\omega t) (1 - \epsilon^2) \\
&= \hat{\mathbf{x}} \frac{c}{4\pi} E_\epsilon^2 (1 - \epsilon^2) \sin(2kx) \sin(2\omega t). \tag{S-13.96}
\end{aligned}$$

Thus  $\mathbf{S} = 0$  if  $\epsilon = 1$ , corresponding to circular polarization. In such a case,  $\mathbf{E}$  is parallel to  $\mathbf{B}$ . In general, also when  $\mathbf{S} \neq 0$ , we have  $\langle \mathbf{S} \rangle = 0$ , and there is no net energy flow.

(b) From the definition of  $T_{ij}$  we find

$$\begin{aligned}
F_x &= T_{xx} = \frac{1}{8\pi} \mathbf{B}^2 (0^-) = \frac{1}{4\pi} E_\epsilon^2 (\cos^2 \omega t + \epsilon^2 \sin^2 \omega t) \\
&= \frac{2I}{c} \left[ 1 + \frac{1 - \epsilon^2}{1 + \epsilon^2} \cos 2\omega t \right]. \tag{S-13.97}
\end{aligned}$$

The oscillating (at  $2\omega$ ) component vanishes for circular polarization. The average of  $F_x$  is the radiation pressure on the mirror (Problem 9.8), which does not depend on polarization.

## S-13.7 Laser Cooling of a Mirror

**(a)** A plane wave of intensity  $I$  exerts a radiation pressure  $2I/c$  on a perfectly reflecting surface. Thus the total force on the mirror, directed along the  $x$  axis of Fig. 13.5, is

$$F = \frac{2A}{c}(I_1 - I_2). \quad (\text{S-13.98})$$

If  $I_1 > I_2$  we have  $F > 0$ .

**(b)** The amplitudes of the electric fields of the two waves, in the mirror rest frame  $S'$ , are

$$E'_1 = \gamma(E_1 - \beta B_1) = \gamma(1 - \beta)E_1 = \sqrt{\frac{1-\beta}{1+\beta}} E_1, \quad (\text{S-13.99})$$

$$E'_2 = \gamma(E_2 + \beta B_2) = \gamma(1 + \beta)E_2 = \sqrt{\frac{1+\beta}{1-\beta}} E_2, \quad (\text{S-13.100})$$

where  $\beta = v/c$ ,  $\gamma = 1/\sqrt{1-\beta^2}$ , and  $E_1 = B_1$ ,  $E_2 = B_2$  in Gaussian units. The intensity of a plane wave is  $I = (c/4\pi)|\mathbf{E} \times \mathbf{B}| = cE^2/4\pi$ , thus we have

$$I'_1 = \frac{1-\beta}{1+\beta} I_1, \quad I'_2 = \frac{1+\beta}{1-\beta} I_2. \quad (\text{S-13.101})$$

Since we have assumed  $I_1 = I_2$ , the total force is

$$F' = \frac{2A}{c}(I'_1 - I'_2) = \frac{2A}{c} \frac{(1-\beta)^2 - (1+\beta)^2}{1-\beta^2} = -8A\beta\gamma^2 \frac{I}{c}. \quad (\text{S-13.102})$$

**(c)** From the answer to point **(b)** we have  $F' < 0$ , the direction of the force is opposite to the direction of  $v$ . At the limit  $v \ll c$ , the force in the laboratory frame is equal to the force in the mirror frame, and we have

$$\mathbf{F} \approx \mathbf{F}' \approx -8A \frac{I}{c^2} \mathbf{v}, \quad (\text{S-13.103})$$

which is a viscous force. Under the action of this force, the mirror velocity will decrease exponentially in time

$$v(t) = v(0)e^{-t/\tau}, \quad \text{where} \quad \tau = \frac{Mc^2}{8AI}. \quad (\text{S-13.104})$$

This effect has some analogies with the “laser-cooling” techniques, used in order to cool atoms down to temperatures of the order of  $10^{-6}$  K. These include, for

instance, Doppler cooling and Sisyphus cooling. The cooling of a macroscopic mirror by radiation pressure has also been studied [1] for possible applications in experiments of optical interferometry of ultra-high precision, e.g., for the detection of gravitational waves.

### S-13.8 Radiation Pressure on a Thin Foil

**(a)** It is instructive to solve this problem by three different methods. For definiteness we assume a linearly polarized incident wave, with electric field  $\mathbf{E}_i = \hat{\mathbf{y}} E_i e^{ik_i x - i\omega t}$ , where  $k_i = \omega/c$ ; generalization to arbitrary polarization is straightforward.

First method (heuristic): we assume the incident plane wave to be a square pulse of arbitrary but finite duration  $\tau$ , and thus length  $c\tau$ . The momentum of the wave packet impinging on the surface  $A$  of the foil is, neglecting boundary effects,

$$\mathbf{p}_i = \frac{\langle \mathbf{S}_i \rangle}{c^2} c\tau A = \hat{\mathbf{x}} \frac{\langle |\mathbf{E}_i|^2 \rangle}{4\pi c} \tau A = \hat{\mathbf{x}} \frac{E_i^2}{8\pi c} \tau A = \hat{\mathbf{x}} \frac{I}{c} \tau A, \quad (\text{S-13.105})$$

where  $I = \langle |\mathbf{S}_i| \rangle = c \langle |\mathbf{E}_i|^2 \rangle / (4\pi) = c E_i^2 / (8\pi)$  is the intensity of the incident wave. The reflected and transmitted wave packets have momenta

$$\mathbf{p}_r = \frac{\langle \mathbf{S}_r \rangle}{c^2} c\tau A = -\hat{\mathbf{x}} R \frac{E_i^2}{8\pi c} \tau A = -\hat{\mathbf{x}} R \frac{I}{c} \tau A, \quad (\text{S-13.106})$$

$$\mathbf{p}_t = \frac{\langle \mathbf{S}_t \rangle}{c^2} c\tau A = +\hat{\mathbf{x}} T \frac{E_i^2}{8\pi c} \tau A = +\hat{\mathbf{x}} T \frac{I}{c} \tau A, \quad (\text{S-13.107})$$

respectively, where  $R = |\mathbf{r}|^2$ ,  $T = |\mathbf{t}|^2$ , and  $R + T = 1$  because of energy conservation. The amount of momentum transferred from the incident wave packet to the foil is

$$\Delta \mathbf{p} = \mathbf{p}_i - (\mathbf{p}_r + \mathbf{p}_t), \quad (\text{S-13.108})$$

resulting in a pressure pushing the foil toward positive  $x$  values (because  $\Delta \mathbf{p} > 0$ )

$$P_{\text{rad}} = \frac{|\Delta \mathbf{p}|}{\tau A} = [1 - (-R + T)] \frac{I}{c} = 2R \frac{I}{c}. \quad (\text{S-13.109})$$

Second method: we calculate the average force on the foil, parallel to  $\hat{\mathbf{x}}$ , directly as

$$\langle \mathbf{F} \rangle = \int_0^d \langle \mathbf{J} \times \mathbf{B} \rangle A dx, \quad (\text{S-13.110})$$

where we have assumed the left surface of the foil located at  $x = 0$ , and the right surface located at  $x = d$ . For a very small thickness  $d$  we can write

$$\begin{aligned} A \int_0^d \langle \mathbf{J} \times \mathbf{B} \rangle dx &= \frac{1}{2} Ad \langle J(t) [B(0^+) + B(0^-)] \rangle \\ &= -\frac{Ac}{8\pi} \langle [B^2(0^+) - B^2(0^-)] \rangle, \end{aligned} \quad (\text{S-13.111})$$

where we have substituted  $J(t) = -[B(0^+) - B(0^-)] c/(4\pi d)$ . Since we have  $|B(0^+)| = |E_t| = |\mathbf{t} E_i|$ , and  $|B(0^-)| = |E_i - E_r| = |(1 - \mathbf{r}) E_i|$ , we can write the radiation pressure on the foil as

$$P_{\text{rad}} = \frac{\langle |\mathbf{F}| \rangle}{A} = -\frac{E_i^2}{16\pi c} (|\mathbf{t}|^2 - |1 - \mathbf{r}|^2) = -\frac{I}{2c} (|\mathbf{t}|^2 - |1 - \mathbf{r}|^2). \quad (\text{S-13.112})$$

Introducing the shorthand  $\alpha = (\omega_p^2 d)/(2\omega c)$  in (13.5), so that  $\eta = i\alpha$ , we have

$$\mathbf{t} = \frac{1}{1 + i\alpha}, \quad T = |\mathbf{t}|^2 = \frac{1}{1 + \alpha^2}, \quad (\text{S-13.113})$$

$$\mathbf{r} = -\frac{i\alpha}{1 + i\alpha}, \quad R = |\mathbf{r}|^2 = \frac{\alpha^2}{1 + \alpha^2}, \quad (\text{S-13.114})$$

$$|1 - \mathbf{r}|^2 = \frac{1 + 5\alpha^2 + 4\alpha^4}{(1 + \alpha^2)^2}, \quad |\mathbf{t}|^2 - |1 - \mathbf{r}|^2 = -\frac{4\alpha^2}{1 + \alpha^2}, \quad (\text{S-13.115})$$

and finally

$$P_{\text{rad}} = \frac{I}{2c^2} \frac{4\alpha^2}{1 + \alpha^2} = 2R \frac{I}{c}. \quad (\text{S-13.116})$$

Third method: we calculate the flow of EM momentum directly using Maxwell's stress tensor  $T_{ij}$ . The theorem of EM momentum conservation states that

$$\frac{d\mathbf{p}_i}{dt} = \oint_S T_{ij} n_j dS \quad (\text{S-13.117})$$

(summation over the repeated index is implied), where  $\hat{\mathbf{n}}$  is the unit vector normal to the surface  $S$  which envelops the thin foil, and  $\mathbf{p}$  is the total momentum (EM and mechanical) of the foil. Since in a steady state the EM contribution is constant, the RHS of (S-13.117) equals the variation of mechanical momentum, i.e., the force.

Taking into account that the electric field has only the component  $E_y$  and the magnetic field only the component  $B_z$ , and that  $\hat{\mathbf{n}} = \mp\hat{\mathbf{x}}$  on the left ( $x = 0^-$ ) and right ( $x = 0^+$ ) surfaces, respectively, the only relevant component of  $T_{ij}$  is  $T_{xx}$ , and

$$\frac{dp_x}{dt} = [T_{xx}(0^+) - T_{xx}(0^-)] A. \quad (\text{S-13.118})$$

For  $T_{xx}(0^+)$  and  $T_{xx}(0^-)$  we have

$$\begin{aligned} T_{xx}(0^+) &= -\frac{1}{8\pi} \langle E^2(0^+) + B^2(0^+) \rangle = -\frac{1}{4\pi} \langle |E_i|^2(0^+) \rangle \\ &= -T \frac{E_i^2}{8\pi}, \\ T_{xx}(0^-) &= -\frac{1}{8\pi} \langle E^2(0^-) + B^2(0^-) \rangle = -\frac{\langle E_i^2 \rangle}{8\pi} [|(1+r)|^2 + |(1-r)|^2] \\ &= -\frac{E_i^2}{16\pi} (1 + |r|^2 + rr^* + 1 + |r|^2 - rr^*) \\ &= -(1+R) \frac{E_i^2}{8\pi}. \end{aligned} \quad (\text{S-13.119})$$

Thus

$$\frac{dp_x}{dt} = \frac{E_i^2}{8\pi} (-T + 1 + R) A = 2R \frac{I}{c} A, \quad (\text{S-13.120})$$

which yields (13.6) again.

**(b)** From the Lorentz transformation of the fields we obtain the intensity of the incident wave in the  $S'$  frame, where the foil is at rest,

$$I' = \frac{1-\beta}{1+\beta} I, \quad (\text{S-13.121})$$

and the force on the foil in  $S'$  is  $F' = 2AI'/c$ . Since for a force parallel to  $v$  we have  $F = F'$ , in the laboratory frame  $S$  we can write

$$F = F' = 2 \frac{1-\beta}{1+\beta} \frac{I}{c} A. \quad (\text{S-13.122})$$

**(c)** The radiation pressure must be multiplied by a factor  $R = R(\omega')$  in the frame  $S'$ , where the frequency is  $\omega' = \sqrt{(1-\beta)/(1+\beta)}\omega$ . Thus

$$F = 2 \frac{1-\beta}{1+\beta} R(\omega') \frac{I}{c} A, \quad \omega' = \sqrt{\frac{1-\beta}{1+\beta}} \omega. \quad (\text{S-13.123})$$

## S-13.9 Thomson Scattering in the Presence of a Magnetic Field

(a) We write the fields in the complex notation. Within our assumptions, the equation of motion for the electron is

$$m_e \frac{d\mathbf{v}}{dt} = -e \left( \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}_0 \right), \quad (\text{S-13.124})$$

where  $-e$  and  $m_e$  are the charge and mass of the electron, respectively. The solution has already been evaluated in Problem 7.10, and is

$$v_x = \frac{\omega_c}{\omega_c^2 - \omega^2} \frac{e}{m_e} E_i e^{-i\omega t}, \quad v_y = \frac{i\omega}{\omega_c^2 - \omega^2} \frac{e}{m_e} E_i e^{-i\omega t}, \quad (\text{S-13.125})$$

and  $v_z = 0$ .

(b) The cycle-averaged radiated power is

$$\langle P \rangle = \frac{e^4}{3m_e^2 c^3} |E_i|^2 \frac{\omega^2}{(\omega_c^2 - \omega^2)^2} (\omega_c^2 + \omega^2), \quad (\text{S-13.126})$$

which is maximum at the cyclotron resonance,  $\omega = \omega_c$ . At the low-frequency limit  $\omega/\omega_c \ll 1$  we have  $\langle P \rangle \propto \omega^2/\omega_c^2$ , while at the high-frequency limit  $\omega/\omega_c \gg 1$  the power is independent of frequency (“white” spectrum).

The orbit of the electron is elliptical, consequently the angular distribution and polarization of the scattered radiation are analogous to what found for an electron in the presence of an elliptically polarized wave, in the absence of external magnetic fields, as discussed in Problem 10.9. According to (S-13.125) we have  $v_x/v_y = -i\omega_c/\omega$ . At the limit  $\omega \ll \omega_c$  we have  $\langle |v_x| \rangle \gg \langle |v_y| \rangle$ , the major axis of the elliptical orbit of the electron is thus parallel to  $\hat{x}$ , and the strongest radiation intensity is observed on the  $yz$  plane. At the opposite limit,  $\omega \gg \omega_c$ , we have  $\langle |v_x| \rangle \ll \langle |v_y| \rangle$ , the major axis of the orbit is parallel to  $\hat{y}$ , and the strongest radiation intensity is observed on the  $xz$  plane.

## S-13.10 Undulator Radiation

(a) According to Maxwell's equation  $\nabla \cdot \mathbf{B} = 0$ , we must have

$$\partial_x B_x = -\partial_y B_y = -(\partial_y b) \cos(kx), \quad (\text{S-13.127})$$

which, after integration in  $dx$ , leads to

$$B_x = -(\partial_y b) \frac{\sin(kx)}{k}, \quad (\text{S-13.128})$$

where we have set to zero the integration constant. In static conditions, and in the absence of electric currents, we have  $\nabla \times \mathbf{B} = 0$ , thus we must also have

$$0 = \partial_x B_y - \partial_y B_x = -kb(y) \sin(kx) + (\partial_y^2 b) \frac{\sin(kx)}{k}, \quad (\text{S-13.129})$$

which, divided by  $\sin(kx)$ , reduces to

$$\partial_y^2 b(y) = k^2 b(y). \quad (\text{S-13.130})$$

The even solution (S-13.130) is

$$b(y) = B_0 \cosh(ky), \quad (\text{S-13.131})$$

where  $B_0$  is a constant to be determined. Thus the two nonzero components of  $\mathbf{B}$  are

$$B_x = -B_0 \sin(kx) \sinh(ky), \quad B_y = B_0 \cos(kx) \cosh(ky), \quad (\text{S-13.132})$$

and on the  $z$  axis, where  $x = 0$  and  $y = 0$ , we have

$$\mathbf{B}(0, 0, z) = \hat{\mathbf{y}} B_0. \quad (\text{S-13.133})$$

**(b)** The Lorentz transformations from the laboratory frame  $S$  to  $S'$  give for the fields in  $S'$

$$B'_x = B_x[x(x', t'), y'] = -B_0 \sinh(ky') \sin[k\gamma(x' + vt')], \quad (\text{S-13.134})$$

$$B'_y = \gamma B_y[x(x', t'), y'] = \gamma B_0 \cosh(ky') \cos[k\gamma(x' + vt')], \quad (\text{S-13.135})$$

$$E'_z = \gamma v B_y[x(x', t'), y'] = \gamma v B_0 \cosh(ky') \cos[k\gamma(x' + vt')]. \quad (\text{S-13.136})$$

where  $\gamma = 1/\sqrt{1 - v^2/c^2}$ . Since the boost is parallel to the  $x$  axis, we have  $y' = y$ .

Disregarding the magnetic force in  $S'$ , the electron oscillates along  $\hat{\mathbf{z}}'$  under the action of the electric field  $E' = E'_z(0, 0, t') = \gamma v B_0 \cos(\omega' t')$ , where  $\omega' = k\gamma v$ . Thus, in  $S'$ , we observe a Thomson scattering, and the electron emits electric-dipole radiation of frequency  $\omega'$ .

**(d)** Transforming back to  $S$ , the frequencies of the radiation emitted in the forward (+) and backward (-) directions are

$$\omega_{\pm} = \gamma(1 \pm \beta)\omega' = \gamma(1 \pm \beta)k\gamma v = kc\gamma^2\beta(1 \pm \beta), \quad (\text{S-13.137})$$

where  $\beta = v/c$ .

In  $S'$ , the electron does not emit radiation along its direction of oscillation, i.e., along  $\hat{\mathbf{z}}'$ . This corresponds to two “forbidden” wavevectors  $\mathbf{k}' \equiv (0, 0, \omega'/c)$  and  $\mathbf{k}' \equiv (0, 0, -\omega'/c)$ . By a back transformation to  $S$  we obtain

$$k_x = \gamma \left( k'_x \pm \frac{\omega'}{c} \beta \right) = \pm \gamma \beta \frac{\omega'}{c}, \quad k_y = 0, \quad k_z = k'_z = \frac{\omega'}{c}, \quad (\text{S-13.138})$$

thus, in  $S$ , we have no radiation emission at the angles  $\pm\theta$  in the  $xz$  plane such that

$$\tan \theta = \frac{k_z}{k_x} = \frac{1}{\gamma \beta}. \quad (\text{S-13.139})$$

The “undulator radiation”, emitted by high-energy electrons injected along a periodically modulated magnetic field, is at the basis of free-electron lasers emitting coherent radiation in the X-ray frequency range.

## S-13.11 Electromagnetic Torque on a Conducting Sphere

(a) We can write the electric field of the wave as

$$\begin{aligned} \mathbf{E}(z, t) &= E_0 [\hat{\mathbf{x}} \cos(kz - \omega t) - \hat{\mathbf{y}} \sin(kz - \omega t)] \\ &= \text{Re}[E_0(\hat{\mathbf{x}} + i\hat{\mathbf{y}}) e^{i(kz - \omega t)}], \end{aligned} \quad (\text{S-13.140})$$

where  $k = \omega/c = 2\pi/\lambda$ . Since  $a \ll \lambda$ , we can consider the electric and magnetic fields of the wave as uniform over the volume of the sphere, and neglect the magnetic induction effects. Thus, the sphere can be considered as located in a uniform rotating electric field

$$\mathbf{E}_0(t) = \text{Re}(\tilde{\mathbf{E}}_0 e^{-i\omega t}), \quad \text{where} \quad \tilde{\mathbf{E}}_0 = E_0(\hat{\mathbf{x}} + i\hat{\mathbf{y}}). \quad (\text{S-13.141})$$

In the presence of oscillating fields, the complex electric permittivity of a medium of real conductivity  $\sigma$  is defined as

$$\tilde{\epsilon}(\omega) = 1 + \frac{4\pi i\sigma}{\omega}. \quad (\text{S-13.142})$$

Thus, our problem is analogous to Problem 3.4, where we considered a dielectric sphere in a uniform external electric field. The internal electric field and the dipole moment of the sphere are

$$\tilde{\mathbf{E}}_{\text{int}} = \frac{3}{\tilde{\varepsilon} + 2} \tilde{\mathbf{E}}_0 = \frac{3\tilde{\mathbf{E}}_0}{3 + 4\pi i\sigma/\omega} = -\frac{3i\omega t_d}{1 - 3i\omega t_d} \tilde{\mathbf{E}}_0, \quad (\text{S-13.143})$$

$$\begin{aligned} \tilde{\mathbf{p}} &= \mathbf{P}V = \chi \tilde{\mathbf{E}}_{\text{int}} V = \frac{3V}{4\pi} \frac{\tilde{\varepsilon} - 1}{\tilde{\varepsilon} + 2} \tilde{\mathbf{E}}_0 = \frac{3V}{4\pi} \frac{4\pi i\sigma/\omega}{3 + 4\pi i\sigma/\omega} \tilde{\mathbf{E}}_0, \\ &= \frac{3V}{4\pi} \frac{4\pi i\sigma}{3\omega + 4\pi i\sigma} \tilde{\mathbf{E}}_0 = \frac{3V}{4\pi} \frac{i/t_d}{3\omega + i/t_d} \tilde{\mathbf{E}}_0 = \frac{3V}{4\pi} \frac{i}{3\omega t_d + i} \tilde{\mathbf{E}}_0 \\ &= \frac{3V}{4\pi} \frac{1 + 3i\omega t_d}{(3\omega t_d)^2 + 1} \tilde{\mathbf{E}}_0 \end{aligned} \quad (\text{S-13.144})$$

where  $V = 4\pi a^3/3$  is the volume of the sphere, and  $t_d = 1/(4\pi\sigma)$ . By writing the complex numerator in terms of its modulus and argument we have

$$1 + 3i\omega t_d = \sqrt{1 + (3\omega t_d)^2} e^{i\phi}, \quad \text{where } \phi = \arctan(3\omega t_d), \quad (\text{S-13.145})$$

and, substituting into (S-13.144) we obtain

$$\tilde{\mathbf{p}} = \frac{3V}{4\pi} \frac{\tilde{\mathbf{E}}_0}{\sqrt{1 + (3\omega t_d)^2}} e^{i\phi}, \quad (\text{S-13.146})$$

and, for the real quantity,

$$\begin{aligned} \mathbf{p} &= \text{Re} \left[ \frac{3V}{4\pi} \frac{E_0}{\sqrt{1 + (3\omega t_d)^2}} (\hat{\mathbf{x}} + i\hat{\mathbf{y}}) e^{-i(\omega t - \phi)} \right] \\ &= \frac{3V}{4\pi} \frac{E_0}{\sqrt{1 + (3\omega t_d)^2}} [\hat{\mathbf{x}} \cos(\omega t - \phi) + \hat{\mathbf{y}} \sin(\omega t - \phi)]. \end{aligned} \quad (\text{S-13.147})$$

Thus the dipole moment of the sphere rotates with a phase delay  $\phi$  relative to the electric field of the wave.

**(b)** The torque acting on an electric dipole  $\mathbf{p}$  in the presence of an electric field  $\mathbf{E}$  is  $\boldsymbol{\tau} = \mathbf{p} \times \mathbf{E}$ . In our case, the angle between  $\mathbf{p}$  and  $\mathbf{E}_0$  is constant in time and equal to  $\phi$ , thus the torque is

$$\boldsymbol{\tau} = \hat{\mathbf{z}} |\mathbf{p}| |\mathbf{E}_0| \sin \phi = \hat{\mathbf{z}} \frac{3V}{4\pi} \frac{E_0^2}{\sqrt{1 + (3\omega t_d)^2}} \sin \phi. \quad (\text{S-13.148})$$

The same result can be obtained by evaluating

$$\begin{aligned} \boldsymbol{\tau} &= \frac{1}{2} \text{Re}(\tilde{\mathbf{p}} \times \tilde{\mathbf{E}}^*) = \text{Re} \left[ \frac{3V}{8\pi} \frac{E_0^2 e^{i\phi}}{\sqrt{1 + (3\omega t_d)^2}} (\hat{\mathbf{x}} + i\hat{\mathbf{y}}) \times (\hat{\mathbf{x}} - i\hat{\mathbf{y}}) \right] \\ &= \text{Re} \left[ \frac{3V}{8\pi} \frac{E_0^2 (\cos \phi + i \sin \phi)}{\sqrt{1 + (3\omega t_d)^2}} (-2i\hat{\mathbf{z}}) \right] = \hat{\mathbf{z}} \frac{3V}{4\pi} \frac{E_0^2}{\sqrt{1 + (3\omega t_d)^2}} \sin \phi. \end{aligned} \quad (\text{S-13.149})$$

## S-13.12 Surface Waves in a Thin Foil

(a) As an educated guess, we search for solutions for the unknown quantities  $E_x$ , and  $B_z$  of the form

$$E_x(x, y, t) = \tilde{E}_x(x) e^{iky - i\omega t}, \quad B_z(x, y, t) = \tilde{B}_z(x) e^{iky - i\omega t}, \quad (\text{S-13.150})$$

where  $\tilde{E}_x(x)$ , and  $\tilde{B}_z(x)$  are complex functions to be determined. According to (13.10),  $E_y$  is symmetric (even) for reflection across the  $x = 0$  plane. Since in vacuum we have  $\nabla \cdot \mathbf{E} = \partial_x E_x + \partial_y E_y = 0$ , we obtain

$$\partial_x \tilde{E}_x = -ik E_0 e^{-q|x|}, \quad (\text{S-13.151})$$

which, after integration in  $dx$ , leads to

$$\tilde{E}_x(x) = \operatorname{sgn}(x) \frac{ik}{q} E_0 e^{-q|x|} = \begin{cases} -\frac{ik}{q} E_0 e^{qx}, & x < 0, \\ \frac{ik}{q} E_0 e^{-qx}, & x > 0. \end{cases} \quad (\text{S-13.152})$$

Thus, the  $E_x$  component is antisymmetric (odd) for reflection across the  $x = 0$  plane. Since our fields are independent of  $z$ , Maxwell's equation  $\nabla \times \mathbf{E} = -\partial_t \mathbf{B}/c$  reduces to

$$-\frac{1}{c} \partial_t B_z = \partial_x E_y - \partial_y E_x = \operatorname{sgn}(x) \left( \frac{k^2}{q} - q \right) E_0 e^{-q|x|} e^{i(ky - \omega t)}, \quad (\text{S-13.153})$$

which, after integration in  $dt$  and division by  $-e^{i(ky - \omega t)}/c$ , leads to

$$\tilde{B}_z = \operatorname{sgn}(x) \frac{ic}{q\omega} (q^2 - k^2) E_0 e^{-q|x|}, \quad (\text{S-13.154})$$

thus  $\tilde{B}_z$ , like  $\tilde{E}_x$ , is an odd function of  $x$ .

We can obtain the surface charge density  $\sigma(y, t)$  and the surface current density  $\mathbf{K}(y, t)$  on the foil from the boundary conditions at the  $x = 0$  plane. Figure S-13.6 shows the surface current  $\mathbf{K}$  and the magnetic field close to the foil.

$$\begin{aligned} \sigma(y, t) &= \frac{1}{4\pi} [E_x(x = 0^+, y, t) - E_x(x = 0^-, y, t)] \\ &= i \frac{2k}{q} E_0 e^{i(ky - \omega t)}, \end{aligned} \quad (\text{S-13.155})$$

$$K_y(y, t) = -\frac{c}{4\pi} [B_z(x = 0^+, y, t) - B_z(x = 0^-, y, t)]$$

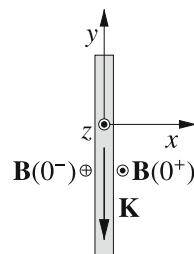


Fig. S-13.6

$$-\mathrm{i} \frac{c^2}{2\pi q\omega} (q^2 - k^2) E_0 e^{\mathrm{i}(ky - \omega t)}, \quad (\text{S-13.156})$$

while the  $z$  component of  $\mathbf{K}$  is zero because its presence would imply a nonzero  $y$  component of  $\mathbf{B}$ .

**(b)** The time-averaged Poynting vector can be written as

$$\langle \mathbf{S} \rangle = \frac{c}{4\pi} \langle \mathbf{E} \times \mathbf{B} \rangle = \frac{c}{8\pi} \left[ \hat{\mathbf{x}} \operatorname{Re}(\tilde{E}_y \tilde{B}_z^*) - \hat{\mathbf{y}} \operatorname{Re}(\tilde{E}_x \tilde{B}_z^*) \right], \quad (\text{S-13.157})$$

where

$$\tilde{E}_y \tilde{B}_z^* = -\operatorname{sgn}(x) \frac{\mathrm{i}c}{q\omega} (q^2 - k^2) |E_0|^2 e^{-2q|x|}, \quad (\text{S-13.158})$$

$$\tilde{E}_x \tilde{B}_z^* = \frac{kc}{q^2\omega} (q^2 - k^2) |E_0|^2 e^{-2q|x|}. \quad (\text{S-13.159})$$

We thus obtain  $\langle S_x \rangle = 0$  because  $\tilde{E}_y \tilde{B}_z^*$ , is purely imaginary, and the energy flow is in the  $\hat{\mathbf{y}}$  direction only:

$$\langle \mathbf{S} \rangle = -\hat{\mathbf{y}} \frac{kc}{8\pi q^2 \omega} (q^2 - k^2) |E_0|^2 e^{-2q|x|}. \quad (\text{S-13.160})$$

**(c)** Form Helmholtz's equation, we obtain

$$q^2 - k^2 + \frac{\omega^2}{c^2} = 0. \quad (\text{S-13.161})$$

**(d)** From (S-13.156) we can write, within our approximations,

$$\mathbf{J} = \frac{\mathbf{K}}{\ell} = -\hat{\mathbf{y}} \mathrm{i} \frac{c^2}{2\pi q\ell\omega} (q^2 - k^2) E_0 e^{\mathrm{i}(ky - \omega t)} \quad (\text{S-13.162})$$

and, combining with (13.11), we obtain

$$\begin{aligned} -\mathrm{i} \frac{c^2}{2\pi q\ell\omega} (q^2 - k^2) E_0 e^{\mathrm{i}(ky - \omega t)} &= 4\pi \mathrm{i} \frac{\omega_p^2}{\omega} E_0 e^{\mathrm{i}(ky - \omega t)}, \\ q^2 - k^2 &= -8\pi^2 \frac{\omega_p^2}{c^2} q\ell. \end{aligned} \quad (\text{S-13.163})$$

where  $\omega_p = \sqrt{4\pi n_e e^2/M_e}$  is the plasma frequency of the foil material. The product  $2n_e \ell$ , appearing in the expression  $\omega_p^2 \ell = 4\pi n_e \ell e^2$ , is the surface number density of the electrons in the foil, which is the relevant parameter in this problem.

(e) By comparing (S-13.161) and (S-13.163) we obtain

$$\omega^2 = 8\pi^2 \omega_p^2 q \ell = \Omega q c, \quad \text{where} \quad \Omega = \frac{8\pi^2 \omega_p^2 \ell}{c}. \quad (\text{S-13.164})$$

Solving (S-13.161) for  $q$  yields

$$cq^2 + \Omega q - k^2 c = 0 \quad \Rightarrow \quad q = \frac{\sqrt{\Omega^2 + (2kc)^2} - \Omega}{2c}, \quad (\text{S-13.165})$$

where the root sign has been chosen so to have  $q > 0$ , as required by the boundary conditions, and in agreement with (13.10). Eventually, we obtain the dispersion relation:

$$\omega^2 = c^2 k^2 - c^2 q^2 = \frac{1}{2} \left[ \Omega \sqrt{\Omega^2 + (2kc)^2} - \Omega^2 - (2kc)^2 \right]. \quad (\text{S-13.166})$$

### S-13.13 The Fizeau Effect

(a) In the rest frame of the medium,  $S'$ , we have  $\omega'/k' = c/n$ . The Lorentz transformations from the laboratory frame  $S$  to  $S'$  lead to

$$\omega' = \gamma(\omega - uk) \approx (\omega - uk), \quad k' = \gamma \left( k - \frac{u\omega}{c^2} \right) \approx \left( k - \frac{u\omega}{c^2} \right), \quad (\text{S-13.167})$$

since  $\gamma \approx 1$  up to the first order in  $\beta = u/c$ . Dividing the two equations side by side we obtain

$$\frac{c}{n} = \frac{\omega'}{k'} \approx \frac{\omega - uk}{k - u\omega/c^2} = \frac{v_\varphi - u}{1 - v_\varphi u/c^2}, \quad (\text{S-13.168})$$

where, in the last step, we have divided numerator and denominator by  $k$ , and substituted the phase velocity in the laboratory frame,  $v_\varphi = \omega/k$ . Multiplying the first and last term by  $1 - v_\varphi u/c^2$  we obtain

$$\begin{aligned} \frac{c}{n} - u \frac{v_\varphi}{cn} &= v_\varphi - u \quad \Rightarrow \quad v_\varphi \left( 1 + \frac{u}{cn} \right) = \frac{c}{n} + u \quad \Rightarrow \quad v_\varphi = \frac{c(c + nu)}{cn + u} \\ &\Rightarrow v_\varphi = c \frac{1 + n\beta}{n + \beta} \approx c \left( \frac{1}{n} + \frac{n^2 - 1}{n^2} \beta \right), \end{aligned} \quad (\text{S-13.169})$$

where, in the last step, we have approximated the fraction by its first-degree Taylor polynomial in  $\beta$ . The phase velocity in the laboratory frame  $S$  is thus

$$v_\varphi = \frac{c}{n} + u \left( 1 - \frac{1}{n^2} \right). \quad (\text{S-13.170})$$

The experiment was performed in 1851, with light propagating in flowing water parallel to the water velocity. Fizeau expected to measure a phase velocity equal to the phase velocity of light in water,  $c/n$ , plus the flow velocity of water,  $u$ , i.e.,  $v_\varphi = (c/n) + u$ , while the experimental result was in agreement with (S-13.170). This found a satisfactory explanation only 54 years later, in 1905, when Einstein published his theory of special relativity.

**(b)** Equation (S-13.170) takes into account the first-order correction to  $v_\varphi$  in  $\beta = u/c$  for a non-dispersive medium. If the medium is dispersive according to a known law  $n = n(\omega)$ , we must also take into account that the frequency  $\omega'$  observed in the rest-frame of the medium is different from the radiation frequency  $\omega$  in the laboratory frame. We want to calculate the first-order correction to (S-13.170) in  $\Delta\omega = \omega' - \omega$ . We need to correct only the first term of the right-hand side of (S-13.170), since the second term is already first-order, and a correction to it would be second-order. The first-order Doppler effect gives us

$$\Delta\omega = \omega' - \omega \simeq -\omega \frac{n(\omega)u}{c}, \quad (\text{S-13.171})$$

since the light velocity in the medium is  $c/n(\omega)$ , and the medium is traveling away from the light source. Thus we have

$$\begin{aligned} \frac{c}{n(\omega')} &\simeq \frac{c}{n(\omega)} + \Delta\omega \partial_\omega \left( \frac{c}{n(\omega)} \right) \\ &= \frac{c}{n(\omega)} + \left( -\omega \frac{n(\omega)u}{c} \right) \left[ -\frac{c}{n^2(\omega)} \partial_\omega n(\omega) \right] \\ &= \frac{c}{n(\omega)} + \omega \frac{u}{n(\omega)} \partial_\omega n(\omega), \end{aligned} \quad (\text{S-13.172})$$

and the first-order expression for the phase velocity in the case of a dispersive medium is

$$v_\varphi(\omega) = \frac{c}{n(\omega)} + u \left[ 1 - \frac{1}{n^2(\omega)} + \frac{\omega}{n(\omega)} \partial_\omega n(\omega) \right] + O(u^2). \quad (\text{S-13.173})$$

**(c)** The refractive index of the free electron medium is  $n(\omega) = (1 - \omega_p^2/\omega^2)^{1/2}$ , where  $\omega_p$  is the plasma frequency. Thus we have inside the square brackets of (S-13.173)

$$1 - \frac{1}{n^2(\omega)} = -\frac{1}{1 - \omega_p^2/\omega^2} = \frac{\omega_p^2}{\omega^2 - \omega_p^2}, \quad (\text{S-13.174})$$

and

$$\frac{\omega}{n(\omega)} \partial_\omega n(\omega) = -\frac{\omega}{(1 - \omega_p^2/\omega^2)^{1/2}} \frac{1}{(1 - \omega_p^2/\omega^2)^{3/2}} \frac{\omega_p^2}{\omega^3} = -\frac{\omega_p^2}{\omega^2 - \omega_p^2}, \quad (\text{S-13.175})$$

so that the two first-order corrections to  $v_\varphi(\omega)$  cancel out, and the phase velocity is independent of the flow velocity of the medium up to the second order in  $\beta$ .

## S-13.14 Lorentz Transformations for Longitudinal Waves

(a) The Lorentz transformations for the wave frequency and wavevector are, in the case of a boost along  $\hat{x}$ ,

$$\omega'_L = \gamma(\omega_L - V k_L), \quad k'_L = \gamma \left( k_L - \frac{V \omega_L}{c^2} \right), \quad (\text{S-13.176})$$

where  $\gamma = 1/\sqrt{1 - V^2/c^2}$ . In the special case where the boost velocity equals the phase velocity,  $V = v_\varphi = \omega_L/k_L$ , we have  $\omega'_L = 0$ , and the fields are independent of time (*static*) in  $S'$ . Further, recalling that  $k_L = \omega_L/v_\varphi$ , we have

$$\begin{aligned} k'_L &= \frac{1}{\sqrt{1 - v_\varphi^2/c^2}} \left( \frac{\omega_L}{v_\varphi} - \frac{v_\varphi \omega_L}{c^2} \right) = \frac{1}{\sqrt{1 - v_\varphi^2/c^2}} \frac{\omega_L}{v_\varphi} \left( 1 - \frac{v_\varphi^2}{c^2} \right) \\ &= \frac{\omega_L}{v_\varphi} \sqrt{1 - v_\varphi^2/c^2} = \frac{k_L}{\gamma}. \end{aligned} \quad (\text{S-13.177})$$

If  $S'$  moves with velocity  $\hat{x}V = \hat{x}v_\varphi$  relative to  $S$ , the fields in  $S'$  are obtained from (9.3) and are

$$\mathbf{E}' = \mathbf{E}'(x') = \hat{x} E_0 e^{ik'_L x'}, \quad \mathbf{B}' = 0, \quad (\text{S-13.178})$$

i.e.,  $\mathbf{E}'$  is constant in time. The charge and current densities in  $S'$  can be obtained either by Lorentz transformations or directly from the equations

$$\varrho' = \frac{1}{4\pi} \mathbf{\nabla}' \cdot \mathbf{E}' = \frac{1}{4\pi} \partial_{x'} E'_x \quad \text{and} \quad 4\pi \mathbf{J}' + \partial'_t \mathbf{E}' = 0, \quad (\text{S-13.179})$$

which lead to

$$\varrho' = \frac{ik'_L}{4\pi} E_0 e^{ik'_L x'}, \quad \mathbf{J}' = 0. \quad (\text{S-13.180})$$

**(b)** The Lorentz transformations for the case  $V = c^2/v_\varphi = c^2 k_L/\omega_L$ , and  $v_\varphi > c$ , lead to the following values for  $k'_L$  and  $\omega'_L$

$$\begin{aligned} k'_L &= \gamma \left( k_L - \frac{V\omega_L}{c^2} \right) = 0, \\ \omega'_L &= \gamma \left( \omega_L - \frac{k_L^2 c^2}{\omega} \right) = \gamma \omega_L \left( 1 - \frac{V^2}{c^2} \right) = \frac{\omega_L}{\gamma}, \end{aligned} \quad (\text{S-13.181})$$

which imply that the fields propagate in space with infinite phase velocity, oscillating with uniform phase at frequency  $\omega'_L$ . The fields are

$$\mathbf{E}' = \mathbf{E}'(t') = \hat{\mathbf{x}} E_0 e^{-i\omega' t'}, \quad \mathbf{B}' = 0, \quad (\text{S-13.182})$$

i.e.,  $\mathbf{E}'$  is uniform in space. We also obtain  $\varrho' = 0$ , and  $\mathbf{J}' = \mathbf{J}/\gamma$ .

**(c)** The Lorentz transformations of the wavevector and the frequency for a boost along the  $y$  axis are

$$\begin{aligned} k'_{Lx} &= k_{Lx} = k_L, \\ k'_{Ly} &= \gamma \left( k_{Ly} - \frac{V\omega_L}{c^2} \right) = -\gamma \frac{V\omega_L}{c^2}, \\ \omega'_L &= \gamma(\omega_L - V k_{Ly}) = \gamma \omega_L. \end{aligned} \quad (\text{S-13.183})$$

All fields and currents depend on space and time through a factor  $e^{i(k'_{Lx}x' + k'_{Ly}y' - \omega'_L t')}$ , thus, the propagation direction forms an angle

$$\theta' = \arctan(k'_{Ly}/k'_{Lx}) = -\arctan(\gamma V v_\varphi / c^2) \quad (\text{S-13.184})$$

with the  $x'$  axis. The wave has field amplitudes

$$E'_x = \gamma \left( E_x + \frac{V}{c} B_z \right) = \gamma E_0, \quad (\text{S-13.185})$$

$$B'_z = \gamma \left( B_z - \frac{V}{c^2} E_x \right) = -\gamma \frac{V}{c^2} E_0, \quad (\text{S-13.186})$$

all other field components being zero. Thus, in a frame moving transversally to the propagation direction, the wave is no longer purely longitudinal and electrostatic.

## S-13.15 Lorentz Transformations for a Transmission Cable

**(a)** The continuity equation for a linear charge density is written  $\partial_t \lambda = -\partial_z I$ . Inserting the expressions for  $\lambda$  and  $I$  of (13.14) we obtain

$$-i\omega\lambda_0 = -ikI_0, \quad \Rightarrow \quad I_0 = \frac{\omega}{k} \lambda_0 = v_\varphi \lambda_0. \quad (\text{S-13.187})$$

(b) The dispersion relation is

$$\omega = v_\varphi k = \frac{c}{n} k = \frac{c}{\sqrt{\epsilon}} k, \quad (\text{S-13.188})$$

with

$$v_\varphi = \frac{c}{\sqrt{\epsilon}} \quad \text{and} \quad k = \frac{\omega}{v_\varphi} = \frac{\omega\sqrt{\epsilon}}{c}. \quad (\text{S-13.189})$$

The electric field can be evaluated by applying Gauss's law to a cylindrical surface coaxial to the wire, of radius  $r$  and height  $h$ . Since the field is transverse, and we have cylindrical symmetry around the wire, the only nonzero component of  $\mathbf{E}$  is  $E_r$

$$\mathbf{E}(r, z, t) = \hat{\mathbf{r}} \frac{2\lambda}{\epsilon r} = \hat{\mathbf{r}} E_r(r) e^{ikz-i\omega t}, \quad \text{where} \quad E_r(r) = \frac{2\lambda_0}{\epsilon r}. \quad (\text{S-13.190})$$

The magnetic field can be evaluated by applying Stokes' theorem to a circle of radius  $r$ , coaxial to the wire. Because of symmetry, the only nonzero component of  $\mathbf{B}$  is the azimuthal component  $B_\phi$

$$\mathbf{B}(r, z, t) = \hat{\phi} \frac{2I}{rc} = \hat{\phi} B_\phi(r) e^{ikz-i\omega t}, \quad \text{where} \quad B_\phi(r) = \frac{2I_0}{rc} = \frac{2\omega\lambda_0}{krc}, \quad (\text{S-13.191})$$

that can be rewritten as

$$B_\phi(r) = \frac{\epsilon\omega}{kc} E_r(r) = \frac{\epsilon v_\varphi}{c} E_r(r) = \frac{c}{v_\varphi} E_r(r). \quad (\text{S-13.192})$$

(c) The wave frequency  $\omega'$  in the frame  $S'$ , moving at the phase velocity  $\hat{\mathbf{z}} v_\varphi$  relative to the laboratory frame  $S$ , is

$$\omega' = \gamma(\omega - v_\varphi k) = 0, \quad (\text{S-13.193})$$

where we have used the second of (S-13.189). Thus the fields are static in  $S'$ . For our Lorentz boost we have

$$\beta = \hat{\mathbf{z}} \frac{v_\varphi}{c} = \frac{\hat{\mathbf{z}}}{\sqrt{\epsilon}}, \quad \gamma = \frac{1}{\sqrt{1-1/\epsilon}} = \sqrt{\frac{\epsilon}{\epsilon-1}}, \quad (\text{S-13.194})$$

and the wave vector  $k'$  in  $S'$  can be written

$$k' = \gamma \left( k - \frac{v_\varphi \omega}{c^2} \right) = \sqrt{\frac{\epsilon}{\epsilon-1}} \left( \frac{\omega\sqrt{\epsilon}}{c} - \frac{\omega}{c\sqrt{\epsilon}} \right) = \frac{\omega}{c} \sqrt{\epsilon-1}. \quad (\text{S-13.195})$$

The  $(z', t')$ -dependence (actually, only  $z'$ -dependence) of our physical quantities in  $S'$  will thus be through a factor  $e^{ik'z'}$ . The amplitude of linear charge density in  $S'$  is

$$\lambda'_0 = \gamma \left( \lambda_0 - \frac{v_\varphi}{c^2} I_0 \right) = \gamma \left( \lambda_0 - \frac{v_\varphi^2}{c^2} \lambda_0 \right) = \gamma \left( \frac{1}{\gamma} \right)^2 \lambda_0 = \frac{\lambda_0}{\gamma}. \quad (\text{S-13.196})$$

The amplitude of the current in  $S'$  is

$$I'_0 = \gamma \left( I_0 - v_\varphi \lambda_0 \right) = 0. \quad (\text{S-13.197})$$

The field amplitudes transform according to (9.3), thus we have

$$E'_r = \gamma \left( E_r - \beta B_\phi \right) = \gamma \left( E_r - \frac{v_\varphi}{c} \frac{c}{v_\varphi} E_r \right) = 0, \quad (\text{S-13.198})$$

$$B'_\phi = \gamma \left( B_\phi - \frac{v_\varphi^2}{c^2} E_r \right) = \gamma \left( B_\phi - \frac{v_\varphi}{c} \frac{v_\varphi}{c} B_\phi \right) = \frac{B_\phi}{\gamma}. \quad (\text{S-13.199})$$

It might seem surprising that, in  $S'$ , we have  $\lambda' \neq 0$  and  $E'_r = 0$ , while  $I' = 0$  and  $B'_\phi \neq 0$ . The reason is that we must take into account also the polarization charge of the medium in contact with the wire,  $\lambda_p(z, t)$ , the presence of a polarization current,  $\mathbf{J}_p(r, z, t)$ , and their Lorentz transformations. In the laboratory frame  $S$  we must have  $\lambda(z, t) + \lambda_p(z, t) = \lambda(z, t)/\epsilon$ , thus

$$\lambda_p(z, t) = -\frac{\epsilon - 1}{\epsilon} \lambda_0 e^{ikz - i\omega t} = -\frac{\lambda_0}{\gamma^2} e^{ikz - i\omega t} = \lambda_0^{(p)} e^{ikz - i\omega t}, \quad (\text{S-13.200})$$

where  $\lambda_0^{(p)} = -\lambda_0/\gamma^2$ . The electric field (S-13.190) generates a polarization of the medium

$$\mathbf{P}(z, r, t) = \hat{\mathbf{r}} \frac{\epsilon - 1}{4\pi} E_r(r) e^{ikz - i\omega t} = \hat{\mathbf{r}} P_r(r) e^{ikz - i\omega t}, \quad (\text{S-13.201})$$

where

$$P_r(r) = \frac{\epsilon - 1}{4\pi} \frac{2\lambda_0}{\epsilon r} = \frac{\epsilon - 1}{\epsilon} \frac{\lambda_0}{2\pi r} = \frac{1}{\gamma^2} \frac{\lambda_0}{2\pi r}. \quad (\text{S-13.202})$$

A time-dependent polarization is associated to a polarization current density

$$\mathbf{J}_p = \partial_t \mathbf{P} = -\hat{\mathbf{r}} i\omega P_r(r) e^{ikz - i\omega t} = \hat{\mathbf{r}} J_r(r) e^{ikz - i\omega t} \quad (\text{S-13.203})$$

where

$$J_r(r) = -i\omega P_r(r) = -i \frac{\omega}{\gamma^2} \frac{\lambda_0}{2\pi r}. \quad (\text{S-13.204})$$

Thus,  $\mathbf{J}_p$  is radial in  $S$ . According to the first of (9.1), we have a polarization four-current

$$J_\mu^{(p)}(r, z, t) = \left[ c\varrho_0^{(p)}(r), \hat{\mathbf{f}} J_r(r) \right] e^{ikz-i\omega t}, \quad (\text{S-13.205})$$

where, for instance

$$\varrho_0^{(p)}(r) = \begin{cases} \frac{\lambda_0^{(p)}}{\pi r_0^2}, & \text{if } r < r_0 \\ 0, & \text{if } r > r_0 \end{cases}, \quad \text{so that} \quad \lambda_0^{(p)} = \int_0^\infty \varrho_0^{(p)}(r) 2\pi r dr, \quad (\text{S-13.206})$$

and we are interested in the limit  $r_0 \rightarrow 0$ . We can thus write

$$J_\mu^{(p)}(r, z, t) = G_\mu(r) e^{ikz-i\omega t}, \quad \text{where} \quad G_\mu = \left[ c\varrho_0^{(p)}, \hat{\mathbf{f}} J_r(r) \right]. \quad (\text{S-13.207})$$

The four-vector  $G_\mu$  transforms according to (9.2), thus we have in  $S'$

$$G'_0 = \gamma(G_0 - \beta \cdot \mathbf{G}) = \gamma G_0, \quad (\text{S-13.208})$$

since the spacelike component of  $G_\mu$ , being radial, is perpendicular to  $\beta$ . The amplitude of the linear polarization charge density in  $S'$  is

$$\lambda_0^{(p)\prime} = \gamma \int_0^\infty \frac{G_0}{c} 2\pi r dr = \gamma \lambda_0^{(p)} = -\gamma \frac{\lambda_0}{\gamma^2} = -\frac{\lambda_0}{\gamma}, \quad (\text{S-13.209})$$

which cancels (S-13.196), therefore we have  $\mathbf{E}' = 0$ . The radial component of  $\mathbf{J}_p$  does not contribute to the magnetic field, thus we are interested in

$$G'_{||} = \gamma(G_{||} - \beta G_0) = -\gamma\beta G_0 = -\gamma\beta\varrho_0^{(p)}(r)c, \quad (\text{S-13.210})$$

which corresponds to a polarization current in  $S'$  of amplitude

$$I_0^{(p)\prime} = \int_\infty G'_{||} 2\pi r' dr' = -\gamma v_\varphi \lambda_0^{(p)} = \gamma v_\varphi \frac{\lambda_0}{\gamma^2} = \frac{I_0}{\gamma} \quad (\text{S-13.211})$$

in agreement with (S-13.199).

## S-13.16 A Waveguide with a Moving End

(a) The electric field of the TE<sub>10</sub> must be parallel to the two conducting planes, thus it must vanish on them, and be of the form  $\mathbf{E}(x, y, t) = \hat{\mathbf{z}} E_0 \cos(\pi y/a) f(x, t)$ . The dispersion relation is

$$\omega^2 = \omega_{\text{co}}^2 + k^2 c^2, \quad \text{where} \quad \omega_{\text{co}} = \frac{\pi c}{a} \quad (\text{S-13.212})$$

is the cutoff frequency of the waveguide. In our terminated waveguide, the global electric field is the superposition of the fields of the wave incident on the terminating wall at  $x = 0$ , and of the reflected wave. Incident and reflected wave have equal amplitudes, thus

$$\mathbf{E}(x, y, t) = \hat{\mathbf{z}} E_0 \cos\left(\frac{\pi y}{a}\right) \sin(kx) e^{-i\omega t}, \quad (\text{S-13.213})$$

where the phase has been chosen so that  $\mathbf{E}(0, y, t) = 0$ . The magnetic field can be obtained from the relation  $\partial_t \mathbf{B} = -c \nabla \times \mathbf{E}$ , and has the components

$$B_x = -\frac{ic}{\omega} \partial_y E_z = \frac{i\pi c}{\omega a} E_0 \sin\left(\frac{\pi y}{a}\right) \sin(kx) e^{-i\omega t}, \quad (\text{S-13.214})$$

$$B_y = \frac{ic}{\omega} \partial_x E_z = -\frac{kc}{\omega} E_0 \cos\left(\frac{\pi y}{a}\right) \cos(kx) e^{-i\omega t}. \quad (\text{S-13.215})$$

Notice that  $B_x(0, y, t) = 0$ , as required.

**(b)** In the frame  $S'$  where the waveguide termination is at rest ( $v' = 0$ ), the incident wave has frequency and wavevector

$$\omega'_i = \gamma(\omega - \beta kc), \quad k'_i = \gamma(k - \beta \omega/c), \quad (\text{S-13.216})$$

where  $\beta = v/c$ . Since we assumed  $\beta < kc/\omega$ , we have  $k'_i > 0$  (notice that  $\omega'_i > 0$  anyway because  $k < \omega/c$ ). In  $S'$  the reflected wave has frequency and wavevector

$$\omega'_r = \omega'_i, \quad k'_r = -k'_i. \quad (\text{S-13.217})$$

By transforming back into the laboratory frame  $S$  we obtain

$$\omega_r = \gamma(\omega'_r + \beta k'_r c) = \gamma^2 \left[ (1 + \beta^2) \omega - 2\beta kc \right], \quad (\text{S-13.218})$$

$$k_r = \gamma \left( -k + \beta \frac{\omega}{c} \right) = \gamma^2 \left[ -(1 + \beta^2) k + 2\beta \frac{\omega}{c} \right], \quad (\text{S-13.219})$$

As a check, at the limit  $a \rightarrow \infty$  we have  $\omega_{\text{co}} \rightarrow 0$  and  $k \rightarrow \omega/c$ , and we obtain (S-9.54) of Problem 9.6 for the frequency reflected by a moving mirror. With some algebraic manipulations we obtain

$$\begin{aligned}
\omega_r^2 - c^2 k_r^2 &= \gamma^4 \left\{ \left[ (1 + \beta^2) \omega - 2\beta k c \right]^2 - c^2 \left[ -(1 + \beta^2) k + 2\beta \frac{\omega}{c} \right]^2 \right\} \\
&= \gamma^4 \left[ (1 + \beta^2) \omega - 2\beta k c + (1 + \beta^2) k c - 2\beta \omega \right] \times \\
&\quad \times \left[ (1 + \beta^2) \omega - 2\beta k c - (1 + \beta^2) k c + 2\beta \omega \right] \\
&= \gamma^2 \left[ (1 - \beta^2) \omega + (1 - \beta^2) k c \right] \left[ (1 + \beta)^2 \omega + (1 + \beta)^2 k c \right] \\
&= (\omega + k c)(\omega - k c) = \omega^2 - k^2 c^2. \tag{S-13.220}
\end{aligned}$$

(c) If  $v > kc^2/\omega$ , in  $S'$  we have  $k'_i < 0$ , the incident wave propagates parallel to  $-\hat{x}'$ , and cannot reach the waveguide termination. In these conditions there is no reflected wave. The condition is equivalent to  $v > v_g$ , the group velocity in the waveguide.

### S-13.17 A “Relativistically” Strong Electromagnetic Wave

(a) The equations of motion for  $p_x$ ,  $p_y$ , and  $p_z$  in the presence of the electromagnetic fields of the wave are

$$\frac{dp_x}{dt} = -eE_x + \frac{e}{c} v_z B_y, \tag{S-13.221}$$

$$\frac{dp_y}{dt} = -eE_y - \frac{e}{c} v_z B_x, \tag{S-13.222}$$

$$\frac{dp_z}{dt} = -\frac{e}{c} v_x B_y + \frac{e}{c} v_y B_z. \tag{S-13.223}$$

In general the magnetic contribution is not negligible, since  $v_z$  is not necessarily much smaller than  $c$ . However, if we assume  $v_z = 0$ , the magnetic force vanishes. In these conditions the solutions of (S-13.221-S-13.222) are

$$\frac{dp_x}{dt} = -eE_0 \cos \omega t, \quad \frac{dp_y}{dt} = -eE_0 \sin \omega t, \tag{S-13.224}$$

$$p_x = -\frac{eE_0}{\omega} \sin \omega t, \quad p_y = +\frac{eE_0}{\omega} \cos \omega t. \tag{S-13.225}$$

Inserting these solutions into (S-13.223) we have

$$\begin{aligned}
\frac{dp_z}{dt} &= \frac{e}{m_e \gamma} (-p_x B_y + p_y B_x) = \\
&= \frac{e}{m_e \gamma} \frac{e E_0^2}{\omega c} (-\sin \omega t \cos \omega t + \cos \omega t \sin \omega t) = 0, \tag{S-13.226}
\end{aligned}$$

so that a  $p_z$  is constant in time. Either assuming  $v_z = 0$  as initial condition or by a proper change of reference frame,  $v_z = 0$  is a self-consistent assumption.

**(b)** Since  $\mathbf{p}^2 = p_x^2 + p_y^2 = (eE_0/\omega)^2$  does not depend on time, the Lorentz factor  $\gamma = \sqrt{1 + p^2/(m_e c)^2} = \sqrt{1 + [eE_0/(m_e \omega c)]^2}$  is a constant. This implies

$$\frac{d\mathbf{p}}{dt} = m_e \frac{d(\gamma v)}{dt} = \gamma m_e \frac{dv}{dt}. \quad (\text{S-13.227})$$

The equations of motion have the same form as in the non-relativistic case if we make the replacement  $m_e \rightarrow \gamma m_e$ . The relativistic behavior can be obtained by attributing an “effective mass”  $\gamma m_e$ , dependent on the wave intensity, to the electron.

**(c)** Accordingly, the refractive index for the relativistic case can be simply obtained by replacing  $m_e \rightarrow m_e \gamma$  into the non-relativistic expression, so that  $\omega_p = \sqrt{(4\pi n_e e^2/m_e)} \rightarrow \omega_p / \sqrt{\gamma}$ . We thus obtain

$$n^2(\omega) = 1 - \frac{4\pi n_e e^2}{m_e \gamma \omega^2} = 1 - \frac{\omega_p^2}{\gamma \omega^2}. \quad (\text{S-13.228})$$

**(d)** The dispersion relation corresponding to  $n^2(\omega)$  in (S-13.228) is

$$\omega^2 = k^2 c^2 + \frac{\omega_p^2}{\gamma}. \quad (\text{S-13.229})$$

The cutoff frequency is  $\omega_{co} = \omega_p / \sqrt{\gamma}$  and depends on the wave amplitude. Since  $\gamma > 1$ , a plasma can be opaque to a low-intensity wave for which  $\omega_p > \omega$ , but transparent to a high-intensity wave of the same frequency if  $\gamma > \omega_p / \omega$ .

It should be stressed, however, that the concept of a refractive index dependent on the wave intensity deserves some care. What we have discussed above is just a special case of “relativistically induced transparency”, applying to a plane, monochromatic, infinite wave. In the case of a real light beam, of finite duration and extension, different parts of the beam can have different amplitudes, and thus can have different phase velocities, resulting in a complicated nonlinear dispersion.<sup>1</sup> However, (S-13.229) can be of help to a *qualitative* discussion of some important nonlinear effects observed for a relativistically strong wave. An important example is the propagation of a strong beam of finite width, for which the effective refractive index is higher at the boundaries (where the intensity is lower and  $\gamma$  is smaller) than on the beam axis. This can compensate diffraction, analogously to what occurs in an optical fiber (see Problem 12.6), and can cause *self-focusing*.

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<sup>1</sup>In some cases, nonlinearity effects can compensate dispersion for particular wavepacket shapes, these special solutions can propagate without changing their envelope shape, and are known as *solitons*.

## S-13.18 Electric Current in a Solenoid

(a) This problem originated from the question: “can the electric field in a solenoid have circular field lines, as seems to be required for driving the current in each turn of the coil?” The answer is obviously *no* in static conditions, since  $\nabla \times \mathbf{E}$  must be zero. But a *uniform* field is sufficient to drive the current, since the coil of a real solenoid does not consist of single circular circular loops perpendicular to the axis (each loop would require its own current source, in this case!) The winding of a real solenoid is actually a helix, of small, but nonzero pitch. The current is driven by the component of  $\mathbf{E}$  *parallel* to the wire, equal to (assuming  $\mathbf{E} = E\hat{\mathbf{z}}$ )

$$E_{||} = E \sin \theta \simeq E \frac{a}{\pi b}. \quad (\text{S-13.230})$$

The perpendicular component  $E_{\perp}$  is compensated by the electrostatic fields generated by the surface charge distribution of the wire, analogously to Problem 3.11. Thus, the current density and intensity in the wire are

$$J = \sigma E_{||} \simeq \sigma E \frac{a}{\pi b}, \quad I = J \pi a^2 \simeq \frac{\sigma a^3}{b} E. \quad (\text{S-13.231})$$

Neglecting boundary effects, the current generates a uniform field  $\mathbf{B}^{(\text{int})} = \hat{\mathbf{z}} B_z$ , with

$$B_z = \frac{4\pi nI}{c} = \frac{4\pi I}{2ac} \simeq \frac{2\pi a^2 \sigma}{bc} E, \quad (\text{S-13.232})$$

inside the solenoid, since  $n = 1/(2a)$  is the number of turns per unit length. The field outside the solenoid,  $\mathbf{B}^{(\text{ext})}$ , is generated by the total current  $I$  flowing parallel to  $\hat{\mathbf{z}}$ . Thus in the external central region  $b < r \ll h$ ,  $|z| \ll h$ , the field is azimuthal,  $\mathbf{B}^{(\text{ext})} = \hat{\phi} B_{\phi}$ , with

$$B_{\phi} \simeq \frac{2I}{cr} = \frac{2\pi J a^2}{cr} \simeq \frac{2\sigma a^3}{bcr} E, \quad b < r \ll h, \quad |z| \ll h, \quad (\text{S-13.233})$$

where the  $z$  origin is located at the center of the solenoid.

(b) In the external central region  $b < r \ll h$ ,  $|z| \ll h$ , the fields  $E_z$  and  $B_{\phi}$  are associated to a Poynting vector

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} = -\hat{\mathbf{r}} \frac{c}{4\pi} E_z B_{\phi} = -\hat{\mathbf{r}} \frac{\sigma a^3}{2\pi br} E^2, \quad (\text{S-13.234})$$

with an entering flux through the lateral surface of a coaxial cylinder of length  $\ell$

$$\Phi_{\text{in}} = 2\pi r \ell \frac{\sigma a^3}{2\pi br} E^2 = \frac{\sigma a^3 \ell}{b} E^2. \quad (\text{S-13.235})$$

The power dissipated by Joule heating in a solenoid portion of length  $\ell$  is obtained by multiplying the power dissipated in single turn

$$W_{\text{turn}} = I^2 R = \left( \frac{\sigma a^3}{b} E \right)^2 \frac{2\pi b}{\pi a^2 \sigma} = \frac{2\sigma a^4}{b} E^2, \quad (\text{S-13.236})$$

by the number of turns  $\ell/(2a)$

$$W(\ell) = \frac{2\sigma a^4}{b} E^2 \frac{\ell}{2a} = \frac{\sigma a^3 \ell}{b} E^2, \quad (\text{S-13.237})$$

in agreement with Poynting's theorem.

### S-13.19 An Optomechanical Cavity

(a) In the following we omit the vector notation for the electric fields, since the results are independent of the polarization. The general expression for the electric field of a monochromatic plane wave propagating along  $x$  is, in complex notation,

$$E(x, t) = E(x) e^{-i\omega t} = (E_1 e^{+ikx} + E_2 e^{-ikx}) e^{-i\omega t}, \quad (\text{S-13.238})$$

where  $k = \omega/c$ . The boundary conditions at the two perfectly conducting walls are  $E(\pm d/2) = 0$ , thus we must have

$$E_1 e^{+ikd/2} + E_2 e^{-ikd/2} = 0, \quad E_1 e^{-ikd/2} + E_2 e^{+ikd/2} = 0. \quad (\text{S-13.239})$$

This system of two equations has nontrivial solutions for  $E_1$  and  $E_2$  only if the determinant is zero,

$$e^{ikd} - e^{-ikd} = 2i \sin(kd) = 0, \quad (\text{S-13.240})$$

from which we obtain

$$kd = n\pi \quad (n = 1, 2, 3, \dots) \quad \omega = kc = n \frac{\pi c}{d}, \quad (\text{S-13.241})$$

$$E_2 = -E_1 e^{in\pi} = (-1)^{n+1} E_1. \quad (\text{S-13.242})$$

Thus, the electric field of the  $n$ -th mode is

$$E_n(x) = \frac{E_0}{2} \left[ e^{in\pi x/d} + (-1)^{n+1} e^{-in\pi x/d} \right]. \quad (\text{S-13.243})$$

The magnetic field can be obtained from  $\partial_t \mathbf{B} = -\nabla \times \mathbf{E}$ :

$$B_n(x) = \frac{E_0}{2} \left[ e^{in\pi x/d} - (-1)^{n+1} e^{-in\pi x/d} \right]. \quad (\text{S-13.244})$$

**(b)** The field is thus the superposition of two plane monochromatic waves of equal frequency  $\omega$  and amplitude  $E_0/2$ , propagating in opposite directions. The radiation pressure on each reflecting wall is thus the pressure exerted by a normally incident wave of intensity  $I = c|E_0/2|^2/8\pi$ , evaluated in Problem 8.5,

$$P = \frac{2I}{c} = \frac{|E_0|^2}{16\pi}. \quad (\text{S-13.245})$$

**(c)** The energy per unit surface inside the cavity is independent of time and can be evaluated as

$$U = \int_{-d/2}^{+d/2} \frac{1}{8\pi} (|\mathbf{E}|^2 + |\mathbf{B}|^2) dx. \quad (\text{S-13.246})$$

We have

$$|\mathbf{E}|^2 = \frac{|E_0|^2}{2} \left[ 1 + (-1)^{n+1} \cos\left(\frac{2n\pi x}{d}\right) \right], \quad (\text{S-13.247})$$

$$|\mathbf{B}|^2 = \frac{|E_0|^2}{2} \left[ 1 - (-1)^{n+1} \cos\left(\frac{2n\pi x}{d}\right) \right]. \quad (\text{S-13.248})$$

Integrating over  $x$ , the oscillating terms of both expressions average to zero, and we finally have

$$U = \frac{|E_0|^2 d}{16\pi} \pi = Pd. \quad (\text{S-13.249})$$

**(d)** At mechanical equilibrium, the force due to the radiation pressure on the walls must balance the recoil force of the springs. Assuming that each wall is displaced by  $\delta$  from its equilibrium position in the absence of fields, we have

$$PS = K\delta = M\Omega^2\delta, \quad (\text{S-13.250})$$

where  $\Omega = \sqrt{K/M}$  is the free oscillation frequency of the walls. Thus

$$\frac{M\Omega^2\delta}{S} = P = \frac{|E_0|^2}{16\pi}, \quad (\text{S-13.251})$$

from which we obtain  $\delta = \alpha |E_0|^2$  where

$$\alpha = \frac{S}{16\pi M\Omega^2} = \frac{S}{16\pi K}. \quad (\text{S-13.252})$$

The length of the cavity is now  $d + 2\delta$ , and the resonance condition is

$$d + 2\delta = n \frac{\lambda_n}{2} = n \frac{\pi c}{\omega_n}, \quad (n = 1, 2, \dots), \quad (\text{S-13.253})$$

where the mode frequencies are

$$\omega_n = \frac{2\pi c}{\lambda_n} = \frac{n\pi c}{d + 2\alpha|E_0|^2}. \quad (\text{S-13.254})$$

This is a simple classical example of a resonant cavity where the frequency and amplitude of the wave depend on each other (and on the cavity length), the link being due to radiation pressure effects; this is called an *optomechanical* cavity [2].

## S-13.20 Radiation Pressure on an Absorbing Medium

We assume the incident wave to be linearly polarized parallel to  $\hat{\mathbf{y}}$  for definiteness (the generalization to a different polarization is straightforward). The electric field of the wave is thus  $\mathbf{E}(x, t) = \hat{\mathbf{y}} E_y(x, t)$ , with

$$E_y(x, t) = \begin{cases} \operatorname{Re} \left( E_i e^{ikx - i\omega t} + E_r e^{-ikx - i\omega t} \right), & (x < 0), \\ \operatorname{Re} \left( E_t e^{iknx - i\omega t} \right), & (x > 0), \end{cases} \quad (\text{S-13.255})$$

where  $E_i = \sqrt{8\pi I_i/c}$ , and

$$E_r = \frac{1-n}{1+n} E_i, \quad E_t = \frac{2}{1+n} E_i \quad (\text{S-13.256})$$

(Fresnel formulas at normal incidence). The magnetic field of the wave can be obtained from  $\partial_t \mathbf{B} = -\nabla \times \mathbf{E}$ , we have  $\mathbf{B}(x, t) = \hat{\mathbf{z}} B_z(x, t)$ , with

$$B_z(x, t) = \begin{cases} \operatorname{Re} \left( E_i e^{ikx - i\omega t} - E_r e^{-ikx - i\omega t} \right), & (x < 0), \\ \operatorname{Re} \left( n E_t e^{iknx - i\omega t} \right), & (x > 0). \end{cases} \quad (\text{S-13.257})$$

The field for  $x > 0$  is exponentially decaying, since

$$E_t e^{ik(n_1 + in_2)x - i\omega t} = E_t e^{ikn_1 - i\omega t} e^{-kn_2 x}, \quad (\text{S-13.258})$$

the decay length being  $(kn_2)^{-1} = \lambda/(2\pi n_2) \gg \lambda/n_1$ , where  $\lambda = 2\pi c/\omega$  is the wavelength in vacuum.

The cycle-averaged value of the Poynting vector at the  $x = 0$  plane gives the flux of electromagnetic energy entering the medium. Since the field decays with increasing  $x$ , there is no net flux of energy for  $x \rightarrow \infty$ , and all the energy entering

the medium is eventually absorbed. Using (S-13.255) and (S-13.257) we find

$$\begin{aligned}\langle S_x(0^+) \rangle &= \left\langle \frac{c}{4\pi} E_y(0^+, t) B_z(0^+, t) \right\rangle = \frac{1}{2} \frac{c}{4\pi} \operatorname{Re}(E_t n^* E_t^*) = \frac{c}{8\pi} |E_t|^2 \operatorname{Re}(n^*) \\ &= \frac{c}{8\pi} n_1 |E_t|^2 = \frac{c}{8\pi} n_1 \frac{4}{|1+n|^2} |E_i|^2 = \frac{4n_1}{|1+n|^2} I_i \equiv A I_i.\end{aligned}\quad (\text{S-13.259})$$

The reflection coefficient  $R = |E_r/E_i|^2 = |1-n|^2/|1+n|^2$ . Thus

$$1-R = 1 - \left| \frac{1-n}{1+n} \right|^2 = \frac{2(n+n^*)}{|1+n|^2} = \frac{4\operatorname{Re}(n)}{|1+n|^2} = \frac{4n_1}{|1+n|^2} = A.\quad (\text{S-13.260})$$

**(b)** The pressure on the medium is the flow of electromagnetic momentum through the  $x = 0$  surface. Such flow is given, in the present conditions, by  $P_{\text{rad}} = -\langle T_{xx}(x=0) \rangle$  where  $T_{ij}$  is Maxwell stress tensor (see Problem 8.5). Since

$$T_{xx}(0, t) = T_{11}(0, t) = -\frac{1}{8\pi} (\mathbf{E}^2(0, t) + \mathbf{B}^2(0, t))\quad (\text{S-13.261})$$

we obtain

$$\begin{aligned}\langle T_{11}(0, t) \rangle &= -\frac{1}{16\pi} |E_t|^2 (1+|n|^2) = -\frac{1}{4\pi} |E_i|^2 \frac{1+|n|^2}{|1+n|^2} = \frac{1}{4\pi} |E_i|^2 \frac{|1+n|^2 - 2\operatorname{Re}(n)}{|1+n|^2} \\ &= -\frac{1}{8\pi} |E_i|^2 \left( 2 - \frac{4n_1}{|1+n|^2} \right) = -\frac{I_i}{c} (1+R) \equiv -P_{\text{rad}}.\end{aligned}\quad (\text{S-13.262})$$

The same result can also be obtained by calculating the total average force per unit surface exerted on the medium by the electromagnetic field

$$P_{\text{EM}} = \int_0^{+\infty} \langle (\mathbf{J} \times \mathbf{B})_x \rangle dx,\quad (\text{S-13.263})$$

since the electric term gives no contribution. The current density  $\mathbf{J}$  inside the medium can be obtained from the equation  $\mathbf{J} = (c\mathbf{v} \times \mathbf{B} - \partial_t \mathbf{E})/4\pi$ , obtaining

$$\begin{aligned}J_y &= \operatorname{Re} \left( -\frac{ikcn}{4\pi} \frac{n}{c} E_t e^{-iknx-i\omega t} + \frac{i\omega}{4\pi} E_t e^{-iknx-i\omega t} \right) \\ &= \operatorname{Re} \left( \frac{i\omega}{4\pi} (1-n^2) E_t e^{-iknx-i\omega t} \right).\end{aligned}\quad (\text{S-13.264})$$

A further way to obtain this result is recalling the relation between conductivity and dielectric permittivity for complex fields, i.e.,

$$\sigma(\omega) = -\frac{i\omega}{4\pi} [\epsilon_r(\omega) - 1] = -\frac{i\omega}{4\pi} [n^2(\omega) - 1].\quad (\text{S-13.265})$$

We thus have

$$\begin{aligned}\langle J_y B_z \rangle &= \frac{1}{2} \frac{\omega}{4\pi} \operatorname{Re} \left[ i(1-n^2) E_t e^{(-ikn_1 - kn_2)x} \right] \left[ n^* E_t^* e^{(+ikn_1 - kn_2)x} \right] \\ &= \frac{1}{8\pi} |E_t|^2 \operatorname{Re} \left[ i(1-n^2) n^* \right] e^{-2kn_2 x}.\end{aligned}\quad (\text{S-13.266})$$

Now

$$\begin{aligned}\operatorname{Re} \left[ i(1-n^2) n^* \right] &= \operatorname{Re} \left[ (1-n_1^2 + n_2^2 - 2in_1 n_2)(in_1 + n_2) \right] = n_2 (1+n_1^2 + n_2^2) \\ &= n_2 (1+|n|^2),\end{aligned}\quad (\text{S-13.267})$$

thus, by substituting in (S-13.263) and comparing to (S-13.262) we obtain

$$\begin{aligned}P_{\text{EM}} &= \int_0^\infty \langle J_y B_z \rangle dx = \frac{1}{2} \frac{1}{4\pi} |E_t|^2 n_2 (1+|n|^2) \int_0^\infty e^{-2kn_2 x} dx \\ &= \frac{1}{2\pi} |E_t|^2 \frac{1+|n|^2}{1-|n|^2} n_2 \frac{1}{2n_2} = P_{\text{rad}}.\end{aligned}\quad (\text{S-13.268})$$

### S-13.21 Scattering from a Perfectly Conducting Sphere

(a) Since the radius of the sphere,  $a$ , is much smaller than the radiation wavelength,  $\lambda$ , we can consider the electric field of the incident wave as uniform over the whole volume of the sphere. As shown in Problems 1.1 and 2.1, the “electron sea” is displaced by an amount  $\delta$  with respect to the ion lattice in order to keep the total electric field equal to zero inside the sphere. According to (S-2.2) we have

$$\delta = -\hat{\mathbf{y}} \frac{3}{4\pi\varrho_0} E_0 \cos(\omega t), \quad (\text{S-13.269})$$

where  $\varrho_0$  is the volume charge density of the ion lattice. This corresponds to a volume polarization  $\mathbf{P}$

$$\mathbf{P} = -\rho\delta = \hat{\mathbf{y}} \frac{3}{4\pi} E_0 \cos(\omega t), \quad (\text{S-13.270})$$

and to a total dipole moment of the conducting sphere

$$\mathbf{p} = \frac{4\pi}{3} a^3 \mathbf{P} = \mathbf{E}_0 a^3 = \hat{\mathbf{y}} a^3 E_0 \cos(\omega t). \quad (\text{S-13.271})$$

The scattered, time-averaged power is thus

$$W_{\text{scatt}}^{(\text{el})} = \frac{1}{3c^3} |\vec{\mathbf{p}}|^2 = \frac{\omega^4 a^6}{3c^3} E_0^2. \quad (\text{S-13.272})$$

The intensity of the incident wave is  $I = (c/8\pi)E_0^2$ , so we obtain for the scattering cross section

$$\sigma_{\text{scatt}}^{(\text{el})} = \frac{W_{\text{scatt}}^{(\text{el})}}{I} = \frac{8\pi}{3} \frac{\omega^4 a^6}{c^4} = 128\pi^4 (\pi a^2) \left(\frac{a}{\lambda}\right)^4. \quad (\text{S-13.273})$$

**(b)** Due to the condition  $a \ll \lambda$ , also the magnetic field of the wave can be considered as uniform inside the sphere

$$\mathbf{B}(t) = \hat{\mathbf{z}} B_0 \cos(\omega t) = \hat{\mathbf{z}} E_0 \cos(\omega t). \quad (\text{S-13.274})$$

Analogously to what seen above for the electric polarization, the sphere must acquire also a uniform magnetization  $\mathbf{M}$  in order to cancel the magnetic field of the wave at its interior. As shown by (S-5.72) of Problem 5.10, we must have

$$\mathbf{M}(t) = -\hat{\mathbf{z}} \frac{3}{8\pi} B_0 \cos(\omega t) = -\hat{\mathbf{z}} \frac{3}{8\pi} E_0 \cos(\omega t), \quad (\text{S-13.275})$$

corresponding to a magnetic dipole moment of the sphere

$$\mathbf{m} = \frac{4\pi a^3}{3} \mathbf{M} = -\hat{\mathbf{z}} \frac{a^3}{2} E_0 \cos(\omega t), \quad (\text{S-13.276})$$

Thus the power scattered by the magnetic dipole is one fourth of the electric dipole contribution:

$$W_{\text{scatt}}^{(\text{magn})} = \frac{1}{3c^3} |\dot{\mathbf{m}}|^2 = \frac{\omega^4 a^6}{12c^3} E_0^2. \quad (\text{S-13.277})$$

The total cross section is thus  $5/4$  times the value due to the electric dipole only:

$$\sigma_{\text{scatt}}^{(\text{el,magn})} = 160\pi^4 (\pi a^2) \left(\frac{a}{\lambda}\right)^4. \quad (\text{S-13.278})$$

A discussion on how the magnetic dipole term contributes to the angular distribution of the scattered radiation can be found in Reference [3].

## S-13.22 Radiation and Scattering from a Linear Molecule

**(a)** At the initial time  $t = 0$ , we assume the center of mass of the molecule to be at rest at the origin of our Cartesian coordinate system. The center of mass will remain at rest, since the net force acting on the molecule is zero. However, the field  $\mathbf{E}_0$  exerts a torque  $\tau_0 = \mathbf{p}_0 \times \mathbf{E}_0$ , and the molecule rotates around the  $z$  axis. The equation of motion is  $I\ddot{\theta} = \tau_0$ , or

$$\mathcal{I}\ddot{\theta} = -p_0 E_0 \sin \theta, \quad (\text{S-13.279})$$

where  $\theta = \theta(t)$  is the angle between  $\mathbf{p}_0$  and the  $x$  axis. The potential energy of the molecule is

$$V(\theta) = -\mathbf{p}_0 \cdot \mathbf{E}_0 + C = -p_0 E_0 \cos \theta + C, \quad (\text{S-13.280})$$

where  $C$  is an arbitrary constant. The molecule has two equilibrium positions, at  $\theta = 0$  (stable), and  $\theta = \pi$  (unstable), respectively. For small oscillations around the stable equilibrium position we can approximate  $\sin \theta \approx \theta$ , and (S-13.279) turns into the equation for the harmonic oscillator

$$\ddot{\theta} \approx -\frac{p_0 E_0}{\mathcal{I}} \theta \equiv -\omega_0^2 \theta, \quad \text{where } \omega_0^2 = \frac{E_0 p_0}{\mathcal{I}}. \quad (\text{S-13.281})$$

Thus, if the molecule starts at rest at a small initial angle  $\theta(0) = \theta_0$ , we have  $\theta(t) \approx \theta_0 \cos \omega_0 t$ . The potential energy of the molecule can be approximated as

$$V(\theta) \approx -p_0 E_0 \left(1 - \frac{\theta^2}{2}\right) + C = \frac{1}{2} p_0 E_0 \theta^2 = \frac{1}{2} \mathcal{I} \omega_0^2 \theta^2, \quad (\text{S-13.282})$$

where we have chosen  $C = p_0 E_0$ , in order to have  $V(0) = 0$ . The kinetic energy of the molecule is

$$K(\dot{\theta}) = \frac{1}{2} \mathcal{I} \dot{\theta}^2. \quad (\text{S-13.283})$$

**(b)** In our coordinate system the instantaneous dipole moment has components

$$p_x = p_0 \cos \theta \approx p_0, \quad p_y = p_0 \sin \theta \approx p_0 \theta_0 \cos(\omega_0 t), \quad (\text{S-13.284})$$

so that, for small oscillations, the radiation emitted by the molecule is equivalent to the radiation of an electric dipole parallel to  $\hat{\mathbf{y}}$ , and of frequency  $\omega_0$ . The radiation is linearly polarized, and the angular distribution of the emitted power is  $\sim \cos^2 \alpha$ , where  $\alpha$  is the observation angle relative to  $\mathbf{E}_0$ . Thus, the radiated power per unit solid angle is maximum in the  $xz$  plane and vanishes in the  $\hat{\mathbf{y}}$  direction. The time-averaged total emitted power is

$$P_{\text{rad}} = \frac{1}{3c^3} |\ddot{\mathbf{p}}|^2 = \frac{1}{3c^3} \omega_0^4 p_0^2 \theta_0^2. \quad (\text{S-13.285})$$

We assume that the decay time is much longer than the oscillation period, so that we can write

$$\theta(t) \approx \theta_s(t) \cos \omega_0 t, \quad (\text{S-13.286})$$

with  $\theta_s(0) = \theta_0$ , and  $\theta_s(t)$  decaying in time so slowly that it is practically constant over a single oscillation. In these conditions the total energy of the molecule during a single oscillation period can be written

$$U(t) = K(\dot{\theta}) + V(\theta) \simeq \frac{1}{2} I \omega_0^2 \theta_s^2(t). \quad (\text{S-13.287})$$

The rate of energy loss due to the emitted radiation is

$$\frac{dU}{dt} = \omega_0^2 I \theta_s \frac{d\theta_s}{dt} = -P_{\text{rad}}(\theta_s), \quad (\text{S-13.288})$$

from which we obtain

$$\frac{d\theta_s}{dt} = -\frac{1}{3c^3} \frac{\omega_0^2 p_0^2}{I} \theta_s. \quad (\text{S-13.289})$$

Thus the oscillation amplitude decays exponentially in time

$$\theta_s(t) = \theta_0 e^{-t/\tau}, \quad \text{with} \quad \tau = \frac{3Ic^3}{\omega_0^2 p_0^2}. \quad (\text{S-13.290})$$

(c) Since  $kd \ll 1$ , the electric field of the wave can be considered as uniform over the molecule, and we can write  $\mathbf{E}_1(0, t) \simeq \mathbf{E}_1 e^{-i\omega t}$  in complex notation. The torque exerted by the wave is  $\tau_1 = \mathbf{p}_0 \times \mathbf{E}_1$ . The complete equation of motion for the molecule is thus

$$I\ddot{\theta} = -p_0 E_0 \sin \theta - p_0 E_1 \cos \theta e^{-i\omega t}, \quad (\text{S-13.291})$$

which, at the limit of small oscillations ( $\sin \theta \simeq \theta, \cos \theta \simeq 1$ ) becomes

$$\ddot{\theta} = -\omega_0^2 \theta - \omega_1^2 e^{-i\omega t}, \quad \text{with} \quad \omega_1^2 = \frac{E_1 p_0}{I} = \omega_0^2 \frac{E_1}{E_0}. \quad (\text{S-13.292})$$

The general solution of (S-13.292) is the sum of the homogeneous solution considered at point (a), which describes free oscillations, and of a particular solution of the complete equation. A particular solution can be found in the form  $\theta(t) = \theta_f e^{-i\omega t}$ , which, substituted into (S-13.292), gives

$$\theta_f = \frac{\omega_1^2}{\omega^2 - \omega_0^2}. \quad (\text{S-13.293})$$

For simplicity, we neglected the possible presence of friction in (S-13.281). However, in principle a friction term such as  $-\dot{\theta}/\tau$  should appear because of the energy loss by radiation. In the presence of the plane wave the friction term is relevant only close to the  $\omega = \omega_0$  resonance, because  $\tau^{-1} \ll \omega_0$ .

(d) After a transition time of the order of  $\tau$  possible initial oscillations at  $\omega_0$  are damped, and the molecule reaches a steady state where it oscillates at frequency  $\omega$ . Assuming, as in (b), small-amplitude oscillations, we have an oscillating dipole

component  $p_y \simeq p_0 \theta_f e^{-i\omega t}$ . The scattered power is

$$P_{\text{scatt}} = \frac{1}{3c^3} |\vec{p}_y|^2 = \frac{p_0^2}{3c^3} \frac{\omega^4 \omega_1^4}{(\omega^2 - \omega_0^2)^2} = \frac{p_0^4 E_1^2}{3I^2 c^3} \frac{\omega^4}{(\omega^2 - \omega_0^2)^2}. \quad (\text{S-13.294})$$

The intensity of the wave is  $I = (c/4\pi)E_1^2$ , thus the scattering cross section is

$$\sigma_{\text{scatt}} = \frac{P_{\text{scatt}}}{I} = \frac{4\pi p_0^4}{3I^2 c^4} \frac{\omega^4}{(\omega^2 - \omega_0^2)^2}. \quad (\text{S-13.295})$$

An order-of-magnitude estimate for a simple molecule such as H<sub>2</sub> can be performed by noticing that  $p_0 \sim ed$  and  $I \sim md$ , with  $m \sim m_p$  the mass of the nuclei, so that  $(p_0^4/I^2 c^4) \sim (e^2/m_p c^2)^2$ .

### S-13.23 Radiation Drag Force

**(a)** The electric field of the wave in complex notation is

$$\mathbf{E} = \hat{\mathbf{y}} \operatorname{Re}(E_0 e^{ikx-i\omega t}). \quad (\text{S-13.296})$$

Neglecting the magnetic field, the particle oscillates in the  $\hat{\mathbf{y}}$  direction without changing its  $x$  and  $z$  coordinates. Thus, assuming the particle to be initially located at the origin of our Cartesian system, and looking for a solution of the form  $\mathbf{v} = \mathbf{v}_0 e^{-i\omega t}$ , we obtain by substitution into (13.22):

$$\mathbf{v}_0 = \hat{\mathbf{y}} \frac{iq}{m(\omega + i\nu)} E_0. \quad (\text{S-13.297})$$

**(b)** The power developed by the electromagnetic force is  $q\mathbf{E} \cdot \mathbf{v}$ . Thus

$$P_{\text{abs}} = \langle q\mathbf{E} \cdot \mathbf{v} \rangle = \frac{q}{2} \operatorname{Re}(E_0 v_0^*) = \frac{q^2}{2m} \frac{\nu}{\omega^2 + \nu^2} |E_0|^2. \quad (\text{S-13.298})$$

**(c)** The electric dipole moment of the particle is  $\mathbf{p} = q\mathbf{r}$ . Using Larmor's formula for the radiated power we obtain

$$P_{\text{rad}} = \frac{2}{3c^3} \langle \dot{\mathbf{p}}^2 \rangle = \frac{2q^2}{3c^3} \langle \dot{\mathbf{r}}^2 \rangle = \frac{q^4}{3m^2 c^3} \frac{\omega^2}{\omega^2 + \nu^2} |E_0|^2. \quad (\text{S-13.299})$$

Assuming  $P_{\text{rad}} = P_{\text{abs}}$ , we obtain

$$\nu = \frac{2q^2\omega^2}{3mc^3}. \quad (\text{S-13.300})$$

(d) We must evaluate

$$F_x = \left( \frac{q}{c} v_y B_z \right), \quad (\text{S-13.301})$$

where for  $v_y$  we use the result of (a), while the amplitude of the magnetic field is  $B_0 = E_0$ . Thus we have

$$F_x = \frac{q}{2c} \operatorname{Re}(v_0 E_0^*) = \frac{P_{\text{abs}}}{c}. \quad (\text{S-13.302})$$

Thus, the ratio between the energy and the momentum absorbed by the particle from the electromagnetic field equals  $c$ .

(e) The radiation from a cluster smaller than one wavelength is coherent and thus scales as  $N^2$ , so does the total force. The cluster mass scales as  $N$ , thus the acceleration scales as  $N^2/N = N$ . In other terms, a cluster of many particles may be accelerated much more efficiently than a single particle: the higher the number of particles (within the limits of our approximations), the stronger the acceleration. This is the basis of a concept of “coherent” acceleration using electromagnetic waves, formulated by V. I. Veksler. [4]

## References

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# Appendix A

## Some Useful Vector Formulas

### A.1 Gradient, Curl, Divergence and Laplacian

Vector equations are independent of the coordinate system used. Cartesian coordinates are used very often because they are the most convenient when the problem has no particular symmetry. However, in the case of particular symmetries, calculations can be greatly facilitated by a suitable choice of the coordinate system. Apart from the elliptical coordinates, used only in Problem 2.14, The only two special systems used in this book are the cylindrical and spherical coordinates.

A cylindrical coordinate system  $(r, \phi, z)$  specifies a point position by the distance  $r$  from a chosen reference (*longitudinal*) axis  $z$ , the angle  $\phi$  that  $r$  forms with a chosen reference plane  $\phi = 0$  containing the  $z$  axis, and the distance, positive or negative, from a chosen reference plane perpendicular to the axis. The origin is the point where  $r$  and  $z$  are zero, for  $r = 0$  the value of  $\phi$  is irrelevant. Fig. A.1 shows a cylindrical coordinate system, superposed to a Cartesian system sharing the same origin, with the  $z$  axes of the two systems are superposed, the  $xz$  plane corresponding to the  $\phi = 0$  plane of the cylindrical system. We have the conversion relations

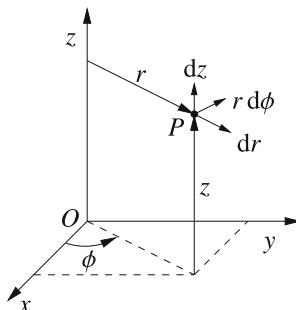


Fig. A.1

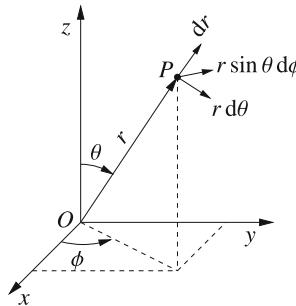


Fig. A.2

**Table A.1** Gradient, curl, divergence and Laplacian in cylindrical and spherical coordinates

Cylindrical Coordinates	Spherical Coordinates
Components of the gradient of a scalar function $V$	
$\hat{\mathbf{r}} \frac{\partial V}{\partial r}$	$\hat{\mathbf{r}} \frac{\partial V}{\partial r}$
$\hat{\phi} \frac{1}{r} \frac{\partial V}{\partial \phi}$	$\hat{\theta} \frac{\partial V}{\partial \theta}$
$\hat{\mathbf{z}} \frac{\partial V}{\partial z}$	$\hat{\phi} \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi}$
Components of the curl of a vector function $\mathbf{A}$	
$\hat{\mathbf{r}} \left( \frac{1}{r} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right)$	$\hat{\mathbf{r}} \frac{1}{r \sin \theta} \left[ \frac{\partial}{\partial \theta} \left( A_\phi \sin \theta - \frac{\partial A_\theta}{\partial \phi} \right) \right]$
$\hat{\phi} \left( \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right)$	$\hat{\theta} \left[ \frac{1}{r \sin \theta} \frac{\partial A_r}{\partial \phi} - \frac{1}{r} \frac{\partial (r A_\phi)}{\partial r} \right]$
$\hat{\mathbf{z}} \frac{1}{r} \left[ \frac{\partial (r A_\phi)}{\partial r} - \frac{\partial A_r}{\partial \phi} \right]$	$\hat{\phi} \frac{1}{r} \left[ \frac{\partial (r A_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta} \right]$
Divergence of a vector function $\mathbf{A}$	
$\frac{1}{r} \frac{\partial (r A_r)}{\partial r} + \frac{1}{r} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$	$\frac{1}{r^2} \frac{\partial (r^2 A_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (A_\theta \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi}$
Laplacian of a scalar function $V$	
$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$	$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2}$

$$x = r \cos \phi, \quad y = r \sin \phi, \quad z = z. \quad (\text{A.1})$$

The orthogonal line elements are  $dr$ ,  $r d\phi$  and  $dz$ , and the infinitesimal volume element is  $r dr d\phi dz$ .

A spherical coordinate system  $(r, \theta, \phi)$  specifies a point position by the radial distance  $r$  from a fixed origin, a polar angle  $\theta$  measured from a fixed zenith direction, and the azimuth angle  $\phi$  of the orthogonal projection of  $r$  on a reference plane that passes through the origin and is orthogonal to the zenith, measured from a fixed reference direction on that plane. Fig. A.2 shows a spherical coordinate system, superposed to a Cartesian system sharing the same, origin, with the  $z$  axis superposed to the zenith axis, and the  $xz$  plane corresponding to the  $\phi = 0$  plane of the spherical system. We have the conversion relations

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta. \quad (\text{A.2})$$

The orthogonal line elements are  $dr$ ,  $r d\theta$ , and  $r \sin \theta d\phi$ , and the infinitesimal volume element is  $r^2 \sin \theta d\theta d\phi dr$  (Table A.1).

## A.2 Vector Identities

Quantities **A**, **B**, and **C** are vectors or vector functions of the coordinates,  $f$  and  $g$  are scalar functions of the coordinates.

$$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{A} \times \mathbf{B} \cdot \mathbf{C} = \mathbf{B} \cdot \mathbf{C} \times \mathbf{A} = \mathbf{C} \cdot \mathbf{A} \times \mathbf{B} ; \quad (\text{A.3})$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{C} \times \mathbf{B}) \times \mathbf{A} = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} ; \quad (\text{A.4})$$

$$\nabla(fg) = f\nabla g + g\nabla f ; \quad (\text{A.5})$$

$$\nabla \cdot (f\mathbf{A}) = f\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla f ; \quad (\text{A.6})$$

$$\nabla \times (f\mathbf{A}) = f\nabla \times \mathbf{A} + \nabla f \times \mathbf{A} ; \quad (\text{A.7})$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B} ; \quad (\text{A.8})$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} ; \quad (\text{A.9})$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} ; \quad (\text{A.10})$$

$$\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A}) , \quad (\text{A.11})$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} . \quad (\text{A.12})$$

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