FEM simulation for rotating disk and stress analysis

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TODO: validate results written in Ashraf2010 - Analytical and Numerical Solutions for a Rotating Annular.pdf

TODO: bayat2009-axisymmetric, heat, validate h

TODO: pure neumann validation, peng2010 - Thermal stress in rotating functionally graded hollow circular disks

TODO: **canical disk vivio2007 - Elastic stress analysis of rotating converging conical, thermal + axisymmetric + numerical solution + h profile**

TODO: write about Larange Multiplication

TODO: write about Larange Multiplication

TODO: citation bibtex (cang nhieu cang tot, nhung de o luc sau)

TODO: model from the book

TODO: Fenics Code

TODO: Select journal

TODO: Hoi anh Dang stabilization ko duoc

TODO: non-axisymmetric $\dfrac{\partial}{\partial\theta} \neq 0$ from the book

TODO: add numerical resultslib

TODO: add picture

TODO: idea bai bao: motivation:

moi nguoi cung su dung, FEM

lam khoa hoc tren ma nguon mo toan bo, google colab (nhan nut chay),

ma nguon mo, so sanh lai nhung gi ho da lam, ket qua tot

TODO: dang journal nao?

Publish o dau cung duoc hoac dua len archive (cho dang bao), di phong van cung la ok, de nguoi ta khong an cap y tuong, roi gui di cac journal, cho publish ArXiV, ai ma dang cai gi la quang vo day truoc.

lam 1 cai ma di rong, di nhieu huong,

minh hoa bai toan nay, reproducible science

chay tren web luon, google cloud 1 phan, tren may tinh 1 phan. chay tren browser het.

Esilever, hoi canh tranh, viet ngon lanh

submit tra tien: https://www.mdpi.com/journal/computation

https://www.journals.elsevier.com/journal-of-computational-science https://www.journals.elsevier.com/journal-of-computational-and-applied-mathematics Van-Dang Nguyen22:01 https://www.journals.elsevier.com/journal-of-computational-science/recent-articles https://www.journals.elsevier.com/journal-of-computational-physics

TODO: lam dai, up len ArXiV roi di xin viec.

TODO: $\theta = 0$ truoc, $\theta != 0$ sau, de ket qua so thoi

TODO: nhiet

TODO: variation of the thickness $h = h(r)$,

TODO: package, de len github

TODO: viet bao quan trong la than thai, cach viet bao rat quan trong, motivation, review phai lam cho ky. cite nhung thang khac, pp cua minh hay o cho nao, vuot troi hon o cho nao. Ra ket qua dep truoc da, roi quay lai viet sau.

TODO: Dinh huong, cach thuc moi, ket qua khong moi, cach tiep can moi, tao ra moi truong moi de nhung nguoi nghien cuu nguoi ta lam chuyen tuong tu. Lam package, pho bien cho moi nguoi, nhan nut la chay. Lam tren clound het.

Tham khao bai bao cua anh Dang, https://arxiv.org/pdf/1908.01719.pdf

Quang cv o achive de xin viec.

Update thuong xuyen, upload ban moi neu thay co sai

TODO: upload len arXvi, phai dung format. File hinh la file gi? ko up pdf truc tiep duoc. build tu source. main.tex, xoa file pdf. Review, dung chu de khong? 1 tuan sau de online. Lam 1 thuc muc khac, sach se, de file hinh vo, nen lai, roi up len. Vay la xong.

1 Introduction

###Rotating members known as rotors are used in turbomachines and many powergenerating or power-consuming thermal and hydraulic machines (turbines, turbopumps, turbochargers, centrifugal pumps, centrifugal compressors, fans, molecular pumps, centrifuges, ultracentrifuges, etc.), turbogenerators, turboalternators, turbo-dynamos and many synchronous and asynchronous electrical machines; certain types of aircraft (helicopters, autogyros or gyroplanes, gyrogliders, rotoplanes, etc.); propulsion units for turboprop, turbofan, and turbojet airplanes and those of rotor ships and turbine-driven tankers; the drives of various ground vehicles featuring mechanical energy storage (flywheels) and of wind generators; many machine tools (spindles, flywheels, etc.); as well as other power-consuming machines such as turbo extractors and turboexpanders. ###

2 Axisymmetric Governing equations

The governing equations is a system of three Partial Differential Equations (PDEs) written in polar coordinate that takes into account the plane stress equilibrium equation in the radial direction, the Kirchhoff strain-displacement relations under axisymmetric assumption such that the variation in the circumferential axis is zero, $\partial/\partial\theta = 0$, and the stress-strain relations, [2, 5],

$$\begin{cases} \dfrac{\partial}{\partial r}(hr\sigma_r) - h\sigma_\theta = -h\rho\omega^2 r^2 \quad \text{in} \quad \Omega^{polar} \\ \epsilon_r(u) = \dfrac{\partial u}{\partial r} \\ \epsilon_\theta(u) = \dfrac{u}{r} \\ \sigma_r(r,\theta) = \dfrac{E}{1-\nu^2}(\epsilon_r + \nu\epsilon_\theta) \\ \sigma_\theta(r,\theta) = \dfrac{E}{1-\nu^2}(\nu\epsilon_r + \epsilon_\theta) \\ h = h_0\left(\dfrac{r}{a}\right)^\alpha \end{cases} \tag{1}$$

where $\Omega^{polar} = \{(r,\theta)|R_{int} \le r \le R_{ext}, 0 \le \theta \le 2\pi\}$ is the domain representing the disk, r and θ are radius and angle variables, and R_{ext} $[m]$ is exterior radius, R_{int} $[m]$ is interior radius, h $[m]$ is disk thickness, h_0 is thickness at R_{ext}, ρ $[kg/m^3]$ is material density, ω $[rpm]$ is rotating speed, $\sigma_r(r)$ is radial stress, $\sigma_\theta(\theta)$ is circumferential stress, $u(r,\theta)$ is radial displacement, $\epsilon_r(r,\theta)$ is radial strain, $\epsilon_\theta(r,\theta)$ is circumferential strain, E is Young modulus and ν is Poisson ratio. We consider the following boundary conditions for the cases of clamped-clamped (C-C), clamped-free (C-F), free-free (F-F),

$$\text{(C-C)} \begin{cases} u = 0 \quad \text{on} \quad \Gamma_{int} \\ u = 0 \quad \text{on} \quad \Gamma_{ext} \end{cases} \text{(C-F)} \begin{cases} u = 0 \quad \text{on} \quad \Gamma_{int} \\ \sigma_r = 0 \quad \text{on} \quad \Gamma_{ext} \end{cases} \text{(F-F)} \begin{cases} \sigma_r = 0 \quad \text{on} \quad \Gamma_{int} \\ \sigma_r = 0 \quad \text{on} \quad \Gamma_{ext} \end{cases} \tag{2}$$

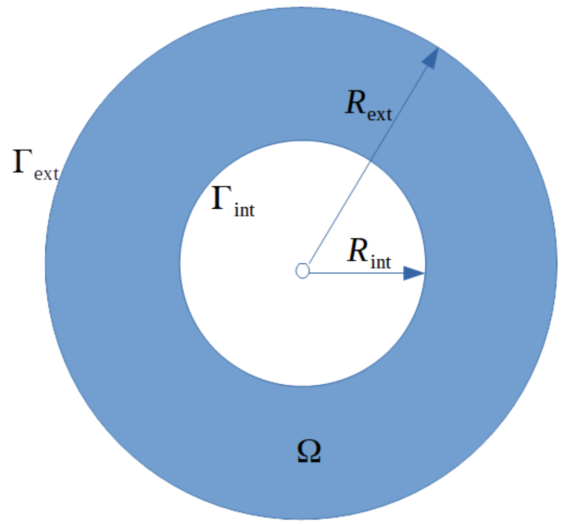

Figure 1: The geometry Ω

3 Simulation strategy

Weak Form:

Multiplying the govening equation with test function $v \in H^1_{\Gamma_{int}}(\Omega)$, i.e. v is vanished on Γ_{int} and integrating over the domain Ω^{polar}, we get

$$\int_{\Omega^{polar}} \frac{\partial}{\partial r}(hr\sigma_r)\overline{u}\ \mathrm{d}\Omega^{polar} - \int_{\Omega^{polar}} h\sigma_\theta \overline{u}\ \mathrm{d}\Omega^{polar} + \int_{\Omega^{polar}} h\rho\omega^2 r^2 \overline{u}\ \mathrm{d}\Omega^{polar} = 0$$

where $\mathrm{d}\Omega^{polar} = r\mathrm{d}r\mathrm{d}\theta$. Applying integral by parts to the first term, we get

$$\int_{\Omega^{polar}} \frac{\partial}{\partial r}(hr\sigma_r)\overline{u}\ \mathrm{d}\Omega^{polar} = \int_{\partial\Omega^{polar}} hr\sigma_r v\ \mathrm{d}S - \int_{\Omega^{polar}} hr\sigma_r \frac{\partial \overline{u}}{\partial r}\ \mathrm{d}\Omega^{polar}$$

The integrating term over the boundary is vannished since either v is vannished on Γ_{int} or σ_r is vannished on Γ_{out}. Subtituting it back to the above equation, we get the week form

$$-\int_{\Omega^{polar}} hr\sigma_r \frac{\partial \overline{u}}{\partial r}\ \mathrm{d}\Omega^{polar} - \int_{\Omega^{polar}} h\sigma_\theta \overline{u}\ \mathrm{d}\Omega^{polar} + \int_{\Omega^{polar}} h\rho\omega^2 r^2 \overline{u}\ \mathrm{d}\Omega^{polar} = 0$$

One can rewrite it as the derivative over r of the test function v equals to the radial strain of the test function v

$$-\int_{\Omega^{polar}} hr\sigma_r \epsilon_r(\overline{u})\ \mathrm{d}\Omega^{polar} - \int_{\Omega^{polar}} h\sigma_\theta \overline{u}\ \mathrm{d}\Omega^{polar} + \int_{\Omega^{polar}} h\rho\omega^2 r^2 \overline{u}\ \mathrm{d}\Omega^{polar} = 0$$

Transfrom the model in to Cartesian coordinate: if the domain is given in xy-plane, the unknow $u^{rec}(x,y) \in \Omega^{rec} = \{(x,y) | R^2_{int} \le x^2 + y^2 \le R^2_{ext}\}$, r is replaced by $\sqrt{x^2+y^2}$, $\mathrm{d}\Omega^{rec} = \mathrm{d}x\mathrm{d}y$ and the radial strain is rewritten according to the chain rule

$$\epsilon_r(u^{rec}) = \frac{\partial u^{rec}}{\partial r} = \frac{1}{r}\left(x\frac{\partial u^{rec}}{\partial x} + y\frac{\partial u^{rec}}{\partial x}\right)$$

Compute von Mises Stress: the von Mises Stress is given by

$$\sigma_{vm} = \sqrt{\sigma_r^2 + \sigma_\theta^2 - \sigma_r\sigma_\theta} \tag{3}$$

Compute Maximum Principle Stress: The Maximum Principle Stress is given by

$$\sigma_{mp} = max(|\sigma_r|, |\sigma_\theta|, |\sigma_z|)$$

Stabilization:
Strong residual

$$\begin{aligned} R(u) &= \frac{\partial}{\partial r}(tr\sigma_r) - t\sigma_\theta + t\rho\omega^2 r^2 \\ L(u) &= \frac{\partial}{\partial r}(tr\sigma_r) - t\sigma_\theta \\ R(u) &= L(u) + t\rho\omega^2 r^2 \end{aligned}$$

stabilization term

$$stb(u, v) = \int_\Omega P(v)\tau R(u)$$

Galerkin Least Square (GLS)

$$P(v) = L(v)$$

where

$$\tau = \left(\frac{2D_{conv}}{cs} + \frac{4D_{diff}}{cs^2} + D_{rec}\right)^{-1}$$

$$Pe = \frac{|drift_{velocity}|cs}{2Diffcoef}$$

Model without tConsider the orignal model and boundary condition of clamping inside, free outside, given by [1] and written in polar coordinate

$$\begin{cases} -\dfrac{\partial}{\partial r}(r\sigma_r) + \sigma_\theta = \rho\omega^2 r^2 \quad \text{in} \quad \Omega^{polar} \\ \epsilon_r(u^{polar}) = \dfrac{\partial u^{polar}}{\partial r} \\ \epsilon_\theta(u^{polar}) = \dfrac{u^{polar}}{r} \\ \sigma_r(r,\theta) = \dfrac{E}{1-\nu^2}(\epsilon_r + \nu\epsilon_\theta) \\ \sigma_\theta(r,\theta) = \dfrac{E}{1-\nu^2}(\nu\epsilon_r + \epsilon_\theta) \end{cases}$$

$$\begin{cases} u^{polar} = 0 \quad \text{on} \quad \Gamma_{in} \\ \sigma_r = 0 \quad \text{on} \quad \Gamma_{out} \end{cases}$$

where $\Omega = \{(r,\theta)|b \le r \le a, 0 \le \theta \le 2\pi\}$ and
week form

$$F = \int_\Omega r\sigma_r(u)\epsilon_r(\overline{u})\ dx + \int_\Omega \sigma_\theta(u)\overline{u}\ dx - \int_\Omega \rho\omega^2 r^2\overline{u}\ dx$$

4 Pure Neuman BC for Poisson

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega \\ \nabla u \cdot n = g & \quad\quad \partial\Omega \end{cases}$$

week form

$$\int \nabla\nabla u\overline{u}\ dx = -\int \nabla u \nabla \overline{u}\ dx + \int_{\partial\Omega} f\overline{u}\ dx$$

$$\int_\Omega \nabla u \nabla \overline{u}\ dx - \int_{\partial\Omega} g\overline{u}\ dx - \int_\Omega f\overline{u} = 0$$

constraint

$$\begin{aligned} -\int_\Omega \Delta u\ dx &= \int_\Omega f\ dx \\ -\int_\Omega 1\Delta u\ dx &= \int_\Omega f\ dx \\ -\int_{\partial\Omega} 1\nabla u \cdot n\ ds + \int_\Omega \nabla 1 \cdot \nabla u\ dx &= \int_\Omega f\ dx \\ -\int_{\partial\Omega} g\ ds &= \int_\Omega f\ dx \end{aligned}$$

c is a constant

$$\begin{cases} -\nabla \cdot \nabla u + c = f \\ \quad u = 0 \end{cases}$$

$$F = \int_\Omega \nabla u \nabla \overline{u}\ dx + \int_\Omega c\overline{u}\ dx + \int_\Omega u\overline{c}\ dx - \int_{\partial\Omega} g\ dx - \int_\Omega f\overline{u} = 0$$

Lagrange multiplier is introduced to the problem to make it well-posed to .... Larson, Bengzon - The Finite Element Method: Theory, Implementation, and Applications (p95)

TODO: write Lagrange multiplier

5 Pure Neuman BC for disk

Applied to the problem

$$\begin{cases} -\dfrac{\partial}{\partial r}(r\sigma_r) + \sigma_\theta + c = \rho\omega^2 r^2 & \text{in} \quad \Omega^{polar} \\ u = 0 & \text{in} \quad R \\ \epsilon_r(u) = \dfrac{\partial u}{\partial r} \\ \epsilon_\theta(u) = \dfrac{u}{r} \\ \sigma_r(r,\theta) = \dfrac{E}{1-\nu^2}(\epsilon_r + \nu\epsilon_\theta) \\ \sigma_\theta(r,\theta) = \dfrac{E}{1-\nu^2}(\nu\epsilon_r + \epsilon_\theta) \end{cases}$$

Find solution $(u, c) \in W = V \times \mathbb{R}$ such that Larange multiplier (viet ra het, khong can giai thich nhieu thi cat bo)

$$\begin{aligned} F_u &= \int_\Omega r\sigma_r(u)\epsilon_r(\overline{u})\ \mathrm{d}\Omega_1 + \int_\Omega \sigma_\theta(u)\overline{u}\ \mathrm{d}\Omega_1 + \int_\Omega c\overline{u}\ \mathrm{d}\Omega_1 - \int_\Omega \rho\omega^2 r^2 \overline{u}\ \mathrm{d}\Omega_1 \\ F_c &= \int_R u\overline{c}\ d\Omega_2 \\ F &= F_u + F_c \end{aligned}$$

Parameter	value	unit
α		
R_{ext}		m
R_{int}		m
ν		
E		
ρ		
ω	15000	
h_0		

Table 1: Table of parameters

6 Numerical Results

7 Axisymetric model with heat load vivio2007-model

Conical profile geometry
$h = h_0(1-t) = h_0(1-\frac{r}{R})$

$$\frac{\partial}{\partial r}(\sigma_r h r) - \sigma_\theta h + \rho\omega^2 r^2 h = 0$$

$$\epsilon_r = \frac{\partial u}{\partial r}$$
$$\epsilon_\theta = \frac{u}{r}$$

$$\sigma_r = \frac{E}{1-\nu^2}\left[(\epsilon_r - \alpha T) + \nu(\epsilon_\theta - \alpha T)\right]$$
$$\sigma_\theta = \frac{E}{1-\nu^2}\left[(\epsilon_\theta - \alpha T) + \nu(\epsilon_r - \alpha T)\right]$$

week form

$$\begin{aligned} F &= \int_\Omega \frac{\partial}{\partial r}(\sigma_r h r)\,\overline{u}\,\mathrm{d}x - \int_\Omega \sigma_\theta h \overline{u}\,\mathrm{d}x + \int_\Omega \rho\omega^2 r^2 h\overline{u}\,\mathrm{d}x \\ &= -\int_\Omega \sigma_r h r \frac{\partial \overline{u}}{\partial r}\,\mathrm{d}x - \int_\Omega \sigma_\theta h\overline{u}\,\mathrm{d}x + \int_\Omega \rho\omega^2 r^2 h\overline{u}\,\mathrm{d}x \end{aligned}$$

8 Thickness profile

$h = h(r)$

1. exponential function: $h = h_0 e^{-n\rho^k}$
2. stodola's hyperbolic function: $h = h_0(1+\rho)^a$
3. elliptical function: $h = h_0(1-n\rho^2)^{1/2}$
4. parapolic function: $h = h_0(1-n\rho^k)$

9 Full model with heat

$$\begin{cases} \frac{\partial}{\partial r}(\sigma_r h r) + \frac{\partial}{\partial \theta}(\tau_{r\theta} h) - \sigma_\theta h + \rho\omega^2 r^2 h = 0 & \text{in} \quad \Omega \\ \frac{\partial}{\partial r}(\tau_{r\theta} h r) + \frac{\partial}{\partial \theta}(\sigma_\theta h) + \tau_{r\theta} h + \rho\dot{\omega} r^2 h = 0 & \text{in} \quad \Omega \end{cases}$$

where

ω is angular velocity

$\dot{\omega}$ is angular acceleration

γ is mass per unit volume, or density, of the material

τ_{rt} is shear stress component

γ_{rt} is shear strain component

centrifugal loading: $\omega \neq 0, \dot{\omega} = 0$ ω is constant

$h = h(r)$ constant thickness disk, uniform strength disk, hyperbolic disk

Hook law for **homogeneous and isotropic**

$$\begin{cases} \epsilon_r = \dfrac{\partial u}{\partial r} \\ \epsilon_\theta = \dfrac{u}{r} + \dfrac{1}{r}\dfrac{\partial v}{\partial \theta} \\ \gamma_{r\theta} = \dfrac{\partial v}{\partial r} - \dfrac{v}{r} + \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} \end{cases}$$

$$\begin{aligned} \sigma_r &= \frac{E}{1-\nu^2}\left[(\epsilon_r - \alpha T) + \nu\left(\epsilon_\theta - \alpha T\right)\right] \\ \sigma_\theta &= \frac{E}{1-\nu^2}\left[(\epsilon_\theta - \alpha T) + \nu\left(\epsilon_r - \alpha T\right)\right] \\ \tau_{r\theta} &= G\gamma_{r\theta} \end{aligned}$$

$$\left\{ \begin{array}{c} \epsilon_r \\ \epsilon_\theta \\ \gamma_{r\theta} \end{array} \right\} = \begin{bmatrix} \dfrac{1}{E} & \dfrac{-\nu}{E} & 0 \\ \dfrac{-\nu}{E} & \dfrac{1}{E} & 0 \\ 0 & 0 & \dfrac{1}{G} \end{bmatrix} \left\{ \begin{array}{c} \sigma_r \\ \sigma_\theta \\ \tau_{r\theta} \end{array} \right\} + \alpha T \left\{ \begin{array}{c} 1 \\ 1 \\ 0 \end{array} \right\}$$

$$\begin{aligned} \epsilon_r &= \frac{1}{E}\sigma_r - \frac{\nu}{E}\sigma_\theta + \alpha T \\ \epsilon_\theta &= -\frac{\nu}{E}\sigma_r + \frac{1}{E}\sigma_\theta + \alpha T \\ \gamma_{r\theta} &= \frac{1}{G}\tau_{r\theta} \end{aligned}$$

ν is Poisson ratio

G is rigidity

α is heat expansion coefficient

which leads to

$$\begin{aligned} \sigma_r &= \frac{E}{1-\nu^2}\left[(\epsilon_r - \alpha T) + \nu\left(\epsilon_\theta - \alpha T\right)\right] \\ \sigma_\theta &= \frac{E}{1-\nu^2}\left[(\epsilon_\theta - \alpha T) + \nu\left(\epsilon_r - \alpha T\right)\right] \\ \tau_{r\theta} &= G\gamma_{r\theta} \end{aligned}$$

Boundary conditions

$$\text{(C-C)} \begin{cases} u = 0 & \text{on} \quad \Gamma_{int} \\ u = 0 & \text{on} \quad \Gamma_{ext} \end{cases} \text{(C-F)} \begin{cases} u = 0 & \text{on} \quad \Gamma_{int} \\ \sigma_r = \tau_{r\theta} = 0 & \text{on} \quad \Gamma_{ext} \end{cases} \text{(F-F)} \begin{cases} \sigma_r = 0 & \text{on} \quad \Gamma_{int} \\ \sigma_r = \tau_{r\theta} = 0 & \text{on} \quad \Gamma_{ext} \end{cases} \quad (4)$$

Thermal load:
chapter 3
week form

$$
\begin{aligned}
F_\sigma &= \int_\Omega \frac{\partial}{\partial r}(\sigma_r h r)\,\overline{u}\;\mathrm{d}x + \int_\Omega \frac{\partial}{\partial \theta}(\tau_{r\theta} h)\,\overline{u}\;\mathrm{d}x - \int_\Omega \sigma_\theta h \overline{u}\;\mathrm{d}x + \int_\Omega \rho\omega^2 r^2 h \\
&= -\int_\Omega (\sigma_r h r)\,\frac{\partial \overline{u}}{\partial r}\;\mathrm{d}x - \int_\Omega (\tau_{r\theta} h)\,\frac{\partial \overline{u}}{\partial \theta}\;\mathrm{d}x - \int_\Omega \sigma_\theta h \overline{u}\;\mathrm{d}x + \int_\Omega \rho\omega^2 r^2 h \overline{u}\;\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
F_\tau &= \int_\Omega \frac{\partial}{\partial r}(\tau_{r\theta} h r)\,\overline{v}\;\mathrm{d}x + \int_\Omega \frac{\partial}{\partial \theta}(\sigma_\theta h)\,\overline{v}\;\mathrm{d}x + \int_\Omega \tau_{r\theta} h \overline{v}\;\mathrm{d}x + \int_\Omega \rho\dot{\omega} r^2 h \overline{v}\;\mathrm{d}x \\
&= -\int_\Omega (\tau_{r\theta} h r)\,\frac{\partial \overline{v}}{\partial r}\;\mathrm{d}x - \int_\Omega (\sigma_\theta h)\,\frac{\partial \overline{v}}{\partial \theta}\;\mathrm{d}x + \int_\Omega \tau_{r\theta} h \overline{v}\;\mathrm{d}x + \int_\Omega \rho\dot{\omega} r^2 h \overline{v}\;\mathrm{d}x
\end{aligned}
$$

$$F = F_\tau + F_\sigma$$

REF: stress, heat load, axisymmetric, numerical [1]
conical disks, thermal load, [3]

Elastic stress analysis of non-linear variable thickness rotating disks subjected to thermal load and having variable density along the radius [4]

Elastic stress analysis of rotating converging conical disks subjected to thermal load and having variable density along the radius [3]

10 Conclution and perspective

11 Appendix

$$
\begin{aligned}
r\frac{\partial u}{\partial r} &= x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \\
\frac{\partial u}{\partial \theta} &= -y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y}
\end{aligned}
$$

References

References

[1] Mehdi Bayat, M Saleem, BB Sahari, AMS Hamouda, and E Mahdi. Mechanical and thermal stresses in a functionally graded rotating disk with variable thickness due to radially symmetry loads. *International Journal of Pressure Vessels and Piping*, 86(6):357–372, 2009.

[2] Mohammad Hadi Jalali and Behrooz Shahriari. Elastic stress analysis of rotating functionally graded annular disk of variable thickness using finite difference method. *Mathematical Problems in Engineering*, 2018, 2018.

[3] Francesco Vivio and Vincenzo Vullo. Elastic stress analysis of rotating converging conical disks subjected to thermal load and having variable density along the radius. *International Journal of Solids and Structures*, 44(24):7767–7784, 2007.

[4] Vincenzo Vullo and Francesco Vivio. Elastic stress analysis of non-linear variable thickness rotating disks subjected to thermal load and having variable density along the radius. *International Journal of Solids and Structures*, 45(20):5337–5355, 2008.

[5] Vincenzo Vullo and Francesco Vivio. *Rotors: Stress analysis and design.* Springer Science & Business Media, 2013.

Ref:
[1] Larson, Bengzon - The Finite Element Method: Theory, Implementation, and Applications (p95)
[2] https://fenicsproject.org/docs/dolfin/1.6.0/python/demo/documented/neumann-poisson/python/documentation.html
Ref:
[1] Hassan2011 - Stress analysis of functionally graded rotating discs: analytical and numerical solutions
[3] Chu - Analytical Calculation of Maximum Mechanical Stress on the Rotor of Interior Permanent-Magnet Synchronous Machines