Slice and Blockwise Well-Composed Sets

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Abstract

An infinite or closed continuous surface partitions space $\mathbb{R}^2$ or $\mathbb{R}^3$ into two disjoint sub-spaces, an "inside" and an "outside". Notions of voxel set separability describe an analogous partitioning of discrete space $\mathbb{Z}^2$ or $\mathbb{Z}^3$ by a surface voxelisation. Similar concepts, 2D and 3D well-composed sets, define the manifold nature of the boundary between a voxel set and its complement embedded in $\mathbb{R}^2$ or $\mathbb{R}^3$. Cohen-Or and Kaufman [1] define separating sets and present theorems for slicewise construction of 3D separating voxel sets from a group of 2D separating slices. This paper presents similar theorems for 3D well-composed sets. This allows slicewise construction to be applied in a wider range of situations, for example, where the manifold nature of a voxel set boundary is of vital importance or where we are considering solid voxelisations. Theorems for blockwise construction of 2D and 3D well-composed sets from a pair of smaller well-composed sets are also presented, providing further tools for piecewise analysis of voxel sets.

1. Introduction

Voxel sets play a central role in medical and other 3D imaging, storage and rendering of virtual objects [7], and as intermediate representations in many processing tasks (e.g. mesh repair and simplification [6, 13, 16]). Generally speaking, a voxel set, or 3D digital set, is a 3D raster image whose individual elements are known as voxels. A voxel is the 3D equivalent of the well known pixel in 2D digital images. For the sake of brevity a pixel is referred to as a 2D voxel in this paper. Voxelisation is the process of generating an approximating voxel set for a virtual object from its geometric representation [7], usually with some emphasis on geometric and topological properties. The field of knowledge concerned with topological properties of digital sets is known as digital topology [15] and is the main focus of this paper. More specifically, the descriptive capabilities of two digital topology concepts, separability and well-composedness, are examined and some rules defined for separating sets are extended to the context of well-composed sets.

Cohen-Or and Kaufman [1] present separability as a topological property of surface voxelisations that describes the segmenting nature of a voxelisation on discrete space $\mathbb{Z}^2$ or $\mathbb{Z}^3$. Generally speaking, if a surface voxelisation S is a subset of a set of voxels X, and $X \setminus S$ contains two disconnected[1] subsets, then S is said to be separating in X. This type of description is valuable when the voxel set S represents a continuous surface (resp. curve in 2D) with two distinct sides (a surface voxelisation). However, it is not well suited to describing voxelisations of solid objects (solid voxelisations), which do not exhibit such partitioning of continuous space $\mathbb{R}^3$ (resp. $\mathbb{R}^2$)(fig.1).

Another problem with separability is that it does not describe whether the boundary between a voxel set and its complement is a manifold when placed in continuous space. That is, when embedded in $\mathbb{R}^3$ (resp. $\mathbb{R}^2$) the neighbourhood of every point on the boundary is topological equivalent to a disk (resp. line segment). This may be an important property to know in some situations, for it is this boundary that often represents the surface of the object for solid voxelisations. In such a setting faces between black and white voxels are used to represent the surface [4], while the voxels themselves [12] are used for surface voxelisations (fig.1). Nonetheless, we may still wish to know this property for surface voxelisations, for whatever the reason.

A more suitable concept for describing the topological properties of a voxelisation with respect to this boundary is well-composed sets. Well-composed sets were first proposed for two dimensions by Latecki *et al.* [9] and later extended to three dimensions by Latecki [10]. They do not incorporate the same partitioning

[1] Formal definitions of "connected" will be given in §2

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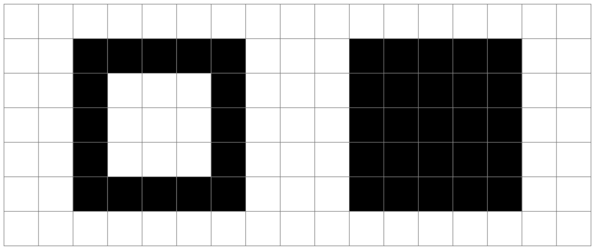

Figure 1. Two-dimensional cross-sections of the 3D surface voxelisation of a cube (left) and solid voxelisation of a block (right). The voxels themselves represent the surface in former, while the faces between black and white voxels do in the latter

restrictions on a set as separability does and ensure that the boundary between a voxel set and its complement is manifold. The use of well-composed sets is then more appropriate than separability if solid voxelisations are being considered or the manifold nature of the voxel set boundary is of vital importance.

It is often desirable to express a problem as a combination of many simpler problems, making it easier to solve [14, 2]. A common approach in computer graphics is to reduce a three-dimensional problem to a number of two-dimensional problems, or to analyse two-dimensional properties of a three-dimensional object [3, 5, 11, 16]. Cohen-Or and Kaufman [1] present a number of useful theorems for slicewise construction of separating sets that can be used to this effect. The theorems describe how a 3D separating set can be constructed from 2D slices containing 2D separating sets. It would be advantageous to have similar rules for the construction of well-composed sets, so that slicewise analysis can be applied when notions of well-composedness are favoured over separability.

In this paper theorems for slicewise construction of 3D well-composed sets from a set of 2D well-composed slices are presented. This makes it possible to ensure the well-composed structure of a 3D set by analysing its 2D slices. These slices can be considered analogous to the 2D contours or cross-sections of a 3D surface or object. A number of blockwise well-composed theorems are also presented. These allow us to combine a pair of 2D or 3D voxel sets containing well-composed subsets to obtain larger well-composed sets.

Before presenting the theorems, notions of voxel sets, separability, and well-composedness are reviewed and terminology used in the introduction is formalised (§2). The theorems for slice and blockwise well-composed sets are presented in sections 3 and 4, and concluding remarks are given in section 5.

2. Voxel Sets, Separability, and Well-Composedness

The terminology used in [1] and [10] will be adopted when discussing digital (voxel) sets and digital topology below. For a fundamental introduction to digital sets see [8].

The set of points with integer coordinates in 3D space $\mathbb{R}^3$ is known as $\mathbb{Z}^3$. A *voxel* is a closed axis-align unit cube centred at a point in $\mathbb{Z}^3$. The union of all voxels fills $\mathbb{R}^3$ space, with the interior of all voxels being disjoint. A 3D *digital (voxel) set* is the union of all points, or voxels centred at points, $x \in X : X \subseteq \mathbb{Z}^3$. Both 2D digital sets and 2D slices of 3D voxel sets can be described similarly. A 2D *slice* of a 3D voxel set is the set off all voxels with one constant coordinate component value. An l-slice, $l \in \{x, y, z\}$, is a slice perpendicular to the l axis, that is, each voxel in the slice has a constant l component. An l-slice $\mathcal{S}^l$ can be uniquely identified using $\mathcal{S}^{l_i}$, where $i \in \mathbb{Z}$ and equals the l component value of the slice.

Latecki [10] formally defines the correspondence between points in $\mathbb{Z}^3$ and cubes (voxels) in $\mathbb{R}^3$ with the function $CA : \mathscr{P}(\mathbb{Z}^3) \rightarrow \mathscr{P}(\mathbb{R}^3)$, the *continuous analog* of elements in $\mathbb{Z}^3$. For the point $p \in \mathbb{Z}^3$, $CA(p)$ is the closed unit cube centred at p with faces orthogonal to coordinate axis. For the digital set $X \subseteq \mathbb{Z}^3$, $CA(X) = \bigcup CA(x) : x \in X$. The *boundary* of $CA(X)$, denoted $bdCA(X)$, is the union of the set of closed faces that are common to a cube in $CA(X)$ and a cube in $CA(X^c)$, where $X^c = \mathbb{Z}^3 \setminus X$ is the complement of X.

In this paper only *binary voxel images* are consider, where a voxel can have either the value 1 (black) or 0 (white). When $X \subseteq Y$ is the set of all voxels in Y with value 1 we say $X_1 = X$ and $X_0 = X^c = Y \setminus X$, the relative complement of X in Y. We will assume this is the case for all our discussions unless otherwise specified. Notice that X^c is used in the paper to denote both absolute and relative complement, but the meaning should be clear within its context. The use of the term “voxel” is preferred herein over “digital” or “point”, so interpret $CA(x)$ of voxel x as the continuous analog of the point centred at x, similarly for $CA(X)$ and $bdCA(X)$. Thus, $CA(x)$ performs an identity operation on x.

Two voxels w and x are said to be *face-adjacent* if cubes $CA(w)$ and $CA(x)$ share a face, *edge-adjacent* if they share an edge only and *vertex-adjacent* if they share a vertex only. Voxels w, x are 6-*adjacent* if $CA(w)$ and $CA(x)$ share a common face, 18-*adjacent*

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if they share a common edge or face, and 26-*adjacent* if they share a common vertex, edge, or face. Thus 6-adjacent voxels are also 18- and 26-adjacent, while 18-adjacent voxels are also 26-adjacent. The N-*neighbourhood* of a voxel x is the set containing x and all voxels N-adjacent to x, where $N \in \{6, 18, 26\}$. An N-*path* in a set X is a sequence of voxels $S \subseteq X$ where consecutive pairs in S are N-adjacent. A set X is N-*connected* if $\forall w, x \in X$ an N-path exists in X between w and x. In 2D digital sets pixels can be edge- or vertex-adjacent (not face) and 4- or 8-adjacent. When considering a 2D slice of a 3D voxel set 6- and 18-adjacency is equivalent to 4- and 8-adjacency, while face- and edge-adjacency is equivalent to 2D edge- and vertex-adjacency respectively.

Cohen-Or and Kaufman [1] define a 3D voxel set X to be N-*separating* in Y if $Y \setminus X$ contains two disconnected components I and O (inside, outside), such that any N-path in Y from a voxel of I to a voxel of O meets X. Or equivalently, $I \cup O$ is not N-connected. In their work X is a surface voxelisation of surface S and its separability property is related to the segmenting property of S on continuous space. That is, I and O are considered the voxelisations of disjoint components of $CA(Y) - S$. One could use the same terminology, though, if X was any voxel set that segmented Y into more than two disconnected components. In three dimensions we have $N \in \{6, 18, 26\}$, in two dimensions $N \in \{4, 8\}$. Figure 2 shows an example of separating sets in two dimensions.

Latecki [10] defines a 3D voxel set X to be *well-composed* if $bdCA(X)$ is a 2-manifold. That is, X is well-composed *iff* the critical configurations (1) and (2) in figure 3 (modulo rotations and symmetries) do not occur in $CA(X)$ or $CA(X^c)$. The same applies for 2D voxels sets and critical configuration (3) [9]. Note that configuration (1) and (3) are identical minus one

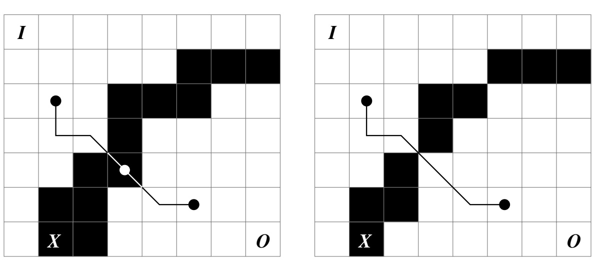
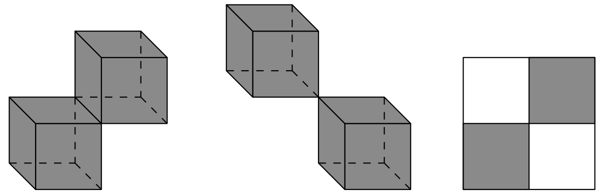


Figure 2. Two-dimensional separating sets. The set X on the left is 8-separating while the one on the right is 4-separating but not 8-separating. The 8-path meets a voxel of X on the left but not on the right



(1) (2) (3)

Figure 3. A 3D voxel set X_k is well-composed if the configurations (1) and (2) modulo rotations and symmetries do not occur in $CA(X_k)(k = 0, 1)$ [10]. A 2D voxel set is well-composed if the configuration (3) does not occur [9]

dimension. Configuration (1) will be named 18-critical, configuration (2) 26-critical, and 2D configuration (3) 8-critical. Cohen-Or and Kaufman [1] also name configuration (2) H^{26}. Following this convention configuration (1) and (3) will be named H^{18} and H^{8} similarly.

Definition 1. An *18(8)-critical configuration* ($H^{18(8)}$) is a 2×2 block of 3D(2D) voxels where "exactly two [cubes(squares) that] do not share a face[(edge)] are contained in $CA(X_k)$ and the other two are not contained in $CA(X_k)$" [10, p.166], where $k = 0, 1$.

Definition 2. A *26-critical configuration* (H^{26}) is a $2 \times 2 \times 2$ block of voxels where "exactly two [cubes that do not share a face or an edge] are contained in $CA(X_k)$ and the other six are not" [10, p.166], where $k = 0, 1$.

In 2D digital sets there is only one type of critical configuration when considering well-composedness [9]. In 3D, however, there are two. Thus, it is possible for a 3D voxel set to be void of one type of critical configuration but not necessarily the other. The following definition aids in describing these situations.

Definition 3. A *semi*^{N}*-well-composed* set X is a set of voxels that contains no H^N configurations in $CA(X)$ or $CA(X^c)$, where $N \in \{18, 26\}$.

It follows from the description of well-composedness above that if a set is both semi^{18}-well-composed and semi^{26}-well-composed it is well-composed.

Finally, it is obvious from the description of separating sets above that the concept is not well suited to describing solid voxelisations. This is because solids and their voxelisations may not partition $\mathbb{R}^3$ or $\mathbb{Z}^3$ into two or more disconnected components.

3. Slicewise Well-Composed Sets

Theorems for slicewise construction of well-composed sets, similar to those presented by Cohen-Or and Kaufman [1] for separating sets, will now be presented. It will be shown that any 3D well-composed set X consists entirely of 2D well-composed slices.

Theorem 1. *Let X be a set of voxels. If for every l-slice $\mathcal{S}^l$ such that $l \in \{x, y, z\}$, $X \cap \mathcal{S}^l$ is 2D well-composed, then X is semi^{18}-well-composed.*

Proof. Assume to the contrary that X is not semi^{18}-well-composed, and $CA(X)$ and/or $CA(X^c)$ contain a H^{18}. The four voxels that form this H^{18} have one coordinate component k with a common value i, thus, they must all belong to slice $\mathcal{S}^{k_i}$. The definition of a H^{18} and H^8 are identical minus one dimension. The H^{18} therefore appears in $\mathcal{S}^{k_i}$ as a H^8, contradicting the fact that all l-slices are 2D well-composed. Thus, X must be semi^{18}-well-composed. □

Theorem 2. *Let X be a semi^{18}-well-composed set of voxels. If no $2 \times 2 \times 2$ subsets of $X \cup X^c$ form a H^{26}, then X is 3D well-composed.*

Proof. Immediately from the definition of semi^{18}-well-composed, H^{26}, and well-composed. □

As explained in [1], the last theorem is not stated explicitly in terms of slices because a H^{26} cannot be characterised with respect to them (vertex-adjacent voxels do not belong to a common slice). We have then only one slice based theorem for well-composed sets as opposed to three in [1]. There, two theorems are given for 6-separability, while in well-composed sets 6-connectivity of X^c is not considered critical, hence fewer theorems.

4. Blockwise Well-Composed Sets

In addition to slice based construction we can also consider blockwise construction of well-composed sets. This allows us to take two 2D or 3D well-composed sets and combine them together to create a larger well-composed set. The following theorems present blockwise construction for a pair of well-composed sets with matching dimensions that share a specified set of common voxels. It should be fairly straight forward to develop more general theorems for blockwise construction, though. For example, to consider sets with different dimensions and different regions of shared voxels (overlap).

To simplify discussions in the theorems and proofs below the following definitions are given:

- The set $Y \subseteq \mathbb{Z}^{3(2)}$ with binary values is well-composed if $X_1 \subseteq Y$ is well-composed. This is considered reasonable as X_1 is well-composed *iff* X_0 is well-composed [10], and $X_1 \cup X_0 = Y$ (§2).
- For an $n \times m \times o$ subset Y of $\mathbb{Z}^3$, voxel $p \in Y$ is denoted $Y_{i,j,k}$. Here $(i, j, k) = (x-u+1, y-v+1, z-w+1)$, where (x, y, z) are the coordinates of p and (u, v, w) are the coordinates of voxel $q \in Y$ with the lowest value in all three coordinate components. Thus, q is denoted $Y_{1,1,1}$ and the voxel in Y with the highest value in all there coordinate components is denoted $Y_{n,m,o}$ (similarly for $\mathbb{Z}^2$).

The proofs of the theorems will be presented for one orientation in 2D or 3D, but can be expressed for others simply by changing the indices used over the sets. For example, using $A_{i,m} = B_{i,1}$ instead of $A_{n,i} = B_{1,i}$ for theorem 3.

Theorem 3. *Let A and B be $n \times m$ subsets of $\mathbb{Z}^2$ with binary values, and let $X_1 = X \subseteq A$ and $Y_1 = Y \subseteq B$ be well-composed sets. If $A_{n,i} = B_{1,i}$, where $i = 1, ..., m$, then the set $A \cup B$ is well-composed.*

Proof. If $A \cup B$ is *not* well-composed it must contain a H^8 configuration. Because X and Y are well-composed X^c and Y^c are too [10]. So individually A and B contain only well-composed like valued components, and can not contain the H^8. Therefore, for $A \cup B$ to contain a H^8 two like valued voxels, $a \in A$ and $b \in B$, must be 2D vertex-adjacent and have mutually common edge neighbours with a different value to that of a and b. Since

$$A_{n,i} = B_{1,i}, \forall i \in \{1, ..., m\},$$

and A and B share no other voxels, this pairing can only occur in the following cases, when:

$$a \in \bigcup_{i=1}^{m} A_{n-1,i} \wedge b \in \bigcup_{i=1}^{m} B_{1,i}$$

or

$$a \in \bigcup_{i=1}^{m} A_{n,i} \wedge b \in \bigcup_{i=1}^{m} B_{2,i}.$$

However, in the first case $a, b \in A$, since $B_{1,i} = A_{n,i}$, and in the second case $a, b \in B$, since $A_{n,i} = B_{1,i}, \forall i \in \{1, ..., m\}$ (fig.4). So any two voxels that could form a H^8 are either both in A or both in B. Because A and B are already well-composed no such neighbour relationship can exist. Thus, $A \cup B$ is well-composed. □

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In other words, because A and B share common voxels along one of their boundary columns, any voxel of A that meets a voxel exclusive to B is also in B (fig.4). The same holds for the inverse. Since A and B are already well-composed the H^8 can not exist. So $A \cup B$ is well-composed.

Theorem 4. *Let A and B be $n \times m \times o$ subsets of $\mathbb{Z}^3$ with binary values. If A and B contain no H^{18} configurations and $A_{n,i,j} = B_{1,i,j}$, where $i = 1, ..., m$ and $j = 1, ..., o$, then the set $A \cup B$ contains no H^{18} configurations.*

Proof. The proof of theorem 3 is restated for 3D. Individually A and B do not contain H^{18} configurations and are semi^{18}-well-composed by definition. So for $A \cup B$ to contain a H^{18} two like valued voxels, $a \in A$ and $b \in B$, must be 3D edge-adjacent and have mutually common face neighbours with a different value to that of a and b. Since

$$A_{n,i,j} = B_{1,i,j}, \forall i, j,\ i \in \{1, ..., m\}, j \in \{1, ..., o\},$$

and A and B share no other voxels, this pairing can only occur in the following cases, when:

$$a \in \bigcup_{i=1}^{m} \bigcup_{j=1}^{o} A_{n-1,i,j} \wedge b \in \bigcup_{i=1}^{m} \bigcup_{j=1}^{o} B_{1,i,j}$$

or

$$a \in \bigcup_{i=1}^{m} \bigcup_{j=1}^{o} A_{n,i,j} \wedge b \in \bigcup_{i=1}^{m} \bigcup_{j=1}^{o} B_{2,i,j}.$$

However, in the first case $a, b \in A$, since $B_{1,i,j} = A_{n,i,j}$, and in the second case $a, b \in B$, since $A_{n,i,j} = B_{1,i,j}, \forall i, j, i \in \{1, ..., m\}, j \in \{1, ..., o\}$. So any two voxels that could form a H^{18} are either both in A or both in B. Because A and B are already semi^{18}-well-composed no such neighbour relationship can exist. Thus, $A \cup B$ contains no H^{18} configurations. $\square$

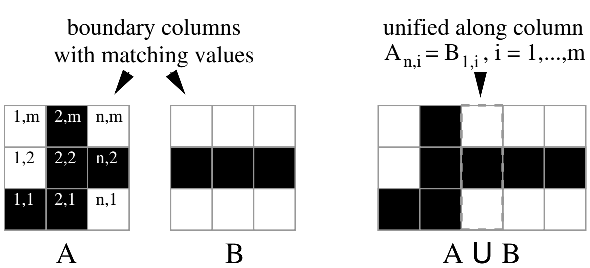

Figure 4. Support of theorem 3. Two $n \times m$ voxel sets unified along column with common values

Theorem 5. *Let A and B be $n \times m \times o$ subsets of $\mathbb{Z}^3$ with binary values. If A and B contain no H^{26} configurations and $A_{n,i,j} = B_{1,i,j}$, where $i = 1, ..., m$ and $j = 1, ..., o$, then the set $A \cup B$ contains no H^{26} configurations.*

Proof. Details of the proof are as in theorem 4. In summary, for $A \cup B$ to contain a H^{26}, a pair of like valued voxels, one from A and one from B, must be 3D vertex-adjacent while their common neighbours have values different to that of their own. However, any voxel of A that meets a voxel exclusive to B is also in B. The same holds for the inverse. Since A and B contain no H^{26} configurations individually, neither does $A \cup B$. $\square$

Theorem 6. *Let A and B be $n \times m \times o$ subsets of $\mathbb{Z}^3$ with binary values, and let $X_1 = X \subseteq A$ and $Y_1 = Y \subseteq B$ be well-composed sets. If $A_{n,i,j} = B_{1,i,j}$, where $i = 1, ..., m$, and $j = 1, ..., o$, then the set $A \cup B$ is well-composed.*

Proof. Because A and B are well-composed they will each contain no H^{18} or H^{26} configurations by definition. Thus, by theorem 4 $A \cup B$ will not contain any H^{18} configurations, and by theorem 5 it will not contain any H^{26} configurations. By definition $A \cup B$ is then well-composed, since it contains no H^{18} or H^{26} configurations. $\square$

Theorems 4, 5 and 6 only consider two subsets of $\mathbb{Z}^3$ that already shared common voxels on a slice of their boundaries. The theorems can be extended to handle disjoint subsets of $\mathbb{Z}^3$ by defining a translation operation on the subsets. Interpret the *translation* of a set Y by the triple $t = (t_x, t_y, t_z) \in \mathbb{Z}^3$, to be an operation that assigns the value of all voxels $v \in Y$ with coordinates (x, y, z) to the voxel w with coordinates $(x + t_x, y + t_y, z + t_z)$. After translation $Y = (Y - x) + w$ for all x, w pairs. Any two $n \times m \times o$ subsets A and B of $\mathbb{Z}^3$ can now be combined if

$$\lambda(A_{n,i,j}) = \lambda(B_{1,i,j}), \forall i, j,\ i \in \{1, ..., m\}, j \in \{1, ..., o\},$$

where $\lambda : \mathbb{Z}^3 \to \{0, 1\}$, as long as A or B is translated such that $A_{n,i,j} = B_{1,i,j}$. Similar reasoning can be extended to theorem 3 in 2D.

5. Conclusion

Theorems for slicewise construction of well-composed sets [9, 10] were presented that complement those previously presented for separating sets [1].

Describing 3D digital sets with respect to well-composedness instead of separability can be advantageous in a number of situations. Particularly when addressing solid voxelisations or when the manifold nature of the voxel set boundary is of primary interest. These theorems allow us to apply slicewise analysis using well-composed sets when notations of separability are inappropriate or difficult to apply. Owing to strong similarities between the properties of well-composed and separating sets it was possible to express proofs for the theorems in a similar manner to those of Cohen-Or and Kaufman [1].

The paper also presented theorems for blockwise construction of well-composed sets that allow 2D and 3D well-composed sets to be unified along boundaries with common values, producing larger well-composed sets. The theorems are currently restricted to "rectangular" subsets of $\mathbb{Z}^2$ or $\mathbb{Z}^3$ with common dimension and common boundary values on opposing boundary lines or slices. Further analysis is likely to produce more general theorems and could be an avenue for future work.

While voxel sets obtained through the process of voxelisation where emphasised in this paper the theories discussed here are applicable to volumetric data from a number of sources, such as real world scanners (e.g. medical imaging) or scientific simulation. As such, the theorems presented could be used in a wide range of applications. The use of well-composed sets in place of separating sets was also emphasised, however, there is no reason why a combination of the concepts could not be used. For example, in a situation where we were working with a surface voxelisations but also considered the manifold nature of the voxel set boundary important.

In ongoing related work these theorems have been applied in solving a real-world computer graphics problem. It was found that expressing the problem using well-composed set terminology, as opposed to separability, provided a more accurate and concise description of the solution that was easier to understand. In particular, definition and use of complicated "local" separability rules was avoided. Such rules added localised topological constraints to solid voxelisations, effectively describing the manifold structure of a voxel set boundary using the concept of separability. Finalisation of this related work is a primary focus of future work.

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