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SURVEY

Digital Topology: Introduction and Survey

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Digital topology deals with the topological properties of digital images; or, more generally, of discrete arrays in two or more dimensions. It provides the theoretical foundations for important image processing operations such as connected component labeling and counting, border following, contour filling, and thinning—and their generalizations to three- (or higher-) dimensional "images." This paper reviews the fundamental concepts of digital topology and surveys the major theoretical results in the field. A bibliography of almost 140 references is included. © 1989 Academic Press, Inc.

1. BASIC CONCEPTS

Digital topology is the study of the topological properties of image arrays. Its results provide a sound mathematical basis for image processing operations such as image thinning, border following, contour filling and object counting.

The image arrays considered in this paper are binary arrays, all of whose elements have value 0 or 1. However, some parts of digital topology can be generalized to *fuzzy digital topology*, which deals with unsegmented gray-scale image arrays whose elements lie in the range $0 \leq x \leq 1$—see [105, 109, 110]. A binary image array is typically obtained by thresholding a gray-scale image array.

For the most part we confine our attention to two- and three-dimensional binary image arrays, although some authors have considered higher dimensional arrays (e.g., [129, 134, 93]). The elements of a two-dimensional image array are called *pixels*; the elements of a three-dimensional image array are called *voxels*. To avoid having to consider the border of the image array we assume that the array is unbounded in all directions.

We associate each pixel or voxel with a lattice point (i.e., a point with integer coordinates) in the plane or in 3-space. Two lattice points in the plane are said to be 8-*adjacent* if they are distinct and each coordinate of one differs from the corresponding coordinate of the other by at most 1; two lattice points are 4-*adjacent* if they are 8-adjacent and differ in at most one of their coordinates. In 3-space two lattice points are said to be 26-*adjacent* if they are distinct and each coordinate of one differs from the corresponding coordinate of the other by at most 1; two lattice

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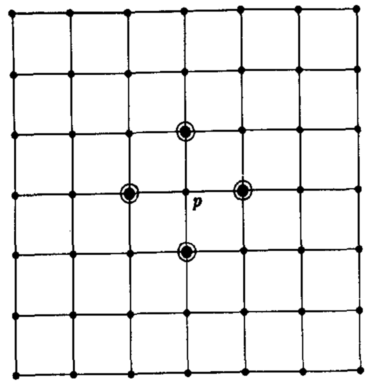

FIG. 1. The 4-neighbors of a point p.

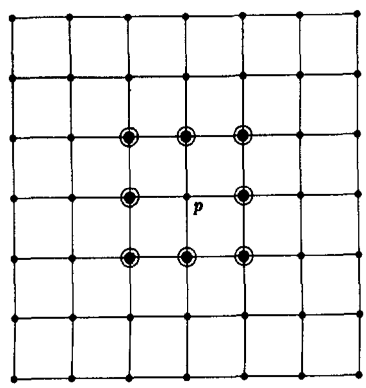

FIG. 2. The 8-neighbors of a point p.

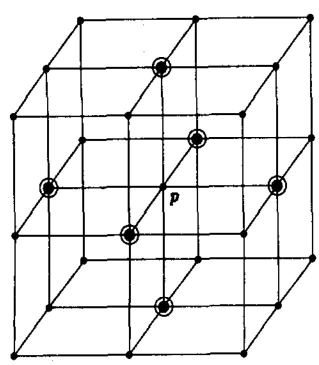

FIG. 3. The 6-neighbors of a point p.

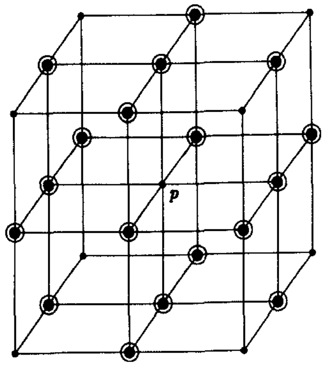

FIG. 4. The 18-neighbors of a point p.

points are *18-adjacent* if they are 26-adjacent and differ in at most two of their coordinates; two lattice points are *6-adjacent* if they are 26-adjacent and differ in at most one coordinate. For $n = 4, 8, 6, 18$, or 26 an *n-neighbor* of a lattice point p is a point that is n-adjacent to p. Figures 1–5 show the 4-, 8-, 6-, 18-, and 26-neighbors of a lattice point.

We say a set S of lattice points is *n-connected* if S cannot be partitioned into two subsets that are not n-adjacent to each other. An *n-component* of a set of lattice points S is a non-empty n-connected subset of S that is not n-adjacent to any other point in S. These concepts are illustrated in Fig. 6.

A 2-dimensional (3-dimensional) *unit cell* is a square (cube) with sides of length one whose corners are all lattice points. Occasionally we refer to 0-dimensional and 1-dimensional unit cells. A 1-dimensional unit cell is a straight line segment of length 1 joining two lattice points. A 0-dimensional unit cell is just a lattice point.

If p is a lattice point in the plane then $N(p)$ denotes the set consisting of p and its 8-neighbors. If p is a lattice point in 3-space then $N(p)$ denotes the set consisting of p and its 26-neighbors.

A lattice point associated with a pixel or voxel that has value 1 is called a *black point*; a lattice point associated with a pixel or voxel with value 0 is called a *white point*. In Figs. 7–20 and 23–28 black points and white points are shown as ● and ⊙,

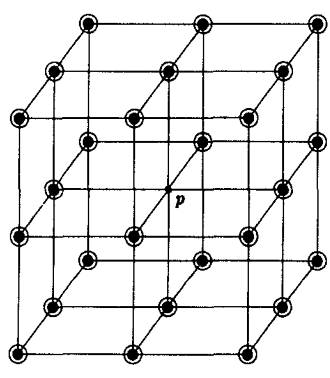

FIG. 5. The 26-neighbors of a point p.

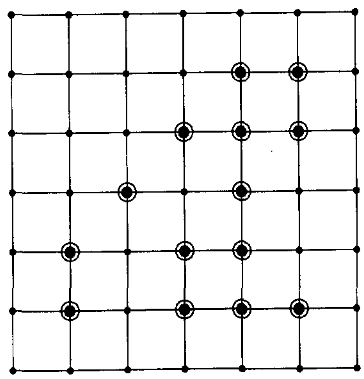

FIG. 6. An 8-connected set with exactly three 4-components.

respectively. Each figure shows only a finite subset of the set of all lattice points. In most cases it does not matter which of the other lattice points are white points and which are black points, but when it does matter we use the convention that all lattice points not shown in the figure are white points.

2. DIGITAL PICTURES

Arguably, the starting point of research on digital topology was the simple but important idea of using different adjacency relations for black and white points, a device which as far as we know was first recommended by Duda, Hart, and Munson [32]. The reason for this at first sight rather bizarre decision was to avoid paradoxes such as those pointed out in [115]: in Fig. 7 if 4-adjacency is used for all pairs of points then the black points are totally disconnected but still separate the set of white points into two components, while if 8-adjacency is used for all pairs of points then the black points form the discrete analog of a Jordan curve but they do not separate the white points.

The difficulty is resolved if we use 8-adjacency for the white points and 4-adjacency for the black, or vice versa. In three dimensions analogous paradoxes are similarly avoided if 6-adjacency is used for the white points and either 18- or 26-adjacency for the black, or vice versa.

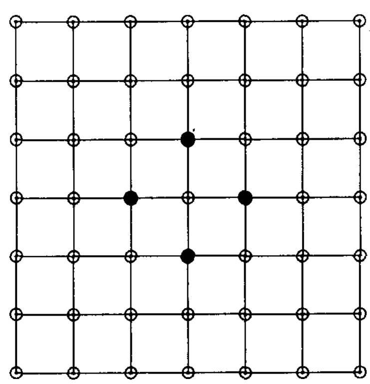

FIG. 7. Connectivity paradoxes.

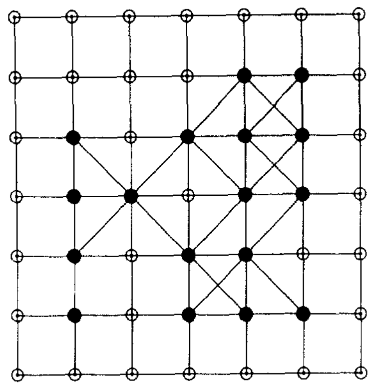

FIG. 8. The adjacencies in an (8, 4) digital picture. Note that the black point set is the same as in Fig. 9.

The idea of using different kinds of adjacency for black and white points motivates our definition of a digital picture. We write $\mathbb{Z}^2$ for the set of lattice points in the plane and $\mathbb{Z}^3$ for the set of lattice points in 3-space.

DEFINITION 2.1. A (conventional)[1] *digital picture* is a quadruple (V, m, n, B), where $V = \mathbb{Z}^2$ or $V = \mathbb{Z}^3$, $B \subseteq V$, and where $(m, n) = (4, 8)$ or $(8, 4)$ if $V = \mathbb{Z}^2$, and $(m, n) = (6, 26), (26, 6), (6, 18)$, or $(18, 6)$ if $V = \mathbb{Z}^3$.

The digital picture $\mathscr{P} = (V, m, n, B)$ is called *two-dimensional* or *three-dimensional* according as $V = \mathbb{Z}^2$ or $V = \mathbb{Z}^3$. The elements of V are called the *points* of the digital picture. The points in B are called the *black points* of the picture; the points in $V - B$ are called the *white points* of the picture. Usually B is a finite set; if so then $\mathscr{P}$ is said to be *finite*.

Two black points in a digital picture (V, m, n, B) are said to be *adjacent* if they are m-adjacent, and two white points or a white point and a black point are said to be *adjacent* if they are n-adjacent. A straight line segment that joins two adjacent points is called an *adjacency*.

A digital picture (V, m, n, B) will also be called an (m, n) *digital picture*. Figures 8 and 9 show all the adjacencies in an $(8, 4)$ and a $(4, 8)$ digital picture. Notice that the two digital pictures have the same black point set.

A point p is said to be *adjacent* to a set of points S if p is adjacent to some point in S. A set of points S is said to be *adjacent* to a set of points T if some point in S is adjacent to some point in T.

We say a set S of black and/or white points in a digital picture is *connected* if S cannot be partitioned into two subsets that are not adjacent to each other. A *component* of a set of black and/or white points S is a non-empty connected subset of S which is not adjacent to any other point in S. Thus in an (m, n) digital picture a component of a set of black points is an m-component, whereas a component of a set of white points is an n-component. See Fig. 20 below.

A component of the set of all black points of a digital picture is called a *black component*, and a component of the set of all white points is called a *white*

[1]Other kinds of digital picture are described in Section 11.

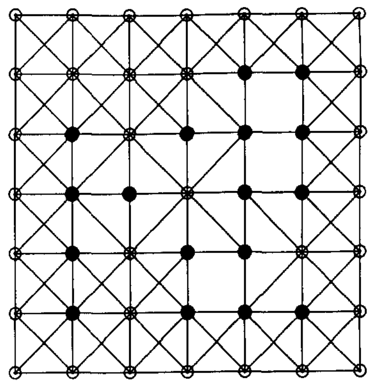

FIG. 9. The adjacencies in a (4, 8) digital picture. Note that the black point set is the same as in Fig. 8.

component. In a finite digital picture there is a unique infinite white component, which is called the *background*.

For any set of points S a *path in* S is a sequence $\langle p_i | 0 \leq i \leq n \rangle$ of points in S such that p_i is adjacent to p_{i+1} for all $0 \leq i < n$. A path $\langle p_i | 0 \leq i \leq n \rangle$ is said to be a path *from* p_0 *to* p_n, and is said to be a *closed* path if $p_n = p_0$. A degenerate one-point path $\langle p_0 \rangle$ is a special case of a closed path. Clearly, a set of points S is connected if and only if for any two points p and q in S there is a path in S from p to q.

If X and Y are sets of points in a digital picture (V, m, n, B) and X is connected then we say X *surrounds* Y if each point in Y is contained in a finite component of $V - X$. It is easy to verify that "X surrounds Y" is an asymmetric and transitive relation—i.e., a partial order—on the connected subsets of V. In a finite digital picture the background surrounds the set of all black points.

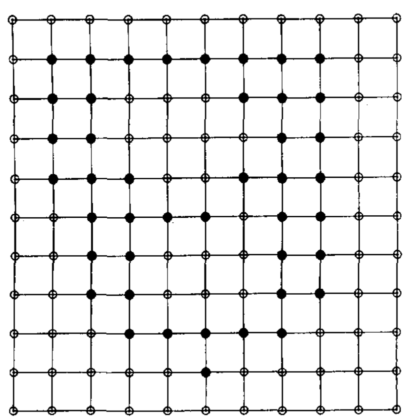

FIG. 10. The black component shown here has two holes if it is regarded as part of an (8, 4) digital picture. The black component has just one hole if it is regarded as part of a (4, 8) digital picture.

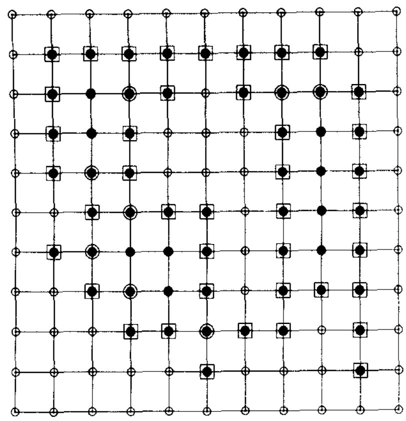

FIG. 11. In an (8, 4) digital picture the boxed points ▣ would be border points; in a (4, 8) digital picture both the boxed points ▣ and the ringed points ◉ would be border points.

In a digital picture $\mathscr{P}$, a white component that is adjacent to and surrounded by a black component C is called a *hole* in C if $\mathscr{P}$ is a two-dimensional digital picture, and a *cavity* in C if $\mathscr{P}$ is a three-dimensional digital picture. By a *hole of* $\mathscr{P}$ (*cavity of* $\mathscr{P}$) we mean a hole (cavity) in some black component of $\mathscr{P}$. Figure 10 illustrates the concept of a hole.

A black component of a digital picture $\mathscr{P}$ normally corresponds to an object, a hole or cavity of $\mathscr{P}$ to a hole or cavity in an object. So component labeling and counting are important operations. This is an active research area (e.g., [86, 118, 73, 127, 99, 48, 117, 133, 2]).

A black point is said to be *isolated* if it is not adjacent to any other black point. See Fig. 13. A black point is called a *border point* if it is adjacent to one or more white points; otherwise it is called an *interior* point. The *border* (*interior*) of a black component C of $\mathscr{P}$ is the set of all border points (resp., all interior points) in C. The

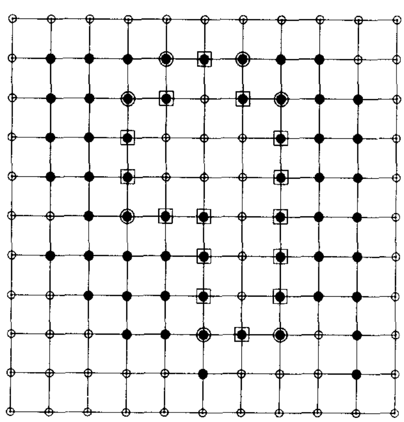

FIG. 12. Same notation as in Fig. 11, except that only those border points which belong to the border of the black component with respect to its (unique) hole are marked.

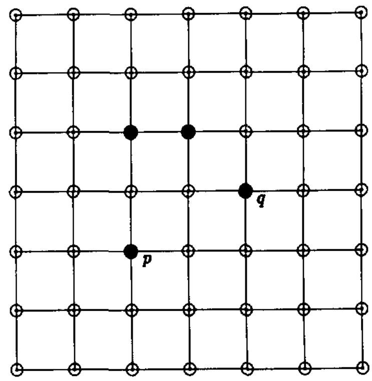

FIG. 13. In a (4, 8) digital picture both p and q would be isolated points. In an (8, 4) digital picture p would be an isolated point but q would not.

border of a black component C *with respect to* a white component D is the set of points in C that are adjacent to D. Recall that in this context *adjacent* means n-adjacent if $\mathscr{P}$ is an (m, n) digital picture. See Figs. 11 and 12.

A *simple closed black curve* in a digital picture is a connected set of black points each of which is adjacent to exactly two other points in the set. Figures 14 and 15 show simple closed black curves in an (8,4) and a (4,8) digital picture. A *simple black arc* is a connected set of black points each of which is adjacent to just two other points in the set, with the exception of two points—the *end points* of the arc—that are each adjacent to just one other point in the set. If we remove any point from a simple closed black curve then the remaining points will form a simple black arc.

Let $\mathscr{P} = (V, m, n, B)$ and $\mathscr{P}' = (V, m, n, B - D)$ be digital pictures, where $D \subseteq B$. Then we say that $\mathscr{P}'$ is obtained from $\mathscr{P}$ by *deleting* the points in D. Alternatively, we may say that $\mathscr{P}$ is obtained from $\mathscr{P}'$ by *adding* the points in D. In the next section we shall discuss a class of point deletion algorithms that are much used in image processing.

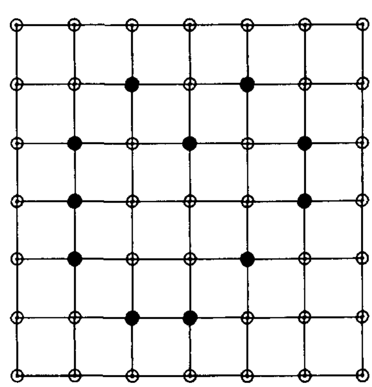

FIG. 14. A simple closed black curve in an (8, 4) digital picture.

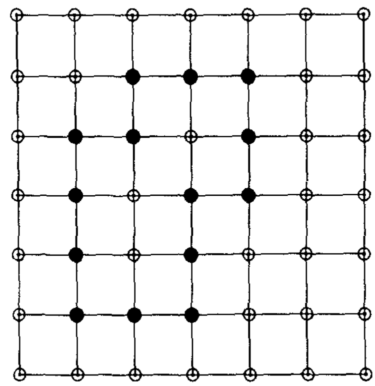

FIG. 15. A simple closed black curve in a (4, 8) digital picture.

3. TWO-DIMENSIONAL THINNING

Image thinning is a common pre-processing operation in pattern recognition. Its goal is to reduce the set of black points to a "skeleton" in a "topology-preserving" way. Figure 16 shows the effect of a thinning algorithm on an (8, 4) digital picture; the boxed black points ▣ are the black points of the output digital picture.

The medial axis transformation proposed by Blum [20] is an analog of thinning for general shapes in the Euclidean plane. Indeed, the output of a thinning algorithm is sometimes called a medial line, as in [10, 16]. But thinning is not the same as the discrete medial axis transformation, a different digital approximation of Blum's transformation which does not preserve topology. Some authors have

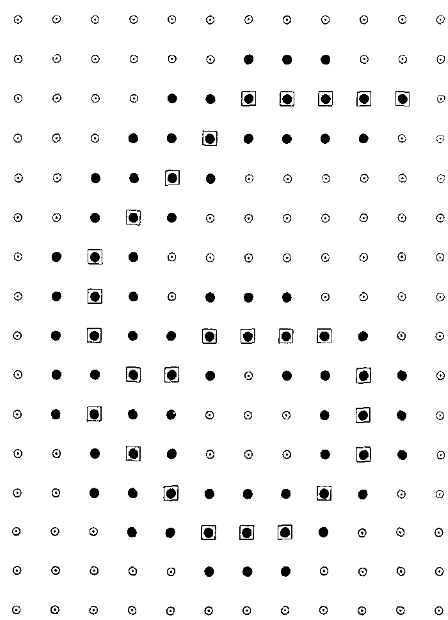

FIG. 16. Effect of a thinning algorithm on an (8, 4) digital picture.

incorporated elements of the discrete medial axis transformation into their thinning algorithms (e.g., [9, 27, 131]).

A non-topological requirement of a thinning algorithm is that each elongated part of the input black point set should be represented by a black digital arc in the output skeleton. An algorithm which does not meet this condition, but merely deletes black points while preserving the topology of the image, is called a *shrinking* algorithm. (Note that this notion of shrinking is different from that of [70, 95], in which holes are not preserved. Our usage is that of [101, 90].) The difference between shrinking and thinning can be seen from the fact that a digital picture whose black point set is a simple black arc ought to be left unchanged by an ideal thinning algorithm, although a shrinking algorithm could reduce the black point set of such a digital picture to a single point. However, a digital picture whose black point set is a simple closed black curve that is not contained in a single unit cell is invariant under shrinking, since deletion of any black point would eliminate the hole.

In discussing her algorithm for thinning (8, 4) digital pictures, Hilditch [43] used the following notion of topology preservation. She identified each black point p with a closed unit square whose sides are parallel to the coordinate axes and whose center is at p, and considered a point deletion algorithm to preserve *connectivity* if the black point set of the input digital picture could always be continuously deformed[2] into the black point set of the output digital picture (each black point set being regarded as a union of unit squares).

In [123], Stefanelli and Rosenfeld verified that their parallel thinning algorithm did indeed preserve topology by showing that in deleting black points the algorithm would never cause a black component to vanish completely, that it would never split a black component into two or more components, that it would never create a new white component, and that it would never merge two white components. In effect they were using the following criterion:

CRITERION 3.1. Let $\mathscr{P} = (\mathbb{Z}^2, m, n, B)$ be a two-dimensional digital picture. Then deletion of the points in a subset D of B preserves topology if and only if

1. each black component of $\mathscr{P}$ contains exactly one black component of $\mathscr{P}'$, and

2. each white component of $\mathscr{P}'$ contains exactly one white component of $\mathscr{P}$,

where $\mathscr{P}'$ is the digital picture $(\mathbb{Z}^2, m, n, B - D)$.

An equivalent criterion was explicitly stated by Ronse in [98]. For (8, 4) digital pictures Criterion 3.1 is equivalent to Hilditch's concept of connectivity preservation. Criterion 3.1 is appropriate for (4, 8) digital pictures too. However, it is usual to thin a binary image as an (8, 4) digital picture rather than as a (4, 8) digital picture, because thinner skeletons are then obtained. In fact, Criterion 3.1 is also applicable to well-behaved digital pictures on non-orthogonal two-dimensional grids (see Section 11). But the criterion is only valid in two dimensions.

The subtle non-topological aspects of the theory of image thinning are beyond the scope of the present paper. More general surveys of the literature on two-dimen-

[2] In this context an appropriate concept of continuous deformation is deformation retraction, which is precisely defined in textbooks on algebraic or geometric topology (e.g., [45, p. 31]).

sional thinning are to be found in [126, 27, 84]. See also [44]. For the thinning of gray-scale—as opposed to binary—images see [43, 34, 44].

4. SIMPLE POINTS IN 2D DIGITAL PICTURES

A black point p in a two-dimensional digital picture is called a *simple* point if its deletion preserves topology in the sense of Criterion 3.1. The notion of a simple point is important in the theory of image shrinking and thinning, as will be seen in Section 5.

By Criterion 3.1, p is a simple point of an (m, n) digital picture (where $(m, n) = (4, 8)$ or $(8, 4)$) if and only if both the number of black components and the number of white components stay the same when p is deleted. In particular, only non-isolated border points can be simple. Figure 17 illustrates the concept of a simple point.

Rosenfeld [100, Section 3; 101, Proposition 4.1] and Mylopoulos and Pavlidis [83, Proposition 2] established useful characterizations of simple points which we now state.

THEOREM 4.1. *Let p be a non-isolated border point in an $(8, 4)$ or $(4, 8)$ digital picture. Let B be the black point set of the digital picture and let $B' = B - \{p\}$. Then the following are equivalent*:

1. *p is a simple point.*
2. *p is adjacent to just one component of $N(p) \cap B'$.*
3. *p is adjacent to just one component of $N(p) - B$.*

This theorem shows that one need only examine the 3 by 3 neighborhood $N(p)$ to determine whether or not a point p is simple. Another easy consequence of Theorem 4.1 is that a point p is a simple point of a digital picture $(\mathbb{Z}^2, m, n, B)$ if and only if p is a simple point of the digital picture $(\mathbb{Z}^2, n, m, (\mathbb{Z}^2 - B) \cup \{p\})$. The latter digital picture is obtained from the former by swapping the black and white point sets and their associated adjacency relations, except that p itself remains a black point.

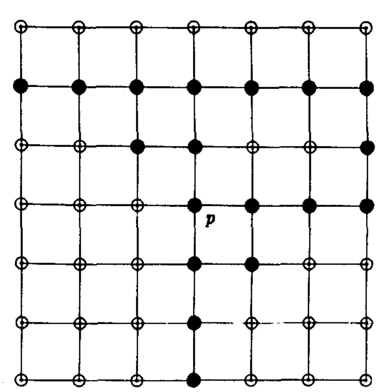

FIG. 17. In an (8, 4) digital picture the point p would be a simple point. In a (4, 8) digial picture the point p would not be simple, because deletion of p would merge two white components and so violate condition 2 of Critrion 3.1.

Gray [38] gave another characterization of simple points in (4, 8) and (8, 4) digital pictures—see Section 7.

Let $\mathscr{P}_0, \mathscr{P}_1, \ldots, \mathscr{P}_n$ be a sequence of digital pictures. If for each $1 \leq i < n$ the picture $\mathscr{P}_{i+1}$ is obtained by deleting a simple point of $\mathscr{P}_i$ from $\mathscr{P}_i$ then we say that $\mathscr{P}_n$ is obtainable from $\mathscr{P}_0$ (or that $\mathscr{P}_0$ can be transformed into $\mathscr{P}_n$) by *sequential deletion* of simple points. (Note that a simple point of $\mathscr{P}_i$ need not be a simple point of $\mathscr{P}_0$; conversely a black point of $\mathscr{P}_i$ that is a simple point of $\mathscr{P}_0$ need not be a simple point of $\mathscr{P}_i$.) If for each $1 \leq i < n$ the picture $\mathscr{P}_{i+1}$ is obtained either by deleting a simple point of $\mathscr{P}_i$ from $\mathscr{P}_i$ or by adding a simple point of $\mathscr{P}_{i+1}$ to $\mathscr{P}_i$ then we say that $\mathscr{P}_n$ is obtainable from $\mathscr{P}_0$ by *sequential addition and deletion* of simple points.

Rosenfeld showed in [100] that any finite (4, 8) or (8, 4) digital picture whose black point set is non-empty and connected and has no holes can be transformed by sequential deletion of simple points to a digital picture with just one black point. In [101] Rosenfeld further established that a finite (4, 8) digital picture whose black point set is connected and has just one hole can be transformed by sequential deletion of simple points to a digital picture whose black point set is a simple closed black curve.

Ronse showed in [98] that for finite (4, 8) and (8, 4) digital pictures topology preservation in the sense of Criterion 3.1 is equivalent to the condition that $\mathscr{P}'$ be obtainable from $\mathscr{P}$ by sequential deletion of simple points.

Alexander and Thaler proved in [5] that in any finite (4, 8) digital picture sequential deletion of simple points will eventually produce a digital picture whose black point set does not contain any 2 by 3 arrays of black points.

Mylopoulos and Pavlidis showed [83] that given two finite (4, 8) digital pictures whose black point sets are connected, and which have the same number of holes, it is possible to transform one to the other by sequential addition and deletion of simple points.

An alternative approach to the notion of a simple point was taken in [43]. Here a *crossing number* $X(p)$ was defined to be the number of times one crosses over from a white point to a black point when the 8-neighbors of p are visited in cyclic order, starting at a 4-neighbor of p and returning to the starting point, always cutting the corner between 8-adjacent black 4-neighbors of p. In a digital picture $(\mathbb{Z}^2, 8, 4, B)$ the Hilditch crossing number $X(p)$ is equal to the number of components of $B \cap N(p) - \{p\}$, except that if p is an interior point then $X(p) = 0$. So it follows from Theorem 4.1 that in an (8, 4) digital picture a black point p is simple if and only if $X(p) = 1$. Hilditch's crossing number $X(p)$ is equal to the connectivity number $N_c^8(p)$ defined by Yokoi *et al.* [138], where the analogous quantity $N_c^4(p)$ for (4, 8) digital pictures was also defined.

A precursor of Hilditch's crossing number was proposed by Rutovitz in [116]. This was defined in the same way as $X(p)$, except that it counted crossings from a black point to a white point as well as crossings from a white point to a black point, and the neighbors of p were visited without cutting the corner between 8-adjacent black 4-neighbors of p. The Rutovitz crossing number of a point p in a digital picture $(\mathbb{Z}^2, 8, 4, B)$ is equal to twice the number of 4-components of $B \cap N(p) - \{p\}$, unless every point in $N(p) - \{p\}$ is a black point in which case it is equal to zero. Thus in an (8, 4) digital picture if the Rutovitz crossing number of a border point p is 2 then p is simple (by Theorem 4.1).

But an (8, 4) digital picture in which no black point has Rutovitz crossing number equal to 2 may still contain diagonal segments that are two black points thick. So a thinning algorithm for (8, 4) digital pictures will produce the thinnest possible skeletons only if it deletes some simple points whose Rutovitz crossing number is greater than 2. This observation is essentially due to Deutsch [29].

Rutovitz's crossing number is of some historical interest as "the first important measure of connectivity which produced a considerable impact on succeeding thinning algorithms..." [126]. Indeed, it has been used directly or indirectly in the point deletion criteria of a number of thinning algorithms over the years (see [30, 31, 10, 11, 87, 139, 72, 46, 39]).

5. RESULTS ON 2D PARALLEL THINNING

It is generally quite tricky to prove that a proposed parallel thinning algorithm satisfies Criterion 3.1. For an example of such a proof see [123].

Unfortunately such proofs often have to be done from first principles, as there are still very few general theorems giving sufficient conditions for an algorithm to satisfy Criterion 3.1. In this section we discuss general theorems of this kind.

A black point with coordinates (x, y) is said to be a *north border point* if the point $(x, y + 1)$ is a white point. An *end point* of a two-dimensional digital picture is a black point that is adjacent to just one other black point.

THEOREM 5.1. *Let P be an* (8, 4) *or a* (4, 8) *digital picture. Then (parallel) deletion of any number of simple north border non-end points of* $\mathcal{P}$ *preserves topology in the sense of Criterion* 3.1.

For (8, 4) digital pictures this theorem is essentially the same as the second author's main result in [103].

In Fig. 18 all the simple north border non-end points are ringed; it is readily confirmed that deletion of any number of those points preserves topology, as asserted by Theorem 5.1.

Theorem 5.1 obviously remains valid if "east," "south," or "west" is substituted for "north." However, the restriction that border points are deleted from just one side (north) is necessary. To see this, consider the (8, 4) digital picture shown in Fig.

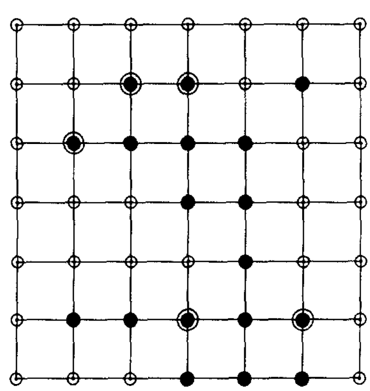

FIG. 18. Simple north-border non-end points in an (8, 4) digital picture.

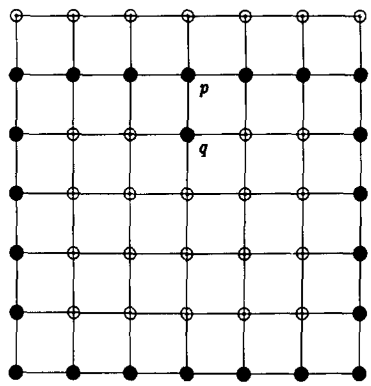

FIG. 19. Simple non-end points in an (8, 4) digital picture whose parallel deletion will merge two white components.

19. Here p is a north border point and q is a west border point. Plainly both p and q are simple. But deletion of both points will merge two white components.

Theorem 5.1 is applicable to algorithms which delete points in parallel from each side in turn (e.g., in the order N, S, E, W). Such algorithms have been called *border sequential* [44]. A large proportion of parallel thinning algorithms work in this way, because of the need to deal correctly with 2 by n rectangles, or, more generally, with portions of the black point set which are just two pixels thick.

Theorem 5.1 can also be applied to so-called *border parallel* algorithms [44]. These algorithms are built up of passes each consisting of four subcycles: in each subcycle points are removed from just one side, but a point p is deleted only if the point deletion criterion is satisfied both by p's current 3 by 3 neighborhood $N(p)$ and by $N(p)$ as it was at the start of the current pass.

Theorem 5.1 has a partial converse. To state this, call a predicate on black points a *3 by 3 predicate* if its truth or falsity for a black point p in a digital picture depends only on the black/white point configuration in p's 3 by 3 neighborhood $N(p)$.[3] (Thus "is a north border point" is a 3 by 3 predicate, but "is the black point with greatest x coordinate" is not. The predicate "is a simple point" is a 3 by 3 predicate, by Theorem 4.1.)

Now consider the operation of deleting (in parallel) all black points of a digital picture that satisfy some specific 3 by 3 predicate P. It is quite easy to show that if the predicate P is satisfiable by a non-simple point of some (8, 4) digital picture then there is an (8, 4) digital picture for which the operation fails to preserve topology in the sense of Criterion 3.1. If the predicate P is satisfiable by an end point of an (8, 4) digital picture then not every (8, 4) digital picture whose black point set is a simple black arc will be invariant under the operation.

So if the operation is to preserve the topology of all (8, 4) digital pictures in the sense of Criterion 3.1 and is to leave unchanged every (8, 4) digital picture whose

[3]One fast way of determining whether a black point p satisfies a given 3 by 3 predicate is by table look-up, using a look-up table of size $2^8 = 256$ indexed by 8-bit codes that represent all possible black/white point configurations of the 8-neighbors of a point. See [120].

black point set is a simple black arc, then the point deletion predicate P must be satisfiable only by simple non-end points. The same result holds for (4, 8) digital pictures. This is a partial converse of Theorem 5.1.

Morgenthaler [80] worked on the problem of generalizing Theorem 5.1 to three dimensions, but further research is needed on this. (See Sections 9 and 10 for further discussion of this topic.)

Arcelli showed in [6] that a non-isolated north-border black non-end point in an (8, 4) digital picture is simple if and only if

$$W \cdot \neg S \cdot E + \neg W \cdot NW \cdot \neg N + \neg N \cdot NE \cdot \neg E + \neg E \cdot SE \cdot \neg S + \neg S \cdot SW \cdot \neg W = 0.$$

Here "+," "·," and "¬" denote boolean disjunction, conjunction, and negation, respectively; $W = 0$ or 1 according as the west neighbor of p is white or black, and similarly for the other compass directions (SW refers to the south-west neighbor, etc.). In fact $\neg N \equiv 1$, since this rule applies only to north border points.

A result due to Tsao and Fu [131] is related to Theorem 5.1, but is applicable to algorithms that delete points from more than one side in parallel. For all black points p let $B(p)$ denote the set of non-simple black points in $N(p)$.

THEOREM 5.2. *Let P be an (m, n) digital picture, where $(m, n) = (4, 8)$ or $(8, 4)$, and let R be a set of simple black points of P such that the following conditions hold for all p in R*:

1. *$B(p)$ contains at least two points and is connected, and*
2. *$B(p)$ is adjacent to p and to every simple black point that is adjacent to p.*

Then deletion of the points in R preserves topology (in the sense of Criterion 3.1).

In [131] Tsao and Fu proposed repeated parallel deletion of all simple black points satisfying conditions 1 and 2 of Theorem 5.2 (until there are no such points) as a parallel thinning algorithm. This algorithm is isotropic, but not particularly fast because conditions 1 and 2 cannot be checked rapidly. Note that neither condition 1 nor condition 2 is a 3 by 3 predicate: it is impossible to determine which points are in $B(p)$ from the 3 by 3 neighborhood of p. The algorithm leaves 2 by n rectangles unchanged; it is plain that no isotropic thinning algorithm can do better.

A result established by Rao *et al.* [94] is also reminiscent of Rosenfeld's theorem. To state this result, define a *template* to be any 3 by 3 array of black, white, or gray points, in which the central point is black. Each template defines a 3 by 3 predicate: a black point p satisfies the predicate if and only if the template matches $N(p)$, where black points match black points, white points match white points and a gray point in the template is a "don't care" which matches either a white point or a black point in $N(p)$. No reflection or rotation is allowed in matching the template to $N(p)$. Call a 3 by 3 predicate a *single-template predicate* if it is the predicate defined by some template.

THEOREM 5.3. *An algorithm which deletes in parallel all black points that satisfy a given single-template predicate preserves the topology of all (8, 4) digital pictures if and only if the predicate is such that every point which satisfies it is a simple point.*

Proof. By Corollary 1 and Theorem 2 in [94]. □

Reference [94] also gave a boolean expression (rather like Arcelli's expression given above) for determining whether or not a single-template predicate is satisfied only by simple points.

In a recent article Hall [39] established two theorems which give sufficient conditions for any member of a large class of parallel thinning algorithms on $(8,4)$ digital pictures to satisfy Criterion 3.1. The class of parallel thinning algorithms in question have the property that all points deleted in each iteration are non-end border points with Rutovitz crossing number (see Section 4) equal to 2. This class includes the algorithms considered in [46, 72, 139].

6. ADJACENCY TREES, BORDER TRACKING, AND JORDAN CURVE THEOREMS

Border and boundary following algorithms have applications in object detection and display [99, 13, 134], and image thinning [90, 7, 136, 67].

Rosenfeld showed in [100] that in a finite $(4,8)$ digital picture with one black component and no holes the set of border points is 8-connected and can be tracked by a border following algorithm—but the border points were defined to be the black points that are 4-adjacent to a white point.

It is now more usual to define border points as we did in Section 2. Thus in a $(4,8)$ digital picture the border of a black component C with respect to a white component D is the set of points in C that are 8-adjacent to D, rather than 4-adjacent to D as in [100].

The following theorem is a fundamental result on the connectedness of borders:

THEOREM 6.1. *In a two-dimensional* $(4,8)$ *or* $(8,4)$ *digital picture the border of a black component C with respect to a white component D is connected.*

Proof. See [106, Chap. 2; 61; 63]. The last two references contain substantial generalizations of this result to digital pictures on non-orthogonal lattices and to three-dimensional digital pictures. □

Figure 12 illustrates the validity of this theorem. The theorem shows that border following algorithms (such as the one in [113]) will indeed find *all* points on the border of a black component with respect to a given white component.

Alexander and Thaler [5] invented the notion of a *boundary curve* in a $(4,8)$ digital picture, defined to be a (4-connected) sequence of border points visited by a precisely specified "left-hand-on-wall" border following algorithm. Different boundary curves can be detected by starting the algorithm off from different pairs of 8-adjacent black and white points, but two boundary curves are considered to be the same if one is a cyclic permutation of the other.

The notion of a boundary curve as a *sequence* of black points should be contrasted with the concept of a border, which is a *set* of black points. Thus if the black points form a simple closed black curve C then there are two distinct boundary curves, one going clockwise around C, the other going anti-clockwise around C. On the other hand if the black points are arranged in an n by n square then there is just one boundary curve—this goes anti-clockwise around the border because the tracking algorithm is "left-hand-on-wall."

An ordered pair (p, q) where p is a black point and q is a white 4-neighbor of p is called an *edge* [100], because it can be identified with the common edge of the unit squares centered at p and q. In three dimensions, if q is a white 6-neighbor of the black point p then (p, q) is appropriately called a *face* (cf. [13]).

An approach to boundaries which is closely related to that of Alexander and Thaler but is more readily generalized to higher dimensions defines the *boundary* between a black component X and an adjacent white component Y in a two-dimensional (three-dimensional) digital picture to be the set of all edges (faces) (p, q) such that p is in X and q is in Y. In the two-dimensional case the boundary between a black component and an adjacent white component is a *bicurve* in the sense of [102].

Artzy, Frieder, and Herman [13] presented an efficient algorithm for tracking such boundaries in three-dimensional (18, 6) digital pictures. Herman and Webster proved that the algorithm is correct in [42]. The algorithm was used in medical image processing software to produce displays of isolated internal organs from three-dimensional computed tomographic data.

Udupa, Srihari, and Herman [134] gave a general boundary detection algorithm that can be used in any number of dimensions. The four-dimensional instance of their algorithm can be used in the detection of dynamically moving surfaces, as explained in [41].

Buneman [22] introduced the *adjacency tree* of a digital picture. This is a graph each of whose vertices represents a distinct black or white component of the digital picture, where two vertices are joined by an edge if and only if the corresponding components are adjacent to each other. References [102; 61; 63] contain proofs that the graph is indeed always a tree, in the 2D and the 3D cases, respectively.

Let $\mathscr{P}$ be an arbitrary finite digital picture. Then the adjacency tree of $\mathscr{P}$ has a natural root vertex, namely the vertex which represents the background component. It is easy to see that a (black or white) component X of $\mathscr{P}$ surrounds another (black or white) component Y of $\mathscr{P}$ if and only if in $\mathscr{P}$'s adjacency tree a path from the vertex that represents Y to the root vertex must pass through the vertex that represents X. Thus if a white component of $\mathscr{P}$ is adjacent to a black component of $\mathscr{P}$ then one of the components must surround the other. If we orient the edges of the adjacency tree of $\mathscr{P}$ so that each edge points away from the root then we get the covering graph (also called the Hasse diagram) of the partial order "X surrounds Y" on the set of all black and white components.

Figure 20 can be interpreted as an (8, 4) or a (4, 8) digital picture. The adjacency trees of those digital pictures are shown in Figs. 21 and 22. In Figs. 21 and 22 the black vertices ● represent black components and the white vertices ○ represent white components. In both cases the topmost vertex is the root of the tree.

For a typical application of the adjacency tree see [137]. In [125] Suzuki and Abe described an algorithm, based on border following, that constructs the adjacency tree of a digital picture. In [15] Atkinson *et al.* presented and analyzed an algorithm that inputs a (4, 8) digital picture represented as a linear quadtree and constructs a simplified version of the adjacency tree, namely the covering graph of the relation "X surrounds Y" on the set of all black components (but not white components). This is an acyclic graph, but in general it is not connected and so it is not a tree.

One drawback of the adjacency tree of a 3D digital picture is that it contains no information on the "tunnels" in the picture. Thus a 3D digital picture whose black

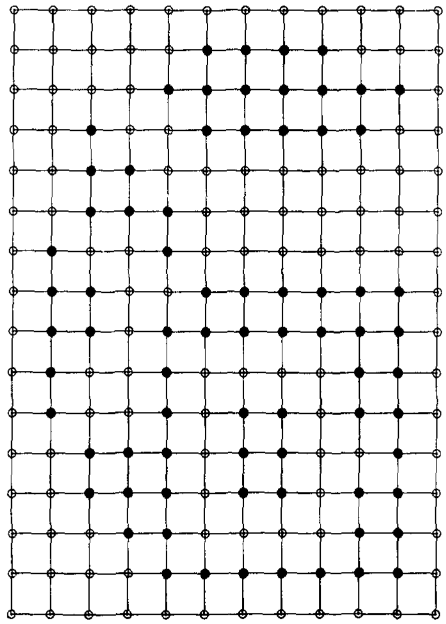

FIG. 20. As an (8, 4) digital picture this has 3 black components and 3 white components. As a (4, 8) digital picture this has 5 black components and 2 white components. See Figs. 21 and 22.

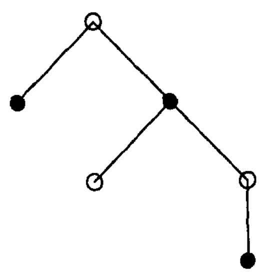

FIG. 21. The adjacency tree of the (8, 4) digital picture given by Fig. 20. The topmost vertex represents the background.

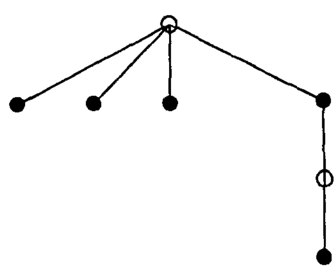

FIG. 22. The adjacency tree of the (4, 8) digital picture given by Fig. 20. The topmost vertex represents the background.

point set is a digitized solid torus has the same adjacency tree as a digital picture whose black point set is a digitized solid ball. Now a 3D digital picture $\mathscr{P}$ may be regarded as a collection of 2D digital pictures $\mathscr{P}_i$ $(i = \ldots -2, -1, 0, 1, 2, 3 \ldots)$, where $\mathscr{P}_i$ represents the "slice" of $\mathscr{P}$ whose points have z-coordinate equal to i. Let T_i be the adjacency tree of $\mathscr{P}_i$. Say that a vertex of T_j is *compatible* with a vertex of T_k (here j may be equal to k) if the two vertices represent points that are in the same black or white component of the 3D digital picture $\mathscr{P}$. Compatibility is plainly an equivalence relation. From the list of trees T_i and the equivalence classes of compatible vertices one can obtain all the topological information about $\mathscr{P}$ that is provided by $\mathscr{P}$'s adjacency tree, and also much information on the tunnels in $\mathscr{P}$. This representation of the topological structure of a 3D digital picture was described by Srihari in [121]. It is not a purely topological representation, as the T_i are not invariant under topology-preserving operations such as thinning. When a thinning algorithm is applied to $\mathscr{P}$ it may for example delete all the black points in one of the $\mathscr{P}_i$, which would leave the corresponding T_i with just one vertex.

The fact that the adjacency tree is indeed a tree is closely related to the above theorem on the connectedness of borders (Theorem 6.1). Both results express global properties of Euclidean space, and each can be deduced from the other.

Another global result is the discrete Jordan curve theorem. In [100] it was proved by a direct "digital" argument that if the black points of a $(4, 8)$ digital picture form a simple closed black curve that is not contained in a unit cell then there are at least two white components. In [101] this was strengthened to assert that there must be just two white components. The same result holds for $(8, 4)$ digital pictures [107]. For interesting new proofs of these theorems see [124].

Rosenfeld established a converse of the discrete Jordan curve theorem in [104]. Morgenthaler, Reed, and Rosenfeld established a Jordan surface theorem—a 3D analog of the discrete Jordan curve theorem—for $(6, 26)$ and $(26, 6)$ digital pictures in a series of papers [81, 97, 96]. In [60], Kong and Roscoe used continuous analogs (see Section 7) to prove the Jordan surface theorem for $(6, 26)$, $(26, 6)$, $(18, 6)$, $(6, 18)$ and several other kinds of digital picture. Khalimsky gave a Jordan curve theorem for his digital pictures (see Section 11) in [40]; for a new proof of this see [54]. In [64] Kopperman *et al.* proved a 3D Jordan surface theorem for Khalimsky's digital pictures by means of continuous analogs.

Discrete Jordan curve theorems can be used to justify connectivity-based contour filling algorithms. These algorithms are used in computer graphics and picture analysis to find the interior of a region when its boundary is given (see [92, Chap. 8]).

7.THE EULER CHARACTERISTIC; CONTINUOUS ANALOGS

A subset of the plane or 3-space is said to be *polyhedral* if it is expressible as a finite union of points, closed straight line segments, closed triangles and closed tetrahedra. The *Euler characteristic* of a polyhedral set S, denoted by $\chi(S)$, can be defined by the following axioms (which can be proved to be consistent):

1. $\chi(\varnothing) = 0$.
2. $\chi(S) = 1$ if S is non-empty and convex.
3. for all polyhedral X and Y, $\chi(X \cup Y) = \chi(X) + \chi(Y) - \chi(X \cap Y)$.

In fact, $\chi(S)$ is equal to the value of the following alternating sum for an arbitrary triangulation of S:

$$(\text{no. of points}) - (\text{no. of edges}) + (\text{no. of triangles}) - (\text{no. of tetrahedra}).$$

The Euler characteristic can be shown to a topological invariant. It turns out that if S is a plane polyhedral set then $\chi(S)$ is equal to the number of connected components of S minus the number of holes in S. (This is a possible alternative definition of the Euler characteristic of a plane polyhedral set.)

If S is a polyhedral set in 3-space then $\chi(S)$ is equal to the number of components of S plus the number of cavities in S minus the number of "tunnels" in S. (Examples: the Euler characteristic of a hollow cube is 2—it has 1 component, 1 cavity, and no tunnels; the Euler characteristic of the boundary of a square is 0—it has 1 component, no cavities, and 1 tunnel). It is actually quite hard to define a "tunnel" in such a way that the term "number of tunnels" has a precise meaning. One solution is to not to define a tunnel at all, but to define the *number* of tunnels in a polyhedral set to be the rank of its first homology group, as in [62].

The basic properties of the Euler characteristic (including the properties mentioned above) are established in textbooks on algebraic topology (e.g., [45, 76]).

We now introduce a quantity that is analogous to the Euler characteristic, but which is defined for a digital picture $\mathscr{P}$. We call this quantity the Euler characteristic of $\mathscr{P}$ and denote it by $\chi(\mathscr{P})$. One way to define $\chi(\mathscr{P})$ is to associate every digital picture $\mathscr{P}$ with a polyhedral set $C(\mathscr{P})$—a *continuous analog* of $\mathscr{P}$—and to define $\chi(\mathscr{P}) = \chi(C(\mathscr{P}))$.

Three important properties of $C(\mathscr{P})$ are that the set of lattice points in each component of $C(\mathscr{P})$ should be a black component of $\mathscr{P}$, that the set of lattice points in each component of the complement of $C(\mathscr{P})$ should be a white component of $\mathscr{P}$, and that a black component D of $\mathscr{P}$ should be adjacent to a white component E of $\mathscr{P}$ if and only if the boundaries of the components of $C(\mathscr{P})$ and its complement that contain D and E meet. It follows from these properties that if $\mathscr{P}$ is a two-dimensional digital picture then $\chi(\mathscr{P}) = \chi(C(\mathscr{P}))$ = (no. of components of $C(\mathscr{P})$) − (no. of holes in $C(\mathscr{P})$) = (no. of black components of $\mathscr{P}$) − (no. of holes in $\mathscr{P}$).

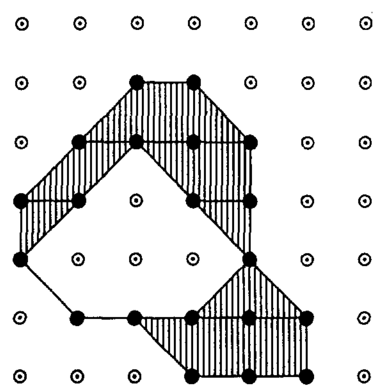

FIG. 23. The continuous analog of an (8, 4) digital picture.

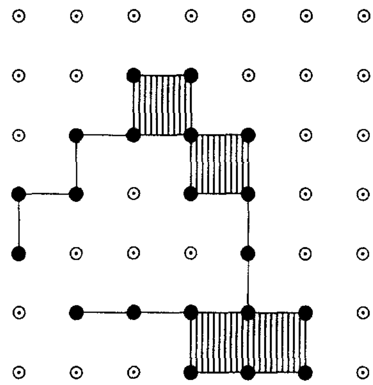

FIG. 24. The continuous analog of a (4, 8) digital picture. The black point set is the same as in Fig. 23.

The precise definition of $C(\mathscr{P})$ is given in [63]. When $\mathscr{P}$ is a two-dimensional (4, 8) or (8, 4) digital picture that definition can be stated as follows:

DEFINITION 7.1. Let $\mathscr{P}$ be an (m, n) digital picture, where $(m, n) = (4, 8)$ or $(8, 4)$. Let C_0 be the black point set of $\mathscr{P}$, let C_1 be the union of all straight line segments whose endpoints are adjacent black points of $\mathscr{P}$, and let C_2 be the union of all unit squares and, if $(m, n) = (8, 4)$, all $(1, 1, \sqrt{2})$ triangles, whose sides are contained in C_1. Then we define $C(\mathscr{P}) = C_0 \cup C_1 \cup C_2$.

Figures 23 and 24 illustrate this definition for (8, 4) and (4, 8) digital pictures, respectively. Observe that the digital pictures in Figs. 23 and 24 have the same black point set. Assuming that all points outside the 7 by 7 window shown in the figures are white points, the Euler characteristics of the two digital pictures are 0 and 1, respectively, by the "no. of black components − no. of holes" rule.

The corresponding definition for three-dimensional (m, n) digital pictures is similar, but somewhat more elaborate, especially when one of m and n is equal to 18. The interested reader may consult [63 or 57] for the details of the definition.

The definition $\chi(\mathscr{P}) = \chi(C(\mathscr{P}))$ illustrates the use of the continuous analog to transfer concepts of polyhedral topology to digital pictures. Techniques and theorems of polyhedral topology can also be applied to digital pictures via continuous analogs. We have already mentioned the use of continuous analogs to establish discrete Jordan curve and Jordan surface theorems. For other applications of continuous analogs see [61, 59, 68].

Gray [38] characterized the simple points in (4, 8) and (8, 4) digital pictures as the points whose deletion does not change the Euler characteristic of the digital picture. Yokoi *et al.* [138, Eqs. (9) and (10)] gave a more explicit statement and proof of this result. In Section 9 we will give a characterization of simple points in 3D digital pictures (due to Morgenthaler and Tsao and Fu) that involves $\chi(\mathscr{P})$.

Reference [135] gives an informal discussion of an application of the Euler characteristic to computer graphics—specifically, to the validation of boundary geometric models of 3D objects. Analogous applications of $\chi(\mathscr{P})$ to 3D voxel graphics can be envisioned. For an entirely different application of the Euler characteristic see [74].

8. COMPUTING EULER CHARACTERISTICS OF DIGITAL PICTURES

Many authors have given formulae for computing Euler characteristics of digital pictures. Gray, for instance, gave the following formulae for two-dimensional (4, 8) and (8, 4) digital pictures [38]:

$$4W = n(Q_1) - n(Q_3) - 2n(Q_D)$$
$$4Z = n(Q_1) - n(Q_3) + 2n(Q_D).$$

Here $n(Q_i)$ denotes the number of two-dimensional unit cells containing just i black points, while $n(Q_D)$ denotes the number of two-dimensional unit cells of type $\begin{smallmatrix}0&1\\1&0\end{smallmatrix}$ or $\begin{smallmatrix}1&0\\0&1\end{smallmatrix}$. W denotes the Euler characteristic of an (8, 4) digital picture and Z the Euler characteristic of a (4, 8) digital picture. Analogous but necessarily more complicated formulae for the "surface characteristic" (equal to twice the Euler characteristic—see 76, p. 195; or 62]) were given by Lobregt *et al.* [71] for (6, 26) and (26, 6) digital pictures. Somewhat similar methods were proposed in [78, 88, 79].

One way to derive formulae of this kind for computing Euler characteristics of digital pictures is as follows [56, 63]. For any two- or three-dimensional unit cell K, let K^0 be the set of corners of K, let K^1 be the union of the edges of K, and if K is a three-dimensional unit cell let K^2 be the union of the six faces of K. For all plane polyhedral sets P define

$$\chi(P; K) = \chi(P \cap K) - \chi(P \cap K^1)/2 - \chi(P \cap K^0)/4$$

and for 3D polyhedral sets define

$$\chi(P; K) = \chi(P \cap K) - \chi(P \cap K^2)/2 - \chi(P \cap K^1)/4 - \chi(P \cap K^0)/8.$$

Note that $\chi(P \cap K^0)$ is just the number of corners of K that lie in P.

It is not difficult to show from the Inclusion–Exclusion Principle that, for any two-dimensional (three-dimensional) polyhedral set P, $\chi(P)$ is just the sum of the $\chi(P; K)$ over all two-dimensional (three-dimensional) unit cells K. One can deduce from this that, for a two-dimensional (three-dimensional) digital picture $\mathscr{P}$, $\chi(\mathscr{P})$ is the sum of $\chi(C(\mathscr{P}); K)$ over all two-dimensional (three-dimensional) unit cells K.

One of the properties of the continuous analog $C(\mathscr{P})$ is that for any unit cell K the set $C(\mathscr{P}) \cap K$ is completely determined by the configuration of black and white points of $\mathscr{P}$ in K and does not depend on the rest of the black and white points of $\mathscr{P}$. So for any given (m, n) it is straightforward to evaluate $\chi(C(\mathscr{P}); K)$ for all possible configurations of black and white points of an (m, n) digital picture $\mathscr{P}$ in K. (Thus if $\mathscr{P}$ is three-dimensional and K contains just one black point then $\chi(C(\mathscr{P})\colon K) = 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} = \frac{1}{8}$. Again, if $\mathscr{P}$ is a three-dimensional (18, 6) or (26, 6) digital picture and K contains just two black points, and those points are diagonally opposite corners of a face of K, then $\chi(C(\mathscr{P})\colon K) = 1 - \frac{1}{2} - \frac{2}{4} - \frac{2}{8} = -\frac{1}{4}$.)

The value of $\chi(C(\mathscr{P}); K)$ for each possible configuration might be stored in a look-up table, of size $2^4 = 16$ and $2^8 = 256$ in the 2D and the 3D cases, respectively. One natural representation of the black/white point configuration in a unit cell is by a 4-bit code for 2D unit cells and an 8-bit code for 3D unit cells. The value of $\chi(\mathscr{P})$ for any (m, n) digital picture $\mathscr{P}$ could be computed by looking up

$\chi(C(\mathscr{P}); K)$ for each K and adding all the results together (cf. [71]). It is plain that this algorithm has very efficient parallel implementations.

A rather different method for computing Euler characteristics of (4, 8) and (8, 4) digital pictures was described in [38]. The method is based on the following identity for two-dimensional polyhedral sets P: $\chi(P)$ = (no. of northwest-facing convexities) − (no. of northwest-facing concavities). (Examples. A square has exactly one northwest-facing convexity and no concavities, so its Euler characteristic is $1 - 0 = 1$. The boundary of a square has one northwest-facing convexity and one northwest-facing concavity, the latter being on the "inside" of the hole, so the Euler characteristic of the boundary of a square is $1 - 1 = 0$.)

In a (4, 8) digital picture, each $\begin{smallmatrix}0&1\\1&1\end{smallmatrix}$ cell corresponds to a northwest-facing concavity, and a northwest-facing convexity arises from each cell of type $\begin{smallmatrix}0&0\\0&1\end{smallmatrix}$ or $\begin{smallmatrix}1&0\\0&1\end{smallmatrix}$. Thus the Euler characteristic of a (4, 8) digital picture is equal to the number of $\begin{smallmatrix}0&0\\0&1\end{smallmatrix}$ and $\begin{smallmatrix}1&0\\0&1\end{smallmatrix}$ cells minus the number of $\begin{smallmatrix}0&1\\1&1\end{smallmatrix}$ cells.

A similar formula can be derived for Euler characteristics of (8, 4) digital pictures. This method can be generalized to three dimensions—see [37; 69].

Yokoi *et al.* [138] developed another method for computing Euler characteristics of (4, 8) and (8, 4) digital pictures. A *coefficient of curvature* was defined for each black point p in terms of the black/white point configuration in its 3 by 3 neighborhood $N(p)$. Yokoi *et al.* showed that the coefficient of curvature at a border point p was always equal to $1/(2\pi)$ times the signed angle change (positive = anti-clockwise) experienced by a "left-hand on wall" border following algorithm as it passed through p (we add up all the angle changes if the algorithm passes through p more than once). One can deduce from this that the Euler characteristic of the digital picture is equal to the sum of the coefficients of curvature at all the border points.

This method can also be generalized to three dimensions by appealing to the following well known formula: $\chi(S) = (1/2\pi) \cdot$ (sum of the angle deficiencies at the vertices of S). Here S is an arbitrary 3D polyhedral surface, and the angle deficiency at a vertex of S is defined to be 2π minus the sum of the face angles incident upon that vertex. As an example, the angle deficiency at each vertex of the surface of a cube is $2\pi - 3(\pi/2) = \pi/2$, so the sum of the angle deficiencies at all eight vertices is 4π. It follows that the Euler characteristic of the surface of a cube is $4\pi/2\pi = 2$.

An entirely different approach was proposed by Bieri and Nef in [19], where two recursive algorithms were given for computing the Euler characteristic of any union of d-dimensional unit cells, based respectively upon the identities

$$f_k(P) = \sum_m f_k(P \cap H_m) + \sum_m f_{k-1}(P \cap H_m^*)$$

and

$$\chi(P) = \sum_m \chi(P \cap H_m) - \sum_m \chi(P \cap H_m^*).$$

Here H_m is the plane $x = m$ and H_m^* is the plane $x = m + \frac{1}{2}$, for all integers m. In the first identity $f_k(P)$ is the number of k-dimensional unit cells in P.

More recently, Bieri [18] has given an algorithm for computing the Euler characteristic of a digital picture from its bintree representation.

9. SIMPLE POINTS IN 3D DIGITAL PICTURES

There has been some interest in shrinking and thinning algorithms for 3D digital pictures (e.g., [122, 71, 80, 121, 130, 128, 132]). For a practical application of 3D thinning see [122].

Whereas the topological aspects of the theory of 2D shrinking and thinning are now fairly well understood (see Sections 3 and 5), the same cannot be said of 3D shrinking and thinning. Criterion 3.1 is not stringent enough in three dimensions, as it would allow a shrinking algorithm to convert any digital picture whose black point set was a simple closed black curve to a digital picture with just one black point. At present there is no generally accepted definition of topology preservation for three-dimensional parallel shrinking and thinning algorithms, though a number of inequivalent definitions have been proposed—see Section 10.

One part of the topology of 2D thinning that has been quite successfully generalized to three dimensions is the theory of simple points. As in the two-dimensional case (see Section 4), we define a simple point in a three-dimensional digital picture to be a black point whose deletion preserves topology. We have just mentioned that the various definitions of topology preservation that have been proposed in this context are not equivalent to each other. However, it turns out that exactly the same points are simple whichever definition is used [56, 59]. Hence the different definitions of topology preservation are equivalent for purely sequential thinning algorithms in which black points are deleted one at a time and the image array is updated every time a point is deleted. A sequential thinning algorithm preserves topology if each point deleted is simple with respect to the digital picture resulting from all previous point deletions.

A characterization of simple points which applies in any number of dimensions was given by Tourlakis and Mylopoulos [129], who used the term *deletable* rather than *simple*. Although Tourlakis and Mylopoulos only considered $(4,8)$ and $(6,26)$ digital pictures, and their higher dimensional analogs, the same approach can be generalized to digital pictures based on other adjacency relations.

A simpler characterization was given by Morgenthaler [80]. We now state this characterization in a further simplified form which is essentially due to Tsao and Fu (e.g., [132]).

THEOREM 9.1. *Let p be a black point in a digital picture $\mathcal{P} = (\mathbb{Z}^3, m, n, B)$, where $(m, n) = (6,26)$ or $(26,6)$. Let $B' = B - \{p\}$. Then p is a simple point if and only if the following conditions all hold*:

1. *p is adjacent to just one component of $N(p) \cap B'$.*
2. *p is adjacent to just one component of $N(p) - B$.*
3. $\chi((\mathbb{Z}^3, m, n, B \cap N(p))) = \chi((\mathbb{Z}^3, m, n, B' \cap N(p)))$.

It is shown in [56, 59] that this characterization is also valid when $(m, n) = (18,6)$ or $(6,18)$. Moreover, in the cases $(m, n) = (26,6)$ or $(18,6)$, conditions 1 and 3 together imply condition 2 (cf. [132]). In the cases $(m, n) = (6,26)$ or $(6,18)$, conditions 2 and 3 together imply condition 1 [56, 59].

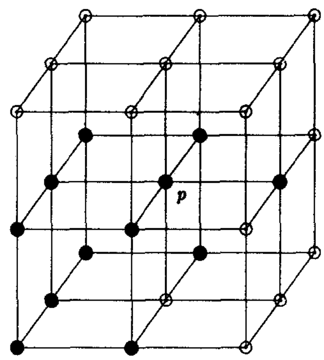

FIG. 25. A non-simple point in an (18,6) or (26,6) digital picture; deletion of p creates a tunnel, which may grow as a result of thinning.

When conditions 1 and 2 of Theorem 9.1 are satisfied, condition 3 stipulates that deletion of p "should not change the number of tunnels in $N(p)$." In each of Figs. 25–28 the point p and its 3 by 3 by 3 neighborhood $N(p)$ satisfy conditions 1 and 2. In Figs. 25 and 27 condition 3 is violated, so p is non-simple. In Fig. 25 deletion of p creates a small tunnel. (Note that such a small tunnel may grow as a result of thinning.) In Fig. 27 deletion of p removes a small tunnel. Deletion of p in Figs. 26 and 28 does not create or remove tunnels: condition 3 is satisfied and p is simple.

It is clear from this characterization of simple points that as in the 2D case one can determine whether or not a point p is simple just by looking at the configuration of black and white points in $N(p)$.

We observed in Section 4 that a point p in B is a simple point of $(\mathbf{Z}^2, m, n, B)$ if and only if it is a simple point of $(\mathbf{Z}^2, n, m, (\mathbf{Z}^2 - B) \cup \{p\})$. One can deduce from Theorem 9.1 that 3D simple points have an analogous property: a point p in B is a simple point of $(\mathbf{Z}^3, m, n, B)$ if and only if it is a simple point of $(\mathbf{Z}^3, n, m, (\mathbf{Z}^3 - B) \cup \{p\})$.

Now suppose the point p in the digital picture $\mathscr{P}$ satisfies condition 3 of Theorem 9.1. Let the coordinates of p be (x_0, y_0, z_0), and let P_x, P_y, and P_z denote the three coordinate planes $x = x_0$, $y = y_0$, and $z = z_0$ that contain p. Then $\mathscr{P}$

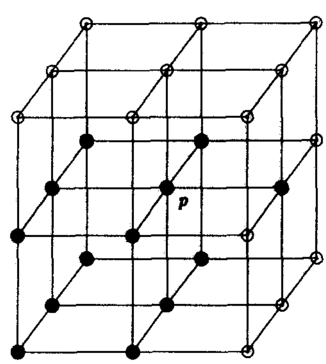

FIG. 26. A simple point in an (18,6) or (26,6) digital picture.

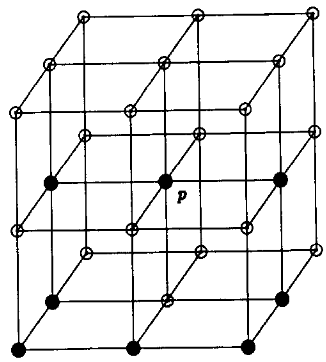

FIG. 27. A non-simple point in a (6,26) or (6,18) digital picture: deletion of p removes a tunnel.

meets P_x in a two-dimensional digital picture $\mathscr{P}_x = (\mathbb{Z}^2, m', n', B_x)$, where $B_x = \{(y, z) \in \mathbb{Z}^2 | (x_0, y, z) \in B\}$ and where $(m', n') = (8,4)$ if $(m, n) = (26,6)$ or $(18,6)$ and $(m', n') = (4,8)$ if $(m, n) = (6,26)$ or $(6,18)$. Similarly, $\mathscr{P}$ meets P_y and P_z in two-dimensional digital pictures $\mathscr{P}_y$ and $\mathscr{P}_z$. Tsao and Fu noted in [132] that if p is a simple point of at least two of $\mathscr{P}_x$, $\mathscr{P}_y$, and $\mathscr{P}_z$ then p is a simple point of $\mathscr{P}$. In this context p is considered to be a simple point of $\mathscr{P}_x$ if and only if the point (y_0, z_0) is a simple point of $\mathscr{P}_x$; and similarly for $\mathscr{P}_y$ and $\mathscr{P}_z$. Tsao and Fu also showed in [132] that the converse is false. (In fact [132] gave these results only for the cases $(m, n) = (6,26)$ and $(26,6)$, but it is not difficult to prove that the results are still true when $(m, n) = (6,18)$ or $(18,6)$.)

10. CRITERIA FOR TOPOLOGY PESERVATION IN 3D THINNING

In the literature there can be found a number of different definitions of what it means for a 3D thinning algorithm to "preserve topology." But the precise logical relationships between the alternative definitions have not yet been determined.

Tourlakis and Mylopoulos [129] gave a sound definition of topological equivalence for finite (4,8) and (6,26) digital pictures and their higher dimensional analogs: two finite digital pictures are topologically equivalent if one can be obtained from the other by sequential addition and deletion of simple points.

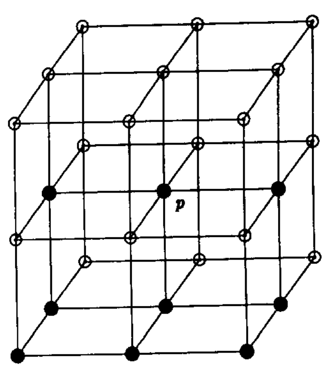

FIG. 28. A simple point in a (6,26) or (6,18) digital picture.

Now one cannot say that a shrinking or thinning algorithm preserves topology just because the output digital picture is always topologically equivalent to the input digital picture in this sense. For if two digital pictures $\mathscr{P}_1$ and $\mathscr{P}_2$ are topologically equivalent in the sense of Tourlakis and Mylopoulos then while it is true that they must have the same *number* of black components, tunnels, and cavities, there need not be any relationship between the *positions* of the black components, tunnels, and cavities of $\mathscr{P}_1$ and those of $\mathscr{P}_2$.

However, a mathematically satisfactory definition of topology preservation can be obtained from Tourlakis and Mylopoulos's notion of topological equivalence by disallowing the *addition* of simple points. This definition of topology preservation requires that the output digital picture be obtainable from the input digital picture by sequential deletion of simple points. Here simple points must be defined independently of the notion of topology preservation—they might be defined as the points that satisfy the three conditions in Theorem 9.1.

Morgenthaler [80] used a less stringent and more natural notion of topology preservation in his discussion of 3D shrinking and thinning. Morgenthaler did not explicitly define his concept of topology preservation, but the following definition is based on his remarks in [80].[4]

CRITERION 10.1. Let $\mathscr{P}_1 = (\mathbb{Z}^3, m, n, B)$ be a three-dimensional digital picture. Then deletion of the points in a subset D of B preserves topology if and only if the resulting digital picture $\mathscr{P}_2 = (\mathbb{Z}^3, m, n, B - D)$ satisfies the following conditions:

1. each black component of $\mathscr{P}_1$ contains exactly one black component of $\mathscr{P}_2$,
2. each white component of $\mathscr{P}_2$ contains exactly one white component of $\mathscr{P}_1$,
3. each closed path in B can be digitally deformed in B to a closed path in $B - D$, and
4. whenever one closed path in $B - D$ can be digitally deformed in B to another closed path in $B - D$, the first closed path can also be digitally deformed in $B - D$ to the second closed path.

The phrase *can be digitally deformed* used in conditions 3 and 4 can be given a precise meaning, though the definition of digital deformation in [80][5] is, as it stands, only appropriate for (6,26) digital pictures. This notion of digital deformation allows all points of a closed path to move: there is no base point that must remain fixed.

Conditions 1 and 2 in Criterion 10.1 are just the familiar conditions from Stefanelli and Rosenfeld's criterion (our Criterion 3.1). The geometric motivation for conditions 3 and 4 is that if a shrinking or thinning algorithm output a digital picture $\mathscr{P}_2$ from an input digital picture $\mathscr{P}_1$ then we should expect the tunnels of $\mathscr{P}_1$ and $\mathscr{P}_2$ to be in 1–1 correspondence, and we should expect each tunnel of $\mathscr{P}_2$ to be "in the same place as" the corresponding tunnel of $\mathscr{P}_1$.

Condition 3 in Criterion 10.1 ensures that a digital picture whose black point set is a simple closed black curve which is not contained in a 3D unit cell is invariant under shrinking. Condition 4 ensures that a shrinking algorithm cannot convert a

[4] See, for example, the last two paragraphs before Proposition 7 and the first paragraph after the proof of Proposition 8 in [80].

[5] This refers to the definition of strongly equivalent closed paths in [80].

digital picture whose black point set is a "digital solid cube" to a digital picture whose black point set is a simple closed black curve that is not contained in a 3D unit cell.

If $\mathscr{P} = (\mathbb{Z}^3, m, n, B)$ is a digital picture then we write $\bar{\mathscr{P}}$ for the digital picture $(\mathbb{Z}^3, n, m, \mathbb{Z}^3 - B)$.

In [57] the first author proposed a criterion that is closely related to Criterion 10.1, but is based on a *digital fundamental group* which is an analog for digital pictures of the fundamental group used in topology:

CRITERION 10.2. Let $\mathscr{P}_1 = (\mathbb{Z}^3, m, n, B)$ be a three-dimensional digital picture. Then deletion of the points in a subset D of B preserves topology if and only if the resulting digital picture $\mathscr{P}_2 = (\mathbb{Z}^3, m, n, B - D)$ satisfies the following conditions:

1. each black component of $\mathscr{P}_1$ contains exactly one black component of $\mathscr{P}_2$,
2. each white component of $\mathscr{P}_2$ contains exactly one white component of $\mathscr{P}_1$,
3. for each point p in $B - D$ (i.e., for each black point of $\mathscr{P}_2$), the inclusion map i: $B - D \to B$ induces a group isomorphism i_*: $\pi(\mathscr{P}_2, p) \to \pi(\mathscr{P}_1, p)$, and
4. for each point q in $\mathbb{Z}^3 - B$ (i.e., for each white point of $\mathscr{P}_1$), the inclusion map j: $\mathbb{Z}^3 - B \to \mathbb{Z}^3 - (B - D)$ induces a group isomorphism j_*: $\pi(\bar{\mathscr{P}}_1, q) \to \pi(\bar{\mathscr{P}}_2, q)$,

where $\pi(\mathscr{P}, x)$ denotes the digital fundamental group of the digital picture $\mathscr{P}$ with base point x.

Criterion 10.2 is more stringent than Criterion 10.1. This is because the first two conditions of both criteria are the same, and one can show that condition 1 and condition 3 of Criterion 10.2 together imply the third and fourth conditions of Criterion 10.1. However, one can construct examples in which all four conditions of Criterion 10.1 are satisfied but condition 4 of Criterion 10.2 is not.

A fourth possible definition of topology preservation applies Hilditch's concept of connectivity preservation (see Section 3) to the continuous analog $C(\mathscr{P})$ described in Section 7. This definition states that if a digital picture $\mathscr{P}_2$ can be obtained from a digital picture $\mathscr{P}_1$ by deleting a set of black points, then the conversion of $\mathscr{P}_1$ to $\mathscr{P}_2$ preserves topology if and only if the continuous analog $C(\mathscr{P}_2)$ can be obtained by continuous deformation (more precisely, by deformation retraction) of the continuous analog $C(\mathscr{P}_1)$.

Topology preservation in this sense implies topology preservation in the sense of Criterion 10.1, but does not imply condition 4 of Criterion 10.2. At present we do not know whether either of Criterion 10.1 and Criterion 10.2 implies that $C(\mathscr{P}_2)$ is a deformation retract of $C(\mathscr{P}_1)$.

In [56] the first author gave a definition of topology preservation that was motivated by Tourlakis and Mylopoulos's definition of a deletable point in [129]. This definition is the same as the definition we have just given, except that it does not allow $C(\mathscr{P}_2)$ to be an arbitrary deformation retract of $C(\mathscr{P}_1)$; only a very simple kind of deformation retraction, called *collapsing* [47], is permitted. This definition only applies to finite digital pictures, and is certainly more stringent than the second, third, and fourth definitions. It may turn out to be equivalent to the first definition, which required that the output digital picture $\mathscr{P}_2$ be obtainable from the input digital picture $\mathscr{P}_1$ by sequential deletion of simple points. If so, then this would be a three-dimensional analog of Ronse's result in [98] for Criterion 3.1, which we stated in Section 4 above.

All five definitions are considerably more complicated than Criterion 3.1. Theorems giving simpler conditions that are ***sufficient*** for topology preservation, such as a three-dimensional version of Theorem 5.1, could be very useful for verifying the topological soundness of a 3D parallel thinning algorithm.

11. OTHER KINDS OF DIGITAL PICTURE

Digital pictures on non-orthogonal grids have been considered by a number of authors. On a non-orthogonal grid the analogs of the 4-, 8-, 6-, 18-, and 26-adjacency relations are the adjacency relations based on the *Voronoi neighborhoods* of the grid points.

The Voronoi neighborhood of a grid point p in a two-dimensional (three-dimensional) grid is the set of all points in the Euclidean plane (Euclidean 3-space) that are at least as close to p as to any other grid point. It is always a closed and convex polyhedral set. For instance, the Voronoi neighborhood of each grid point in the standard 3D cubic grid is a unit cube centered on that point with faces parallel to the x, y, and z planes. Different adjacency relations between grid points are induced by vertex, edge and, in the case of 3D grids, face adjacency of Voronoi neighborhoods; but for some grids two or all three of these are the same. (Though we are mainly concerned with regularly spaced grids, these Voronoi adjacency relations exist for arbitrary grids. References [3, 4] discuss applications of a Voronoi adjacency relation on a random dot pattern.) On the cubic grid the Voronoi adjacency relations are just the 6-, 18-, and 26-adjacency relations. In general, a Voronoi adjacency relation in which each grid point is adjacent to exactly n other grid points is referred to as the *n-adjacency* relation on the grid.

Many authors have suggested the use of a 2D isometric hexagonal grid (e.g., [36, 43, 38, 31, 93]). The only Voronoi adjacency relation on this grid is 6-adjacency—see Fig. 29. A 2D triangular grid was considered in [31] and arbitrary 2D mosaics in [119].

The points (x, y, z) in which x, y, and z are integers such that $x + y + z$ is even are the grid points of a *face-centered cubic grid*. Toriwaki *et al.* [128] considered digital pictures on such a grid. There are two Voronoi adjacency relations on a face-centered cubic grid: 12-adjacency and 18-adjacency. The 12-neighbors of a grid point p are the twelve grid points that are closest to p—they are at a distance of $\sqrt{2}$ from p. The 18-neighbors of p are the 12-neighbors together with the six grid points that are at a distance of 2 from p. Figure 30 shows the 12-neighbors and Fig. 31

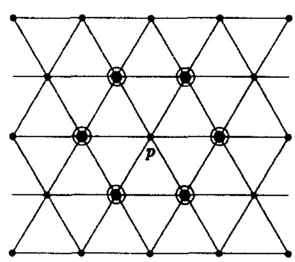

FIG. 29. The 6-neighbors of a point p on a hexagonal grid.

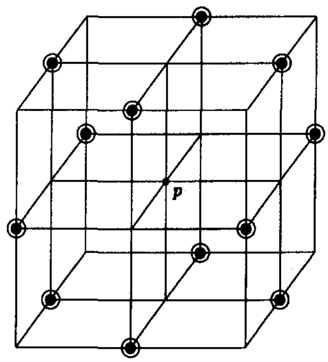

FIG. 30. The 12-neighbors of a point p on a face-centered cubic grid.

shows the 18-neighbors of a grid point. Different adjacency relations may be used for the black and the white points, or 12-adjacency may be used for both. But the use of 18-adjacency for both black and white points leads to paradoxes.

Kovalevsky [65] considered digital pictures derived from a tessellation of three-space by truncated octahedra. This is equivalent to using a *body-centered cubic grid* whose grid points are the points (x, y, z) in which x, y, and z are integers and $x \equiv y \equiv z \pmod 2$. The only Voronoi adjacency relation on the body-centered cubic grid is 14-adjacency. The 14-neighbors of a grid point p consist of the eight grid points that are closest to p (at a distance of $\sqrt{3}$ from p) together with the six grid points that are at a distance of 2 from p. Figure 32 shows the 14-neighbors of a grid point.

Non-Voronoi adjacency relations have also been considered. Mylopoulos and Pavlidis showed in [82, 83] that many concepts of digital geometry could be applied to the Cayley diagram[6] of any finite presentation of an abelian group. An infinite number of such Cayley diagrams can be obtained by choosing different abelian groups and different sets of generators. In fact the graph of any "translation invariant" adjacency relation on the grid points of an arbitrary plane or space lattice —this includes most of the adjacency relations we have discussed—is such a Cayley diagram.

A very interesting kind of digital picture that is not covered by the theory of Mylopoulos and Pavlidis has been introduced by Khalimsky (e.g., [40]), and more recently rediscovered by Kovalevsky (see [66]). Khalimsky's digital pictures are based on the usual cartesian grids. The same adjacency relation is used for both the black and the white points. However, the adjacency relation is non-uniform in the sense that some grid points have more neighbors than others. (This is why Khalimsky's digital pictures are not covered by the theory of Mylopoulos and Pavlidis.)

Following Khalimsky call a lattice point a *pure* point if both or all three of its coordinates are even numbers, or if both or all three are odd numbers. Call the other lattice points *mixed* points. Then two lattice points in a two- or three-dimensional Khalimsky digital picture are adjacent if they are 8-adjacent or 26-adjacent and at least one of them is a pure point, or (in the 3D case) if they are 6-adjacent mixed

[6]See [21, Chap. 8] for a readable introduction to Cayley diagrams.

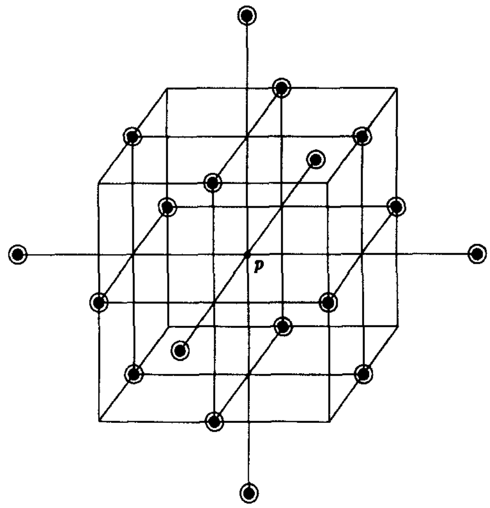

FIG. 31. The 18-neighbors of a point p on a face-centered cubic grid.

points. Figure 33 shows the pattern of adjacencies in a two-dimensional Khalimsky digital picture—the points with eight neighbors are the pure points.

The nicest feature of Khalimsky's adjacencies is that they are *topology generated*. Now every adjacency relation induces a metric on the grid points (e.g., 4-adjacency induces the city block metric considered in [77]). But the topology derived from the metric is not useful—it is just the discrete topology and so there are no non-trivial topologically connected sets. For Khalimsky's adjacencies there is a quite different topology on the set of grid points with the property that a set of grid points is connected with respect to the adjacency relation if and only if it is topologically connected. Moreover, that topology is a natural open quotient of the usual topology

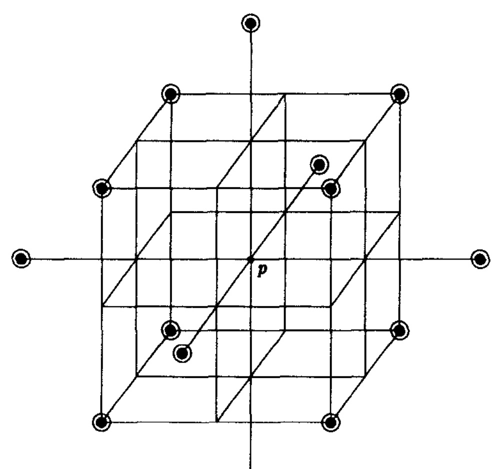

FIG. 32. The 14-neighbors of a point p on a body-centered cubic grid.

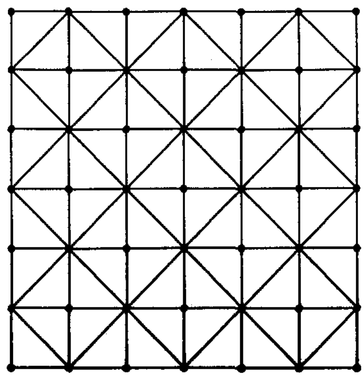

FIG. 33. Two-dimensional Khalimsky adjacencies.

on Euclidean space (see [66]). As a result of this Khalimsky's digital pictures are mathematically very well behaved [50, 51, 52, 54, 64]. Their principal drawback is that the adjacency relation is not translation invariant.

Digital pictures of Khalimsky's type can also be defined for non-orthogonal grids. Indeed, Kovalevsky's construction of Khalimsky's digital pictures in [66] generalizes easily to Sklansky and Kibler's arbitrary mosaics [119] and to the analogs of such mosaics in three and higher dimensions.

In [26] Das and Chatterji establish some results relating to the curious "knight's move" adjacency relation on the 2D square grid, in which two lattice points (x_1, y_1) and (x_2, y_2) are adjacent if and only if $|(x_2 - x_1)(y_2 - y_1)| = 2$.

References [61, 63] consider digital pictures on the square and cubic grids with almost arbitrary adjacency relations. Khalimsky's digital pictures (in two and three dimensions) are included as special cases. Also, adjacency relations on non-orthogonal lattices can in very many cases be mapped by affine transformation to adjacency relations on the square or cubical lattice that are anisotropic but nonetheless covered by the general theories of [61, 63].

12. CONCLUDING REMARKS

Two-dimensional digital topology is fairly well understood; most of the basic concepts were identified in the 1960s and early 1970s. Where different approaches have been taken, the relationships between those approaches have for the most part been established.

The same cannot be said of three-dimensional digital topology, which is considerably harder for two reasons. First, the topology of polyhedral sets in three-space is far more difficult than the topology of polyhedral sets in the plane—indeed, the subject is still being actively investigated by topologists. Second, arguments about two-dimensional digital pictures which involve case checking are often hard to generalize to three dimensions, as there are usually many more cases to consider. The topology of image thinning is one prominent area in which well-known two-dimensional results have not yet been successfully extended to three dimensions.

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