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Bayes estimation of $P(Y < X)$ for the Weibull distribution with arbitrary parameters

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ABSTRACT

In the model of $R = P(Y < X)$, X and Y usually represent the strength of a system and stress applied to it. Then, R is the measure of system reliability. In this paper, Bayes estimation of $R = P(Y < X)$ is studied under the assumption that X and Y are independent Weibull random variables with arbitrary scale and shape parameters. We show here for the first time how to compute the Bayes estimates and credible intervals for R in that case. First, a closed form expression for R is derived. Prior distributions are assumed for Weibull parameters, and the posterior distribution is presented. Next, by proposing an universal sample-based method according to the Monte Carlo Markov Chain (MCMC) method, we draw samples and compute the Bayes estimates and credible intervals for R. Through Monte Carlo simulations and two real data examples, the proposed method is demonstrated to be robust and satisfactory.

1. Introduction

Let X and Y be independent random variables representing the strength of a system and stress applied to it. Then, $R = P(Y < X)$ is the measure of system reliability. Hence, estimation of $R = P(Y < X)$ is important.

The Weibull distribution and its modified forms display various shapes for the density and hazard rate functions [1], making the Weibull model suitable to describe failure data due to fatigue [2]. This model has been widely studied [3–5]. Therefore, the estimation of $R = P(Y < X)$ is even more significant for the case that X and Y are independent Weibull random variables.

Many authors have actually studied estimation of $R = P(Y < X)$ (especially Bayesian estimation of R) for the case that X and Y are independent Weibull random variables [6–10]. But all of these studies have been limited. Valiollahi et al. [6] considered estimation of R when the scale parameters of X and Y are equal. Other researchers [7–10] considered estimation of R when the shape parameters of X and Y are equal. Further references can be found in [11].

We are aware of no studies on Bayesian estimation of R for arbitrary values of the shape and scale parameters. In practice, there is no reason why the shape parameters must be equal or the scale parameters must be equal. There may be particular situations where the shape parameters are not significantly different and / or the scale parameters are not significantly different. In general, however, the shape and scale parameters should be free parameters. Hence, it is important that Bayesian estimation procedures are developed for arbitrary values of these parameters. That is the aim of this paper.

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The contents of the paper are organized as follows. In Section 2, we present a closed form expression for R and a sample-based method to compute Bayes estimates and credible intervals for R. The Monte Carlo simulations are conducted in Section 3 and two real data examples are illustrated in Section 4. The paper is concluded in Section 5.

2. Bayes inference

2.1. The closed form for R

The probability density function (PDF) of the two-parameter Weibull distribution is

$$f(t; m, \eta) = \frac{m}{\eta}\left(\frac{t}{\eta}\right)^{m-1} \exp\left[-\left(\frac{t}{\eta}\right)^{m}\right],$$

for $t > 0$, $m > 0$ and $\eta > 0$, where m and η are the shape and scale parameters. For ease of inference, by introducing $\lambda = \eta^{-m}$, we can rewrite the PDF as

$$f(t; m, \lambda) = \lambda m t^{m-1} \exp(-\lambda t^{m}). \tag{1}$$

The corresponding reliability function for the Weibull distribution is

$$R(t; m, \lambda) = \exp(-\lambda t^{m}). \tag{2}$$

Let X and Y be independent Weibull random variables with parameters m_x, λ_x and m_y, λ_y, respectively. Then, $R = P(Y < X)$ can be expressed as

$$\begin{aligned} R &= \int_0^{+\infty} f(x; m_x, \lambda_x) \int_0^{x} f(y; m_y, \lambda_y) dy dx \\ &= 1 - \int_0^{+\infty} f(x; m_x, \lambda_x) R(x; m_y, \lambda_y) dx \\ &= 1 - m_x\lambda_x \int_0^{+\infty} x^{m_x-1} \exp(-\lambda_x x^{m_x} - \lambda_y x^{m_y}) dx. \end{aligned} \tag{3}$$

If $m_x = m_y = m$, (3) reduces to

$$R = \frac{\lambda_y}{\lambda_x + \lambda_y}, \tag{4}$$

same as the expression given by McCool [7]. If $\lambda_x = \lambda_y = \lambda$, (3) reduces to

$$R = 1 - \lambda m_x \int_0^{+\infty} x^{m_x-1} \exp[-\lambda(x^{m_x} + x^{m_y})] dx,$$

same as the expression given by Asgharzadeh et al. [9]. In the general case, (3) can be rewritten as

$$\begin{aligned} R &= 1 - m_x\lambda_x \int_0^{+\infty} x^{m_x-1} \exp(-\lambda_x x^{m_x} - \lambda_y x^{m_y}) dx \\ &= 1 - m_x\lambda_x \int_0^{+\infty} x^{m_x-1} \exp(-\lambda_x x^{m_x}) \left[\sum_{k=0}^{+\infty} \frac{(-\lambda_y x^{m_y})^k}{k!}\right] dx \\ &= 1 - m_x\lambda_x \sum_{k=0}^{+\infty} \frac{(-\lambda_y)^k}{k!} \int_0^{+\infty} x^{k m_y + m_x - 1} \exp(-\lambda_x x^{m_x}) dx \\ &= 1 - \sum_{k=0}^{+\infty} \frac{(-\lambda_y)^k}{k!} \lambda_x^{-\frac{k m_y}{m_x}} \int_0^{+\infty} z^{\frac{k m_y}{m_x}} \exp(-z) dz \\ &= 1 - \sum_{k=0}^{+\infty} \frac{(-\lambda_y)^k}{k!} \lambda_x^{-\frac{k m_y}{m_x}} \Gamma\left(\frac{k m_y}{m_x} + 1\right). \end{aligned} \tag{5}$$

Using the function

$$\phi(\rho, \varphi; z) = \sum_{k=0}^{+\infty} \frac{z^k}{k!} \Gamma(\rho k + \varphi),$$

a particular case of the Fox–Wright function [12], (5) can be reduced to the closed form

$$R = 1 - \phi\left(\frac{m_y}{m_x}, 1; -\lambda_y \lambda_x^{-\frac{m_y}{m_x}}\right). \tag{6}$$

2.2. The posterior distribution of the Weibull parameters

Suppose the collected data are $x_1, x_2 \dots, x_n$ and $y_1, y_2, \dots, y_h$. If x_i or y_j is the failure time, we denote them as $x_i \in F_x$ or $y_j \in F_y$. Otherwise, we denote them as $x_i \notin F_x$ or $y_j \notin F_y$. Moreover, let d_x and d_y denote the numbers of failure times in the samples $x_1, x_2, \dots, x_n$ and $y_1, y_2, \dots, y_h$. To compute the Bayes estimate of R, we need the posterior distribution $\pi(R|data)$ of R.

Based on the samples $x_1, x_2, \dots, x_n$ and $y_1, y_2, \dots, y_h$, the likelihood function is

$$L = \prod_{x_i \in F_x} f(x_i; m_x, \lambda_x) \prod_{x_j \notin F_x} R\big(x_j; m_x, \lambda_x\big) \prod_{y_u \in F_y} f(y_u; m_y, \lambda_y) \prod_{y_v \notin F_y} R(y_v; m_y, \lambda_y),$$

where $f(\cdot)$ and $R(\cdot)$ are given by (1) and (2), respectively. The likelihood function can be simplified to

$$L = c_x m_x^{d_x} \lambda_x^{d_x} c_y m_y^{d_y} \lambda_y^{d_y} \exp\left(-\lambda_x x_s^{m_x} - \lambda_y y_s^{m_y}\right), \tag{7}$$

where

$$x_s^{m_x} = \sum_{i=1}^{n} x_i^{m_x}, \quad y_s^{m_y} = \sum_{i=1}^{h} y_i^{m_y},$$

$$c_x = \begin{cases} \prod_{i \in F_x} x_i^{m_x - 1}, & d_x > 0, \\ 1, & d_x = 0, \end{cases}, \quad c_y = \begin{cases} \prod_{i \in F_y} y_i^{m_y - 1}, & d_y > 0, \\ 1, & d_y = 0. \end{cases}.$$

Next, the prior distribution of $(m_x, m_y, \lambda_x, \lambda_y)$ should be set. The Bayes inference here involves two Weibull distributions, which is similar to the Weibull competing failure model [13] or the accelerated life test problem [14] or the combination of the two problems [15]. As X and Y are independent, we could set the prior distributions for (m_x, λ_x) and (m_y, λ_y) separately. Therefore, the problem reduces to how to determine the prior distribution of (m, λ). Since the conjugate continuous joint prior distribution of (m, λ) does not exist [5], we further assume that the prior distributions of m and λ are independent. The prior distribution for λ is selected as the conjugate gamma distribution $\Gamma(\lambda; a, b) \propto \lambda^{a-1}\exp(-b\lambda)$ [16–18]. So, we select

$$\pi(\lambda_x; a_x, b_x) = \Gamma(\lambda_x; a_x, b_x) \propto \lambda_x^{a_x - 1} \exp(-b_x \lambda_x),$$
$$\pi(\lambda_y; a_y, b_y) = \Gamma(\lambda_y; a_y, b_y) \propto \lambda_y^{a_y - 1} \exp(-b_y \lambda_y).$$

For the prior distribution of m, there are different choices: simulated [5], gamma [16], non-informative [19], Jeffrey [20], discrete [21] and uniform [22] priors. Hence, we just set them as $\pi(m_x; \theta_x)$ and $\pi(m_y; \theta_y)$ with no specified forms.

According to Bayes theory, the posterior distribution of $(m_x, m_y, \lambda_x, \lambda_y)$ is

$$\pi(m_x, m_y, \lambda_x, \lambda_y) \propto \pi(m_x)\pi(m_y)\pi(\lambda_x)\pi(\lambda_y)L,$$

where L is the likelihood function in (7). After simplification, we see

$$\pi(m_x|\text{data}) \propto \frac{c_x m_x^{d_x}}{\left(b_x + x_s^{m_x}\right)^{a_x + d_x}} \pi(m_x; \theta_x), \tag{8}$$

$$\pi(\lambda|m_x, \text{data}) = \Gamma(\lambda_x; a_x + d_x, b_x + x_s^{m_x}), \tag{9}$$

$$\pi(m_y|\text{data}) \propto \frac{c_y m_y^{d_y}}{\left(b_y + y_s^{m_y}\right)^{a_y + d_y}} \pi(m_y; \theta_y), \tag{10}$$

$$\pi(\lambda|m_y, \text{data}) = \Gamma\left(\lambda_y; a_y + d_y, b_y + y_s^{m_y}\right) \tag{11}$$

and

$$\pi(m_x, m_y, \lambda_x, \lambda_y|\text{data}) \propto \pi(\lambda|m_x, \text{data})\pi(m_x|\text{data})\pi(\lambda|m_y, \text{data})\pi(m_y|\text{data}). \tag{12}$$

2.3. The Bayes estimate of R

The Bayes estimates and credible intervals for R are wanted. Based on Bayes theory, we need to further derive the posterior distribution $\pi(R|\text{data})$ through the posterior distribution $\pi(m_x, m_y, \lambda_x, \lambda_y|\text{data})$ in (12) and the expression for R in (6). This is also the core idea of Ventura and Racugno [23]. However, it is difficult to obtain a closed form for $\pi(R|\text{data})$ here. Hence, we propose a sample-based method to obtain a sample of R from the distribution $\pi(R|\text{data})$ to yield the Bayes estimate of R. This is also the main idea used in [6,8,9]. Supposing there are samples on $(m_x, m_y, \lambda_x, \lambda_y)$ of size s from $\pi(m_x, m_y, \lambda_x, \lambda_y|\text{data})$, these samples could be used to estimate R by (6). The estimates of R of size s could be regarded as samples from the posterior distribution $\pi(R|\text{data})$. Hence, the problem is transformed to how to draw samples from $\pi(m_x, m_y, \lambda_x, \lambda_y|\text{data})$.

By (8)–(12), $\pi(m_x, m_y, \lambda_x, \lambda_y|\text{data})$ could be given by the posterior distribution function of the Weibull parameters. It is rather convenient to employ the Gibbs sampling to draw samples of $(m_x, m_y, \lambda_x, \lambda_y)$. That is, we could first draw the samples of (m_x, m_y) and generate the samples of (λ_x, λ_y) later. The key step is to draw the samples of (m_x, m_y).

For this step, Kundu and Gupta [8] assumed that the prior distribution $\pi(m;\, \theta)$ is log-concave. Both Asgharzadeh et al. [9] and Valiollahi et al. [6] applied the Metropolis–Hastings method with a normal distribution. Here, we propose an universal and robust sampling algorithm based on the MCMC method to generate samples of (m_x, m_y). This algorithm has been used previously in Jia and Guo [22] to compute Bayes estimates of the reliability for a cold-standby system. The details of the algorithm are as follows.

Algorithm. Given the posterior distribution $\pi(m_x, m_y, \lambda_x, \lambda_y|\text{data})$ and sample size s:

1: Initialize $l = 1$ and generate m_x^1 and m_y^1 from the prior distributions $\pi(m_x^1; \theta_x)$ and $\pi(m_y^1; \theta_y)$, respectively.

2: Generate λ_x^1 and λ_y^1 from $\Gamma(\lambda_x; a_x + d_x, b_x + x_s^{m_x^1})$ and $\Gamma(\lambda_y; a_y + d_y, b_y + y_s^{m_y^1})$, respectively.

3: Update $l = l + 1$ and generate m_x^c and m_y^c from $\pi(m_x^c; \theta_x)$ and $\pi(m_y^c; \theta_y)$, respectively. According to (8) and (10), compute

$$\alpha_x = min(1, \beta_x), \quad \alpha_y = min(1, \beta_y),$$

where

$$\beta_x = \frac{\pi(m_x^c|\text{data})}{\pi\left(m_x^{l-1}|\text{data}\right)} = \left(\frac{m_x^c}{m_x^{l-1}}\right)^{d_x} \frac{\pi(m_x^c; \theta_x)}{\pi\left(m_x^{l-1}; \theta_x\right)} \left(\frac{b_x + x_s^{m_x^{l-1}}}{b_x + x_s^{m_x^c}}\right)^{a_x+d_x},$$

$$\beta_y = \frac{\pi\left(m_y^c|\text{data}\right)}{\pi\left(m_y^{l-1}|\text{data}\right)} = \left(\frac{m_y^c}{m_y^{l-1}}\right)^{d_y} \frac{\pi\left(m_y^c; \theta_y\right)}{\pi\left(m_y^{l-1}; \theta_y\right)} \left(\frac{b_y + y_s^{m_y^{l-1}}}{b_y + y_s^{m_y^c}}\right)^{a_y+d_y}.$$

4: Generate μ_x, μ_y from the uniform distribution $U(0, 1)$ and compare them with α_x and α_y, respectively. Then set

$$m_x^l = \begin{cases} m_x^c, & \alpha_x \geq \mu_x, \\ m_x^{l-1}, & \alpha_x < \mu_x, \end{cases}, \quad m_y^l = \begin{cases} m_y^c, & \alpha_y \geq \mu_y, \\ m_y^{l-1}, & \alpha_y < \mu_y. \end{cases}.$$

5: Generate λ_x^l and λ_y^l from $\Gamma(\lambda_x; a_x + d_x, b_x + x_s^{m_x^l})$ and $\Gamma(\lambda_y; a_y + d_y, b_y + y_s^{m_y^l})$, respectively.

6: Repeat steps 3–5 until $l = s$.

Then we obtain the samples $(m_x^i, m_y^i, \lambda_x^i, \lambda_y^i),\ i = 1, 2, \ldots, s$. These samples can be used to estimate R by (6), resulting in samples of size s from the posterior distribution $\pi(R|\text{data})$. Discarding the early iterations, i.e. burn-in with length $0.1s$, we sort the remaining samples in ascending order. Finally, the Bayes estimate of R is

$$R_B = \frac{1}{0.9s} \sum_{i=1}^{0.9s} R_{(i)}, \tag{13}$$

and the $100(1-\alpha)\%$ Bayes credible interval is

$$\left[R_{(l)},\ R_{(u)}\right], \tag{14}$$

where $l = 0.9s\frac{\alpha}{2}$ and $u = 0.9s\left(1 - \frac{\alpha}{2}\right)$.

3. Simulation study

In this section, we conduct Monte Carlo simulations to examine the performance of the Bayes estimators. Since X and Y are Weibull random variables taking arbitrary shape and scale parameters, we set λ_x, λ_y, m_x and m_y differently. In the simulations, we take $\lambda_x = 0.1$ and $\lambda_y = 1$. The values of (m_x, m_y) are taken to be (0.5, 0.8), (0.5, 3) and (2, 3). For convenience, the sample sizes n and h are chosen to be one of 10, 30 or 50 equally. The sample size s is adopted as 5000, as in Jia and Guo [22]. The prior distributions $\pi(m_x)$ and $\pi(m_y)$ are taken as gamma (G) and uniform (U) distributions. All the parameter settings are described in Table 1.

Under each parameter setting, the complete data for X and Y were first generated. Then, we drew samples of $(m_x, m_y, \lambda_x, \lambda_y)$ by the algorithm in Section 2.3 and produced the Bayes estimates and 90% credible intervals for R through (13) and (14). Further, we replicated this process 10,000 times. Based on these 10,000 Bayes estimates and credible intervals, the bias, mean squared error (MSE), coverage probability (CP) and average interval width were computed. For ease of comparison, the ratios of bias, MSE and average width to the true values of R were computed. These results are reported in Tables 2–5.

Table 1
All the parameter settings for the simulation.

Case index	1	2	3	4	5	6	7	8	9
$\pi(m_x)$, $\pi(m_y)$	G	G	G	G	G	G	G	G	G
n, h	10	30	50	10	30	50	10	30	50
m_x	0.5	0.5	0.5	0.5	0.5	0.5	2	2	2
m_y	0.8	0.8	0.8	3	3	3	3	3	3
Case index	10	11	12	13	14	15	16	17	18
$\pi(m_x)$, $\pi(m_y)$	U	U	U	U	U	U	U	U	U
n, h	10	30	50	10	30	50	10	30	50
m_x	0.5	0.5	0.5	0.5	0.5	0.5	2	2	2
m_y	0.8	0.8	0.8	3	3	3	3	3	3

Table 2
The ratio of bias to the true value of R.

Case index	1	2	3	4	5	6	7	8	9
Ratio	−0.0213	−0.0083	−0.0025	−0.0103	−0.0050	−0.0027	−0.0808	−0.0240	−0.0136
Case index	10	11	12	13	14	15	16	17	18
Ratio	−0.0063	−0.0033	−0.0023	0.0016	−0.0011	0.0039	−0.0063	−0.0030	-8.3584×10^{-4}

Table 3
The ratio of MSE to the true value of R.

Case index	1	2	3	4	5	6	7	8	9
Ratio	0.0043	0.0013	7.6677×10^{-4}	0.0039	0.0015	8.1277×10^{-4}	0.0024	0.0012	7.2307×10^{-4}
Case index	10	11	12	13	14	15	16	17	18
Ratio	0.0039	0.0012	5.4970×10^{-4}	0.0038	0.0014	6.0429×10^{-4}	0.0033	0.0013	7.703×10^{-4}

Table 4
The CP.

Case index	1	2	3	4	5	6	7	8	9
CP	0.877	0.910	0.90	0.856	0.895	0.914	0.894	0.912	0.926
Case index	10	11	12	13	14	15	16	17	18
CP	0.854	0.913	0.923	0.824	0.89	0.915	0.895	0.89	0.888

Table 5
The ratio of average width to the true value of R.

Case index	1	2	3	4	5	6	7	8	9
Ratio	0.2152	0.1252	0.0953	0.2116	0.13	0.1018	0.2857	0.1373	0.1037
Case index	10	11	12	13	14	15	16	17	18
Ratio	0.1953	0.1216	0.1045	0.1958	0.1273	0.1029	0.1975	0.1207	0.0939

From Tables 2–5, the Bayes estimates and credible intervals based on uniform and gamma priors of (m_x, m_y) can be compared. The following conclusions could be found.

For the Bayes estimate:

1. All the biases are acceptable. The absolute values of biases for the uniform prior are less than those for the gamma prior.
2. All the MSEs are satisfactory. Generally, the MSEs for the uniform prior are smaller than those for the gamma prior.
3. As the sample size increases, the absolute value of bias and MSE for both the priors decrease.

For the Bayes credible interval:

1. All the CPs agree with the nominal level 0.9 and all the average interval widths are excellent.
2. The average interval widths for the uniform prior are narrower than those for the gamma prior generally.
3. As the sample size increases, the average interval widths for both the priors decrease.

This demonstrates that the proposed Bayes estimates and credible intervals for R perform well. Different prior distributions $\pi(m_x, m_y)$ were tested and the simulation study was rerun under different combinations of parameter settings. All the results were satisfactory. Therefore, the proposed method is universal and robust. For more preferable estimation, larger sample sizes and the uniform prior are recommended.

Table 6
Data for the gauge lengths of 20 mm.

1.312	1.314	1.479	1.552	1.700	1.803	1.861	1.865	1.944	1.958
1.966	1.997	2.006	2.021	2.027	2.055	2.063	2.098	2.140	2.179
2.224	2.240	2.253	2.270	2.272	2.274	2.301	2.301	2.359	2.382
2.382	2.426	2.434	2.435	2.478	2.490	2.511	2.514	2.535	2.554
2.566	2.570	2.586	2.629	2.633	2.642	2.648	2.684	2.697	2.726
2.770	2.773	2.800	2.809	2.818	2.821	2.848	2.880	2.954	3.012
3.067	3.084	3.090	3.096	3.128	3.233	3.433	3.585	3.585	

Table 7
Data for the gauge lengths of 10 mm.

1.901	2.132	2.203	2.228	2.257	2.350	2.361	2.396	2.397	2.445
2.454	2.474	2.518	2.522	2.525	2.532	2.575	2.614	2.616	2.618
2.624	2.659	2.675	2.738	2.740	2.856	2.917	2.928	2.937	2.937
2.977	2.996	3.030	3.125	3.139	3.145	3.220	3.223	3.235	3.243
3.264	3.272	3.294	3.332	3.346	3.377	3.408	3.435	3.493	3.501
3.537	3.554	3.562	3.628	3.852	3.871	3.886	3.971	4.024	4.027
4.225	4.395	5.020							

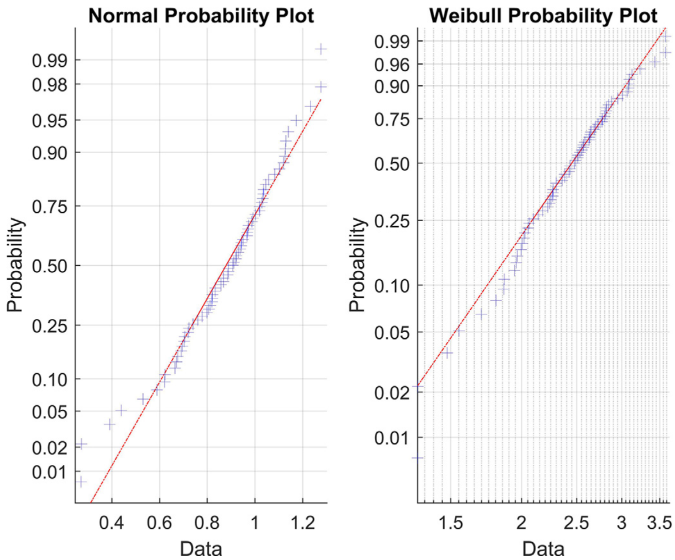

Fig. 1. The lognormal and Weibull probability plots for the data on X.

4. Real data examples

In this section, the application of the proposed method is illustrated by two published real data sets.

4.1. Example 1

4.1.1. The illustration of the proposed methods

This data set was originally reported by Badar and Priest [24] and it represents the strength measured in GPA for single carbon fibers and impregnated 1000-carbon fiber tows. The gauge lengths at which the single fibers were tested were 1, 10, 20, and 50 mm. The impregnated tows were tested at lengths of 20, 50, 150, and 300 mm [25]. Valiollahi et al. [6], Kundu and Gupta [8], Asgharzadeh et al. [9] and Surles and Padgett [25] have all studied these data sets. Here, we infer on $R = P(Y < X)$ by taking X to represent the strength of a 20 mm fiber and taking Y to represent the strength of a 10 mm fiber. This assumption has been commonly adopted in other studies, see Valiollahi et al. [6], Kundu and Gupta [8], Asgharzadeh et al. [9] and Surles and Padgett [25].

The data about the single fibers under tension at gauge lengths of 20 mm and 10 mm with sample sizes $n = 69$ and $h = 63$, respectively, are given in Tables 6 and 7. The probability plot method was used to compare the fits of Weibull and lognormal models. For both data, the plots are displayed in Figs. 1 and 2. Both the lognormal and Weibull models fit the data well. However, the errors resulting from the fits for the data on X are 0.982 and 0.9558 for the lognormal and Weibull

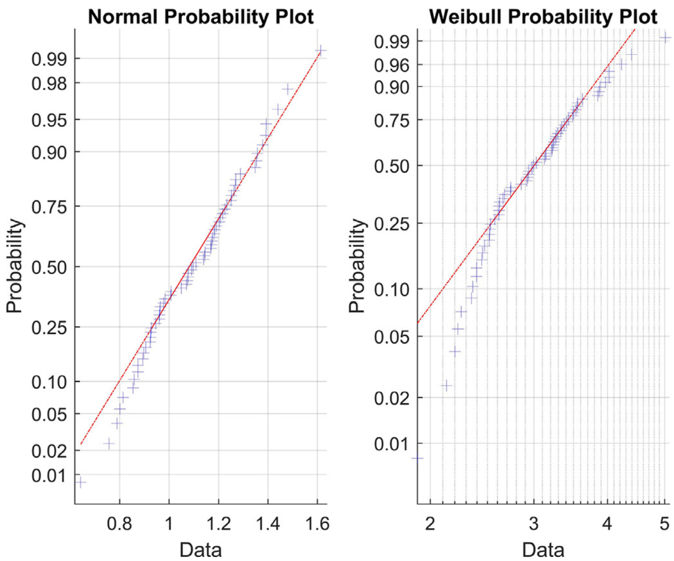

Fig. 2. The lognormal and Weibull probability plots for the data on Y.

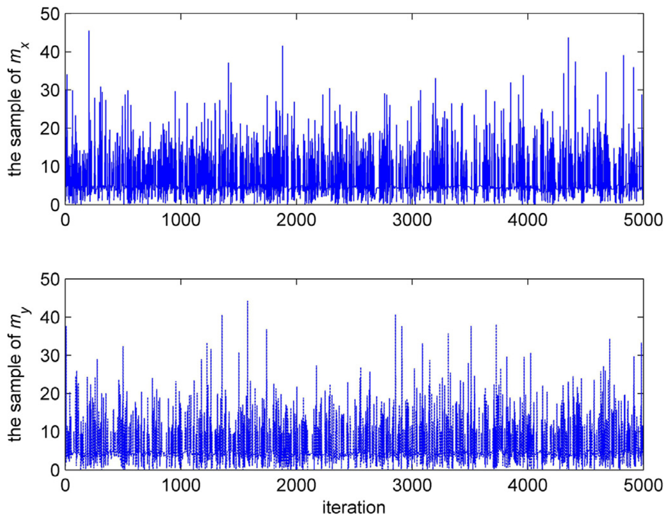

Fig. 3. The samples of m_x and m_y for the gamma prior.

models, respectively. The errors for the data on Y are 0.9944 and 0.9833 for the lognormal and Weibull models, respectively. As the errors for the Weibull model are smaller, the Weibull distribution is used to describe the two data sets.

Under the Weibull model, the maximum likelihood estimates (MLEs) of m_x and m_y are $\widehat{m_x} = 5.5049$ and $\widehat{m_y} = 5.0494$. By (6), the MLE of $R = P(Y < X)$ is $\widehat{R} = 0.2424$. For comparison, the MLEs $\widehat{m_x}$, $\widehat{m_y}$ and $\widehat{R}$ are regarded as the true values of m_x, m_y and R.

In this example, the gamma priors of (λ_x, λ_y) are set with $a_x = b_x = a_y = b_y = 0$ (the non-informative priors). Though the priors of (λ_x, λ_y) are set equally, the posterior distribution of (λ_x, λ_y) would still be different. The priors are set equally just for convenience. It would not affect the Bayes estimates of (λ_x, λ_y) and R.

The priors of (m_x, m_y) are assumed to be the gamma distributions $\Gamma(m_x; 1, \widehat{m_x})$ and $\Gamma\left(m_y; 1, \widehat{m_y}\right)$, where $\widehat{m_x}$ and $\widehat{m_x}$ are the MLEs of m_x and m_y. Samples of $(m_x, \lambda_x, m_y, \lambda_y)$ with size $s = 5000$ are drawn using the proposed algorithm. The samples of (m_x, m_y) using gamma priors are plotted in Fig. 3. The PDFs of these samples are compared with the posterior distributions $\pi(m_x|\text{data})$ and $\pi(m_y|\text{data})$ in Fig. 4. These samples are used to estimate R from the posterior distribution $\pi(R|\text{data})$ and the sample of estimates of R is plotted in Fig. 5. The Bayes estimate and 90% credible interval are computed using (13) and (14). Next, the priors of (m_x, m_y) are changed to be the uniform distribution $U(0, 6)$ and the process is repeated again. In this case, the samples are shown in Figs. 6–8. All the Bayes estimates and credible intervals for R are given in Table 8.

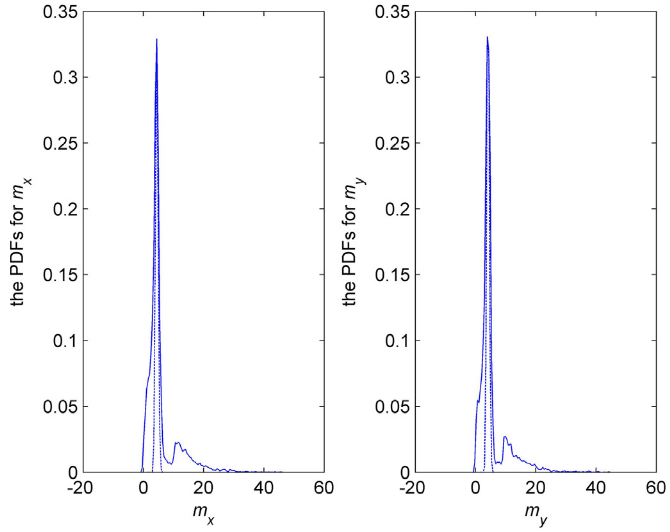

Fig. 4. The PDFs of the sample (solid line) and posterior (dash line) distributions for the gamma prior.

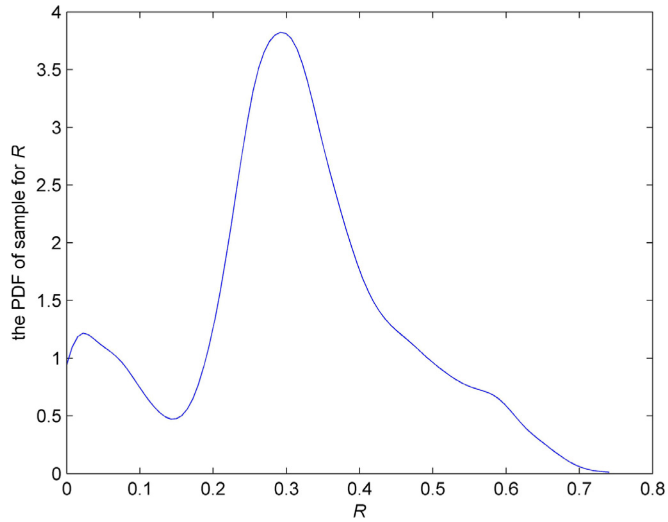

Fig. 5. The PDF of the sample distribution of R for the gamma prior.

Table 8
Estimates of R for example 1.

	Estimate	Credible interval	Interval width
Gamma prior	0.3057	[0.0238, 0.5682]	0.5444
Uniform prior	0.2445	[0.1837, 0.3124]	0.1287

By comparing Figs. 3 and 6, we find that the samples of (m_x, m_y) with the gamma prior fluctuate more drastically than the samples with the uniform prior. The samples of (m_x, m_y) using the uniform prior are closer to the true values of (m_x, m_y). The PDFs of the samples (m_x, m_y) with the uniform prior are closer to the posterior distributions $\pi(m_x|\text{data})$ and $\pi(m_y|\text{data})$, see Figs. 4 and 7. Besides, the credible interval width from the uniform prior is narrower than that from the gamma prior, see Table 8. Hence, the Bayes estimate and credible interval for R with the uniform prior are more convincing. These conclusions agree with the simulation results.

4.1.2. The comparison with other results

Here, we compute the Bayes estimate and credible interval for R by assuming that the shape parameters of X and Y are equal as $m_x = m_y = 5.0494$. We take the priors of m_x and m_y as $\Gamma\left(m_x; 1, \widehat{m_y}\right)$ and $\Gamma\left(m_y; 1, \widehat{m_y}\right)$, where $\widehat{m_y}$ is the MLE of m_y. The uniform priors are still taken as $U(0, 6)$. The Bayes estimate and credible interval for R with equal shape parameters

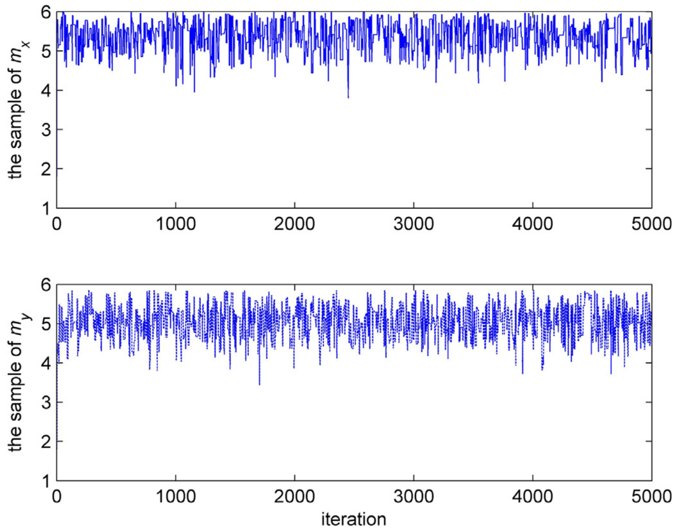

Fig. 6. The samples of m_x and m_y for the uniform prior.

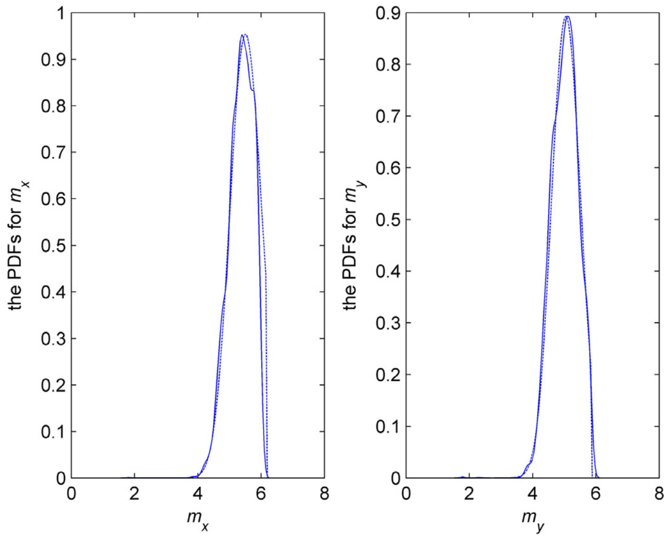

Fig. 7. The PDFs of the sample (solid line) and posterior (dash line) distributions for the uniform prior.

Table 9
Estimates of R for example 1 if shape parameters are equal.

	Estimate	Credible interval	Interval width
Gamma prior	0.4171	$[8.5728 \times 10^{-8}, 1]$	1
Uniform prior	0.3261	[0.1224, 0.5852]	0.4627

are tabulated in Table 9. It is evident that both the Bayes estimates are different from the true value of R. The comparison demonstrates that the shape parameters of X and Y should not be taken to be equal. It also shows the importance of the study in this paper.

Next, we apply the bootstrap method to compute the confidence interval for $R = P(Y < X)$ as this method is widely used to construct confidence intervals in applications. The bootstrap confidence interval is [0.1745,0.3100] with width equal to 0.1355. It is clear that the bootstrap interval is wider than the Bayes credible interval with the uniform prior. The comparison proves that the proposed Bayes method is superior.

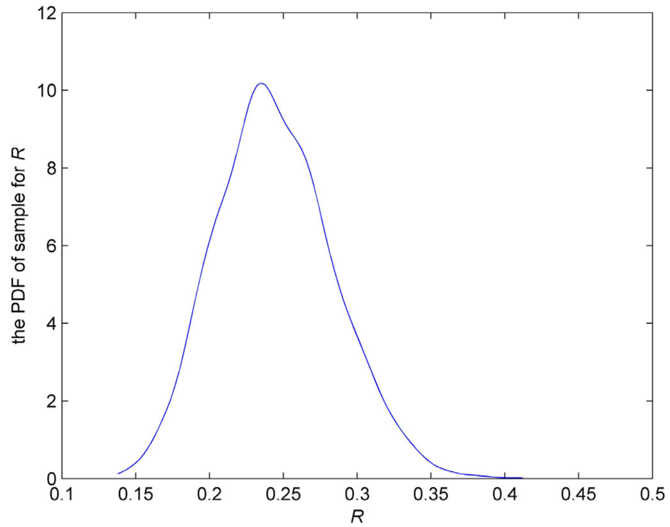

Fig. 8. The PDF of the sample distribution of R for the uniform prior.

Table 10
Estimates of R for example 2.

		Estimate	Credible interval	Interval width
$m_x \neq m_y$	Gamma prior	0.2045	[0.002, 0.474]	0.472
	Uniform prior	0.034	[0.0089, 0.0849]	0.076
$m_x = m_y$	Gamma prior	0.3733	$[5.3665 \times 10^{-42}, 1]$	1
	Uniform prior	0.1622	$[2.7379 \times 10^{-9}, 0.9951]$	0.9951
Bootstrap confidence interval			$[4.5798 \times 10^{-4}, 0.0492]$	0.0487

4.2. Example 2

Here, we use another data set from Kundu [26] to illustrate the proposed methods again. This data set are measurements of shear strength of spot welds for two different gauges of steel. Let X denote the weld strengths for 0.040 - 0.040- in and let Y denote the weld strengths for 0.060 - 0.060- in steel. The shear strength data for X are 350, 380, 385, 450, 465, 185, 535, 555, 590 and 605 (in units of 0.001 in). For Y, they are 680, 800, 780, 885, 975, 1025, 1100, 1030, 1175 and 1300.

Kundu [26] showed that both data sets can be modeled reasonably by the Weibull distribution. Hence, we suppose X and Y are Weibull distributed. The MLEs of m_x and m_y are $\widehat{m_x} = 4.4436$ and $\widehat{m_y} = 5.9654$. The MLE of $R = P(Y < X)$ is $\widehat{R} = 0.0131$. Similar to Section 4.1, the MLEs are regarded as the true values and the gamma priors of (λ_x, λ_y) are set with $a_x = b_x = a_y = b_y = 0$.

Firstly, the Bayes estimate and credible interval for R are computed by assuming that m_x and m_y have the gamma priors $\Gamma(m_x; 1, \widehat{m_x})$ and $\Gamma\left(m_y; 1, \widehat{m_y}\right)$, where $\widehat{m_x}$ and $\widehat{m_y}$ are the MLEs of m_x and m_y. The uniform priors are still taken to be $U(0, 6)$. Secondly, the Bayes estimate and credible interval for R are computed by setting m_x and m_y to be equal as $m_x = m_y = 5.1335$. In this case, the gamma priors of m_x and m_y are $\Gamma(m;\ 1,\ 5.1335)$. The uniform priors of m_x and m_y are still $U(0, 6)$. Thirdly, the bootstrap confidence interval for R is computed. All the Bayes estimates and credible intervals for R are listed in Table 10.

The Bayes estimate of R using the uniform prior with unequal m_x and m_y is closer to the true value of R. In this case, the Bayes credible interval is narrow. When m_x and m_y are equal, the differences between the true value and both the Bayes estimates are large. These findings coincide with the results for example 1 very well.

5. Conclusions

We have considered Bayes inference on $R = P(Y < X)$ by assuming X and Y are absolutely different Weibull random variables. First, the expression, especially the closed form of R was derived. Assuming prior distributions for the Weibull parameters, the posterior distribution of the Weibull parameters was presented. Next, by proposing an universal sample-based method according to the MCMC method, we drew samples of the Weibull parameters and computed Bayes estimates and credible intervals for R. Through a Monte Carlo simulation study, the proposed sample-based method was shown to be robust. Finally, two real data examples were presented to show the application of the proposed method. In both the examples, the fitted results with unequal shape parameters were compared with the ones with equal shape parameters. The comparisons showed that if the true values of the shape parameters for X and Y are different, they should not be simplified to be equal as this leads to the estimate of R being different from the true value of R. Both data examples demonstrate the significance and necessity of our study.

There is still room for improvement in our study. Different priors lead to different Bayes estimates, as we saw in the simulation study and real data examples. Some future questions to address are: how to choose the appropriate prior for a specific problem? can closed form expressions be obtained for the Bayes estimates? can the study be extended if X and Y are dependent Weibull random variables?

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