Signal Processing

Airat Galiamov

November 2024

1 Fourier transform

1.1 Definitions

If $f(t)$ is a function of time, then $F(\omega)$ is its Fourier transform $f(t) \xrightarrow{\mathcal{F}} F(\omega)$ and spectral density if:

$$F(\omega) = \mathcal{F}[f(t)] = \int\limits_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

Inverse transform $F(\omega) \xrightarrow{\mathcal{F}^{-1}} f(t)$:

$$f(t) = \mathcal{F}^{-1}[F(\omega)] = \int\limits_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega$$

1.2 Properties

Linearity Follows from the linearity of the integration:

$$af(t) + bg(t) \xrightarrow{\mathcal{F}} aF(\omega) + bG(\omega)$$

Scaling Wider in time - narrower in frequency. Follows from the change of variables:

$$f(at) \xrightarrow{\mathcal{F}} \frac{1}{|a|}F(\frac{\omega}{a})$$

$$\int\limits_{-\infty}^{\infty} f(at)e^{-i\omega t}dt =$$

$$= \left| u = at;\ dt = \frac{1}{a}du; \text{ change of limits} \rightarrow \text{absolute value of } a \right| =$$

$$= \frac{1}{|a|}\int\limits_{-\infty}^{\infty} f(u)e^{-i\frac{\omega}{a}u}du =$$

$$= \frac{1}{|a|}F(\frac{\omega}{a})$$

Time shift Shift in time - phase shift in frequency. Follows from the change of variables:

$$f(t-t_0) \xrightarrow{\mathcal{F}} e^{-i\omega t_0}F(\omega)$$

$$\int\limits_{-\infty}^{\infty} f(t-t_0)e^{-i\omega t}dt = |u = t - t_0| =$$

$$= \int\limits_{-\infty}^{\infty} f(u)e^{-i\omega u - i\omega t_0}du =$$

$$= e^{-i\omega t_0}\int\limits_{-\infty}^{\infty} f(u)e^{-i\omega u}du = e^{-i\omega t_0}F(\omega)$$

Differentiation Derivative in time - multiplication by $i\omega$ in frequency. Useful property for solving differential equations as it transform differential equation into algebraic. Follows from the inverse transform:

$$\frac{df(t)}{dt} \xrightarrow{\mathcal{F}} i\omega F(\omega)$$

$$\frac{df(t)}{dt} = \frac{d}{dt}\int\limits_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega =$$

$$= \int\limits_{-\infty}^{\infty} F(\omega)\frac{de^{i\omega t}}{dt}d\omega =$$

$$= \int\limits_{-\infty}^{\infty} \left[i\omega F(\omega)\right] e^{i\omega t}d\omega$$

Convolution Convolution in time - multiplication in frequency. Useful in working with Linear Time Invariant systems, discretization, optics etc. Effect of the LTI systems is described as a convolution of the input signal with its impulse response. Follows from the changing the order of integration:

$$f(t) * g(t) \xrightarrow{\mathcal{F}} F(\omega)G(\omega)$$

$$\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau e^{-i\omega t}dt =$$

Symmetry Follows from the definitions of the transform and it's inverse:

$$f(t) \xrightarrow{\mathcal{F}} F(\omega)$$

$$F(t) \xrightarrow{\mathcal{F}} f(-\omega)$$

Multiplication Multiplication in time - convolution in frequency. Follows from the symmetry:

$$f(t)g(t) \xrightarrow{\mathcal{F}} F(\omega) * G(\omega)$$

Integration Reverse of the differentiation. The term with Dirac delta function takes care of the constant of integration with a constant in time part of the function.

$$\int\limits_{-\infty}^{t} f(\tau)d\tau \xrightarrow{\mathcal{F}} \frac{F(\omega)}{i\omega} + \frac{1}{2}F(0)\delta(\omega)$$

Modulation Multiplication with a sine wave in time - shift in frequency. Basis of AM modulation. Follows from the time-shift and symmetry:

$$e^{i\omega_0 t} f(t) \xrightarrow{\mathcal{F}} F(\omega - \omega_0)$$

Conjugate signal

$$f^*(t) \xrightarrow{\mathcal{F}} F^*(-\omega)$$

1.3 Important functions

Dirac delta function Dirac delta function $\delta(t)$ is defined as:

$$\delta(t) = \begin{cases} 0, \text{ if } t = 0 \\ \infty, \text{ overwise} \end{cases}$$

and

$$\int\limits_{-\infty}^{\infty} \delta(t) dt = 1$$

So it has infinite value at 0 but its "energy" is finite

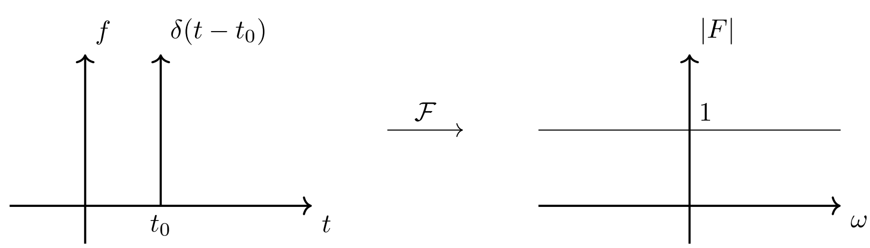
Important property of the Dirac delta function - it samples the functions it is getting integrated with:

$$\int\limits_{-\infty}^{\infty} f(t)\delta(t - t_0) dt = f(t_0)$$

Fourier transform of the Dirac delta function:

$$\delta(t - t_0) \xrightarrow{\mathcal{F}} = e^{-i\omega t_0}$$

$$\int\limits_{-\infty}^{\infty} \delta(t - t_0) e^{-i\omega t} dt = e^{-i\omega t_0}$$



Inverse Fourier transform of Dirac delta function on frequency:

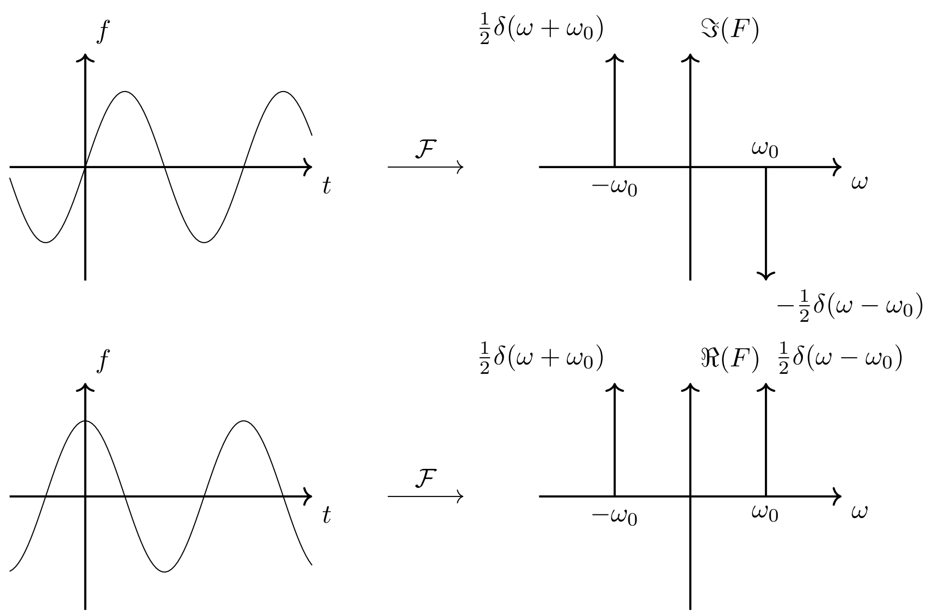
$$\delta(\omega - \omega_0) \xrightarrow{\mathcal{F}^{-1}} = e^{i\omega_0 t}$$
$$\int\limits_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{i\omega t} d\omega = e^{i\omega_0 t}$$

sin and cos Using Euler's formula $sin(\omega_0 t)$ and $cos(\omega_0 t)$ can be represented as a sum of complex exponential functions:

$$e^{ix} = cos(x) + isin(x)$$
$$sin(\omega_0 t) = \frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2i}$$
$$cos(\omega_0 t) = \frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2}$$

From these and properties of Dirac delta function the Fourier transforms of $sin(\omega_0 t)$ and $cos(\omega_0 t)$ are:

$$sin(\omega_0 t) \xrightarrow{\mathcal{F}} \frac{1}{2i} \left(\delta(\omega - \omega_0) - \delta(\omega + \omega_0)\right)$$
$$cos(\omega_0 t) \xrightarrow{\mathcal{F}} \frac{1}{2} \left(\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right)$$



Heaviside step function Heaviside step function is defined as:

$$H(t) = \begin{cases} 0, \text{ if } t < 0 \\ 1, \text{ overwise} \end{cases}$$

Also step function is an integral (antiderivative) of the Dirac delta function:

$$H(t) = \int\limits_{-\infty}^{t} \delta(\tau) d\tau$$

Fourier transform of the step function follows from its link to the Dirac delta function and integration property of the transform:

$$H(t) \xrightarrow{\mathcal{F}} \frac{1}{i\omega} + \frac{1}{2}\delta(\omega)$$

rect Rectangular function is defined as:

$$rect(t) = \begin{cases} 0, \text{ if } |t| > \frac{1}{2} \\ 1, \text{ if } |t| \leq \frac{1}{2} \end{cases}$$

Rectangular function is also a combination of Heaviside step functions:

$$rect(t) = H(t + \frac{1}{2}) - H(t - \frac{1}{2})$$

Fourier transform of the rectangular function is a *sinc* functions:

$$rect(t) \xrightarrow{\mathcal{F}} \frac{sin(\frac{\omega}{2})}{\frac{\omega}{2}} = sinc(\frac{\omega}{2})$$

$$\mathcal{F}[rect(t)] = \int\limits_{-\infty}^{\infty} rect(t) e^{-i\omega t} dt = \int\limits_{-\frac{1}{2}}^{\frac{1}{2}} 1 e^{-i\omega t} dt =$$

$$= \frac{1}{-i\omega} e^{-i\omega t}\Big|_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{e^{i\frac{\omega}{2}} - e^{-i\frac{\omega}{2}}}{2i\frac{\omega}{2}} = \frac{\sin(\frac{\omega}{2})}{\frac{\omega}{2}} = sinc(\frac{\omega}{2})$$

also

$$\frac{1}{2} rect(\frac{t}{2}) \xrightarrow{\mathcal{F}} sinc(\omega)$$

sinc sinc function is defined as:

$$sinc(t) = \begin{cases} \frac{\sin(t)}{t}, \text{ if } t \neq 0 \\ 1, \text{ if } t = 0 \end{cases}$$

Fourier transform of the sinc function os a rectangular function, which follows from the symmetry of the transform:

$$sinc(t) \xrightarrow{\mathcal{F}} \frac{1}{2} rect(\frac{\omega}{2})$$

Dirac comb Dirac comb is a function with periodic delta impulses. Multiplication of some function with the train performs a discrete sampling of the function.

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Fourier transform of the Dirac comb is a Dirac comb:

$$\delta_T(t) \xrightarrow{\mathcal{F}} \frac{2\pi}{T} \delta_{\frac{2\pi}{T}}(\omega)$$

2 Nyquist-Shannon-Kotelnikov theorem

Paraphrased from [1, 2, 3]:

> Any function $f(t)$ consisting of frequencies from 0 to f_1 (band-limited) can be reconstructed with arbitrary accuracy from discrete samples at intervals of no higher than $\frac{1}{2f_1}$.

Aliasing is an effect in discrete signals when frequencies of the initial continuous signal higher than twice the sampling frequency get shifted to the lower frequency after sampling and change the signal.

Let $f(t)$ be band limited continuous signal with its spectra $F(\omega)$ and the highest frequency in its spectra f_1. Let $s(t)$ be the $f(t)$ signal sampled with a period T, sampling frequency $f_s = \frac{1}{T}$ and angular sampling frequency $\omega_s = \frac{2\pi}{T}$. Then the $s(t)$ is the multiplication of the $f(t)$ and the Dirac comb:

$$s(t) = f(t)\delta_T(t)$$

Multiplication in time domain is a convolution in frequency domain. Convolving with a Dirac delta impulse shifts the function to the shift if teh Dirac impulse.

$$\mathcal{F}[s(t)] = \mathcal{F}[f(t)\delta_T(t)] = F(\omega) * \omega_s \delta_{\omega_s}(\omega) =$$
$$= \sum_{n=-\infty}^{\infty} \omega_s F(\omega - n\omega_s)$$

3 Signal energy

Factor of $\frac{1}{2\pi}$ is required for the normalization and holding of Plancherel's theorem for energy relation.

4 Definitions

Complex numbers:

$$z = a + ib = re^{i\phi}$$
$$z^* = a - ib$$
$$|z|^2 = zz^* = a^2 + b^2$$

Inner product of complex functions:

$$\langle f(t), g(t) \rangle = \int_{-\infty}^{\infty} f(t) g^*(t) dx$$

Convolution:

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau) g(t - \tau) d\tau$$

Dirac delta function:

$$\delta(t) = \begin{cases} \infty, \text{ if } t = 0; \\ 0, \text{ otherwise} \end{cases}$$

References

[1] H. Nyquist, “Certain topics in telegraph transmission theory,” vol. 47, no. 2, pp. 617–644. [Online]. Available: http://ieeexplore.ieee.org/document/5055024/

[2] C. Shannon, “Communication in the presence of noise,” vol. 37, no. 1, pp. 10–21. [Online]. Available: http://ieeexplore.ieee.org/document/1697831/

[3] V. Kotel’nikov, “On the transmission capacity of ’ether’ and wire in electric communications,” vol. 176, no. 7, p. 762. [Online]. Available: http://ufn.ru/ru/articles/2006/7/h/