Signal Processing

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1 Fourier transform

1.1 Definitions

If $f(t)$ is a function of time, then $F(\omega)$ is its Fourier transform $f(t) \xrightarrow{\mathcal{F}} F(\omega)$ and spectral density if:

$$F(\omega) = \mathcal{F}(f(t)) = \int\limits_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

Inverse transform $F(\omega) \xrightarrow{\mathcal{F}^{-1}} f(t)$:

$$f(t) = \mathcal{F}^{-1}(F(\omega)) = \int\limits_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega$$

1.2 Properties

Linearity Follows from the linearity of the integration:

$$af(t) + bg(t) \xrightarrow{\mathcal{F}} aF(\omega) + bG(\omega)$$

Scaling Wider in time - narrower in frequency. Follows from the change of variables:

$$f(at) \xrightarrow{\mathcal{F}} \frac{1}{|a|}F(\frac{\omega}{a})$$

$$\int_{-\infty}^{\infty} f(at)e^{-i\omega t}dt =$$

$$= \left| u = at;\ dt = \frac{1}{a}du; \text{ change of limits} \rightarrow \text{absolute value of } a \right| =$$

$$= \frac{1}{|a|}\int_{-\infty}^{\infty} f(u)e^{-i\frac{\omega}{a}u}du =$$

$$= \frac{1}{|a|}F(\frac{\omega}{a})$$

Time shift Shift in time - phase shift in frequency. Follows from the change of variables:

$$f(t-t_0) \xrightarrow{\mathcal{F}} e^{-i\omega t_0}F(\omega)$$

$$\int_{-\infty}^{\infty} f(t-t_0)e^{-i\omega t}dt = |u = t - t_0| =$$

$$= \int_{-\infty}^{\infty} f(u)e^{-i\omega u - i\omega t_0}du =$$

$$= e^{-i\omega t_0}\int_{-\infty}^{\infty} f(u)e^{-i\omega u}du = e^{-i\omega t_0}F(\omega)$$

Differentiation Derivative in time - multiplication by $i\omega$ in frequency. Useful property for solving differential equations as it transform differential equation into algebraic. Follows from the inverse transform:

$$\frac{df(t)}{dt} \xrightarrow{\mathcal{F}} i\omega F(\omega)$$

$$\frac{df(t)}{dt} = \frac{d}{dt} \int\limits_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega =$$

$$= \int\limits_{-\infty}^{\infty} F(\omega) \frac{de^{i\omega t}}{dt} d\omega =$$

$$= \int\limits_{-\infty}^{\infty} \left[i\omega F(\omega) \right] e^{i\omega t} d\omega$$

Convolution Convolution in time - multiplication in frequency. Useful in working with Linear Time Invariant systems, discretization, optics etc. Effect of the LTI systems is described as a convolution of the input signal with its impulse response. Follows from the changing the order of integration:

$$f(t) * g(t) \xrightarrow{\mathcal{F}} F(\omega)G(\omega)$$

$$\int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau e^{-i\omega t} dt =$$

Symmetry Follows from the definitions of the transform and it's inverse:

$$f(t) \xrightarrow{\mathcal{F}} F(\omega)$$

$$F(t) \xrightarrow{\mathcal{F}} f(\omega)$$

Multiplication Multiplication in time - convolution in frequency. Follows from the symmetry:

$$f(t)g(t) \xrightarrow{\mathcal{F}} F(\omega) * G(\omega)$$

Integration Reverse of the differentiation. The term with Dirac delta function takes care of the constant of integration with a constant in time part of the function.

$$\int\limits_{-\infty}^{t} f(\tau) d\tau \xrightarrow{\mathcal{F}} \frac{F(\omega)}{i\omega} + \frac{1}{2} F(0) \delta(t)$$

Modulation Multiplication with a sine wave in time - shift in frequency. Basis of AM modulation. Follows from the time-shift and symmetry:

$$e^{i\omega_0 t} f(t) \xrightarrow{\mathcal{F}} F(\omega - \omega_0)$$

Conjugate signal

$$f^*(t) \xrightarrow{\mathcal{F}} F^*(-\omega)$$

1.3 Important functions

Dirac delta function Dirac delta function $\delta(t)$ is defined as:

$$\delta(t) = \begin{cases} 0, \text{ if } t = 0 \\ \infty, \text{ overwise} \end{cases}$$

and

$$\int\limits_{-\infty}^{\infty} \delta(t) dt = 1$$

So it has infinite value at 0 but its "energy" is finite

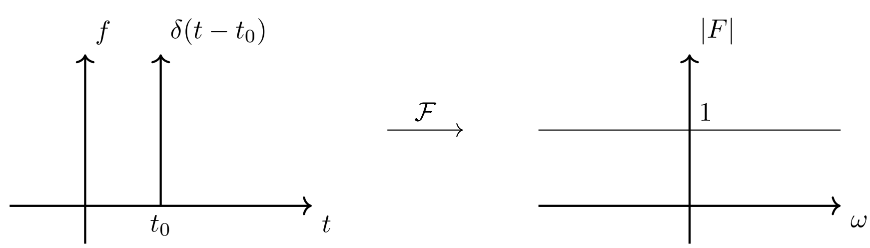
Important property of the Dirac delta function - it samples the functions it is getting integrated with:

$$\int\limits_{-\infty}^{\infty} f(t)\delta(t - t_0) dt = f(t_0)$$

Fourier transform of the Dirac delta function:

$$\delta(t - t_0) \xrightarrow{\mathcal{F}} = e^{-i\omega t_0}$$

$$\int\limits_{-\infty}^{\infty} \delta(t - t_0) e^{-i\omega t} dt = e^{-i\omega t_0}$$



Inverse Fourier transform of Dirac delta function on frequency:

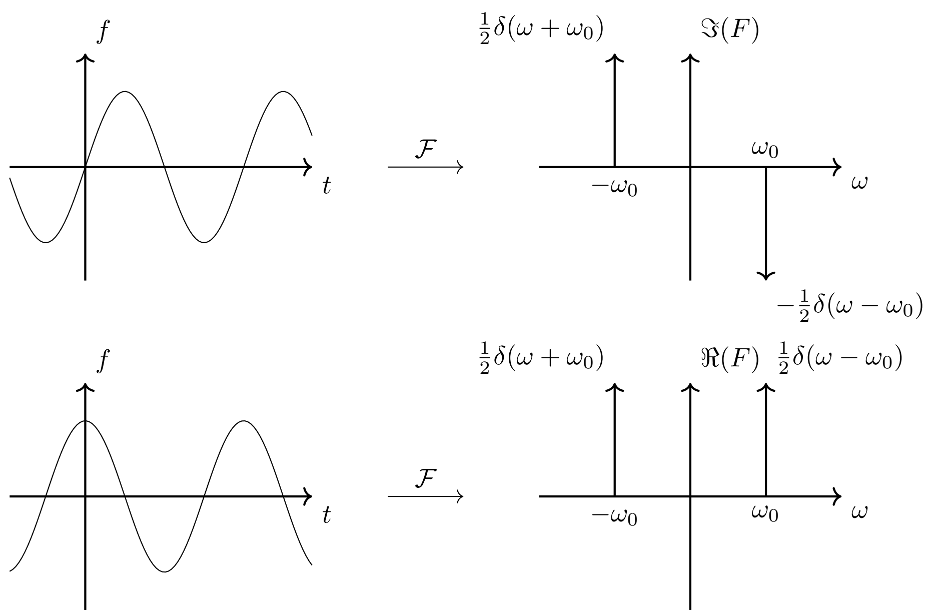
$$\delta(\omega - \omega_0) \xrightarrow{\mathcal{F}^{-1}} = e^{i\omega_0 t}$$
$$\int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{i\omega t} d\omega = e^{i\omega_0 t}$$

sin and cos Using Euler's formula $sin(\omega_0 t)$ and $cos(\omega_0 t)$ can be represented as a sum of complex exponential functions:

$$e^{ix} = cos(x) + isin(x)$$
$$sin(\omega_0 t) = \frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2i}$$
$$cos(\omega_0 t) = \frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2}$$

From these and properties of Dirac delta function the Fourier transforms of $sin(\omega_0 t)$ and $cos(\omega_0 t)$ are:

$$sin(\omega_0 t) \xrightarrow{\mathcal{F}} \frac{1}{2i} \left(\delta(\omega - \omega_0) - \delta(\omega + \omega_0)\right)$$
$$cos(\omega_0 t) \xrightarrow{\mathcal{F}} \frac{1}{2} \left(\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right)$$



Heaviside step function

rect

sinc

Grid of delta functions

2 Nyquist-Shannon-Kotelnikov theorem

Paraphrased from [1, 2, 3]:

> Any function $f(t)$ consisting of frequencies from 0 to f_1 (band-limited) can be reconstructed with arbitrary accuracy from discrete samples at intervals of no higher than $\frac{1}{2f_1}$.

3 Signal energy

Factor of $\frac{1}{2\pi}$ is required for the normalization and holding of Plancherel's theorem for energy relation.

4 Definitions

Complex numbers:

$$\begin{aligned} z &= a + ib = re^{i\phi} \\ z^* &= a - ib \\ |z|^2 &= zz^* = a^2 + b^2 \end{aligned}$$

Inner product of complex functions:

$$\langle f(t), g(t) \rangle = \int_{-\infty}^{\infty} f(t) g^*(t) dx$$

Convolution:

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau) g(t - \tau) d\tau$$

Dirac delta function:

$$\delta(t) = \begin{cases} \infty, \text{ if } t = 0; \\ 0, \text{ otherwise} \end{cases}$$

4.1 Fourier series

Functions $e^{in\omega t}$ for $n \in \mathbb{Z}$ form an orthonormal basis in Hilbert space. So Fourier transform is a basis expansion of a $f(t)$ function. And in Fourier series coefficients are the inner products of a $f(t)$ with basis functions. Fourier series expansion is:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t}, \text{ where}$$

$$c_n = \langle f(t), e^{in\omega t} \rangle = \int_{-\infty}^{\infty} f(t) e^{-in\omega t} dt$$

References

[1] H. Nyquist, "Certain topics in telegraph transmission theory," vol. 47, no. 2, pp. 617–644. [Online]. Available: http://ieeexplore.ieee.org/document/5055024/

[2] C. Shannon, "Communication in the presence of noise," vol. 37, no. 1, pp. 10–21. [Online]. Available: http://ieeexplore.ieee.org/document/1697831/

[3] V. Kotel'nikov, "On the transmission capacity of 'ether' and wire in electric communications," vol. 176, no. 7, p. 762. [Online]. Available: http://ufn.ru/ru/articles/2006/7/h/