Signal Processing

Airat Galiamov

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1 Fourier transform

1.1 Definitions

If $f(t)$ is a function of time, then $F(\omega)$ is its Fourier transform $f(t) \xrightarrow{\mathcal{F}} F(\omega)$ and spectral density if:

$$F(\omega) = \mathcal{F}(f(t)) = \int\limits_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

Inverse transform $F(\omega) \xrightarrow{\mathcal{F}^{-1}} f(t)$:

$$f(t) = \mathcal{F}^{-1}(F(\omega)) = \int\limits_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega$$

1.2 Properties

Linearity   Follows from the linearity of the integration:

$$af(t) + bg(t) \xrightarrow{\mathcal{F}} aF(\omega) + bG(\omega)$$

Scaling Wider in time - narrower in frequency. Follows from the change of variables:

$$f(at) \xrightarrow{\mathcal{F}} \frac{1}{|a|} F(\frac{\omega}{a})$$

$$\int\limits_{-\infty}^{\infty} f(at) e^{-i\omega t} dt =$$

$$= \left| u = at;\ dt = \frac{1}{a} du;\ \text{change of limits} \rightarrow \text{absolute value of } a \right| =$$

$$= \frac{1}{|a|} \int\limits_{-\infty}^{\infty} f(u) e^{-i\frac{\omega}{a} u} du =$$

$$= \frac{1}{|a|} F(\frac{\omega}{a})$$

Time shift Shift in time - phase shift in frequency. Follows from the change of variables:

$$f(t - t_0) \xrightarrow{\mathcal{F}} e^{-i\omega t_0} F(\omega)$$

$$\int\limits_{-\infty}^{\infty} f(t - t_0) e^{-i\omega t} dt = |u = t - t_0| =$$

$$= \int\limits_{-\infty}^{\infty} f(u) e^{-i\omega u - i\omega t_0} du =$$

$$= e^{-i\omega t_0} \int\limits_{-\infty}^{\infty} f(u) e^{-i\omega u} du = e^{-i\omega t_0} F(\omega)$$

Differentiation Derivative in time - multiplication by $i\omega$ in frequency. Useful property for solving differential equations as it transform differential equation

into algebraic. Follows from the inverse transform:

$$\frac{df(t)}{dt} \xrightarrow{\mathcal{F}} i\omega F(\omega)$$

$$\frac{df(t)}{dt} = \frac{d}{dt} \int\limits_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega =$$

$$= \int\limits_{-\infty}^{\infty} F(\omega)\frac{de^{i\omega t}}{dt}d\omega =$$

$$= \int\limits_{-\infty}^{\infty} [i\omega F(\omega)] \, e^{i\omega t}d\omega$$

Convolution   Convolution in time - multiplication in frequency. Useful in working with Linear Time Invariant systems, discretization, optics etc. Effect of the LTI systems is described as a convolution of the input signal with its impulse response. Follows from the changing the order of integration:

$$f(t) * g(t) \xrightarrow{\mathcal{F}} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau e^{-i\omega t}dt =$$

Symmetry   Follows from the definitions:

$$f(t) \xrightarrow{\mathcal{F}} F(\omega)$$

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Multiplication   Multiplication in time - convolution in frequency. Follows from the symmetry:

$$f(t)g(t) \xrightarrow{\mathcal{F}} F(\omega) * G(\omega)$$

Integration   Reverse of the differentiation. The term with Dirac delta function takes care of the constant of integration with a constant in time part of the function.

$$\int\limits_{-\infty}^{t} f(\tau)d\tau \xrightarrow{\mathcal{F}} \frac{F(\omega)}{i\omega} + \frac{1}{2}F(0)\delta(t)$$

Modulation   Multiplication with a sine wave in time - shift in frequency. Basis of AM modulation. Follows from the time-shift and symmetry:

$$e^{i\omega_0 t}f(t) \xrightarrow{\mathcal{F}} F(\omega - \omega_0)$$

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Conjugate signal

$$f^*(t) \xrightarrow{\mathcal{F}} F^*(-\omega)$$

1.3 Important functions

Dirac delta function

sin and cos

rect

sinc

Grid of delta functions

1.4 Fourier series

Functions $e^{in\omega t}$ for $n \in \mathbb{Z}$ form an orthonormal basis in Hilbert space. So Fourier transform is a basis expansion of a $f(t)$ function. And in Fourier series coefficients are the inner products of a $f(t)$ with basis functions. Fourier series expansion is:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t}, \text{ where}$$

$$c_n = \langle f(t), e^{in\omega t} \rangle = \int_{-\infty}^{\infty} f(t) e^{-in\omega t} dt$$

2 Nyquist-Shannon-Kotelnikov theorem

Paraphrased from [1, 2, 3]:

> Any function $f(t)$ consisting of frequencies from 0 to f_1 (band-limited) can be reconstructed with arbitrary accuracy from discrete samples at intervals of no higher than $\frac{1}{2f_1}$.

3 Signal energy

Factor of $\frac{1}{2\pi}$ is required for the normalization and holding of Plancherel's theorem for energy relation.

4 Definitions

Complex numbers:

$$z = a + ib = re^{i\phi}$$
$$z^* = a - ib$$
$$|z|^2 = zz^* = a^2 + b^2$$

Inner product of complex functions:

$$\langle f(t), g(t) \rangle = \int\limits_{-\infty}^{\infty} f(t)g^*(t)dx$$

Convolution:

$$f(t) * g(t) = \int\limits_{-\infty}^{\infty} f(\tau)g(t - \tau)d\tau$$

Dirac delta function:

$$\delta(t) = \begin{cases} \infty, \text{ if } t = 0; \\ 0, \text{ otherwise} \end{cases}$$

References

[1] H. Nyquist, "Certain topics in telegraph transmission theory," vol. 47, no. 2, pp. 617–644. [Online]. Available: http://ieeexplore.ieee.org/document/5055024/

[2] C. Shannon, "Communication in the presence of noise," vol. 37, no. 1, pp. 10–21. [Online]. Available: http://ieeexplore.ieee.org/document/1697831/

[3] V. Kotel'nikov, "On the transmission capacity of 'ether' and wire in electric communications," vol. 176, no. 7, p. 762. [Online]. Available: http://ufn.ru/ru/articles/2006/7/h/