Vectors and Functions

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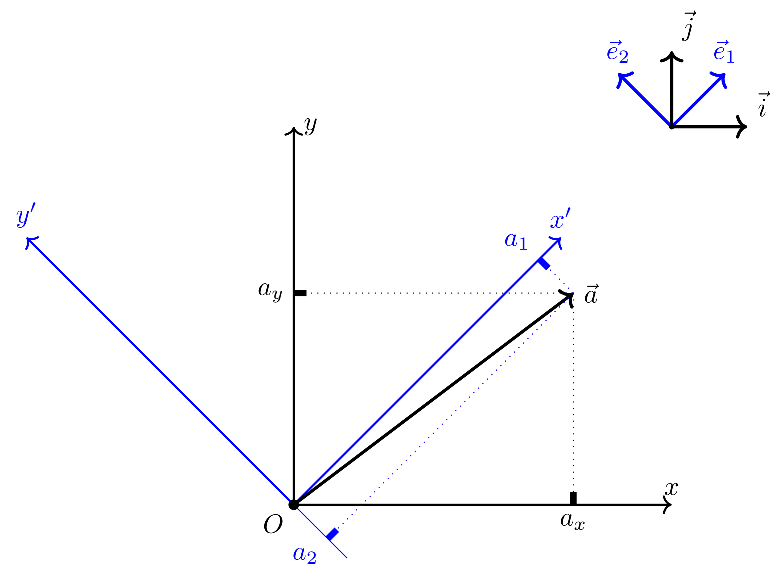
1 Vector basis



Inner product: $\vec{a} \cdot \vec{b} = \left\langle \vec{a}, \vec{b} \right\rangle = \vec{a}^T \cdot \vec{b}$.

Orthonormal basis $\vec{i}, \vec{j}$:

$$\left\langle \vec{i}, \vec{j} \right\rangle = 0$$
$$\left|\vec{i}\right|^2 = 1$$
$$\left|\vec{j}\right|^2 = 1$$

Orthonormal basis $\vec{e}_1, \vec{e}_2$:

$$\vec{e}_1 = e_{x,1} \cdot \vec{i} + e_{y,1} \cdot \vec{j} = \begin{bmatrix} e_{x,1} \\ e_{y,1} \end{bmatrix}$$
$$\vec{e}_2 = e_{x,2} \cdot \vec{i} + e_{y,2} \cdot \vec{j} = \begin{bmatrix} e_{x,2} \\ e_{y,2} \end{bmatrix}$$
$$\langle \vec{e}_1, \vec{e}_2 \rangle = [e_{x,1}\, e_{y,1}] \cdot \begin{bmatrix} e_{x,2} \\ e_{y,2} \end{bmatrix} = e_{x,1} \cdot e_{x,2} + e_{y,1} \cdot e_{y,2} = 0$$
$$|\vec{e}_n|^2 = \langle \vec{e}_n, \vec{e}_n \rangle = [e_{x,n}\, e_{y,n}] \cdot \begin{bmatrix} e_{x,n} \\ e_{y,n} \end{bmatrix} = e_{x,n}^2 + e_{y,n}^2 = 1$$

It follows from orthonormality:

$$\begin{cases} e_{x,1} = e_{y,2} \\ e_{x,2} = -e_{y,1} \end{cases} \quad \text{or} \quad \begin{cases} e_{x,1} = -e_{y,2} \\ e_{x,2} = e_{y,1} \end{cases} \tag{1}$$

Example:

$$\vec{e}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\vec{e}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Basis expansion - linear combination of basis vectors: Example for a given vector $\vec{a}$:

$$\vec{a} = a_x \cdot \vec{i} + a_y \cdot \vec{j} = \begin{bmatrix} a_x \\ a_y \end{bmatrix}$$

Changing basis to $\vec{e}_n$:

$$\vec{a} = a_1 \cdot \vec{e}_1 + a_2 \cdot \vec{e}_2 = a_1 \cdot \begin{bmatrix} e_{x,1} \\ e_{y,1} \end{bmatrix} + a_2 \cdot \begin{bmatrix} e_{x,2} \\ e_{y,2} \end{bmatrix} =$$
$$= \begin{bmatrix} a_1 \cdot e_{x,1} + a_2 \cdot e_{x,2} \\ a_1 \cdot e_{y,1} + a_2 \cdot e_{y,2} \end{bmatrix} = \begin{bmatrix} a_x \\ a_y \end{bmatrix}$$
$$a_x = a_1 \cdot e_{x,1} + a_2 \cdot e_{x,2}$$
$$a_y = a_1 \cdot e_{y,1} + a_2 \cdot e_{y,2}$$

Solve to find a_1 and a_2. Remember in orthonormal basis:

$$a_1 = \langle \vec{a}, \vec{e}_1 \rangle = [a_x\, a_y] \cdot \begin{bmatrix} e_{x,1} \\ e_{y,1} \end{bmatrix} =$$
$$= a_x \cdot e_{x,1} + a_y \cdot e_{y,1}$$
$$a_2 = \langle \vec{a}, \vec{e}_2 \rangle = [a_x\, a_y] \cdot \begin{bmatrix} e_{x,2} \\ e_{y,2} \end{bmatrix} =$$
$$= a_x \cdot e_{x,2} + a_y \cdot e_{y,2}$$

Check it is correct (remember 1):

$$\begin{aligned}
a_x &= a_1 \cdot e_{x,1} + a_2 \cdot e_{x,2} = \\
&= (a_x \cdot e_{x,1} + a_y \cdot e_{y,1}) \cdot e_{x,1} + (a_x \cdot e_{x,2} + a_y \cdot e_{y,2}) \cdot e_{x,2} = \\
&= a_x \cdot e_{x,1}^2 + a_y \cdot e_{y,1} \cdot e_{x,1} + a_x \cdot e_{x,2}^2 + a_y \cdot e_{y,2} \cdot e_{x,2} = \\
&= a_x \cdot (e_{x,1}^2 + e_{x,2}^2) + a_y \cdot (e_{y,1} \cdot e_{x,1} + e_{y,2} \cdot e_{x,2}) = \\
&= a_x \\
a_y &= ...
\end{aligned}$$

For example vector:

$$\begin{aligned}
&\vec{a} = \begin{bmatrix} 37 \\ 28 \end{bmatrix} \\
&\vec{e}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{and}\, \vec{e}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\
&a_1 = a_x \cdot e_{x,1} + a_y \cdot e_{y,1} = 37\frac{1}{\sqrt{2}} + 28\frac{1}{\sqrt{2}} = 65\frac{1}{\sqrt{2}} \\
&a_2 = a_x \cdot e_{x,2} + a_y \cdot e_{y,2} = -37\frac{1}{\sqrt{2}} + 28\frac{1}{\sqrt{2}} = -9\frac{1}{\sqrt{2}} \\
&\vec{a} = 65\frac{1}{\sqrt{2}} \cdot \vec{e}_1 - 9\frac{1}{\sqrt{2}}\vec{e}_2
\end{aligned}$$

	Vectors	Complex Functions
Elements	$\vec{a} = \begin{bmatrix} a_x \\ a_y \\ \vdots \\ a_z \end{bmatrix}$	$\mathbf{F}(x)$
Inner product	$\vec{a} \cdot \vec{b} = \left\langle \vec{a}, \vec{b} \right\rangle = \vec{a}^T \cdot \vec{b} = a_x \cdot b_x + \ldots + a_z \cdot b_z = \sum_{i=x}^{z} a_i \cdot b_i$	$\langle \mathbf{F} \vert \mathbf{G} \rangle = \int_{-\infty}^{\infty} \mathbf{F} \cdot \mathbf{G}^* dx$
Linear	$(k\vec{a} + t\vec{b}) \cdot \vec{c} = k(\vec{a} \cdot \vec{c}) + t(\vec{b} \cdot \vec{c})$	$\langle k\mathbf{F} + t\mathbf{G} \vert \mathbf{H} \rangle = k \langle \mathbf{F} \vert \mathbf{H} \rangle + t \langle \mathbf{G} \vert \mathbf{H} \rangle$
Orthogonal	$\vec{a} \perp \vec{b} \quad \text{if } \vec{a} \cdot \vec{b} = 0$	$\mathbf{F} \perp \mathbf{G} \quad \text{if } \langle \mathbf{F} \vert \mathbf{G} \rangle = 0$
Norm^2	$\vert\vec{a}\vert^2 = \vec{a} \cdot \vec{a}$	$\vert\mathbf{F}\vert^2 = \langle \mathbf{F} \vert \mathbf{F} \rangle$
Normalized	$\vert\vec{a}\vert^2 = 1$	$\vert\mathbf{F}\vert^2 = 1$
Orthonormal basis	$\vec{e}_n; \quad \vert\vec{e}_n\vert^2 = 1; \quad \vec{e}_n \cdot \vec{e}_m = 0, \quad \text{if } n \neq m$	$\mathbf{\Phi}_n; \quad \vert\mathbf{\Phi}_n\vert^2 = 1; \quad \langle \mathbf{\Phi}_n \vert \mathbf{\Phi}_m \rangle = 0, \quad \text{if } n \neq m$
Basis expansion, function series	$\vec{a} = c_1\vec{e}_1 + \ldots + c_n\vec{e}_n = \sum_{i=1}^{n} c_i\vec{e}_i$	$\mathbf{F}(x) = c_1\mathbf{\Phi}_1 + \ldots + c_n\mathbf{\Phi}_n = \sum_{i=1}^{n} c_i\mathbf{\Phi}_i$
Series coefficients	$c_i = \vec{a} \cdot \vec{e}_i$	$c_i = \langle \mathbf{F} \vert \mathbf{\Phi}_i \rangle = \int_{-\infty}^{\infty} \mathbf{F} \cdot \mathbf{\Phi}_i^* dx$
	$\begin{aligned} \vec{a} \cdot \vec{e}_i \quad &= (c_1\vec{e}_1 + \ldots + c_n\vec{e}_n) \cdot \vec{e}_i = \\ &= c_1\vec{e}_1 \cdot \vec{e}_i + \ldots + c_i\vec{e}_i \cdot \vec{e}_i + \ldots + c_n\vec{e}_n \cdot \vec{e}_i = \\ &= c_i \end{aligned}$	$\begin{aligned} \langle \mathbf{F} \vert \mathbf{\Phi}_i \rangle \quad &= \langle (c_1\mathbf{\Phi}_1 + \ldots + c_n\mathbf{\Phi}_n) \vert \mathbf{\Phi}_i \rangle = \\ &= c_1 \langle \mathbf{\Phi}_1 \vert \mathbf{\Phi}_i \rangle + \ldots + c_i \langle \mathbf{\Phi}_i \vert \mathbf{\Phi}_i \rangle + \ldots c_n \langle \mathbf{\Phi}_n \vert \mathbf{\Phi} \\ &= c_i \end{aligned}$